

Sharp entropy bounds for plane curves and dynamics of the curve shortening flow

JULIUS BALDAUF AND AO SUN

We prove that a closed immersed plane curve with total curvature $2\pi m$ has entropy at least m times the entropy of the embedded circle, as long as it generates a type I singularity under the curve shortening flow (CSF). We construct closed immersed plane curves of total curvature $2\pi m$ whose entropy is less than m times the entropy of the embedded circle. As an application, we extend Colding-Minicozzi's notion of a generic mean curvature flow to closed immersed plane curves by constructing a piecewise CSF whose only singularities are embedded circles and type II singularities.

1	Introduction	595
2	Properties of entropy	602
3	Entropy and turning number	608
4	Entropy index, F-index and Morse index	610
5	Entropy instability of CSF singularities	615
6	Generic CSF	620
	References	622

1. Introduction

Huisken [18] conjectured that for mean curvature flow (MCF) from generic initial embedded hypersurfaces, all singularities are spheres or cylinders. In a recent fundamental paper [11], Colding and Minicozzi made an important

step towards establishing Huisken’s genericity conjecture. In that paper, they define the *entropy* of an immersed hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ to be

$$(1.1) \quad \lambda(\Sigma) = \sup_{x_0, t_0} (4\pi t_0)^{-n/2} \int_{\Sigma} e^{-\frac{|x-x_0|^2}{4t_0}} d\mu,$$

where the supremum is taken over all translations $x_0 \in \mathbb{R}^{n+1}$ and rescalings $t_0 > 0$ of Σ . Entropy is nonincreasing along MCF and is constant along the self-shrinking flows, which are known as *shrinkers*. Entropy is used in the following important argument: if Σ is a shrinker arising as the limit of a blowup sequence of a singular point of the MCF starting at the initial hypersurface Σ_0 then $\lambda(\Sigma) \leq \lambda(\Sigma_0)$. Entropy thus provides a vital tool for ruling out certain singularities for a MCF.

The first main theorem of this paper gives entropy lower bounds for plane curves in terms of a topological quantity, the *turning number*, which is the total curvature divided by 2π . We denote by $\Gamma_m \subset \mathbb{R}^2$ the *m-covered circle* with radius $\sqrt{2}$, i.e. Γ_m is an immersion $S^1 \rightarrow \mathbb{R}^2$ whose turning number is m and whose image is the circle of radius $\sqrt{2}$. It is easy to see that Γ_m is a shrinker for all $m \geq 1$.

Theorem A. *Let $\Gamma \subset \mathbb{R}^2$ be a closed immersed curve with turning number m .*

A.1 *If the CSF starting at Γ generates a type I singularity, then $\lambda(\Gamma) \geq \lambda(\Gamma_m)$.*

A.2 *There exist closed immersed curves $\Gamma' \subset \mathbb{R}^2$ with turning number m , but $\lambda(\Gamma') < \lambda(\Gamma_m)$. Any such curve generates only type II singularities under the CSF.*

Straightforward calculations show that

$$(1.2) \quad \lambda(\Gamma_m) = m \cdot \lambda(\Gamma_1) = m\sqrt{2\pi/e}.$$

In other words, the entropy of the m -covered circle is m times the entropy of the embedded circle.

An important special case of Theorem A.1 is that $\Gamma \subset \mathbb{R}^2$ is a closed shrinker of turning number m . Closed shrinkers for the CSF are called *Abresch-Langer curves* [1]. For any pair of relatively prime integers $m, n \in \mathbb{N}$ satisfying

$$(1.3) \quad \frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2},$$

there exists a closed convex shrinker $\Gamma_{m,n}$ with turning number m and whose curvature has $2n$ critical points. Moreover, every closed shrinker is of this form (other than the round circles). Theorem A.1 implies that

$$(1.4) \quad \lambda(\Gamma_{m,n}) \geq \lambda(\Gamma_p)$$

for every Abresch-Langer curve $\Gamma_{m,n}$ and every p -covered circle Γ_p with $m \geq p$. The normalization constant $(4\pi)^{-n/2}$ in the definition of entropy (1.1) is chosen so that hyperplanes have entropy 1. Moreover,

$$(1.5) \quad \lambda(\Sigma) = \lambda(\Sigma \times \mathbb{R})$$

for all hypersurfaces $\Sigma \subset \mathbb{R}^{n+1}$. Using this fact, Theorem A.1 immediately implies entropy bounds on higher dimensional shrinkers $\Sigma \subset \mathbb{R}^{n+1}$ of the form $\Sigma = \Gamma \times \mathbb{R}^{n-1}$, where Γ is a closed planar shrinker.

The key idea for proving Theorem A.2 is the notion of *entropy instability*. A shrinker $\Sigma \subset \mathbb{R}^{n+1}$ is *entropy unstable* if there exists a variation $(\Sigma_\epsilon)_{\epsilon \in (-\delta, \delta)}$ of Σ which decreases entropy. Entropy stability of embedded shrinkers was first introduced by Colding and Minicozzi in [11]. Our method to construct closed curves with turning number m but entropy less than that of the m -covered circle Γ_m is to show that Γ_m is entropy unstable for all $m \geq 2$. We not only show that multiply-covered circles are entropy unstable, but also that *all* closed shrinkers $\Gamma \subset \mathbb{R}^2$ other than the embedded circle are entropy unstable. Furthermore, we calculate their *F-index*, which measures the number of linearly independent variations which reduce the entropy of the shrinker (see Liu [21]).

Based on their work on the stability of self-shrinkers, Colding and Minicozzi defined a *piecewise MCF* starting at a closed hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ as a finite collection of MCFs Σ_t^i on time intervals $[t_i, t_{i+1}]$ so that each $\Sigma_{t_{i+1}}^{i+1}$ is a graph over $\Sigma_{t_{i+1}}^i$ of a function u_{i+1} and

$$(1.6) \quad \lambda(\Sigma_{t_{i+1}}^{i+1}) \leq \lambda(\Sigma_{t_{i+1}}^i).$$

Piecewise MCF provides an ad hoc notion of generic MCF that is important in many applications: it provides a method of continuing a flow through unstable singularities. If a MCF reaches an entropy unstable singularity, the flow is slightly perturbed to decrease entropy so that the singularity can never re-occur along the flow. Colding and Minicozzi proposed a piecewise MCF for closed embedded surfaces in \mathbb{R}^3 which becomes extinct in a round point [11]. We realize a similar result by constructing a piecewise CSF for closed immersed curves in \mathbb{R}^2 .

Theorem (Theorem B). *Let $\Gamma \subset \mathbb{R}^2$ be a closed immersed curve. Then there exists a piecewise CSF starting at Γ and defined up to a time $T > 0$ at which the flow either becomes extinct in an embedded circle, or has type II singularities. Moreover, if Γ has turning number greater than 1, the latter case holds.*

It is worth noting that the break-points of the piecewise CSF constructed in Theorem B can be made arbitrarily small in the C^∞ norm. Furthermore, a result of Gage-Hamilton [13] and Grayson [14] states that the CSF starting at a closed embedded plane curve becomes convex and eventually extincts in a “round point”. See [2] for an elegant direct proof. It follows that the piecewise CSF of Theorem B starting at a closed embedded plane curve reduces to the usual CSF whose only singularity is modelled by an embedded circle.

A singularity of a MCF defined up to time T is of *type I* if the curvature blows up no faster than $(T - t)^{-1/2}$; otherwise, the singularity is of *type II*. Type I singularities are important in the singularity analysis of MCF because Huisken [16] showed that a blow up limit of a type I singularity is a smooth shrinker. More generally, the limit of a blowup sequence of a MCF at a fixed spacetime point is modeled on a possibly singular shrinker [16],[17]. The study of shrinkers is therefore central in the analysis of singularity formation of the MCF. Shrinkers can equivalently be defined as immersed hypersurfaces $\Sigma \subset \mathbb{R}^n$ satisfying the equation

$$(1.7) \quad H = \frac{1}{2} \langle x, \mathbf{n} \rangle.$$

Straightforward calculations show that the hyperplanes passing through the origin, the round spheres of radius $\sqrt{2n}$ centered at the origin, and the cylinders of the form $S^k \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$ are all shrinkers. Much work has concerned the construction and classification of shrinkers [3],[7],[19], and for the case $n = 1$, a complete classification is known [1].

In [23], Stone computed the entropy of embedded spheres (and thus also of shrinking cylinders), implying that

$$(1.8) \quad 2 > \lambda(S^1) > \lambda(S^2) > \dots \rightarrow \sqrt{2}.$$

In [6], Brakke proved that hyperplanes have the least entropy among all shrinkers, and that there is a gap between the entropy of a plane and the entropy of the next lowest shrinker. Colding-Ilmanen-Minicozzi-White [9] proved that the sphere is the shrinker with lowest entropy among all closed

embedded shrinkers. This result has been generalized by Bernstein-Wang [5], Zhu [25], and Ketover-Zhou [20]: Bernstein-Wang proved that the sphere has lowest entropy among all closed embedded hypersurfaces in \mathbb{R}^{n+1} where $2 \leq n \leq 6$; Ketover-Zhou proved the same result by using an alternative min-max method; Zhu removed the dimension restriction. Very recently, Hershkovits and White [15] proved a lower bound and rigidity theorem for shrinkers in higher codimension: a closed embedded shrinker $\Sigma \in \mathbb{R}^{n+1}$ with non-trivial k -th homology has entropy greater than or equal to the entropy of S^k , and equality holds if and only if $\Sigma = S^k \times \mathbb{R}^{n-k}$.

Our results can be viewed as an extension of these results to the case of closed *immersed* curves, providing the first lower entropy bounds for immersed shrinkers. The main difference between the embedded case and the immersed case is that immersed MCFs lack many of the nice properties of embedded MCFs. For example, the avoidance principle implies that embeddedness is preserved under MCF. Also, the level set flow weak solutions are easier to handle in the embedded case. Fortunately, the immersed CSF has simpler geometry, allowing us to control the behavior of the immersed curve under CSF.

