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# Motion of level sets by inverse anisotropic mean curvature

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In this paper we consider the weak formulation of the inverse anisotropic mean curvature flow, in the spirit of Huisken-Ilmanen [15]. By using approximation method involving Finsler-$p$-Laplacian, we prove the existence and uniqueness of weak solutions.

## 1. Introduction

Let $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a Minkowski norm in $\mathbb{R}^n$, i.e.,

(i) $F$ is a norm in $\mathbb{R}^n$, i.e., $F$ is a convex, even, 1-homogeneous function satisfying $F(\xi) > 0$ when $\xi \neq 0$;

(ii) $F$ satisfies a uniformly elliptic condition: $D^2(\frac{1}{2}F^2)$ is positive definite in $\mathbb{R}^n \setminus \{0\}$.

Let $X(\cdot, t)\colon M \times [0, T) \to \mathbb{R}^n$ be a family of smooth embeddings from a closed manifold $M$ in $\mathbb{R}^n$ satisfying the evolution equation

$$\frac{\partial}{\partial t} X(x,t) = \frac{1}{H_F(x,t)} \nu_F(x,t), \tag{1}$$

where $H_F(x,t) > 0$ is the anisotropic mean curvature function of the hypersurface $N_t = X(M,t)$ and $\nu_F(x,t)$ is the unit anisotropic outer normal.

The anisotropic type curvature flows have been used by Angenent and Gurtin to modeling the motion of the interface with external force, see for example [3, 4, 14] and the reference therein. Geometrically, they can be thought of as curvature flows in Minkowski geometry. Different kind of anisotropic type curvature flows have been studied, see for example [1, 2, 20]. For the anisotropic mean curvature flow, there are works concerning with weak solutions and their regularity issue, as well as its numerical analysis, see [8, 12]

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and the reference therein. For more references on anisotropic type curvature flows we refer to [20] and the reference therein.

In this paper, we look at (1), the inverse anisotropic mean curvature flow (IAMCF). When $F$ is the Euclidean norm, $\nu_F$ and $H_F$ reduce to the unit outer normal and the mean curvature respectively, and in turn (1) reduces to the classical inverse mean curvature flow (IMCF).

Gerhardt [11] and Urbas [22] proved that the classical IMCF which initiated from a star-shaped and strictly mean convex hypersurface exists for all time and converge to a round sphere after rescaling. For general initial data, the IMCF may develop singularity. Huisken-Ilmanen [15] has developed a theory of weak solutions for the IMCF of hypersurfaces in Riemannian manifolds by its level-set formulation and applied it to show the validity of the Riemannian Penrose inequality.

For the anisotropic counterpart, recently the third author [20] has studied the IAMCF which initiated from a star-shaped and strictly $F$-mean convex hypersurface and proved the long time existence and convergence result analogous to Gerhardt and Urbas' result. As a direct application, he has proved the anisotropic Minkowski inequality between the anisotropic mean curvature integral and the anisotropic area for star-shaped and strictly $F$-mean convex hypersurfaces. This anisotropic Minkowski inequality for convex hypersurfaces is a classical result in the theory of convex geometry. One asks naturally what happens if the initial hypersurface is no longer star-shaped. Analogous to Huisken-Ilmanen [15], we are able to develop a theory of weak solutions for the IAMCF by its level-set formulation. This is the aim of this paper.

Suppose the evolving hypersurfaces $N_t$ are given by level sets of a function $u\colon \mathbb{R}^n \to \mathbb{R}$, that is

$$E_t = \{x \in \mathbb{R}^n : u(x) < t\}, \quad N_t = \partial E_t.$$

If $u$ is smooth and $\nabla u \neq 0$, then (1) is equivalent to the degenerate elliptic equation

$$\operatorname{div}\left(F_\xi(\nabla u)\right) = F(\nabla u). \tag{2}$$

See Section 2. When $F$ is Euclidean, it is clear (2) reduces to

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u|. \tag{3}$$

As Huisken-Ilmanen [15], we define a weak solution of (2) by the following minimization principle.

**Definition 1.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $u \in C^{0,1}_{loc}(\Omega)$ is called a weak solution to* (2) *if*

$$J_{F,u}(u) \leq J_{F,u}(\varphi) \tag{4}$$

*for every precompact set $K \subset \Omega$ and for every test function $\varphi \in C^{0,1}_{loc}(\Omega)$ with $\varphi = u$ in $\Omega \setminus K$, and where*

$$J_{F,u}(\varphi) := \int_K \big[F(\nabla \varphi) + \varphi F(\nabla u)\big] dx. \tag{5}$$

*Moreover, $u$ is a proper solution if in addition*

$$\lim_{|x| \to +\infty} u(x) = +\infty.$$

Our main result of this paper is the following existence result.

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^n$ be an open set with smooth boundary such that $\Omega^c = \mathbb{R}^n \setminus \Omega$ is bounded. There exists a unique proper weak solution $u \in C^{0,1}_{loc}(\overline{\Omega})$ of* (2)*, in the sense of Definition 1, such that $u = 0$ on $\partial\Omega$. Moreover, $u$ satisfies*

$$F(\nabla u(x)) \leq \sup_{\partial\Omega} H_F^+, \quad x \in \overline{\Omega}, \tag{6}$$

$$F(\nabla u(x)) \leq H_F^+(x), \quad x \in \partial\Omega, \tag{7}$$

*where $H_F^+(x) = \max\{H_F(x), 0\}$ and $H_F$ is the anisotropic mean curvature of $\partial\Omega$.*

Huisken-Ilmanen's approach in the classical IMCF case to prove the existence is studying an approximate equation of (3), known as elliptic regularization. One of the key feature of this elliptic regularization is that it corresponds to a family of translating graphs which solves the IMCF in $\mathbb{R}^n \times \mathbb{R}$. It seems that such elliptic regularization is not available in the anisotropic case. This is due to the presence of the high nonlinearity $F(\nabla u)$ in $\mathbb{R}^n$, for which it is not immediate to understand what is the correct AIMCF in a higher dimension, preserving the correct equation in the limit.

Later, Moser [17] found another approximate equation of (3) involving the $p$-Laplacian. It turns out that this approximate equation is also effective to prove the existence of weak solutions for IAMCF.

