Communications in
Analysis and Geometry
Volume 31, Number 1, 31–67, 2023

Avoidance for set-theoretic solutions of mean-curvature-type flows

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We give a self-contained treatment of set-theoretic subsolutions to flow by mean curvature, or, more generally, to flow by mean curvature plus an ambient vector field. The ambient space can be any smooth Riemannian manifold. Most importantly, we show that if two such set-theoretic subsolutions are initially disjoint, then they remain disjoint provided one of the subsolutions is compact; previously, this was only known for Euclidean space (with no ambient vectorfield). We also give a simple proof of a version of Ilmanen's interpolation theorem.

The first author was partially supported by an AMS-Simons Travel Grant.
The second author was partially supported by NSF grants DMS-1404282 and DMS-1711293.

1. Introduction

Under mean curvature flow, an initially smooth compact hypersurface in $\mathbf{R}^{n+1}$ must become singular in finite time. Singularities typically occur before the surface disappears, that is, before its area tends to zero. Thus it is desirable to have weak notions of mean curvature flow that allow the flow to extend past singularities.

Level set flow, introduced simultaneously in [3] and [5], is one such notion. It is very natural and has proved to be very useful. Under mild hypotheses on the ambient space, there is a unique level set flow starting with **any** compact initial set; for a smoothly embedded initial surface, it agrees with the classical solution as long as the classical solution exists (i.e., up until the first singular time). However, the definition has the unfortunate feature that a limit of level set flows need not be a level set flow.

Partly to get around that feature, Ilmanen [8, 9] introduced a weaker notion, that of a "set-theoretic subsolution to mean curvature flow" or (in the terminology of [11]) a "weak set flow". Roughly speaking, a one-parameter family of closed subsets of a Riemannian manifold is a weak set flow provided it does not bump into any smoothly embedded, closed hypersurface moving by mean curvature flow.

A key feature of weak set flows is that not only do they not bump into smooth mean curvature flows, they also cannot bump into other weak set flows. More precisely, they satisfy the following avoidance principle: two initially disjoint weak set flows remain disjoint as long as at least one of them remains compact. (Under the mild hypothesis that the ambient space is complete with Ricci curvature bounded below, any initially compact weak set flow remains compact.) Ilmanen gave a very elegant proof of the avoidance principle in Euclidean space, but it strongly relied on invariance of mean curvature flow under spatial translations, and thus it did not seem to extend to other Riemannian manifolds. One of the main contributions of this paper is modifying Ilmanen's proof so that it works in arbitrary Riemannian manifolds, and, more generally, for closed sets (in a Riemannian manifold) moving by mean curvature plus an ambient vectorfield.

Weak set flows and level set flows are related by a containment theorem (Theorem 22): the level set flow starting from a given set is a weak set flow, and it contains every other weak set flow starting from that set. Ilmanen [8, 4H] proved that the containment theorem follows from the avoidance principle. But since the avoidance principle was only known in Euclidean space, likewise the containment theorem was only known in that case.

The organization of this paper as follows. Section 2 gives the basic definitions. We have found it convenient to use a definition of weak set flow that differs from, but is equivalent to, Ilmanen's original definition. In Section 3, we derive some elementary properties of weak set flows. In Section 4, we prove some technical results about modifying barriers to get barriers with additional desirable properties. In Sections 5 and 6, the barrier modification theorems are used to prove the avoidance principle. In Section 7, we show that our definition of weak set flow (Definition 2) agrees with Ilmanen's original definition. In Section 8, we show that there is a biggest weak set flow with any given initial set, and we prove (under mild hypotheses) that this biggest flow coincides with the level set flow. In Sections 9 and 10, we show that limits of weak set flows and boundaries of level set flows are weak set flows. In Section 11, we explain how the discussion in this paper extends to motion by mean curvature plus an ambient vectorfield. In Section 12, we present the basic facts about surfaces that move in one direction under the flow. In Section 13, we consider varifolds flowing by mean curvature plus an ambient vectorfield, and we show that the support of such a varifold flow is a weak set flow. In the appendix, we give a simple proof of a version of Ilmanen's interpolation theorem, a key tool in the proof of the avoidance theorem.

2. Basic definitions

Definition 1. Let N be a smooth Riemannian manifold. A family $t \in [a,b] \mapsto K(t)$ of closed subsets of N is called a **smooth barrier** in N provided it is a smooth, one-parameter family of closed regions with smooth boundary. Equivalently, it is a smooth barrier provided there exists a smooth function $f : N \times [a,b] \to \mathbf{R}$ such that $K(t) = \{x : f(x,t) \leq 0\}$ and such that

$\nabla f(x,t)$ is nonzero at all points of $\partial K(t)$. We say that the barrier is **compact** if $\cup_{t\in[a,b]}K(t)$ is a compact subset of N, or, equivalently, if

$$K := \{(p,t) : t \in [a,b],\ p \in K(t)\}$$

is a compact subset of $N \times \mathbf{R}$.

If $t \in [a,b] \mapsto K(t)$ is a smooth barrier and if $x \in \partial K(t)$, we let $\nu_K(x,t)$ be the unit normal to $\partial K(t)$ that points out from $K(t)$, we let $\mathrm{H}_K(x,t)$ denote the dot product of $\nu_K(x,t)$ and the mean curvature vector of $\partial K(t)$ at x, and we let $\mathbf{v}_K(x,t)$ denote the normal velocity of $\tau \mapsto \partial K(\tau)$ at (x,t) in the direction of ν_K. In terms of a function f as in Definition 1,

$$\begin{aligned} \nu_K &= \frac{\nabla f}{|\nabla f|}, \\ \mathrm{H}_K &= -\operatorname{Div}\left(\frac{\nabla f}{|\nabla f|}\right), \\ \mathbf{v}_K &= -\frac{1}{|\nabla f|}\frac{\partial f}{\partial t}. \end{aligned} \tag{1}$$

Alternatively, we can describe $\mathbf{v}_K$ as follows. Let $I \subset \mathbf{R}$ be an interval containing t and $\gamma : I \to N$ be a smooth map such that $\gamma(t) = x$ and such that $\gamma(\tau) \in \partial K(\tau)$ for all $\tau \in I$. Then

$$\mathbf{v}_K(x,t) = \gamma'(t) \cdot \nu_K(x,t).$$

For $x \in \partial K(t)$, we define $\Phi_K = \Phi_K(x,t)$ by

$$\Phi_K = \mathbf{v}_K - \mathrm{H}_K.$$

Thus $\Phi_K \le 0$ everywhere if and only if $t \mapsto \partial K(t)$ is a subsolution of mean curvature flow, and $\Phi_K \ge 0$ if and only it is a supersolution.

For example, let $\lambda > 0$, and for $t < 0$, let

$$K(t) = \{x \in \mathbf{R}^{m+1} : |x| \ge (-\lambda t)^{1/2}\}.$$

Thus $\partial K(t)$ is the sphere of radius $(-\lambda t)^{1/2}$ centered at the origin. At a point $x \in \partial K(t)$, $\mathbf{v}_K(x,t) = \frac{1}{2}\lambda^{1/2}|t|^{-1/2}$ and $\mathrm{H}_K(x,t) = m(\lambda|t|)^{-1/2}$. Consequently, Φ_K is positive, zero, or negative according to whether λ is greater than, less than, or equal to $(2m)^{1/2}$.

Definition 2. Let Z be a closed subset of $N \times [T_0, \infty)$, and for each $t \in [T_0, \infty)$, let

$$Z(t) := \{x \in N : (x, t) \in Z\}.$$

We say that Z is a **weak set flow** (for mean curvature flow) with **starting time** T_0 provided the following holds: if

$$t \in [a, b] \mapsto K(t)$$

is a smooth compact barrier with $a \geq T_0$, if $K(t)$ is disjoint from $Z(t)$ for all $t \in [a, b)$, and if p is in the intersection of $K(b)$ and $Z(b)$, then $p \in \partial K(b)$ and

$$\Phi_K(p, b) \geq 0.$$

If the starting time is not specified, we take it to be 0.

For example, consider a smooth barrier $t \in [0, T] \mapsto K(t)$. Then K is a weak set flow if and only if $\Phi_K \leq 0$ at every (p, t) with $p \in \partial K(t)$. Similarly, consider a smooth one-parameter family $t \in [0, T] \mapsto M(t)$ of smooth, properly embedded hypersurfaces in N. Then $t \mapsto M(t)$ is a weak set flow if and only it is a classical mean curvature flow. (These facts follow easily from the definition of weak set flow.)

Note that if Z is a weak set flow and $a \in \mathbf{R}$, then

$$\widetilde{Z}(t) := \begin{cases} Z(t) & \text{if } t \leq a, \\ \emptyset & \text{if } t > a \end{cases}$$

is also a weak set flow. Thus weak set flows are allowed to suddenly vanish at any time.

Definition 2 differs from Ilmanen's original definition, but we will show that the two definitions are equivalent in Section 7.

3. Elementary properties of weak set flows

Theorem 3. *Let $m = \dim N - 1$ and $c > 2m$. Given $p \in N$, there exists an $\epsilon > 0$ with the following property.*

(i) *If $0 < \delta \leq \epsilon$ and if $0 < \tau < \delta^2/c$, then*

$$t \in [0, \tau] \mapsto K(t) := \{x : \operatorname{dist}(x, p) \leq (\delta^2 - ct)^{1/2}\}$$

is a smooth compact barrier, and $\Phi_K(x, t) < 0$ for all $t \in [0, \tau]$ and $x \in \partial K(t)$.

(ii) *If* $Z : [T_0, \infty) \mapsto Z(t)$ *is a weak set flow in* N*, then*

$$f(t)^2 + ct$$

is a non-decreasing function of $t \in [T_0, \infty)$*, where*

$$f(t) = \min\{\epsilon, \operatorname{dist}(Z(t), p)\}.$$

Proof. For $r > 0$, let $B(r) = \{x : \operatorname{dist}(x, p) \le r\}$. Choose $\epsilon > 0$ so that for $r \in (0, \epsilon]$, the geodesic sphere $\partial B(r)$ is smooth and compact, and

$$\mathrm{H}(B(r)) > -\frac{c}{2r}. \tag{2}$$

(This is possible since $\mathrm{H}(B(r)) = -(m/r) + o(r)$.) Assertion (i) follows immediately.

