Communications in
Analysis and Geometry
Volume 30, Number 10, 2267–2283, 2022

Isoperimetric relations for inner parallel bodies

M. A. Hernández Cifre and E. Saorín Gómez

We analyze aspects of the behavior of the family of inner parallel bodies of a convex body for the isoperimetric quotient and deficit of arbitrary quermassintegrals. By means of technical boundary properties of the so-called form body of a convex body and similar constructions for inner parallel bodies, we point out an erroneous use of a relation between the latter bodies in two different works. We correct these results, limiting them to convex bodies having a very precise boundary structure.

1. Introduction

Inner parallel bodies of convex bodies have been object of recent studies with different flavors [6, 12–17]. More classical existing literature on them (e.g. [1, 3, 4, 9, 10, 21]) along with their role in the proofs of fundamental results in the theory of convex bodies, make inner parallel bodies essential objects not only within classical Convex Geometry (see [22, Section 7.5]), but also in other related fields (see e.g. [7, 11, 20] and the references in [22, Note 3 for Section 3.1] and [6]).

In [6] and [16] the authors study the behavior of the isoperimetric quotient for the family of inner parallel bodies, and provide a lower bound for the perimeter of the inner parallel bodies of a convex body, respectively. However, both articles make an erroneous use of the relation $K \subset K_\lambda + |\lambda|\, K_\lambda^{\mathrm{f}}$ between the inner parallel bodies K_λ of a convex body, their form bodies K_λ^{f} and the original convex body K (see (2.4) and Section 2 for the proper definitions). This relation, which holds, for example, under technical properties

This research is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Science and Technology Agency of the Región de Murcia. The work is supported by PID2021-124157NB-I00, funded by MCIN/AEI/10.13039/501100011033/ “ERDF A way of making Europe”, and Fundación Séneca projects 19901/GERM/15 and 21899/PI/22.

of the boundary of the involved convex bodies (see (2.3)), is, however, not true without further conditions. To the best of the authors' knowledge, a full characterization of the conditions under which the above inclusion holds is not known.

The purpose of this paper is twofold. On the one hand, we describe the error contained in the two mentioned references, providing examples proving these have to be adjusted with further hypotheses. On the other hand, we provide alternative proofs to those results under suitable restrictions of the boundaries of the involved convex bodies, and further, we extend the results concerning inner parallel bodies in [6] to a more general setting.

The paper is organized as follows. In Section 2 we introduce the notions and basic results, which are needed throughout the paper. In Section 3 we analyze the problems in the proof of the main result in [6], providing an example where the used methods do not hold. In Section 4 we obtain new results concerning the behavior of the isoperimetric quotient and deficit under assumptions on the boundary of the involved convex bodies. Finally in Section 5 we point out an error -of the same spirit of the one found in [6]- in one of the proofs of [16] and discuss it.

2. Background

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., nonempty compact convex subsets of the Euclidean space $\mathbb{R}^n$, and let $\mathcal{K}^n_n$ the subset of convex bodies having interior points. A convex body K is called *regular* if all its boundary points are regular, i.e., the supporting hyperplane to K at any boundary point is unique. Let B^n be the n-dimensional Euclidean unit ball and $\mathbb{S}^{n-1}$ the corresponding unit sphere. The volume of a measurable set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$, and the measure of its boundary, i.e., its surface area (also called perimeter), is represented by $\mathrm{S}(M)$. Furthermore, the closure of M is denoted by $\mathrm{cl}\, M$. For $K \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1}$, $h(K,u) = \sup\bigl\{\langle x, u\rangle : x \in K\bigr\}$ stands for the *support function* of K (see e.g. [22, Section 1.7]).

The vectorial or Minkowski addition of two sets $K, L \subset \mathbb{R}^n$ is given by

$$K + L = \{x + y : x \in K,\, y \in L\},$$

whereas the Minkowski difference of $K, L \subset \mathbb{R}^n$ is given by

$$K \sim L = \{x \in \mathbb{R}^n \,:\, x + L \subset K\}.$$

Note that $(K \sim L) + L \subset K$, and the inequality may be strict.

Let $K \in \mathcal{K}^n$ and $E \in \mathcal{K}^n_n$. The *inradius* $\mathrm{r}(K;E)$ of K relative to E is the radius of one of the largest dilations of E which fits inside K, i.e.,

$$\mathrm{r}(K;E) = \sup\{r \geq 0 : \exists\, x \in \mathbb{R}^n \text{ with } x + rE \subset K\}.$$

For $-\mathrm{r}(K;E) \leq \lambda \leq 0$ the *inner parallel body of K at distance $|\lambda|$* is the Minkowski difference of K and $|\lambda|\, E$, i.e.,

$$K_\lambda := K \sim |\lambda|\, E = \bigl\{x \in \mathbb{R}^n : x + |\lambda|\, E \subset K\bigr\} \in \mathcal{K}^n.$$

Notice that if $E = B^n$, then $K_{-\mathrm{r}(K;B^n)}$ is the set of incenters of K, which is usually called the *kernel* of K, and its dimension is strictly less than n (see [2, p. 59]). Equivalently (see [22, Section 3.1]), the inner parallel body K_λ of K, $-\mathrm{r}(K;E) \leq \lambda \leq 0$, can be defined using the support functions of K and E as

$$K_\lambda = \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(K,u) - |\lambda|\, h(E,u), u \in \mathbb{S}^{n-1}\bigr\}. \tag{2.1}$$

A vector $u \in \mathbb{S}^{n-1}$ is a 0*-extreme normal vector* (or just *extreme vector*) of K if it cannot be written as a linear combination of two linearly independent normal vectors at one and the same boundary point of K. We denote by $\mathcal{U}(K)$ the set of 0-extreme normal vectors of K, which play a key role in the study of convex bodies. Indeed, the dual of the Krein-Milman theorem (see e.g. [22, Corollary 1.4.5]) yields

$$K = \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(K,u), u \in \mathcal{U}(K)\bigr\}, \tag{2.2}$$

and thus, the inner parallel bodies of K can be expressed as (cf. (2.1))

$$K_\lambda = \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(K,u) - |\lambda|\, h(E,u), u \in \mathcal{U}(K)\bigr\}$$

for $-\mathrm{r}(K;E) \leq \lambda \leq 0$.