Inspired by Moser's approach, we consider the approximate equation of (2) involving the Finsler-$p$-Laplacian, that is,

$$\begin{cases} \operatorname{div}\left(F^{p-1}(\nabla u)F_\xi(\nabla u)\right) = F(\nabla u)^p & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \\ u \to \infty & \text{as } x \to \infty. \end{cases} \tag{8}$$

We have the following

**Theorem 1.2.** *Let $\Omega \subset \mathbb{R}^n$ be an open set with smooth boundary such that $\Omega^c = \mathbb{R}^n \setminus \Omega$ is bounded. For every $p > 1$, there exists a unique solution $u_p \in C^{1,\alpha}_{loc}(\overline{\Omega})$. Moreover, for every $\varepsilon > 0$, there exists $p_0 = p_0(\varepsilon) > 1$ such that if $u_p \in C^{1,\alpha}_{loc}(\overline{\Omega})$ is the solution to (8) for $1 < p \leq p_0$, then*

$$F(\nabla u_p(x)) \leq \sup_{\partial\Omega} H_F^+ + \varepsilon, \quad x \in \overline{\Omega} \tag{9}$$

$$F(\nabla u_p(x)) \leq H_F^+(x) + \varepsilon, \quad x \in \partial\Omega. \tag{10}$$

Theorem 1.1 follows from Theorem 1.2 by approximation.

The rest of this paper is organized as follows. In Section 2, we recall some fundamentals on anisotropic functions and anisotropic mean curvature. In Section 3, we study Huisken-Ilmanen type weak formulation of IAMCF and its properties. In Section 4, we study the approximate equation involving the Finsler-$p$-Laplacian and show the gradient estimate and the existence of weak solution of IAMCF.

## 2. Notation and preliminaries

### 2.1. Minkowski norm and Wulff shape

Let $F$ be a Minkowski norm on $\mathbb{R}^n$. The polar function $F^o\colon \mathbb{R}^n \to [0, +\infty[$ of $F$, defined as

$$F^o(x) = \sup_{\xi \neq 0} \frac{\langle \xi, x\rangle}{F(\xi)},$$

is again a Minkowski norm on $\mathbb{R}^n$. Furthermore,

$$F(\xi) = \sup_{x \neq 0} \frac{\langle \xi, x\rangle}{F^o(x)}.$$

Denote

$$\mathcal{W} = \{x \in \mathbb{R}^n \colon F^o(x) < 1\}.$$

This is the so-called Wulff shape centered at the origin. More generally, we denote by $\mathcal{W}_r(x_0)$ the set $r\mathcal{W} + x_0$, that is the Wulff shape centered at $x_0$ with radius $r$ and $\mathcal{W}_r = \mathcal{W}_r(0)$.

The following properties of $F$ and $F^o$ hold true: for any $x, \xi \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{gathered} \langle F_\xi(\xi), \xi\rangle = F(\xi), \quad \langle F^o_x(x), x\rangle = F^o(x) \\ F(F^o_x(x)) = F^o(F_\xi(\xi)) = 1, \\ F^o(x)F_\xi(F^o_x(x)) = x, \quad F(\xi)F^o_x\left(F_\xi(\xi)\right) = \xi. \end{gathered}$$

See e.g. [21], Chapter 2.

## 2.2. Anisotropic mean curvature and anisotropic area functional

Let $N$ be a smooth closed hypersurface in $\mathbb{R}^n$ and $\nu$ be the unit Euclidean outer normal of $N$. The anisotropic outer normal of $N$ is defined by

$$\nu_F = F_\xi(\nu).$$

The anisotropic mean curvature of $N$ is defined by

$$H_F = \mathrm{div}_N(\nu_F).$$

Here $\mathrm{div}_N$ is the tangential divergence on $N$. See [20].

In this paper we are interested in the case when $N$ is given by a level set of a smooth function $u$, namely,

$$N = N_t = \partial E_t, \text{ where } E_t = \{x \in \mathbb{R}^n : u(x) < t\}.$$

When $\nabla u \neq 0$, it is clear that $\nu = \frac{\nabla u}{|\nabla u|}$ and

$$\nu_F = F_\xi(\nabla u). \tag{11}$$

It was proved that in [23] that

$$H_F = \mathrm{div}(F_\xi(\nabla u)). \tag{12}$$

Here div is the standard divergence on $\mathbb{R}^n$. If $N_t$ satisfies the IAMCF, we see that $u(x(t)) = t$ and by taking derivative about $t$, we get

$$\left\langle \nabla u, \frac{1}{H_F}\nu_F \right\rangle = 1.$$

By virtue of (11) and (12), we arrive at (2).

The anisotropic area functional of $N$ is defined as

$$\sigma_F(N) = \int_N F(\nu)\, d\mathcal{H}^{n-1},$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

It is well-known that a variational characterization of $H_F$ is given by the first variational formula of $\sigma_F$, see for instance [6, 18, 20]. More precisely, we have

**Proposition 2.1 (Reilly [18], Bellettini-Novaga-Riey [6]).**

*Let $N$ be a smooth closed hypersurface given by an embedding $X_0 : M \to \mathbb{R}^n$. Let $N_s$ be a variation of $N$ given by $X(\cdot, s) : M \to \mathbb{R}^n$, $s \in (-\varepsilon, \varepsilon)$, whose variational vector field $\frac{\partial}{\partial s}|_{s=0} X(\cdot, s) = V$. Then*

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} \sigma_F(N_s) &= \int_N \mathrm{div}_{F,N}(V) F(\nu) d\mathcal{H}^{n-1} \\ &= \int_N H_F(X_0) \langle V, \nu \rangle \, d\mathcal{H}^{n-1}, \end{aligned} \tag{13}$$

*where*

$$\mathrm{div}_{F,N}(V) := \mathrm{div} V - \left\langle \nabla_{\nu_F} V, \frac{\nu}{F(\nu)} \right\rangle.$$

*Proof.* We refer to [20] for the proof of the second equality. For completeness, we prove the first equality here. We denote $\nu_s$ and $d\sigma_s$ be the unit outer normal and the area element of $N_s$ respectively. It is well-known that

$$\frac{d}{ds}\bigg|_{s=0} \nu_s = -\langle \nu, \nabla_{e_i} V \rangle e_i,$$

where $\{e_i\}$ is an orthonormal basis of $TN$.