We highlight that our theorem detects long time behaviour of CSF from the information of the initial data. Although the monotonicity formula for entropy can be used to rule out high entropy shrinkers as singularity models of a mean curvature flow, it does not tell us whether the singularities are type I or type II. Our theorem provides a criterion to show that the type II singularities must occur from the initial data.

The stability of Abresch-Langer curves has been studied in a different context by Epstein-Weinstein [12]. Our results show that the Jacobi operator of an Abresch-Langer curve has the same number of negative eigenvalues as the linear stability operator studied by Epstein-Weinstein. Their analysis does not incorporate the action of the group generated by rigid motions and dilations.

Outline of the proofs. We now outline the proof of Theorem A.1.

The key result we use to conclude the proof is due to Au [4]. Recall that Au proved the following theorem in [4].

Theorem 1.9 (Main Theorem of [4], part (a)). *Let $\Gamma_{m,n}$ be an Abresch-Langer curve with turning number m and $\gamma_0 = \Gamma + \epsilon \mathbf{n}$ a perturbation with $|\epsilon|$ small. Then if $\epsilon > 0$, the curve shortening flow γ_t starting from γ_0 is asymptotic to an m -covered circle.*

Given a closed curve Γ with turning number m , we run the CSF starting from this curve. If the flow hits a type I singularity, then after rescaling it must be an Abresch-Langer curve. The continuity of entropy shows that the entropy of Γ is bounded from below by the entropy of this Abresch-Langer curve. Using Au's result and the monotonicity property of entropy, we show that this Abresch-Langer curve has entropy bounded from below by the entropy of the m -covered circle. Thus we conclude the proof.

The main technical challenge is showing that the entropy functional is continuous under small perturbations near a closed shrinker Γ . Note that the entropy functional is not continuous in general: a sequence of rescalings at a point on a sphere converges to a hyperplane in the limit, however, a hyperplane has entropy 1, while a sphere has entropy at least $\sqrt{2}$. We prove that the entropy functional is continuous near a closed shrinker $\Gamma \subset \mathbb{R}^2$ in two steps.

Firstly, Proposition 2.1 is a general result giving a sufficient condition for the F -functional being continuous at Σ . More precisely, when (x_0, t_0) lies in a compact subset $K \subset \mathbb{R}^{n+1} \times (0, \infty)$ and variations of Σ lie in a bounded subset $\mathcal{B} \subset C^{2,\alpha}(\Sigma)$, the F -functional is continuous.

The second step is to provide a quantitative version of [11, Lemma 7.7] to show that we may in fact apply Proposition 2.1 to closed shrinkers $\Gamma \subset \mathbb{R}^2$. To that end, Theorem 2.3 shows that the entropy of the perturbed curve $\Gamma + f\mathbf{n}$ with $f \in \mathcal{B}$ is achieved in $B_R(0) \times [T_0, T_1]$ for some $R > 0$ and some $0 < T_0 < T_1 < \infty$. The main technical challenge is to show that $T_0 > 0$. We discuss the details in Section 2.

Next, we give an outline of the proof of Theorem A.2. Our method to construct closed curves with turning number m but entropy less than that of the m -covered circle Γ_m is to show that Γ_m is entropy unstable for all $m \geq 2$. Recall that the F -functional is defined to be

$$(1.10) \quad F_{x_0, t_0}(\Sigma) = (4\pi t_0)^{-n/2} \int_{\Sigma} e^{-\frac{|x-x_0|^2}{4t_0}} d\mu.$$

The entropy $\lambda(\Sigma)$ is then the supremum over all translations $x_0 \in \mathbb{R}^{n+1}$ and dilations $t_0 > 0$ of the F -functional. It follows from Huisken's monotonicity formula [16] that shrinkers are precisely the critical points of the $F_{0,1}$ -functional. The $F_{0,1}$ -functional is called the *Gaussian area*. Therefore, shrinkers are precisely the minimal hypersurfaces of \mathbb{R}^{n+1} equipped with the conformally changed metric $g_{ij} = e^{-\frac{|x|^2}{2n}} \delta_{ij}$. We show that the entropy index of a shrinker is bounded below by the difference between the Morse

index¹ of the shrinker (considered as a minimal surface of $(\mathbb{R}^{n+1}, e^{-\frac{|x|^2}{2n}} \delta_{ij})$) and the dimension of the space spanned by the mean curvature function and the component functions of the normal vector field of the shrinker. Consequently, we reduce the calculation of the entropy index of a shrinker to the calculation of the Morse index. For 1-dimensional shrinkers in \mathbb{R}^2 , the Jacobi operator reduces to a 1-dimensional Sturm-Liouville operator. Using the well-developed theory of spectra of Sturm-Liouville operators, we are able to determine exactly how many negative eigenvalues the Jacobi operator of a 1-dimensional closed shrinker has, and thus also the entropy index.

Lastly, we outline the proof of Theorem B. As mentioned above, the possible singularities of the CSF starting at Γ are classified into two categories: type I and type II. Huisken [16, Theorem 3.5] shows that any rescaling of a type I singularity is a shrinker. By Abresch and Langer's classification of shrinkers for the CSF [1], any rescaling of a type I singularity must therefore be an Abresch-Langer curve, or a (multiply-covered) circle. Corollary 5.9 states that the only entropy stable closed singularity for CSF is the embedded circle.

Let Γ be any closed immersed plane curve, which is the initial data for a CSF. If the singularity of the CSF is of type I, any blowup gives an Abresch-Langer curve or a circle, say $\tilde{\Gamma}_\infty$. If $\tilde{\Gamma}_\infty$ is not an embedded circle, Lemma 6.1 shows that the time slice of the flow can be slightly perturbed to a curve Γ' with $\lambda(\Gamma') < \lambda(\tilde{\Gamma}_\infty)$. Consequently, $\tilde{\Gamma}_\infty$ can never appear as a singularity for the flow starting from Γ' . The above process is then repeated with Γ' instead of Γ . Since the perturbations can be made with arbitrarily small C^∞ -norm, the perturbations preserve turning number. Therefore piecewise CSF preserves turning number, and since there are only finitely many closed shrinkers of a given turning number, the piecewise flow terminates after finitely many perturbations.

Organization of the paper. In Section 2 we prove some basic properties of the entropy functional for immersed hypersurfaces and shrinkers. In particular we prove the continuity Theorem 2.3 for entropy, which is central to our argument.

In Section 3, we prove Theorem A.1.

In Section 4, we introduce and study some basic properties of index of the shrinkers. The results of this section hold for shrinkers of any dimension.

¹Recall that the Morse index of a minimal surface $\Sigma \subset (M, g)$ is defined as the number of negative eigenvalues of the Jacobi operator determined by (M, g) and Σ (see [10, p. 41]).

In Section 5, we compute the precise entropy index of closed immersed shrinkers for the CSF. As a corollary, we obtain Theorem A.2.

In Section 6 we combine our results to prove Theorem B.

Notation. Throughout, all hypersurfaces $\Sigma \subset \mathbb{R}^{n+1}$ are assumed to be immersed and orientable, so that there exists a globally defined unit normal vector field $\mathbf{n} : \Sigma \rightarrow S^n$. We denote by $x : \Sigma \rightarrow \mathbb{R}^{n+1}$ the given immersion and by $\Sigma + f\mathbf{n}$ the normal variation of Σ by a function $f : \Sigma \rightarrow \mathbb{R}$. The mean curvature of Σ is denoted by $H : \Sigma \rightarrow \mathbb{R}$. One can show (see [22], for example) that MCF is a geometric flow which is invariant under tangential reparameterization, so we may also use the image $(\Sigma_t)_{t \in [0, T]}$ to denote the flow. In the case of curves, we use the notation $(\Gamma_t)_{t \in [0, T]}$. Finally, angle brackets $\langle \cdot, \cdot \rangle$ denote the standard Euclidean inner product.

Acknowledgments. The authors thank Professor W. Minicozzi for suggesting this topic and for his support, as well as Professor D. Maulik, Professor A. Moitra and Dr. S. Gerovitch for supporting this project via the Summer Program in Undergraduate Research. The first author is indebted to C. Mantoulidis for introducing him to the mean curvature flow and for many interesting discussions.

2. Properties of entropy

In this section, we will prove that the entropy functional is continuous near a closed planar shrinker under certain conditions. In general, the entropy functional is not a continuous functional on the space of immersed hypersurfaces (with the C^∞ topology, say). The standard example is given by blowing up a sphere at a point, giving a hyperplane in the limit. Each element of the blowup sequence is a sphere, thus having entropy $\sqrt{2}$. However, a hyperplane has entropy 1. The results of this section will be used in the Section 3 to prove Theorem A.1.

The following proposition shows that, when a closed hypersurface is slightly perturbed, the F -functional cannot change too much under the perturbation, as long as (x_0, t_0) are confined to a compact subset of $\mathbb{R}^{n+1} \times (0, \infty)$. From now on, in this paper, we fix $\alpha \in (0, 1)$ once and for all.

Proposition 2.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed hypersurface and $K \subset \mathbb{R}^{n+1} \times (0, \infty)$ be a compact subset. Then there exist $\epsilon_0 > 0$ and a constant C , both depending only on Σ and K , such that, for all $f \in C^{2, \alpha}(\Sigma)$ with $\|f\|_{C^{2, \alpha}} \leq 1$,*

all $\epsilon \geq 0$ with $\epsilon < \epsilon_0$, and all $(x_0, t_0) \in K$, the following inequality holds:

$$(2.2) \quad |F_{x_0, t_0}(\Sigma_\epsilon) - F_{x_0, t_0}(\Sigma)| \leq C\epsilon,$$

where $\Sigma_\epsilon = \Sigma + \epsilon f \mathbf{n}$.

