Communications in
Analysis and Geometry
Volume 30, Number 10, 2315–2346, 2022

Refined position estimates for surfaces of Willmore type in Riemannian manifolds

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In this paper we consider surfaces which are critical points of the Willmore functional subject to constrained area. In the case of small area we calculate the corrections to the intrinsic geometry induced by the ambient curvature. These estimates together with the choice of an adapted geometric center of mass lead to refined position estimates in relation to the scalar curvature of the ambient manifold.

This research was supported by the DFG in form of grant ME 3816/1-2.

1. Introduction

Let (M, g) be a three dimensional Riemannian manifold and let $\Sigma \subset M$ be a smooth, compact, two-sided, immersed surface. The Willmore energy of Σ is defined as

$$\mathcal{W}(\Sigma) = \frac{1}{4} \int_\Sigma H^2 \,\mathrm{d}\mu$$

where H is the mean curvature of the immersion and $\mathrm{d}\mu$ denotes the induced surface measure on Σ. We consider surfaces Σ which are critical points of $\mathcal{W}$ subject to the constraint of prescribed area $|\Sigma|$. These surfaces satisfy the Euler-Lagrange equation

$$\Delta H + H|\mathring{A}|^2 + H \operatorname{Ric}(\nu, \nu) + \lambda H = 0, \tag{1}$$

where $\lambda \in \mathbf{R}$ is the Lagrange parameter, Δ is the Laplace-Beltrami operator of the induced metric γ on Σ, $\mathring{A} = A - \frac{1}{2}H\gamma$ is the trace free part of the second fundamental form A, and ν denotes (one choice of) the normal vector to Σ. Furthermore, Ric is the Ricci curvature of ambient metric g.

Concerning the existence of minimizers of the area constrained problem in compact manifolds, we have

Theorem 1.1. *Let (M, g) be a three dimensional Riemannian manifold. Then there exists $a_{min} \in (0, \infty)$ and for each $a \in (0, a_{min})$ there exists a smooth closed embedded surface Σ_a^{min} such that*

$$\mathcal{W}(\Sigma_a^{min}) = \inf\{\mathcal{W}(\Sigma) \mid \Sigma \textit{ smooth closed immersion and } |\Sigma| = a\}$$

and $|\Sigma_a^{min}| = a$.

This was shown by Chen and Li [1] in the class of $W^{2,2}$-conformal immersions and by Lamm and the author [7] as well as by Mondino and Rivière [11] with the additional assertion of smoothness of the minimizing surfaces.

Critical points for this minimization problem can be constructed by perturbing geodesic spheres centered at a non-degenerate point of the scalar curvature. Independently Ikoma, Malchiodi, and Mondino [4] as well as Lamm, Schulze and the author [9] have shown the following:

Theorem 1.2. *Let (M, g) be a three dimensional Riemannian manifold and let $p \in M$ be such that $\nabla \operatorname{Sc}(p) = 0$ and such that $\nabla^2 \operatorname{Sc}(p)$ is non-degenerate. Then there exist $a_{pert} \in (0, \infty)$, a neighborhood U of p, and for each $a \in$*

$(0, a_{pert})$ a spherical surface Σ_a^{pert} which satisfies (1) *for some $\lambda \in \mathbf{R}$ and $|\Sigma_a| = a$. The Σ_a are mutually disjoint and $\bigcup_{(0,a_{pert})} \Sigma_a = U \setminus \{p\}$.*

The shape and position of critical points, that is solutions to equation (1) was studied by Lamm and the author [6, 7] and with more general assumptions by Laurain and Mondino [10]. The position estimates implied by combining these three papers are the following:

Theorem 1.3. *Let (M, g) be a three dimensional Riemannian manifold. Then there exist $a_0 \in (0, \infty)$ and constants $C_1, C_2 \in (0, \infty)$ with the following property. Let $\Sigma \subset M$ be a surface satisfying equation* (1) *for some $\lambda \in \mathbf{R}$ such that $|\Sigma| \le a_0$ and $\mathcal{W}(\Sigma) < 4\pi + a_0$. Then $\operatorname{diam}(\Sigma) \le C_1 |\Sigma|^{1/2}$ and $|\nabla \operatorname{Sc}| \le C_2 |\Sigma|^{1/2}$ on Σ.*

A consequence of this Theorem is that the surfaces $\Sigma_a^{\min}$ concentrate near critical points of the scalar curvature of M. From the expansion of the Willmore functional in [6, Theorem 5.1] it follows in addition that the minimizers $\Sigma_a^{\min}$ concentrate near the points in M where the scalar curvature is maximal as $a \to 0$.

The previously cited results were to a large extend based on the observation, that surfaces satisfying (1) with small area and Willmore energy close to 4π behave like their Euclidean counterparts. The aim of this paper is to provide a more precise description of the shape and position of solutions to (1), that take into account the perturbations induced by the ambient geometry.

As an application of the estimates we derive an improved position estimate.

Theorem 1.4. *Let (M, g) be a three dimensional Riemannian manifold with C_B-bounded geometry. Then there exists $a_0 \in (0, \infty)$ and a constant $C \in (0, \infty)$ with the following property. For every surface $\Sigma \subset M$ satisfying equation* (1) *for some $\lambda \in \mathbf{R}$ with $|\Sigma| \le a_0$ and $\mathcal{W}(\Sigma) < 4\pi + a_0$ there exists a point p_0 contained in the region enclosed by Σ such that for all $p \in \Sigma$ we have $\operatorname{dist}(p_0, p) < \frac{3}{4} \operatorname{diam}(\Sigma)$ and $|\nabla \operatorname{Sc}(p_0)| \le C|\Sigma|$.*

For the definition of C_B-bounded geometry, refer to definition 2.1.

The main use for this improvement of Theorem 1.3 is to further narrow down the position of the surfaces Σ as in the theorem. To this end assume that the critical points of the scalar curvature of (M, g) are such that the Hessian there is non-degenerate, in other words, that the scalar curvature on M is a Morse function. Let $\bar{p}$ be a critical point for Sc and p some point near $\bar{p}$. By non-degeneracy $|\nabla \operatorname{Sc}(p)| \geq c \operatorname{dist}(p, \bar{p})$ and Theorem 1.3 implies

$$c \operatorname{dist}(p, \bar{p}) \leq CR(\Sigma).$$

From this it follows that for any neighborhood U of the critical points of Sc there is $a_0 > 0$ such that if $|\Sigma| < a_0$ then $\Sigma \subset U$. However, it is not clear that a critical point of Sc lies in the region enclosed by Σ. This can be derived as a consequence of Theorem 1.4. The assertions of the following corollaries are also included in the paper [4], as step 2 in the proof of their Theorem 1.1.

Corollary 1.5. *Let (M, g) be a compact three dimensional Riemannian manifold with C_B bounded geometry. Let*

$$Z := \{x \in M \mid \nabla \operatorname{Sc}(x) = 0\}$$

and assume that the Hessian $\operatorname{Hess}\operatorname{Sc}(x)$ *is non-degenerate for every $x \in Z$.*

Then there exists an $a_0 \in (0, \infty)$ depending only on (M, g) such that for every surface Σ that satisfies the Euler-Lagrange equation (1) for some λ, with $|\Sigma| \leq a_0$ and $\mathcal{W}(\Sigma) \leq 4\pi + a_0$ the region enclosed by Σ intersects Z in a single point.

Note that for the $\Sigma_a^{\min}$ it is automatic that $\mathcal{W}(\Sigma) \leq 4\pi + O(a)$ by comparison with geodesic spheres. Hence, Theorem 1.4 applies in particular to these surfaces and we can be more precise:

Corollary 1.6. *Let (M, g) be a compact three dimensional Riemannian manifold. Let*

$$Z^{max} := \{x \in M \mid \operatorname{Sc}(x) = \max_M \operatorname{Sc}\}$$

and assume that the Hessian $\operatorname{Hess}\operatorname{Sc}(x)$ *is non-degenerate for every $x \in Z^{max}$.*

Then there exists $a_0 \in (0, \infty)$ with the following property. For every $a \in (0, a_0)$ the surface Σ_a^{min} from Theorem 1.1 is such that it encloses a region that intersects Z^{max} in a single point.

The paper is organized as follows. In section 2 we collect some estimates from the literature and combine them to an L^∞-estimate for $\mathring{A}$. In section 3 we introduce a geometric center of mass for small surfaces $\Sigma \subset M$. We use this to select the point p_0 in the position estimate Theorem 1.4. In sections 4 and 5 we compute the top order contributions in the expansion of certain geometric quantities on a solution Σ of (1). Section 6 provides a calculation of geometric identities necessary in section 7. Theorem 1.4 follows from a slightly more precise version, Theorem 7.1, is carried out in section 7. Finally, section 8 contains the proof of Corollary 1.5.

2. Preliminaries

Recall the Gauss equation relating the scalar curvature ${}^{\Sigma}\mathrm{Sc}$ of γ and the scalar curvature Sc of g:

$$ {}^{\Sigma}\mathrm{Sc} = \mathrm{Sc} - 2\,\mathrm{Ric}(\nu,\nu) + \tfrac{1}{2}H^2 - |\mathring{A}|^2. $$

Denote the genus of Σ by $q(\Sigma)$. Integrating the Gauss equation and using Gauss-Bonnet yields that

$$ \mathcal{W}(\Sigma) = 4\pi(1 - q(\Sigma)) + \frac{1}{2}\mathcal{U}(\Sigma) + \mathcal{V}(\Sigma) \tag{2} $$

where

$$ \mathcal{U}(\Sigma) = \int_\Sigma |\mathring{A}|^2 \,\mathrm{d}\mu \quad \text{and} \quad \mathcal{V}(\Sigma) = \int_\Sigma \mathrm{Ric}(\nu,\nu) - \tfrac{1}{2}\,\mathrm{Sc}\,\mathrm{d}\mu. \tag{3} $$

Equation (2) implies that for bounded area $|\Sigma|$ and bounded ambient curvature $|\,\mathrm{Ric}\,| + |\,\mathrm{Sc}\,| \le C_B$ a bound for $\mathcal{U}$ is equivalent to bounding $\mathcal{W}$, regardless of the topology of Σ:

$$ \mathcal{W}(\Sigma) \le 4\pi + \tfrac{1}{2}\mathcal{U}(\Sigma) + C_B|\Sigma|. \tag{4} $$

A similar bound holds in the other direction for surfaces Σ with bounded genus $q(\Sigma) \le q_0$:

$$ \mathcal{U}(\Sigma) \le \mathcal{W}(\Sigma) + 4\pi(q_0 - 1) + C_B|\Sigma|. $$

For the rest of the paper we will use these estimates for spherical surfaces. As a consequence, an a priori bound on $\mathcal{W}$ (or $\mathcal{U}$) and on $|\Sigma|$ will yields an a priori bound for the L^2-norm of the second fundamental form $\|A\|^2_{L^2(\Sigma)} = \mathcal{U}(\Sigma) + 2\mathcal{W}(\Sigma)$.