$$\frac{d}{ds}\bigg|_{s=0} d\sigma_s = \mathrm{div}_N(V) d\sigma.$$

Thus

$$\begin{aligned}\left.\frac{d}{ds}\right|_{s=0} F(\nu_s)d\sigma_s &= \langle F_\xi(\nu), -\langle \nu, \nabla_{e_i} V\rangle e_i\rangle d\sigma + F(\nu)\,\mathrm{div}_N(V)d\sigma \\ &= \left[-\langle \nu, \nabla_{e_i} V\rangle \langle F_\xi(\nu), e_i\rangle + (\mathrm{div}\, V - \langle \nabla_\nu V, \nu\rangle)F(\nu)\right] d\sigma \\ &= \left[\mathrm{div} V - \left\langle \nabla_{\nu_F} V, \frac{\nu}{F(\nu)}\right\rangle\right] F(\nu)d\sigma\end{aligned}$$

The last line follows from

$$\nu_F = F_\xi(\nu) = F(\nu)\nu + \langle F_\xi(\nu), e_i\rangle e_i.$$

□

## 3. IAMCF: a variational formulation

In this section, we review the weak formulation of IAMCF developed by Huisken-Ilmanen [15] by using a minimizing principle. We follow closely Huisken-Ilmanen's strategy in [15], Section 1.

### 3.1. Weak formulation of IAMCF

Recall (Definition 1) that $u$ is called a weak solution (subsolution, supersolution resp.) of (2) in $\Omega$ if $u \in C^{0,1}_{\mathrm{loc}}(\Omega)$ and $J_{F,u}(u) \le J_{F,u}(\varphi)$ for every precompact set $K \subset \Omega$ and for every test function $\varphi \in C^{0,1}_{\mathrm{loc}}(\Omega)$ ($\varphi \le u$, $\varphi \ge u$ resp.) with $\varphi = u$ in $\Omega \setminus K$, where $J_{F,u}$ is defined in (5).

The fact that

$$J_{F,u}(\min\{\varphi, u\}) + J_{F,u}(\max\{\varphi, u\}) = J_{F,u}(\varphi) + J_{F,u}(u)$$

whenever $\{u \ne \varphi\}$ is precompact implies $u$ is a solution if and only if it is both a weak supersolution and a weak subsolution.

There is an equivalent weak formulation by set functional. For $K \subseteq \Omega$ and $u \in C^{0,1}_{\mathrm{loc}}(\Omega)$, define

$$\mathcal{J}_{F,u}(G) = \mathcal{J}^K_{F,u}(G) := \int_{\partial^* G \cap K} F(\nu) d\mathcal{H}^{n-1} - \int_{G \cap K} F(\nabla u) dx,$$

for a set $G$ of locally finite perimeter, and $\partial^* G$ denotes the reduced boundary of $G$.

**Definition 2.** *We say that $E$ minimizes $J_{F,u}$ in a set $A$ (minimizes on the outside, minimizes on the inside, resp.) if*

$$\mathcal{J}_{F,u}(E) \leq \mathcal{J}_{F,u}(G)$$

*for any $G$ such that $E\Delta G \subset\subset A$ ($G \supseteq E$, $G \subseteq E$ resp.) and any compact set $K$ containing $E\Delta G$. Here $E\Delta G = (E \setminus G) \cup (G \setminus E)$.*

The fact that

$$\mathcal{J}_{F,u}(E \cap G) + \mathcal{J}_{F,u}(E \cup G) \leq \mathcal{J}_{F,u}(E) + \mathcal{J}_{F,u}(G)$$

whenever $E\Delta G$ is precompact guarantees that $E$ minimizes $\mathcal{J}_{F,u}$ in $\Omega$ if and only if $E$ is minimizes $\mathcal{J}_{F,u}$ both on the inside and on the outside in $\Omega$.

The Definitions 1 and 2 are equivalent in the following sense.

**Proposition 3.1.** *Let $\Omega$ be an open set and $u \in C^{0,1}_{loc}(\Omega)$, then $u$ is a weak solution of (2) in $\Omega$ if and only if for each $t$, $E_t = \{x \in \Omega \colon u < t\}$ minimizes $\mathcal{J}_{F,u}$ in $\Omega$.*

*Proof.* By the co-area formula, we have for a choice of $a < b$ such that $a < u < b$ and $a < \varphi < b$ in $K$, that

$$\begin{aligned} J_{F,u}(\varphi) &= \int_K (F(\nabla\varphi) + \varphi F(\nabla u))dx \\ &= \int_a^b dt \int_{(\partial^*\{\varphi<t\})\cap K} F\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right) + \int_K \varphi F(\nabla u)dx \\ &= \int_a^b dt \int_{(\partial^*\{\varphi<t\})\cap K} F(\nu)\, d\sigma - \int_K \int_a^b \chi_{\{\varphi<t\}} F(\nabla u)dxdt \\ &\quad + b\int_K F(\nabla u)\, dx \\ &= \int_a^b \mathcal{J}^K_{F,u}(\{\varphi < t\}) + b\int_K F(\nabla u)\, dx. \end{aligned} \tag{14}$$

Then, if for any $t$, $E_t$ is a minimizer of the set functional $J_{F,u}$, then

$$J_{F,u}(\varphi) \geq J_{F,u}(u),$$

that gives the minimality of $u$.

The viceversa can be proved exactly as in the proof of Lemma 1.1 in [15]. □

Next we study the weak formulation of IAMCF with initial condition.

**Definition 3.** *Let $E_0$ be an open set with smooth boundary. Let $\{E_t\}_{t>0}$ be a nested family of open sets in $\mathbb{R}^n$.*

(i) *$u$ is called a weak solution of (2) with initial condition $E_0$ if $u \in C^{0,1}_{loc}(\mathbb{R}^n)$, $E_0 = \{u < 0\}$ and $u$ is a weak solution of (2) in $\mathbb{R}^n \setminus \overline{E_0}$.*

(ii) *Define $u$ by the characterization $E_t = \{u < t\}$. $\{E_t\}_{t>0}$ is called a weak solution of (1) with initial condition $E_0$ if $u \in C^{0,1}_{loc}(\mathbb{R}^n)$ and $E_t$ minimizes $\mathcal{J}_{F,u}$ in $\mathbb{R}^n \setminus E_0$ for each $t > 0$.*

From Proposition 3.1, it is easy to see the above two definitions are also equivalent.

**Proposition 3.2.** *$u$ is a weak solution of (2) with initial condition $E_0$ if and only if $\{u < t\}_{t>0}$ is a weak solution of (1) with initial condition $E_0$.*

The weak solution is unique.