The (relative) *form body* of a convex body $K \in \mathcal{K}^n_n$ *with respect to* $E \in \mathcal{K}^n_n$, denoted by K^{f}, is defined as (see e.g. [3])

$$K^{\mathrm{f}} = \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(E,u), u \in \mathcal{U}(K)\bigr\}.$$

Note that K^{f} strongly depends on the body E. Nevertheless, for the sake of simplicity, we omit E in the notation.

The form body of $K \in \mathcal{K}^n_n$ (with respect to an arbitrary $E \in \mathcal{K}^n_n$) is always a tangential body of E. We recall that a convex body K containing a convex body E, is called a *tangential body* of E, if through each boundary point of K there exists a support hyperplane to K that also supports E. Note that if K is a tangential body of E, then $\mathrm{r}(K;E)=1$.

There is also a very close connection between inner parallel bodies and tangential bodies. The next result explains it.

Theorem 2.1 ([22, Lemma 3.1.14]). *Let $K,E\in\mathcal{K}^n_n$ and let $-\mathrm{r}(K;E)<\lambda<0$. Then K_λ is homothetic to K if and only if K is homothetic to a tangential body of E.*

Remark 2.2. The proof of Theorem 2.1 shows that if K is a tangential body of E then $K_\lambda=(1+\lambda)K$ for $-1<\lambda\leq 0$.

In the following, we collect some standard properties of inner parallel bodies, form bodies and extreme vectors, together with other relations through the Minkowski sum, which will be needed later on. There exist further relations, in a stronger form, through the so-called Riemann-Minkowski integral, for which we refer to [5] and [21, Lemma 3.2].

Lemma 2.3. *Let $K,L\in\mathcal{K}^n$ and $E\in\mathcal{K}^n_n$. The following properties hold:*

(i) $\mathcal{U}(K_\lambda)\subset\mathcal{U}(K)$ *for $-\mathrm{r}(K;E)<\lambda\leq 0$ (see [21, Lemma 4.5]).*
(ii) *If $K\in\mathcal{K}^n_n$ and E is regular then $\mathrm{cl}\,\mathcal{U}(K)=\mathcal{U}\big(K^{\mathrm{f}}\big)$ (see [21, Lemma 2.6] and [12, Lemma 2.1]).*
(iii) $\mathcal{U}(K)\cup\mathcal{U}(L)\subset\mathcal{U}(K+L)=\mathcal{U}(K+\mu L)$ *for $\mu>0$. The inclusion may be strict (see [21, Lemma 2.4] and [14, Lemma 3.1]).*
(iv) $K_\lambda+|\lambda|\,E\subset K$ *for $-\mathrm{r}(K;E)\leq\lambda\leq 0$ (see [21, (4.1)]).*
(v) *If $K\in\mathcal{K}^n_n$ then $K_\lambda+|\lambda|\,K^{\mathrm{f}}\subset K$ for $-\mathrm{r}(K;E)\leq\lambda\leq 0$ (see [21, Lemma 4.8]).*

Remark 2.4. The equality cases in Lemma 2.3 (iv) and (v) are well known:

(i) Equality holds in (iv) for all $-\mathrm{r}(K;E)\leq\lambda\leq 0$ if and only if $K=K_{-\mathrm{r}(K;E)}+\mathrm{r}(K;E)E$ (see [21, p. 81]).

(ii) If E is regular, equality holds in (v) for all $-\mathrm{r}(K;E)\leq\lambda\leq 0$ if and only if K is a tangential body of $K_{-\mathrm{r}(K;E)}+\mathrm{r}(K;E)E$ satisfying $\mathcal{U}(K)=\mathcal{U}\big(K_\lambda+K^{\mathrm{f}}\big)$ for all $-\mathrm{r}(K;E)\leq\lambda\leq 0$ (see [14, Theorem 2.2]).

Let $K,E\in\mathcal{K}^n_n$. From now on we will write $K^{\mathrm{f}}_\lambda=(K_\lambda)^{\mathrm{f}}$ to denote the form body of the inner parallel body of K at distance $|\lambda|$, $-\mathrm{r}(K;E)<\lambda\leq 0$.

The following counterpart of the relations contained in Lemma 2.3 (v), can be found in [21, Corollary to Lemma 4.8] (see also Lemma 2.3 (ii)).

Proposition 2.5. *Let $K, E \in \mathcal{K}^n_n$, with E regular. Assume that, for some $-\mathrm{r}(K;E) < \lambda < 0$, the relation*

$$\mathcal{U}\big(K^{\mathrm{f}}_\lambda\big) = \mathcal{U}\big(K_\lambda + K^{\mathrm{f}}_\lambda\big) \tag{2.3}$$

holds. Then,

$$K \subset K_\lambda + |\lambda|\, K^{\mathrm{f}}_\lambda. \tag{2.4}$$

For $n = 2$ there is equality in (2.4) *for all $K \in \mathcal{K}^2_2$.*

Condition (2.3) deserves further observations. On the one hand, it is similar to the identity $\mathcal{U}\big(K_\lambda + K^{\mathrm{f}}\big) = \mathcal{U}\big(K^{\mathrm{f}}\big)$, which is a direct consequence of the relation $\mathcal{U}\big(K_\lambda + K^{\mathrm{f}}\big) = \mathcal{U}(K)$ needed in [14, Theorem 2.2] (see Remark 2.4), together with Lemma 2.3 (ii) and (iii). However, since examples of convex bodies for which $\mathcal{U}(K_\lambda) \subsetneq \mathcal{U}(K)$ are easily constructed, both conditions are different. On the other hand, Lemma 2.3 (iii) yields $\mathcal{U}\big(K^{\mathrm{f}}_\lambda\big) \subset \mathcal{U}\big(K_\lambda + K^{\mathrm{f}}_\lambda\big)$; however, the inclusion may be strict (see Section 3).