2.1. A priori estimates for small surfaces of Willmore type

Here we quote some estimates from the papers [6, 7]. They require uniform bounds on the geometry of (M, g) in the following sense.

Definition 2.1 (cf. Definition 2.1 in [7]). *Let (M, g) be a complete Riemannian manifold and $C_B \in (0, \infty)$. We say that (M, g) has C_B-bounded geometry if for every $p \in M$ we have $\operatorname{inj}(M, g, p) \geq C_B^{-1}$ and $|\operatorname{Rm}(p)| + |\nabla \operatorname{Rm}(p)| \leq C_B$.*

To phrase the estimates quoted below in a geometric way, we use the area radius of a surface defined as

$$R(\Sigma) = \sqrt{\frac{|\Sigma|}{4\pi}}.$$

The following lemma that if we assume small enough area and bounded Willmore energy, then the area radius of a surface is comparable to its diameter.

Lemma 2.2. *Let (M, g) be of C_B-bounded geometry as in Definition 2.1 and $E_0 \in (0, \infty)$. Then there exist constants $a_0 \in (0, \infty)$ and C depending only on C_B and E_0 with the following property: If $\Sigma \subset M$ is a smooth closed hypersurface with $\mathcal{W}(\Sigma) \leq E_0$ and $|\Sigma| \leq a_0$ then*

$$C^{-1} \operatorname{diam}(\Sigma) \leq R(\Sigma) \leq C \operatorname{diam}(\Sigma).$$

Proof. The right estimate follows from Lemma 2.2 in [6] whereas the left estimate is a direct consequence of Lemma 2.5 in [7]. □

Theorem 2.3 (cf. Theorem 5.4 from [7]). *Let (M, g) be of C_B-bounded geometry as in Definition 2.1. Then there exist constants $a_0 \in (0, \infty)$ and C depending only on C_B such that for every surface Σ satisfying (1), with $|\Sigma| \leq a_0$ and $\mathcal{W}(\Sigma) \leq 4\pi + a_0$ we have the estimate*

$$\int_\Sigma |\nabla^2 H|^2 + H^2 |\nabla A|^2 + H^4 |\mathring{A}|^2 \, \mathrm{d}\mu \leq C.$$

Lemma 2.4 (cf. Corollary 5.5 from [7]). *Under the assumptions of theorem 2.3 we have that*

$$\|\mathring{A}\|_{L^2(\Sigma)} \leq C R(\Sigma)^2 \quad \text{and} \quad \left\| H - \tfrac{2}{R(\Sigma)} \right\|_{L^\infty(\Sigma)} \leq C R(\Sigma).$$

This lemma shows that for small enough area $|\Sigma|$ the mean curvature $H > 0$ and thus the estimates from [6] hold under the assumptions of Theorem 2.3. It also follows that $\|A\|_{L^2(\Sigma)} \leq C$.

For $r \in (0, \infty)$, denote the intrinsic ball centered at $p \in M$ by

$$\mathcal{B}_r(p) = \{q \in M \mid \operatorname{dist}_g(p, q) \le r\}.$$

Proposition 2.5 (cf. Corollary 3.6 from [6]). *Let (M, g) and Σ be as in Theorem 2.3. Let $p_0 \in M$ be a point with $\Sigma \subset \mathcal{B}_{2\,\mathrm{diam}(\Sigma)}(p_0)$, then*

$$\left|\lambda + \tfrac{1}{3}\operatorname{Sc}(p_0)\right| \le CR(\Sigma).$$

2.2. General inequalities

The Bochner identity for surfaces can be stated as follows.

Lemma 2.6. *For all functions $f \in C^\infty(\Sigma)$ we have that*

$$\int_\Sigma 2|\nabla^2 f|^2 + \tfrac{1}{2}H^2|\nabla f|^2\,\mathrm{d}\mu \le \int_\Sigma 2|\Delta f|^2 + \left(2\operatorname{Ric}(\nu,\nu) - \operatorname{Sc} + |\mathring{A}|^2\right)|\nabla f|^2\,\mathrm{d}\mu.$$

Proof. The Bochner identity states that

$$\int_\Sigma |\nabla^2 f|^2\,\mathrm{d}\mu = \int_\Sigma (\Delta f)^2 - {}^\Sigma\mathrm{Rc}(\nabla f, \nabla f)\,\mathrm{d}\mu.$$

Since Σ is a surface, its Ricci curvature satisfies ${}^\Sigma\mathrm{Rc} = \frac{1}{2}{}^\Sigma\mathrm{Sc}\gamma$ and the scalar curvature ${}^\Sigma\mathrm{Sc}$ of Σ can be expressed via the the Gauss equation

$${}^\Sigma\mathrm{Sc} = \operatorname{Sc} - \operatorname{Ric}(\nu,\nu) + \tfrac{1}{2}H^2 - |\mathring{A}|^2.$$

$\square$

Let C and a_0 be the constants from lemma 2.2. Hence, if Σ is such that $|\Sigma| \le a_0$ and $p \in M$ is some point such that there exists $x \in \Sigma$ with $\operatorname{dist}(p, x) \le R(\Sigma)$, then $\Sigma \subset \mathcal{B}_{(C+1)R(\Sigma)}(p)$. Hence, there exists $a_0' \in (0, a_0)$ such that if $|\Sigma| \le a_0'$ and $\rho = (C+1)R(\Sigma)$ then $\rho \le \operatorname{inj}(M, g, p)$. In this case there are normal coordinates $x : \mathcal{B}_\rho(p) \to B_\rho(0) \subset \mathbf{R}^3$. The metric $(x^{-1})^*g$ on $B_\rho(0)$ has the expansion $(x^{-1})^*g = \delta + h$ with

$$\sup\left(|x|^{-2}|h| + |x|^{-1}|\partial h| + |\partial^2 h|\right) \le h_0.$$

Here h_0 is a constant depending only on C_B and ∂ denotes partial derivatives in the coordinate system given by x. In particular, we can apply the following two estimates on surfaces Σ as in section 2.1 with uniform constants, that is constants independent of Σ, provided $|\Sigma| \le a_0$ for some constant $a_3 \in (0, \infty)$ depending only on C_B.

Lemma 2.7. *Let $g = g^E + h$ on B_ρ and C_0 be given. Then there exists $\rho_0 \in (0, \rho)$ and a constant C depending only on ρ, h_0 and C_0 such that for all surfaces $\Sigma \subset B_{\rho_0}$ with $\|A\|_{L^2(\Sigma)} \leq C_0$ and all $f \in C^\infty(\Sigma)$ we have*

$$\left(\int_\Sigma f^2 \, \mathrm{d}\mu\right)^{1/2} \leq C \int_\Sigma |\nabla f| + |Hf| \, \mathrm{d}\mu.$$

In addition, for surfaces as in Theorem 2.3 we have a Poincaré inequality of the following form:

Lemma 2.8. *Let (M, g) be of C_B-bounded geometry as in Definition 2.1. Then there exist constants $a_0 \in (0, \infty)$, $\varepsilon \in (0, \infty)$ and C depending only on C_B with the following property. If $\Sigma \subset M$ is a smooth closed hypersurface with $\mathcal{U}(\Sigma) \leq \varepsilon$ and $|\Sigma| \leq a_0$ then for all $f \in C^\infty(\Sigma)$ we have*

$$\int_\Sigma |f - \bar{f}|^2 \, \mathrm{d}\mu \leq C|\Sigma| \int_\Sigma |\nabla f|^2 \, \mathrm{d}\mu.$$

Here

$$\bar{f} = |\Sigma|^{-1} \int_\Sigma f \, \mathrm{d}\mu$$

is the mean value of f.

Proof. This holds without assuming an upper bound for the area of Σ if (M, g) is Euclidean space in view of the eigenvalue estimates of DeLellis and Müller [3, Corollary 1.3] for nearly umbilical surfaces.

For general (M, g) we first use inequality (4) to bound $\mathcal{W}(\Sigma)$ in terms of $\mathcal{U}(\Sigma)$ and $|\Sigma|$. Then by Lemma 2.2 we infer that small area $|\Sigma|$ implies small diameter. Using normal coordinates covering Σ implies that we are in a nearly Euclidean setting. It is then straight forward to deduce the desired Poincaré inequality on Σ with respect to the metric induced by g from the one with respect to the Euclidean metric in the normal coordinate system. $\square$

The following estimate follows from the Michael-Simon-Sobolev inequality from Lemma 2.7 and can be proved exactly as [5, Lemma 2.8]. This form appears in [6, Lemma 3.7].

Lemma 2.9. *Assume that the metric $g = g^E + h$ on B_ρ is given. Then there exist $\rho_0 \in (0, \rho)$ and a constant $C < \infty$ such that for all surfaces $\Sigma \subset$*

B_r with $r \in (0, \rho_0)$ and for all smooth forms ϕ on Σ we have

$$\|\phi\|^4_{L^\infty(\Sigma)} \le C\|\phi\|^2_{L^2(\Sigma)} \int_\Sigma |\nabla^2\phi|^2 + |H|^4|\phi|^2 \,\mathrm{d}\mu.$$

2.3. A L^∞ estimate for $\mathring{A}$

For the later exposition we need two more estimates not present in [6, 7].

Lemma 2.10. *Let (M, g) be as in definition 2.1. Then there exist constants $a_0 \in (0, \infty)$ and $C \in (0, \infty)$ depending only on C_B such that if Σ satisfies* (1) *for some $\lambda \in \mathbf{R}$ with $|\Sigma| \le a_0$ and $\mathcal{W}(\Sigma) \le 4\pi + a_0$ then*

$$\|\mathring{A}\|_{L^\infty(\Sigma)} \le CR(\Sigma)$$

and

$$\|H^{-1} - R(\Sigma)/2\|_{L^\infty(\Sigma)} \le CR(\Sigma)^2.$$

Proof. We assume that a_0 is so small that the estimates from theorem 2.3 apply. Then the second estimate is an immediate consequence from the second estimate in 2.4. To show the first estimate, proceed as in the proof of [8, Lemma 15]. In view of the Bochner identity it suffices to estimate $\Delta\mathring{A}$ in the L^2-norm. The L^∞ estimate then follows from lemma 2.9. To derive this estimate, recall the Simons identity [13] in the form of [8, eq. (8)]

$$\begin{aligned}\Delta\mathring{A}_{ij} &= (\nabla^2 H)^\circ_{ij} + H\mathring{A}^k_i\mathring{A}_{kj} + \tfrac{1}{2}H^2\mathring{A}_{ij} - |\mathring{A}|^2\mathring{A}_{ij} - \tfrac{1}{2}H|\mathring{A}|^2\gamma_{ij} \\ &\quad + \mathring{A}^k_j\gamma_{lm}\,\mathrm{Rm}_{likm} + \mathring{A}^{kl}\,\mathrm{Rm}_{ikjl} + 2\nabla_i\omega_j - \operatorname{div}\omega\gamma_{ij}.\end{aligned} \tag{5}$$

Here $\omega = \mathrm{Ric}(\nu, \cdot)^T$ denotes the tangential 1-form obtained from projecting the 1-from $\mathrm{Ric}(\nu, \cdot)$ to the tangent space of Σ. From the calculation in section 4.2 we get that

$$|\nabla\omega| \le |\nabla\,\mathrm{Ric}\,| + |A||\,\mathrm{Ric}\,|.$$

This yields

$$\|\nabla\omega\|^2_{L^2(\Sigma)} \le C|\Sigma| + \int_\Sigma |A|^2 \,\mathrm{d}\mu \le C.$$

Together with equation (5) this gives

$$\begin{aligned}\|\Delta\mathring{A}\|_{L^2(\Sigma)} &\leq c\big(\|\nabla^2 H\|_{L^2(\Sigma)} + \|H^2\mathring{A}\|_{L^2(\Sigma)} + \|\mathring{A}\|^2_{L^6} \\ &\qquad + \|A\|_{L^2}\|\operatorname{Rm}\|_{L^\infty(\Sigma)} + \|\nabla\operatorname{Ric}\|_{L^2(\Sigma)}\big). \\ &\leq C + c\|\mathring{A}\|^2_{L^\infty(\Sigma)}\|\mathring{A}\|_{L^2(\Sigma)}.\end{aligned}$$

Here c denotes a purely numerical constant, and we used the estimates from section 2.1 in the second step.