**Proposition 3.3 (Uniqueness of the weak solutions).**

(i) *Let $u$ and $v$ be weak solutions to (2) in $\Omega$ in the sense of Definition 1, and $\{v > u\} \subset\subset \Omega$. Then $v \leq u$ in $\Omega$;*

(ii) *if $\{E_t\}_{t>0}$ and $\{F_t\}_{t>0}$ solve (1) in the sense of Definition 3, with initial data $E_0, F_0$ respectively, and $E_0 \subseteq F_0$, then $E_t \subseteq F_t$ as long as $E_t$ is precompact. In particular, for a given $E_0$ there exists at most one solution $\{E_t\}_{t>0}$ of (1) such that $E_t$ is precompact.*

*Proof.* The proof runs exactly in the same way of the Euclidean case, contained in Huisken-Ilmanen [15, Theorem 2.2], and the only modifications have to be done in point (i), by replacing the Euclidean norm of $\nabla u$ with $F(\nabla u)$, and using the fact that $F$ is a norm in $\mathbb{R}^n$. In particular, the proof of (i) consists in showing that if $u$ is weak supersolution to (2), in the sense that if $w \geq u$ is a Lipschitz function such that $\{w \neq u\} \subset\subset \Omega$, we have

$$\int_K F(\nabla u) + uF(\nabla u) \leq \int_K F(\nabla w) + wF(\nabla u),$$

where the integration is performed over any compact set $K$ containing $u \neq w$, it holds that $v \leq u$. □

### 3.2. Properties of weak IAMCF

**Definition 4.** *Let $\Omega$ be an open set.*

(i) *A set $E$ is called an $F$-minimizing hull in $\Omega$ if*

$$\sigma_F(\partial^* E \cap K) \leq \sigma_F(\partial^* G \cap K) \tag{15}$$

*for any $G$ containing $E$ such that $G \setminus E \subset\subset \Omega$ and any compact set $K$ containing $G \setminus E$.*

(ii) *A set $E$ is called a strictly $F$-minimizing hull in $\Omega$ if it is an $F$-minimizing hull in $\Omega$ and equality holds in* (15) *if and only if*

$$G \cap \Omega = E \cap \Omega \text{ a.e.}$$

(iii) *Given a measurable set $E$, the set $E'$ is defined to be the intersection of all the strictly $F$-minimizing hulls in $\Omega$ that contain $E$.*

One has the following properties for weak solutions of IAMCF.

**Proposition 3.4.** *Let $u$ be a weak solution of* (2) *with initial condition $E_0$. Set*

$$E_t = \{u < t\}, \quad E_t^+ = \operatorname{int}\{u \leq t\}.$$

*Then*

(i) *For $t > 0$, $E_t$ is an $F$-minimizing hull in $\mathbb{R}^n$;*

(ii) *Fot $t \geq 0$, $E_t^+$ is a strictly $F$-minimizing hull in $\mathbb{R}^n$ and $E_t' = E_t^+$ if it is precompact;*

(iii) *For $t > 0$, $\sigma_F(\partial E_t) = \sigma_F(\partial E_t^+)$ provided $E_t^+$ is precompact. This holds true for $t = 0$ if $E_0$ is an $F$-minimizing hull.*

(iv) *$\sigma_F(\partial E_t) = e^t \sigma_F(\partial E_0)$ provided $E_0$ is an $F$-minimizing hull.*

*Proof.* (i)-(iii) runs exactly the same way of the Euclidean case, contained in Huisken-Ilmanen [15, Property 1.4] . The equivalence of the two definitions and the fact that $F(\nabla u) = 0$ a.e. on $\{u = t\}$ are essentially used. One needs just to use $\int_{E_t} F(\nabla u)$ and $\sigma_F(\partial^* E_t \cap K)$ instead of $\int_{E_t} |\nabla u|$ and $|\partial^* E_t \cap K|$.

For (iv), the minimizing property implies that $J_{F,u}(E_t)$ is independent of $t$. By co-area formula, we get

$$J_{F,u}(E_t) = \sigma_F(\partial E_t) - \int_0^t \sigma_F(\partial E_s)ds = const.$$

which implies $e^{-t}\sigma_F(\partial E_t)$ is constant for $t > 0$. The assertion now follows from (iii). □

Analogous to the classical case, we define the weak anisotropic mean curvature by the first variational formula, Proposition 2.1.

**Definition 5.** *Let $N \subset \mathbb{R}^n$ be a hypersurface of $C^1$ or $C^1$ with a small singular set and locally finite Hausdorff measure. A locally integrable function $H_F$ on $N$ is called weak anisotropic mean curvature provided it satisfies the second equality in* (13) *for every $V \in C_c^\infty(\mathbb{R}^n)$.*

For smooth IAMCF given by $\{u = t\}$, one sees $H_F = F(\nabla u)$. We show next weak solutions of (2) still have this property.

**Proposition 3.5.** *Let $u$ be a weak solution of* (2) *with initial condition $E_0$ and let $N_t = \partial E_t = \partial\{u < t\}$. Then for a.e. $t$, the weak anisotropic mean curvature $H_F$ of $N_t$ satisfies*

$$H_F = F(\nabla u) \ a.e. \ x \in N_t.$$

*Proof.* Let $V \in C_c^\infty(\mathbb{R}^n)$ and $\Phi^s : \mathbb{R}^n \to \mathbb{R}^n, s \in (-\varepsilon, \varepsilon)$, be the flow of diffeomorphisms generated by $V$ and $\Phi^0 = Id$. Let $W$ be any precompact open set containing supp$(V)$.

Because $u$ be a weak solution of (2) in $\mathbb{R}^n \setminus \overline{E_0}$, we see $J_{F,u}(u \circ \Phi^s) \le J_{F,u}(u)$. Thus $\frac{d}{ds}\Big|_{s=0} J_{F,u}(u \circ \Phi^s) = 0$. Next we derive $\frac{d}{ds}\Big|_{s=0} J_{F,u}(u \circ \Phi^s)$.