3. Convex bodies not satisfying the inclusion $\boldsymbol{K \subset K_\lambda + |\lambda|\, K^{\mathrm{f}}_\lambda}$

Let $K \in \mathcal{K}^n$ be a convex body. In [6] the authors study the isoperimetric quotient $I(K_{-\lambda}) := \mathrm{vol}(K_{-\lambda})/\mathrm{S}(K_{-\lambda})^{n/(n-1)}$ of the family of inner parallel bodies $K_{-\lambda}$, $0 \le \lambda < \mathrm{r}(K;B^n)$, when $E = B^n$, and analyze the behavior of the function $I(\lambda) = I(K_{-\lambda})$: in [6, Theorem 1] they prove that the isoperimetric quotient function $I(\lambda)$ is non-increasing in $0 \le \lambda < \mathrm{r}(K;B^n)$ *for all convex bodies*. However, as we mentioned in the introduction, the proof of this result is erroneous.

The main idea of the proof is to bound from below the quotient defining $I(\lambda)$. To this end, the numerator $\mathrm{vol}(K_{-\lambda})$ is bounded using Lemma 2.3 (iv) and the property $(K_{-\lambda})_{-\mu} = K_{-\lambda-\mu}$ for $0 \le \lambda, \mu \le \lambda + \mu < \mathrm{r}(K;B^n)$ (see [22, (3.17)]). More precisely, and following the notation in [6], for $0 \le t \le t_0 < \mathrm{r}(K;B^n)$,

$$\mathrm{vol}(K_{-t}) \ge \mathrm{vol}\big(K_{-t_0} + |t - t_0|\, B^n\big).$$

In order to bound the denominator $\mathrm{S}(K_{-t})$, the authors make use of the monotonicity of the surface area applied to the content (2.4), namely,

$$K_{-t} \subset K_{-t_0} + |t - t_0|\, K^{\mathrm{f}}_{-t_0}$$

for $0 \le t \le t_0 < \mathrm{r}(K; B^n)$. However, this inclusion is not true without further conditions (as, for instance, the equality $\mathcal{U}\big(K^{\mathrm{f}}_{-t_0}\big) = \mathcal{U}\big(K_{-t_0} + K^{\mathrm{f}}_{-t_0}\big)$, see Proposition 2.5). Indeed, in Proposition 3.3 we prove that the content $K \subset K_\lambda + |\lambda|\, K^{\mathrm{f}}_\lambda$, $-\mathrm{r}(K; E) < \lambda \le 0$, is not valid in its full generality.

For $K \in \mathcal{K}^n_n$ and $\mu \ge 0$, we consider the following convex body:

$$K(\mu) := \big\{x \in \mathbb{R}^n : \langle x, u\rangle \le h(K, u) + \mu h(E, u), u \in \mathcal{U}(K)\big\}. \tag{3.1}$$

This construction appeared already in [16, 17, 21]. Indeed, in [21] the following result was proved.

Proposition 3.1 ([21]). *Let $K, E \in \mathcal{K}^n_n$ and let $\mu \ge 0$. Then*

(i) $K + \mu E \subset K + \mu K^{\mathrm{f}} \subset K(\mu)$.
(ii) $\mathrm{r}\big(K(\mu); E\big) = \mu + \mathrm{r}(K; E)$.
(iii) *For* $-\mathrm{r}(K; E) - \mu \le \lambda \le 0$ *we have*

$$K(\mu)_\lambda = \begin{cases} K(\mu + \lambda) & \text{for } -\mu \le \lambda \le 0,\\ K_{\lambda+\mu} & \text{for } -\mathrm{r}(K; E) - \mu \le \lambda \le -\mu. \end{cases}$$

We will also need the following additional result.

Lemma 3.2. *Let $K \in \mathcal{K}^n$ and let $E \in \mathcal{K}^n_n$ be regular. Then, for any $\mu \ge 0$,*

$$\mathcal{U}\big(K(\mu)\big) = \mathrm{cl}\,\mathcal{U}(K). \tag{3.2}$$

Proof. It is enough to observe that, from the definition of $K(\mu)$, it follows that $K(\mu)$ is the form body of K with respect to $E' = K + \mu E$. Since E is regular, so is $K + \mu E = E'$, and hence the identity follows from Lemma 2.3 (ii). □

We are now in a position to prove the announced non-validity of the inclusion (2.4) without further assumptions. The next result will be proved when $E = B^n$, although it holds true for any regular $E \in \mathcal{K}^n_n$.

Proposition 3.3. *There exists $K \in \mathcal{K}^n_n$ such that $K \supsetneq K_\lambda + |\lambda|\, K^{\mathrm{f}}_\lambda$ for some $-\mathrm{r}(K; B^n) < \lambda < 0$.*

Proof. For any $K \in \mathcal{K}_n^n$ and $\mu \geq 0$, since $K = K(\mu)_{-\mu}$ is an inner parallel body of $K(\mu)$ (see Proposition 3.1 (iii)), Proposition 3.1 (i) yields

$$K(\mu)_{-\mu} + \mu K(\mu)^{\mathrm{f}}_{-\mu} = K + \mu K^{\mathrm{f}} \subset K(\mu). \tag{3.3}$$

So, we have to find a convex body K (and $\mu > 0$) such that the above inclusion is strict.