From the Bochner identity 2.6 (more precisely a variant for two tensors) we obtain that

$$\begin{aligned}&\|\nabla^2\mathring{A}\|_{L^2(\Sigma)} + \|H\nabla\mathring{A}\|_{L^2(\Sigma)} \\ &\quad\leq c\|\Delta\mathring{A}\|_{L^2(\Sigma)} + C(1+\|\mathring{A}\|_{L^\infty(\Sigma)})\|\nabla\mathring{A}\|_{L^2(\Sigma)} \\ &\quad\leq C + c\|\mathring{A}\|^2_{L^\infty(\Sigma)}\|\mathring{A}\|_{L^2(\Sigma)} + C(1+\|\mathring{A}\|_{L^\infty(\Sigma)})\|\nabla\mathring{A}\|_{L^2(\Sigma)}\end{aligned}$$

Note that the last term on the right hand side can be absorbed to the left, if R is small enough.

This yields in view of lemma 2.9 that

$$\begin{aligned}\|\mathring{A}\|^4_{L^\infty(\Sigma)} &\leq C\|\mathring{A}\|^2_{L^2(\Sigma)}\big(\|\nabla^2\mathring{A}\|^2_{L^2(\Sigma)} + \|H^2\mathring{A}\|_{L^2(\Sigma)}\big) \\ &\leq CR(\Sigma)^4(C + C\|\mathring{A}\|^4_{L^\infty(\Sigma)}R(\Sigma)^2).\end{aligned}$$

If $R(\Sigma)$ is small enough, this gives

$$\|\mathring{A}\|_{L^\infty(\Sigma)} \leq CR(\Sigma).$$

Note that by choosing a_0 small enough, we can ensure that $R(\Sigma)$ is so small, that the above steps apply to Σ as in the assumption. $\square$

2.4. Approximately spherical surfaces

In this section we discuss the approximation of a given surface $\Sigma \subset \mathbf{R}^3$ by spheres. The main tool here are the estimates from DeLellis and Müller [2, 3]. We quote their estimates in the form needed here from [6, Theorem 2.4]. These results are purely Euclidean. To distinguish geometric quantities computed with respect to the Euclidean metric we use the superscript ^{E}.

Theorem 2.11. *There exists a universal constant C with the following properties. Assume that $\Sigma \subset \mathbf{R}^3$ is a surface with $\|\mathring{A}^E\|^2_{L^2(\Sigma,\gamma^E)} <$*

8π. *Let* $R^E := \sqrt{|\Sigma|^E/4\pi}$ *be the Euclidean area radius of* Σ *and* $a^E := |\Sigma|_E^{-1} \int_\Sigma x \,\mathrm{d}\mu^E$ *be the Euclidean center of gravity. Then there exists a conformal map* $F : S := S_{R^E}(a^E) \to \Sigma \subset \mathbf{R}^3$ *with the following properties. Let* γ^S *be the standard metric on* S, N *the Euclidean normal vector field and* ϕ *the conformal factor, that is* $F^*\gamma^E = \phi^2\gamma^S$. *Then the following estimates hold*

$$\begin{aligned} \|H^E - 2/R^E\|_{L^2(\Sigma,\gamma^E)} &\leq C\|\mathring{A}^E\|_{L^2(\Sigma,\gamma^E)} \\ \|F - \mathrm{id}_S\|_{L^\infty(S)} &\leq CR^E\|\mathring{A}^E\|_{L^2(\Sigma,\gamma^E)} \\ \|\phi^2 - 1\|_{L^\infty(S)} &\leq C\|\mathring{A}^E\|_{L^2(\Sigma,\gamma^E)} \\ \|N - \nu^E \circ F\|_{L^2(S)} &\leq CR^E\|\mathring{A}^E\|_{L^2(\Sigma,\gamma^E)}. \end{aligned}$$

These estimates can be applied in our situation by choosing appropriate normal coordinates near a small surfaces as described in section 2.2 and compare geometric quantities in the given metric to the Euclidean background. In particular we have:

Lemma 2.12 (cf. [6, Lemma 2.5]). *Let* $g = g^E + h$ *on* B_ρ *be given. Then there exists* $0 < \rho_0 < \rho$ *and a constant* C *depending only on* ρ *and* h_0 *such that for all surfaces* $\Sigma \subset B_r$ *with* $r < \rho_0$ *we have*

$$\|\mathring{A}^E\|^2_{L^2(\Sigma,\gamma^E)} \leq C\|\mathring{A}\|^2_{L^2(\Sigma,\gamma)} + C\rho^4\|H\|^2_{L^2(\Sigma,\gamma)}$$

3. A geometric center of mass

The calculations in section 7 require that the normal coordinates in which we look at our surfaces Σ are well adapted to Σ. In this section we propose one way to assign a geometric center of mass to our surfaces. Centering the normal coordinates there gives good control on the center of mass of the image of the surface in the coordinate picture.

Let (M, g) be a Riemannian manifold and $\Sigma \subset M$ a closed smooth hypersurface with extrinsic diameter $d = \mathrm{diam}(\Sigma) = \max\{\mathrm{dist}(x, y) \mid x, y \in \Sigma\}$ where dist denotes the distance function in (M, g). Assume that $2d < \mathrm{inj}(M, g)$. For $p \in M$ let $d_p(x) := \mathrm{dist}(p, x)$ and set

$$w(p) := \int_\Sigma d_p(x)^2 \,\mathrm{d}\mu.$$

Then w is a smooth, positive, proper function on M which attains its global infimum on the compact set

$$K := \{p \in M \mid \operatorname{dist}(p, \Sigma) \le d\}.$$

This follows from comparing values of w outside of K with $w(p)$ for some $p \in \Sigma$. Let $p_0 \in K$ be a point where w attains its minimum. Since $p_0 \in K$ we have that $\Sigma \subset \mathcal{B}_{2d}(p_0)$ and since $2d < \operatorname{inj}(M, g)$ we find that Σ is completely contained in a normal coordinate neighborhood centered at p_0. Let $\psi : \mathcal{B}_\rho(p_0) \to B_\rho(0) \subset \mathbf{R}^n$ be such normal coordinates where $\rho > 2d$ denotes the injectivity radius on (M, g) at p_0. Let $x \in B_\rho(0)$ and $p = \psi^{-1}(x)$. Then

$$\tilde{w}(x) := w(p) = \int_{\psi(\Sigma)} \operatorname{dist}_g(x, y)^2 \, \mathrm{d}\mu_g(y)$$

where dist_g is the distance function induced by the pull-back metric $(\psi^{-1})^* g$ to $B_\rho(0)$ and $\mathrm{d}\mu_g$ denotes the induced surface measure.

Since w is critical at p_0 also $\tilde{w}$ is critical at 0 and we compute

$$0 = \frac{\partial}{\partial x^\alpha} \tilde{w}(0) = 2 \int_{\psi(\Sigma)} y^\alpha \, \mathrm{d}\mu_g$$

since in normal coordinates $\operatorname{dist}_g(x, y)^2 = |x - y|^2 + O(|x|^2)$.

We can also change the surface measure to the Euclidean one, recording the error term:

$$\left| \int_{\psi(\Sigma)} y^\alpha \, \mathrm{d}\mu_g - \int_{\psi(\Sigma)} y^\alpha \, \mathrm{d}\mu^E \right| \le C d^3 |\Sigma|.$$

Here and in the following we use y to refer to the position vector on $\psi(\Sigma)$.

Summarizing, we arrive at the following:

Lemma 3.1. *Let (M, g) be of C_B-bounded geometry. Then there exists a constant C depending only on C_B with the following property: For every closed smooth hypersurface $\Sigma \subset M$ with extrinsic diameter $d = \operatorname{diam}(\Sigma) = \max\{\operatorname{dist}(x, y) \mid x, y \in \Sigma\} < \frac{1}{2} \operatorname{inj}(M, g)$ there exists a point $p_0 \in M$ such that $\operatorname{dist}(p_0, \Sigma) \le d$ such that in normal coordinates centered at p_0 we have that*

$$\int_{\psi(\Sigma)} y^\alpha \, \mathrm{d}\mu_g = 0 \quad \text{and} \quad \left| \int_{\psi(\Sigma)} y^\alpha \, \mathrm{d}\mu^E \right| \le C d^3 |\Sigma|.$$

Combined with theorem 2.11 and lemma 2.12 we obtain the following estimate in the case where Σ is a surface in a 3-dimensional manifold:

Lemma 3.2. *Let (M, g) be three dimensional and of C_B-bounded geometry. Then there exist constants C and $a_0 \in (0, \infty)$ depending only on C_B with the following property: For every closed smooth surface $\Sigma \subset M$ with $|\Sigma| \leq a_0$ and $\mathcal{U}(\Sigma) \leq a_0$ there exists a point $p_0 \in M$, normal coordinates $\psi : \mathcal{B}_\rho(p_0) \to B_\rho(0) \subset \mathbf{R}^3$ and in these coordinates we have that*

$$\|\tfrac{y}{R} - \nu\|_{L^2(\Sigma)} \leq C\big(R^3 + R\|\mathring{A}\|_{L^2(\Sigma)}\big) \tag{6}$$

and

$$\|\operatorname{dist}(p_0, \cdot) - R\|_{L^\infty(\Sigma)} \leq C\big(R^3 + R\|\mathring{A}\|_{L^2(\Sigma)}\big). \tag{7}$$

Here $R = R(\Sigma)$ denotes the area radius of Σ.