Proof. Let $x : \Sigma \rightarrow \mathbb{R}^{n+1}$ be the given immersion and let $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \leq 1$. In local coordinates around a point $p \in \Sigma$, we can write

$$\begin{aligned} \tilde{g}_{ij} &= \left\langle \partial_i(x + \epsilon f \mathbf{n}), \partial_j(x + \epsilon f \mathbf{n}) \right\rangle \\ &= g_{ij} + 2\epsilon f h_{ij} + \epsilon^2 f^2 h_{ik} h^k_j + \epsilon^2 (\partial_i f)(\partial_j f), \end{aligned}$$

where g_{ij} is the metric induced on Σ as a hypersurface in \mathbb{R}^{n+1} , and \tilde{g}_{ij} is the metric induced on Σ_ϵ . Since Σ is closed, there exists $\epsilon_0 > 0$, such that, for any $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \leq 1$ and any $\epsilon \in \mathbb{R}$ satisfying $|\epsilon| < \epsilon_0$, the tensor \tilde{g} defines a metric on Σ_ϵ and the volume measure $\sqrt{\tilde{g}}$ and the derivative $\partial \tilde{g} / \partial \epsilon$ are uniformly bounded on compact sets. Define

$$G_{x_0, t_0}(\beta, x) = e^{-\frac{|x - x_0 + \beta f \mathbf{n}|^2}{4t_0}} \sqrt{\tilde{g}}.$$

Then Jacobi's formula gives

$$\frac{\partial G_{x_0, t_0}(\beta, x)}{\partial \beta} = \frac{1}{2} \left(\text{Tr} \left(\tilde{g}^{-1} \frac{\partial \tilde{g}}{\partial \beta} \right) - \frac{f \langle x - x_0, \mathbf{n} \rangle + \beta f^2}{t_0} \right) e^{-\frac{|x - x_0 + \beta f \mathbf{n}|^2}{4t_0}} \sqrt{\tilde{g}},$$

which is uniformly bounded on K by a constant depending only on ϵ_0 and K . In particular, the constant is independent of the choice of $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \leq 1$. Therefore

$$\begin{aligned} |F_{x_0, t_0}(\Sigma_\epsilon) - F_{x_0, t_0}(\Sigma)| &= \left| \frac{1}{\sqrt{4\pi t_0}} \int_\Sigma (G_{x_0, t_0}(\epsilon, x) - G_{x_0, t_0}(0, x)) dx \right| \\ &\leq \frac{1}{\sqrt{4\pi t_0}} \int_\Sigma |G_{x_0, t_0}(\epsilon, x) - G_{x_0, t_0}(0, x)| dx \\ &= \frac{\epsilon}{\sqrt{4\pi t_0}} \int_\Sigma \left| \int_0^1 \frac{\partial G_{x_0, t_0}(\epsilon u, x)}{\partial \beta} du \right| dx \\ &\leq \frac{\epsilon}{\sqrt{4\pi t_0}} \int_\Sigma \int_0^1 \left| \frac{\partial G_{x_0, t_0}(\epsilon u, x)}{\partial \beta} \right| du dx \\ &\leq \epsilon C \end{aligned}$$

as desired. □

In the rest of this section, we will focus on the case of curves in the plane. The main result of this section, below, shows that there exists a compact subset of $\mathbb{R}^2 \times (0, \infty)$ such that the entropy of any curve obtained by perturbing a given closed plane shrinker is attained in this compact set. As a consequence, we may apply the previous proposition to conclude that the entropy functional is continuous under small perturbations of a closed plane shrinker. Theorem 2.3 may be viewed as a quantitative version of [11, Lemma 7.7] for immersed curves.

Theorem 2.3. *Let $\Gamma \subset \mathbb{R}^2$ be a closed shrinker. Then there exist $\epsilon_0 > 0$ and a compact subset $K \subset \mathbb{R}^2 \times (0, \infty)$, both depending only on Γ , such that, for any variation $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \leq 1$ and any $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_0$, the following holds:*

$$\lambda(\Gamma_\epsilon) = \sup_{(x_0, t_0) \in K} F_{x_0, t_0}(\Gamma_\epsilon),$$

where $\Gamma_\epsilon = \Gamma + \epsilon f \mathbf{n}$.

Before proving Theorem 2.3, let us recall an application of Huisken's monotonicity to self-shrinkers. Suppose Σ is a self-shrinker in \mathbb{R}^{n+1} (not necessarily embedded), then for any $y \in \mathbb{R}^{n+1}$ and $a \in \mathbb{R}$, the function

$$g(s) = F_{sy, 1+as^2}(\Sigma)$$

satisfies $g'(s) \leq 0$ for all $s > 0$ with $1 + as^2 > 0$. See [11, Eq. 7.13]. Moreover, $g'(s) = 0$ if and only if $asx^\perp + y^\perp = 0$ for any $x \in \Sigma$, see [11, Eq. 7.27]. As a corollary of this monotonicity formula, we have the following Lemma, see section 7.2 of [11]. Again, Σ is not necessarily embedded, because the monotonicity formula also holds for an immersed mean curvature flow.

Lemma 2.4. *Then entropy of a self-shrinker Σ is only achieved at $F_{0,1}(\Sigma)$ if Σ does not split along a line.*

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Throughout, we will denote by Γ_ϵ^f the perturbed curve $\Gamma_\epsilon^f = \Gamma + \epsilon f \mathbf{n}$ if $f \in C^{2,\alpha}(\Gamma)$. As stated in the proof of the previous proposition, there exists an $\epsilon_0 > 0$ such that Γ_ϵ^f is a closed immersed curve for any $f \in C^{2,\alpha}(\Gamma)$ with $\|f\|_{C^{2,\alpha}} \leq 1$ and any $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_0$. Let \mathcal{B} be the set consisting of those Γ_ϵ^f with $f \in C^{2,\alpha}(\Sigma)$ such that $\|f\|_{C^{2,\alpha}} \leq 1$ and $\epsilon \in \mathbb{R}$ such that $|\epsilon| < \epsilon_0$.

Step (1): Bound on $|x_0|$. By the compactness of Γ and the uniform $C^{2,\alpha}$ bound of the variational functions, there exists $R > 0$ (independent of f and ϵ) such that Γ_ϵ^f is contained in the closed ball $B_R(0) \subset \mathbb{R}^2$ of radius R centered at the origin, which is compact. Colding and Minicozzi [11] showed that for any closed hypersurface $M \subset \mathbb{R}^{n+1}$ and each fixed $t_0 > 0$, $\sup_{x_0 \in \mathbb{R}^{n+1}} F_{x_0,t_0}(M)$ is achieved inside the convex hull of M . The convex hull of M is compact since M is closed. Therefore, for each fixed t_0 and all $\Gamma_\epsilon^f \in \mathcal{B}$, $\sup_{x_0 \in \mathbb{R}^2} F_{x_0,t_0}(\Gamma_\epsilon^f)$ is attained in $B_R(0) \subset \mathbb{R}^2$.

Step (2): Upper bound on t_0 . Let L be the length of Γ . When ϵ_0 is sufficiently small, the length of $\Gamma_\epsilon^f \in \mathcal{B}$ is bounded by $2L$. Let $T_1 = (4\pi)^{-1}(3L)^2$. Then for any $t_0 > T_1$, any $x_0 \in B_R(0)$, and any curve $\Gamma_\epsilon^f \in \mathcal{B}$, we have

$$F_{x_0,t_0}(\Gamma_\epsilon^f) \leq (4\pi t_0)^{-1/2} \text{Length}(\Gamma_\epsilon^f) < (4\pi T_1)^{-1/2} 2L = 2/3 < 1.$$

However, we know that $\lambda(\Gamma_\epsilon^f) \geq 1$. This can be seen by zooming in at any point on the curve. We conclude that

$$\sup_{(x_0,t_0) \in B_R(0) \times (0,\infty)} F_{x_0,t_0}(\Gamma_\epsilon^f) = \sup_{(x_0,t_0) \in B_R(0) \times (0,T_1]} F_{x_0,t_0}(\Gamma_\epsilon^f),$$

for all $\Gamma_\epsilon^f \in \mathcal{B}$.

Step (3): Lower bound on t_0 . We observe that for ϵ_0 small enough, Γ_ϵ^f has a uniform curvature bound. This follows from the expression of curvature in terms of the first and second derivatives of the immersion. Therefore, we may assume that $|k| \leq k_0$ for all $\epsilon \leq \epsilon_0$. Also, if ϵ_0 is small, then the multiplicity of each self-intersection points of Γ_ϵ^f are the same as the multiplicity of the self-intersection points of Γ . Let $R_0 > 0$ be a constant whose value will be determined. Now for any fixed x_0 and any $t_0 \leq R_0^{-2}$, we have

$$\begin{aligned} F_{x_0,t_0}(\Gamma_\epsilon^f) &= (4\pi t_0)^{-1/2} \int_{\{|x-x_0| \geq R_0^{-1}\} \cap \Gamma_\epsilon^f} e^{-\frac{|x-x_0|^2}{4t_0}} ds \\ &\quad + (4\pi t_0)^{-1/2} \int_{\{|x-x_0| \leq R_0^{-1}\} \cap \Gamma_\epsilon^f} e^{-\frac{|x-x_0|^2}{4t_0}} ds. \end{aligned}$$

The first term on the right hand side is bounded by

$$(4\pi t_0)^{-1/2} e^{-\frac{R_0^{-2}}{4t_0}} \text{Length}(\Gamma_\epsilon^f) \leq C R_0^{-1/2}$$

for a universal constant C .

Now we estimate the second term on the right hand side. By change of variables, this term equals

$$\int_{\{|x-x_0| \leq R_0^{-1}t_0^{-1/2}\} \cap (t_0^{-1/2}\Gamma_\epsilon^f)} e^{-\frac{|x-x_0|^2}{4}} ds,$$

where $t_0^{-1/2}\Gamma_\epsilon^f$ is the curve Γ_ϵ^f rescaled by $t_0^{-1/2}$ with center at x_0 . Suppose $\{|x - x_0| \leq R_0^{-1}t_0^{-1/2}\} \cap (t_0^{-1/2}\Gamma_\epsilon^f)$ consists of the connected components $\gamma_1, \dots, \gamma_m$. Then each γ_i is a curve with curvature uniformly bounded by k_0t_0 . In particular, for a given $\delta > 0$, when R_0 is large enough depending on k_0 , the curvature of γ_i is small enough and therefore

$$\int_{\gamma_i} e^{-\frac{|x-x_0|^2}{4}} ds \leq 1 + \delta.$$

This can be seen by a blow-up argument. In fact as $R_0 \rightarrow \infty, t_0 \rightarrow 0$, the uniform curvature bound implies that all γ_i would converge to a straight line, and the Gaussian integral would converge to 1.