If we assume, to the contrary, that $K(\mu)_{-\mu} + \mu K(\mu)^{\mathrm{f}}_{-\mu} = K(\mu)$ for all $K \in \mathcal{K}_n^n$ and $\mu \geq 0$, then we have, in particular, that $K(\mu) = K + \mu K^{\mathrm{f}}$. Then, since $\mathcal{U}(K^{\mathrm{f}}) = \mathrm{cl}\,\mathcal{U}(K) \supset \mathcal{U}(K)$ (Lemma 2.3 (ii)), we can use Lemma 2.3 (iii) and Lemma 3.2 to get

$$\mathcal{U}(K^{\mathrm{f}}) = \mathcal{U}(K) \cup \mathcal{U}(K^{\mathrm{f}}) \subset \mathcal{U}(K + K^{\mathrm{f}}) = \mathcal{U}(K(\mu)) = \mathrm{cl}\,\mathcal{U}(K) = \mathcal{U}(K^{\mathrm{f}}).$$

Hence

$$\mathcal{U}(K^{\mathrm{f}}) = \mathcal{U}(K + K^{\mathrm{f}}). \tag{3.4}$$

Now, it will be enough to find a convex body for which the latter equality does not hold. The following polytope P serves for the purpose (see Figure 1; note that it coincides with the polytope $P(12)$ used in [17, Proposition 5.1]). Let

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \begin{array}{rcl} \pm 12x_1 + 35x_3 & \leq & 432, \\ \pm 12x_2 + 5x_3 & \leq & 60, \\ x_3 & \geq & 0 \end{array} \right\}. \tag{3.5}$$

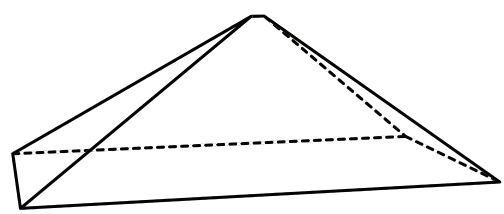

Figure 1: A polytope such that $P(\mu) \supsetneq P(\mu)_{-\mu} + \mu P(\mu)^{\mathrm{f}}_{-\mu}$ for $\mu > 0$.

On the one hand, since the extreme vectors of P are just the unit outer normal vectors to its facets, namely,

$$\mathcal{U}(P)=\left\{\left(\pm\frac{12}{37},0,\frac{35}{37}\right)^{\intercal},\left(0,\pm\frac{12}{13},\frac{5}{13}\right)^{\intercal},(0,0,-1)^{\intercal}\right\},$$

and since $P^{\mathrm{f}}=\bigl\{x\in\mathbb{R}^n:\langle x,u\rangle\leq 1, u\in\mathcal{U}(P)\bigr\}$, we have

$$P^{\mathrm{f}}=\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\in\mathbb{R}^3: \begin{array}{rcl}\pm 12x_1+35x_3 & \leq & 37,\\ \pm 12x_2+5x_3 & \leq & 13,\\ x_3 & \geq & -1,\end{array}\right\}$$

and also $\mathcal{U}\bigl(P^{\mathrm{f}}\bigr)=\mathcal{U}(P)$. On the other hand, the polytope $P+P^{\mathrm{f}}$ has a facet with unit outer normal vector $(0,0,1)^{\intercal}\notin\mathcal{U}\bigl(P^{\mathrm{f}}\bigr)$, which arises from the edge of P determined by the straight line $\{x_2=0,x_3=12\}$, and the edge of P^{f} corresponding to the line $\{x_1=0,x_3=37/35\}$. Therefore $\mathcal{U}\bigl(P^{\mathrm{f}}\bigr)\subsetneq\mathcal{U}\bigl(P+P^{\mathrm{f}}\bigr)$, which contradicts (3.4) and concludes the proof. □

Remark 3.4. Unfortunately, the previous polytope P does not provide us with a counterexample for the non-increasing behavior of the isoperimetric quotient function $I(\lambda)$. Thus, except for particular families of convex bodies (see Section 4) it is not known yet whether the isoperimetric quotient function is non-increasing for an arbitrary convex body.

4. Isoperimetric quotients and deficits

Unlike the authors in [6], we shall consider the family of inner parallel bodies defined in the range $-\mathrm{r}(K;E)\leq\lambda\leq 0$, which will reverse the behavior of the isoperimetric quotient function (in [6] the range is $0\leq\lambda\leq\mathrm{r}(K;B^n)$ what makes the behavior of $I(\lambda)$ non-increasing). Furthermore, since we will also work with the isoperimetric deficit, we will consider the isoperimetric quotient in the usual way, namely, $\mathrm{S}(K_\lambda)^n/\mathrm{vol}(K_\lambda)^{n-1}$, in order to compare the behavior in both cases.

In this section we obtain new results concerning the behavior of the isoperimetric quotient (and also of the isoperimetric deficit) under assumptions on the boundary of the convex bodies involved. The first condition we can impose is, actually, the boundary condition necessary to validate the proof of Theorem 1 in [6], namely:

Theorem 4.1 ([6, Theorem 1] revised). *Let $K \in \mathcal{K}^n_n$. If*

$$\mathcal{U}\big(K_\lambda^{\mathrm{f}}\big) = \mathcal{U}\big(K_\lambda + K_\lambda^{\mathrm{f}}\big) \quad \textit{for } -\mathrm{r}(K;B^n) \le \lambda \le 0,$$

then the isoperimetric quotient $\mathrm{S}(K_\lambda)^n/\mathrm{vol}(K_\lambda)^{n-1}$ *is non-increasing.*

The proof of this result is exactly the proof of [6, Theorem 1], where the use of (2.4) is justified by assuming (2.3).

Next we prove that under different conditions to (2.3), the isoperimetric quotient function is also non-decreasing. In fact, we will get a more general result for all the quermassintegrals of a convex body K (relative to an arbitrary $E \in \mathcal{K}^n_n$), which we define next.

Given $K \in \mathcal{K}^n$ and $E \in \mathcal{K}^n_n$, the so-called *relative Steiner formula* states that the volume of the Minkowski addition $K + \mu E$, $\mu \ge 0$, is a polynomial of degree n in μ,

$$\mathrm{vol}(K + \mu E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\mu^i.$$

The coefficients $\mathrm{W}_i(K;E)$ are called *(relative) quermassintegrals* of K, and they are just a special case of the more general *mixed volumes*, for which we refer to [22, s. 5.1]. In particular, we have $\mathrm{W}_0(K;E) = \mathrm{vol}(K)$ and $\mathrm{W}_n(K;E) = \mathrm{vol}(E)$. Moreover, if $E = B^n$, the polynomial in the right hand side becomes the classical *Steiner polynomial*, see [23], and $n\mathrm{W}_1(K;B^n) = \mathrm{S}(K)$ is the usual surface area of K.