Proof. We choose $a_0 \in (0, 1]$ in a moment. By (4) this gives the a priori bound $\mathcal{W}(\Sigma) \leq 4\pi + \frac{1}{2} + C_B$. In view of the diameter bound from Lemma 2.2 we can choose $a_0 \in (0, 1]$ so small that Lemma 3.1 holds for Σ as in the assumption. Let $\psi : \mathcal{B}_\rho(p_0) \to B_\rho(0)$ denote the coordinates from there. In view of the diameter estimate the quantities $d = \operatorname{diam}(\Sigma)$ from Lemma 3.1 and R are comparable. Hence also $\max_{p\in\Sigma} \operatorname{dist}(p, p_0) \leq CR$ so that the estimate from Lemma 2.12 can be rephrased as

$$\|\mathring{A}^E\|^2_{L^2(\Sigma,\gamma^E)} \leq C(\mathcal{U}(\Sigma) + R^4) \tag{8}$$

where we also used that the Willmore functional is a priori bounded.

To prove (6), it is thus sufficient to prove the Euclidean inequality

$$\|\tfrac{y}{R} - \nu^E\|_{L^2(\Sigma,\gamma^E)} \leq C\big(R^3 + R\|\mathring{A}^E\|_{L^2(\Sigma,\gamma^E)}\big)$$

since in the previous coordinates we have that $\nu - \nu^E = O(R^2)$ and due to equation (8). Compute

$$\nabla^E \tfrac{y}{R} = R^{-1}\operatorname{Id} \quad \text{and} \quad \nabla^E \nu^E = \tfrac{H^E}{2}\operatorname{Id} + \mathring{A}^E.$$

Here we denote the tangential derivative along Σ by ∇^E, Id denotes the identity endomorphism field in the tangent bundle on Σ and we slightly abuse notation by not distinguishing $\mathring{A}^E$ from its associated endomorphism.

This gives the estimate

$$
\begin{aligned}
\|\nabla\left(\tfrac{y}{R}-\nu^E\right)\|_{L^2(\Sigma,\gamma^E)} &\leq \tfrac{1}{2}\|\tfrac{2}{R}-H^E\|_{L^2(\Sigma,\gamma^E)}+\|\mathring{A}^E\|_{L^2(\Sigma,\gamma^E)} \\
&\leq C\|\mathring{A}^E\|_{L^2(\Sigma,\gamma^E)}.
\end{aligned}
$$

The last inequality follows from Theorem 2.11 if $a_0 \in (0,1]$ is chosen so small that equation (8) implies $\|\mathring{A}^E\|^2_{L^2(\Sigma,\gamma^E)} \leq 6\pi$.

By choosing $a_0 \in (0,1]$ even smaller, we can ensure that the Poincaré inequality from Theorem 2.8 holds. This gives

$$
\|\tfrac{y}{R}-\nu^E-m\|_{L^2(\Sigma,\gamma^E)} \leq CR\|\mathring{A}^E\|_{L^2(\Sigma,\gamma^E)}.
$$

where

$$
m = |\Sigma|^{-1}\int_{\psi(\Sigma)}(\tfrac{y}{R}-\nu^E)\,\mathrm{d}\mu^E = |\Sigma|^{-1}\int_{\psi(\Sigma)}\tfrac{y}{R}\,\mathrm{d}\mu^E.
$$

By Lemma 3.1 we have $|m| \leq CR^2$ that is $\|m\|_{L^2(\Sigma)} \leq CR^3$ and thus the first of the claimed estimate follows.

To show equation (7) observe that the Euclidean center of gravity a^Σ of $\psi(\Sigma)$ satisfies

$$
|a^E| = |\Sigma|_E^{-1}\left|\int_{\psi(\Sigma)} y\,\mathrm{d}\mu^E\right| \leq CR^3.
$$

Parameterizing $\psi(\Sigma)$ with a map $F : S_{R^E}(a^E) \to \psi(\Sigma)$ as in Theorem 2.11, we get that for every $y \in S_{R^E}(a^E)$

$$
|y-a^E|-|a^E|-|F(y)-y| \leq |F(y)| \leq |y-a^E|+|a^E|+|F(y)-y|
$$

so that

$$
\big||F(y)|-R^E\big| \leq |a^E|+|F(y)-y| \leq CR^3+CR^E\|\mathring{A}^E\|_{L^2(\Sigma,\gamma^E)}.
$$

Since we are in normal coordinates around p_0 we have that for all $x \in \Sigma$ that $\operatorname{dist}(x,p_0) = |\psi(x)| = |F(F^{-1}(\psi(x)))|$ and the second claim follows. $\square$

Corollary 3.3. *Let (M,g) and Σ satisfy the assumptions of Lemma 3.2 and let ψ be as there. Then for every $k \in \mathbf{N}\cup\{0\}$ there is a constant C_k*

depending only on the constant C in Lemma 3.2 and on k such that

$$\left|\int_{\psi(\Sigma)} \prod_{l=1}^{2k+1} \rho_l \,\mathrm{d}\mu\right| \le C_k\big(R^3 + R\|\mathring{A}\|_{L^2(\Sigma)}\big).$$

Here, for every $l \in \{1, \ldots, 2k+1\}$ we can choose ρ_l freely from the functions $\{\nu^\alpha, \frac{y^\alpha}{R} \mid \alpha = 1, 2, 3\}$.

Proof. Note that if $k = 0$ then the claim directly follows from Lemma 3.1 if $\rho_1 = \frac{y^\alpha}{R}$ and from the fact that $\int_{\psi(\Sigma)} (\nu^E)^\alpha \,\mathrm{d}\mu^E = 0$ if $\rho_1 = \nu^\alpha$ for some $\alpha = 1, 2, 3$. For brevity, we indicate the proof only in the case $k = 1$ below. Also note that it is sufficient to consider the Euclidean setting, that is with $\rho_l \in \{(\nu^E)^\alpha, \frac{y^\alpha}{R} \mid \alpha = 1, 2, 3\}$ and with $\|\mathring{A}\|_{L^2(\Sigma)}$ replaced by $\|\mathring{A}^E\|_{L^2(\Sigma,\gamma^E)}$ using the same reduction as in the proof of Lemma 3.2.

We proceed in two steps: in the first step, we use Theorem 2.11 to prove the case $\rho_l \in \{(\nu^E)^\alpha \mid \alpha = 1, 2, 3\}$, in the second step we use Lemma 3.2 to conclude.

Step 1: Let $\rho_l = (\nu^E)^{\alpha_l}$ for $l = 1, 2, 3$ and $\alpha_l \in \{1, 2, 3\}$. Let R^E and a^E as in Theorem 3.2 and denote $S := S_{R^E}(a^E)$. Let $F :\to \Sigma$ be the parameterization from Theorem 3.2, $N : S \to S^2$ be the normal of S and $\tilde{\rho}_l := \rho_l \circ F$. Then we can write

$$\begin{aligned}
\int_{\psi(\Sigma)} \rho_1\rho_2\rho_3 \,\mathrm{d}\mu^E &= \int_S \tilde{\rho}_1\tilde{\rho}_2\tilde{\rho}_3\phi^2 \,\mathrm{d}\mu^E \\
&= \int_S N^{\alpha_1}N^{\alpha_2}N^{\alpha_3} + N^{\alpha_1}N^{\alpha_2}N^{\alpha_3}(\phi^2 - 1) + (\tilde{\rho}_1 - N^{\alpha_1})N^{\alpha_2}N^{\alpha_3}\phi^2 \\
&\qquad + \tilde{\rho}_1(\tilde{\rho}_2 - N^{\alpha_2})N^{\alpha_3}\phi^2 + \tilde{\rho}_1\tilde{\rho}_2(\tilde{\rho}_3 - N^{\alpha_3})\phi^2 \,\mathrm{d}\mu^E.
\end{aligned}$$

Since S is a sphere $\int_S N^{\alpha_1}N^{\alpha_2}N^{\alpha_3} \,\mathrm{d}\mu^E = 0$ and thus, using Cauchy-Schwarz in the first inequality and Theorem 2.11 in the last inequality we conclude

$$\begin{aligned}
&\left|\int_{\psi(\Sigma)} \rho_1\rho_2\rho_3 \,\mathrm{d}\mu^E\right| \\
&\quad \le CR\|N - \nu^E \circ R\|_{L^2(S,\gamma^E)} + CR^2\|\phi^2 - 1\|_{L^\infty(S)} \le CR^2\|\mathring{A}^E\|_{L^2(\Sigma,\gamma^E)}.
\end{aligned}$$

Note that this implies the claimed inequality.

Step 2: Assume that $\rho_l = (\nu^E)^{\alpha_l}$ or $\rho_l = \frac{y^{\alpha_l}}{R}$ for $l = 1, 2, 3$. We can use a telescope sum as above and Cauchy-Schwarz to estimate

$$\left|\int_{\psi(\Sigma)} \rho_1\rho_2\rho_3 - (\nu^E)^{\alpha_1}(\nu^E)^{\alpha_2}(\nu^E)^{\alpha_3} \,\mathrm{d}\mu\right| \le CR\sum_{l=1}^{3} \|\rho_l - (\nu^E)^{\alpha_l}\|_{L^2(\Sigma,\gamma^E)}.$$

Note that the terms in the sum on the right either vanish or can be bounded using Lemma 3.2. We thus arrive at the claimed inequality. $\square$

Note that products of an even number of factors can be treated in a similar fashion as above and equal the respective integrals on a centered round sphere up to the same error term as above.

4. Geometric identities

Throughout this section we assume that (M, g) has C_B-bounded geometry and that $\Sigma \subset M$ is a closed, immersed, smooth surface such that

1) Σ satisfies equation (1).
2) $|\Sigma| \leq a_0$ and $\mathcal{W}(\Sigma) \leq 4\pi + a_0$.

Here we assume that a_0 is so small that the estimates from Theorem 2.3 and Lemmas 2.4 and 2.10 hold.

To shorten the exposition, we augment the big-O notation as follows. If f is some quantity defined on a surface Σ as above, we say $f = O_{L^p}(R^k)$ if

$$\int_\Sigma f^p \, \mathrm{d}\mu \leq C R^{pk+2},$$

where $R = R(\Sigma)$ refers to the area radius of Σ. We also use this for $p = \infty$, that is $f = O_{L^\infty}(R^k)$ denotes

$$\|f\|_{L^\infty(\Sigma)} \leq C R^k.$$

Using this notation, the a priori estimates from section 2 can be stated as follows:

$$\mathring{A} = O_{L^\infty}(R), \qquad \nabla A = O_{L^2}(1), \quad \text{and} \quad \nabla^2 H = O_{L^2}(R^{-1}).$$

Lemmas 2.4 and 2.10 imply that $H = O_{L^\infty}(R^{-1})$ and $H^{-1} = O_{L^\infty}(R)$.

The following computations are done in abstract index notation, where Latin indices $i, j, k, \ldots \in \{1, 2\}$ refer to an local orthonormal frame $\{e_1, e_2\}$ on Σ and ν denotes a choice of normal to $\Sigma \subset M$ so that $A_{ij} := A(e_i, e_j) = g({}^M\nabla_{e_i}\nu, e_j)$ where ${}^M\nabla$ denotes the Levi-Civita connection on (M, g). Tangential derivatives to Σ are denoted by ∇.