First, we assume $u$ is smooth. Then

$$\begin{aligned}
&\frac{d}{ds}\Big|_{s=0} \int_K F(\nabla(u \circ \Phi^s))dx \\
&= \int_W F_{\xi_i}(\nabla u)\nabla_i(\frac{d}{ds}\Big|_{s=0}(u \circ \Phi^s))dx \\
&= \int_W F_{\xi_i}(\nabla u)\nabla_i(\nabla_j u V^j)dx \\
&= \int_W F_{\xi_i}(\nabla u)\nabla^2_{ji} u V^j + F_{\xi_i}(\nabla u)\nabla_j u \nabla_i V^j dx
\end{aligned}$$

$$= \int_W -F_{\xi_i\xi_k}(\nabla u)\nabla^2_{kj}u\nabla_i u V^j - F_{\xi_i}(\nabla u)\nabla_i u\nabla_j V^j + F_{\xi_i}(\nabla u)\nabla_j u\nabla_i V^j dx$$
$$= \int_W -F(\nabla u)\operatorname{div} V + \langle \nabla u, \nabla_{F_\xi(\nabla u)} V\rangle dx.$$

By co-area formula,

$$\begin{aligned}&\int_W -F(\nabla u)\operatorname{div} V + \langle \nabla u, \nabla_{F_\xi(\nabla u)} V\rangle dx\\ &\quad = \int_{-\infty}^{\infty}\int_{N_t\cap W} -\operatorname{div} V F(\nu) + \langle \nu, \nabla_{\nu_F} V\rangle d\sigma_t dt\\ &\quad = \int_{-\infty}^{\infty}\int_{N_t\cap W} -\operatorname{div}_{F,N} V F(\nu) d\sigma_t dt.\end{aligned}$$

Thus

$$\frac{d}{ds}\Big|_{s=0}\int_K F(\nabla(u\circ\Phi^s))dx = \int_{-\infty}^{\infty}\int_{N_t\cap W} -\operatorname{div}_{F,N} V F(\nu) d\sigma_t dt. \tag{16}$$

By an approximation argument, we see that the formula (16) is still true for $u$ only locally Lipschitz.

On the other hand, it is easy to see

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}\int_K (u\circ\Phi^s)F(\nabla u)dx &= \int_W \langle \nabla u, V\rangle F(\nabla u)dx\\ &= \int_{-\infty}^{\infty}\int_{N_t\cap W} \langle \nu, V\rangle F(\nabla u) d\sigma_t dt.\end{aligned} \tag{17}$$

Combining (16) and (17), we get

$$\begin{aligned}0 &= \frac{d}{ds}\Big|_{s=0} J_{F,u}(u\circ\Phi^s)\\ &= \int_{-\infty}^{\infty}\int_{N_t\cap W} -\operatorname{div}_{F,N} V F(\nu) + \langle \nu, V\rangle F(\nabla u) d\sigma_t dt.\end{aligned} \tag{18}$$

Finally, by the definition of the weak anisotropic mean curvature, we conclude from (18) that

$$H_F = F(\nabla u) \text{ a.e. } x\in N_t \text{ a.e. } t.$$

□

## 4. Existence of solutions and gradient estimates

For any $p > 1$, we will consider the following auxiliary problem

$$(19)\qquad \begin{cases} \operatorname{div}\left(F^{p-1}(\nabla v) F_\xi(\nabla v)\right) = 0 & \text{in } \Omega, \\ v = 1 & \text{in } \Omega^c. \\ v \to 0 & \text{as } x \to \infty. \end{cases}$$

**Proposition 4.1.** *If $1 < p < n$, then there exists a unique positive solution $v_p \in C^{1,\alpha}_{loc}(\overline{\Omega}) \cap C^\infty(\overline{\Omega} \setminus \{\nabla v_p = 0\})$ of (19). If $\mathcal{W}_r(x_0) \subset \Omega^c \subset \mathcal{W}_s(y_0)$, then*

$$(20)\qquad \left(\frac{r}{F^o(x - x_0)}\right)^{\frac{n-p}{p-1}} \le v_p(x) \le \left(\frac{s}{F^o(x - y_0)}\right)^{\frac{n-p}{p-1}}, \quad \forall x \in \Omega \setminus \{y_0\};$$

*Moreover, $v_p$ verifies*

$$(21)\qquad \lim_{|x| \to \infty} \frac{F(\nabla v_p)}{v_p} = 0.$$

*Proof.* The proof of existence, uniqueness, regularity, as well as (20), follow by nowadays standard arguments; we refer the reader to [7, Theorem 3.3] for the general anisotropic case we consider.

Finally we prove (21). We argue as in [17]. Let $\eta \in C^\infty_0(\Omega)$ be a suitable cut-off function. Taking $\psi = v_p \eta^p$ as test function in the weak formulation for (19) and using the Hölder inequality, it easily follows that:

$$\int_\Omega \eta^p F(\nabla v_p)^p dx \le p^p \int_\Omega v_p^p F(\nabla \eta)^p dx.$$

By Harnack inequality (see for instance [19]) we get

$$r^{p-n} \int_{B_{r/4}(x_0)} F(\nabla v_p)^p dx \le C(n, p) \inf_{B_{r/2}(x_0)} v_p^p,$$

where $C(n, p)$ is a positive constant depending on $n$ and $p$. By applying the result contained in [10], we have

$$\|F(\nabla v_p)\|_{L^\infty(B_{r/8}(x_0))} \le \frac{C(n, p)}{r} \inf_{B_{r/2}(x_0)} v_p,$$

which implies (21), and the proof is completed. □

It is direct to see that

$$u_p = (1-p)\log v_p \in C^{1,\alpha}(\overline{\Omega})$$

solves (8) (we refer the reader, e.g., also to [9] for problems involving equations as in (8)). Next we show the gradient estimate in Theorem 1.2, which is based on the following Lemma.

**Lemma 4.2.** *Let $1<p<n$ and $u_p \in C^{1,\alpha}_{loc}(\overline{\Omega})$ be a solution to (8), then*

$$\sup_{\bar{\Omega}} F(\nabla u_p) = \sup_{\partial\Omega} F(\nabla u_p). \tag{22}$$

*Proof.* We omit the subscript $p$ in $u_p$ in the proof. Let $\tau = \sup_{\partial\Omega} F(\nabla u)$. We consider the following set

$$\Omega_\beta = \{x \in \Omega \colon F(\nabla u) > \beta\},$$

with $\beta > \tau \geq 0$. From (21) we see $F(\nabla u)$ vanishes at infinity by (21), then $\Omega_\beta$ is a bounded, open set such that $\overline{\Omega}_\beta \cap \partial\Omega = \emptyset$ and $F(\nabla u) = \beta$ on $\partial\Omega_\beta$.

In order to prove (22), we will prove that $\Omega_\beta = \emptyset$.