The number m is bounded by the multiplicity of the self-intersection points of γ_i when ϵ_0 is sufficiently small and R_0 sufficiently large. So we conclude that

$$F_{x_0,t_0}(\Gamma_\epsilon^f) \leq CR_0^{-1/2} + m(1 + \delta) = m + \delta m + CR_0^{-1/2}.$$

Lemma 2.4 implies that the entropy of Γ is achieved at $F_{0,1}(\Gamma)$, rather than blowing up a self-intersection point. That is, $\lambda(\Gamma) > m + \beta$ for some $\beta > 0$. Therefore when ϵ_0 is small, Proposition 2.1 gives that $F_{0,1}(\Gamma_\epsilon^f) > m + \beta/2$. If we pick δ small enough and R_0 large enough, for $t_0 \leq R_0^{-3} =: T_0$ we have

$$F_{x_0,t_0}(\Gamma_\epsilon^f) \leq m + \beta/4 < m + \beta/2 < \lambda(\Gamma).$$

Combining all of the steps above, the proposition follows, with $K := B_R(0) \times [T_0, T_1]$. □

Remark 2.5. We believe that the self-intersection points of an Abresch-Langer curve actually have multiplicity 2, i.e. an Abresch-Langer curve passes a point in the plane at most twice. It should follow from the fact that the Abresch-Langer curves are all convex.

Corollary 2.6. *Let $\Gamma \subset \mathbb{R}^2$ be a closed shrinker and let $(\Gamma_i)_{i \in \mathbb{N}}$ be a sequence of plane curves which C^∞ converges to Γ . Then*

$$\lim_{i \rightarrow \infty} \lambda(\Gamma_i) = \lambda(\Gamma).$$

Proof. It follows from the convergence assumption that there exist an $N' \in \mathbb{N}$, such that, for $i > N'$, Γ_i can be written as a graph

$$\Gamma_i = \Gamma + \epsilon_i f_i \mathbf{n},$$

for some $f_i \in C^\infty(\Gamma)$ normalized to $\|f_i\|_{C^\infty} = 1$ and some $\epsilon_i > 0$ with $\epsilon_i \rightarrow 0$ as $i \rightarrow \infty$. Applying the previous theorem implies that there exists an $N'' \geq N'$ and a compact subset $K \subset \mathbb{R}^2 \times (0, \infty)$ such that, for all $i > N''$, the entropy $\lambda(\Gamma_i)$ is achieved at some $(x_i, t_i) \in K$. Then by Proposition 2.1, there exist an $N \geq N''$ and a constant C depending only on K such that, for any $(x_0, t_0) \in K$ and any $i > N$, we have

$$F_{x_0, t_0}(\Gamma_i) - C\epsilon_i \leq F_{x_0, t_0}(\Gamma) \leq F_{x_0, t_0}(\Gamma_i) + C\epsilon_i.$$

Consequently,

$$\sup_K F_{x_0, t_0}(\Gamma_i) - C\epsilon_i \leq \sup_K F_{x_0, t_0}(\Gamma) \leq \sup_K F_{x_0, t_0}(\Gamma_i) + C\epsilon_i.$$

Since the entropy $\lambda(\Gamma_i)$ is achieved in K for all $i > N$, it follows immediately that

$$\lambda(\Gamma_i) - C\epsilon_i \leq \lambda(\Gamma) \leq \lambda(\Gamma_i) + C\epsilon_i.$$

Now taking the limit $i \rightarrow \infty$ gives the desired result. □

The following corollary will be central to our construction of a generic CSF. The previous corollary shows that the entropy is continuous *at* a closed plane shrinker in the space of immersed curves. However, for the construction of the piecewise CSF in Theorem B, we also need the entropy to be continuous *near* a closed plane shrinker in the space of immersed curves. The following corollary ensures that the latter statement is satisfied.

Corollary 2.7. *Let $\Gamma \subset \mathbb{R}^2$ be a closed shrinker, $f \in C^{2,\alpha}(\Gamma)$ be a variation, and $(g_i)_{i \in \mathbb{N}} \subset C^\infty(\Gamma)$ be a sequence of functions which C^∞ -converges to 0. Then there exists an $\epsilon_0 > 0$ such that, for all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_0$, we have*

$$\lim_{i \rightarrow \infty} \lambda(\Gamma_\epsilon + g_i \mathbf{n}) = \lambda(\Gamma_\epsilon),$$

where $\Gamma_\epsilon = \Gamma + \epsilon f \mathbf{n}$.

Proof. By the C^∞ convergence assumption, there exists $N > 0$ such that $\|g_i\|_{C^{2,\alpha}} \leq 1/2$ for all $i > N$. Since Γ is compact and $f \in C^{2,\alpha}(\Gamma)$, there exists

$\epsilon_1 > 0$ such that $\|\epsilon f\|_{C^{2,\alpha}} \leq 1/2$ for all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_1$. Consequently, $\|\epsilon f + g_i\|_{C^{2,\alpha}} \leq 1$ for all $i > N$ and all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_1$. By Theorem 2.3, there exists an $\epsilon_2 > 0$ with $\epsilon_2 \leq \epsilon_1$, an $N' \geq N$, and a compact subset $K \subset \mathbb{R}^2 \times (0, \infty)$ such that, for all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_2$ and all $i > N'$, the entropy of the curves Γ_ϵ and $\Gamma_\epsilon + g_i \mathbf{n}$ is attained in K . By Proposition 2.1, for all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_0$ and all $(x_0, t_0) \in K$, we have

$$\lim_{i \rightarrow \infty} F_{x_0, t_0}(\Gamma_\epsilon + g_i \mathbf{n}) = F_{x_0, t_0}(\Gamma_\epsilon).$$

Since the curves Γ_ϵ and $\Gamma_\epsilon + g_i \mathbf{n}$ attain their entropies in K for all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_0$ and all $i > N'$, arguing as in the proof of the previous corollary concludes the proof. □

3. Entropy and turning number

In this section we will prove Theorem A.1, which gives lower bounds for the entropy of closed plane curves which generate type I singularities under the CSF. We will first prove a particular case, in which we only consider the closed plane shrinkers themselves.

Theorem 3.1. *Let $\Gamma \subset \mathbb{R}^2$ be a closed shrinker with turning number m . Then $\lambda(\Gamma) \geq \lambda(\Gamma_m)$.*

The proof is based on a result by Au [4] together with the idea from [9]. Let us first recall the Theorem proved by Au.

Theorem 3.2 (Main Theorem of [4], part (a)). *Let $\Gamma_{m,n}$ be an Abresch-Langer curve with turning number m and $\gamma_0 = \Gamma + \epsilon \mathbf{n}$ be a perturbation with $|\epsilon|$ small. Then if $\epsilon > 0$, the curve shortening flow γ_t starting from γ_0 is asymptotic to an m -covered circle.*

Now we give the proof of Theorem 3.1.

Proof of Theorem 3.1. We assume $\Gamma \neq \Gamma_m$, since otherwise the result holds trivially. Let $x : S^1 \rightarrow \mathbb{R}^2$ be the given immersion of Γ . There are two cases to consider.

Case (1): Suppose that the degree of the map $x : S^1 \rightarrow x(S^1)$ is 1 (in other words, Γ is not multiply-covered). Define the constant normal variation

function $f = 1$ along Γ . By Corollary 2.6, if $\Gamma_\epsilon = \Gamma + \epsilon \mathbf{n}$, then

$$\lim_{\epsilon \rightarrow 0} \lambda(\Gamma_\epsilon) = \lambda(\Gamma).$$

In particular, there exists a constant $C > 0$ such that

$$\lambda(\Gamma) - C\epsilon \leq \lambda(\Gamma_\epsilon) \leq \lambda(\Gamma) + C\epsilon,$$

for all $\epsilon > 0$ small enough. Au [4] showed that the rescaled CSF starting at Γ_ϵ converges to the m -covered circle. By the monotonicity property of entropy, we therefore have

$$\lambda(\Gamma_m) \leq \lambda(\Gamma_\epsilon) \leq \lambda(\Gamma) + C\epsilon.$$

Taking the limit $\epsilon \rightarrow 0$ shows that $\lambda(\Gamma) \geq \lambda(\Gamma_m)$.

Case (2): Suppose that the degree of the map $x : S^1 \rightarrow x(S^1)$ is greater than 1 (in other words, Γ is a multiply-covered Abresch-Langer curve). Then Γ is a k -covered Abresch-Langer curve (with $k \geq 2$) Γ' , where Γ' has turning number p satisfying $kp = m$. By the previous case and (1.2),

$$\lambda(\Gamma) = k \cdot \lambda(\Gamma') \geq k \cdot \lambda(\Gamma_p) = kp \cdot \lambda(\Gamma_1) = m \cdot \lambda(\Gamma_1) = \lambda(\Gamma_m).$$

□

Lemma 3.3. *The turning number of a closed immersed curve $\Gamma \subset \mathbb{R}^2$ is preserved under the CSF and rescaled CSF.*

Proof. CSF and rescaled CSF preserve immersedness and are therefore regular homotopies. Regular homotopy classes of immersions $S^1 \rightarrow \mathbb{R}^2$ are classified by their turning number, by the Whitney-Graustein theorem [24, Theorem 1]. In other words, two closed immersed planar curves are regularly homotopic if and only if they have the same turning number. □

Proof of Theorem A.1. We assume that Γ is not a shrinker, since otherwise the theorem follows immediately from Theorem 3.1. Let $(\Gamma_t)_{t \in [0, T]}$ be the CSF starting at Γ . Since all type I singularities are closed shrinkers, they are compact, so there exists a unique singular point $x_0 \in \mathbb{R}^2$. Without loss of generality, we may assume x_0 is the origin. Let $(\tilde{\Gamma}_\tau)_{\tau \in [\log T^{-1/2}, \infty)}$ be a rescaled CSF [16, Section 2] around $0 \in \mathbb{R}^2$ starting at Γ . By [16, Theorem 3.5], for any sequence $\tau_i \rightarrow \infty$, there exists a subsequence also denoted τ_i such that the rescaled curves $\tilde{\Gamma}_{\tau_i}$ will smoothly converge to a shrinker $\tilde{\Gamma}_\infty$

as $\tau_i \rightarrow \infty$. By the classification of 1-dimensional shrinkers [1], $\tilde{\Gamma}_\infty$ must be a multiply-covered circle or a multiply-covered Abresch-Langer curve.