4.1. The Hessian of $\operatorname{Ric}(\nu,\nu)$

We begin by calculating the gradient

$$
\begin{aligned}
\nabla_i \operatorname{Ric}(\nu,\nu) &= ({}^M\nabla_{e_i} \operatorname{Ric})(\nu,\nu) + 2A_{ik} \operatorname{Ric}(e_k,\nu) \\
&= ({}^M\nabla_{e_i} \operatorname{Ric})(\nu,\nu) + H \operatorname{Ric}(e_i,\nu) + 2\mathring{A}_{ik} \operatorname{Ric}(e_k,\nu) \\
&= H\omega_i + O_{L^\infty}(1).
\end{aligned} \tag{9}
$$

As before $\omega = \operatorname{Ric}(\nu,\cdot)^T$ denotes the tangential projection of the 1-form $\operatorname{Ric}(\nu,\cdot)$ to Σ. In the second step we used the splitting

$$A_{ij} = \mathring{A}_{ij} + \tfrac{1}{2}H\gamma_{ij}.$$

Differentiating further yields

$$
\begin{aligned}
\nabla^2_{i,j} \operatorname{Ric}(\nu,\nu) &= ({}^M\nabla^2_{i,j} \operatorname{Ric})(\nu,\nu) - ({}^M\nabla_\nu \operatorname{Ric})(\nu,\nu)A_{ij} \\
&\quad + 2({}^M\nabla_{e_i} \operatorname{Ric})(e_k,\nu)A_{jk} + 2({}^M\nabla_{e_j} \operatorname{Ric})(e_k,\nu)A_{ik} \\
&\quad - 2\operatorname{Ric}(\nu,\nu)A_{kj}A^k_j + 2\operatorname{Ric}(e_k,e_l)A^k_i A^l_j \\
&\quad + 2\operatorname{Ric}(e_k,\nu)\nabla_{e_i} A^k_j \\
&= -2\operatorname{Ric}(\nu,\nu)A^k_i A_{kj} + 2\operatorname{Ric}(e_k,e_l)A^k_i A^l_j + O_{L^2}(R^{-1}) \\
&= -\tfrac{1}{2}H^2 \operatorname{Ric}(\nu,\nu)\gamma_{ij} + \tfrac{1}{2}H^2 T_{ij} + O_{L^2}(R^{-1}).
\end{aligned} \tag{10}
$$

Here $T_{ij} = \operatorname{Ric}(e_i,e_j)$ denotes the tangential projection of the Ricci-Tensor. The last step uses Lemma 2.10 to discard the terms containing $\mathring{A}$ into the error term.

Taking the trace in equation (10) yields that

$$\Delta \operatorname{Ric}(\nu,\nu) = -\tfrac{3}{2}H^2 \operatorname{Ric}(\nu,\nu) + \tfrac{1}{2}H^2 \operatorname{Sc} + O_{L^2}(R^{-1}). \tag{11}$$

We thus infer that the trace free part of the Hessian of $\operatorname{Ric}(\nu,\nu)$ is given by

$$
\begin{aligned}
\big(\nabla^2 \operatorname{Ric}(\nu,\nu)\big)^\circ_{ij} &= \nabla^2_{i,j} \operatorname{Ric}(\nu,\nu) - \tfrac{1}{2}\Delta \operatorname{Ric}(\nu,\nu)\gamma_{ij} \\
&= \tfrac{1}{2}H^2\big(\tfrac{1}{2}\operatorname{Ric}(\nu,\nu)\gamma_{ij} - \tfrac{1}{2}\operatorname{Sc}\gamma_{ij} + T_{ij}\big) + O(R^{-1}) \\
&= \tfrac{1}{2}H^2 \mathring{T}_{ij} + O_{L^2}(R^{-1}).
\end{aligned} \tag{12}
$$

Here we used that

$$\mathring{T}_{ij} = T_{ij} - \tfrac{1}{2}\operatorname{tr} T\gamma_{ij} = T_{ij} + \tfrac{1}{2}\operatorname{Ric}(\nu,\nu)\gamma_{ij} - \tfrac{1}{2}\operatorname{Sc}\gamma_{ij}.$$

4.2. The covariant derivative of ω

In a calculation similar to equation (10), we derive

$$\begin{aligned}\nabla_i\omega_j &= ({}^M\nabla_i \operatorname{Ric})(\nu, e_j) + A_i^k \operatorname{Ric}(e_k, e_j) - A_{ij}\operatorname{Ric}(\nu,\nu)\\ &= \tfrac12 H(T_{ij} - \operatorname{Ric}(\nu,\nu)\gamma_{ij}) + O_{L^\infty}(1).\end{aligned}\tag{13}$$

Taking the trace yields

$$\operatorname{div}\omega = \tfrac12 H \operatorname{Sc} - \tfrac32 H \operatorname{Ric}(\nu,\nu) + O_{L^\infty}(1).\tag{14}$$

Later we will also use the following combination

$$\begin{aligned}2\nabla_i\omega_j - \operatorname{div}\omega\gamma_{ij} &= H(T_{ij} + \tfrac12 \operatorname{Ric}(\nu,\nu)\gamma_{ij} - \tfrac12 \operatorname{Sc}\gamma_{ij}) + O_{L^\infty}(1)\\ &= H\mathring{T}_{ij} + O_{L^\infty}(1).\end{aligned}\tag{15}$$

4.3. The Laplacian of $\mathring{T}$

First calculate the Hessian of T. Neglecting the lower order terms yields

$$\begin{aligned}\nabla^2_{k,l}T_{ij} = \tfrac14 H^2\big(&\gamma_{ki}\gamma_{lj}\operatorname{Ric}(\nu,\nu) + \gamma_{kj}\gamma_{li}\operatorname{Ric}(\nu,\nu)\\ &- \gamma_{ki}\operatorname{Ric}(e_l,e_j) - \gamma_{kj}\operatorname{Ric}(e_l,e_i)\big) + O_{L^2}(R^{-1}).\end{aligned}$$

Taking the trace gives

$$\Delta T_{ij} = \tfrac12 H^2(\operatorname{Ric}(\nu,\nu)\gamma_{ij} - T_{ij}) + O_{L^2}(R^{-1}).$$

Thus we can calculate further

$$\Delta\mathring{T}_{ij} = \Delta T_{ij} - \tfrac12\Delta\big(\operatorname{Sc} - \operatorname{Ric}(\nu,\nu)\big)\gamma_{ij}.$$

In view of the fact that

$$\Delta\operatorname{Sc} = {}^M\Delta\operatorname{Sc} - {}^M\nabla^2_{\nu,\nu}\operatorname{Sc} + Hg({}^M\nabla\operatorname{Sc},\nu) = O_{L^2}(R^{-1}),$$

and the expression for $\Delta\operatorname{Ric}(\nu,\nu)$ in (11), we infer that

$$\Delta\mathring{T}_{ij} = -\tfrac12 H^2\mathring{T}_{ij} + O_{L^2}(R^{-1}).\tag{16}$$

5. Expansion of the curvature

In this section we consider the crucial geometric quantities on Σ as in section 4 and derive the top order deviations from their Euclidean value.

5.1. Curvature corrections to H^2

Combining equations (11) and (1) with the curvature estimates we infer that

$$
\begin{aligned}
&\Delta\big(\tfrac{1}{2}H^2 - \tfrac{2}{3}\operatorname{Ric}(\nu,\nu)\big)\\
&\quad= H\Delta H + |\nabla H|^2 - \tfrac{2}{3}\Delta\operatorname{Ric}(\nu,\nu)\\
&\quad= -H^2\operatorname{Ric}(\nu,\nu) - H^2\lambda\\
&\qquad - \tfrac{2}{3}\big(- \tfrac{3}{2}H^2\operatorname{Ric}(\nu,\nu) + \tfrac{1}{2}H^2\operatorname{Sc}\big) + O(R^{-1})\\
&\quad= -H^2\big(\lambda + \operatorname{Sc}\big) + O_{L^2}(R^{-1})\\
&\quad= O_{L^2}(R^{-1}).
\end{aligned}
\tag{17}
$$

This identity leads to the following estimate.

Proposition 5.1. *Let (M,g) be of C_B-bounded geometry. Then there exist constants $a_0 \in (0,\infty)$ and C depending only on C_B such that for every surface Σ satisfying (1), with $|\Sigma| \le a_0$ and $\mathcal{W}(\Sigma) \le 4\pi + a_0$ we have the estimate*

$$\big\|\tfrac{1}{2}H^2 - 8\pi|\Sigma|^{-1} - \tfrac{2}{3}\operatorname{Ric}(\nu,\nu) + \tfrac{5}{9}\operatorname{Sc}(0)\big\|_{L^\infty} \le CR(\Sigma).$$

Proof. Let $w = \frac{1}{2}H^2 - \frac{2}{3}\operatorname{Ric}(\nu,\nu)$. The Bochner identity from Lemma 2.6 implies that

$$\int_\Sigma 2|\nabla^2 w|^2 + H^2|\nabla w|^2 \,\mathrm{d}\mu \le \int_\Sigma |\Delta w|^2 + \big(\operatorname{Ric}(\nu,\nu) - \operatorname{Sc} + |\mathring{A}|^2\big)|\nabla w|^2 \,\mathrm{d}\mu.$$

Note that Ric and Sc are bounded by a constant, that $\|\mathring{A}\|_{L^\infty} \le CR(\Sigma)$ by lemma 2.10, and that $H \ge C^{-1}R(\Sigma)^{-1}$ if a_0 is chosen small enough. If necessary we can decrease a_0 further so that the gradient term on the right can be absorbed to the left. In view of equation (17), this yields

$$\|\nabla^2 w\|_{L^2} \le C \quad\text{and}\quad \|\nabla w\|_{L^2} \le CR(\Sigma).$$

Consequently, the Poincaré inequality implies the estimate

$$\|w - \bar{w}\|_{L^2} \le CR(\Sigma)^2.$$

Plugging this into the estimate from lemma 2.9, we infer that

$$\|w - \bar{w}\|_{L^\infty} \le CR(\Sigma). \tag{18}$$

It remains to calculate $\bar{w}$. To this end recall [6, Theorem 5.1]. This implies that

$$\left|\int_\Sigma \tfrac{1}{2} H^2 \,\mathrm{d}\mu - 8\pi + \frac{|\Sigma|}{3}\operatorname{Sc}(0)\right| \leq CR(\Sigma)^3.$$

From [6, Lemma 3.3] it follows that in addition

$$\left|\int_\Sigma \operatorname{Ric}(\nu,\nu)\,\mathrm{d}\mu - \frac{|\Sigma|}{3}\operatorname{Sc}(0)\right| \leq CR(\Sigma)^3.$$

In combination, this implies that for $\bar{w} = |\Sigma|^{-1}\int_\Sigma w\,\mathrm{d}\mu$ we have.

$$\left|\bar{w} - 8\pi + \tfrac{5}{9}\operatorname{Sc}(0)\right| \leq CR.$$

In view of (18) this yields the claim. □

Remark 5.2. Note that this is *not* the expansion of H^2 on geodesic spheres, which can be found in [12, Lemma 2.4] for example. This is due to the fact that geodesic spheres do not satisfy (1) on the order on which we do these calculations. In other words, if a surface satisfies (1), then its shape differs from that of a geodesic sphere in a way visible in the lower order correction terms of the mean curvature.