Note that in $\overline{\Omega_\beta}$, $\nabla u \neq 0$ and hence $u \in C^\infty(\overline{\Omega_\beta})$. By writing

$$G(\xi) = \frac{1}{2}F^2(\xi),$$

the equation in (8) becomes

$$\operatorname{div}\left(G^{\frac{p}{2}-1}(\nabla u) G_\xi(\nabla u)\right) = G^{\frac{p}{2}}(\nabla u). \tag{23}$$

Hereafter we will adopt the Einstein convention on the repeated indices, and use the notations

$$G = G(\nabla u), \quad G_i = G_{\xi_i}(\nabla u), \quad G_{ij} = G_{\xi_i\xi_j}(\nabla u),$$
$$u_i = u_{x_i}, \quad u_{ij} = u_{x_i x_j}, \cdots$$

Differentiating (23) with respect to $x_i$, we get

$$\partial_{x_k}\left(\partial_{x_i}[G^{\frac{p}{2}-1}G_k]\right) = \frac{p}{2}G^{\frac{p}{2}-1}G_j u_{ij},$$

and

$$\partial_{x_k}\left(G_i\partial_{x_i}[G^{\frac{p}{2}-1}G_k]\right) = \frac{p}{2}G^{\frac{p}{2}-1}G_iG_j u_{ij} + G_{il}u_{lk}\partial_{x_i}[G^{\frac{p}{2}-1}G_k].$$

Then

$$\partial_{x_k}\left(\left[\frac{p-2}{2}G^{\frac{p}{2}-2}G_jG_k+G^{\frac{p}{2}-1}G_{kj}\right]G_iu_{ij}\right)$$
$$=\frac{p}{2}G^{\frac{p}{2}-1}G_iG_ju_{ij}+\left[\frac{p-2}{2}G^{\frac{p}{2}-2}G_jG_k+G^{\frac{p}{2}-1}G_{kj}\right]G_{il}u_{lk}u_{ij};$$

hence

$$(24)\quad \partial_{x_k}\left(\left[\frac{p-2}{2}G^{\frac{p}{2}-2}G_iG_jG_k+G^{\frac{p}{2}-1}G_iG_{kj}\right]u_{ij}\right)$$
$$=\frac{p}{2}G^{\frac{p}{2}-1}G_iG_ju_{ij}+\left[\frac{p-2}{2}G^{\frac{p}{2}-2}G_jG_kG_{il}u_{lk}u_{ij}+G^{\frac{p}{2}-1}G_{kj}G_{il}u_{lk}u_{ij}\right]$$
$$=\frac{p}{2}G^{\frac{p}{2}-1}G_iG_ju_{ij}+\frac{p-1}{2}G^{\frac{p}{2}-2}G_jG_kG_{il}u_{lk}u_{ij}$$
$$+G^{\frac{p}{2}-2}\left[-\frac{1}{2}G_jG_kG_{il}u_{lk}u_{ij}+G\,G_{kj}G_{il}u_{lk}u_{ij}\right].$$

The Kato type inequality (see [24, Lemma 2.2]) implies that

$$GG_{il}G_{jk}u_{ij}u_{kl}\geq\frac{1}{2}G_{il}G_jG_ku_{ij}u_{kl}.$$

Hence, from (24) we get

$$\partial_{x_k}\left(\left[\frac{p-2}{2}G^{\frac{p}{2}-2}G_iG_jG_k+G^{\frac{p}{2}-1}G_iG_{kj}\right]u_{ij}\right)$$
$$\geq\frac{p}{2}G^{\frac{p}{2}-1}G_iG_ju_{ij}+\frac{p-1}{2}G^{\frac{p}{2}-2}G_jG_kG_{il}u_{lk}u_{ij}.$$

The above inequality can be read as

$$\operatorname{div}\left[G^{\frac{p}{2}-1}\left(G_{\xi\xi}\nabla_xG+\frac{p-2}{2}\frac{(G_\xi\cdot\nabla_xG)}{G}G_\xi\right)\right]-\frac{p}{2}G^{\frac{p}{2}-1}(G_\xi\cdot\nabla_xG)$$
$$\geq\frac{p-1}{2}(G_{\xi\xi}\nabla_xG)\cdot\nabla_xG,$$

that is

$$(25)\qquad \operatorname{div}\left[G^{\frac{p}{2}-1}A\nabla_xG\right]-\frac{p}{2}G^{\frac{p}{2}-1}G_\xi\cdot\nabla_xG$$
$$\geq\frac{p-1}{2}G^{\frac{p}{2}-2}\left(G_{\xi\xi}\nabla_xG\right)\cdot\nabla_xG\geq 0,$$

where

$$A = G_{\xi\xi} + \frac{p-2}{2}\frac{G_\xi \otimes G_\xi}{G}$$

is a uniformly positive definite matrix. Hence, the functional in the left-hand side of (25) can be seen as a linear elliptic operator acting on $G(\nabla u)$, and by the maximum principle we have that

$$\sup_{\Omega_\beta} G(\nabla u) \le \sup_{\partial\Omega_\beta} G(\nabla u) = \frac{\beta^2}{2}.$$

This implies that $\Omega_\beta = \left\{x \in \Omega \colon G(\nabla u) > \frac{\beta^2}{2}\right\}$ is empty, and the proof is completed. □

*Proof of Theorem 1.2.* We are remained to prove the boundary gradient estimate (10). The global gradient estimate (9) follows from Lemma 4.2 and (10).