Let m be the turning number of $\tilde{\Gamma}_\infty$. By the lemma, the turning number of $\tilde{\Gamma}_\tau$ is m for all $\tau \in [\log T^{-1/2}, \infty)$. By the C^∞ -convergence $\tilde{\Gamma}_{\tau_i} \rightarrow \tilde{\Gamma}_\infty$, the curvature of $\tilde{\Gamma}_{\tau_i}$ smoothly converges to the curvature of $\tilde{\Gamma}_\infty$. Therefore the total curvatures satisfy

$$n = \lim_{i \rightarrow \infty} \frac{1}{2\pi} \int_{\tilde{\Gamma}_{\tau_i}} \tilde{k}_i = m.$$

This shows that $\tilde{\Gamma}_\infty$ is a shrinker of turning number m . By the monotonicity of entropy,

$$\lambda(\tilde{\Gamma}_\tau) \leq \lambda(\Gamma)$$

for all $\tau \in [\log T^{-1/2}, \infty)$. By the C^∞ -convergence $\tilde{\Gamma}_{\tau_i} \rightarrow \tilde{\Gamma}_\infty$, Corollary 2.6 implies that

$$\lim_{i \rightarrow \infty} \lambda(\tilde{\Gamma}_{\tau_i}) = \lambda(\tilde{\Gamma}_\infty),$$

and since this holds for any blowup sequence, it follows thus that

$$\lambda(\Gamma) \geq \lambda(\tilde{\Gamma}_\infty).$$

Since we assume that Γ is not a shrinker and since entropy is a constant along the CSF only for shrinkers, the inequality is in fact strict. The theorem now follows from Theorem 3.1. □

4. Entropy index, F -index and Morse index

Recall that the shrinkers $\Sigma \subset \mathbb{R}^{n+1}$ are precisely the minimal surfaces of \mathbb{R}^{n+1} equipped with the conformally changed metric $g_{ij} = e^{-\frac{|x|^2}{2n}} \delta_{ij}$, and are critical points of the Gaussian area functional $F = F_{0,1}$. In this section, we will determine the relationship between three types of stability of a shrinker: entropy stability, F -stability, and Morse stability (when a shrinker is considered as a critical point of $F = F_{0,1}$). Theorem 4.12, shows that a closed shrinker $\Sigma \subset \mathbb{R}^{n+1}$ is entropy unstable if the Morse index of Σ is greater than $n + 2$. Consequently, to prove Theorem A.2 we will only need to show that the m -covered circle has Morse index greater than 3. The results of this section apply to closed shrinkers $\Sigma^n \subset \mathbb{R}^{n+1}$ of any dimension.

Entropy is hard to compute. Therefore it is not easy to characterize the stability of entropy by variational methods. In order to overcome this issue, Colding-Minicozzi [11] introduced the notion of F -stability, and studied the

connections between F -stability and entropy stability. Later Liu [21] studied further properties of these connections. Let us recall some definitions in [11] and [21]. For the purpose of this paper, we will only discuss the case of closed shrinkers.

Definition 4.1 (Stability, see [11]). Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a closed shrinker and Σ_ϵ a variation of Σ in the direction $f \in C^{2,\alpha}(\Sigma)$.

- (i): The variation Σ_ϵ is *entropy unstable* if $\lambda(\Sigma_\epsilon) < \lambda(\Sigma)$ for all sufficiently small $\epsilon \neq 0$; otherwise, the variation is *entropy stable*.
- (ii): The variation Σ_ϵ is *F-stable* if there exist some variations x_ϵ of 0 and t_ϵ of 1 such that

$$\frac{\partial^2}{\partial \epsilon^2} \Big|_{\epsilon=0} F_{x_\epsilon, t_\epsilon}(\Sigma_\epsilon) \geq 0;$$

otherwise, the variation is *F-unstable*.

- (iii): The variation Σ_ϵ is *Morse stable* if

$$\frac{\partial^2}{\partial \epsilon^2} \Big|_{\epsilon=0} F(\Sigma_\epsilon) \geq 0;$$

otherwise, the variation is *Morse unstable*.

Definition 4.2 (Index, see [21]). Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a closed shrinker.

- (i): The *entropy index* of Σ , denoted by $\text{ind}_\lambda(\Sigma)$, is the maximum dimension of all subspaces $V \subset C^{2,\alpha}(\Sigma)$ such that, for any nonzero $g \in V$, the variation Σ_ϵ in the direction of g is entropy unstable. The shrinker Σ is *entropy stable* if its entropy index is zero; otherwise, Σ is *entropy unstable*.
- (ii): The *F-index* of Σ , denoted by $\text{ind}_F(\Sigma)$, is the maximum dimension of all subspaces $V \subset C^{2,\alpha}(\Sigma)$ such that, for any nonzero $g \in V$, the variation Σ_ϵ in the direction of g is F -unstable. The shrinker Σ is *F-stable* if its F -index is zero; otherwise, Σ is *F-unstable*.
- (iii): The *Morse index* of Σ , denoted by $\text{ind}_M(\Sigma)$, is the maximum dimension of all subspaces $V \subset C^{2,\alpha}(\Sigma)$ such that, for any nonzero $g \in V$, the variation Σ_ϵ in the direction of g is Morse unstable. The shrinker Σ is *Morse stable* if its Morse index is zero; otherwise, Σ is *Morse unstable*.

Remark 4.3. In the definition of the indexes, the variations are assumed to be $C^{2,\alpha}$. However, from standard elliptic theory (also see the discussion after

(4.6)), one can define the indexes as the dimension of unstable variations with other regularities, such as $W^{1,2}$, and the indexes would be the same.

In [11, Theorem 0.15], Colding-Minicozzi showed that an F -unstable self-shrinker must be entropy unstable, which provides the link between entropy and the F -functional. Their theorem was stated for embedded shrinkers, but their proof also works for closed immersed shrinkers.

Theorem 4.4 ([11]). *Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a closed shrinker. Then the entropy index of Σ is bounded below by the F -index of Σ , i.e.*

$$(4.5) \quad \text{ind}_\lambda(\Sigma) \geq \text{ind}_F(\Sigma).$$

This theorem is just a re-formulation of Colding-Minicozzi's [11, Theorem 0.15]. For a proof, we refer the readers to the proof of [11, Theorem 0.15] (see [11, page 789]).

In [11], Colding-Minicozzi introduced the elliptic operator L , defined by

$$(4.6) \quad L = \Delta + |A|^2 - \frac{1}{2}\langle x, \nabla(\cdot) \rangle + \frac{1}{2},$$

see [11, (4.13)]. This operator is the linearized operator of a self-shrinker, and it is also the stability operator for the Gaussian area functional. Therefore, the Morse index of a self-shrinker can be also defined to be the number (count multiplicity) of negative eigenvalues of the operator L .

We point out that there are two eigenfunctions to the operator L on a self-shrinker. It was proved by Colding-Minicozzi in [11, Lemma 5.5] that

$$(4.7) \quad LH = H, \quad L\langle \mathbf{n}, v \rangle = \frac{1}{2}\langle \mathbf{n}, v \rangle,$$

where v is a constant vector field in \mathbb{R}^{n+1} .

We will use two lemmas in the proof of Theorem 4.12:

Lemma 4.8. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed shrinker and let $f \in C^{2,\alpha}(\Sigma)$ be contained in the subspace of variations spanned by the mean curvature H and the components $\mathbf{n}_1, \dots, \mathbf{n}_{n+1}$ of the normal vector field $\mathbf{n} : \Sigma \rightarrow \mathbb{R}$. Then f is an F -stable variation.*

Proof. We refer the reader to [11, Section 4] for properties and notations related to the second variation of the F -functional. Let $f = aH + \langle Y, \mathbf{n} \rangle$ for

some $a \in \mathbb{R}$ and some $Y \in \mathbb{R}^{n+1}$. Choose any $y \in \mathbb{R}^{n+1}$ and any $h \in \mathbb{R}$ and set $x_\epsilon = \epsilon y$, $t_\epsilon = 1 + \epsilon h$ and $\Sigma_\epsilon = \Sigma + \epsilon f \mathbf{n}$. Then

$$\begin{aligned} & \left. \frac{\partial^2}{\partial \epsilon^2} \right|_{\epsilon=0} F_{x_\epsilon, t_\epsilon}(\Sigma_\epsilon) \\ &= \int_\Sigma \left(-fLf + 2hfH - h^2H^2 + f\langle y, \mathbf{n} \rangle - \frac{1}{2}\langle y, \mathbf{n} \rangle^2 \right) e^{-\frac{|x|^2}{4}} d\mu \\ &= \int_\Sigma \left(-a^2H^2 - \frac{1}{2}\langle Y, \mathbf{n} \rangle^2 + 2ahH^2 \right. \\ &\quad \left. - h^2H^2 + \langle Y \mathbf{n}, \rangle \langle y, \mathbf{n} \rangle - \frac{1}{2}\langle y, \mathbf{n} \rangle^2 \right) e^{-\frac{|x|^2}{4}} d\mu \\ &= - \int_\Sigma \left((a-h)^2H^2 + \frac{1}{2}(\langle Y, \mathbf{n} \rangle - \langle y, \mathbf{n} \rangle)^2 \right) e^{-\frac{|x|^2}{4}} d\mu. \end{aligned}$$

If we choose $h = a$ and $y = Y$, then $\left. \frac{\partial^2}{\partial \epsilon^2} \right|_{\epsilon=0} F_{x_\epsilon, t_\epsilon}(\Sigma_\epsilon) = 0$. Therefore f is an F -stable variation. □

In the following, we will suppose x_i 's are the standard coordinate functions in \mathbb{R}^{n+1} . We will use ∂_{x_i} to denote the vector field generated by the coordinate function x_i .