Let $\mathrm{W}_i(\lambda) := \mathrm{W}_i(K_\lambda;E)$ for $-\mathrm{r}(K;E) \le \lambda \le 0$. From the concavity of the family of inner parallel bodies (see [22, Lemma 3.1.13]) and the general Brunn-Minkowski theorem for relative quermassintegrals (see e.g. [22, Theorem 7.4.5]), we obtain

$$'\mathrm{W}_i(\lambda) \ge \mathrm{W}_i'(\lambda) \ge (n-i)\mathrm{W}_{i+1}(\lambda)$$

for $i = 0, \dots, n-1$ and for $-\mathrm{r}(K;E) \le \lambda \le 0$. Here $'\mathrm{W}_i$ and W_i' denote, respectively, the left and right derivatives of the function $\mathrm{W}_i(\lambda)$, and for $\lambda = -\mathrm{r}(K;E)$ (respectively, $\lambda = 0$) only the right (left) derivative is considered (W_i' will also denote the full derivative of W_i when the function is differentiable). In [12] the following definition was introduced.

Definition 4.2. Let $E \in \mathcal{K}^n_n$ and let $0 \le p \le n-1$. A convex body $K \in \mathcal{K}^n$ *belongs to the class* $\mathcal{R}_p$ if, for all $0 \le i \le p$ and for $-\mathrm{r}(K;E) \le \lambda \le 0$,

$$'\mathrm{W}_i(\lambda) = \mathrm{W}_i'(\lambda) = (n-i)\mathrm{W}_{i+1}(\lambda). \tag{4.1}$$

Notice that the class $\mathcal{R}_p$ depends on the fixed convex body E. Nevertheless, for the sake of simplicity, we will also omit E in the notation.

Since the volume is always differentiable with respect to λ and $\mathrm{vol}'(\lambda) = n\mathrm{W}_1(\lambda)$ (see e.g. [1, 18]), the class $\mathcal{R}_0$ consists of all convex bodies, i.e., $\mathcal{R}_0 = \mathcal{K}^n$. From the definition we get $\mathcal{R}_p \supset \mathcal{R}_{p+1}$, $p = 0, \ldots, n-2$, and all these inclusions are strict (particular tangential bodies show it; see [12]). The problem of determining the convex bodies belonging to the class $\mathcal{R}_p$ was studied by Bol [1] and Hadwiger [9] in the 3-dimensional case when $E = B^n$. In [12] and [15] the general classes $\mathcal{R}_{n-1}$ and $\mathcal{R}_{n-2}$, respectively, were characterized. The cases $p = 1, \ldots, n-3$ remains open.

Finally, we recall the following inequalities for quermassintegrals, which can be deduced from the well-known Aleksandrov-Fenchel inequalities for mixed volumes (see e.g. [22, Sections 7.3 and 7.4]). They motivate and are also needed to prove our results. Let $K \in \mathcal{K}^n$ and $E \in \mathcal{K}^n_n$. Then

$$\mathrm{W}_i(K;E)\mathrm{W}_j(K;E) \geq \mathrm{W}_k(K;E)\mathrm{W}_l(K;E), \quad 0 \leq l < i \leq j < k \leq n, \tag{4.2}$$

and

$$\mathrm{W}_j(K;E)^{n-i} \geq \mathrm{W}_i(K;E)^{n-j}\mathrm{vol}(E)^{j-i}, \quad 0 \leq i \leq j \leq n. \tag{4.3}$$

Note that the last inequality, for $E = B^n$ and $i = 0$, $j = 1$, yields the well-known isoperimetric inequality $\mathrm{S}(K)^n \geq n^n \mathrm{vol}(B^n)\mathrm{vol}(K)^{n-1}$.

4.1. Non-decreasing isoperimetric quotients

Inspired by the families of inequalities (4.3), we consider the isoperimetric quotient (up to the constant $\mathrm{vol}(E)^{j-i}$) $\mathrm{W}_j(K;E)^{n-i}/\mathrm{W}_i(K;E)^{n-j}$ and study its behavior for the family of inner parallel bodies.

We start by proving that for convex bodies lying in the suitable class $\mathcal{R}_p$, the above isoperimetric quotients for inner parallel bodies are non-increasing in the range $-\mathrm{r}(K;E) < \lambda \leq 0$.

Proposition 4.3. *Let $0 \leq i < j < n$, and let $K \in \mathcal{R}_j$ and $E \in \mathcal{K}^n_n$. Then the isoperimetric quotient function $\mathrm{W}_j(\lambda)^{n-i}/\mathrm{W}_i(\lambda)^{n-j}$ is non-increasing for $-\mathrm{r}(K;E) < \lambda \leq 0$. In particular,*

$$\frac{\mathrm{W}_j(K_\lambda;E)^{n-i}}{\mathrm{W}_i(K_\lambda;E)^{n-j}} \geq \frac{\mathrm{W}_j(K;E)^{n-i}}{\mathrm{W}_i(K;E)^{n-j}}.$$

Proof. We consider the function

$$\phi(\lambda) := \frac{\mathrm{W}_j(\lambda)^{n-i}}{\mathrm{W}_i(\lambda)^{n-j}} \quad \text{for} \quad -\mathrm{r}(K;E) < \lambda \leq 0.$$

Taking derivatives with respect to λ, and since $K \in \mathcal{R}_j \subsetneq \mathcal{R}_i$ because $i < j$, we can use the relations $\mathrm{W}_i'(\lambda) = (n-i)\mathrm{W}_{i+1}(\lambda)$ and $\mathrm{W}_j'(\lambda) = (n-j)\mathrm{W}_{j+1}(\lambda)$ to get

$$\begin{aligned}\phi'(\lambda) &= \frac{\mathrm{W}_j(\lambda)^{n-i-1}}{\mathrm{W}_i(\lambda)^{n-j+1}}\Big[(n-i)\mathrm{W}_i(\lambda)\mathrm{W}_j'(\lambda) - (n-j)\mathrm{W}_j(\lambda)\mathrm{W}_i'(\lambda)\Big]\\ &= \frac{(n-i)(n-j)\mathrm{W}_j(\lambda)^{n-i-1}}{\mathrm{W}_i(\lambda)^{n-j+1}}\Big[\mathrm{W}_i(\lambda)\mathrm{W}_{j+1}(\lambda) - \mathrm{W}_j(\lambda)\mathrm{W}_{i+1}(\lambda)\Big].\end{aligned}$$