Suppose Assertion (ii) is false. Then there exist $T < T + \tau$ such that $f(T)^2 + cT$ is greater than $f(T+\tau)^2 + c(T+\tau)$. That is,

$$f(T)^2 > f(T+\tau)^2 + c\tau.$$

By relabeling, it suffices to consider the case $T = 0$:

$$f(0)^2 > f(\tau)^2 + c\tau.$$

Thus $f(\tau) < f(0) \le \epsilon$, so $f(\tau) = \operatorname{dist}(Z(t), p)$. Choose δ with with $f(0) > \delta > (f(\tau)^2 + c\tau)^{1/2}$. Thus

$$\min\{\epsilon, \operatorname{dist}(Z(0), p)\} > \delta > \left(\operatorname{dist}(Z(\tau), p)^2 + c\tau\right)^{1/2}. \tag{3}$$

Define $K(\cdot)$ by

$$t \in [0, \tau] \mapsto K(t) := B((\delta^2 - ct)^{1/2}).$$

Note by (3) that the radius of the ball $K(t)$ is strictly between 0 and $\delta < \epsilon$ for all $t \in [0, \tau]$. Thus K is a smooth compact barrier. By Assertion (i), $\Phi_K < 0$ at all points of $\partial K(\cdot)$. On the other hand, from (3) we see that $K(t)$ and $Z(t)$ are disjoint at time 0 but not at time τ, a contradiction. □

Corollary 4. *Suppose* $T > T_0$*.*

(i) *If* $p \in Z(T)$*, then* $\operatorname{dist}(Z(t), p)^2 \le c(T - t)$ *for* $t < T$ *close to* T*.*

(ii) *If* $u : N \times \mathbf{R} \to \mathbf{R}$ *is continuous, then*

$$\tilde{u}(T) \geq \limsup_{t \uparrow T} \tilde{u}(t),$$

where $\tilde{u}(t) := \inf_{x \in Z(t)} u(x,t)$.

Proof. In the notation of Theorem 3, $f(T) = 0$, so $f(t)^2 + ct \leq CT$ for $t \leq T$, and therefore

$$\min\{\epsilon, \operatorname{dist}(Z(t), p)\}^2 = f(t)^2 \leq c(T-t),$$

which proves Assertion (i).

To prove Assertion (ii), let $p \in Z(T)$. By Assertion (i), if $t < T$ is sufficiently close to T, then there exists a point $p(t) \in Z(t)$ closest to p, and $p(t) \to p$ as $t \to T$. Now $\tilde{u}(t) \leq u(p(t), t)$, so

$$\limsup_{t \uparrow T} \tilde{u}(t) \leq \limsup_{t \uparrow T} u(p(t), t) = u(p, T).$$

Assertion (ii) follows by taking the infimum over all $p \in Z(T)$. □

Theorem 5. *For every* $r > 0$, $\lambda \in \mathbf{R}$, *and positive integer* n, *there is a constant* $h = h(r, \lambda, n) > 0$ *with the following property. Suppose that* N *is a smooth Riemannian* n*-manifold, that* $R > r$, *that the geodesic ball* $\overline{B}(p, R)$ *in* N *is compact, and that the Ricci curvature of* N *is* $\geq \lambda$ *on* $\overline{B}(p, R)$. *If* $t \in [0, T] \mapsto Z(t)$ *is a weak set flow in* N *and if* $\operatorname{dist}(Z(0), p) > R$, *then*

$$\operatorname{dist}(Z(t), p) > R - ht \tag{*}$$

for all $t \in [0, T]$ *with* $t \leq (R - r)/h$.

Proof. Let $\mathcal{H}$ be a complete n-dimensional manifold that has the same dimension as N, that has constant sectional curvature, and that has Ricci curvature equal to the minimum of 0 and λ. Let $h > 0$ be the mean curvature of a sphere of radius $r/2$ in $\mathcal{H}$.

Suppose, contrary to the theorem, that (15) fails for some time $t \leq (R - r)/h$. Let τ be the first such time. By Corollary 4 (applied to $\tilde{u}(t) :=$

$\mathrm{dist}(p, Z(t)))$,

$$\mathrm{dist}(Z(\tau), p) = R - h\tau.$$

Since $0 < \tau \le (R - r)/h$,

$$r \le \mathrm{dist}(Z(\tau), p) < R.$$

Let q be a point in $Z(\tau)$ with $\mathrm{dist}(p, q) = \mathrm{dist}(p, Z(\tau)) = R - m\tau$. Let γ be a unit-speed, shortest geodesic from p to q, prolonged to be a geodesic of length R:

$$\begin{aligned}&\gamma : [0, R] \to N,\\ &\gamma(0) = p,\\ &\gamma(R - h\tau) = q.\end{aligned}$$

Let

$$K : t \in [0, \tau] \mapsto K(t) := \overline{B}(\gamma(R - ht - r/2), r/2).$$

Since γ is length minimizing on $[0, R - h\tau]$, it follows that the function

$$(x, y) \in N \times N \mapsto \mathrm{dist}(x, y)$$

is smooth in a small neighborhood of (x, y) if x and y are points in $\gamma((0, R - h\tau))$.

Thus K is smooth in a spacetime neighborhood of (q, τ).

Note that $K(t)$ is disjoint from $Z(t)$ for $t < \tau$ and that $K(\tau) \cap Z(\tau) = \{q\}$.

Thus

$$\Phi_K(q, \tau) := \mathbf{v}_K(q, \tau) - \mathrm{H}_K(q, \tau) \ge 0,$$

so $\mathbf{v}_K(q, \tau) \ge \mathrm{H}_K(q, \tau)$.[1] However, $\mathbf{v}_q = -h$, and $\mathrm{H}_K(q, \tau) > -h$ by mean curvature comparison (see [10, Lemma 7.1.2] or [4, Theorem 1.2.2]). Thus $\mathbf{v}_K(q, \tau) < \mathrm{H}_K(q, \tau)$, a contradiction. □

Theorem 6 (Ilmanen [7], Theorem 6.4). *Suppose that $Z : [0, \infty) \mapsto Z(t)$ is a weak set flow in a complete Riemannian n-manifold with Ricci*

[1] To conclude above that $\Phi_K(q, \tau) \ge 0$ using the definition of weak set flow, the barrier K should be smooth everywhere. However, it suffices for K to be smooth in a spacetime neighborhood of (q, τ). See Theorem 8.

curvature bounded below by λ. *Then*

$$Y(t) \subset \{x : \mathrm{dist}(x, Y(0)) \le r + ht\}$$

for all $t > 0$. *Here* r *can be any positive number, and* $h = h(r, \lambda, n)$ *is as in Theorem 5.*

In particular, if $Z(0)$ *is empty, then* $Z(t)$ *is empty for all* t, *and if* $Z(0)$ *is compact, then* $\cup_{t \le T} Z(t)$ *is compact for* $T < \infty$.

Proof. If $\mathrm{dist}(p, Y(0)) > r + ht$, then $\mathrm{dist}(p, Y(t)) \ge r$, and so $p \notin Y(t)$. Thus if $p \in Y(t)$, then $\mathrm{dist}(p, Y(0)) \le r + ht$. □

The "in particular" assertions of Theorem 6 are false (in general) without the lower bound on Ricci curvature. For example, let Σ be a compact manifold with Riemannian metric σ, and let $N = \mathbf{R} \times \Sigma$ with the complete metric $dx^2 + (\exp(-x - x^3/3))^2 \sigma$. Then $t \mapsto M(t) := \{\tan t\} \times \Sigma$ is a mean curvature flow with $M(0)$ compact and $\cup_{t \in [0, \pi/2]} M(t)$ noncompact, and $t \mapsto M(t - \pi/2)$ is a mean curvature flow with $M(t)$ empty for $t = 0$ but nonempty for $t \in (0, \pi)$.

4. Barrier modification

Lemma 7. *Suppose that* U *is an open subset of* N, *that* $t \in [a, b] \mapsto K(t)$ *is a smooth barrier in* U, *and that* $p \in U \in \partial K(b)$. *Then there is* $\hat{a} \in [a, b)$ *and a smooth compact barrier*

$$t \in [\hat{a}, b] \mapsto \widehat{K}(t)$$

in U *such that*

$$\widehat{K}(t) \subset \mathrm{interior}\, K(t) \quad \textit{for } t \in [\hat{a}, b),$$
$$\widehat{K}(b) \cap \partial K(b) = \{p\}, \textit{ and}$$
$$\Phi_{\widehat{K}}(p, b) = \Phi_K(p, b).$$

Proof. Let $f : U \times [a, b] \to \mathbf{R}$ be as in Definition 1. By postcomposing with a smooth bounded function, we can assume that f is bounded. Let $\phi : U \times [a, b] \to \mathbf{R}$ be a smooth proper function such that ϕ vanishes to infinite order at (p, b) and such that $\phi > 0$ at all other points. By Sard's Theorem, almost every c is a non-critical value of

$$(x,t)\in (U\times[a,b])\setminus(p,b)\mapsto -\frac{f(x,b)}{\phi(x,b)}.$$

Choose such a $c>0$, and let

$$\widehat{f}(q,t):=f(q,t)+c\phi(q,t).$$

Since f is bounded and ϕ is proper,

$$\{(x,t)\in U\times[a,b]:\widehat{f}(x,t)\le 0\}$$

is compact. By choice of c, $\nabla\widehat{f}$ does not vanish anywhere on $\{x\in U:\widehat{f}(x,b)=0\}$. Now choose $\hat{a}\in[a,b)$ sufficiently close to a that $\nabla\widehat{f}$ does not vanish anywhere on $\{(x,t):t\in[\hat{a},b],\,\widehat{f}(x,t)=0\}$. □

Theorem 8 (Noncompact, nonsmooth barriers). *Suppose that $f:N\times[a,b]\to\mathbf{R}$ is continuous, and let $K(t)=\{x:f(x,t)\le 0\}$ for $t\in[a,b]$. Suppose that Z is a weak set flow in N with starting time $T_0<b$, that $Z(t)$ is disjoint from the interior of $K(t)$ for all $t<b$, and that $p\in Z(b)\cap\partial K(b)$,*

If f is smooth in a spacetime neighborhood of (p,b) and if $\nabla f(p,b)$ is nonzero, then $\Phi_K(p,b)\ge 0$.

Proof. Choose U and ϵ small enough that $t\in[b-\epsilon,b]\mapsto K(t)\cap U$ is a smooth barrier in U. By Lemma 7, there is smooth compact barrier $t\in[\hat{a},b]\mapsto\widehat{K}(t)\subset U\cap K(t)$ such that $p\in\partial\widehat{K}(b)$ and $\Phi_{\widehat{K}}(p,b)=\Phi_K(p,b)$. By definition of weak set flow, $\Phi_{\widehat{K}}(p,b)\ge 0$. □

Theorem 9 (Barrier Modification Theorem). *Suppose that $t\in[a,b]\mapsto K(t)$ is a smooth compact barrier in N, that $p\in\partial K(b)$, and that*

$$\Phi_K(p,b):=\mathbf{v}_K(p,b)-\mathrm{H}_K(p,b)<\eta.$$

Then there is an $\hat{a}\in[a,b)$ and a smooth compact barrier $t\in[\hat{a},b]\mapsto\widehat{K}(t)$ with the following properties:

(1) *$\widehat{K}(t)$ is contained in $K(t)$ for all $t\in[\hat{a},b]$.*

(2) *$p\in\partial\widehat{K}(b)$.*

(3)

$$\lim_{x\in\partial K(b),\,x\to p}\frac{\operatorname{dist}(x,\widehat{K}(b))}{\operatorname{dist}(x,p)^2}>0.$$

(4) $\Phi_{\widehat{K}}(x,t) := \mathbf{v}_{\widehat{K}}(x,t) - \mathrm{H}_{\widehat{K}}(x,t) < \eta$ *for all* (x,t) *with* $t \in [\hat{a}, b]$ *and* $x \in \partial \widehat{K}(t)$.