Lemma 4.9. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed, orientable hypersurface. Then the component functions $\mathbf{n}_i = \langle \mathbf{n}, \partial_{x_i} \rangle : \Sigma \rightarrow \mathbb{R}$ are linearly independent.*

Proof. We claim that if the Gauss map $G : \Sigma \rightarrow S^n$ is surjective, then the lemma holds. Suppose that the Gauss map is surjective, and suppose for contradiction that the \mathbf{n}_i are linearly dependent, i.e. there exists $y \in \mathbb{R}^{n+1}$ such that $\langle y, \mathbf{n}(p) \rangle = 0$ for all $p \in \Sigma$. Then since the Gauss map is surjective, y is orthogonal to every vector in \mathbb{R}^{n+1} , which is true if and only if y is the zero vector.

Now we show that the Gauss map of any closed orientable hypersurface $x : M^n \rightarrow \mathbb{R}^{n+1}$ is surjective. Pick any vector $v \in S^n$. We wish to show that there exists some $p \in M$ such that $\mathbf{n}(p) = v$. By the compactness of M , the smooth function $\langle v, x \rangle : M \rightarrow \mathbb{R}$ attains a maximum at some point $p \in M$. Let $\{e_i\}$ be a basis of T_pM . Then

$$0 = \nabla_{e_i} \langle v, x(p) \rangle = \langle v, e_i \rangle$$

for all $i = 1, \dots, n$. Consequently, v is orthogonal to M at p , so $v = \pm \mathbf{n}(p)$. Since we have chosen p to be the maximum, and since $\langle v, x \rangle$ is not constant (M is closed), we must in fact have $v = \mathbf{n}(p)$, as desired. □

Corollary 4.10. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed shrinker. Then*

$$(4.11) \quad \dim \operatorname{span}\{H, \mathbf{n}_1, \dots, \mathbf{n}_{n+1}\} = n + 2,$$

where $\mathbf{n}_i = \langle \mathbf{n}, \partial_{x_i} \rangle$.

Proof. Colding-Minicozzi have shown that, for all i , the functions \mathbf{n}_i and H are eigenfunctions of the stability operator L determined by Σ , see [11, Lemma 5.5] and (4.7). Moreover, \mathbf{n}_i and H have distinct eigenvalues. It follows that the mean curvature H is linearly independent of \mathbf{n}_i , for any i . The proposition now follows from Lemma 4.9. \square

We are now able to prove the main result of this section, which will be used in the next section to calculate the entropy index of Abresch-Langer curves and m -covered circles.

Theorem 4.12. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed shrinker. Then the F -index and Morse index are related by*

$$(4.13) \quad \operatorname{ind}_F(\Sigma) = \operatorname{ind}_M(\Sigma) - n - 2.$$

Consequently, if

$$(4.14) \quad \operatorname{ind}_M(\Sigma) > n + 2,$$

then Σ is entropy unstable.

Proof. By standard elliptic theory, the Morse index of an elliptic integrand is finite. In particular, $\operatorname{ind}_M(\Sigma)$ is finite. In what follows, $m = \operatorname{ind}_M(\Sigma)$.

First we show that $\operatorname{ind}_F(\Sigma) \leq m - n - 2$. Combining with Lemma 4.8 and Corollary 4.10 shows that it suffices to show that any F -unstable variation is a Morse unstable variation. Suppose f is an F -unstable variation of Σ . Then, by definition, for any variations x_ϵ of $0 \in \mathbb{R}^{n+1}$ and t_ϵ of 1, we have $(F_{x_\epsilon, t_\epsilon}(\Sigma_\epsilon))''|_{\epsilon=0} < 0$. In particular, this holds for the trivial variations $x_\epsilon = 0$ and $t_\epsilon = 1$:

$$(F_{0,1}(\Sigma_\epsilon))''|_{\epsilon=0} = - \int_{\Sigma} f L f e^{-\frac{|x|^2}{4}} d\mu < 0.$$

This shows that f is an unstable variation of Σ when Σ is considered as a critical point to the Gaussian area functional F . This shows that the F -index is bounded from above by $m - n - 2$.

Now we will show that the F -index is at least $m - n - 2$. By definition of the Morse index, there exist m linearly independent eigenfunctions $u_1, \dots, u_m \in C^{2,\alpha}(\Sigma)$ of L corresponding to the eigenvalues

$$\mu_1 < \mu_2 \leq \dots \leq \mu_m < 0.$$

Colding and Minicozzi [11, Corollary 5.15] have shown that there exists an orthonormal basis of eigenfunctions of L for the weighted L^2 space. Therefore, without loss of generality, we may assume that u_1, \dots, u_m are orthonormal in the weighted L^2 space. The number of functions in $\{u_1, \dots, u_m\}$ which are orthogonal to H and the components $\mathbf{n}_1, \dots, \mathbf{n}_{n+1}$ of \mathbf{n} is $m - n - 2$. Pick any such function u_i (if none exists, the result holds vacuously) and let Σ_ϵ be a variation of Σ by $u_i \mathbf{n}$. Choose any $y \in \mathbb{R}^{n+1}$ and any $h \in \mathbb{R}$ and set $x_\epsilon = \epsilon y$ and $t_\epsilon = 1 + \epsilon h$. Then

$$\begin{aligned} (F_{x_\epsilon, t_\epsilon}(\Sigma_\epsilon))''|_{\epsilon=0} &= \int_\Sigma \left(-u_i L u_i - h^2 H^2 - \frac{1}{2} \langle y, \mathbf{n} \rangle^2 \right) e^{-\frac{|x|^2}{4}} d\mu \\ &\leq \int_\Sigma (-u_i L u_i) e^{-\frac{|x|^2}{4}} d\mu \\ &= \mu_i \int_\Sigma u_i^2 e^{-\frac{|x|^2}{4}} d\mu \\ &< 0 \end{aligned}$$

by orthogonality and the assumption that $\mu_i < 0$. Since there are $m - n - 2$ possible linearly independent choices for such a u_i , 4.13 follows.

If $\text{ind}_M > n - 2$, then Colding-Minicozzi's Theorem 4.4 implies that Σ has positive entropy index, meaning that Σ is entropy unstable. \square

5. Entropy instability of CSF singularities

Having established in the previous section the relationship between the entropy index and Morse index of a shrinker, we will now calculate the Morse index of m -covered circles Γ_m and the Abresch-Langer curves $\Gamma_{m,n}$. For 1-dimensional closed shrinkers, the Jacobi operator is a Sturm-Liouville operator.

Proposition 5.1. *Let $\Gamma \subset \mathbb{R}^2$ be a shrinker. Then the Jacobi operator L (see (4.6)) of Γ is a Sturm-Liouville operator. In particular, for all $j \geq 1$, the eigenfunctions u_{2j-1} and u_{2j} of L have exactly $2j$ zeros.*

Proof. We refer the reader to [11, Section 5] for the properties of the Jacobi operator of a shrinker. Differentiate the relation $k = \frac{1}{2}\langle x, \mathbf{n} \rangle$ to obtain $2k_s/k = \langle x, \mathbf{t} \rangle$ and note that $\nabla(\cdot) = \partial_s(\cdot)\mathbf{t}$. Therefore

$$\begin{aligned} L &= \Delta - \frac{1}{2}\langle x, \nabla(\cdot) \rangle + k^2 + \frac{1}{2} \\ &= \partial_{ss} - \frac{1}{2}\langle x, \partial_s(\cdot)\mathbf{t} \rangle + k^2 + \frac{1}{2} \\ &= \partial_{ss} - \frac{k_s}{k}\partial_s + k^2 + \frac{1}{2} \\ &= k \left(\partial_s \left(\frac{1}{k}\partial_s \right) + \frac{1}{k}(k^2 + \frac{1}{2}) \right). \end{aligned}$$

Since the Abresch-Langer curves are convex (i.e. they have $k > 0$, see [1]), the latter part of the theorem follows from a general Sturm-Liouville theory [8, Theorem 8.3.1]. \square

For m -covered circles, the Jacobi operator reduces even further, since the curvature is a constant. In this case, we are able to calculate the spectrum and the entropy unstable variations precisely (see Remark 5.3).

Theorem 5.2. *Let Γ_m be an m -covered circle of radius $\sqrt{2}$. Then the F -index of Γ_m is*

$$\text{ind}_F(\Gamma_m) = 2\lceil\sqrt{2}m\rceil - 4,$$

where $\lceil \cdot \rceil$ is the ceiling function. In particular, Γ_1 is the only entropy stable circle.