5.2. Curvature corrections for H and its derivatives

By a slight variation of terms, one can also derive estimates for H instead of $\frac{1}{2}H^2$. Alternatively one can proceed as follows. Recall that for functions $f, g \in C^\infty(\Sigma)$ with $g \neq 0$ we have the identity

$$\nabla^2_{i,j}\tfrac{u}{v} = -\tfrac{1}{v^2}\big(\nabla_i v\nabla_j u + \nabla_i u\nabla_j v\big) + \tfrac{1}{v}\nabla^2_{i,j}u - \tfrac{u}{v^2}\nabla^2_{i,j}v + 2\tfrac{u}{v^3}u\nabla_i v\nabla_j v.$$

Using the a priori estimates for H, ∇H and $\nabla^2 H$ as before, we find that

$$\nabla^2_{i,j}\big(H^{-1}\operatorname{Ric}(\nu,\nu)\big) = H^{-1}\nabla^2_{i,j}\operatorname{Ric}(\nu,\nu) + O_{L^2}(1),$$

so that equation (10) yields

$$\nabla^2_{i,j}\big(H^{-1}\operatorname{Ric}(\nu,\nu)\big) = -\tfrac{1}{2}H\operatorname{Ric}(\nu,\nu)\gamma_{ij} + \tfrac{1}{2}HT_{ij} + O_{L^2}(1).$$

Splitting into trace part and trace-free part we get

$$\Delta\big(H^{-1}\operatorname{Ric}(\nu,\nu)\big) = -\tfrac{3}{2}H\operatorname{Ric}(\nu,\nu) + \tfrac{1}{2}H\operatorname{Sc} + O_{L^2}(1) \tag{19}$$

and

$$\left[\nabla^2\big(H^{-1}\operatorname{Ric}(\nu,\nu)\big)\right]^\circ_{ij} = \tfrac{1}{2}H\mathring{T}_{ij} + O_{L^2}(1). \tag{20}$$

Let $v := H - \frac{2}{3}H^{-1}\operatorname{Ric}(\nu,\nu)$. Combining equations (19) and (1), with the estimate from theorem 2.5 as in section 5.1 we find that

$$\Delta v = O_{L^2}(1).$$

Arguing as before, the Bochner identity implies:

$$\|\nabla^2 v\|_{L^2} \leq CR \quad \text{and} \quad \|\nabla v\|_{L^2} \leq CR^2. \tag{21}$$

These considerations imply the following estimate.

Proposition 5.3. *Assume that (M,g) and Σ are as in Proposition 5.1. Then*

$$\|(\nabla^2 H)^\circ - \tfrac{1}{3}H\mathring{T}\|_{L^2} \leq CR(\Sigma) \quad \textit{and} \quad \|\nabla H - \tfrac{2}{3}\omega\|_{L^2} \leq CR(\Sigma)^2.$$

Proof. The proof follows directly from the estimates (21) in combination with formulas (9) and (12). □

5.3. Curvature corrections for $\mathring{A}$

To estimate the corrections of the curvature to $\mathring{A}$ recall the Simons-Identity on Σ as in equation (5). In view of the a priori estimates from theorem 2.3 and the conventions in in section 4, on surfaces as in theorem 2.3 this yields

$$\Delta\mathring{A}_{ij} = (\nabla^2 H)^0_{ij} + \tfrac{1}{2}H^2\mathring{A}_{ij} + 2\nabla_i\omega_j - \operatorname{div}\omega\gamma_{ij} + O_{L^\infty}(R). \tag{22}$$

In view of proposition 5.3 and equation (15) we thus infer

$$\Delta\mathring{A}_{ij} = \tfrac{4}{3}H\mathring{T}_{ij} + \tfrac{1}{2}H^2\mathring{A}_{ij} + O_{L^2}(1).$$

In view of the a priori estimates and equation (16) we find that

$$\Delta(H^{-1}\mathring{T})_{ij} = -\tfrac{1}{2}H\mathring{T}_{ij} + O_{L^2}(1),$$

so that the tensor

$$S_{ij} := \mathring{A}_{ij} + \frac{4}{3}H^{-1}\mathring{T}_{ij}$$

satisfies

$$\Delta S_{ij} = \tfrac{1}{2} H^2 S_{ij} + O_{L^2}(1). \tag{23}$$

Multiplying (23) by S^{ij} and integrating by parts implies

$$\int_\Sigma |\nabla S|^2 + \tfrac{1}{2} H^2 |S|^2 \,\mathrm{d}\mu \leq \int_\Sigma |S| \,\mathrm{d}\mu \leq \frac{1}{4} \int_\Sigma H^2 |S|^2 \,\mathrm{d}\mu + \int_\Sigma H^{-2} \,\mathrm{d}\mu.$$

Absorbing the first term on the right to the left yields the following estimate.

Proposition 5.4. *Assume that (M, g) and Σ are as in Proposition 5.1. Then*

$$\|\mathring{A} + \tfrac{4}{3} H^{-1} \mathring{T}\|_{L^2} \leq CR(\Sigma)^3 \quad \textit{and} \quad \|\nabla \mathring{A} + \tfrac{4}{3} H^{-1} \nabla \mathring{T}\|_{L^2} \leq CR(\Sigma)^2.$$

6. Expansion of the metric

It is well known[1] that the metric in normal coordinates has the expansion

$$g_{\alpha\beta}(y) = \delta_{\alpha\beta} + \tfrac{1}{3} \operatorname{Rm}_{\alpha\mu\beta\nu} y^\mu y^\nu + O(|y|^2).$$

Here we denote $\operatorname{Rm}_{\alpha\mu\beta\nu} = \operatorname{Rm}_{\alpha\mu\beta\nu}(0)$ and all other curvature quantities are evaluated at 0 as well. From this we calculate that

$$g_{\alpha\beta,\mu\nu}(0) = \tfrac{1}{3} (\operatorname{Rm}_{\alpha\mu\beta\nu} + \operatorname{Rm}_{\alpha\nu\beta\mu}).$$

The Christoffel symbols thus satisfy

$$\begin{aligned}
&\Gamma^\nu_{\alpha\beta,\gamma}(0) \\
&\quad = \tfrac{1}{2} \delta^{\nu\mu} \big(g_{\alpha\mu,\beta\gamma} + g_{\beta\mu,\alpha\gamma} - g_{\alpha\beta,\mu\gamma} \big) \\
&\quad = \tfrac{1}{6} \delta^{\nu\mu} (\operatorname{Rm}_{\alpha\beta\mu\gamma} + \operatorname{Rm}_{\alpha\gamma\mu\beta} + \operatorname{Rm}_{\beta\alpha\mu\gamma} + \operatorname{Rm}_{\beta\gamma\mu\alpha} - \operatorname{Rm}_{\alpha\mu\beta\gamma} - \operatorname{Rm}_{\alpha\gamma\beta\mu}) \\
&\quad = \tfrac{1}{3} \delta^{\nu\mu} (\operatorname{Rm}_{\alpha\gamma\mu\beta} + \operatorname{Rm}_{\beta\gamma\mu\alpha}) \\
&\quad = -\tfrac{1}{3} \delta^{\nu\mu} (\operatorname{Rm}_{\alpha\gamma\beta\mu} + \operatorname{Rm}_{\beta\gamma\alpha\mu}) = -\tfrac{1}{3} (\operatorname{Rm}^\nu_{\alpha\gamma\beta} + \operatorname{Rm}^\nu_{\beta\gamma\alpha}).
\end{aligned}$$

From this we get that in normal coordinates

$$\begin{aligned}
\Gamma^\nu_{\alpha\beta}(y) &= \Gamma^\nu_{\alpha\beta}(0) + \Gamma^\nu_{\alpha\beta,\gamma}(0) y^\gamma + O(|y|^2) \\
&= -\tfrac{1}{3} (\operatorname{Rm}^\nu_{\alpha\gamma\beta} + \operatorname{Rm}^\nu_{\beta\gamma\alpha}) y^\gamma + O(|y|^2).
\end{aligned}$$

---

[1] We use the convention for the curvature tensor from [8, Section 2] that is $\operatorname{Rm}_{\alpha\beta\gamma\nu} = \langle (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) \partial_\gamma, \partial_\nu \rangle$ and $\operatorname{Rm}^\nu_{\alpha\beta\gamma} = g^{\nu\mu} \operatorname{Rm}_{\alpha\beta\gamma\mu}$.

This implies that for a constant vector field $b = b^\nu \partial_\nu \in \mathbf{R}^3$ we have

$$\nabla_\alpha b^\nu = \partial_\alpha b^\nu + \Gamma^\nu_{\alpha\beta} b^\beta = -\tfrac{1}{3}(\mathrm{Rm}^\nu_{\alpha\gamma\beta}(0) + \mathrm{Rm}^\nu_{\beta\gamma\alpha}(0)) y^\gamma b^\beta + O(|y|^2) \tag{24}$$

and

$$\operatorname{div} b = \nabla_\alpha b^\alpha = -\tfrac{1}{3}\operatorname{Ric}_{\beta\gamma}(0) y^\gamma b^\beta + O(|y|^2). \tag{25}$$

Terms like these will show up in the position estimates. Here we explicitly included the point at which to evaluate the curvature for later reference.

7. The position estimates revisited

The basic idea of the position estimates for small area constrained Willmore surfaces in [6] is to test the Euler-Lagrange-Equation

$$\delta_f \mathcal{W}(\Sigma) = \lambda \int_\Sigma f H \,\mathrm{d}\mu \tag{26}$$

with the function $f = H^{-1} g(b,\nu)$. The main result of [6] is the estimate

$$|\nabla \operatorname{Sc}(p)| \le C R(\Sigma) \tag{27}$$

where p is a point with $\operatorname{dist}(p,\Sigma) \le \operatorname{diam}(\Sigma)$ and C is a constant depending only on C_B. With this choice of point, we have that $r(x) = \operatorname{dist}(p,x)$ for $x \in \Sigma$ is comparable to the area radius $R(\Sigma)$.

To improve this estimate further we have to carefully choose the center point p of the above coordinates. The main result of the paper in this section is:

Theorem 7.1. *Let (M,g) be a 3-manifold with C_B-bounded geometry. Then there exist constants $a_0 \in (0,\infty)$ and $C \in (0,\infty)$ depending only on C_B with the following property. Let $\Sigma \subset M$ be a surface satisfying the Euler-Lagrange equation* (1) *for some $\lambda \in \mathbf{R}$, with $|\Sigma| \le a_0$, and $\mathcal{W}(\Sigma) \le 4\pi + a_0$. Then there exists a point $p_0 \in M$ such that*

i) $|\operatorname{dist}(p_0,x) - R(\Sigma)| \le C R(\Sigma)^3$ *for all $x \in \Sigma$,*

ii) with respect to normal coordinates $\psi : \mathcal{B}_\rho(p_0) \to B_\rho(0) \subset \mathbf{R}^3$ centered at p_0 we have

$$\int_{\psi(\Sigma)} y^\alpha \,\mathrm{d}\mu_g(y) = 0,$$

iii) and $|\nabla \operatorname{Sc}(p_0)| \leq CR(\Sigma)^2$.

