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Deformation theory of nearly G_2 manifolds

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We study the deformation theory of nearly G_2 manifolds. These are seven dimensional manifolds admitting real Killing spinors. We show that the infinitesimal deformations of nearly G_2 structures are obstructed in general. Explicitly, we prove that the infinitesimal deformations of the homogeneous nearly G_2 structure on the Aloff–Wallach space are all obstructed to second order. We also completely describe the cohomology of nearly G_2 manifolds.

1. Introduction

Given a 7-dimensional smooth manifold M, a nearly G_2 structure on M is a non-degenerate (or positive) 3-form φ such that for some non-zero real constant τ_0,

$$d\varphi = \tau_0 *_\varphi \varphi \tag{1.1}$$

where the metric and the orientation and hence the Hodge star $*$ are all induced by φ. The existence of a nearly G_2 structure was shown to be equivalent to the existence of a *real Killing spinor* in [3]. A Killing spinor on a Riemannian spin manifold (M^n, g) is a section of the spinor bundle $\mu \in \Gamma(\slashed{S}(M))$ such that

$$\nabla_X \mu = \alpha X \cdot \mu \tag{1.2}$$

for any vector field X on M and some $\alpha \in \mathbb{C}$. Here $\cdot$ is the Clifford multiplication. It was proved by Friedrich [8] that any manifold with a Killing spinor is Einstein with $\mathrm{Ric}(g) = 4(n-1)\alpha^2 g$ and one of the three cases must hold:

- $\alpha = 0$ in which case μ is a parallel spinor and M has holonomy contained in $\mathrm{SU}(\frac{n}{2})$, $\mathrm{Sp}(\frac{n}{4})$, G_2 or $\mathrm{Spin}(7)$.
- α is non-zero and is purely imaginary.
- α is non-zero and real, in which case μ is a real Killing spinor and if M is complete then since it is positive Einstein, it is compact with $\pi_1(M)$ finite.

Given a nearly G_2 structure φ on M that satisfies equation (1.1), there exists a real Killing spinor μ that satisfies equation (1.2) with $\alpha = -\frac{1}{8}\tau_0$ and vice-versa. See [3] for more details.

Using the equivalence with real Killing spinors, nearly G_2 structures on homogeneous spaces, excluding the case of the round 7-sphere, were classified in [9]. Their classification is based on the dimension of the space of Killing spinors $K\slashed{S}$. They showed that 3 different types can occur:

1) $\dim(K\slashed{S}) = 1$ - nearly G_2 structures of type 1.
2) $\dim(K\slashed{S}) = 2$ - nearly G_2 structures of type 2.
3) $\dim(K\slashed{S}) = 3$ - nearly G_2 structures of type 3.

A 7-dimensional manifold (M, φ) with a nearly G_2 structure φ is a nearly G_2 manifold (see §2 for more details). Other examples apart from the round S^7 include the squashed S^7, Aloff–Wallach spaces $N(k,l)$, the Berger space $\mathrm{SO}(5)/\mathrm{SO}(3)$ and the Stiefel manifold $V_{5,2}$. Another important aspect of nearly G_2 manifolds is that the Riemannian cone $C(M)$ over M has holonomy contained in the Lie group $\mathrm{Spin}(7)$. In that case, the possible holonomies are $\mathrm{Spin}(7)$, $\mathrm{SU}(4)$ or $\mathrm{Sp}(2)$ depending on whether the link of the cone is a nearly G_2 manifold of type 1, 2 or 3 respectively.

In this paper, we study the deformation theory of nearly G_2 manifolds. The infinitesimal deformations of nearly G_2 manifolds were studied by

Alexandrov–Semmelmann in [1] where they identified the space of infinitesimal deformations with an eigenspace of the Laplacian acting on co-closed 3-forms on M of type Ω^3_{27}. We address the question of whether nearly G_2 manifolds have smooth obstructed or unobstructed deformations, i.e., whether infinitesimal deformations can be integrated to genuine deformations. This could potentially give new examples of nearly G_2 manifolds. Another applicability of studying the deformation theory of nearly G_2 manifolds can be to develop the deformation theory of Spin(7) *conifolds* which are asymptotically conical and conically singular Spin(7) manifolds, similar to the theory developed by Karigiannis–Lotay [15] for G_2 conifolds. Lehmann [17] studies the deformation theory of asymptotically conical Spin(7)−manifolds.

The study of deformation theory of special algebraic structures is not new. Deformations of Einstein metrics were studied by Koiso where he showed [16, Theorem 6.12] that the infinitesimal deformations of Einstein metrics is in general obstructed, by exhibiting certain Einstein symmetric spaces which admit non-trivial infinitesimal Einstein deformations which cannot be integrated to second order. The deformation theory of nearly Kähler structures on homogeneous 6-manifolds was studied by Moroianu–Nagy–Semmelmann in [18]. They identified the space of infinitesimal deformations with an eigenspace of the Laplacian acting on co-closed primitive $(1,1)$-forms. Using this, they proved that the nearly Kähler structures on $\mathbb{CP}^3$ and $S^3 \times S^3$ are rigid and the flag manifold $\mathbb{F}_3$ admits an 8-dimensional space of infinitesimal deformations. Later, Foscolo proved [7, Theorem 5.3] that the infinitesimal deformations of the flag manifold $\mathbb{F}_3$ are all obstructed.

Nearly G_2 manifolds are in many ways similar to nearly Kähler 6-manifolds. Both admit real Killing spinors and hence are positive Einstein. The minimal hypersurfaces in both nearly Kähler 6-manifolds and nealy G_2 manifolds behave in a similar way [6]. It was proved in [1] that the nearly G_2 structures on the squashed S^7 and the Berger space SO(5)/SO(3) are rigid while the space of infinitesimal nearly G_2 deformations of the Aloff–Wallach space $X_{1,1}$ is 8-dimensional. It is therefore natural to ask whether these infinitesimal deformations are obstructed to second order.

To address this question, we use a Dirac-type operator on nearly G_2 manifolds (cf. equation (3.6)). The use of Dirac operators to study deformation theory has been very useful. Nordström in [20] used Dirac operators to study the deformation theory of compact manifolds with special holonomy from a different point of view than Joyce [11]. In particular, the mapping properties of the Dirac type operators can be used to prove slice theorems for the action of the diffeomorphism group. This approach has also been very effective in studying the deformation theory of non-compact manifolds with special holonomy, most notably by Nordström [20] for asymptotically cylindrical manifolds with exceptional holonomy and by Karigiannis–Lotay [15] for G_2 conifolds. Dirac-type operators, in a way very close to the use made by the authors in this paper, were also used by Foscolo [7] to study the deformation theory of nearly Kähler 6-manifolds.

We follow a strategy similar to [7] in this paper. After introducing the Dirac operator and a modified Dirac operator on nearly G_2 manifolds in §3, we use their properties and the Hodge decomposition theorem to completely describe the cohomology of a complete nearly G_2 manifold. We prove our first two main results of the paper which characterize harmonic forms. These are the following.

Theorem 3.8. *Let (M, φ, ψ) be a complete nearly G_2 manifold, not isometric to round S^7. Then every harmonic 4-form lies in Ω^4_{27}. Equivalently, every harmonic 3-form lies in Ω^3_{27}.*

Theorem 3.9 *Let (M, φ, ψ) be a complete nearly G_2 manifold, not isometric to round S^7. Then every harmonic 2-form lies in Ω^2_{14}. Equivalently, every harmonic 5-form lies in Ω^5_{14}.*

We note that Theorem 3.9 was originally proved by Ball–Oliveira [2, Remark 15]. We give a different proof in this paper.

We use the properties of the modified Dirac operator, explicitly we use Proposition 3.7, to prove a slice theorem for the action of the diffeomorphism group on the space of nearly G_2 structures on M in Proposition 4.2. Using this, in Theorem 4.3 we obtain a new proof of the identification of the space of infinitesimal nearly G_2 deformations with an eigenspace of the Laplacian acting on co-closed 3-forms of type Ω^3_{27}, a result originally due to Alexandrov–Semmelmann [1].

To study higher order deformations of nearly G_2 manifolds, we use the point of view of Hitchin [10] where he interprets nearly G_2 structures as constrained critical points of a functional defined on the space $\Omega^3 \times \Omega^4_{\text{exact}}$. This approach is inspired from the work of Foscolo [7] where he used similar ideas to study second order deformations of nearly Kähler structures on 6-manifolds. The advantage of this approach is that it allows us to view the nearly G_2 equation (2.24) as the vanishing of a smooth map (cf. equation (4.9))

$$\Phi : \Omega^4_{+,\text{exact}} \times \Gamma(TM) \longrightarrow \Omega^4_{\text{exact}}$$

where $\Omega^4_{+,\mathrm{exact}}$ denotes the space of exact *positive* 4-forms on M. Thus the obstructions on the first order deformations of a nearly G_2 structure to be integrated to higher order deformations can be characterized by $\mathrm{Im}(D\Phi)$ which we do in Proposition 4.6.

Finally, we use the general deformation theory of nearly G_2 structures developed in the first part of the paper to study the infinitesimal deformations of the Aloff–Wallach space $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$. It was expected in [7] that the infinitesimal deformations of the Aloff–Wallach space might be obstructed to higher orders. In §5 we confirm this expectation. More precisely, we prove the following.

Theorem 5.1. *The infinitesimal deformations of the homogeneous nearly* G_2 *structure on the Aloff–Wallach space* $X_{1,1} \cong \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$ *are all obstructed.*

The proof of the above theorem is inspired from the ideas in [7]. However, we note that since in the nearly G_2 case we only have one *stable* form and the other is the dual of it, unlike the nearly Kähler case, the expressions and computations involved are more complicated and the proof of the theorem is computationally much more involved.

The paper is organized as follows. We discuss some preliminaries on G_2 and nearly G_2 structures in §2. We discuss the decomposition of space of differential forms on manifolds with a G_2 structure. We describe some first order differential operators in §2.1 which appear throughout the paper. In §2.2, we prove many important identities for 2-forms and 3-forms on manifolds with nearly G_2 structures. Some of these appear to be new, at least in the present form and we believe that they will be useful in other contexts as well. We introduce the Dirac and the modified Dirac operator in §3 and use the mapping properties of the latter to prove Theorem 3.8 and Theorem 3.9. We begin the discussion on infinitesimal deformations in §4.1. We prove a slice theorem and use that to obtain a new proof of the result of Alexandrov–Semmmelmann on infinitesimal nearly G_2 deformations. We interpret the nearly G_2 equation as the vanishing of a smooth map and prove the characterization for a first order deformation of a nearly G_2 structure to be integrated to second order in Proposition 4.6. Finally, in §5, we prove Theorem 5.1.

Note. The almost simultaneous preprint [19] by Semmelmann–Nagy has some overlap with the present paper and some of the ideas involved are the same. We also characterize the cohomology of nearly G_2 manifolds. The second version of their paper also contains a discussion of the deformations of the Aloff–Wallach spaces.

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2. Preliminaries on G_2 geometry

We start this section by defining G_2 structures and nearly G_2 structures on a seven dimensional manifold and also discuss the decomposition of space of differential forms on such a manifold. We also collect together various identities which will be used throughout the paper.

Let M^7 be a smooth manifold. A G_2 structure on M is a reduction of the structure group of the frame bundle from $GL(7, \mathbb{R})$ to the Lie group $G_2 \subset SO(7)$. Such a structure exists on M if and only if the manifold is orientable and spinnable, conditions which are respectively equivalent to the vanishing of the first and second Stiefel–Whitney classes. From the point of view of differential geometry, a G_2 structure on M is equivalently defined by a 3-form φ on M that satisfies a certain pointwise algebraic non-degeneracy condition. Such a 3-form nonlinearly induces a Riemannian metric g_φ and an orientation vol_φ on M and hence a Hodge star operator $*_\varphi$. We denote the Hodge dual 4-form $*_\varphi\varphi$ by ψ. Pointwise we have $|\varphi| = |\psi| = 7$, where the norm is taken with respect to the metric induced by φ.

Notations and conventions. Throughout the paper, we compute in a local orthonormal frame, so all indices are subscripts and any repeated indices are summed over all values from 1 to 7. Our convention for labelling the Riemann curvature tensor is

$$R_{ijkm}\frac{\partial}{\partial x^m} = (\nabla_i\nabla_j - \nabla_j\nabla_i)\frac{\partial}{\partial x^k},$$

in terms of coordinate vector fields. With this convention, the Ricci tensor is $R_{jk} = R_{ljkl}$, and the Ricci identity is

$$\nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k = -R_{ijkl} X_l. \tag{2.1}$$

We will use the metric to identify the vector fields and 1-forms by the musical isomorphisms. As such, throughout the paper, we will use them interchangeably without mention.

We have the following contraction identities between φ and ψ, whose proofs can be found in [13].

$$\varphi_{ijk}\varphi_{abk} = g_{ia}g_{jb} - g_{ib}g_{ja} + \psi_{ijab}, \tag{2.2}$$
$$\varphi_{ijk}\varphi_{ajk} = 6g_{ia} \tag{2.3}$$

and

$$\varphi_{ijk}\psi_{abck} = g_{ja}\varphi_{ibc} + g_{jb}\varphi_{aic} + g_{jc}\varphi_{abi} - g_{ia}\varphi_{jbc} - g_{ib}\varphi_{ajc} - g_{ic}\varphi_{abj}, \tag{2.4}$$
$$\varphi_{ijk}\psi_{abjk} = 4\varphi_{iab}, \tag{2.5}$$
$$\psi_{ijkl}\psi_{abkl} = 4g_{ia}g_{jb} - 4g_{ib}g_{ja} + 2\psi_{ijab} \tag{2.6}$$
$$\psi_{ijkl}\psi_{ajkl} = 24g_{ia}. \tag{2.7}$$

A G_2 structure on M induces a splitting of the spaces of differential forms on M into irreducible G_2 representations. The space of 2-forms $\Omega^2(M)$ and 3-forms $\Omega^3(M)$ decompose as

$$\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M), \tag{2.8}$$
$$\Omega^3(M) = \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M) \tag{2.9}$$

where Ω^k_l has pointwise dimension l. More precisely, we have the following description of the space of forms:

$$\Omega^2_7(M) = \{X \lrcorner \varphi \mid X \in \Gamma(TM)\} = \{\beta \in \Omega^2(M) \mid *(\varphi \wedge \beta) = 2\beta\}, \tag{2.10}$$
$$\begin{aligned}\Omega^2_{14}(M) &= \{\beta \in \Omega^2(M) \mid \beta \wedge \psi = 0\} \\ &= \{\beta \in \Omega^2(M) \mid *(\varphi \wedge \beta) = -\beta\}.\end{aligned} \tag{2.11}$$

In local coordinates, the above conditions can be re-written as

$$\beta \in \Omega^2_7 \iff \beta_{ij}\psi_{abij} = 4\beta_{ab}, \tag{2.12}$$
$$\beta \in \Omega^2_{14} \iff \beta_{ij}\psi_{abij} = -2\beta_{ab} \iff \beta_{ij}\varphi_{ijk} = 0. \tag{2.13}$$

Similarly, for 3-forms

$$\Omega^3_1 = \{f\varphi \mid f \in C^\infty(M)\}, \tag{2.14}$$
$$\Omega^3_7 = \{X \lrcorner \psi \mid X \in \Gamma(TM)\} = \{*(\alpha \wedge \varphi) \mid \alpha \in \Omega^1\}, \tag{2.15}$$
$$\Omega^3_{27} = \{\eta \in \Omega^3 \ \mid \ \eta \wedge \varphi = 0 = \eta \wedge \psi\}. \tag{2.16}$$

Moreover, the space Ω^3_{27} is isomorphic to the space of sections of $S^2_0(T^*M)$, the traceless symmetric 2-tensors on M, where the isomorphism i_φ is given explicitly as

$$\begin{aligned} \eta = \frac{1}{6}\eta_{ijk}dx^i \wedge dx^j \wedge dx^k \in \Omega^3_{27} \\ \overset{i_\varphi}{\longleftrightarrow} \quad h_{ab}dx^a dx^b \in C^\infty(S^2_0(T^*M)) \\ \text{where } \ \eta_{ijk} = h_{ip}\varphi_{pjk} + h_{jp}\varphi_{ipk} + h_{kp}\varphi_{ijp}. \end{aligned} \tag{2.17}$$

The decompositions of $\Omega^4(M) = \Omega^4_1(M) \oplus \Omega^4_7(M) \oplus \Omega^4_{27}(M)$ and $\Omega^5(M) = \Omega^5_7(M) \oplus \Omega^5_{14}(M)$ are obtained by taking the Hodge star of (2.9) and (2.8) respectively.