The Aleksandrov-Fenchel inequalities (4.2) yield $\phi'(\lambda) \leq 0$, i.e., $\phi(\lambda)$ is non-increasing when $-\mathrm{r}(K;E) < \lambda \leq 0$. □

Note that if K is a tangential body of E then, since $K_\lambda = (1+\lambda)K$ (Remark 2.2) and the i-th quermassintegral is homogeneous of degree $n-i$ in its first argument (see e.g. [8, Theorem 6.13]), the isoperimetric quotient function

$$\phi(\lambda) = \frac{\mathrm{W}_j(\lambda)^{n-i}}{\mathrm{W}_i(\lambda)^{n-j}} = \frac{\mathrm{W}_j\big((1+\lambda)K;E\big)^{n-i}}{\mathrm{W}_i\big((1+\lambda)K;E\big)^{n-j}} = \frac{\mathrm{W}_j(K;E)^{n-i}}{\mathrm{W}_i(K;E)^{n-j}}$$

is constant in $-1 < \lambda \leq 0$ for all $0 \leq i < j < n$ (without additional assumptions on the classes $\mathcal{R}_p$).

Remark 4.4. For $\lambda \geq 0$, setting $\mathrm{W}_i(\lambda) = \mathrm{W}_i(K + \lambda E)$, $0 \leq i \leq n-1$, one has $\mathrm{W}_i'(\lambda) = (n-i)\mathrm{W}_{i+1}(\lambda)$ for all $i = 0, \ldots, n-1$ directly from the Steiner formula for quermassintegrals (see [22, (5.29) and p. 225]); here, for $\lambda = 0$ only the right derivative is considered. This yields that the isoperimetric quotient function ϕ defined for $\lambda \geq 0$ satisfies $\phi'(\lambda) \leq 0$ too, and thus, ϕ is non-increasing in the full range $\big(-\mathrm{r}(K;E), \infty\big)$. We observe that, when $\lambda \geq 0$, no examples of constant ϕ, apart from $K = E$, are known to the authors.

The case $i = 0$, $j = 1$ (and $E = B^n$) in Proposition 4.3 provides us with an alternative result on the monotonicity of the classical isoperimetric quotient with respect to the family of inner parallel bodies, now under a different assumption on K (cf. Theorem 4.1):

Corollary 4.5. *Let $K \in \mathcal{K}_n^n$. If $K \in \mathcal{R}_1$ then the isoperimetric quotient $\mathrm{S}(K_\lambda)^n/\mathrm{vol}(K_\lambda)^{n-1}$ is a non-increasing function for $-\mathrm{r}(K;B^n) < \lambda \le 0$. In particular,*

$$\frac{\mathrm{S}(K_\lambda)^n}{\mathrm{vol}(K_\lambda)^{n-1}} \ge \frac{\mathrm{S}(K)^n}{\mathrm{vol}(K)^{n-1}}.$$

4.2. Isoperimetric deficit

Next we consider the isoperimetric deficit, instead of the quotient, of the inequality (4.3). As we will see, the behavior is the opposite.

Proposition 4.6. *Let $0 \le i < j < n$, and let $K \in \mathcal{R}_i$ and $E \in \mathcal{K}_n^n$. Then the isoperimetric deficit function $\mathrm{W}_j(\lambda)^{n-i} - \mathrm{W}_i(\lambda)^{n-j}\mathrm{vol}(E)^{j-i}$ is non-decreasing for $-\mathrm{r}(K;E) < \lambda \le 0$. In particular,*

$$\begin{aligned}\mathrm{W}_j(K_\lambda;E)^{n-i} &- \mathrm{W}_i(K_\lambda;E)^{n-j}\mathrm{vol}(E)^{j-i}\\ &\le \mathrm{W}_j(K;E)^{n-i} - \mathrm{W}_i(K;E)^{n-j}\mathrm{vol}(E)^{j-i}.\end{aligned}$$

Proof. We consider the function

$$\psi(\lambda) = \mathrm{W}_j(\lambda)^{n-i} - \mathrm{W}_i(\lambda)^{n-j}\mathrm{vol}(E)^{j-i} \quad \text{for } -\mathrm{r}(K;E) < \lambda \le 0.$$

Since $K \in \mathcal{R}_i$, we know that $\mathrm{W}_i(\lambda)$ is differentiable and $\mathrm{W}_i'(\lambda) = (n-i)\mathrm{W}_{i+1}(\lambda)$; however, for $\mathrm{W}_j'(\lambda)$ we can only take one-side derivatives, which satisfy ${}'\mathrm{W}_j(\lambda) \ge \mathrm{W}_j'(\lambda) \ge (n-j)\mathrm{W}_{j+1}(\lambda)$. Thus, taking the right derivative of $\psi(\lambda)$ with respect to λ and using the above relations, we obtain that

$$\begin{aligned}\psi'(\lambda) &= (n-i)\mathrm{W}_j(\lambda)^{n-i-1}\mathrm{W}_j'(\lambda) - (n-j)\mathrm{W}_i(\lambda)^{n-j-1}\mathrm{W}_i'(\lambda)\mathrm{vol}(E)^{j-i}\\ &\ge (n-j)(n-i)\Big[\mathrm{W}_j(\lambda)^{n-i-1}\mathrm{W}_{j+1}(\lambda) - \mathrm{W}_i(\lambda)^{n-j-1}\mathrm{W}_{i+1}(\lambda)\mathrm{vol}(E)^{j-i}\Big].\end{aligned}$$