Proof. Since Γ_m has a constant curvature, $k_s = 0$. Thus the Jacobi operator reduces to

$$L = \partial_{ss} + 1,$$

and the spectrum can be calculated explicitly:

$$\mu_j = \frac{j^2}{2m^2} - 1$$

for $j \in \mathbb{Z}$. Straightforward calculation using the above formula gives that the number of negative eigenvalues is $2\lceil\sqrt{2}m\rceil - 1$. The result now follows from Theorem 4.12. Note that the variation function corresponding to $j = 0$ is constant and hence proportional to k . Furthermore, the variation functions corresponding to $|j| = m$ are in the span of the component functions of the normal \mathbf{n} , since $\mu_j = -\frac{1}{2}$ in these cases. \square

Remark 5.3. The unstable variation functions of the m -covered circle can be determined explicitly. A unit speed parameterization $x : [0, 2\sqrt{2}\pi m] \rightarrow \mathbb{R}^2$ of the m -covered circle Γ_m of radius $\sqrt{2}$ is given by

$$x(\theta) = \sqrt{2}(\cos(\theta/\sqrt{2}), \sin(\theta/\sqrt{2})).$$

Straightforward calculation shows that the unstable variations of Γ_m are given by the collection of functions $f_j, g_j : [0, 2\sqrt{2}\pi m] \rightarrow \mathbb{R}$ defined by

$$(5.4) \quad f_j(\theta) = \sin\left(\frac{j}{\sqrt{2}m}\theta\right) \quad \text{and} \quad g_j = \cos\left(\frac{j}{\sqrt{2}m}\theta\right),$$

for $j \in \mathbb{N}$ such that $1 \leq j < \sqrt{2}m$ and $j \neq m$. By Theorem 5.2, these are all of the F -unstable variations. Geometrically, varying Γ_m by these functions corresponds to enlarging some circles of Γ_m , while contracting the others. The CSF will amplify this perturbation.

The proof of Theorem A.2 now follows easily using Theorem 5.2.

Proof of Theorem A.2. Theorem 5.2 shows that an m -covered circle is entropy unstable for $m \geq 2$. By Theorem 4.4, the m -covered circle Γ_m can be perturbed to a curve Γ' with lower entropy. Moreover, since the perturbation can be chosen to be arbitrary C^∞ -small, the perturbed curve Γ' can be chosen close enough to Γ_m so that the perturbed curve also has turning number m . The second part of Theorem A.2 follows immediately from Theorem A.1. □

Entropy instability means that we can perturb a shrinker slightly to decrease the entropy. Thus the singularity corresponding to the given shrinker will never occur along the flow starting from the perturbed shrinker. In Section 6, our construction of a piecewise CSF for closed immersed curves, whose only singularities are embedded circles and type II singularities, rests upon the fact that the embedded circle is the only entropy stable closed plane shrinker. The latter fact will follow from

Theorem 5.5. *An Abresch-Langer curve $\Gamma_{m,n}$ has F -index*

$$(5.6) \quad \text{ind}_F(\Gamma_{m,n}) = 2n - 5.$$

Proof. Let L be the Jacobi operator of $\Gamma := \Gamma_{m,n}$. We claim that k_s/k is an eigenfunction of L with eigenvalue 0, i.e. that $L(k_s/k) = 0$. Indeed, differentiating the shrinker equation (1.7) gives $2k_s/k = \langle x, \mathbf{t} \rangle$. Therefore

$$\begin{aligned} 2L(k_s/k) &= \partial_{ss}\langle x, \mathbf{t} \rangle - \frac{k_s}{k}\partial_s\langle x, \mathbf{t} \rangle + (k^2 + \frac{1}{2})\langle x, \mathbf{t} \rangle \\ &= \partial_s(1 - k\langle x, \mathbf{n} \rangle) - \frac{1}{2}\langle x, \mathbf{t} \rangle(1 - k\langle x, \mathbf{n} \rangle) + (k^2 + \frac{1}{2})\langle x, \mathbf{t} \rangle \\ &= -k_s\langle x, \mathbf{n} \rangle - k^2\langle x, \mathbf{t} \rangle + \frac{1}{2}k\langle x, \mathbf{t} \rangle\langle x, \mathbf{n} \rangle + k^2\langle x, \mathbf{t} \rangle \\ &= -k_s\langle x, \mathbf{n} \rangle + k_s\langle x, \mathbf{n} \rangle \\ &= 0. \end{aligned}$$

Since k is strictly positive and k_s has exactly $2n$ zeros on $[0, 2\pi m)$, k_s/k is either the $(2n - 1)^{\text{st}}$ or $2n^{\text{th}}$ eigenfunction of L . It remains to show that k_s/k is the $(2n - 1)^{\text{st}}$ eigenfunction and that zero is a simple eigenvalue. To do so, we reparameterize Γ using the variable

$$\theta = -\cos^{-1}\langle e_1, \mathbf{n} \rangle,$$

where $e_1 \in \mathbb{R}^2$ is a constant unit vector. This is possible since Γ is convex (however, θ is only locally continuous as a discontinuity appears after one round). It follows from the shrinker equation that if we define the operator \tilde{L} by

$$\tilde{L} = k^2\partial_{\theta\theta} + (k^2 + 1/2),$$

then $\frac{\partial\theta}{\partial s} = k$ and $\tilde{L}f = Lf$. The theorem now follows from the linear analysis in [12, Proposition 2.1], however, we provide a proof here for the convenience of the reader.

Note that $k_\theta = k_s/k$ and

$$\tilde{L}k_\theta = \tilde{L}(k_s/k) = L(k_s/k) = 0.$$

In other words, k_θ is an eigenfunction of \tilde{L} with eigenvalue 0. We denote by $\{\mu_j\}$ and $\{\nu_j\}$, respectively, the Dirichlet and Neumann eigenvalues of \tilde{L} on $[0, \pi m/n]$. (By our conventions, these sequences are nondecreasing.) Let f_j solve the Neumann problem $\tilde{L}f_j = -\nu_j f_j$ on $[0, \pi m/n]$. Since k is an even function, f_j may be extended by reflection and then periodically to an eigenfunction \tilde{f}_j of \tilde{L} on the circle of length $2\pi m$ with the same eigenvalue

ν_j . Precisely, the function \bar{f}_j is defined on $[0, 2\pi m/n]$ by

$$(5.7) \quad \bar{f}_j(\theta) = \begin{cases} f_j(\theta) & \text{for } \theta \in [0, \frac{\pi m}{n}] \\ f_j(\frac{2\pi m}{n} - \theta) & \text{for } \theta \in [\frac{\pi m}{n}, \frac{2\pi m}{n}] \end{cases},$$

and is extended periodically to the circle of length $2\pi m$. The eigenfunction \bar{f}_2 will have exactly $2n$ zeros on $[0, 2\pi m)$ and k_θ is the lowest Dirichlet eigenfunction of \tilde{L} on $[0, \pi m/n]$, so $\mu_1 = 0$. From the standard fact that $\mu_1 \geq \nu_2$, it follows that k_θ and \bar{f}_2 are, respectively, the $(2n - 1)^{\text{st}}$ and $2n^{\text{th}}$ eigenfunctions of \tilde{L} on $[0, 2\pi m]$. Moreover, since \tilde{L} and L have the same eigenfunctions, $k_s/k = k_\theta$ is the $(2n - 1)^{\text{st}}$ eigenfunction of L .

Next we show that zero is a simple eigenvalue. Note that \tilde{L} is a linear second order differential operator so the equation $\tilde{L}f = 0$ can have at most two linearly independent solutions. If zero were not a simple eigenvalue, then there would exist a nontrivial $2\pi m$ periodic solution w of $\tilde{L}w = 0$. Consequently, any solution of $\tilde{L}f = 0$, being a linear combination of k_θ and w , which are both $2\pi m$ periodic, would have to be $2\pi m$ periodic. As a result, to show that zero is a simple eigenvalue, it is sufficient to produce a solution to $\tilde{L}f = 0$ which is not $2\pi m$ periodic.

We will now produce such a solution. It follows from the shrinker equation that k solves the ODE

$$(5.8) \quad k_{\theta\theta} + k - \frac{1}{2k} = 0.$$

This equation has first integral

$$E = k_\theta^2 + k^2 - \log k$$

and the general solution can be expressed as $k(\theta + a, E)$. Let

$$u = \frac{\partial k}{\partial E} \Big|_{E=E_{m,n}},$$

where $E_{m,n}$ is the constant corresponding to $\Gamma_{m,n}$. Straightforward calculation using (5.8) shows that $\tilde{L}u = 0$. Abresch-Langer [1, Proposition 3.2] show that u is not $2\pi m$ periodic. As stated above, it follows that zero is a simple eigenvalue of \tilde{L} and thus also of L .

Summarizing, we have shown that $L(k_s/k) = 0$, that k_s/k is the $(2n - 1)^{\text{st}}$ eigenfunction of L , and that zero is a simple eigenvalue of L . Consequently, L has $2n - 2$ negative eigenvalues. By Theorem 4.12, the Abresch-Langer curve $\Gamma_{m,n}$ has F -index $2n - 5$. □

Corollary 5.9. *The embedded circle is the only entropy stable closed shrinker of the CSF for closed immersed curves.*

6. Generic CSF

In this section, we prove Theorem B. The following lemma shows that, as a consequence of Corollary 5.9, we can perturb an unstable closed plane shrinker to satisfy the entropy conditions (1.6) for a piecewise CSF. For the piecewise flow in Theorem B, we will actually require the entropy inequality (1.6) to be a strict inequality so that we can exclude a shrinker from appearing as a singularity at later times of the flow.