Remark 7.2. Note that appealing to theorem 2.11, lemma 2.12 and estimate 2.3, we automatically have that Σ is $W^{2,2}$-close to the geodesic sphere of radius R around p_0 in the following sense. Denote by $h_R : \mathbf{R}^3 \to \mathbf{R}^3 : y \mapsto \frac{y}{R}$ the scaling vector field. Let $\Sigma_R := h_R(\psi(\Sigma) \subset B_{\frac{\rho}{R}}(0)$. Then there is a map $F : S^2 \to \Sigma_R$, conformal with respect to metric on Σ_R induced by the Euclidean metric such that

$$\|F\|_{W^{2,2}(S^2)} \leq CR(\Sigma)^2.$$

For the proof of Theorem 7.1 assume that Σ is as in the statement of Theorem 7.1 and that a_0 is chosen so small that all the estimates from sections 2 to section 6 are applicable. In particular, Lemma 3.1 gives a point $p_0 \in M$ such that $|\operatorname{dist}(p_0, x) - R(\Sigma)| \leq CR(\Sigma)^3$ for all $x \in \Sigma$ and such that if $\psi : \mathcal{B}_\rho(p_0) \to B_\rho(0) \subset \mathbf{R}^3$ are normal coordinates at p_0 then $\int_{\psi(\Sigma)} y \,\mathrm{d}\mu_g = 0$. Then the first two assertions of the Theorem directly follow. The estimate for $|\nabla \operatorname{Sc}(p_0)|$ follows from the calculations in the remainder of this section. All these calculations are done in the normal coordinates centered at p_0.

Recall the splitting

$$\delta_f \mathcal{W}(\Sigma) = \delta_f \mathcal{U}(\Sigma) + \delta_f \mathcal{V}(\Sigma) \tag{28}$$

that was used with (26) for the test function $f = H^{-1} g(b, \nu)$. Here $b \in \mathbf{R}^3$ is a constant vector in the normal coordinate neighborhood. The computations below use the same f and the same splitting.

7.1. The right hand side

For $f = H^{-1} g(b, \nu)$ we have that

$$\int_\Sigma fH \,\mathrm{d}\mu = \int_\Sigma g(b,\nu) \,\mathrm{d}\mu = \int_\Omega \operatorname{div} b \,\mathrm{d}V$$

where Ω is the region enclosed by Σ. In the integral on the right, we replace the volume form of g by the Euclidean volume form of the normal coordinates at p_0 and obtain an error of the form

$$\left| \int_\Omega \operatorname{div} b \,\mathrm{d}V - \int_\Omega \operatorname{div} b \,\mathrm{d}V^E \right| \leq \operatorname{Vol}(\Omega) \sup_\Omega |\nabla b| R^2 \leq C|\Sigma|^3$$

since $|\nabla b| = O(R)$, $|\,\mathrm{d}V - \mathrm{d}V^E| = O(R^2)\,\mathrm{d}V^E$ and $\mathrm{Vol}(\Omega) = O(R^3)$ by [6, Eq. (4.7)]. As usual we abbreviate $R = R(\Sigma)$. At this point we do not care about errors of the order $O(R^5)$ but want to compute the top order term which is $O(R^4)$. In section 6 we computed that

$$\operatorname{div} b = -\tfrac{1}{3}\,\mathrm{Ric}_{\alpha\beta}\, y^\alpha b^\beta + O(R^2).$$

Note that the volume integral of the error term is $O(R^5)$, and that Ric here is evaluated at p_0, the origin of the normal coordinates y. We thus get

$$\begin{aligned}\int_\Omega \operatorname{div} b \,\mathrm{d}\mu &= -\tfrac{1}{3}\,\mathrm{Ric}_{\alpha\beta}\, b^\beta \int_\Omega y^\alpha \,\mathrm{d}\mu^E + O(R^5)\\ &= -\tfrac{1}{6}\,\mathrm{Ric}_{\alpha\beta}\, b^\beta \int_\Sigma |y|^2 (\nu^E)^\alpha \,\mathrm{d}\mu^E + O(R^5).\end{aligned}$$

Here ν_E denotes the normal to Σ with respect to the Euclidean metric in our coordinates. From Corollary 3.3 with $k=1$, $\rho_1 = \rho_2 = \frac{y^\beta}{R}$ and $\rho_3 = (\nu^E)^\alpha$ it follows that

$$\int_\Sigma (y^\beta)^2 (\nu^E)^\alpha \,\mathrm{d}\mu^E \le CR^5$$

and after summation over β we arrive at

$$\left| \int_\Omega \operatorname{div} b \,\mathrm{d}\mu \right| \le CR^5$$

or, in combination with the estimates above, with (26) and using Theorem 2.5 this gives

$$|\delta_f \mathcal{W}(\Sigma)| \le CR^5. \tag{29}$$

Note that this improves the estimate from [6] by one power of R.

7.2. The variation of $\mathcal{U}(\Sigma)$

From [6] we have

$$\delta_f \mathcal{U}(\Sigma) = -\int_\Sigma 2\langle \mathring{A}, \nabla^2 f\rangle + 2f\langle \mathring{A}, \mathring{T}\rangle + fH|\mathring{A}|^2 \,\mathrm{d}\mu \tag{30}$$

and that for our choice $f = H^{-1} g(b,\nu)$ we have

$$\begin{aligned}\nabla^2_{ij} f = &-A_i^k A_{jk} f + H^{-1} g(\nabla_i b, e_k) A^{kj} - H^{-2}\nabla_i H g(b, e_k) A_j^k\\ &+ \nabla_i\big(H^{-1} g(\nabla_j b, \nu) - H^{-2}\nabla_j H g(b,\nu)\big)\end{aligned}$$

Since

$$A_i^k A_{jk} = \mathring{A}_i^k \mathring{A}_{jk} + H\mathring{A}_{ij} + \tfrac{1}{4}H^2 g_{ij}$$

and since the first and the last term give zero when contracted with $\mathring{A}$ we get that

$$\begin{aligned}\int_\Sigma \langle \nabla^2 f, \mathring{A}\rangle \,\mathrm{d}\mu = \int_\Sigma &\operatorname{div} \mathring{A}^j \big(H^{-2}\nabla_j H g(b,\nu) - H^{-1} g(\nabla_j b,\nu)\big) + fH|\mathring{A}|^2 \\ &- H^{-2}\nabla_i H g(b,e_k) A_j^k \mathring{A}^{ij} + H^{-1} g(\nabla_i b, e_k) A_j^k \mathring{A}^{ij} \,\mathrm{d}\mu.\end{aligned}$$

Plugging into (30) we get that

$$\begin{aligned}\delta_f \mathcal{U}(\Sigma) = 2\int_\Sigma &\operatorname{div} \mathring{A}^j \big(H^{-1} g(\nabla_j b,\nu) - H^{-2}\nabla_j H g(b,\nu)\big) - \tfrac{3}{2} fH|\mathring{A}|^2 \\ &+ H^{-2}\nabla_i H g(b,e_k) A_j^k \mathring{A}^{ij} - H^{-1} g(\nabla_i b, e_k) A_j^k \mathring{A}^{ij} - f\langle \mathring{A}, \mathring{T}\rangle \,\mathrm{d}\mu.\end{aligned}$$

We shall only keep the top order parts of the first two terms in the second line. In view of the L^∞ estimates for $\mathring{A}$, H^{-1} and the L^2-estimates for ∇H, we have that

(31)

$$\begin{aligned}\delta_f \mathcal{U}(\Sigma) = \int_\Sigma &2\operatorname{div} \mathring{A}^j \big(H^{-1} g(\nabla_j b,\nu) - H^{-2}\nabla_j H g(b,\nu)\big) - 3fH|\mathring{A}|^2 \\ &+ H^{-1}\nabla_i H g(b,e_j)\mathring{A}^{ij} - g(\nabla_i b, e_j)\mathring{A}^{ij} - 2f\langle \mathring{A}, \mathring{T}\rangle \,\mathrm{d}\mu + O(R^5).\end{aligned}$$

In view of the estimates in section 5 all the above terms can be replaced with their highest order parts. The error terms are then of order $O(R^5)$ or better. To be specific, we recall that to top order

$$H^{-1} \approx \tfrac{R}{2}, \quad \mathring{A} \approx -\tfrac{4}{3}H^{-1}\mathring{T}, \quad \nabla H \approx \tfrac{2}{3}\omega, \quad \text{and} \quad \operatorname{div}\mathring{A} = \tfrac{4}{3}\omega.$$

This yields that

$$\begin{aligned}\delta_f \mathcal{U}(\Sigma) = \frac{2}{3}\int_\Sigma &2R\omega_j g(\nabla_j b,\nu) - \tfrac{2}{3}R^2|\omega|^2 g(b,\nu) - R^2 g(b,\nu)|\mathring{T}|^2 \\ &- \tfrac{1}{3}R^2\omega_i g(b,e_j)\mathring{T}^{ij} + Rg(\nabla_i b, e_j)\mathring{T}^{ij} \,\mathrm{d}\mu + O(R^5)\end{aligned} \tag{32}$$

Note that all the previous terms can be expanded into integrals that can individually be estimated using Corollary 3.3. Consider for example the first term on the right of (32):

$$
\begin{aligned}
&\frac{4R}{3}\int_\Sigma \sum_{j=1}^{2} \omega(e_j) g(\nabla_{e_j} b, \nu)\,\mathrm{d}\mu \\
&= \frac{4R}{3}\int_\Sigma \sum_{\beta=1}^{3} (\operatorname{Ric}_{\alpha\beta} \nu^\alpha g_{\eta\mu} \nabla_\beta b^\eta \nu^\mu) - \operatorname{Ric}_{\alpha\beta} \nu^\alpha \nu^\beta g_{\eta\mu} \nabla_\kappa b^\eta \nu^\mu \nu^\kappa \,\mathrm{d}\mu.
\end{aligned}
$$

After replacing ∇b using the expansion (24), along with

$$
g_{\eta\mu} = \delta_{\eta\mu} + O(R^2), \quad \text{and} \quad \operatorname{Ric} = \operatorname{Ric}(p_0) + O(R),
$$

and noting that the resulting error terms are of order $O(R^5)$ we can use Corollary 3.3 to see that the whole term is $O(R^5)$. Inspecting the other terms of (32) shows that they can be treated similarly. Indeed all the tangential contractions in these terms can be resolved as above and the remaining terms are products of an odd number of factors ν or y/R. To show the pattern note that

$$
\begin{aligned}
|\omega|^2 &= |\operatorname{Ric}(\nu,\cdot)^T|^2 = |\operatorname{Ric}(\nu,\cdot)|^2 - \operatorname{Ric}(\nu,\nu)^2 \\
&= \sum_{\beta=1}^{3} g_{\beta\kappa} \operatorname{Ric}_{\alpha\beta} \operatorname{Ric}{\eta\kappa} \nu^\alpha \nu^\eta - (\operatorname{Ric}_{\alpha\beta} \nu^\alpha \nu^\beta)^2
\end{aligned}
$$

Both terms on the right have an even number of factors ν so that multiplied with $g(b,\nu)$ in the second term on the right of (32) yields an odd number. The third term can be treated by computing with $\tau = \operatorname{tr} T = \operatorname{Sc} - \operatorname{Ric}(\nu,\nu)$ that

$$
\begin{aligned}
|\mathring{T}|^2 &= |T|^2 - \tfrac{1}{2}\tau^2 \\
&= |\operatorname{Ric}|^2 - 2|\omega|^2 - \operatorname{Ric}(\nu,\nu)^2 - \tfrac{1}{2}\operatorname{Sc}^2 - \operatorname{Sc}\operatorname{Ric}(\nu,\nu) - \tfrac{1}{2}\operatorname{Ric}(\nu,\nu)^2.
\end{aligned}
$$

Note that all terms on the right contain an even number of factors ν, so the third term in (32) is also done. The remaining two terms have a similar structure. We infer the estimate

$$
|\delta_f \mathcal{U}(\Sigma)| \leq CR^5 \tag{33}
$$

for the particular choice of f above.