Proof. It suffices to consider the case $[a,b] = [a,0]$. Let $f : N \times [a,0] \to \mathbf{R}$ be as in Definition 1. By multiplying f by a constant, we can assume that $|\nabla f(p,0)| = 1$. We can also assume that f is proper, i.e., that $f(p_i) \to \infty$ provided p_i is a divergent sequence in N.

Let $\delta(\cdot)$ be a smooth bounded function on N that is positive on $N \setminus \{p\}$ and that coincides with $\frac{1}{2}\operatorname{dist}(\cdot, p)^2$ in a neighborhood of p. Let

$$\begin{aligned} &\widetilde{f} : N \times [a,0] \to \mathbf{R}, \\ &\widetilde{f}(x,t) = f(x,t) + c\,(\delta(x) - t)\,, \end{aligned}$$

and let

$$\widetilde{K}(t) = \{(x,t) : \widetilde{f} \le 0\},$$

where c is a positive constant that will be specified below.

Since 0 is a regular value of $f(\cdot,0)$, there is an $\epsilon > 0$ such that 0 is a regular value of $\widetilde{f}(\cdot,0)$ provided $c \in [0,\epsilon]$. Fix a $c \in (0,\epsilon]$ such that $\Phi_{\widetilde{K}}(p,0) < \eta$. (This is possible since $\Phi_{\widetilde{K}}(p,0)$ depends continuously on c.)

Since $\widetilde{f} \ge f$ with strict inequality except at $(p,0)$, we see that

$$\widetilde{K}(t) \subset \operatorname{interior}(K(t)) \quad \text{for } t \in [a,0) \tag{4}$$

and

$$\widetilde{K}(0) \setminus \operatorname{interior}(K(0)) = \{p\}.$$

Note also that

$$\lim_{x \in \partial\widetilde{K}(0),\, \operatorname{dist}(x,p) \to 0} \frac{\operatorname{dist}(x, \partial K(0))}{\operatorname{dist}(x,p)^2} = c.$$

Let $\psi : N \to \mathbf{R}$ be a smooth, bounded, nonnegative function such that ψ vanishes on an open set U containing p and such that $\psi > 0$ at all points of the set

$$\Sigma := \{x \in \partial\widetilde{K}(0) : \Phi_{\widetilde{K}}(x,0) \ge \eta\}.$$

Now let

$$\begin{aligned} &\widehat{f} : N \times [a,0] \to \mathbf{R}, \\ &\widehat{f}(x,t) = \widetilde{f}(x,t) + t\Lambda\psi(x), \end{aligned}$$

and

$$\widehat{K}(t) = \{x : \widehat{f}(x,t) \le 0\} \qquad (t \in [a,0]),$$

where Λ is a positive constant that will be specified below.

Note that $\widehat{f}(\cdot,0) = \widetilde{f}(\cdot,0)$, so $\widehat{K}(0) = \widetilde{K}(0)$. Thus for $x \in \partial\widehat{K}(0)$,

$$\mathrm{H}_{\widehat{K}}(x,0) = \mathrm{H}_{\widetilde{K}}(x,0)$$

and

$$\mathbf{v}_{\widehat{K}}(x,0) = \mathbf{v}_{\widetilde{K}}(x,0) - \Lambda w(x)$$

(by (1)), where

$$w(x) = \frac{\psi(x)}{|\nabla \widehat{f}(x,0)|} = \frac{\psi(x)}{|\nabla \widetilde{f}(x,0)|}.$$

Consequently,

$$\Phi_{\widehat{K}}(x,0) = \Phi_{\widetilde{K}}(x,0) - \Lambda w(x).$$

Thus if $x \notin \Sigma$, then

$$\Phi_{\widehat{K}}(x,0) \le \Phi_{\widetilde{K}}(x,0) < \eta,$$

and if $x \in \Sigma$, then

$$\Phi_{\widehat{K}}(x,0) < (\max_{y\in\Sigma} \Phi_{\widetilde{K}}(y,0)) - \Lambda(\min_{\Sigma} w).$$

Choose $\Lambda > 0$ large enough that this last expression is $< \eta$. Then

$$\Phi_{\widehat{K}}(x,0) < \eta \qquad (x \in \partial\widehat{K}(0)).$$

Since

$$\widehat{K}(0) \setminus U \subset \mathrm{interior}(K(0)),$$

there is an $\hat{a} \in [a,0)$ such that

$$\widehat{K}(t) \setminus U \subset \mathrm{interior}(K(t)) \quad \text{for all } t \in [\hat{a},0].$$

On the other hand, since ψ vanishes on U,

$$\widehat{K}(t) \cap U = \widetilde{K}(t) \cap U \subset \mathrm{interior}(K(t)) \quad \text{for all } t \in [\hat{a},0)$$

by (4). Thus

$$\widehat{K}(t) \subset \mathrm{interior}\, K(t) \quad \text{for all } t \in [\hat{a},0).$$

Finally, since 0 is a regular value of $\widehat{f}(\cdot,0)$ and since $\Phi_{\widehat{K}}(\cdot,0) < \eta$ everywhere on $\partial\widehat{K}(0)$, we can choose $\hat{a}$ close enough to 0 to guarantee that 0 is a regular value of $\widehat{f}(\cdot,t)$ for all $t \in [\hat{a},0]$ and that $\Phi_K(\cdot,t) < \eta$ everywhere on $\partial\widehat{K}(t)$ for all $t \in [\hat{a},0]$. □

5. Bounds on the distance function

Lemma 10. *Suppose that N is a smooth Riemannian manifold, that $Z(\cdot)$ is a weak set flow in N with starting time T_0, and that*

$$t \in [a,b] \mapsto K(t)$$

is a smooth barrier with $a \ge T_0$. Suppose that $\lambda \in \mathbf{R}$, that

$$t \in [a,b] \mapsto e^{-\lambda t} \operatorname{dist}(K(t), Z(t))$$

attains a positive minimum at time $t = b$, and that there is a geodesic

$$\gamma : [0,L] \to N$$

parametrized by arclength such that

$$\begin{aligned} p &:= \gamma(0) \in K(b), \\ q &:= \gamma(L) \in Z(b), \quad \textit{and} \\ L &= \operatorname{dist}(K(b), Z(b)). \end{aligned}$$

If $\operatorname{Ric}(\gamma',\gamma') > \lambda$ on $[0,L]$, then

$$\Phi_K(p,b) > 0.$$

Proof. We may assume that $b = 0$. Unfortunately, the signed distance function to $\partial K(0)$ need not be smooth at the point $q = \gamma(L)$. (It is smooth in a neighborhood of each $\gamma(s)$ with $s \in [0,L)$.) We will use the Barrier Modification Theorem 9 to get around the lack of smoothness.

Let λ_0 be the minimum of $\operatorname{Ric}(\gamma',\gamma')$ on $[0,L]$. Thus $\lambda_0 > \lambda$. We will prove the lemma by proving that

$$\Phi_K(p,0) \ge (\lambda_0 - \lambda)L. \tag{5}$$

Suppose that (5) does not hold. Then by the Barrier Modification Theorem 9, there is a smooth, compact barrier

$$t \in [-\epsilon, 0] \mapsto \widehat{K}(t)$$

such that:

$$
\begin{aligned}
&\widehat{K}(t) \subset K(t) \quad (t \in [-\epsilon, 0]), \\
&\widehat{K}(t) \subset \operatorname{interior}(K(t)) \quad (t < 0), \\
&\widehat{K}(0) \cap \partial K(0) = \{p\}, \\
&\liminf_{x \in \partial K(0),\, x \to p} \frac{\operatorname{dist}(x, \widehat{K}(0))}{\operatorname{dist}(x,p)^2} > 0, \\
&\Phi_{\widehat{K}}(p,0) < (\lambda_0 - \lambda) L.
\end{aligned} \tag{6}
$$

Note that $\operatorname{dist}(\cdot, \widehat{K}(0))$ is smooth on an open set containing q, and thus

$$
(t,x) \mapsto \operatorname{dist}(x, e^{\lambda t} \widehat{K}(t))
$$

is smooth on an open spacetime set containing $(q,0)$. For $t \in [-\epsilon, 0]$, let

$$
\widetilde{K}(t) = \{x \in W : e^{-\lambda t} \operatorname{dist}(x, \partial \widehat{K}(t)) \le L\}.
$$

By (6), $Z(t) \cap W$ and $\widetilde{K}(t)$ are disjoint for $t < 0$ and $Z(0) \cap \widetilde{K}(0) = \{q\}$. Thus

$$
\Phi_{\widetilde{K}}(q,0) \ge 0 \tag{7}
$$

by Theorem 8.

A standard computation (cf. [13, Lemma 12.2]) shows that

$$
\begin{aligned}
\mathrm{H}_{\widetilde{K}}(q,0) &\ge \mathrm{H}_{\widehat{K}}(p,0) + \int_0^L \operatorname{Ric}_{\gamma(s)}(\gamma'(s), \gamma'(s))\, ds \\
&\ge \mathrm{H}_{\widehat{K}}(p,0) + \lambda_0 L.
\end{aligned} \tag{8}
$$

Note also that

$$
\mathbf{v}_{\widetilde{K}}(q,0) = \mathbf{v}_{\widehat{K}}(p,0) + \lambda L. \tag{9}
$$

By (7), (8), and (9),

$$
\begin{aligned}
0 &\le \Phi_{\widetilde{K}}(q,0) \\
&= \mathbf{v}_{\widetilde{K}}(q,0) - \mathrm{H}_{\widetilde{K}}(q,0) \\
&\le \mathbf{v}_{\widehat{K}}(p,0) - \mathrm{H}_{\widehat{K}}(p.0) + (\lambda - \lambda_0) L \\
&= \Phi_{\widehat{K}}(p,0) + (\lambda - \lambda_0) L,
\end{aligned}
$$

contradicting (6). □

Proposition 11. *Suppose $t \in [0,T] \mapsto K(t)$ is a compact smooth barrier such that $\Phi_K(p,t) \le 0$ for all $t \in [0,T]$ and $p \in \partial K(t)$. Suppose that $\eta > 0$, that $\lambda \in \mathbf{R}$, that the set*

$$Q := \{(p,t) : t \in [0,T],\, \mathrm{dist}(p, K(t)) \le e^{\lambda t}\eta\}$$

is compact, and that λ is a strict lower bound for Ricci curvature on $\cup_{t\in[0,T]} Q(t)$. If $t \in [0,T] \mapsto Z(t)$ is a weak set flow and if

$$\mathrm{dist}(Z(0), K(0)) > \eta,$$

then

$$\mathrm{dist}(Z(t), K(t)) > e^{\lambda t}\eta$$

for all $t \in [0,T]$.