Given a G_2 structure φ on M, we can decompose $d\varphi$ and $d\psi$ according to (2.8) and (2.9). This defines the *torsion forms*, which are unique differential forms $\tau_0 \in \Omega^0(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$ such that (see [13])

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *_\varphi\tau_3, \tag{2.18}$$
$$d\psi = 4\tau_1 \wedge \psi + *_\varphi\tau_2. \tag{2.19}$$

Let ∇ denote the Levi-Civita connection of the metric induced by the G_2 structure. The *full torsion tensor* T of a G_2 structure is a 2-tensor satisfying

$$\nabla_i\varphi_{jkl} = T_{im}\psi_{mjkl}, \tag{2.20}$$
$$T_{lm} = \frac{1}{24}(\nabla_l\varphi_{abc})\psi_{mabc}, \tag{2.21}$$
$$\nabla_m\psi_{ijkl} = -T_{mi}\varphi_{jkl} + T_{mj}\varphi_{ikl} - T_{mk}\varphi_{ijl} + T_{ml}\varphi_{ijk}. \tag{2.22}$$

The full torsion T is related to the torsion forms by (see [13])

$$T_{lm} = \frac{\tau_0}{4}g_{lm} - (\tau_3)_{lm} - (\tau_1)_{lm} - \frac{1}{2}(\tau_2)_{lm}. \tag{2.23}$$

Remark 2.1. The space Ω^2_7 is isomorphic to the space of vector fields and hence to the space of 1-forms. Thus in (2.23), we are viewing τ_1 as an element of Ω^2_7 which justifies the expression $(\tau_1)_{lm}$.

A G_2 structure φ is called **torsion-free** if $\nabla\varphi = 0$ or equivalently $T = 0$. We can now define nearly G_2 structures.

Definition 2.2. A G_2 structure φ is a **nearly** G_2 structure if τ_0 is the only nonvanishing component of the torsion, that is

$$d\varphi = \tau_0\psi \quad \text{and} \quad d\psi = 0. \tag{2.24}$$

In this case, we see from (2.23) that $T_{ij} = \dfrac{\tau_0}{4} g_{ij}$.

Remark 2.3. If φ is a nearly G_2 structure on M then since $d\varphi = \tau_0\psi$, we can differentiate this to get $d\tau_0 \wedge \psi = 0$ and hence $d\tau_0 = 0$, as wedge product with ψ is an isomorphism from $\Omega^1_7(M)$ to $\Omega^5_7(M)$. Thus τ_0 is a constant, if M is connected.

Given a G_2 structure φ with torsion T_{lm}, we have the expressions for the Ricci curvature R_{ij} and the scalar curvature R of its associated metric g which can be found in [5] or [13] as

$$R_{jk} = (\nabla_i T_{jm} - \nabla_j T_{im})\varphi_{mki} - T_{jl}T_{lk} + \mathrm{tr}(T)T_{jk} - T_{jb}T_{lp}\psi_{lpbk}, \tag{2.25}$$

$$R = -12\nabla_i(\tau_1)_i + \frac{21}{8}{\tau_0}^2 - |\tau_3|^2 + 5|\tau_1|^2 - \frac{1}{4}|\tau_2|^2. \tag{2.26}$$

where $|C|^2 = C_{ij}C_{kl}g^{ik}g^{jl}$ is the matrix norm in (2.26).

In particular, for a manifold M with a nearly G_2 structure φ, we see that

$$R_{ij} = \frac{3}{8}{\tau_0}^2 g_{ij}, \tag{2.27}$$

$$R = \frac{21}{8}{\tau_0}^2. \tag{2.28}$$

Finally, we remark that S^7 with the round metric and also the squashed S^7 are examples of manifolds with nearly G_2 structure (see [9] for more on nearly G_2 structures. The authors in [9] call such structures nearly parallel G_2 structures but we will call them nearly G_2 structures.) In particular, S^7 with radius 1 has scalar curvature 42, so comparing with (2.24) we get that $\tau_0 = 4$.

We use the following identities throughout the paper. They are all proved in [12, Lemma 2.2.1 and Lemma 2.2.3] and we collect them here for the convenience of the reader. First, we note that if α is a k-form and w is a vector field then

$$*(w \lrcorner \alpha) = (-1)^{k+1}(w \wedge *\alpha), \tag{2.29}$$
$$*(w \wedge \alpha) = (-1)^{k}(w \lrcorner * \alpha). \tag{2.30}$$

If α is a 1-form then we have the following identities

$$*(\varphi \wedge *(\varphi \wedge \alpha)) = -4\alpha, \tag{2.31}$$
$$\psi \wedge *(\varphi \wedge \alpha) = 0, \tag{2.32}$$
$$*(\psi \wedge *(\psi \wedge \alpha)) = 3\alpha, \tag{2.33}$$
$$\varphi \wedge *(\psi \wedge \alpha) = 2(\psi \wedge \alpha). \tag{2.34}$$

Suppose w is a vector field then we have the following identities

$$\varphi \wedge (w \lrcorner \psi) = -4 * w, \tag{2.35}$$
$$\psi \wedge (w \lrcorner \psi) = 0, \tag{2.36}$$
$$\psi \wedge (w \lrcorner \varphi) = 3 * w, \tag{2.37}$$
$$\varphi \wedge (w \lrcorner \varphi) = 2 * (w \lrcorner \varphi). \tag{2.38}$$

Let $\Theta : \Omega^3_+ \to \Omega^4_+$ be the non-linear map which associates to any G_2 structure φ, the dual 4-form $\psi = \Theta(\varphi) = *\varphi$ with respect to the metric g_φ. We note that $\Theta^{-1} : \Omega^4_+ \to \Omega^3_+$ is defined only when we fix the orientation on M. See [10, §8] for more details. We will need the following result from [11, Proposition 10.3.5], later.

Proposition 2.4. *Suppose φ be a G_2 structure on M with $\psi = *\varphi$. Let ξ be a 3-form which has sufficiently small pointwise norm with respect to g_φ so that $\varphi + \xi$ is still a positive 3-form and η be a 4-form with small enough pointwise norm so that $\psi + \eta$ is a positive 4-form. Then*

(1) the image of ξ under the linearization of Θ at φ is

$$\Theta(\xi) = *_\varphi\Big(\frac{4}{3}\pi_1(\xi) + \pi_7(\xi) - \pi_{27}(\xi)\Big). \tag{2.39}$$

(2) the image of η under the linearization of Θ^{-1} at ψ is

$$\Theta^{-1}(\eta) = *_\varphi\Big(\frac{3}{4}\pi_1(\eta) + \pi_7(\eta) - \pi_{27}(\eta)\Big). \tag{2.40}$$

2.1. First order differential operators

In this section, we discuss various first order differential operators on a manifold with a nearly G_2 structure and prove some identities involving them.

For $f \in C^\infty(M)$, we have the vector field $\operatorname{grad} f$ given by

$$(\operatorname{grad} f)_k = \nabla_k f$$

and for any vector field X we have the divergence of X which is a function

$$\operatorname{div} X = \nabla_k X_k.$$

On a manifold with a G_2 structure φ, for a vector field $X \in \Gamma(TM)$, we define the *curl* of X, as

$$(\operatorname{curl} X)_k = \nabla_i X_j \varphi_{ijk} \tag{2.41}$$

which can also be written as

$$(\operatorname{curl} X) = *(dX \wedge \psi) \tag{2.42}$$

and so up to G_2-equivariant isomorphisms, the vector field $\operatorname{curl} X$ is the projection of the 2-form dX onto the Ω^2_7 component. In fact, we have the following

Proposition 2.5. *Let X be a vector field on M. The Ω^2_7 component of dX is given by*

$$\pi_7(dX) = \frac{1}{3}(\operatorname{curl} X) \lrcorner \varphi = \frac{1}{3} * (\operatorname{curl} X \wedge \psi). \tag{2.43}$$

Proof. We know that $\pi_7(dX) = W \lrcorner \varphi$ for some vector field W. Using (2.37) we compute

$$\operatorname{curl} X = *(dX \wedge \psi) = *(\pi_7(dX) \wedge \psi) = *((W \lrcorner \varphi) \wedge \psi) = 3W$$

which gives (2.43). □

In the next proposition we state and prove various relations among the first order differential operators described above. We prove the results for *any* G_2 structure and will later state the results for nearly G_2 structures. These formulas are generalizations of the formulas first proved for torsion-free G_2 structures by Karigiannis [14, Proposition 4.4].

Proposition 2.6. *Let $f \in C^\infty(M)$ and X be a vector field on M with a G_2 structure φ. Then*

$$\operatorname{curl}(\operatorname{grad} f) = 0, \tag{2.44}$$

$$\operatorname{div}(\operatorname{curl} X) = \nabla_i X_j (4(\tau_1)_{ij} - (\tau_2)_{ij}) + (\pi_7(\mathrm{Rm}))^j_{jl} X_l, \tag{2.45}$$

$$\begin{aligned} \operatorname{curl}(\operatorname{curl} X)_l = {} & \nabla_l(\operatorname{div} X) + R_{lm}X_m - \Delta X_l - (\operatorname{curl} X)_m T_{ml} \\ & - (\nabla_l X_i - \nabla_i X_l)(\tau_1)_{ms}\varphi_{msi} + \operatorname{tr} T (\operatorname{curl} X)_l \\ & + \nabla_i X_j T_{is}\varphi_{jsl} + \nabla_i X_j T_{js}\varphi_{sil}. \end{aligned} \tag{2.46}$$

Remark 2.7. For fixed i, j, the Riemann curvature tensor R_{ijkl} is skew-symmetric in k and l and hence

$$R_{ijkl} = (\pi_7(\mathrm{Rm}))_{ijkl} + (\pi_{14}(\mathrm{Rm}))_{ijkl}.$$

Explicitly,

$$\begin{aligned} (\pi_7(\mathrm{Rm}))_{ijkl} &= \frac{1}{3}R_{ijkl} + \frac{1}{6}R_{abkl}\psi_{abij}, \\ (\pi_{14}(\mathrm{Rm}))_{ijkl} &= \frac{2}{3}R_{ijkl} - \frac{1}{6}R_{abkl}\psi_{abij}. \end{aligned}$$

Moreover, from [13, eq. (4.17)], we have

$$(\pi_7(\mathrm{Rm}))_{ijkl} = (\pi_7(\mathrm{Rm}))^m_{ij}\varphi_{mkl} \quad \text{where} \quad \pi_7(\mathrm{Rm})^m_{ij} = \frac{1}{6}R_{ijkl}\varphi_{klm}. \tag{2.47}$$

Proof. We compute

$$\operatorname{curl}(\operatorname{grad} f) = \nabla_i(\nabla_j f)\varphi_{ijk} = 0$$

as φ is skew-symmetric, thus proving (2.44). For (2.45) we use the Ricci identity (2.1) to get

$$\begin{aligned}\operatorname{div}(\operatorname{curl} X) &= \nabla_k(\nabla_i X_j \varphi_{ijk}) \\ &= \nabla_k \nabla_i X_j \varphi_{ijk} + \nabla_i X_j \nabla_k \varphi_{ijk} \\ &= \frac{1}{2}(\nabla_k \nabla_i X_j - \nabla_i \nabla_k X_j)\varphi_{ijk} + \nabla_i X_j T_{km} \psi_{mijk} \\ &= -\frac{1}{2} R_{kijl} X_l \varphi_{ijk} + \nabla_i X_j (4(\tau_1)_{ij} - (\tau_2)_{ij}) \\ &= 3(\pi_7(\mathrm{Rm}))^j_{lj} X_l + \nabla_i X_j (4(\tau_1)_{ij} - (\tau_2)_{ij})\end{aligned}$$

where we used (2.12), (2.13) and (2.47). We have also used the fact that the symmetric part of T will vanish when contracted with ψ.

Finally we use the contraction identities (2.2) and (2.4) and the Ricci identity (2.1) to compute

$$\begin{aligned}(\operatorname{curl}(\operatorname{curl} X))_l &= \nabla_m(\nabla_i X_j \varphi_{ijk})\varphi_{mkl} \\ &= (\nabla_m \nabla_i X_j \varphi_{ijk} + \nabla_i X_j T_{ms} \psi_{sijk})\varphi_{lmk} \\ &= \nabla_m \nabla_i X_j (g_{il} g_{jm} - g_{im} g_{jl} + \psi_{ijlm}) \\ &\quad + \nabla_i X_j T_{ms} (g_{ms} \varphi_{lij} + g_{mi} \varphi_{slj} + g_{mj} \varphi_{sil} \\ &\qquad\qquad - g_{ls} \varphi_{mij} - g_{li} \varphi_{smj} - g_{lj} \varphi_{sim}) \\ &= \nabla_j \nabla_l X_j - \Delta X_l + \frac{1}{2}(\nabla_m \nabla_i X_j - \nabla_i \nabla_m X_j)\psi_{ijlm} \\ &\quad + \operatorname{tr} T \nabla_i X_j \varphi_{ijl} + \nabla_i X_j T_{is} \varphi_{slj} + \nabla_i X_m T_{ms} \varphi_{sil} \\ &\quad - \nabla_i X_j T_{ml} \varphi_{mij} - \nabla_l X_j T_{ms} \varphi_{smj} - \nabla_i X_l T_{ms} \varphi_{msi} \\ &= \nabla_l(\operatorname{div} X) + R_{lm} X_m - \Delta X_l + \operatorname{tr} T (\operatorname{curl} X)_l \\ &\quad + \nabla_i X_j T_{is} \varphi_{jsl} + \nabla_i X_m T_{ms} \varphi_{sil} - (\operatorname{curl} X)_m T_{ml} \\ &\quad - \nabla_l X_j (\tau_1)_{ms} \varphi_{msj} + \nabla_i X_l (\tau_1)_{ms} \varphi_{msi}\end{aligned}$$

where we used the fact that $R_{abcd}\psi_{abck} = 0$ for the third term in the fourth equality and (2.13) to cancel the τ_2 components which contract on two indices with φ for the last two terms in the fourth equality. Thus, we get

$$\begin{aligned}(\operatorname{curl}(\operatorname{curl} X))_l &= \nabla_l(\operatorname{div} X) + R_{lm} X_m - \Delta X_l \\ &\quad - (\operatorname{curl} X)_m T_{ml} - (\nabla_i X_l - \nabla_l X_i)(\tau_1)_{ms} \varphi_{msi} \\ &\quad + \operatorname{tr} T (\operatorname{curl} X)_l + \nabla_i X_j T_{is} \varphi_{jsl} + \nabla_i X_j T_{js} \varphi_{sil}.\end{aligned}$$

□

For a nearly G_2 structure we have $T_{ij} = \frac{\tau_0}{4} g_{ij}$ and $R_{ij} = \frac{3{\tau_0}^2}{8} g_{ij}$. Moreover from [13, eq. (4.18)],

$$(\pi_7(\mathrm{Rm}))^j_{jl} = -\nabla_l(\operatorname{tr} T) + \nabla_j(T_{lj}) + T_{la}T_{jb}\varphi_{abj} = 0.$$

Thus using the Weitzenböck formula for X, $\nabla^*\nabla X_l = -\nabla_j\nabla_j X_l = (\Delta_d X)_l + R_{il}X_i$, we get the following

Corollary 2.8. *Let $f \in C^\infty(M)$ and X be a vector field on M with a nearly G_2 structure φ. Then*

$$\operatorname{curl}(\operatorname{grad} f) = 0, \tag{2.48}$$

$$\operatorname{div}(\operatorname{curl} X) = 0, \tag{2.49}$$

$$\operatorname{curl}(\operatorname{curl} X) = \operatorname{grad}(\operatorname{div} X) - \Delta X + \frac{3{\tau_0}^2}{8} X + \tau_0(\operatorname{curl} X), \tag{2.50}$$

$$= \Delta_d X + \operatorname{grad}(\operatorname{div} X) + \tau_0(\operatorname{curl} X). \tag{2.51}$$

2.2. Identities for 2-forms and 3-forms

In this subsection, we prove some identities for 2-forms and 3-forms on a manifold with a nearly G_2 structure. These identities will be used several times in the paper.