7.3. The variation of $\mathcal{V}(\Sigma)$

From [6, Section 4.3] we get that for the given choice of f we have

$$\begin{aligned}\delta_f \mathcal{V}(\Sigma) = \int_\Sigma & -G(b,\nu) - \tfrac{1}{2} g(b,\nu)\,\mathrm{Sc} + 2f\langle \mathring{A}, G^T\rangle \\ & - 2\omega(e_i)\big(H^{-1} g(\nabla_{e_i} b,\nu) + H^{-1}\mathring{A}_i^j g(b,e_j) \\ & - H^{-2}\nabla H g(b,\nu)\big)\,\mathrm{d}\mu.\end{aligned} \tag{34}$$

As in section 7.2 we can estimate

$$\left|\int_\Sigma 2f\langle \mathring{A}, G^T\rangle - 2\omega(e_i)H^{-2}\big(Hg(\nabla_{e_i}b,\nu) + H\mathring{A}_i^j g(b,e_j) - \nabla H g(b,\nu)\big)\,\mathrm{d}\mu\right| \\ \leq CR^5.$$

To see this, use $\nabla H = \frac{2}{3}\omega + O_{L^2}(R)$ and $\mathring{A} = -\frac{4}{3}H^{-1}\mathring{T} + O_{L^2}(R^2)$ from Propositions 5.3 and 5.4. The estimate then follows by inspection as in section 7.2.

Furthermore, as in [6, Section 4.3] let X be the vector field on $\mathcal{B}_\rho(p_0)M$ such that $g(X,Y) = G(b,Y)$ for all vector fields Y on $\mathcal{B}_\rho(p_0)$. Then $\mathrm{div}_M X = \langle G, \nabla b\rangle$ since G is divergence free. Let $\Omega \subset \mathcal{B}_\rho(p_0)$ enclosed by Σ and recall from [6, Section 4.3] that $\mathrm{Vol}(\Omega) \leq CR^3$. Compute

$$\begin{aligned}\int_\Sigma G(b,\nu)\,\mathrm{d}\mu &= \int_\Omega \mathrm{div}_M X\,\mathrm{d}V = \int_\Omega \langle G, \nabla b\rangle\,\mathrm{d}V = \int_\Omega G_{\alpha\beta}\nabla_\kappa b^\alpha g^{\beta\kappa}\,\mathrm{d}V \\ &= -\tfrac{1}{3}G_{\alpha\beta}(p_0)\delta^{\beta\kappa}(\mathrm{Rm}^\alpha_{\kappa\eta\mu}(p_0) + \mathrm{Rm}^\alpha_{\mu\eta\kappa}(p_0))b^\mu \int_\Omega y^\kappa\,\mathrm{d}V^E + O(R^5).\end{aligned}$$

Here we used equation (24) in the last step and replaced all curvature quantities by their values at p_0. Also the integration is now with respect to the Euclidean volume form $\mathrm{d}V^E$. The value of the constant in front of the integral is not important for the following. For $\kappa \in \{1,2,3\}$ consider the vector field $Y := \frac{1}{4}y^\kappa y$. Then $\mathrm{div}_E Y = y^\kappa$ and thus using Stokes in the first equality and Corollary 3.3 in the estimate.

$$\int_\Omega y^\kappa\,\mathrm{d}V^E = \int_\Sigma y^\kappa \langle y, \nu^E\rangle_E\,\mathrm{d}\mu^E = O(R^5). \tag{35}$$

To treat the remaining term, we consider the vector field $X = \mathrm{Sc}\, b$ as in [6, Section 4.3]. Then

$$\int_\Sigma g(b,\nu)\,\mathrm{Sc}\;\mathrm{d}\mu = \int_\Omega \mathrm{div}_M X\,\mathrm{d}V = \int_\Omega g(b,\nabla\,\mathrm{Sc}) + \mathrm{Sc}\,\mathrm{div}_M b\,\mathrm{d}V$$

with Ω as above. Using $g = g^E + O(R^2)$, $\mathrm{Sc} = \mathrm{Sc}(p_0) + O(R)$,

$$\nabla_\beta \mathrm{Sc} = \nabla_\beta \mathrm{Sc}(0) + \nabla^2_{\beta,\kappa} \mathrm{Sc}(0) y^\kappa + O(R^2),$$

and equation (25) for the expansion of $\operatorname{div} b$, we get

$$\begin{aligned}\int_\Sigma g(b,\nu)\,\mathrm{Sc}\,\mathrm{d}\mu = &\int_\Omega g^E(b, \nabla \mathrm{Sc}(p_0))\,\mathrm{d}V^E \\ &+ \big(b^\alpha \nabla^2_{\alpha\kappa} \mathrm{Sc} - \tfrac{1}{3}\mathrm{Sc}(p_0)\,\mathrm{Ric}_{\beta\kappa}(p_0) b^\beta\big) \int_\Omega y^\kappa \,\mathrm{d}V^E + O(R^5).\end{aligned}$$

In view of (35) this gives

$$\int_\Sigma g(b,\nu)\,\mathrm{Sc}\,\mathrm{d}\mu = \mathrm{Vol}(\Omega) g(b, \nabla \mathrm{Sc}(p_0)) + O(R^5).$$

In combination with the above, we arrive at the estimate

$$\big|\delta_f \mathcal{V}(\Sigma) + \tfrac{1}{2}\mathrm{Vol}(\Omega) g(b, \nabla \mathrm{Sc}(p_0))\big| \le C R(\Sigma)^5. \tag{36}$$

7.4. The conclusion

From the splitting (28), estimates (29), (33), and (36) we arrive at

$$\mathrm{Vol}(\Omega) |g(b, \nabla \mathrm{Sc}(p_0))| \le C R(\Sigma)^5.$$

Since $b \in \mathbf{R}^3$ is arbitrary and since $\mathrm{Vol}(\Omega) \ge C^{-1} R(\Sigma)^3$ by [6, Eq. (4.7)] this gives the claimed estimate:

$$|\nabla \mathrm{Sc}(p_0)| \le C R(\Sigma)^2.$$

This concludes the proof of Theorem 7.1.

8. The proof of Corollary 1.5

Corollary 8.1. *Let (M, g) be a compact three dimensional Riemannian manifold with C_B bounded geometry. Let*

$$Z := \{x \in M \mid \nabla \mathrm{Sc}(x) = 0\}$$

and assume that the Hessian $\operatorname{Hess}\mathrm{Sc}(x)$ *is non-degenerate for every $x \in Z$.*

Then there exists an a_0 depending only on (M, g) such that for every surface Σ that satisfies the Euler-Lagrange equation (1) *for some λ, with $|\Sigma| \leq a_0$ and $\mathcal{W}(\Sigma) \leq 4\pi + a_0$ the region enclosed by Σ intersects Z in a single point.*

Proof. Since M is compact and all critical points of Sc are non-degenerate, the set Z is discrete. Let

$$\rho_0 := \frac{1}{2} \min\{\operatorname{dist}(x, y) \mid x \neq y \in Z\}.$$

For $\rho \in (0, \rho_0)$ let

$$Z_\rho := \{x \in M \mid \operatorname{dist}(x, Z) < \rho\}.$$

For $r \in (0, \infty)$ let

$$G_r := \{x \in M \mid |\nabla \operatorname{Sc}(x)| \leq r\}.$$

By the compactness of M and since Sc is a Morse function, there exist $r_0 \in (0, \infty)$ and $c \in (0, \infty)$ such that for every $r \in (0, r_0)$ we have $G_r \subset Z_{cr}$.

Let a_0 and C be the constants from Theorem 7.1 applied to (M, g). By decreasing a_0, we can assume that in addition to the assertion of Theorem 7.1, we also have that $CR(\Sigma)^2 \leq r_0$ and that $\operatorname{diam}(\Sigma) < \rho_0$ whenever Σ satisfies the assumption of this Lemma with the chosen a_0.

Let Σ be such a surface and let $R = R(\Sigma)$. Denote by Ω the open region enclosed by Σ. For $s \in (0, \infty)$ denote

$$\Omega_s := \{x \in \Omega \mid \operatorname{dist}(x, \Sigma) > s\}.$$

Then Ω_s is an open subset of Ω.

Let p_0 be the point from Theorem 7.1. Then $|\operatorname{dist}(p_0, x) - R| \leq CR^3$ for all $x \in \Sigma$ and $|\nabla \operatorname{Sc}(p_0)| \leq CR^2$. The first estimate shows that $p_0 \in \Omega$, in fact $p_0 \in \Omega_{\frac{3}{4}R}$ if we choose a_0 sufficiently small.

Let $\sigma := |\nabla \operatorname{Sc}(p_0)|$. Then $\sigma \leq CR^2$ so that $p_0 \in G_\sigma \subset Z_{c\sigma}$. This implies that there exists a point $p_1 \in Z$ such that $p_1 \in \Omega_{\frac{3}{4}R - c\sigma} \subset \Omega_{\frac{3}{4}R - CR^2}$. By choosing a_0 and thus R smaller again, we can ensure that $\frac{3}{4}R - CR^2 \geq \frac{R}{2}$, so that $Z \cap \Omega_{\frac{R}{2}} \neq \emptyset$. Since $\operatorname{diam}(\Sigma) < \rho_0$ we know that $\Omega_{\frac{R}{2}} \subset \Omega \subset B_{\rho_0}(p_0)$ and by the choice of ρ_0 the ball $B_{\rho_0}(p_0)$ can intersect Z in at most one point. □

From the expansion of the Willmore energy in [7, Corollary 5.6] we know that the minimizers $\Sigma_a^{\min}$ from Theorem 1.1 concentrate near the maxima of the scalar curvature of M. Thus a slight variant of the proof of Corollary 8.1 yields the proof of Corollary 1.6.

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Received May 4, 2020
Accepted July 22, 2020