Proof. Suppose not. Then $Q \cap Z$ is a nonempty compact subset of space time. Thus there is first time t such that $Q(t) \cap Z(t)$ is nonempty. By hypothesis, $t > 0$. Let $q \in Z(t) \cap Q(t)$:

$$\mathrm{dist}(q, K(t)) \le e^{\lambda t}\eta.$$

Thus $\mathrm{dist}(q, K(t)) = e^{\lambda t}\eta$ by Corollary 4 (applied to the function $u(x,\tau) = \mathrm{dist}(x, K(\tau))$). Let $p \in K(t)$ be a point closest to q. By Lemma 10, $\Phi_K(p,t) > 0$, a contradiction. □

Corollary 12. *In Proposition 11, if $t \mapsto Z(t)$ is a weak set flow in N such that* $\mathrm{dist}(Z(0), K(0)) \ge \eta$*, then* $\mathrm{dist}(Z(t), K(t)) \ge e^{\lambda t}\eta$ *for all $t \in [0,T]$.*

Proof. By Proposition 11,

$$\mathrm{dist}(Y(t), V(t)) > e^{\lambda t}\delta \quad (t \in [0,T])$$

holds for all δ with $0 < \delta < \eta$. The result follows immediately. □

Remark 13. Proposition 11 and Corollary 12 remain true if $t \in [0,T] \mapsto K(t)$ is a smooth mean curvature flow of closed hypersurfaces. The proofs are the same except for minor changes of notation. Furthermore, in this case, one sees from the proof that it is not necessary for the flow to be smooth at the initial time: K can be any compact subset of $N \times [0,T]$ such that $t \in (0,T] \mapsto K(t)$ is a smooth mean curvature flow.

Theorem 14. *Suppose $t \in [0,T] \mapsto Y(t)$ and $t \in [0,T] \mapsto Z(t)$ are weak set flows in a smooth Riemannian manifold N. Suppose $\eta > 0$ and λ are such that the set*

$$Y_\lambda^\eta := \cup_{t\in[0,T]}\{p : \operatorname{dist}(p, Y(t)) \le e^{\lambda t}\eta\}$$

is compact. Suppose also that λ is a lower bound for Ricci curvature of N on the set Y_λ^η. If

$$\operatorname{dist}(Z(t), Y(t)) \ge e^{\lambda t}\eta \tag{10}$$

holds for $t = 0$, then it holds for all $t \in [0,T]$.

Proof. **Case 1**: λ is a strict lower bound for Ricci curvature on Y_λ^η. Let $\mathcal{T}$ be the set of times $\tau \in [0,T]$ such that (10) holds for all $t \in [0,\tau]$. By hypothesis, $0 \in \mathcal{T}$. Thus $\mathcal{T}$ is either $[0,b]$ or $[0,b)$ where $b = \sup \mathcal{T}$. Let $y \in Y(b)$ and $z \in Z(b)$. Then for $t < b$,

$$\begin{aligned} e^{\lambda t}\eta &< \operatorname{dist}(Y(t), Z(t)) \\ &\le \operatorname{dist}(Y(t), y) + \operatorname{dist}(y,z) + \operatorname{dist}(z, Z(t)). \end{aligned}$$

Taking the limit as $t \uparrow b$ gives (see Corollary 4)

$$e^{\lambda b}\eta \le \operatorname{dist}(y,z).$$

Taking the infimum over $y \in Y(b)$ and $z \in Z(b)$ gives $e^{\lambda b}\eta \le \operatorname{dist}(Y(b), Z(b))$. Thus $\mathcal{T} = [0,b]$.

Hence it suffices to show that if $\tau < T$ is in $\mathcal{T}$, then $\tau + \epsilon \in \mathcal{T}$ for some $\epsilon > 0$. Consider such a time τ. Let

$$J = \{p \in N : \operatorname{dist}(p, Y(\tau)) \ge e^{\lambda\tau}\eta\}.$$

Since $\tau \in \mathcal{T}$, we see that $Z(\tau) \subset J$. By Theorem A1, there exists a closed C^1 hypersurface M in N such that M separates $Y(\tau)$ and J and such that

$$\operatorname{dist}(Y(\tau), M) = \operatorname{dist}(M, J) = \frac{1}{2}\operatorname{dist}(Y(\tau), J) = \frac{1}{2}e^{\lambda\tau}\eta.$$

Existence of such a hypersurface that is $C^{1,1}$ was sketched in [8, Lemma 4G] and proved in [1]. (See also [6].) Since existence of such an M that is merely C^1 suffices for our application and is simpler to prove, we provide an existence proof in the appendix.

By the Local Regularity Theorem [12], there exists a smooth mean curvature flow

$$t \in (\tau, \tau + \epsilon] \mapsto M(t)$$

such that $M(t)$ converges in C^1 to M as $t \to \tau$. Accordingly, we set $M(\tau) = M$.

Let

$$\widetilde{M} := \cup_{t \in [\tau, \tau+\epsilon]} \{p : \operatorname{dist}(p, M(t)) \leq \frac{1}{2} e^t \eta\}.$$

By replacing ϵ by a smaller $\epsilon > 0$, we can assume that $[\tau, \tau + \epsilon] \subset [0, T]$, that $\widetilde{M}$ is compact, and that λ is a strict lower bound for Ricci curvature of N on $\widetilde{M}$.

Consequently,

$$\begin{aligned} \operatorname{dist}(Y(t), M(t)) &\geq e^{\lambda(t-\tau)} \operatorname{dist}(Y(\tau), M(\tau)) \\ &= e^{\lambda(t-\tau)} \frac{1}{2} e^{\lambda\tau} \eta \\ &= \frac{1}{2} e^{\lambda t} \eta \end{aligned}$$

and

$$\begin{aligned} \operatorname{dist}(Z(t), M(t)) &\geq e^{\lambda(t-\tau)} \operatorname{dist}(Z(\tau), M(\tau)) \\ &\geq e^{\lambda(t-\tau)} \operatorname{dist}(J, M) \\ &= e^{\lambda(t-\tau)} \frac{1}{2} e^{\lambda\tau} \eta \\ &= \frac{1}{2} e^{\lambda t} \eta. \end{aligned}$$

for $t \in [\tau, \tau + \epsilon]$ by Corollary 12 and Remark 13, with the time interval $[\tau, \tau + \epsilon]$ in place of $[0, T]$. Since $M(t)$ separates $Y(t)$ and $Z(t)$,

$$\begin{aligned} \operatorname{dist}(Y(t), Z(t)) &\geq \operatorname{dist}(Y(t), M(t)) + \operatorname{dist}(M(t), Z(t)) \\ &\geq e^{\lambda t} \eta \end{aligned}$$

for $t \in [\tau, \tau + \epsilon]$. Thus $[\tau, \tau + \epsilon] \subset \mathcal{T}$. This proves the theorem in Case 1.

Case 2: λ is any lower bound for Ricci curvature on Y^η_λ. Taking any $\lambda' < \lambda$, then $Y^\eta_{\lambda'} \subset Y^\eta_\lambda$, and thus λ' is a strict lower bound for Ricci curvature on $Y^\eta_{\lambda'}$. Thus by Case 1,

$$\operatorname{dist}(Y(t), Z(t)) \ge e^{\lambda' t}\eta$$

for all $t \in [0, T]$. Since this inequality holds for every $\lambda' < \lambda$, it also holds for $\lambda' = \lambda$. $\square$

Theorem 15. *Suppose N is a complete Riemannian manifold with Ricci curvature bounded below by λ. Suppose $t \in [0, \infty) \mapsto Y(t)$ and $t \in [0, \infty) \mapsto Z(t)$ are weak set flows with $Y(0)$ compact. Then*

$$t \in [0, T] \mapsto e^{-\lambda t} \operatorname{dist}(Y(t), Z(t))$$

is non-decreasing.

Proof. Let $0 < T < \infty$. By Theorem 6, $\cup_{t \le T} Y(t)$ is a compact subset of $N \times [0, T]$. By Theorem 14,

$$\operatorname{dist}(Y(t), Z(t)) \ge e^{\lambda t} \operatorname{dist}(Y(0), Z(0)). \tag{11}$$

for $t \le T$. (Note that the hypotheses of Theorem 14 are satisfied for every $\eta > 0$.) Since T is arbitrary, (11) holds for all $t \ge 0$. The same argument shows that

$$\operatorname{dist}(Y(\tau + t), Z(\tau + t)) \ge e^{\lambda t} \operatorname{dist}(Y(\tau), Z(\tau))$$

for $0 \le \tau < t$. $\square$

6. The avoidance theorem

Theorem 16. *Let $t \in [0, T] \mapsto Y(t)$ and $t \in [0, T] \mapsto Z(t)$ be weak set flows in N such that $C := \cup_{t \in [0,T]} Y(t)$ is compact. If $Y(t)$ and $Z(t)$ are disjoint at time 0, then they are disjoint at every time $t \in [0, T]$.*

Proof. Let U be an open set containing C with $\overline{U}$ compact. Let λ be a lower bound for Ricci curvature on U, and choose η with $0 < \eta < \operatorname{dist}(Y(0), Z(0))$ sufficiently small that

$$\cup_{t \in [0,T]} \{x : \operatorname{dist}(x, Y(t)) \le e^{\lambda t}\eta\}$$

lies in U. Then $\operatorname{dist}(Y(t), Z(t)) \ge e^{\lambda t}\eta$ for all $t \in [0, T]$ by Theorem 14. $\square$

7. Equivalent definitions of weak set flow

A **strong mean-curvature-flow barrier**, or **strong barrier** for short, is a smooth compact barrier $t \in [a,b] \mapsto K(t)$ such that $\Phi_K(x,t) < 0$ for all (x,t) with $x \in \partial K(t)$ and $t \in [a,b]$. (Recall that the barrier K is said to be compact if $\cup_t K(t)$ is compact, or, equivalently, if K is a compact subset of $N \times \mathbf{R}$.)

Theorem 17. *Let Z be a closed subset of $N \times [T_0, \infty)$. The following are equivalent:*

(1) *Z is a weak set flow (as in Definition 2) with starting time T_0.*

(2) *If $t \in [a,b] \mapsto K(t)$ is a strong barrier with $a \geq T_0$ and if $K(a)$ is disjoint from $Z(a)$, then $K(t)$ is disjoint from $Z(t)$ for all $t \in [a,b]$.*

(3) *If $t \in [a,b] \mapsto K(t)$ is a strong barrier with $a \geq T_0$ and if $K(a)$ is disjoint from $Z(a)$, then $K(t)$ is disjoint from $Z(t)$ for all $t \in [a,b)$.*

(4) *If $t \in [a,b] \mapsto M(t)$ is a smooth mean curvature flow of closed, embedded, hypersurfaces with $a \geq T_0$, and if $M(0)$ is disjoint from $Z(0)$, then $M(t)$ is disjoint from $Z(t)$ for all $t \in [a,b]$.*

Proof. Trivially (1) implies (2).