Lemma 2.9. *Let (M, φ) be a manifold with a G_2 structure. If $\beta = \beta_7 + \beta_{14}$ is a 2-form then*

(1) $*(\beta \wedge \varphi) = 2\beta_7 - \beta_{14}$.

(2) $*(\beta \wedge \beta \wedge \varphi) = 2|\beta_7|^2 - |\beta_{14}|^2$.

Proof. The identity in (1) follows from (2.10) and (2.11). For (2) we note that for 7-dimensional manifolds $*^2(\alpha) = \alpha$ for a k-form α, so

$$\beta \wedge \beta \wedge \varphi = \beta \wedge *^2(\beta \wedge \varphi) = \beta \wedge *(2\beta_7 - \beta_{14})$$

and the decomposition of 2-forms is orthogonal. □

Lemma 2.10. *Let (M, φ) be a manifold with a G_2 structure. Let $\sigma = f\varphi + \sigma_7 + \sigma_{27}$ be a 3-form on M and let $\sigma_7 = X \lrcorner \psi$ for some vector field X on M. Then*

(1) $*(\sigma \wedge \varphi) = 4X$.

(2) $*(\sigma \wedge \psi) = 7f$.

Proof. For (1) we have

$$\begin{aligned} *(\sigma \wedge \varphi) &= *((f\varphi + \sigma_7 + \sigma_{27}) \wedge \varphi) = *(\sigma_7 \wedge \varphi) = *((X \lrcorner * \varphi) \wedge \varphi) \\ &= 4X \end{aligned} \tag{2.52}$$

where we have used the fact that $\Omega^3_1 \oplus \Omega^3_{27}$ lies in the kernel of wedge product with φ and (2.35) in the last equality. For (2) we note that $\Omega^3_7 \oplus \Omega^3_{27}$ lies in the kernel of wedge product with ψ and $\varphi \wedge \psi = 7\,\mathrm{vol}$. □

Next, we explicitly derive the expressions for exterior derivative and the divergence of various components of 2-forms and 3-forms on a manifold with a nearly G_2 structure. Some of these identities are new, at least in the present form and we believe that they will be useful in other contexts as well.

Lemma 2.11. *Suppose (M, φ) is a manifold with a nearly G_2 structure. Let $f \in C^\infty(M)$, $\beta \in \Omega^2_{14}$ and $X \in \Gamma(TM)$. Then*

(1) $d(f\varphi) = df \wedge \varphi + \tau_0 f \psi$.

(2) $d^*(f\varphi) = -(df) \lrcorner \varphi$.

(3) $d\beta = \frac{1}{4} * (d^*\beta \wedge \varphi) + \pi_{27}(d\beta)$.

(4) $d(X \lrcorner \varphi) = -\frac{3}{7}(d^*X)\varphi + \frac{1}{2} * \Big(\Big(\frac{3\tau_0}{2}X - \operatorname{curl} X\Big) \wedge \varphi\Big) + i_\varphi\Big(\frac{1}{2}(\nabla_i X_j + \nabla_j X_i) + \frac{1}{7}(d^*X) g_{ij}\Big)$.

(5) $d^*(X \lrcorner \varphi) = \operatorname{curl} X$.

(6) $d(X \lrcorner \psi) = -\frac{4}{7} d^*X \psi - \Big(\frac{1}{2}\operatorname{curl} X + \frac{\tau_0}{4}X\Big) \wedge \varphi - * i_\varphi\Big(\frac{1}{2}(\nabla_i X_j + \nabla_j X_j) + \frac{1}{7}(d^*X) g_{ij}\Big)$.

Proof. We have

$$\begin{aligned} d(f\varphi) &= df \wedge \varphi + f d\varphi \\ &= df \wedge \varphi + \tau_0 f \psi \end{aligned}$$

where we have used (2.24) which proves (1). For part (2) we compute

$$d^*(f\varphi) = -*d*(f\varphi) = -*d(f*\varphi) = -*(df\wedge *\varphi) = -df\lrcorner\varphi$$

as $d\psi = 0$.
We prove part (3). Since $d\beta$ is a 3-form so

$$d\beta = \pi_1(d\beta) + \pi_7(d\beta) + \pi_{27}(d\beta). \tag{2.53}$$

We compute each term on the right hand side of (2.53). We will repeatedly use the identities (2.29)–(2.38). Suppose

$$\pi_1(d\beta) = a\varphi$$

for some $a \in C^\infty(M)$. Since $\Omega^3_7 \oplus \Omega^3_{27}$ lies in the kernel of wedge product with ψ and $\beta\wedge\psi = 0$ for $\beta \in \Omega^2_{14}$, we have

$$0 = d(\beta\wedge\psi) = d\beta\wedge\psi = \pi_1(d\beta)\wedge\psi = 7a\,\mathrm{vol}$$

and hence

$$\pi_1(d\beta) = 0.$$

Suppose $\pi_7(d\beta) = X\lrcorner\psi$ for $X \in \Gamma(TM)$. Using (2.11) and Lemma 2.10 (1), we have

$$d^*\beta = *d*(\beta) = -*d(\beta\wedge\varphi) = -*(d\beta\wedge\varphi) - \tau_0*(\beta\wedge\psi) = -4X.$$

Thus

$$\pi_7(d\beta) = -\frac{1}{4}d^*\beta\lrcorner\psi = \frac{1}{4}*(d^*\beta\wedge\varphi),$$

which proves (3).

Since $d(X\lrcorner\varphi)$ is a 3-form, so we will write

$$d(X\lrcorner\varphi) = \pi_1(d(X\lrcorner\varphi)) + \pi_7(d(X\lrcorner\varphi)) + \pi_{27}(d(X\lrcorner\varphi)) \tag{2.54}$$

and will calculate each term on the right hand side. As before, assume

$$\pi_1(d(X\lrcorner\varphi)) = a\varphi$$

for some $a \in C^\infty(M)$. Then

$$d((X \lrcorner \varphi) \wedge \psi) = \pi_1(d(X \lrcorner \varphi)) \wedge \psi = 7a \operatorname{vol}$$

and hence $7a = *d((X \lrcorner \varphi) \wedge \psi) = *d(3 * X)$. So we get that

$$a = \frac{3}{7} * d * X = -\frac{3}{7} d^* X.$$

Assume that

$$\pi_7(d(X \lrcorner \varphi)) = Y \lrcorner \psi$$

for some $Y \in \Gamma(TM)$. Using the fact that $\Omega^3_1 \oplus \Omega^3_{27}$ lies in the kernel of wedge product with φ we get

$$\begin{aligned} d((X \lrcorner \varphi) \wedge \varphi) &= d(X \lrcorner \varphi) \wedge \varphi + (X \lrcorner \varphi) \wedge d\varphi \\ &= \pi_7(d(X \lrcorner \varphi)) \wedge \varphi + \tau_0 (X \lrcorner \varphi) \wedge \psi = (Y \lrcorner \psi) \wedge \varphi + 3\tau_0 * X. \end{aligned}$$

So we get

$$4 * Y + 3\tau_0 * X = d((X \lrcorner \varphi) \wedge \varphi) = d(2 * (X \lrcorner \varphi)) = 2d(X \wedge \psi) = 2(dX) \wedge \psi$$

which gives

$$Y = \frac{1}{2}\Big(* ((dX) \wedge \psi) - \frac{3\tau_0}{2} X\Big) = \frac{1}{2}\Big(\operatorname{curl} X - \frac{3\tau_0}{2} X\Big)$$

and hence

$$\pi_7(d(X \lrcorner \varphi)) = -\frac{1}{2} * \Big(\Big(\operatorname{curl} X - \frac{3\tau_0}{2} X\Big) \wedge \varphi\Big).$$

Recall the map i_φ from (2.17). To calculate $\pi_{27}(d(X\lrcorner\varphi))$ we have

(2.55)
$$\begin{aligned}
&d(X\lrcorner\varphi)_{imn}\varphi_{jmn} + d(X\lrcorner\varphi)_{jmn}\varphi_{imn}\\
&\quad= \Big[\frac{-3}{7}(d^*X)\varphi_{imn} + \frac{1}{2}\Big(\Big(\operatorname{curl}X - \frac{3\tau_0}{2}X\Big)\lrcorner\psi\Big)_{imn} + i(h_0)_{imn}\Big]\varphi_{jmn}\\
&\qquad+ \Big[\frac{-3}{7}(d^*X)\varphi_{jmn} + \frac{1}{2}\Big(\Big(\operatorname{curl}X - \frac{3\tau_0}{2}X\Big)\lrcorner\psi\Big)_{jmn} + i(h_0)_{jmn}\Big]\varphi_{imn}\\
&\quad= -\frac{36}{7}(d^*X)g_{ij} + 8(h_0)_{ij} + \frac{1}{2}\Big(\operatorname{curl}X - \frac{3\tau_0}{2}X\Big)_s\psi_{simn}\varphi_{jmn}\\
&\qquad+ \Big(\operatorname{curl}X - \frac{3\tau_0}{2}X\Big)_s\psi_{sjmn}\varphi_{imn}\\
&\quad= -\frac{36}{7}(d^*X)g_{ij} + 8(h_0)_{ij}.
\end{aligned}$$

We calculate the left hand side of (2.55). We have

$$\begin{aligned}
&d(X\lrcorner\varphi)_{imn}\varphi_{jmn} + d(X\lrcorner\varphi)_{jmn}\varphi_{imn}\\
&\quad= (\nabla_i(X_l\varphi_{lmn}) - \nabla_m(X_l\varphi_{lin}) + \nabla_n(X_l\varphi_{lim}))\varphi_{jmn}\\
&\qquad+ (\nabla_j(X_l\varphi_{lmn}) - \nabla_m(X_l\varphi_{ljn}) + \nabla_n(X_l\varphi_{ljm}))\varphi_{imn}\\
&\quad= (\nabla_iX_l\varphi_{lmn} - \nabla_mX_l\varphi_{lin} + \nabla_nX_l\varphi_{lim})\varphi_{jmn}\\
&\qquad+ \frac{\tau_0}{4}(X_l\psi_{ilmn} - X_l\psi_{mlin} + X_l\psi_{nlim})\varphi_{jmn}\\
&\qquad+ (\nabla_jX_l\varphi_{lmn} - \nabla_mX_l\varphi_{ljn} + \nabla_nX_l\varphi_{ljm})\varphi_{imn}\\
&\qquad+ \frac{\tau_0}{4}(X_l\psi_{jlmn} - X_l\psi_{mljn} + X_l\psi_{nljm})\varphi_{imn}
\end{aligned}$$

where we have used (2.20) and (2.24). So

$$\begin{aligned}
&d(X\lrcorner\varphi)_{imn}\varphi_{jmn} + d(X\lrcorner\varphi)_{jmn}\varphi_{imn}\\
&\quad= (\nabla_iX_l\varphi_{lmn}\varphi_{jmn} - 2\nabla_mX_l\varphi_{lin}\varphi_{jmn})\\
&\qquad+ \frac{\tau_0}{4}(X_l\psi_{ilmn} - X_l\psi_{mlin} + X_l\psi_{nlim})\varphi_{jmn}\\
&\qquad(\nabla_jX_l\varphi_{lmn}\varphi_{imn} - 2\nabla_mX_l\varphi_{ljn}\varphi_{imn})\\
&\qquad+ \frac{\tau_0}{4}(X_l\psi_{jlmn} - X_l\psi_{mljn} + X_l\psi_{nljm})\varphi_{imn}.
\end{aligned}$$

We use the contraction identities (2.2), (2.3) and (2.4) to get

$$\begin{aligned}
&d(X\lrcorner\varphi)_{imn}\varphi_{jmn} + d(X\lrcorner\varphi)_{jmn}\varphi_{imn}\\
&\quad = 4\nabla_i X_j + 4\nabla_j X_i + 4(\operatorname{div} X)g_{ij}\\
&\qquad + \frac{\tau_0}{4}(-4X_l\varphi_{ilj} + 4X_l\varphi_{lij} + 4X_l\varphi_{lij})\\
&\qquad + \frac{\tau_0}{4}(-4X_l\varphi_{jli} + 4X_l\varphi_{lji} + 4X_l\varphi_{lji})\\
&\quad = 4\nabla_i X_j + 4\nabla_j X_i - 4(d^*X)g_{ij}
\end{aligned}$$

and so from (2.55) we get

$$-\frac{36}{7}(d^*X)g_{ij} + 8(h_0)_{ij} = 4\nabla_i X_j + 4\nabla_j X_i - 4(d^*X)g_{ij}$$

and thus

$$(h_0)_{ij} = \frac{1}{2}(\nabla_i X_j + \nabla_j X_i) + \frac{1}{7}(d^*X)g_{ij}$$

which completes the proof of (4).

We obtain (5) by

$$d^*(X\lrcorner\varphi) = *d*(X\lrcorner\varphi) = *d(X\wedge\psi) = *(dX\wedge\psi) = \operatorname{curl} X.$$

To prove part (6), we notice that since $d\psi = 0$, $d(X\lrcorner\psi) = \mathcal{L}_X\psi$ which is the image of $\mathcal{L}_X\varphi = d(X\lrcorner\varphi) + \tau_0 X\lrcorner\psi$ under the linearization of the map Θ. We then use part (4) of the lemma and (2.39) to get part (6). □

We use the following important lemma on several occasions.