Lemma 6.1. *Let $\Gamma \subset \mathbb{R}^2$ be a closed immersed curve such that the CSF $(\Gamma_t)_{t \in [0, T]}$ starting from Γ has a type I singularity other than the embedded circle and let $\tilde{\Gamma}_\infty \subset \mathbb{R}^2$ be some blowup sequence limit at this singularity. Then there exists a $T_0 \in [0, T)$ and a function $u \in C^\infty(\Gamma)$ such that the graph $\bar{\Gamma} = \Gamma_{T_0} + \mathbf{u}\mathbf{n}$ satisfies*

$$(6.2) \quad \lambda(\bar{\Gamma}) < \lambda(\tilde{\Gamma}_\infty).$$

Proof. If the CSF $(\Gamma_t)_{t \in [0, T]}$ has a type I singularity at time T , any limit of a rescaling sequence is a closed shrinker and thus is compact. Therefore, there is only one singular point $x_1 \in \mathbb{R}^2$ at time T . Without loss of generality, we may assume that $0 \in \mathbb{R}^2$ is a singular point. By the classification of 1-dimensional shrinkers [1], $\tilde{\Gamma}_\infty$ must be a multiply-covered circle or a multiply-covered Abresch-Langer curve (including 1 time covered). Since we assume that the shrinker $\tilde{\Gamma}_\infty$ is not an embedded circle, Corollary 5.9 implies that $\tilde{\Gamma}_\infty$ is entropy unstable. By Theorem 2.3, there exists a variation $f \in C^\infty(\tilde{\Gamma}_\infty)$ and an $\epsilon_0 > 0$ such that

$$\lambda(\tilde{\Gamma}_\infty + \epsilon f \mathbf{n}) < \lambda(\tilde{\Gamma}_\infty),$$

for all $\epsilon \neq 0$ with $|\epsilon| < \epsilon_0$. By assumption, there exists a sequence $\tau_i \rightarrow \infty$ of rescaled times such that the rescaled curves $(\tilde{\Gamma}_{\tau_i})_{i \in \mathbb{N}}$ smoothly converge to $\tilde{\Gamma}_\infty$. Thus, there exists $N' \in \mathbb{N}$ and a sequence $(g_i)_{i \in \mathbb{N}} \subset C^\infty(\tilde{\Gamma}_\infty)$ with $\|g_i\|_{C^\infty} \rightarrow 0$ such that, for $i > N'$, the rescaled curve $\tilde{\Gamma}_{\tau_i}$ can be written as a graph

$$\tilde{\Gamma}_{\tau_i} = \tilde{\Gamma}_\infty + g_i \mathbf{n}.$$

By the monotonicity of entropy and Corollary 2.7, there exists $\epsilon_1 \leq \epsilon_0$ and $N \geq N'$ such that, for all $\epsilon \neq 0$ with $|\epsilon| < \epsilon_1$ and all $i > N$, we have

$$\lambda(\tilde{\Gamma}_\infty + (\epsilon f + g_i)\mathbf{n}) < \lambda(\tilde{\Gamma}_\infty) \leq \lambda(\tilde{\Gamma} + g_i\mathbf{n}).$$

Since entropy is invariant under dilations, when the curve $\tilde{\Gamma}_\infty + (\epsilon f + g_i)\mathbf{n}$ is rescaled back to the original spacetime, the entropy conditions can be satisfied when $|\epsilon| \neq 0$ is small enough, i is large enough. The time T_0 then corresponds to the rescaled time τ_i and the function u is a multiple of the function ϵf . □

By using the classification of singularities for the CSF of closed curves and Lemma 6.1, we now prove Theorem B.

Proof of Theorem B. We will construct a piecewise CSF with a finite number of discontinuities that eventually shrinks to a circular singularity, or has a type II singularity. We perform a smooth jump just before a (entropy unstable) singular time, where we replace a time slice of the flow by a graph over it, and the crucial point is to show that the entropy decreases below the entropy of the unstable singularity. We repeat this until we get a singular point which is either modelled by an embedded circle or type II.

Let $(\Gamma_t)_{t \in [0, T_1]}$ be the CSF starting at Γ . The CSF $(\Gamma_t)_{t \in [0, T_1]}$ has either a type I or a type II singularity at time T_1 . We consider both cases:

Case (I):: If the CSF $(\Gamma_t)_{t \in [0, T_1]}$ has a type I singularity at time T_1 , any limit of a rescaling sequence is either an entropy stable shrinker or an entropy unstable closed shrinker. In particular, any limit of a rescaling sequence is compact, and therefore, there is only one singular point $x_1 \in \mathbb{R}^2$ at time T_1 .

Case (I.a):: If there is some rescaling sequence limit which is an entropy unstable closed shrinker Γ_∞^1 , Lemma 6.1 gives the existence of a $t_2 < T_1$ and a curve $\Gamma_{t_2}^2$ such that $\Gamma_{t_2}^2$ is a graph over $\Gamma_{t_2}^1 := \Gamma_{t_2}$ of a function $u_2 \in C^{2,\alpha}(\Gamma_{t_2}^1)$ satisfying

$$\lambda(\Gamma_{t_2}^2) < \lambda(\Gamma_\infty^1).$$

Consequently, there exists a CSF $(\Gamma_t^2)_{t \in [t_2, T_2]}$ starting at $\Gamma_{t_2}^2$ such that the concatenation of $(\Gamma_t^1)_{t \in [0, t_2]}$ with $(\Gamma_t^2)_{t \in [t_2, T_2]}$ is a piecewise CSF starting at Γ .

Case (I.b):: If all rescaling sequences converge to an entropy stable closed shrinker, it must be an embedded circle, so the theorem holds.

Case (II):: If the CSF $(\Gamma_t)_{t \in [0, T_1]}$ has a type II singularity at time T_1 , the theorem holds.

If *Case (I.a)* applies, we return to cases (I) or (II) for $(\Gamma_t^2)_{t \in [t_2, T_2]}$, and repeat the process. Since the piecewise CSF preserves turning number and there are only finitely many closed planar shrinkers of a given turning number, by the fact that

$$\lambda(\Gamma_{T_i}^i) < \lambda(\Gamma_\infty^{i-1})$$

at each step and the monotonicity of entropy under piecewise CSF, *Case (I.a)* can happen only finitely many times. This proves the first part of the theorem.

It was shown in the proof of Theorem A.1 that any blowup sequence of a type I singularity of the CSF for closed curves preserves the turning number. It follows that a closed curve with turning number greater than 1 cannot have an embedded circle as the limit of a rescaling sequence. This proves the second part of Theorem B. \square

References

- [1] Abresch, U., & Langer, J. The normalized curve shortening flow and homothetic solutions. *Journal of Differential Geometry* **23** (1986), no. 2, 175–196.
- [2] Andrews, B., & Bryan, P. Curvature bound for curve shortening flow via distance comparison and a direct proof of Grayson’s theorem. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, **2011** (2011), no. 653, 179–187.
- [3] Angenent, S. B. Shrinking doughnuts. In *Nonlinear Diffusion Equations and Their Equilibrium States* **3**, pp. 21–38, Birkhäuser, Boston, MA, 1992.
- [4] Au, T. K. K. On the saddle point property of Abresch-Langer curves under the curve shortening flow. *Communications in Analysis and Geometry* **18** (2010), no. 1, 1–21.
- [5] Bernstein, J., & Wang, L. A sharp lower bound for the entropy of closed hypersurfaces up to dimension six. *Inventiones mathematicae* **206** (2016), no. 3, 601–627.
- [6] Brakke, K. The motion of a surface by its mean curvature. *Mathematical Notes* **20** (1978).

- [7] Brendle, S. Embedded self-similar shrinkers of genus 0. *Annals of Mathematics* (2016), 715–728.
- [8] Coddington, E. A., & Levinson, N. *Theory of Ordinary Differential Equations*, Tata McGraw. Hill Publishing Co. LTD., New Delhi, 1987.
- [9] Colding, T. H., Ilmanen, T., Minicozzi, W. P., & White, B. The round sphere minimizes entropy among closed self-shrinkers. *Journal of Differential Geometry* **95** (2013), no. 1, 53–69.
- [10] Colding, T. H., & Minicozzi, W. P. *A Course in Minimal Surfaces*. (Vol. 121). American Mathematical Soc., 2011.
- [11] Colding, T. H., & Minicozzi, W. P. Generic mean curvature flow I; generic singularities. *Annals of Mathematics* (2012), 755–833.
- [12] Epstein, C. L., & Weinstein, M. I. A stable manifold theorem for the curve shortening equation. *Communications on Pure and Applied Mathematics* **40** (1987), no. 1, 119–139.
- [13] Gage, M., & Hamilton, R. S. The heat equation shrinking convex plane curves. *Journal of Differential Geometry* **23** (1986), no. 1, 69–96.
- [14] Grayson, M. A. The heat equation shrinks embedded plane curves to round points. *Journal of Differential Geometry* **26** (1987), no. 2, 285–314.
- [15] Hershkovits, O., & White, B. Sharp entropy bounds for self-shrinkers in mean curvature flow. *Geometry & Topology* **23** (2019), no. 3, 1611–1619.
- [16] Huisken, G. Asymptotic-behavior for singularities of the mean-curvature flow. *Journal of Differential Geometry* **31** (1990), no. 1, 285–299.
- [17] Ilmanen, T. *Elliptic Regularization and Partial Regularity for Motion by Mean Curvature*. (Vol. 520). American Mathematical Soc., 1994.
- [18] Ilmanen, T. *Problems in mean curvature flow*. Preprint (2003).
- [19] Kapouleas, N., Kleene, S. J., & Møller, N. M. Mean curvature self-shrinkers of high genus: non-compact examples. *Journal für die reine und angewandte Mathematik (Crelles Journal)* **2018**, no. 739, 1–39.
- [20] Ketover, D., & Zhou, X. Entropy of closed surfaces and min-max theory. *Journal of Differential Geometry* **110** (2018), no. 1, 31–71.

- [21] Liu, Z. H. The index of shrinkers of the mean curvature flow. [arXiv:1603.06539](#), (2016).
- [22] Mantegazza, C. *Lecture Notes on Mean Curvature Flow* (Vol. 290). Springer Science & Business Media, 2011.
- [23] Stone, A. A density function and the structure of singularities of the mean curvature flow. *Calculus of Variations and Partial Differential Equations* **2** (1994), no. 4, 443–480.
- [24] Whitney, H. On regular closed curves in the plane. *Compositio Math.* **4** (1937), 276–284.
- [25] Zhu, J. J. On the entropy of closed hypersurfaces and singular self-shrinkers. *Journal of Differential Geometry* **114** (2020), no. 3, 551–593.

DEPARTMENT OF MATHEMATICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MA 02139, USA
E-mail address: juliusbl@mit.edu

DEPARTMENT OF MATHEMATICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MA 02139, USA
Current address:

DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY
BETHLEHEM, PA 18015, USA
E-mail address: aos223@lehigh.edu

RECEIVED SEPTEMBER 24, 2018

ACCEPTED DECEMBER 25, 2020