To see that (2) implies (1), suppose to the contrary that (2) holds but that (1) fails. Then there is a smooth compact barrier $[a,b] \mapsto K(t)$ such that $Z(t) \cap K(t)$ is empty for $t \in [a,b)$ and such that $K(t) \cap Z(t)$ contains a point p such that

$$p \in \text{interior}(K(b)), \text{ or}$$
$$p \in \partial K(b) \text{ and } \Phi_K(b) < 0.$$

By Corollary 4 (applied to the function $u(x,t) = \text{dist}(x, N \setminus K(t)))^2$ p must be in $\partial K(b)$, and so $\Phi_K(b) < 0$. By the Barrier Modification Theorem 9, there is a strong barrier $t \in [b-\epsilon, b] \mapsto \widehat{K}(t)$ such that $\widehat{K}(t) \subset K(t)$ for all $t \in [b-\epsilon, b]$ and such that $p \in \partial \widehat{K}(b)$. But that violates (2).

Thus we have proved that (1) and (2) are equivalent.

Trivially (2) implies (3). The reverse implication holds because any strong barrier $t \in [a,b] \mapsto K(t)$ can be prolonged to a strong barrier on a slightly larger time interval $[a, b+\epsilon]$.

---

[2]Corollary 4 was proved for weak set flows. But Corollary 4 is based on Theorem 3, and the only barriers in the proof of that result were strong barriers. Thus Theorem 3 and Corollary 4 also hold for flows having Property (2).

The Avoidance Theorem 16 shows that (1) implies (4) (since any smooth mean curvature flow is a weak set flow).

It remains only to show that (4) implies (2), or, equivalently, that failure of (2) implies failure of (4). Thus suppose that (2) does not hold, i.e., that that there is a strong barrier $t \in [a,b] \mapsto K(t)$ with $a \ge T_0$ such that $K(a)$ is disjoint from $Z(a)$ but $K(t) \cap Z(t)$ is nonempty for some time $t \in (a,b]$. By relabeling, we may assume that b is the first such time.

By replacing a by an $a' < b$ sufficiently close to b, we may suppose that the mean curvature flow starting from $\partial K(a)$ remains smooth and compact for time at least $b-a$. For $t \in [a,b]$, let $M(t)$ be the result of flowing $M(0) = \partial K(a)$ for time $t-a$. Let $\widehat{K}(t)$ be the closed region bounded by $M(t)$ such that

$$K(t) \subset \widehat{K}(t).$$

Since $K(b) \cap Z(b)$ is nonempty and since $K(b) \subset \widehat{K}(b)$, we see that $\widehat{K}(b) \cap Z(b)$ is nonempty. By Lemma 18 below, the first contact of $\widehat{K}(t)$ and $Z(t)$ occurs at a point in $\partial\widehat{K}(t)$, that is, a point in $M(t)$. Thus $t \in [a,b] \mapsto M(t)$ is a smooth mean curvature flow that is disjoint from Z at time a but not at some later time, which violates (4). □

Lemma 18. *Suppose that U is an open subset of N and that $p \in U$.*

(1) *For all sufficiently small $\epsilon > 0$, there is a mean curvature flow*

$$t \in [0,\epsilon] \mapsto M(t)$$

of smoothly embedded, closed hypersurfaces in U such that $p \in M(\epsilon)$.

(2) *Suppose Z has Property (4) in Theorem 17. If $T > 0$ and if $p \in Z(T)$, then $Z(t) \cap U$ is nonempty for all $t \le T$ sufficiently close to T.*

The second assertion implies that at the first time $Z(\cdot)$ bumps into a smooth compact barrier, the contact occurs only at the boundary of the barrier.

Proof. Let $R > 0$ be very small and let q be point with $\operatorname{dist}(p,q) = R$. Choose R sufficiently small that the geodesic spheres $S_r := \partial\mathbf{B}(q,r)$ with $R/2 \le r \le 2R$ are smooth and compact and lie in U. Let $\delta > 0$ be such that, under mean curvature flow, each of those spheres remains smooth and compact and in U during the time interval $[0,\delta]$. For $t \in [0,\delta]$, let $S_r(t)$ be the result of flowing S_r for time t. Choose $\epsilon \in (0,\delta]$ sufficiently small that q lies in the region between $S_{R/2}(\epsilon)$ and $S_{2R}(\epsilon)$. Thus there will be a unique $r \in (R/2, 2R)$ such that $q \in S_r(\epsilon)$. Now let $M(t) = S_r(t)$ for $t \in [0,\epsilon]$. This proves (1).

To prove (2), let $M(\cdot)$ and ϵ by as in (1). Consider the mean curvature flow

$$t \in [T-\epsilon, T] \mapsto \Sigma(t) := M(t-T).$$

Since $p \in Z(T) \cap \Sigma(T)$, we see that $Z(T-\epsilon) \cap \Sigma(T-\epsilon)$ is nonempty since Z has Property (4) in Theorem 17. Thus $Z(T-\epsilon) \cap U$ is nonempty since $\Sigma(T-\epsilon) \subset U$. □

8. The biggest flow

Theorem 19. *Let N be a smooth Riemannian manifold and let C be a closed subset of N. There exists a weak set flow Y in $N \times [0, \infty)$, called the **biggest flow** generated by C, such that*

1) $Y(0) = C$, and

2) If Z is a weak set flow in $N \times [0, \infty)$ with $Z(0) \subset C$, then $Z(t) \subset Y(t)$ for all $t \geq 0$.

Proof. Let

$$\mathcal{Z} := \{Z \text{ is a weak set flow with } Z(0) \subset C\}.$$

Let Y be the closure of $\bigcup_{Z \in \mathcal{Z}} Z$.

Note that $C \times \{0\}$ is an element of $\mathcal{Z}$. Thus $C \subset Y(0)$. Shrinking ball barriers (see Theorem 3) imply that $Y(0) \subset C$. Thus $Y(0) = C$.

It remains to check that Y is indeed a weak set flow. By Theorem 17, it suffices to check that if $t \in [a, b] \mapsto K(t)$ is a strong barrier with $a \geq 0$ and with $Y(a) \cap K(a) = \emptyset$, then $Y(t) \cap K(t)$ is empty for every $t \in (a, b)$. Choose $0 < \epsilon < \operatorname{dist}(Y(a), K(a))$ sufficiently small that the spacetime set

$$K_\epsilon := \{(p, t) : t \in [a, b], \operatorname{dist}(p, K(t)) \leq \epsilon\} \tag{12}$$

is a strong barrier. Let $Z \in \mathcal{Z}$. Then

$$\operatorname{dist}(Z(a), K(a)) \geq \operatorname{dist}(Y(a), K(a)) > \epsilon,$$

so $Z(a)$ is disjoint from $K_\epsilon(a)$. Consequently, $Z(t)$ is disjoint from $K_\epsilon(t)$ for each $t \in [a, b]$. That is,

$$\operatorname{dist}(Z(t), K(t)) > \epsilon \text{ for all } t \in [a, b].$$

Consequently,

$$\operatorname{dist}(Y(t), K(t)) \geq \epsilon \text{ for all } t \in (a, b).$$

This completes the proof that Z is a weak set flow. □

Definition 20. If C is a closed subset of N and if $t \geq 0$, we let

$$F_t(C) = Y(t)$$

where $Y \subset N \times [0, \infty)$ is the biggest flow generated by C.

Proposition 21. *$F_{t+s}(C) = F_t(F_s(C))$ for all $s, t \geq 0$.*

Proof. Suppose that $t \in [0, \infty) \mapsto Y(t)$ is a weak set flow and that $T > 0$. Then $t \in [0, \infty) \mapsto Y(t - T)$ is a weak set flow. Also, if $t \in [0, \infty) \mapsto Z(t)$ is a weak set flow and if $Z(0) \subset Y(T)$, then

$$t \in [0, \infty) \mapsto \begin{cases} Y(t) & \text{if } t \in [0, T], \\ Z(t - T) & \text{if } t > T \end{cases}$$

is a weak set flow. (These facts follow easily from the definition of weak set flow.) Theorem 21 is an immediate consequence. □

We end this section by a characterization of the biggest flow in terms of solutions to the level set equation:

$$\frac{\partial u}{\partial t} = |\nabla u| \operatorname{Div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{13}$$

Theorem 22. *Let $Y \subset N \times [0, T]$ be let $\mathcal{U} \subset N \times [0, t]$ be an open set containing Y. Suppose that there exists a continuous function $u : \mathcal{U} \to \mathbf{R}$ such that u solves (13) in the viscosity sense, and such that $Y_a := u^{-1}(a)$ are compact for $a \in (-\epsilon, \epsilon)$ and such that $Y_0 = Y$. Then $Y(t) = F_t(Y(0))$ for every $t \in [0, T]$.*

Proof. Note that $Y \subset \operatorname{Int}(\bigcup_{a \in (-\epsilon/4, \epsilon/4)} Y_a)$. Letting $\chi : [-1, 1] \to \mathbf{R}$ be a continuous function such that $\chi(x) = x$ for $|x| \leq \epsilon/4$ and $\chi(x) = 1$ when $|x| \geq \epsilon/2$, the relabeling lemma [7, 3.2] implies that $v := \chi(u)$ is a solution of (13) on $N \times [0, T]$. Now, [7, 6.3] and Theorem 17 imply that for each $a \in (-\epsilon/4, \epsilon/4)$, Y_a is a weak set flow. In particular, $Y(t) \subset F_t(Y(0))$.

Assuming that $Y(t) \neq F_t(Y(0))$ for some $t \in [0, T]$, set $t_0 = \inf\{t \in [0, T] \mid Y(t) \neq F_t(Y(0))\}$. Note that Lemma 4 implies that $Y(t_0) =$

$F_{t_0}(Y(0))$ and that if $t > t_0$ is such that $t - t_0$ is sufficiently small, then

$$F_t(Y(0)) \subset \operatorname{Int}\left(\bigcup_{|a|<\epsilon/4} Y_a(t) \right).$$

By the definition of t_0, there exists some $t > t_0$ and $a \neq 0$ such that $Y_a(t) \cap F_t(Y(0)) \neq \emptyset$. But as both Y_a and $F_t(Y(0))$ are compact weak set flows with $F_{t_0}(Y(0)) \cap Y_a(t_0) = \emptyset$, this contradicts Theorem 16. □

9. Limits of weak set flows

Definition 23. Suppose (Q, d) is a metric space. We say that a sequence $Z_n \subset Q$ of closed subsets **Kuratowski-converges** to $Z \subset Q$ if

$$Z = \{x : \limsup_n d(x, Z_n) = 0\} = \{x : \liminf_n d(x, Z_n) = 0\}.$$

Note that Kuratowski-convergence $Z_n \to Z$ is equivalent to: every point in Z is a limit of a sequence of points $p_n \in Z_n$, and no point in $Q \setminus Z$ is a subsequential limit of such points. Thus two metrics on Q that give the same topology also give the same notion of Kuratowski-convergence.