Lemma 2.12. *Let φ be a nearly G_2 structure on M and σ be a 3-form so that*

$$\sigma = f\varphi + *(X\wedge\varphi) + \eta$$

where $\eta \in \Omega^3_{27}$ with $\eta = i_\varphi(h)$ where h is a symmetric traceless 2-tensor. Then

$$\pi_1(d\sigma) = \Big(\tau_0 f + \frac{4}{7}d^*X\Big)\psi, \tag{2.56}$$

$$\pi_7(d\sigma) = \Big(df + \frac{\tau_0}{4}X + \frac{1}{2}\operatorname{curl} X - \frac{1}{2}\operatorname{div} h\Big)\wedge\varphi, \tag{2.57}$$

$$\pi_7(d^*\sigma) = *\Big((-df + \tau_0 X - \frac{2}{3}\operatorname{curl} X - \frac{2}{3}\operatorname{div} h)\wedge\psi\Big). \tag{2.58}$$

Proof. We note that $*\sigma = f\psi + (X \wedge \varphi) + *\eta$ and since φ is a nearly G_2 structure hence

$$d\sigma = df \wedge \varphi + \tau_0 f\psi + d * (X \wedge \varphi) + d\eta \tag{2.59}$$

and

$$d^*\sigma = - * d * \sigma = - * (df \wedge \psi) - *d(X \wedge \varphi) + d^*\eta. \tag{2.60}$$

Now $\pi_1(d\sigma) = \lambda\psi$ for some $\lambda \in C^\infty(M)$. We use Lemma 2.11 (6) to get,

$$\begin{aligned} 7\lambda &= \langle \lambda\psi, \psi \rangle = \langle \pi_1(d\sigma), \psi \rangle = \langle d\sigma, \psi \rangle \\ &= \langle df \wedge \varphi + \tau_0 f\psi + d * (X \wedge \varphi) + d\eta, \psi \rangle \\ &= \langle df \wedge \varphi, \psi \rangle + 7\tau_0 f + 4d^*X + \langle d\eta, \psi \rangle. \end{aligned} \tag{2.61}$$

The first term on the right hand side of (2.61) is 0 as $df \wedge \varphi \in \Omega^4_7$ and $\psi \in \Omega^4_1$. The last term is also 0 as from (2.16)

$$\langle d\eta, \psi \rangle \operatorname{vol} = d\eta \wedge \varphi = d(\eta \wedge \varphi) + \tau_0 \eta \wedge \psi = 0.$$

Thus we get that

$$7\lambda = 7\tau_0 f + 4d^*X \qquad \Longrightarrow \qquad \lambda = \tau_0 f + \frac{4}{7} d^*X$$

which gives (2.56).

To derive (2.57) and (2.58), we will need to contract $\eta \in \Omega^3_{27}$ with φ on two indices and with ψ on three indices. Using (2.17) and the contraction identities (2.2) and (2.5), a short computation gives

$$\eta_{ijk}\varphi_{ajk} = 4h_{ia}, \tag{2.62}$$

$$\eta_{ijk}\psi_{aijk} = 0. \tag{2.63}$$

Suppose $\pi_7(d\sigma) = Y \wedge \varphi$ for some 1-form Y. Note that for an arbitrary 1-form Z we have

$$\begin{aligned} \langle Y \wedge \varphi, Z \wedge \varphi \rangle \operatorname{vol} &= Y \wedge \varphi \wedge *(Z \wedge \varphi) \\ &= -Y \wedge \varphi \wedge (Z \lrcorner \psi) = 4Y \wedge *Z \\ &= 4\langle Y, Z \rangle \operatorname{vol}. \end{aligned}$$

So from (2.59) we have

$$\begin{aligned}
4\langle Y, Z\rangle &= \langle Y \wedge \varphi, Z \wedge \varphi\rangle = \langle \pi_7(d\sigma), Z \wedge \varphi\rangle = \langle d\sigma, Z \wedge \varphi\rangle \\
&= \langle df \wedge \varphi + \tau_0 f \psi + d * (X \wedge \varphi) + d\eta, Z \wedge \varphi\rangle \\
&= 4\langle df, Z\rangle + \langle d * (X \wedge \varphi), Z \wedge \varphi\rangle + \langle d\eta, Z \wedge \varphi\rangle. \qquad (2.64)
\end{aligned}$$

We first use Lemma 2.11 (6) to calculate the second term on the right hand side of (2.64). We have

$$\begin{aligned}
\langle d * (X \wedge \varphi), Z \wedge \varphi\rangle &= \left\langle (\frac{1}{2}\operatorname{curl} X + \frac{\tau_0}{4} X) \wedge \varphi, Z \wedge \varphi \right\rangle \\
&= \langle 2\operatorname{curl} X + \tau_0 X, Z\rangle
\end{aligned}$$

So in (2.64), we have

$$4\langle Y, Z\rangle = \langle 4df + \tau_0 X + 2\operatorname{curl} X, Z\rangle + \langle d\eta, Z \wedge \varphi\rangle. \qquad (2.65)$$

We compute in local coordinates

$$\begin{aligned}
\langle d\eta, Z \wedge \varphi\rangle &= \frac{1}{24}(d\eta)_{ijkl}(Z \wedge \varphi)_{ijkl} \\
&= \frac{1}{24}(\nabla_i \eta_{jkl} - \nabla_j \eta_{ikl} + \nabla_k \eta_{ijl} - \nabla_l \eta_{ijk})(Z \wedge \varphi)_{ijkl} \\
&= \frac{1}{6}(\nabla_i \eta_{jkl})(Z_i \varphi_{jkl} - Z_j \varphi_{ikl} - Z_k \varphi_{jil} - Z_l \varphi_{jki}) \\
&= \frac{1}{6}(Z_i \nabla_i \eta_{jkl} \varphi_{jkl} - 3 Z_j \nabla_i \eta_{jkl} \varphi_{ikl}) \\
&= \frac{1}{6}(Z_i \nabla_i(\eta_{jkl}\varphi_{jkl}) - \frac{\tau_0}{4} Z_i \eta_{jkl} \psi_{ijkl} \\
&\qquad - 3 Z_j \nabla_i(\eta_{jkl}\varphi_{ikl}) + \frac{3\tau_0}{4} Z_j \eta_{jkl} \psi_{iikl}).
\end{aligned}$$

We now use (2.62), (2.63) and the fact that h is traceless to get

$$\begin{aligned}
\langle d\eta, Z \wedge \varphi\rangle &= \frac{1}{6}(Z_i \nabla_i(4\operatorname{tr} h) - 0 - 3 Z_j \nabla_i(4h_{ji})) \\
&= -2\langle \operatorname{div} h, Z\rangle.
\end{aligned}$$

Thus from (2.65) we get

$$\langle Y, Z\rangle = \left\langle df + \frac{\tau_0}{4} X + \frac{1}{2}\operatorname{curl} X - \frac{1}{2}\operatorname{div} h, Z \right\rangle$$

and since Z is arbitrary, we get

$$Y = df + \frac{\tau_0}{4} X + \frac{1}{2} \operatorname{curl} X - \frac{1}{2} \operatorname{div} h$$

which establishes (2.57).

Next, we see from (2.60) and (2.10) that

$$\begin{aligned} d^*\sigma &= - * (df \wedge \psi) - *(dX \wedge \varphi) + *\tau_0(X \wedge \psi) + d^*\eta \\ &= - * (df - \tau_0 X \wedge \psi) - 2\pi_7(dX) + \pi_{14}(dX) + d^*\eta \end{aligned}$$

which on using (2.43) becomes

$$d^*\sigma = - * \left(\left(df - \tau_0 X + \frac{2}{3} \operatorname{curl} X \right) \wedge \psi \right) + \pi_{14}(dX) + d^*\eta. \tag{2.66}$$

Suppose $\pi_7(d^*\sigma) = *(W \wedge \psi)$ for some 1-form W. For any 1-form Z we note that

$$\begin{aligned} \langle *(W \wedge \psi), *(Z \wedge \psi) \rangle \operatorname{vol} &= *(W \wedge \psi) \wedge Z \wedge \psi \\ &= *(W \wedge \psi) \wedge \psi \wedge Z = 3 * W \wedge Z = 3\langle W, Z \rangle \operatorname{vol}. \end{aligned}$$

Thus using (2.66) and the orthogonality of the spaces Ω^2_7 and Ω^2_{14}, we have

$$\begin{aligned} 3\langle W, Z \rangle &= \langle *(W \wedge \psi), *(Z \wedge \psi) \rangle = \langle \pi_7(d^*\sigma), *(Z \wedge \psi) \rangle = \langle d^*\sigma, *(Z \wedge \psi) \rangle \\ &= \langle - * ((df - \tau_0 X + \frac{2}{3} \operatorname{curl} X) \wedge \psi) + \pi_{14}(dX) + d^*\eta, *(Z \wedge \psi) \rangle \\ &= \langle -3df + 3\tau_0 X - 2 \operatorname{curl} X, Z \rangle + \langle d^*\eta, *(Z \wedge \psi) \rangle. \end{aligned} \tag{2.67}$$

Using (2.62) and (2.63), we compute the last term on the right hand side of (2.67), in local coordinates. We have

$$\begin{aligned} \langle d^*\eta, *(Z \wedge \psi) \rangle &= \langle d^*\eta, Z \lrcorner \varphi \rangle = \frac{1}{2} (d^*\eta)_{ij} Z_m \varphi_{mij} = -\frac{1}{2} \nabla_p (\eta_{pij}) Z_m \varphi_{mij} \\ &= -\frac{1}{2} Z_m (\nabla_p (\eta_{pij} \varphi_{mij}) - \frac{\tau_0}{4} \eta_{pij} \psi_{pmij}) \\ &= -\frac{1}{2} Z_m (4 \nabla_p h_{pm} - 0) = -2 \langle \operatorname{div} h, Z \rangle \end{aligned}$$

and hence we get

$$\langle W, Z \rangle = \left\langle - df + \tau_0 X - \frac{2}{3} \operatorname{curl} X - \frac{2}{3} \operatorname{div} h, Z \right\rangle.$$

Since Z is arbitrary we get

$$W = -df + \tau_0 X - \frac{2}{3}\operatorname{curl} X - \frac{2}{3}\operatorname{div} h$$

which gives (2.58). □

Remark 2.13. The main point of the previous lemma is to exhibit a relation between $\pi_7(d\eta)$ and $\pi_7(d^*\eta)$. Such a relation is expected because of the form of the linearization of the map Θ. More precisely, from (2.39), applying the linearization of Θ to Lie derivatives, we have $\pi_{27}(\mathcal{L}_X\psi) = - * \pi_{27}(\mathcal{L}_X\varphi)$, $\langle d\eta, Z \wedge \varphi\rangle_{L^2} = -\langle \eta, *\mathcal{L}_X\psi\rangle_{L^2}$ and $\langle d^*\eta, Z \lrcorner \varphi\rangle_{L^2} = \langle \eta, \mathcal{L}_X\varphi\rangle_{L^2}$. The computations in local coordinates was done to relate $\pi_7(d\eta)$ and $\pi_7(d^*\eta)$ to the divergence of the symmetric 2-tensor h.

Remark 2.14. The previous lemma generalizes Proposition 2.17 from [15] where the G_2 structure was assumed to be torsion-free ($\tau_0 = 0$).

We have the following corollary of Lemma 2.12.

Corollary 2.15. *Let φ be a nearly G_2 structure and let $\eta \in \Omega^3_{27}$. Then*

(1) If η is closed then $d^*\eta \in \Omega^2_{14}$.

(2) If η is co-closed then $d\eta \in \Omega^4_{27}$.

Proof. In the notation of Lemma 2.12 we get that $f = X = 0$ and $\sigma = \eta$. Thus we get that

$$\pi_7(d\eta) = 0 \qquad \Longleftrightarrow \qquad \pi_7(d^*\eta) = 0$$

as from Lemma 2.12, both conditions are equivalent to $\operatorname{div} h = 0$. Now if $d\eta = 0$ then $\pi_7(d^*\eta) = 0$ and hence $d^*\eta \in \Omega^2_{14}$. If $d^*\eta = 0$ then $\pi_7(d\eta) = 0$. Also, since $f = X = 0$, we know from (2.56) that $\pi_1(d\eta) = 0$. So $d\eta \in \Omega^4_{27}$. □

We also have a result similar to Lemma 2.12 for 4-forms which we state below. The proof follows from the proof of Lemma 2.12 by taking $\zeta = *\sigma$ and noting that $*i_\varphi(h) = -i_\varphi(h)$. We expect that both Lemma 2.12 and Lemma 2.16 will be useful in other contexts as well.

Lemma 2.16. *Let φ be a nearly G_2 structure on M and ζ be a 4-form on M so that*

$$\zeta = f\psi + X \wedge \varphi + \zeta_0$$

where $X \in \Omega^1(M)$ and $\zeta_0 \in \Omega^4_{27}$ with $\zeta_0 = *i_\varphi(h)$ where h is a symmetric traceless 2-tensor. Then

$$\pi_7(d\zeta) = W \wedge \psi \quad \text{where} \quad W = df - \tau_0 X + \frac{2}{3}\operatorname{curl} X - \frac{2}{3}\operatorname{div} h, \tag{2.68}$$

$$\pi_1(d^*\zeta) = \Big(\tau_0 f + \frac{4}{7}d^*X\Big)\varphi, \tag{2.69}$$

$$\pi_7(d^*\zeta) = Y \lrcorner \psi \quad \text{where} \quad Y = -df - \frac{1}{2}\operatorname{curl} X - \frac{\tau_0}{4}X - \frac{1}{2}\operatorname{div} h. \tag{2.70}$$

We get the following corollary.

Corollary 2.17. *Let φ be a nearly G_2 structure on M and let $\zeta_0 \in \Omega^4_{27}$. Then*

1) If $d\zeta_0 = 0$ then $d^*\zeta_0 \in \Omega^3_{27}$.

2) If $d^*\zeta_0 = 0$ then $d\zeta_0 \in \Omega^5_{14}$.

3. Hodge theory of nearly G_2 manifolds

3.1. Dirac operators on nearly G_2 manifolds

We begin this section by defining the Dirac operator on (M, φ) with a nearly G_2 structure. We then define a *modified* Dirac operator which is more suitable for our purposes. A G_2 structure on M induces a spin structure, so M admits an associated Dirac operator $\not{D}$ on its spinor bundle $\not{S}(M)$. Since τ_0 is constant, by rescaling the metric induced by the nearly G_2 structure, we can change the magnitude of τ_0 and by changing the orientation, we can change its sign. In the later part of the paper, we study deformations of nearly G_2 structures through nearly G_2 structures φ_t. Since the underlying metric of any nearly G_2 structure is positive Einstein, the family of metrics g_t corresponding to φ_t will be positive Einstein and so by [4, Corollary 2.12], the scalar curvature R_t is constant in t. Thus, by (2.28), τ_0 will be constant through the deformation. Henceforth, we will assume that $\tau_0 = 4$. *The results of the paper do not depend on the value of τ_0 chosen.* Recall the following definition from §1 with $\tau_0 = 4$.

Definition 3.1. A spinor $\eta \in \Gamma(\$(M))$ is called a *Killing* spinor if for any $X \in \Gamma(TM)$

$$\nabla_X \eta = -\frac{1}{2} X \cdot \eta \tag{3.1}$$

where " $\cdot$ " is the Clifford multiplication.