Let $x \in \partial\Omega$ such that $\mathcal{W}_r(x_0) \subset \Omega^c$. Since $u_p = 0$ on $\partial\Omega$, hence if $\nabla u_p(x) \neq 0$, then $\nu_F(x) = F_\xi(\nabla u_p(x))$ and $F(\nabla u_p(x)) = \frac{\partial u_p}{\partial \nu_F}(x)$. On the other hand, since $W_r(x_0)$ and $\partial\Omega$ are tangent at $x$, we see $x - x_0 = r\nu_F(x)$. It follows that

$$\begin{aligned} F(\nabla u_p(x)) &= \frac{\partial u_p}{\partial \nu_F}(x) = \lim_{t\to 0}\frac{u_p(x+t\nu_F)}{t} \\ &\le (n-p)\lim_{t\to 0}\frac{\log F^0(x+t\nu_F - x_0) - \log r}{t} \\ &= \frac{n-p}{r}. \end{aligned}$$

Thus if we define

$$R := \sup\{r > 0 \colon \forall x \in \partial\Omega, \exists \mathcal{W}_r(x_0) \subset \Omega^c \text{ such that } x \in \partial\mathcal{W}_r(x_0)\}. \tag{26}$$

then

$$\|F(\nabla u_p)\|_{L^\infty(\partial\Omega)} \le \frac{n-p}{R}.$$

It follows from Lemma 4.2 that

$$\|F(\nabla u_p)\|_{L^\infty(\overline{\Omega})} \le \frac{n-p}{R}. \tag{27}$$

Next prove the estimate (10). We argue as in [15, 16]. Let $\varepsilon > 0$. Choose $\bar{w} \in C^\infty(\overline{\Omega})$ such that

i) $\bar{w} = 0$ on $\partial\Omega$ and $\bar{w} > 0$ in $\Omega$;

ii) $H_F^+ < F(\nabla \bar{w}) \le H_F^+ + \varepsilon$ on $\partial\Omega$.

Denote

$$\mathcal{Q}_p[\varphi] := \operatorname{div}\left(F^{p-1}(\nabla\varphi)F_\xi(\nabla\varphi)\right) - F(\nabla\varphi)^p. \tag{28}$$

Since $F(\nabla \bar{w}) > 0$ and $\bar{w} = 0$ on $\partial\Omega$, the anisotropic mean curvature of $\partial\Omega$ is given by

$$H_F(x) = \operatorname{div}\left(F_\xi(\nabla \bar{w})\right).$$

Thus

$$\mathcal{Q}_1[\bar{w}](x) = H_F(x) - F(\nabla \bar{w}(x)) < 0 \text{ for } x \in \partial\Omega.$$

Let $\delta > 0$ and denote by $U_\delta$ the components of the set $\{0 \le \bar{w} < \delta\}$ containing $\partial\Omega$. If we choose $\delta > 0$ small enough, we may have $F(\nabla \bar{w}) > 0$ and $\mathcal{Q}_1[\bar{w}] < 0$ in $U_\delta$.

Define $w \in C^\infty(U_\delta)$ by

$$w = \frac{\bar{w}}{1 - \frac{\bar{w}}{\delta}}.$$

Then

$$\nabla w = \frac{\nabla \bar{w}}{(1 - \frac{\bar{w}}{\delta})^2}.$$

A simple computation gives

$$\mathcal{Q}_1[w] = \mathcal{Q}_1[\bar{w}] + \left(1 - \frac{1}{\left(1 - \frac{\bar{w}}{\delta}\right)^2}\right) F(\nabla \bar{w}) < 0 \quad \text{ on } U_\delta.$$

By (27), there exists a constant $C = C(\delta) > 0$, such that $u_p \le C$ in $U_\delta$. Denote by $\tilde{U}_{C+1}$ the component of the set $\{0 \le w \le C+1\}$ in $U_\delta$. Since $u_p = w = 0$ on $\partial\Omega$, we see

$$u_p \le w \quad \text{on } \partial\tilde{U}_{C+1}. \tag{29}$$

In order to compare $u_p$ and $w$ in $\overline{\tilde{U}}_{C+1}$, we compute

$$\mathcal{Q}_p[w] = F^{p-1}(\nabla w)\left(\mathcal{Q}_1[w] + (p-1)\frac{w_{ik}F_{\xi_i}(\nabla w)F_{\xi_k}(\nabla w)}{F(\nabla w)}\right) \text{ in } \tilde{U}_{C+1}.$$

Since $\mathcal{Q}_1[w] < 0$ in $\tilde{U}_{C+1}$, one may choose $p-1$ small enough, depending on $\|\bar{w}\|_{C^2(\tilde{U}_{C+1})}$, $\inf_{\tilde{U}_{C+1}} F(\nabla \bar{w})$ and $C$, such that

$$\mathcal{Q}_p[w] < 0 \text{ in } \tilde{U}_{C+1}.$$

Then since $\mathcal{Q}_p[u_p] = 0 > \mathcal{Q}_p[w]$ in $\tilde{U}_{C+1}$ and (29), by the comparison principle, we have

$$u_p \leq w \text{ on } \overline{\tilde{U}}_{C+1}.$$

It follows that

$$F(\nabla u_p) = \frac{\partial u_p}{\partial \nu_F} \leq \frac{\partial w}{\partial \nu_F} = \frac{\partial \bar{w}}{\partial \nu_F} = F(\nabla \bar{w}) \leq H_F^+ + \varepsilon \quad \text{on } \partial\Omega.$$

The proof of Theorem 1.2 is completed. □

Now we are ready to prove Theorem 1.1.

*Proof of the Theorem 1.1.* Let $u_p$ be the solution of (8) in Theorem 1.2. Then for any precompact set $K \subset \Omega$, $u_p$ is also a minimizer of the functional

$$J_w^p(\varphi) = \int_K \left[\frac{1}{p} F(\nabla\varphi)^p + \varphi F(\nabla w)^p\right] dx, \tag{30}$$

in the sense that

$$J_{u_p}^p(u_p) \leq J_{u_p}^p(\varphi), \quad \forall \varphi \in W^{1,p}_{\text{loc}}(\Omega) \text{ such that } \varphi = u_p \text{ in } \Omega \setminus K. \tag{31}$$

Indeed, being $u_p - \varphi = 0$ outside $K$, and using it a as test function for (8), we get that

$$\begin{aligned}\int_K F^p(\nabla u_p)(u_p - \varphi)dx &= \int_K F^{p-1}(\nabla u_p) F_\xi(\nabla u_p) \cdot \nabla(\varphi - u_p)dx \\ &\leq \frac{1}{p}\int_K (F(\nabla\varphi)^p - F(\nabla u_p)^p)dx.\end{aligned}$$

The inequality above follows from the convexity of $F(\xi)^p$.

On the other hand, from (20), we know $u_p$ has uniform upper bound on any compact set in $\Omega$. Since we also have uniform global gradient estimate (9) for $\nabla u_p$, by Arzela-Ascoli's theorem, we get that, there exists a subsequence $p_k \to 1^+$ and $u \in C^{0,1}_{loc}(\overline{\Omega})$ such that

$$u_{p_k} \to u \text{ uniformly in any compact sets of } \overline{\Omega}. \tag{32}$$

If we can show $u$ is a weak solution of (2), then since the weak solution of (2) is unique, we will get $u_p \to u$ uniformly convergence in any compact sets of $\overline{\Omega}$ as $p \to 1$.