If Q is separable and if Z_n is a sequence of closed subsets of Q, then, after passing to a subsequence, $\operatorname{dist}(\cdot, Z_n)$ converges locally uniformly to a limit function $\delta(\cdot) : Q \to [0, \infty]$ (by Arzela-Ascoli), and thus the Z_n Kuratowski-converge to $\{x : \delta(x) = 0\}$. If Q is complete, then $\delta(\cdot) = \operatorname{dist}(\cdot, Z)$.

Theorem 24. *Let g_n be a sequence of Riemannian metrics on N that converge smoothly to a Riemannian metric g. For $n = 1, 2, \dots,$ let $Z_n \subset N \times [0, \infty)$ be a weak set flow (for the metric g_n) such that the sequence Z_n Kuratowski-converges to Z. Then Z is a weak set flow for the metric g.*

Here the metric space is $N \times [0, \infty)$ with the spacetime metric

$$d((x_1, t_1), (x_2, t_2)) = \max\{\operatorname{dist}_g(x_1, x_2),\ |t_1 - t_2|^{1/2}\}.$$

Note in Theorem 24 that $Z_n \to Z$ does not imply that $Z_n(t) \to Z(t)$ for each t. For example, if $T_n \uparrow 1$, then the shrinking circles

$$Z_n := \{(p, t) \in \mathbf{R}^2 \times [0, T_n] : \frac{1}{2}|p|^2 = T_n - t\}$$

converge to the shrinking circle

$$Z := \{(p,t) \in \mathbf{R}^2 \times [0,1] : \frac{1}{2}|p|^2 = 1-t\}$$

But $Z_n(1) = \emptyset$ does not converge to $Z(1) = \{0\}$.

Proof of Theorem 24. Let $a \ge 0$ and let $t : [a,b] \mapsto K(t)$ be a strong barrier with $K(a)$ disjoint from $Z(a)$. By Theorem 17, it suffices to show that $K(t)$ and $Z(t)$ are disjoint for each $t \in (a,b)$.

Fix a very small $\epsilon > 0$, and let

$$K_\epsilon : t \in [a,b] \mapsto \{x \in N : \operatorname{dist}_g(x, K(t)) \le \epsilon\}.$$

In particular, we choose $\epsilon > 0$ small enough so that K_ϵ is a strong barrier (with respect to g) and such that $K_\epsilon(a)$ is disjoint from $Z(a)$.

For all sufficiently large n, K_ϵ is a strong barrier with respect to g_n and $K_\epsilon(a)$ is disjoint from $Z_n(a)$. Thus by Theorem 17, $K_\epsilon(t)$ is disjoint from $Z_n(t)$ for all $t \in [a,b]$, so $\liminf \operatorname{dist}_{g_n}(K(t), Z_n(t)) \ge \epsilon$. Hence $\operatorname{dist}_g(K(t), Z(t)) \ge \epsilon$ for all $t \in (a,b)$. In particular, $K(t)$ is disjoint from $Z(t)$ for all $t \in (a,b)$. □

Remark 25. The **Kuratowski limsup** of a sequence of closed sets Z_n in Q is defined to be $\{x : \liminf_n \operatorname{dist}(x, Z_n) = 0\}$. In Theorem 24, if we do not assume that the sequence Z_n Kuratowski-converges, the Kuratowski limsup is a weak set flow. The proof is exactly the same.

10. Boundaries

Theorem 26. *Suppose C is a closed subset of a Riemannian manifold N. Let*

$$\mathcal{C} := \{(x,t) : t \ge 0,\ x \in F_t(C)\}$$

be the biggest flow generated by C, and let

$$M = \partial \mathcal{C}$$

Then M is a weak set flow and $M(0) = \partial C$.

In this section, for a subset Q of $N \times [0,\infty)$, terms like "interior" and "boundary" refer to the relative topology in $N \times [0,\infty)$. Thus interior(Q)

is the largest subset of Q that is relatively open in $N \times [0, \infty)$. Correspondingly,

$$\partial Q = \overline{Q} \setminus \text{interior}(Q).$$

For example, the interior of $N \times [0, \infty)$ is (in this context) all of $N \times [0, \infty)$ and therefore the boundary is the empty set. Since the $\mathcal{C}$ in Theorem 26 is closed, $\overline{\mathcal{C}} = \mathcal{C}$, and thus $\partial\mathcal{C} = \mathcal{C} \setminus \text{interior}(\mathcal{C})$.

Proof. Let $\mathcal{F}$ be the family of all strong barriers $K : t \in [a, b] \mapsto K(t)$ such that $K(a) \times \{a\}$ lies in the interior of $\mathcal{C}$. Let $W = \cup_{K \in \mathcal{F}} K$. Since each $K \in \mathcal{F}$ is a weak set flow, $K \subset \mathcal{C}$ (by definition of biggest set flow) and therefore $W \subset \mathcal{C}$.

Also, W is a relatively open subset of $N \times [0, \infty)$. This follows easily from the facts that if $t \in [a, b] \mapsto K(t)$ is a strong barrier, then

(i) K can be prolonged to a strong barrier on a slighty longer time interval $[a, b + \epsilon]$, and

(ii) $t \in [a, b] \mapsto K_\epsilon(t)$ is a strong barrier for all sufficiently small $\epsilon > 0$, where K_ϵ is given by (12).

Using strong barriers consisting of small shrinking balls (as in Theorem 3), one sees that

$$(\text{interior}(C)) \times \{0\} \subset W, \tag{14}$$

and that

$$\text{interior}(\mathcal{C}) \subset W.$$

We have shown that W is an open subset of $\mathcal{C}$ that contains interior($\mathcal{C}$). Thus

$$W = \text{interior}(\mathcal{C}).$$

By (14), $M(0) = \partial C$.

It remains to show that $\partial\mathcal{C}$ is a weak set flow. Let $t \in [a, b] \mapsto K(t)$ be a strong barrier with $a \geq 0$ such that $K(a)$ is disjoint from $M(a)$, i.e., such that $K(a) \times \{a\}$ is disjoint from $\partial\mathcal{C}$. We may assume that K is connected. Thus either $K(a) \times \{a\}$ is disjoint from $\mathcal{C}$ or $K(a) \times \{a\}$ lies in interior($\mathcal{C}$) $= W$. In the first case, K is disjoint from $\mathcal{C}$ since $\mathcal{C}$ is a weak set flow. In the second case, K lies in $W = \text{interior}(\mathcal{C})$ by definition of W. In either case, K is disjoint from $\partial\mathcal{C}$. Thus $\partial\mathcal{C}$ is a weak set flow by Theorem 17. □

11. Mean curvature flow with a transport term

Let N be a smooth Riemannian manifold and let X be a smooth vectorfield on N. A smooth one-parameter family of hypersurfaces in N is said to be an **X-mean-curvature flow** provided the normal component of velocity is everywhere equal to the mean curvature plus the normal component of X. If $t \in [a, b] \mapsto K(t)$ is a smooth barrier and if $x \in \partial K(t)$, we let

$$\mathrm{H}_K^X(x,t) = \mathrm{H}_K(x,t) + X \cdot \nu_K(x,t),$$
$$\Phi_K^X(x,t) = \mathbf{v}_K(x,t) - \mathrm{H}_K^X(x,t),$$

and if v is a tangent vector to N, we let

$$\mathrm{Ric}^X(v,v) = \mathrm{Ric}(v,v) + v \cdot \nabla_v X.$$

We define a **weak set flow for X-mean-curvature flow** (or **weak X-flow** for short) by replacing Φ_K by Φ_K^X in Definition 2. With three exceptions, all the theorems and proofs in this paper remain true provided we make the following changes:

1) Mean curvature flow, H_K, and Φ_K are replaced by X-mean curvature flow, H_K^X, and Φ_K^X.
2) Lower bounds of the form $\mathrm{Ric} > \lambda$ (or $\mathrm{Ric} \ge \lambda$) are replaced by $\mathrm{Ric}^X > \lambda$ (or $\mathrm{Ric}^X \ge \lambda$).
3) All of the global theorems in this paper assume that N is complete with Ricci curvature bounded below. In the case of X-flows, we add the assumption that $|\nabla X|$ is bounded.

The extra hypothesis (3) ensures that compact surfaces remain compact under the flow, and that the empty surface remains empty under the flow. See Theorem 29 below.

The three exceptions (in which there is something new in the statement and/or the proof) are Theorems 5, 6, and 15. However, with very slight modification, those results continue to hold for X-mean-curvature flow:

Theorem 27 (X-flow version of Theorem 5). *For every $r > 0$, $\lambda \in \mathbf{R}$, and positive integer n, there is a constant $h = h(r, \lambda, n) > 0$ with the following property. Suppose that N is a smooth Riemannian n-manifold, that $R > r$, that the geodesic ball $\overline{\mathbf{B}}(p, R)$ in N is compact, and that the Ricci curvature of N is $\ge \lambda$ on $\overline{\mathbf{B}}(p, R)$. If $t \in [0, \infty) \mapsto Z(t)$ is a weak set* *flow in N and if* $\operatorname{dist}(Z(0),p) > R$, *then*

$$\operatorname{dist}(Z(t),p) > R - (h+\chi)t \qquad \textit{for all } t \in [0,(R-r)/(h+\chi)], \tag{15}$$

provided $|X| \le \chi$ *on* $\overline{\mathbf{B}}(p,R)$.

The proof is almost identical to the proof of Theorem 5.

Corollary 28. *If* $r \le \delta \le R$, *then*

$$\operatorname{dist}(p, Z(t)) > \delta \quad \textit{for all } t \in [0,(R-\delta)/(h+\chi)].$$

Theorem 29 (X-flow version of Theorem 6). *Suppose that N is a complete Riemannian manifold with Ricci curvature bounded below, that X is a smooth vectorfield on N with $|\nabla X|$ bounded, and that $t \in [0,\infty) \mapsto Z(t)$ is a weak X-flow in N.*

1) If $Z(0)$ is empty, then $Z(t)$ is empty for every t.

2) If $Z(0)$ is compact, then $\cup_{t\le T} Z(t)$ is compact for every $T < \infty$.

Proof of Theorem 29. It suffices to consider the case that N is connected. Let x_0 be a point in N. Since $|\nabla X|$ is bounded,

$$c := \sup \frac{|X(\cdot)|}{\max\{1, \operatorname{dist}(\cdot, x_0)\}} < \infty.$$

If Q is a closed subset of N, let $t \in [0,\infty) \mapsto F_t^X(Q)$ denote the biggest weak X-flow with $F_0^X(Q) = Q$.

Let $h = h(1,\lambda,n)$ be as in Theorem 27, where λ is a lower bound for Ricci curvature on N.