The real spinor bundle $\$(M)$, as a G_2 representation, is isomorphic to $\Omega^0 \oplus \Omega^1$, where the isomorphism is

$$(f, X) \longrightarrow f \cdot \eta + X \cdot \eta.$$

For comparison with the Dirac-type operator which we define later, let us derive a formula for the Dirac operator $D\!\!\!/$ on a nearly G_2 manifold in terms of this isomorphism.

A unit spinor η on a nearly G_2 manifold M satisfies (3.1). Thus

$$D\!\!\!/(f\eta) = \sum_{i=1}^{7} e_i \cdot \nabla_{e_i}(f\eta) = \nabla f \cdot \eta + \frac{7}{2} f\eta,$$

where we have used the fact that $e_i \cdot e_i = -1$. Also,

$$\begin{aligned} D\!\!\!/(X \cdot \eta) &= \sum_{i=1}^{7} e_i \cdot \nabla_{e_i}(X \cdot \eta) = \sum_{i=1}^{7} (e_i \cdot \nabla_{e_i} X \cdot \eta + e_i \cdot X \cdot \nabla_{e_i} \eta) \\ &= (dX) \cdot \eta + (d^* X)\eta + \sum_{i=1}^{7} e_i \cdot X \cdot \nabla_{e_i} \eta \end{aligned}$$

which on using $X \cdot e_i + e_i \cdot X = -2\langle X, e_i \rangle$ and (3.1) becomes

$$\begin{aligned} D\!\!\!/(X \cdot \eta) &= (dX) \cdot \eta + (d^* X)\eta - \sum_{i=1}^{7} (X \cdot e_i \cdot \nabla_{e_i} \eta + 2\langle X, e_i \rangle \nabla_{e_i} \eta) \\ &= (dX) \cdot \eta + (d^* X)\eta - \frac{7}{2} X \cdot \eta + X \cdot \eta \\ &= (dX) \cdot \eta + (d^* X)\eta - \frac{5}{2} X \cdot \eta. \end{aligned}$$

Thus we get

$$D\!\!\!/(f\eta + X \cdot \eta) = \left(\frac{7}{2} f + d^* X\right)\eta + \left(\nabla f + dX - \frac{5}{2} X\right) \cdot \eta. \tag{3.2}$$

Now dX is a 2-form, hence $dX = \pi_7(dX) + \pi_{14}(dX)$. Since the Lie group G_2 preserves the nearly G_2 structure φ, it preserves the real Killing spinor η induced by φ and $\Omega^2_{14}(M) \cong \mathfrak{g}_2$, the Lie algebra of G_2, we have $\pi_{14}(dX) \cdot \eta = 0$. Also, we know from (2.42) that $\pi_7(dX) = \frac{1}{3}(\operatorname{curl} X) \lrcorner \varphi$ and it follows from the definition of the Clifford multiplication, for instance as in [14, §4.2], that $(Y \lrcorner \varphi) \cdot \eta = 3Y \cdot \eta$ for any $Y \in \Gamma(TM)$, we get that

$$\not{D}(f, X) = \Big(\frac{7}{2}f + d^*X\Big)\eta + \Big(\nabla f + \operatorname{curl} X - \frac{5}{2}X\Big) \cdot \eta$$

which we will write as

$$\not{D}(f, X) = \Big(\frac{7}{2}f + d^*X, \nabla f + \operatorname{curl} X - \frac{5}{2}X\Big). \tag{3.3}$$

Definition 3.2. The *Dirac operator* $\not{D}$ is a first-order differential operator on $\not{S}(M)$ defined as follows. Let $s = (f, X) \in \Gamma(\not{S}(M))$. Then

$$\not{D}(f, X) = \Big(\frac{7}{2}f + d^*X, \nabla f + \operatorname{curl} X - \frac{5}{2}X\Big). \tag{3.4}$$

The Dirac operator is formally self-adjoint, that is, $\not{D}^* = \not{D}$ and is also an elliptic operator.

Consider the *Dirac Laplacian* $\not{D}^2 = \not{D}^* \not{D}$. We relate it to the Hodge Laplacian in the following

Proposition 3.3. *Let $s = (f, X)$ be a section of the spinor bundle $\not{S}(M)$. Then*

$$\not{D}^2(f, X) = \Big(\Delta f + \frac{49}{4}f + d^*X,\ \Delta_d X + \operatorname{curl} X + \frac{25}{4}X + \nabla f\Big). \tag{3.5}$$

Thus $\not{D}^2$ is equal to Δ_d up to lower order terms.

Proof. Using Corollary 2.8, we calculate

$$\begin{aligned} D\!\!\!/^2(f,X) = \Big(&\frac{7}{2}\Big(\frac{7}{2}f + d^*X\Big) + d^*\Big(\nabla f + \operatorname{curl} X - \frac{5}{2}X\Big), \\ &d\Big(\frac{7}{2}f + d^*X\Big) + \operatorname{curl}\Big(\nabla f + \operatorname{curl} X - \frac{5}{2}X\Big) \\ &- \frac{5}{2}\Big(\nabla f + \operatorname{curl} X - \frac{5}{2}X\Big)\Big) \\ = \Big(&\Delta f + \frac{49}{4}f + d^*X,\ \Delta_d X + \operatorname{curl} X + \frac{25}{4}X + \nabla f\Big) \end{aligned}$$

which proves (3.5). □

We need a modification of the Dirac operator defined above. The spinor bundle $\$(M)$ is isomorphic to $\Omega^0_1 \oplus \Omega^1_7$ and hence, via a G_2-equivariant isomorphism, it is also isomorphic to $\Omega^3_1 \oplus \Omega^3_7$. We define the **modified Dirac operator**, which we denote by D, as follows. Consider the map

$$\begin{aligned} D : \Omega^0_1 \oplus \Omega^1_7 &\longrightarrow \Omega^3_1 \oplus \Omega^3_7 \\ (f, X) &\mapsto \frac{1}{2} * d(f\varphi) + \pi_{1\oplus 7}(d(X \lrcorner \varphi)). \end{aligned}$$

Using Lemma 2.11 (4) with $\tau_0 = 4$, we get

$$D(f,X) = \Big(2f - \frac{3}{7}d^*X, \frac{1}{2}df + 6X - \operatorname{curl} X\Big). \tag{3.6}$$

Remark 3.4. We note that D is defined in the same way as in [15] where the authors denote the operator by $\check{D\!\!\!/}$.

We find the kernel of D. Let $(f, X) \in \Omega^0 \oplus \Omega^1$ be in the kernel of D. Then

$$\begin{aligned} 2f - \frac{3}{7}d^*X &= 0, \\ \frac{1}{2}df + 6X - \operatorname{curl} X &= 0. \end{aligned}$$

Taking d^* of the second equation and using the first equation and equation (2.49), we get

$$\Delta f = d^*df = 2d^* \operatorname{curl} X - 12d^*X = -56f.$$

Since Δ is a non-negative operator, $f = 0$. For X, we have

$$d^*X = 0 \qquad \text{and} \qquad \operatorname{curl} X = 6X.$$

We want to prove that X is a Killing vector field. Let $dX = Y \lrcorner \varphi + \pi_{14}(dX)$. Then

$$\begin{aligned} dX \wedge \psi &= (Y \lrcorner \varphi) \wedge \psi \\ &= 3 * Y. \end{aligned}$$

Therefore

$$\pi_7(dX) = \frac{1}{3} * (dX \wedge \psi) \lrcorner \varphi = \frac{1}{3}(\operatorname{curl} X) \lrcorner \varphi = 2X \lrcorner \varphi.$$

From Lemma 2.9 (2), we have

$$\begin{aligned} \int_M dX \wedge dX \wedge \varphi &= 2\|2X \lrcorner \varphi\|^2 - \|\pi_{14}(dX)\|^2 \\ &= 8\langle X \lrcorner \varphi, X \lrcorner \varphi \rangle - \|\pi_{14}(dX)\|^2 \\ &= 8\langle X, *((X \lrcorner \varphi) \wedge \psi)\rangle - \|\pi_{14}(dX)\|^2 \\ &= 24\|X\|^2 - \|\pi_{14}(dX)\|^2. \end{aligned}$$

On the other hand, since M is compact, using integration by parts we have

$$\begin{aligned} \int_M dX \wedge dX \wedge \varphi &= \int_M X \wedge dX \wedge d\varphi \\ &= 4 \int_M X \wedge dX \wedge \psi = 4 \int_M X \wedge (6 * X) = 24\|X\|^2. \end{aligned}$$

Therefore, $\pi_{14}(dX) = 0$ and $dX = \pi_7(dX) = 2X \lrcorner \varphi$. Now using Lemma 2.11 (4), along with the fact that $X \in \ker D$, i.e., $d^*X = 0$ and $\operatorname{curl} X = 6X$, we get

$$0 = d(dX) = d(X \lrcorner \varphi) = \frac{1}{2} i_\varphi(\mathcal{L}_X g),$$

and hence X is a Killing vector field. Therefore $\ker D$ is isomorphic to the set of Killing vector fields X such that $\operatorname{curl} X = 6X$. We denote $\ker D$ by $\mathcal{K}$, that is,

$$\ker D = \mathcal{K} = \{X \in \Gamma(TM) \mid \mathcal{L}_X g = 0 \text{ and } \operatorname{curl} X = 6X\}. \tag{3.7}$$

Remark 3.5. Note that the above can also be proved using the identity $\Delta X = d^* dX = -2d^*(X \lrcorner \varphi) = 12X$, since $\mathrm{Ric}_g = 6g$ for $\tau_0 = 4$.

Remark 3.6. If we also want the vector field $X \in \mathcal{K}$ to preserve the G_2 structure, then

$$\mathcal{L}_X \varphi = d(X \lrcorner \varphi) + X \lrcorner\, d\varphi = 4X \lrcorner \psi = 0,$$

but since $\Omega^1 \cong \Omega^4_7$, this implies $X = 0$. Hence the only vector fields in $\mathcal{K}$ that preserve the G_2 structure are trivial. Note that when φ is a nearly G_2 structure of type-1, that is $\dim(K\$) = 1$, every Killing vector field preserves the G_2 structure and hence $\mathcal{K} = \{0\}$.

The motivation for defining the modified Dirac operator can be understood from the following.

Consider the following operator

$$\begin{aligned} D^+ \ :\ & \Omega^3_1 \oplus \Omega^5_7 \to \Omega^4_{1\oplus 7} \\ & (f\varphi, X \wedge \psi) \mapsto \pi_{1\oplus 7}(d(f\varphi) + d^*(X \wedge \psi)). \end{aligned}$$

From previous calculations and Lemma 2.11 we know that

$$\begin{aligned} d(f\varphi) &= df \wedge \varphi + 4f\psi \in \Omega^4_{1\oplus 7}, \\ \pi_{1\oplus 7}(d^*(X \wedge \psi)) &= \frac{3}{7}(d^* X)\psi + \frac{1}{2}\Big(\operatorname{curl} X - 6X\Big) \wedge \varphi. \end{aligned}$$

Thus

$$D^+(f\varphi, X \wedge \psi) = \Big(4f + \frac{3}{7}(d^* X), df + \frac{1}{2}\Big(\operatorname{curl} X - 6X\Big)\Big).$$

Doing a similar calculation as we did for $\ker D$, we observe that if $(f, X) \in \ker D^+$, then

$$\Delta f = -28f, \quad \operatorname{curl} X = 6X \quad \implies \quad f = 0 = d^* X \quad \text{hence} \quad X \in \mathcal{K}$$

and so $\ker D^+ = \ker D$. Since $\Omega^3_1 \oplus \Omega^5_7 \cong \Omega^4_{1\oplus 7}$ and D, D^+ are self-adjoint operators, we have the following identification

$$\begin{aligned} \Omega^4_{1\oplus 7} &= \operatorname{Im} D^+ \oplus \ker D^+ = \operatorname{Im} D^+ \oplus \ker D \\ &= d\Omega^3_1 \oplus \pi_{1\oplus 7}(d^*\Omega^5_7) \oplus \{X \wedge \varphi | X \in \mathcal{K}\}. \end{aligned} \tag{3.8}$$

This is used in the following important

Proposition 3.7. *Let (M, φ, ψ) be a nearly* G_2 *manifold. Then the following holds.*

1) $\Omega^4 = \{X \wedge \varphi | X \in \mathcal{K}\} \oplus d\Omega^3_1 \oplus d^*\Omega^5_7 \oplus \Omega^4_{27}$.
2) *We have an L^2-orthogonal decomposition* $\Omega^4_{\mathrm{exact}} = \{X \wedge \varphi \mid X \in \mathcal{K}\} \oplus d\Omega^3_1 \oplus \Omega^4_{27,\mathrm{exact}}$.

Proof. The first part of the proposition follows from the decomposition of $\Omega^4_{1\oplus 7}$ in equation (3.8).

For the second part we note that the space $d^*\Omega^5_7$ is L^2-orthogonal to exact 4-forms. To prove the L^2-orthogonality of the remaining summands we proceed term by term. Let $X \in \mathcal{K}$, $d(f\varphi) \in d\Omega^3_1$ and $\gamma \in \Omega^4_{27}$, such that $d\alpha = X \wedge \varphi + d(f\varphi) + \beta$ for some exact 4-form $d\alpha$. Using the pointwise orthogonality of Ω^4_1 and Ω^4_7, we have

$$\begin{aligned}\langle X \wedge \varphi, d(f\varphi)\rangle_{L^2} &= \langle X \wedge \varphi, df \wedge \varphi + 4f\psi\rangle_{L^2} \\ &= \langle X \wedge \varphi, df \wedge \varphi\rangle_{L^2} \\ &= 4\langle X, df\rangle_{L^2} = 4\langle d^*X, f\rangle_{L^2} = 0.\end{aligned}$$

Note that since $X \in \mathcal{K}$, Lemma 2.11 (6) implies that $X \wedge \varphi = d\left(-\frac{1}{4}X \lrcorner \psi\right)$, and hence is exact. Thus, $\beta \in \Omega^4_{27,\mathrm{exact}}$. Let $\beta = d\alpha_0$. The L^2-orthogonality of Ω^4_{27} and Ω^4_1, along with the identity $\varphi \wedge * d\alpha = 0$ implies

$$\begin{aligned}\langle d\alpha_0, d(f\varphi)\rangle_{L^2} &= \langle d\alpha_0, df \wedge \varphi + 4f\psi\rangle_{L^2} \\ &= \langle d\alpha_0, df \wedge \varphi\rangle_{L^2} + \langle d\alpha_0, 4f\psi\rangle_{L^2} = 0.\end{aligned}$$

The orthogonality of $X \wedge \varphi$ and $d\alpha_0$ follows from the L^2-orthogonality of Ω^4_7 and Ω^4_{27}. ☐

Thus, from the previous proposition, we know that any 4-form α on a nearly G_2 manifold can be written as $\alpha = X \wedge \varphi + d(f\varphi) + d^*(Y \wedge \psi) + \alpha_0$, for some $X \in \mathcal{K}$, $f \in C^\infty(M), Y \in \Gamma(TM)$ and $\alpha_0 \in \Omega^4_{27}$. Since for $Y \in \mathcal{K}$, $d^*(Y \wedge \psi) = 0$, one can choose $Y \in \mathcal{K}^{\perp_{L^2}}$ in the previous proposition.