Next we prove that

$$\mathrm{W}_j(\lambda)^{n-i-1}\mathrm{W}_{j+1}(\lambda) \ge \mathrm{W}_i(\lambda)^{n-j-1}\mathrm{W}_{i+1}(\lambda)\mathrm{vol}(E)^{j-i}. \tag{4.4}$$

Since $i < j$, we can use the relation $\mathrm{W}_j(\lambda)^{n-i-1} \ge \mathrm{W}_{i+1}(\lambda)^{n-j}\mathrm{vol}(E)^{j-i-1}$ (cf. (4.3)), and thus, in order to prove (4.4) it suffices to show that

$$\mathrm{W}_{i+1}(\lambda)^{n-j-1}\mathrm{W}_{j+1}(\lambda) \ge \mathrm{W}_i(\lambda)^{n-j-1}\mathrm{vol}(E). \tag{4.5}$$

If $j = n-1$, (4.5) holds trivially; so, we assume that $j \le n-2$.

The family of inequalities given in (4.3) has a more general version, namely, $\mathrm{W}_s^{k-l}(K;E) \geq \mathrm{W}_l^{k-s}(K;E)\mathrm{W}_k^{s-l}(K;E)$ for $0 \leq l \leq s \leq k \leq n$ (see e.g. [22, (7.63)]). Then, since $i < i+1 \leq n-j+i-1$, we can write

$$\mathrm{W}_{i+1}(\lambda)^{n-j-1} \geq \mathrm{W}_i(\lambda)^{n-j-2}\mathrm{W}_{n-j+i-1}(\lambda),$$

and hence, using also the Aleksandrov-Fenchel inequalities (4.2) we get

$$\begin{aligned} \mathrm{W}_{i+1}(\lambda)^{n-j-1}\mathrm{W}_{j+1}(\lambda) &\geq \mathrm{W}_i(\lambda)^{n-j-2}\mathrm{W}_{n-j+i-1}(\lambda)\mathrm{W}_{j+1}(\lambda) \\ &\geq \mathrm{W}_i(\lambda)^{n-j-2}\mathrm{W}_i(\lambda)\mathrm{vol}(E) = \mathrm{W}_i(\lambda)^{n-j-1}\mathrm{vol}(E). \end{aligned}$$

This proves (4.5) and hence (4.4) holds.

So, we have that the right derivative $\psi'(\lambda) \geq 0$ for each $\lambda \in \bigl(-\mathrm{r}(K;E),0\bigr)$. Since ψ is a continuous function in the interval $\bigl[-\mathrm{r}(K;E),0\bigr]$, [19, Theorem 1] yields that $\psi(\lambda)$ is non-decreasing when $-\mathrm{r}(K;E) < \lambda \leq 0$, which concludes the proof. $\square$

We point out that in Proposition 4.3 we need that the convex bodies lie in the class $\mathcal{R}_j$ whereas for Proposition 4.6 the assumption is weaker: the convex body has to lie in $\mathcal{R}_i$, and $\mathcal{R}_j \subsetneq \mathcal{R}_i$ because $i < j$. Therefore, in the case of the classical isoperimetric deficit, i.e., $i = 0$, $j = 1$, since $\mathcal{R}_0 = \mathcal{K}^n$, no hypothesis is needed:

Corollary 4.7. *For every $K \in \mathcal{K}^n$, the isoperimetric deficit $\mathrm{S}(K_\lambda)^n - n^n\mathrm{vol}(B^n)\mathrm{vol}(K_\lambda)^{n-1}$ is a non-decreasing function for $-\mathrm{r}(K;B^n) < \lambda \leq 0$. In particular,*

$$\mathrm{S}(K_\lambda)^n - n^n\mathrm{vol}(B^n)\mathrm{vol}(K_\lambda)^{n-1} \leq \mathrm{S}(K)^n - n^n\mathrm{vol}(B^n)\mathrm{vol}(K)^{n-1}.$$

Again if we define the isoperimetric deficit function for positive values of λ, namely, $\psi(\lambda) = \mathrm{W}_j(K+\lambda E;E)^{n-i} - \mathrm{W}_i(K+\lambda E;E)^{n-j}\mathrm{vol}(E)^{j-i}$, $\lambda \geq 0$, we also get the same monotonicity, and thus $\psi(\lambda)$ is non-decreasing in the full range $\bigl(-\mathrm{r}(K;E),\infty\bigr)$.

5. On the perimeter of inner parallel bodies

In [16, Theorem 1.2] the author provides a lower bound for the surface area of the inner parallel bodies of a convex body K (with respect to B^n); following our notation, it is shown that

$$\mathrm{S}(K_\lambda) \geq \left(1 + \frac{\lambda}{\mathrm{r}(K;B^n)}\right)^{n-1} \mathrm{S}(K). \tag{5.1}$$

For the proof of the above inequality, the author uses an auxiliary result ([16, Lemma 2.1]) which states that, for $K \in \mathcal{K}^n_n$ and $-\mathrm{r}(K;B^n) \leq \lambda \leq 0$, among all convex bodies $L \in \mathcal{K}^n$ satisfying that $L_\lambda = K_\lambda$, the set $L = K_\lambda + |\lambda|\, K^{\mathrm{f}}$ has maximal surface area.

The proof of this lemma makes the implicit assumption of a condition closely related to (2.3), namely,

$$\mathcal{U}\big(K + K^{\mathrm{f}}\big) = \mathcal{U}(K), \tag{5.2}$$

which is necessary to have the following equality in the last step in the proof:

$$\bigcap_{u \in \mathcal{U}(K)} \Big\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h\big(K + |\lambda|\, K^{\mathrm{f}}, u\big)\Big\} = K + |\lambda|\, K^{\mathrm{f}}$$

(cf. (2.2)). Unfortunately, condition (5.2) is not satisfied for all convex bodies $K \in \mathcal{K}^n_n$: indeed, although $\mathcal{U}(K) \subset \mathcal{U}\big(K + K^{\mathrm{f}}\big)$ always holds (Lemma 2.3 (iii)), the reverse inclusion needs not be true in general, as the polytope P given in (3.5) shows; note that $\mathcal{U}(P) = \mathcal{U}\big(P^{\mathrm{f}}\big) \subsetneq \mathcal{U}\big(P + P^{\mathrm{f}}\big)$.