Let $R \ge 2$. On the ball $\overline{\mathbf{B}}(x_0, 2R)$, $|X|$ is bounded by $2Rc$. Since $\operatorname{dist}(x_0, \emptyset) = \infty > 2R$,

$$\operatorname{dist}(x_0, F_t(\emptyset)) > R \qquad \text{for all } t \ge 0 \text{ with } t \le \frac{R}{h+2Rc}.$$

by Corollary 28 (with $r = 1$ and $\delta = R$). Since $R \ge 1$,

$$\frac{R}{h+2Rc} = \frac{1}{(h/R)+2c} \ge \frac{1}{h+2c}.$$

Thus if we let $\tau = 1/(h+2c)$,

$$\operatorname{dist}(x_0, F_t(\emptyset)) > R \qquad \text{for all } t \le \tau.$$

Since this holds for every $R \geq 1$, we see that $F_t^X(\emptyset) = \emptyset$ for all $t \in [0, \tau]$. By iteration, $F_t^X(\emptyset) = \emptyset$ for all $t \geq 0$.

We now prove (2). For $r \geq 0$, let $B_r = \overline{\mathbf{B}}(x_0, r) = \{x : \operatorname{dist}(x, x_0) \leq r\}$. Suppose $a \geq 1$ and $\operatorname{dist}(p, x_0) \geq 4a$. We now derive a lower bound on the first time t (if there is one) such that $p \in F_t^X(B_a)$.

Define R by $\operatorname{dist}(p, x_0) = a + 3R$. Thus $R \geq a \geq 1$.

Now

$$\mathbf{B}(p, 2R) \subset \mathbf{B}(x_0, \operatorname{dist}(x_0, p) + 2R)) = \mathbf{B}(x_0, a + 5R) \subset \mathbf{B}(x_0, 6R).$$

Thus $|X|$ is bounded above by $6Rc$ on $\mathbf{B}(p, 2R)$.

Now B_a is disjoint from $\overline{\mathbf{B}}(p, 2R)$, so by Corollary 28,

$$\operatorname{dist}(p, F_t^X(B_a)) > R \qquad \text{for } t \leq \frac{R}{h + 6Rc}.$$

Now

$$\frac{R}{h + 6Rc} = \frac{1}{(h/R) + 6c} \geq \frac{1}{h + 6c}$$

since $R \geq 1$. Thus if $T := 1/(h + 6c)$, then

$$p \notin \cup_{t \in [0,T]} F_t^X(B_a).$$

Since this holds for all p with $\operatorname{dist}(p, x_0) \geq 4a$,

$$\cup_{t \in [0,T]} F_t^X(B_a) \subset B_{4a}.$$

By iteration,

$$\cup_{t \in [0,kT]} F_t^X(B_a) \subset B_{4^k a}.$$

□

Theorem 30 (X-flow version of Theorem 15). *Suppose that N is a complete, smooth Riemannian manifold with Ricci curvature bounded below, and that X is a smooth vectorfield with $|\nabla X|$ bounded. If $Y, Z \subset N \times [0, \infty)$ are weak X flows with $Y(0)$ compact, then for every $t < \infty$,*

$$\operatorname{dist}(Y(t), Z(t)) \geq e^{\lambda t} \operatorname{dist}(Y(0), Z(0)),$$

where λ is a lower bound for Ric^X.

(Note that Ric^X is bounded below because Ric^X and Ric differ at each point by at most $|\nabla X|$.)

Proof. Given Theorem 29, the proof of Theorem 30 is just like the proof of Theorem 15. ☐

Theorem 31. *Suppose that N is a complete Riemannian manifold with Ricci curvature bounded below and that X is a smooth vectorfield with $|\nabla X|$ bounded. Suppose that M is a smooth, closed, embedded hypersurface in N, and that $t \in [0,T] \mapsto M(t)$ is a smooth X-mean curvature flow with $M(0) = M$ and $\cup_{t\in[0,T]} M(t)$ compact. Then*

$$F_t^X(M) = M(t)$$

for $t \in [0,T]$, where $F_t^X(\cdot)$ is biggest X-flow.

If M bounds a closed region Q, then $F_t^X(Q)$ is the corresponding closed region bounded by $M(t)$.

Proof. Trivially, $t \mapsto M(t)$ is a weak X-flow, so $M(t) \subset F_t^X(M)$ for all $t \le T$. Thus it suffices to show that $F_t^X(M) \subset M(t)$.

For $s > 0$, let $M_s = \{x \in N : \text{dist}(x, M) = x\}$. Let $\epsilon > 0$ be sufficiently small that for all $s \in [0,\epsilon]$, there is a smooth X-mean-curvature-flow

$$t \in [0,T] \mapsto M_s(t)$$

with $M_s(0) = M_s$. Let $s \in (0,\epsilon]$. For each $t \in [0,T]$, let $K_s(t)$ be the union of $M_s(t)$ and the connected components of $N \setminus M_s(t)$ that do not contained $M(t)$. Then $t \in [0,T] \mapsto K_s(t)$ is a weak X-flow, so $F_t^X(M)$ and $K_s(t)$ are disjoint for all $t \in [0,T]$ by Theorem 30. Since this holds for all $x \in (0,T]$, we see that $F_t^X(M) \subset M(t)$ for all $t \in [0,T]$. This completes the proof that $F_t^X(M) = M(t)$.

To prove the assertion about Q, let $K_s = \{x \in N : \text{dist}(x, Q) \ge s\}$. Choose $\epsilon > 0$ sufficiently small that there a smooth, compact X-mean curvature flow on the time interval $[0,T]$ with initial surface ∂K_s. Let $t \in [0,T] \mapsto K_s(t)$ be the corresponding flow of regions.

By Theorem 30, $F_t^X(Q)$ and $\partial K_s(t)$ are disjoint for all $t \in [0,T]$. It follows that $t \in [0,T] \mapsto F_t^X(Q) \cap K_s$ is a weak X-flow that is empty at time 0. Thus it is empty for all $t \in [0,T]$ by Theorem 29. Since this holds for all $s \in (0,\epsilon]$, we see that $F_t^X(Q) \subset Q(t)$ for all $t \in [0,T]$. The reverse inclusion holds trivially (since $t \mapsto Q(t)$ is a weak X-flow.) ☐

12. X-mean-convex flows

Theorem 32. *Suppose that N is a complete Riemannian manifold with Ricci curvature bounded below, and that X is a smooth vectorfield with $|\nabla X|$ bounded. Suppose that Q is a closed region in N bounded by a compact hypersurface M, and suppose that Q is strictly X-mean-convex, i.e., that $\vec{H} + X^\perp$ is nonzero and points into Q at each point of M.*

Then there is a continuous time-of-arrival function $u : Q \to [0, \infty]$ such that for each $t \in [0, \infty)$,

$$\begin{aligned} &F_t^X(M) = \{x : u(x) = t\}, \\ &F_t^X(Q) = \{x : u(x) \geq t\}, \\ &\partial F_t^X(Q) = M(t), \\ &\cup_{\tau \leq t} M(\tau) \text{ is compact,} \end{aligned}$$

where $F_t^X(\cdot)$ denotes biggest X-flow.

Proof. Since M is smooth and compact, there is an $\epsilon > 0$ and a smooth X-mean curvature flow

$$t \in [0, \epsilon] \mapsto M(t)$$

with $M(0) = M$. For $t \in [0, \epsilon]$, let $Q(t)$ be the closed region (corresponding to Q) in N bounded by $M(t)$. Note that

$$Q(t) \subset Q \text{ and } Q(t) \cap M = \emptyset \text{ for } 0 < t \leq \epsilon \tag{16}$$

by the smooth maximum principle. By Theorem 31,

$$F_t^X(M) = M(t) \text{ and } F_t^X(Q) = Q(t) \text{ for all } t \in [0, \epsilon]. \tag{17}$$

By (16) and (17), $F_t^X(Q) \subset Q$ for $t \in [0, \epsilon]$ and thus (since F_τ^X preserves inclusion)

$$F_{\tau+t}^X(Q) \subset F_\tau^X(Q) \quad \text{for all } t \in [0, \epsilon] \text{ and } \tau \geq 0.$$

By transitivity of inclusion, this implies

$$F_T^X(Q) \subset F_t^X(Q) \quad \text{for all } T \geq t \geq 0. \tag{18}$$

By Theorem 29, $\cup_{t \in [0,T]} F_t^X(M)$ is compact for all $T < \infty$, and by (16) and by avoidance (e.g., Theorem 30),

$$F_{\tau+t}^X(Q) \cap F_\tau^X(M) = \emptyset \quad \text{for all } t \in (0, \epsilon] \text{ and } \tau \geq 0.$$

Hence by (18),

$$F_\tau^X(Q) \cap F_t^X(M) = \emptyset \quad \text{for all } \tau > t \geq 0. \tag{19}$$

In particular,

$$F_\tau^X(M) \cap F_t^X(M) = \emptyset \quad \text{for all } \tau > t \geq 0 \tag{20}$$

since $F_t^X(M) \subset F_t^X(Q)$ for all t.

Now define $u : Q \to [0, \infty]$ by

$$u(x) = \begin{cases} t & \text{if } x \in F_t^X(M), \\ \infty & \text{if } x \in Q \setminus \cup_t F_t^X(M). \end{cases}$$

(This is well-defined since the $F_t^X(M)$ are disjoint.)

Since the $F_t^X(M)$ trace out a closed subset of spacetime, $u : Q \to [0, \infty]$ is continuous.

Now suppose that $x \in Q \setminus F_T^X(Q)$. Then the spacetime set

$$\{(x, t) : x \in F_t^X(Q),\, t \geq 0\}, \tag{*}$$

contains the point $(x, 0)$ but not the point (x, T). Thus there is a $t \in [0, T)$ such that (x, t) lies in the boundary $\mathcal{B}$ (relative to $N \times [0, \infty)$) of the set (*). By Theorem 26, $\mathcal{B}$ is a weak X-flow starting from M. Thus $\mathcal{B}$ lies in the biggest such weak X-flow, so

$$(x, t) \in \mathcal{B} \subset \{(y, \tau) : y \in F_\tau^X(M),\, \tau \geq 0\},$$

and therefore $x \in M(t)$. Hence we have shown

$$x \in Q \setminus F_T^X(Q) \implies u(x) < T.$$

On the other hand

$$u(x) < T \implies x \in M_{u(x)} \implies x \notin F_T^X(Q)$$

by (19). Thus $F_T^X(Q) = \{u \geq T\}$.

Finally, if $t > 0$, then every point in $\{u = t\}$ is a limit of points in $\{u < t\}$ by Corollary 4), so $\{u = t\}$ has no interior. Hence $\{u = t\}$ is the boundary of $\{u \geq t\}$. $\square$

Remark 33. Theorem 32 remains true (with the same proof) for any closed set Q and for $M = \partial Q$ (not necessarily smooth) such that

$$F_t^X(Q) \subset Q \setminus M$$

for all t is some small time interval $(0, \epsilon]$.