Thus for every 4-form α there exists unique $X \in \mathcal{K}$, $Y \in \mathcal{K}^{\perp_{L^2}}$, $f \in C^\infty(M)$ and $\alpha_0 \in \Omega^4_{27}$ such that

$$\alpha = X \wedge \varphi + d(f\varphi) + d^*(Y \wedge \psi) + \alpha_0.$$

3.2. Harmonic 2-forms and 3-forms on nearly $\mathbf{G_2}$ manifolds

The above decomposition of 4-forms has a very useful application in determining the cohomology of nearly G_2 manifolds. We first note that since nearly G_2 manifolds are positive Einstein, it follows from Bochner formula and Hodge theory that any harmonic 1-form is 0 and hence $\mathcal{H}^1(M) = \mathcal{H}^6(M) = 0$. The next two theorems describe the degree 3, 4 and degree 2 and 5 cohomology of a nearly G_2 manifold.

Theorem 3.8. *Let (M, φ, ψ) be a complete nearly G_2 manifold. Then every harmonic 4-form lies in Ω^4_{27}. Equivalently, every harmonic 3-form lies in Ω^3_{27}.*

Proof. Let α be a harmonic 4-form that is $d\alpha = d^*\alpha = 0$. From Proposition 3.7 there exists $X \in \mathcal{K}$, $f \in C^\infty(M)$, $Y \in \mathcal{K}^{\perp_{L^2}}$ and $\alpha_0 \in \Omega^4_{27}$ such that

$$\alpha = X \wedge \varphi + d(f\varphi) + d^*(Y \wedge \psi) + \alpha_0.$$

Since $X \in \mathcal{K}$ and hence $6X = \operatorname{curl} X$, by Lemma 2.11 (6), $d^*(X \wedge \varphi) = 4X \lrcorner \psi \in \Omega^3_7$ and since $d(f\varphi) = df \wedge \varphi + 4f\psi \in \Omega^4_{1\oplus 7}$, we have

$$\begin{aligned} 0 = \langle \alpha, d(f\varphi)\rangle_{L^2} &= \langle X \wedge \varphi, d(f\varphi)\rangle_{L^2} + \|d(f\varphi)\|^2_{L^2} \\ &\quad + \langle d^*(Y \wedge \psi), d(f\varphi)\rangle_{L^2} + \langle \alpha_0, d(f\varphi)\rangle_{L^2} \\ &= \langle d^*(X \wedge \varphi), f\varphi\rangle_{L^2} + \|d(f\varphi)\|^2_{L^2} \\ &= \|d(f\varphi)\|^2_{L^2}. \end{aligned}$$

Thus $d(f\varphi) = 0$ and hence $f = 0$.

Now, $0 = d^*\alpha = d^*(X \wedge \varphi) + d^*\alpha_0 = 4X \lrcorner \psi + d^*\alpha_0$. Using the identity, $(X \lrcorner \psi) \wedge \varphi = 4 * X$ we have

$$\begin{aligned} \|d^*\alpha_0\|^2_{L^2} &= 16\langle X \lrcorner \psi, X \lrcorner \psi\rangle_{L^2} \\ &= 16\langle X, *((X \lrcorner \psi) \wedge \varphi)\rangle_{L^2} = 64\|X\|^2_{L^2}. \end{aligned}$$

On the other hand, again by Lemma 2.11 (6)

$$\begin{aligned} \|d^*\alpha_0\|^2_{L^2} &= \langle d^*\alpha_0, d^*\alpha_0\rangle_{L^2} \\ &= -4\langle d^*\alpha_0, X \lrcorner \psi\rangle_{L^2} \\ &= -4\langle \alpha_0, d(X \lrcorner \psi)\rangle_{L^2} = 16\langle \alpha_0, X \wedge \varphi\rangle_{L^2} = 0, \end{aligned}$$

which implies $X = 0$. So $\alpha = d^*(Y \wedge \psi) + \alpha_0$.

Since $d^*\alpha_0 = 0$, applying Corollary 2.17 on α_0 implies $d\alpha_0 \in \Omega^5_{14}$. This identity together with the closedness of α gives us

$$\begin{aligned} 0 = \langle \alpha, d^*(Y \wedge \psi)\rangle_{L^2} &= \|d^*(Y \wedge \psi)\|^2_{L^2} + \langle \alpha_0, d^*(Y \wedge \psi)\rangle_{L^2} \\ &= \|d^*(Y \wedge \psi)\|^2_{L^2} + \langle d\alpha_0, Y \wedge \psi\rangle_{L^2} = \|d^*(Y \wedge \psi)\|^2_{L^2}. \end{aligned}$$

as $Y \wedge \psi \in \Omega^5_7$. Hence $d^*(Y \wedge \psi) = 0$ or equivalently $Y \in \mathcal{K}$, thus $Y = 0$ which implies that $\alpha = \alpha_0$ which completes the proof of the theorem. □

We also describe the degree 2 (and hence degree 5) cohomology on nearly G_2 manifolds below. In combination with Theorem 3.8, this completely describes the cohomology of a nearly G_2 manifold.

Theorem 3.9. *Let (M, φ, ψ) be a complete nearly G_2 manifold with $\tau_0 = 4$. Let β be a 2-form with*

$$\beta = \beta_7 + \beta_{14} = (X \lrcorner \varphi) + \beta_{14} \quad \text{for some } X \in \Gamma(TM).$$

If β is harmonic then $\beta \in \Omega^2_{14}$.

Proof. Suppose $\beta \in \Omega^2(M)$ is harmonic. Then $d\beta = d^*\beta = 0$ and since d and d^* are linear, we have

$$d\beta_7 + d\beta_{14} = 0, \qquad d^*\beta_7 + d^*\beta_{14} = 0$$

which on using Lemma 2.11 (3), (4) and (5) imply

$$\begin{aligned} -\frac{3}{7}(d^*X)\varphi &+ \frac{1}{2} * ((6X - \operatorname{curl} X) \wedge \varphi) \\ &+ i_\varphi\Big(\frac{1}{2}(\mathcal{L}_X g) + \frac{1}{7}(d^*X)g\Big) + \frac{1}{4} * (d^*\beta_{14} \wedge \varphi) + \pi_{27}(d\beta_{14}) = 0 \end{aligned}$$

and

$$d^*\beta_{14} = -\operatorname{curl} X.$$

Thus we get

$$\begin{aligned} -\frac{3}{7}(d^*X)\varphi &+ \frac{1}{2} * ((6X - \operatorname{curl} X - \frac{1}{2}\operatorname{curl} X) \wedge \varphi) \\ &+ i_\varphi\Big(\frac{1}{2}(\mathcal{L}_X g) + \frac{1}{7}(d^*X)g\Big) + \pi_{27}(d\beta_{14}) = 0 \end{aligned}$$

and so

$$d^*X = 0, \quad \operatorname{curl} X = 4X \quad \text{and} \quad \frac{1}{2}(\mathcal{L}_X g) + \pi_{27}(d\beta_{14}) = 0. \tag{3.9}$$

Now $\operatorname{curl} X = 4X$, so taking curl of both sides and using (2.51) with $d^*X = 0$, we get

$$\Delta_d X + 4 \operatorname{curl} X = 4 \operatorname{curl} X \quad \Longrightarrow \quad \Delta_d X = 0.$$

Thus X is harmonic. Since nearly G_2 manifolds are positive Einstein, it follows from Bochner formula and Myers theorem that $X = 0$. Hence $\beta = \beta_{14} \in \Omega^2_{14}$. □

Remark 3.10. Theorem 3.9 was also proved in a very different way in [2, Remark 15]. The theorem has the following interesting interpretation in the context of G_2-instantons on a nearly G_2 manifold, as already described in [2, Corollary 14]. For any $\alpha \in H^2(M, \mathbb{Z})$, by Theorem 3.9, there is a unique G_2-instanton on a complex line bundle L with $c_1(L) = \alpha$.

Remark 3.11. It was brought to the attention of the authors by Uwe Semmelmann and Paul-Andi Nagy that Theorem 3.8 also follows from the description of nearly G_2 manifolds using Killing spinors which is based on an old result of Hijazi saying that the Clifford product of a harmonic form and a Killing spinor vanishes. We also describe degree 2 cohomology by our methods. We believe that the methods and the identities described here, apart from being useful in other contexts, also have the potential to be extended to manifolds with *any* G_2 structure (not necessarily nearly G_2) with suitable modifications. The authors are currently investigating this.

4. Deformations of nearly G_2 structures

Let (M, φ, ψ) be a nearly G_2 manifold with a nearly G_2 structure (φ, ψ). We are interested in studying the deformation problem of (φ, ψ) in the space of nearly G_2 structures. The infinitesimal version of this problem was settled by Alexandrov and Semmelmann in [1]. We will obtain new proofs of some of their results using the results proved in the previous sections.

Let $\mathcal{P}$ be the space of G_2 structures on M, that is, the set of all $(\varphi, \psi) \in \Omega^3_+ \times \Omega^4_+$ with $\Theta(\varphi) = \psi$. Given a point $\mathfrak{p} = (\varphi, \psi) \in \mathcal{P}$ we define the tangent space $T_{\mathfrak{p}}\mathcal{P}$.

Lemma 4.1. *The tangent space $T_{\mathfrak{p}}\mathcal{P}$ is the set of all $(\xi, \eta) \in \Omega^3(M) \times \Omega^4(M)$ such that*

$$\xi = 3f\varphi - X \lrcorner \psi + \gamma$$
$$\eta = 4f\psi + X \wedge \varphi - *\gamma$$

for some $f \in \Omega^0(M)$, $X \in \Gamma(TM)$ and $\gamma \in \Omega^3_{27}$.

Proof. The proof immediately follows from equations (2.39) and (2.40) from Proposition 2.4. □

4.1. Infinitesimal deformations

We want to study deformations of a given nearly G_2 structure φ on a compact manifold M by nearly G_2 structures φ_t. We will only be interested in deformations of the nearly G_2 structures modulo the action of the group $\mathbb{R}^* \times \mathrm{Diff}_0(M)$ where $\mathrm{Diff}_0(M)$ denotes the space of diffeomorphisms of M which are isotopic to the identity. We first use Proposition 3.7 to find a slice for the action of diffeomorphism group on $\mathcal{P}$ which is used to find the space of infinitesimal nearly G_2 deformations, a result originally due Alexandrov–Semmelmann [1].

For the purposes of doing analysis, we consider the Hölder space $\mathcal{P}^{k,\alpha}$ of G_2 structures on M such that φ and ψ are of class $C^{k,\alpha}$, $k \geq 1$, $\alpha \in (0,1)$. Let $\mathfrak{p} = (\varphi, \psi) \in \mathcal{P}^{k,\alpha}$ be a nearly G_2 structure such that the induced metric is not isometric to round S^7. Denote the orbit of $\mathfrak{p}$ under the action of $\mathrm{Diff}_0^{k+1,\alpha}(M)$ – $C^{k+1,\alpha}$ diffeomorphisms isotopic to the identity, by $\mathcal{O}_{\mathfrak{p}}$. The tangent space $T_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}$ is the space of Lie derivatives $\mathcal{L}_X(\varphi, \psi)$ for $X \in \Gamma(TM)$. We are interested in finding a complement $\mathcal{C}$ of $T_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}$ in $T_{\mathfrak{p}}\mathcal{P}$.

If $(\xi, \eta) \in T_{\mathfrak{p}}\mathcal{P}$ then using Proposition 3.7 (1), we can write

$$\eta = X \wedge \varphi + df \wedge \varphi + 4f\psi + d^*(Y \wedge \psi) + \eta_0$$

for unique $X \in \mathcal{K}$ $f \in \Omega^0(M)$, $Y \in \mathcal{K}^{\perp_{L^2}}$ and $\eta_0 \in \Omega^4_{27}$. From Lemma 2.11 (4) we know that

$$\begin{aligned} *d*(Y \wedge \psi) = -*d(Y \lrcorner \varphi) &= \frac{3}{7}(d^*Y)\psi - (3Y - \frac{1}{2}\operatorname{curl} Y) \wedge \varphi \\ &\quad - *i_\varphi\Big(\frac{1}{2}(\nabla_i Y_j + \nabla_j Y_i) + \frac{1}{7}(d^*Y)g_{ij}\Big) \end{aligned}$$

and since

$$\mathcal{L}_Y\psi = d(Y\lrcorner\psi) = -\frac{4}{7}d^*Y\psi - \Big(\frac{1}{2}\operatorname{curl} Y + Y\Big)\wedge\varphi - *i_\varphi\Big(\frac{1}{2}(\nabla_i Y_j + \nabla_j Y_j) + \frac{1}{7}(d^*Y)g_{ij}\Big)$$

from Lemma 2.11 (6), we see that

$$d^*(Y\wedge\psi) = -\frac{1}{7}(d^*Y)\psi + (\operatorname{curl} Y - 2Y)\wedge\varphi - \mathcal{L}_Y\psi.$$

Thus up to an element in $T_\mathfrak{p}\mathcal{O}_\mathfrak{p}$ we get that

$$\eta = \Big(4f - \frac{1}{7}d^*Y\Big)\psi + (X + df + \operatorname{curl} Y - 2Y)\wedge\varphi + \eta_0 \tag{4.1}$$

and hence from Lemma 4.1

$$\xi = (3f - \frac{3}{28}d^*Y)\varphi - (X + df + \operatorname{curl} Y - 2Y)\lrcorner\psi - *\eta_0. \tag{4.2}$$

Now, if $X \in \mathcal{K}$ then from Lemma 2.11 (6) and $\operatorname{curl} X = 6X$ we see that

$$\mathcal{L}_{-\frac{X}{4}}\psi = d\left(-\frac{X}{4}\lrcorner\psi\right) = X\wedge\varphi$$

and hence

$$\eta = \mathcal{L}_{-\frac{X}{4}}\psi + d(f\varphi) + d^*(Y\wedge\psi) + \eta_0$$

which implies that up to an element in $T_\mathfrak{p}\mathcal{O}_\mathfrak{p}$ combined with the above observation, we can write

$$\eta = \Big(4f - \frac{1}{7}d^*Y\Big)\psi + (df + \operatorname{curl} Y - 2Y)\wedge\varphi + \eta_0 \tag{4.3}$$

which implies that

$$\xi = (3f - \frac{3}{28}d^*Y)\varphi - (df + \operatorname{curl} Y - 2Y)\lrcorner\psi - *\eta_0 \tag{4.4}$$

and hence we get a splitting $T_\mathfrak{p}\mathcal{P} = T_\mathfrak{p}\mathcal{O}_\mathfrak{p} \oplus \mathcal{C}$ where $\mathcal{C} \cong \Omega^0(M)\times\mathcal{K}^{\perp_{L^2}}\times\Omega^4_{27}$ which consists of $(\xi,\eta)\in T_\mathfrak{p}\mathcal{P}$ of the form (4.4) and (4.3) respectively. This gives a choice of slice. In fact, as discussed in [20, pg. 49 & Theorem 3.1.4] we have

Proposition 4.2. *There exists an open neighbourhood U of $\mathcal{C}$ of the origin such that the "exponentiation" of U is a slice for the action of* $\mathrm{Diff}_0^{k+1,\alpha}(M)$ *on a sufficiently small neighbourhood of $\mathfrak{p} \in \mathcal{P}^{k,\alpha}$.*

With this choice of slice we determine the infinitesimal deformations of the nearly G_2 structure $\mathfrak{p}$ which gives a new proof of a result of Alexandrov–Semmelmann [1, Theorem 3.5].