Next we show that $u$ is a proper weak solution to problem (2), in the sense of Definition 1. To this aim, we need to prove that

(33) $$|\nabla u_{p_k}|^{p_k} \to |\nabla u| \quad \text{in } L^1_{loc}(\Omega) \text{ for a subsequence } p_k \to 1^+.$$

Indeed from (32) and (33) we can pass to the limit in (30) obtaining that

$$J^K_{u_{p_k}}(\varphi, p_k) \to J^K_u(\varphi) \quad \text{and} \quad J^K_{u_{p_k}}(u_{p_k}, p_k) \to J^K_u(u)$$

In order to prove (33), we argue as in [15, 17]. Let $K \subset \Omega$ be a precompact set and consider the following test function $\psi$ in (31)

$$\psi = \eta\varphi + (1-\eta)u_p,$$

where $\eta \in C^\infty(\Omega)$ is a cut-off function such that $0 \le \eta \le 1$, $\eta \equiv 1$ in $K$, and $\varphi \in C^{0,1}_{\text{loc}}(\Omega)$. Then we get

$$\begin{aligned}
&\int_{\operatorname{supp}\eta} \left( \frac{F^p(\nabla u_p)}{p} + \eta\,(u_p - \varphi)\,F^p(\nabla u_p) \right) dx \\
&\quad \le \frac{1}{p} \int_{\operatorname{supp}\eta} F^p\left(\nabla(\eta\varphi + (1-\eta)u_p)\right) dx \\
&\quad \le \frac{1}{p} \int_{\operatorname{supp}\eta} [(\varphi - u_p)F(\nabla\eta) + \eta F(\nabla\varphi) + (1-\eta)F(\nabla u_p)]^p\, dx \\
&\quad \le \frac{3^{p-1}}{p} \int_{\operatorname{supp}\eta} [|\varphi - u_p|^p F^p(\nabla\eta) + \eta^p F^p(\nabla\varphi) + (1-\eta)^p F^p(\nabla u_p)]\, dx.
\end{aligned}$$

Choosing $\varphi = u$, and letting $p_k \to 1^+$, we obtain

$$\limsup_{p_k \to 1^+} \int_\Omega \eta F(\nabla u_{p_k})^{p_k} dx \le \int_\Omega \eta F(\nabla u)\, dx,$$

which together with Fatou's Lemma gives (33).

The properness of $u$ follows directly from (20). The estimate follows from that in Theorem 1.2. The proof of Theorem 1.1 is completed. □

## References

[1] B. Andrews, Motion of hypersurface by Gauss curvature, Pacific J. Math. **195** (2000), 1–34.

[2] B. Andrews, Volume-preserving anisotropic mean curvature flow. Indiana Univ. Math. J. **50** (2001), no. 2, 783–827.

[3] S. Angenent and M. E. Gurtin, Multiphase thermomechanics with interfacial structure. II. Evolution of an isothermal interface. Arch. Rational Mech. Anal. **108** (1989), no. 4, 323–391.

[4] S. Angenent and M. E. Gurtin, Anisotropic motion of a phase interface. Well-posedness of the initial value problem and qualitative properties of the interface. J. Reine Angew. Math. **446** (1994), 1–47.

[5] G. Bellettini and M. Paolini, Anisotropic motion by mean curvature in the context of Finsler geometry. *Hokkaido Math. J.* **25** (1996), 537–566.

[6] G. Bellettini, M. Novaga and G. Riey, First variation of anisotropic energies and crystalline mean curvature for partitions. Interfaces Free Bound. **5** (2003), 331–356.

[7] C. Bianchini and G. Ciraolo, Wulff shape characterizations in overdetermined anisotropic elliptic problems, Comm. PDE **43** (2018), 790–820.

[8] Y. G. Chen, Y. Giga, and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. J. Differential Geom. **33** (1991), no. 3, 749–786.

[9] F. Della Pietra and N. Gavitone, Sharp estimates and existence for anisotropic elliptic problems with general growth in the gradient, Zeitschrift für Analysis und ihre Anwendungen (ZAA) **35** (2016), 61–80.

[10] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. TMA **7** (1983), no. 8, 827–850.

[11] C. Gerhardt, Flow of nonconvex hypersurfaces into spheres. J. Differential Geom. **32** (1990), no. 1, 299–314.

[12] Y. Giga, Surface evolution equations. A level set approach. Monographs in Mathematics, **99**. Birkhäuser Verlag, Basel, 2006. xii+264 pp.

[13] D. Gilbarg and N. Trudinger, Elliptic partial differential equations of second order. *Classics in Mathematics. Springer-Verlag, Berlin*, 2001.

[14] Gurtin, M. E., Thermomechanics of evolving phase boundaries in the plane. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1993. xi+148 pp.

[15] G. Huisken and T. Ilmanen, The inverse mean curvature flow and the Riemannian Penrose inequality. J. Differential Geom. **59** (2001), no. 3, 353–437.

[16] B. Kotschwar and L. Ni, Local gradient estimates of $p$-harmonic functions, $1/H$-flow, and an entropy formula. Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 1, 1–36.

[17] R. Moser, The inverse mean curvature flow and $p$-harmonic functions, J. Eur. Math. Soc. **9** (2007), 77–83.

[18] R. C. Reilly, The relative differential geometry of nonparametric hypersurfaces. Duke Math. J. **43** (1976), no. 4, 705–721.

[19] N. Trudinger, On Harnack type inequalities and their applications to quasilinear elliptic equations, Comm. Pure Appl. Math. **20** (1967), 721–747.

[20] C. Xia, Inverse anisotropic mean curvature flow and a Minkowski type inequality, Adv. Math., **315** (2017), 102–129.

[21] C. Xia, On a class of anisotropic problem. PhD Thesis, Albert-Ludwigs University Freiburg, 2012.

[22] J. Urbas, On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures. Math. Z. **205** (1990), no. 3, 355–372.

[23] G. Wang and C. Xia, A characterization of the Wulff shape by an overdetermined anisotropic PDE. *Arch. Ration. Mech. Anal.* **199** (2011), no. 1, 99–115.

[24] G. Wang and C. Xia, An optimal anisotropic Poincaré inequality for convex domains. *Pacific J. Math.* **258** (2012), no. 2, 305–326.

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