13. Varifold flows

An m-dimensional **integral Brakke X-flow** in a Riemannian manifold N is a one-parameter family $t \in [0, \infty) \mapsto M(t)$ of Radon measures on N such that for almost every t, $M(t)$ is the radon measure associated to an m-dimensional integral varifold in N, and such that for every C^2, nonnegative, compactly supported function ϕ on $N \times [0, \infty)$,

$$\overline{\mathrm{D}}_t \int \phi \, dM(t) \le \int \left(\frac{\partial \phi}{\partial t} + \nabla\phi^\perp \cdot (X + H) - \phi H \cdot (H + X^\perp) \right) dM(t), \tag{21}$$

where $\bar{D}_t f(t) := \limsup_{h \to 0} (f(t+h) - f(t))/h$. As in the case of Brakke flow, the inequality (21) follows from the special case when ϕ is independent of time; see [9, §6] or [2, 3.5]. Also, as for Brakke flow, the right side of (21) should be interpreted as $-\infty$ if any terms in the the expression do not make sense at time t; see the discussion in [9, §6].

For integral varifolds, $H = H^\perp$, so we can rewrite (21) as

$$\begin{aligned}
&\overline{\mathrm{D}}_t \int \phi \, dM(t) \\
&\quad \le \int \left(\frac{\partial \phi}{\partial t} + \nabla\phi^\perp \cdot X + \nabla\phi \cdot H - \phi H \cdot X - \phi |H|^2 \right) dM(t) \\
&\quad = \int \left(\frac{\partial \phi}{\partial t} + \nabla\phi^\perp \cdot X - \mathrm{Div}_M \nabla\phi + \mathrm{Div}_M(\phi X) - \phi |H|^2 \right) dM(t) \\
&\quad = \int \Bigg(\frac{\partial \phi}{\partial t} + \nabla\phi^\perp \cdot X - \mathrm{Div}_M \nabla\phi \\
&\qquad\qquad + (\nabla\phi)^{\tan} \cdot X + \phi \, \mathrm{Div}_M X - \phi |H|^2 \Bigg) dM(t) \\
&\quad = \int \left(\frac{\partial \phi}{\partial t} + \nabla\phi \cdot X - \mathrm{Div}_M \nabla\phi + \phi \, \mathrm{Div}_M X - \phi |H|^2 \right) dM(t).
\end{aligned} \tag{22}$$

Theorem 34. *Let $t \in [0, \infty) \mapsto M(t)$ be an m-dimensional integral X-Brakke flow in a smooth $(m+1)$-dimensional Riemannian manifold N. Let*

$Z \subset N \times [0,\infty)$ be the spacetime support of the flow (i.e., the closure in $N \times \mathbf{R}$ of $\cup_t(\operatorname{spt} M(t)) \times \{t\})$). Then Z is a weak X-flow.

Theorem 34 was proved in [9, 10.5] for Brakke flows in $\mathbf{R}^{n+1}$ (with $X = 0$).

Proof of Theorem 34. Let

$$t \in [a,b] \subset [0,\infty) \mapsto K(t)$$

be a strong barrier (as in §7) such that $K(t)$ is disjoint from $Z(t)$ for $t \in [a,b)$. By Theorem 17, it suffices to show that $Z(b)$ is disjoint from $K(b)$.

Let $r(\cdot,t)$ be the signed distance to $\partial K(t)$ such that r is positive in the complement of $K(t)$. Then for $x \in \partial K(t)$,

$$0 > \Phi_K = \mathbf{v}_K - H_K^X = -\frac{\partial r}{\partial t} + \Delta r - X \cdot \nabla r$$

by (1) with r in place of f. Consequently, we can choose $\delta > 0$ sufficiently small and $k > 0$ so that wherever $|r| \le \delta$, the function r is smooth and

$$\frac{\partial r}{\partial t} - \Delta r + X \cdot \nabla r \ge k. \tag{23}$$

We also choose δ to be less that $\operatorname{dist}(Z(a), K(a))$.

By (22),

$$\overline{\mathrm{D}}_t \int \phi \, dM(t) \le \int \left(\frac{\partial \phi}{\partial t} + \nabla\phi \cdot X - \operatorname{Div}_M \nabla\phi + C\phi \right) dM(t), \tag{24}$$

where C is m times the maximum of $|\nabla X|$ on a compact set containing the support of ϕ.

Now let $\phi = ((\delta - r)^+)^3$. Note that this function is C^2 on the points of $N \times [a,b]$ in the support of the flow. Letting $s = (\delta - r)^+$, we have

$$\begin{aligned}
\frac{\partial \phi}{\partial t} &= -3s^2 \frac{\partial r}{\partial t}, \\
\nabla \phi &= -3s^2 \nabla r \\
\nabla^2 \phi &= -3s^2 \nabla^2 r + 6s \nabla r \otimes \nabla r, \\
\operatorname{Div}_M(\nabla\phi) &= -3s^2 \operatorname{Div}_M \nabla r + 6s|(\nabla r)^{\tan}|^2 \\
&= -3s^2(\Delta r - \nabla^2 r(\mathbf{n},\mathbf{n})) + 6s(1 - |\mathbf{n}\cdot\nabla r|^2),
\end{aligned}$$

where $\mathbf{n}(x,t)$ is a unit normal to the approximate tangent plane to $M(t)$ at x. Thus by (24) and (23),

$$\begin{aligned}
&\overline{\mathrm{D}}_t \int \phi \, dM(t) \\
&\quad \le \int \Big(-3s^2 \Big(\frac{\partial r}{\partial t} - \Delta r + \nabla r \cdot X + \nabla^2 r(\mathbf{n},\mathbf{n}) \Big) \\
&\qquad\qquad - 6s(1 - |\mathbf{n}\cdot \nabla r|^2) + Cs^3 \Big) \, dM(t) \\
&\quad \le \int \left(-3s^2 k + 3s^2 |\nabla^2 r(\mathbf{n},\mathbf{n})| - 6s(1 - |\mathbf{n}\cdot\nabla r|^2) + C\delta s^2 \right) dM(t) \\
&\quad \le \int \left(3s^2 |\nabla^2 r(\mathbf{n},\mathbf{n})| - 6s(1 - |\mathbf{n}\cdot\nabla r|^2) \right) dM(t)
\end{aligned}$$

provided we choose $\delta < 3k/C$. Now $\nabla^2 r(\cdot,\cdot)$ is a quadratic form that vanishes on ∇r, so

$$|\nabla^2 r(\mathbf{n},\mathbf{n})| \le c(1 - (\mathbf{n}\cdot\nabla r)^2)$$

for some constant c. Thus

$$\begin{aligned}
D_t \int \phi \, dM(t) &\le \int (3s^2 c - 6s)(1 - |\mathbf{n}\cdot\nabla r|^2) \, dM(t) \\
&\le \int 3s(\delta c - 2)(1 - |\mathbf{n}\cdot\nabla r|^2) \, dM(t),
\end{aligned}$$

which is ≤ 0 provided we chose $\delta < 2/c$.

Since $\int \phi \, dM(t)$ is nonnegative, zero at the initial time a, and decreasing, it is zero for all $t \in [a,b]$. Thus $\mathrm{dist}(Z(t), K(t)) \ge \delta$ for all $t \in [a,b]$. □

Appendix A.

Theorem A1. *Suppose that X and Y are closed subsets of N such that*

$$r := \frac{1}{2}\,\mathrm{dist}(X,Y) > 0$$

and such that

$$\{p : \mathrm{dist}(p,X) = r\}$$

is compact. Then there is a compact, C^1 embedded hypersurface surface M separating X and Y such that

$$\mathrm{dist}(X,M) = \mathrm{dist}(Y,M) = r.$$

Proof. Note that there is a $\delta \in (0, r)$ such that

$$\{p : r - \delta \leq \operatorname{dist}(p, X) \leq r + \delta\} \tag{A.1}$$

is compact. By replacing δ be a smaller δ, we can assume that geodesic balls with centers in (A.1) and with radii $\leq \rho$ have smooth boundaries.

Let

$$\begin{aligned} X' &= \{p : \operatorname{dist}(p, X) \leq r - \delta\}, \\ Y' &= \{p : \operatorname{dist}(p, X) \geq r + \delta\}, \end{aligned}$$

and let

$$\begin{aligned} &A = \{p : \operatorname{dist}(p, X) \leq r\} = \{p : \operatorname{dist}(p, X') \leq \delta\}, \\ &B = \{p : \operatorname{dist}(p, Y') \leq \delta\}, \\ &Z = A \cap B, \\ &U = N \setminus (A \cup B). \end{aligned}$$

Note that $\overline{U}$ is compact.

Consider a point $z \in Z$. Let C_1^z and C_2^z be shortest geodesics joining z to X' and to Y'. Then $C_1^z \cup C_2^z$ is a shortest curve joining X' to Y', and thus is a geodesic. Consequently, C_1^z and C_2^z are unique and depend continuously on $z \in Z$. Therefore

$$z \in Z \mapsto \mathbf{v}(z)$$

is continuous, where $\mathbf{v}(z)$ is the unit tangent vector to $C_1^z \cup C_2^z$ at z that points out of C_1^z and into C_2^z.

Let $h : U \to \mathbf{R}$ be the function that minimizes $\int |Dh|^2$ subject to

$$\begin{aligned} h &= -1 \quad \text{on } (\partial A) \setminus B \text{ and} \\ h &= 1 \quad \text{on } (\partial B) \setminus A. \end{aligned}$$

Then h is harmonic (and therefore smooth) on U and continuous on $\overline{U} \setminus Z$.

(The continuity holds because if $p \in \partial U$ and if q is a point in $X' \cup Y'$ closest to p, then $\mathbf{B}(q, \delta) \subset U^c$ and $p \in \partial \mathbf{B}(q, \delta)$.)

Let $c \in (-1, 1)$ be a regular value of h, and let

$$M = h^{-1}(c) \cup Z.$$

To prove that M is C^1, it suffices to show that if $p_i \in M \cap U$ converges to $p \in Z$, then

$$\frac{\nabla h(p_i)}{|\nabla h(p_i)|} \to \mathbf{v}(p).$$

Let $\mathbf{B}(q_i, r_i)$ be the largest ball in U that contains p_i. We work in normal coordinates at the point p. Let

$$U_i = (U - q_i)/r_i$$

and

$$\begin{aligned} &h_i : U_i \to \mathbf{R}, \\ &h_i(x) = h(r_i(q_i + x)). \end{aligned}$$

Note that U_i converges to the slab

$$\{x \in \mathbf{R}^{n+1} : 0 < x \cdot \mathbf{v}(p) < 1\}.$$

Therefore h_i converges smoothly to the harmonic function

$$x \cdot \mathbf{v}(p)$$

and p_i converges (perhaps after passing to a subsequence) to a point p' such that $p' \cdot \mathbf{v}(p) = c$. The result follows immediately. □

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Received September 12, 2018
Accepted August 7, 2020