Theorem 4.3. *Let (M, φ, ψ) be a complete nearly* G_2 *manifold, not isometric to the round S^7. Then the infinitesimal deformations of the nearly* G_2 *structure are in one to one correspondence with $(X, \xi_0) \in \mathcal{K} \times \Omega^3_{27}$ with*

$$*d\xi_0 = -4\xi_0 \qquad \text{and} \qquad \Delta X = 12X. \tag{4.5}$$

Hence ξ_0 is co-closed as well. Moreover, $\Delta_d \xi_0 = 16\xi_0$.

Proof. Let $(\xi, \eta) \in T_{\mathfrak{p}}\mathcal{P}$ be an infinitesimal nearly G_2 deformation of a G_2 structure $\mathfrak{p} \in \mathcal{P}$. So η must be exact and hence from Proposition 3.7 (2), we can remove the $d^*(Y \wedge \psi)$ term, in which case (4.1) and (4.2) become

$$\eta = 4f\psi + (X + df) \wedge \varphi + \eta_0 \quad \text{and} \quad \xi = 3f\varphi - (X + df) \lrcorner \psi - *\eta_0. \tag{4.6}$$

Moreover, for infinitesimal nearly G_2 deformations we must have

$$d\xi = 4\eta$$

and hence (4.6) implies

$$4f\psi + (4X + df) \wedge \varphi + 4\eta_0 + d((X + df) \lrcorner \psi) + d * \eta_0 = 0.$$

Using Lemma 2.11 (6) for the fourth term above and taking inner product with ψ gives

$$28f - 4d^*(X + df) = 0.$$

But since $X \in \mathcal{K} \implies d^*X = 0$ and hence we get $\Delta f = 7f$. Since M is not isometric to round S^7, Obata's theorem then implies that $f = 0$ and hence

$$\eta = X \wedge \varphi + \eta_0 \qquad \text{and} \qquad \xi = -X \lrcorner \psi - *\eta_0 \tag{4.7}$$

which proves the one to one correspondence between the infinitesimal nearly G_2 deformations and $\mathcal{K} \times \Omega^3_{27}$. Since $\mathrm{Ric} = 6g$ and X is a Killing vector field, we have $\Delta X = 12X$ which is the second part of (4.5). Since η_0 is exact, $d\eta_0 = 0$. From (4.7) and the fact that $d\xi = 4\eta$, we get

$$d * \eta_0 = -4\eta_0$$

and hence

$$*d\xi_0 = -4\xi_0.$$

Taking d^* of both sides give $d^*\xi_0 = 0$. Moreover,

$$\Delta_d \xi_0 = d^* d\xi_0 = -4d^* * \xi_0 = -4 * (d\xi_0) = 16\xi_0$$

which completes the proof of the theorem. □

Remark 4.4. From the computations for the proof of Proposition 4.2 we know that for $X \in \mathcal{K}$,

$$-4X \wedge \varphi = \mathcal{L}_X \psi.$$

Thus, from Theorem 4.3 we see that the infinitesimal deformations of a nearly G_2 structure *modulo diffeomorphisms* are in one-to-one correspondence with $\xi_0 \in \Omega^3_{27}$ such that $*d\xi_0 = -4\xi_0$.

Motivated from the study of deformations of nearly Kähler 6-manifolds by Foscolo [7, §4] where he used observations of Hitchin [10], we also want to interpret the nearly G_2 condition (2.24) as the vanishing of a smooth map on the space of exact positive 4-forms. Moreover, in order to study the second order deformations, it will be convenient to enlarge the space by introducing a vector field as an additional parameter which is natural when one considers the action of the diffeomorphism group. We elaborate on this below.

Let $\psi = d\alpha$ be an exact positive 4-form, not necessarily satisfying the nearly G_2 condition. Let $\eta \in \Omega^4_{\text{exact}}$ be the first order deformation of ψ. Hitchin in [10] defined a volume functional for exact 4-form $\rho = d\gamma$ given by

$$V(\rho) = \int_M *\rho \wedge \rho,$$

and a quadratic form

$$W(\rho, \rho') = \int_M \gamma \wedge \rho' = \int_M \rho \wedge \gamma',$$

where $\rho = d\gamma$ and $\rho' = d\gamma'$ are exact 4-forms. We denote $W(\rho,\rho)$ by $W(\rho)$. When M is compact, Hitchin proves [10, Theorem 5] that stable 4-forms (which is the same as a positive 4-form in our case) $\eta \in \Omega^4_{\text{exact}}(M)$ is a critical point of the volume functional V subject to the constraint $W(\eta) = $ constant if and only if η defines a nearly G_2 structure. The linearization of the volume functional at ψ is given by

$$dV(\eta) = \frac{d}{dt}\bigg|_{t=0} V(\psi + t\eta) = \int_M \varphi \wedge \eta + \int_M *\eta \wedge \psi$$
$$= 2\int_M \varphi \wedge \eta.$$

For the linearization of the quadratic form, suppose ψ is exact with $\psi = d\alpha$. We use integration by parts to get

$$dW(\eta) = \frac{d}{dt}\bigg|_{t=0} W(\psi + t\eta) = \int_M \alpha \wedge \eta + \int_M \gamma \wedge \psi$$
$$= 2\int_M \alpha \wedge \eta.$$

Let us define an energy functional E on exact 4-forms by

$$E(\rho) := V(\rho) - 4W(\rho).$$

Then from above calculations

$$dE(\eta) = \int_M (\varphi - 4\alpha) \wedge \eta = \int_M d((\varphi - 4\alpha) \wedge \gamma.$$

Therefore $\psi = d\alpha$ is a critical point of E if and only if $dE(\eta) = 0$ for every $\eta \in \Omega^4_{\text{exact}}$ that is if and only if

$$d\varphi - 4d\alpha = d\varphi - 4\psi = 0.$$

Hence the critical points of the functional E on $\Omega^4_{+,\text{exact}}$ are nearly G_2 structures. Since the energy functional E is diffeomorphism invariant, we can introduce an extra vector field, as dE will vanish in the direction of Lie derivatives. Thus ψ being a stable exact 4-form can be given by the formula

$$\psi = \frac{1}{4} d(\varphi - *d(Z \lrcorner \psi))$$

for some $Z \in \Gamma(TM)$. We use these observations to write the nearly G_2 condition (2.24) as the vanishing of a smooth map. Let us denote by $\widehat{P}$ the space of stable 3 and stable, exact 4-forms, i.e., $(\varphi, \psi) \in \Omega^3_+ \times \Omega^4_{+,\text{exact}}$. We have the following

Proposition 4.5. *Suppose $(\varphi, \psi) \in \widehat{\mathcal{P}}$ satisfies*

$$d\varphi - 4\psi = d * d(Z \lrcorner \psi) \tag{4.8}$$

for some vector field Z and $*$ denotes the Hodge star with respect to a fixed background metric. Then (φ, ψ) is a nearly G_2 structure.

Proof. We will prove that $d(Z \lrcorner \psi) = 0$. We note from (2.32) that

$$(Z \lrcorner \psi) \wedge \psi = 0$$

So from (4.8) we get that

$$\begin{aligned} \|d(Z \lrcorner \psi)\|^2_{L^2} &= \langle d(Z \lrcorner \psi), d(Z \lrcorner \psi) \rangle_{L^2} \\ &= \langle (Z \lrcorner \psi), *d * d(Z \lrcorner \psi) \rangle_{L^2} \\ &= \langle (Z \lrcorner \psi), *(d\varphi - 4\psi) \rangle_{L^2} \\ &= \int_M (Z \lrcorner \psi) \wedge (d\varphi - 4\psi) = \int_M (Z \lrcorner \psi) \wedge d\varphi \end{aligned}$$

Since φ is a G_2 structure and $d\psi = 0$ from (4.8), we know from (2.19) that $\tau_1 = 0$ and hence $d\varphi$ has no component in Ω^4_7. Thus

$$\langle (Z \lrcorner \psi), *d\varphi \rangle = 0$$

which implies that

$$\|d(Z \lrcorner \psi)\|^2_{L^2} = \int_M (Z \lrcorner \psi) \wedge d\varphi = 0$$

which proves the proposition. □

Suppose we want to describe the local moduli space of nearly G_2 structures on a manifold M. If $\mathcal{NP}$ denotes the space of nearly G_2 structures on M then the local moduli space is $\mathcal{M} = \mathcal{NP}/\text{Diff}_0(M)$. A natural way to study this problem is to view the nearly G_2 structures on M as the zero locus of an appropriate function, find the linearization of the function and prove its surjectivity, so that an Implicit Function Theorem argument describes $\mathcal{M}$.

Now let (φ, ψ) be a nearly G_2 structure on M. Let $U \subset \Omega^4_{+,\text{exact}}$ be a small neighborhood of the 4-form ψ. Since the condition of being stable is open we can assume the existence of such a neighborhood. Thus for $\eta \in \Omega^4_{\text{exact}}$ with sufficiently small norm with respect to the metric induced by φ, $\tilde{\psi} = \psi + \eta$ is also a stable exact 4-form. From Proposition 4.5 the pair of stable forms $(\tilde{\varphi}, \tilde{\psi})$ defines a nearly G_2 structure if there exists a $Z \in \Gamma(TM)$ such that

$$d\tilde{\varphi} - 4\tilde{\psi} = d * d(Z \lrcorner \tilde{\psi}).$$

This condition is equivalent to the vanishing of the map

$$\begin{aligned} \Phi : U \times \Gamma(TM) &\to \Omega^4_{\text{exact}} \\ (\tilde{\psi}, Z) &\mapsto d * \tilde{\psi} - 4\tilde{\psi} - d * d(Z \lrcorner \tilde{\psi}). \end{aligned} \tag{4.9}$$

Thus, the nearly G_2 structures are the zero locus of the map Φ modulo diffeomorphisms.

Let ξ be the dual of η under the Hitchin's duality map Θ as in Proposition 2.4. The linearization of the map Φ at the point $(\psi, 0)$ is given by

$$d\xi - 4\eta = d * d(Z \lrcorner \psi).$$

Thus the obstructions on the first order deformations of the nearly G_2 structure (φ, ψ) are given by $\mathrm{Im}(D\Phi)$ which is characterized in the following proposition, whose proof is inspired from a similar theorem in the nearly Kähler case by Foscolo [7, Proposition 4.5].

Proposition 4.6. *Let (φ, ψ) be a nearly G_2 structure and $(\xi, \eta) \in \Omega^3 \times \Omega^4_{\text{exact}}$ be a first order deformation in $\mathcal{P}$. Then $\alpha \in \Omega^4_{\text{exact}}$ lies in the image of $D\Phi$ if and only if*

$$\langle d^*\alpha - 4 * \alpha, \chi \rangle_{L^2} = 0$$

for all co-closed $\chi \in \Omega^3_{27}$ such that $\Delta\chi = 16\chi$.

Proof. From Proposition 3.7 (2), there exists $X \in \mathcal{K}, f \in C^\infty(M)$ and $\eta_0 \in \Omega^4_{27,\text{exact}}$ such that

$$\begin{aligned} \eta &= X \wedge \varphi + d(f\varphi) + \eta_0 \\ &= d\left(-\frac{1}{4} X \lrcorner \psi + f\varphi\right) + \eta_0 \end{aligned}$$

and from Lemma 4.1, the 3-form

$$\xi = 3f\varphi - (df + X) \lrcorner \psi - *\eta_0.$$

By Proposition 3.7, $\alpha = Y \wedge \varphi + d(h\varphi) + \alpha_0$ for some $Y \in \mathcal{K}, h \in C^\infty(M), \alpha_0 \in \Omega^4_{27,\mathrm{exact}}$. Such an α lies in the image of $D\Phi$ if

$$d\xi - 4\eta - d * d(Z \lrcorner \psi) = \alpha = d\Big(-\frac{1}{4} Y \lrcorner \psi + h\varphi \Big) + \alpha_0.$$

From Lemma 2.11 (5)

$$\begin{aligned} d^*(Z \wedge \psi) &= - * d(Z \lrcorner \varphi) \\ &= \frac{3}{7}(d^* Z)\psi - \frac{1}{2}\Big(6Z - \operatorname{curl} Z\Big) \wedge \varphi \\ &\quad - * i_\varphi \Big(\frac{1}{2}(\nabla_i Z_j + \nabla_j Z_i) + \frac{1}{7}(d^* Z) g_{ij} \Big) \end{aligned}$$

Comparing the last term in the above expression with that of $d(Z \lrcorner \psi)$ in Lemma 2.11 we get

$$d(Z \lrcorner \psi) = \frac{1}{7} d^* Z \psi + (2Z - \operatorname{curl} Z) \wedge \varphi + d^*(Z \wedge \psi).$$

Using these expressions for ξ, η and $d(Z \lrcorner \psi)$ we get

$$\begin{aligned} d\xi - 4\eta - d * d(Z \lrcorner \psi) = d((-f - \frac{1}{7} d^* Z)\varphi - (df - 2Z + \operatorname{curl} Z) \lrcorner \psi) \\ - d * \eta_0 - 4\eta_0. \end{aligned}$$

Thus, for finding the $\mathrm{Im}(D\Phi)$, we need to solve the equations

$$\begin{aligned} f + \frac{1}{7} d^* Z &= -h \\ df - 2Z + \operatorname{curl} Z &= \frac{1}{4} Y \\ -d * \eta_0 - 4\eta_0 &= \alpha_0. \end{aligned} \tag{4.10}$$

Let $\alpha_0 = 0$. Then by Implicit Function Theorem, a solution of the first pair of equations always exist if the operator

$$\begin{aligned} \tilde{D} \, : \, \Omega^0 \times \Omega^1 &\to \Omega^0 \times \Omega^1 \\ (f, Z) &\mapsto \Big(f + \frac{1}{7} d^* Z, df - 2Z + \operatorname{curl} Z \Big) \end{aligned}$$

is invertible in a small neighborhood of its zero locus. Since $\tilde{D}$ differs from the modified Dirac operator D in (3.6) only by self-adjoint zeroth-order term, it is self-adjoint and hence $\ker(\tilde{D}) = \operatorname{coker}(\tilde{D})$. A pair (f, Z) is in the kernel of the operator D if and only if

$$f + \frac{1}{7}d^*Z = 0$$
$$df - 2Z + \operatorname{curl} Z = 0.$$

Applying the operator d^* on the second equation and using the fact that $d^*(\operatorname{curl} Z) = 0$ gives

$$0 = d^*df - 2d^*Z = d^*df + 14f.$$

Thus $f = 0$ as Δ is a non-negative operator. The second equation then becomes

$$\operatorname{curl} Z = d^*(Z \lrcorner \varphi) = *(dZ \wedge \psi) = 2Z$$

and Proposition 2.5 implies that $dZ = \frac{2}{3} Z \lrcorner \varphi + \pi_{14}(dZ)$. Using Lemma 2.9 (2) we get that

$$\int_M dZ \wedge dZ \wedge \varphi = \frac{8}{9}\|Z \lrcorner \varphi\|^2 - \|\pi_{14}(dZ)\|^2$$
$$= \frac{8}{3}\|Z\|^2 - \|\pi_{14}(dZ)\|^2.$$