The proof of Lemma 2.1 in [16] actually yields that for $K \in \mathcal{K}^n_n$ and $-\mathrm{r}(K;B^n) \leq \lambda \leq 0$, among all convex bodies $L \in \mathcal{K}^n$ satisfying that $L_\lambda = K_\lambda$, exactly the set $L = K_\lambda\big(|\lambda|\big)$ (cf. (3.1)) has maximal surface area. For the sake of completeness we state it as a result.

Lemma 5.1 ([16, Lemma 2.1] revised). *Let $K \in \mathcal{K}^n_n$ and $-\mathrm{r}(K;B^n) \leq \lambda \leq 0$. Among all convex bodies $L \in \mathcal{K}^n$ satisfying that $L_\lambda = K_\lambda$, exactly the set*

$$K_\lambda\big(|\lambda|\big) = \Big\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(K_\lambda, u) + |\lambda|\, h(E, u), u \in \mathcal{U}(K_\lambda)\Big\}$$

has maximal surface area.

We conclude this note pointing out that, although the proof of Theorem 1.2 in [16] is partially based on an incorrect lemma, the result itself is valid: the proof of (5.1) follows from [21, Lemma 2.9], which states that

$$K_\lambda \supset \left(1 + \frac{\lambda}{\mathrm{r}(K;B^n)}\right) K,$$

and the monotonicity and $(n-1)$-homogeneity of the surface area.

Acknowledgements. The authors would like to thank J. Yepes Nicolás for pointing out the existence of results concerning the behaviour of the isoperimetric quotient, in particular [6]. We also strongly thank the anonymous referees for their very valuable comments and helpful suggestions.

References

[1] G. Bol, Beweis einer Vermutung von H. Minkowski, *Abh. Math. Sem. Univ. Hamburg* **15** (1943), 37–56.

[2] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*. Springer, Berlin, 1934, 1974. English translation: *Theory of convex bodies*. Edited by L. Boron, C. Christenson and B. Smith. BCS Associates, Moscow, ID, 1987.

[3] A. Dinghas, Bemerkung zu einer Verschärfung der isoperimetrischen Ungleichung durch H. Hadwiger, *Math. Nachr.* **1** (1948), 284–286.

[4] A. Dinghas, Über eine neue isoperimetrische Ungleichung für konvexe Polyeder, *Math. Ann.* **120** (1949), 533–538.

[5] A. Dinghas, Zum Minkowskischen Integralbegriff abgeschlossener Mengen, *Math. Zeitschr.* **66** (1956), 173–188.

[6] G. Domokos, Z. Lángi, The isoperimetric quotient of a convex body decreases monotonically under the Eikonal abrasion model, *Mathematika* **65** (1) (2019), 119–129.

[7] R. Gardner, M. Kiderlen, A solution to Hammer's X-ray reconstruction problem, *Adv. Math.* **214** (1) (2007), 323–343.

[8] P. M. Gruber, *Convex and Discrete Geometry*. Springer, Berlin Heidelberg, 2007.

[9] H. Hadwiger, *Altes und Neues über konvexe Körper*. Birkhäuser Verlag, Basel und Stuttgart, 1955.

[10] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.

[11] M. Henk, J. M. Wills, A Blichfeldt-type inequality for the surface area, *Monatsh. Math.* **154** (2008), 135–144.

[12] M. A. Hernández Cifre, E. Saorín, On differentiability of quermassintegrals, *Forum Math.* **22** (1) (2010), 115–126.

[13] M. A. Hernández Cifre, E. Saorín, On the volume of inner parallel bodies, *Adv. Geom.* **10** (2) (2010), 275–286.

[14] M. A. Hernández Cifre, E. Saorín, On inner parallel bodies and quermassintegrals, *Israel J. Math.* **177** (2010), 29–48.

[15] M. A. Hernández Cifre, E. Saorín, Differentiability of quermassintegrals: a classification of convex bodies, *Trans. Amer. Math. Soc.* **366** (2014), 591–609.

[16] S. Larson, A bound for the perimeter of inner parallel bodies, *J. Funct. Anal.* **271** (2016), 610–619.

[17] E. Linke, E. Saorín Gómez, Decomposition of polytopes using inner parallel bodies, *Monatsh. Math.* **176** (4) (2015), 575–588.

[18] G. Matheron, La formule de Steiner pour les érosions, *J. Appl. Prob.* **15** (1978), 126–135.

[19] A. D. Miller, R. Vyborny, Some remarks on functions with one-sided derivatives, *Amer. Mat. Month.* **93** (6) (1986), 471–475.

[20] J. H. Rolfes, F. Vallentin, Covering compact metric spaces greedily, *Acta Math. Hung.* **155** (1) (2018), 130–140.

[21] J. R. Sangwine-Yager, *Inner Parallel Bodies and Geometric Inequalities.* Ph.D. Thesis Dissertation, University of California Davis, 1978.

[22] R. Schneider, *Convex bodies: The Brunn-Minkowski theory*, 2nd expanded ed. Encyclopedia of Mathematics and its Applications, 151. Cambridge: Cambridge University Press, 2014.

[23] J. Steiner, *Über parallele Flächen*, Monatsber. Preuss. Akad. Wiss. (1840), 114–118, [Ges. Werke, Vol II (Reimer, Berlin, 1882) 245–308].

Departamento de Matemáticas, Universidad de Murcia
Campus de Espinardo, 30100-Murcia, Spain
E-mail address: mhcifre@um.es

ALTA institute for Algebra, Geometry, Topology and their Applications, Universität Bremen, 28334-Bremen, Germany
E-mail address: esaoring@uni-bremen.de

Received October 22, 2019
Accepted June 15, 2020