On the other hand

$$\int_M dZ \wedge dZ \wedge \varphi = 4 \int_M Z \wedge dZ \wedge \psi = 8\|Z\|^2.$$

Combining these two equations we get $\frac{16}{3}\|Z\|^2 = -\|\pi_{14}(dZ)\|^2$ and hence $Z = 0$ as well. Thus $\ker(\tilde{D}) = \operatorname{coker}(\tilde{D}) = 0$ and $\tilde{D}$ is invertible when $\alpha_0 = 0$ and we can always solve the first pair of equations in (4.10). Thus there are no restrictions on Y, h to be in the image of $D\Phi$. Moreover if $\alpha_0 \neq 0$ satisfies the third equation in (4.10) then

$$d^*\alpha_0 = -d^*d * \eta_0 - 4d^*\eta_0,$$
$$*\alpha_0 = -d^*\eta_0 - 4 * \eta_0$$

which on using the fact that $*\eta_0$ is co-closed implies

$$d^*\alpha_0 - 4 * \alpha_0 = 16 * \eta_0 - d^*d * \eta_0 = 16 * \eta_0 - \Delta_d * \eta_0.$$

Thus $\alpha_0 \in \Omega^4_{27,\mathrm{exact}}$ is a solution to the equation (4.10) (3) if and only if

$$\langle d^*\alpha_0 - 4 * \alpha_0, \xi_0 \rangle_{L^2} = 0$$

for all co-closed $\xi_0 \in \Omega^3_{27}$ such that $\Delta\xi = 16\xi$. To complete the proof of the proposition we now only need to prove the L^2-orthogonality condition for α. But observe that since $Y \in \mathcal{K}$

$$d^*\alpha = d^*(Y \wedge \varphi) + d^*d(h\varphi) + d^*\alpha_0 = -4Y \lrcorner \psi + d^*d(h\varphi) + d^*\alpha_0,$$

and so $d^*\alpha - 4 * \alpha = d^*d(h\varphi) - 4 * d(h\varphi) + d^*\alpha_0 - 4 * \alpha_0$. Since ξ is co-closed, from Corollary 2.15 $d\xi \in \Omega^4_{27}$ and

$$\langle d^*d(h\varphi), \xi \rangle_{L^2} = \langle d(h\varphi), d\xi \rangle_{L^2} = 0.$$

Similarly

$$\langle *d(h\varphi), \xi \rangle_{L^2} = \langle d^*(h\psi), \xi \rangle_{L^2} = \langle h\psi, d\xi \rangle_{L^2} = 0$$

which completes the proof of the proposition. □

Remark 4.7. Proposition 4.6 puts a very strong restriction on the first order deformations of a nearly G_2 structure to be unobstructed.

4.2. Second-order deformations

Following the work of Koiso [16] on deformations of Einstein metrics and the work of Foscolo [7] on the second order deformations of nearly Kähler structures on 6-manifolds, we define the notion of second order deformations of nearly G_2 structures.

Definition 4.8. Given a nearly G_2 structure (φ_0, ψ_0) and an infinitesimal deformation (ξ_1, η_1), a second order deformation of (φ_0, ψ_0) in the direction of (ξ_1, η_1) is a pair $(\xi_2, \eta_2) \in \Omega^3 \times \Omega^4$ such that

$$\varphi = \varphi_0 + \epsilon\xi_1 + \frac{\epsilon^2}{2}\xi_2, \qquad \psi = \psi_0 + \epsilon\eta_1 + \frac{\epsilon^2}{2}\eta_2$$

is a nearly G_2 structure up to terms of order $O(\epsilon^2)$. An infinitesimal deformation (ξ_1, η_1) is said to be *obstructed to second order* if there exists no second-order deformation in its direction.

Remark 4.9. Second order deformations are the same as the second derivative of a curve of nearly G_2 structures on a manifold M.

Remark 4.10. In a similar way, we can define higher order deformations of a nearly G_2 structure.

Following the discussion in the previous section and in particular Proposition 4.5, in order to find second order deformations of a given nearly G_2 structure (φ_0, ψ_0), we look for formal power series defining positive *exact* 4-form

$$\psi_\epsilon = \psi_0 + \epsilon\eta_1 + \frac{\epsilon^2}{2}\eta_2 + \cdots$$

where $\eta_i \in \Omega^4_{\text{exact}}$ and a vector field

$$Z_\epsilon = \epsilon Z_1 + \frac{\epsilon^2}{2} Z_2 + \cdots$$

which satisfy (4.8), that is

$$d\varphi_\epsilon - 4\psi_\epsilon = d * d(Z_\epsilon \lrcorner \psi_\epsilon) \tag{4.11}$$

where φ_ϵ is the dual of ψ_ϵ. Note that the Hodge star $*$ is taken with respect to φ_ϵ.

Since we are interested in second order deformations, given an infinitesimal nearly G_2 deformation (ξ_1, η_1), we set $Z_1 = 0$ and look for $\eta_2 \in \Omega^4_{\text{exact}}$ such that (4.11) is satisfied upto terms of $O(\epsilon^3)$. Explicitly, we write

$$\varphi_\epsilon = \varphi_0 + \epsilon\xi_1 + \frac{\epsilon^2}{2}(\widehat{\eta_2} - Q_3(\eta_1))$$

where $\widehat{\eta_2}$ denotes the linearization of Hitchin's duality map Θ for stable forms in Proposition 2.4 and $Q_3(\eta_1)$ is the quadratic term of Hitchin's duality map. Since we want solutions to (4.11) up to second order, we look for η_2 such that

$$d\widehat{\eta_2} - 4\eta_2 = d(Q_3(\eta_1)) + d * d(Z_2 \lrcorner \psi_0) \tag{4.12}$$

as $Z_1 = 0$ and $Z_2 \lrcorner \psi_0$ is the only second order term in $Z_\epsilon \lrcorner \psi_\epsilon$. We know from Proposition 4.6 that there are obstructions to finding second order deformations and hence in solving the above equation. We want to establish a one-to-one correspondence between second order deformations of a nearly G_2 structure and solutions to (4.12). We do this in the following lemma.

Lemma 4.11. *Suppose* η_2 *is a solution of* (4.12)*. Then* $d(Z_2 \lrcorner \psi_0) = 0$ *and* $(\widehat{\eta_2} - Q_3(\eta_1), \eta_2)$ *defines a second-order deformation of* (φ_0, ψ_0) *in the direction of* (ξ_1, η_1) *in the sense of Definition 4.8. Conversely, every second order deformation* (ξ_2, η_2) *is a solution to* (4.12)*.*

Proof. We start with

$$\begin{aligned}\|d(Z_2 \lrcorner \psi_0)\|^2_{L^2} &= \langle Z_2 \lrcorner \psi_0, d^* d(Z_2 \lrcorner \psi_0)\rangle_{L^2} \\ &= \langle Z_2 \lrcorner \psi_0, *d * d(Z_2 \lrcorner \psi_0)\rangle_{L^2} \\ &= \langle Z_2 \lrcorner \psi_0, *(d\widehat{\eta_2} - 4\eta_2 - dQ_3(\eta_1))\rangle_{L^2}\end{aligned}$$

Since $d\psi_\epsilon = O(\epsilon^3)$, hence from (2.18) and (2.19) we see that for any vector field Y, $\int d\varphi_\epsilon \wedge (Y \lrcorner \psi_\epsilon) = O(\epsilon^3)$. Thus the terms which are $O(\epsilon^2)$ in $\int d\varphi_\epsilon \wedge (Y \lrcorner \psi_\epsilon)$ vanish, that is

$$\int d\varphi_0 \wedge (Y \lrcorner \eta_2) + d\xi_1 \wedge \eta_1 + d(\widehat{\eta_2} - Q_3(\eta_1)) \wedge (Y \lrcorner \psi_0) = 0.$$

Using the facts that $d\varphi_0 = 4\psi_0$, $d\xi_1 \wedge \eta_1 = 0$, being an 8-form on a seven dimensional manifold and $(Y \lrcorner \eta_2) \wedge \psi_0 = -(Y \lrcorner \psi_0) \wedge \eta_2$ we get that

$$\int d(\widehat{\eta_2} - Q_3(\eta_1)) \wedge (Y \lrcorner \psi_0) - 4\eta_2 \wedge (Y \lrcorner \psi_0) = 0$$

Taking $Y = Z_2$ proves that $d(Z_2 \lrcorner \psi_0) = 0$. From (4.12) we get that

$$d(\widehat{\eta_2} - Q_3(\eta_1)) = 4\eta_2$$

which proves that $((\widehat{\eta_2} - Q_3(\eta_1), \eta_2))$ is a second-order deformation of (φ_0, ψ_0) in the direction of (ξ_1, η_1) in the sense of Definition 4.8. Conversely, suppose that (ξ_2, η_2) is a second-order deformation of (φ_0, ψ_0). Then $d\xi_2 = 4\eta_2$. □

From the previous proposition and Proposition 4.6 we have that if (ξ_2, η_2) is a second order deformation of the nearly G_2 structure (φ_0, ψ_0) in the sense of Definition 4.8 then

$$\langle d^* dQ_3(\eta_1) - 4 * dQ_3(\eta_1), \chi\rangle_{L^2} = 0 \tag{4.13}$$

for all $\chi \in \Omega^3_{27}$ such that $d^*\chi = 0, \Delta\chi = 16\chi$. The above equation simplifies to

$$\langle *Q_3(\eta_1), d\chi - 4 * \chi\rangle_{L^2} = 0.$$

Moreover, if χ is an infinitesimal deformation of (φ_0, ψ_0), then by Theorem 4.3 χ satisfies $d\chi = -4 * \chi$ (which of course implies $d^*\chi = 0$ and $\Delta\chi = 16\chi$) and so the above equation is equivalent to

$$\langle Q_3(\eta_1), \chi\rangle_{L^2} = 0.$$

5. Deformations on the Aloff-Wallach space

In [1, Prop. 8.3] Alexandrov–Semmelmann established that the space of infinitesimal deformations of the nearly G_2 structure on the Aloff–Wallach space $X_{1,1} \cong \frac{SU(3)\times SU(2)}{SU(2)\times U(1)}$ is an eight dimensional space isomorphic to $\mathfrak{su}(3)$, the Lie algebra of SU(3). The rest of the paper is devoted to prove that these deformations are obstructed to second order.

The embedding of $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ in $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$, which we denote by $\mathfrak{su}(2)_d$ and $\mathfrak{u}(1)$, following [1], is given by

$$\mathfrak{su}(2)_d = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, a \right) \mid a \in \mathfrak{su}(2) \right\},$$
$$\mathfrak{u}(1) = \operatorname{span}\{C\} = \operatorname{span}\left\{ \left(\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, 0 \right) \right\}.$$

The Lie algebra $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ splits as

$$\mathfrak{su}(3) \oplus \mathfrak{su}(2) = \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{m}$$

where $\mathfrak{m}$ is the 7-dimensional orthogonal complement of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ with respect to B, the Killing form of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$. The normal nearly G_2 metric on $X_{1,1}$ is then given by $-\frac{3}{40}B$ where the constant $-\frac{3}{40}$ comes from our choice of $\tau_0 = 4$. If we denote by W the standard 2-dimensional complex irreducible representation of SU(2) and by $F(k)$ the 1-dimensional complex irreducible representation of U(1) with highest weight k, then as an $SU(2) \times U(1)$-representation

$$\mathfrak{su}(3)_{\mathbb{C}} \cong S^2W \oplus WF(3) \oplus WF(-3) \oplus \mathbb{C}.$$

Let $\{e_i\}_{i=1}^7$ be the basis of $\mathfrak{m}$. If we define $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, we have

$$e_1 := \frac{1}{3}\left(\begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}, -3I\right), \quad e_2 := \frac{1}{3}\left(\begin{pmatrix} 2J & 0 \\ 0 & 0 \end{pmatrix}, -3J\right),$$
$$e_3 := \frac{1}{3}\left(\begin{pmatrix} 2K & 0 \\ 0 & 0 \end{pmatrix}, -3K\right),$$
$$e_4 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \end{pmatrix}, 0\right), \quad e_5 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & 0 & \sqrt{2}i \\ 0 & 0 & 0 \\ \sqrt{2}i & 0 & 0 \end{pmatrix}, 0\right),$$
$$e_6 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}, 0\right), \quad e_7 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}, 0\right).$$

This basis is orthonormal with respect to the metric $g = -\frac{3}{40}B$. We use the shorthand $e^{i_1 i_2 \cdots i_n}$ to denote the n-form $e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_n}$. The nearly G_2 structure φ is given by

$$\varphi = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}.$$

As an $SU(2) \times U(1)$ representation, $\mathfrak{m}_{\mathbb{C}} \cong S^2W \oplus WF(3) \oplus WF(-3)$ where

$$S^2W = \mathrm{Span}\{e^1, e^2, e^3\}, \quad WF(3) = \mathrm{Span}\{e^4 - ie^5, e^6 - ie^7\},$$
$$WF(-3) = \mathrm{Span}\{e^4 + ie^5, e^6 + ie^7\}.$$

By Theorem 4.3, the space of first order deformations is given by $\{\xi \in \Omega^3_{27} \mid d\xi = -4 * \xi\}$. In this example, it was found to be isomorphic to $\mathfrak{su}(3)$. As an $SU(2) \times U(1)$ representation, $\mathfrak{su}(3)$ is isomorphic to the span of $\{C, e_1, \ldots, e_7\}$. The $SU(2) \times U(1)$-invariant homomorphism from $\mathfrak{su}(3)$ to $\Omega^3_{27}(X_{1,1})$ is given by $\mathrm{Span}\{A\}$ where

$$A(C) = \varphi - 7e^{123}, \quad A(e_1) = \frac{5}{3}(e^{145} + e^{167}),$$
$$A(e_2) = \frac{5}{3}(e^{245} + e^{267}), \quad A(e_3) = \frac{5}{3}(e^{345} + e^{367}),$$
$$A(e_4) = \frac{5}{9}(3e^{467} + e^{137} + e^{126} + e^{234}),$$
$$A(e_5) = \frac{5}{9}(3e^{567} + e^{235} - e^{136} + e^{127}),$$
$$A(e_6) = \frac{5}{9}(3e^{456} - e^{236} - e^{135} + e^{124}),$$
$$A(e_7) = \frac{5}{9}(3e^{457} - e^{237} + e^{125} + e^{134}).$$

Let us fix an $\alpha \in \mathfrak{su}(3)$. The adjoint action of $h = (h_1, h_2) \in \mathrm{SU}(3) \times \mathrm{SU}(2)$ is given by

$$h^{-1}\alpha h = h_1^{-1}\alpha h_1 = \begin{pmatrix} iv_1 & x_1 + ix_2 & x_3 + ix_4 \\ -x_1 + ix_2 & iv_2 & x_5 + ix_6 \\ -x_3 + ix_4 & -x_5 + ix_6 & -i(v_1 + v_2) \end{pmatrix}$$

where $v_1, v_2, x_1, x_2, x_3, x_4, x_5, x_6$ are functions on $X_{1,1}$.

The infinitesimal deformation ξ_α associated to α such that $d\xi_\alpha = -4 * \xi_\alpha$ is given by

$$\xi_\alpha = \frac{v_1 + v_2}{2} A(C) + \frac{v_1 - v_2}{2} A(e_1) + \sum_{i=1}^{6} x_i A(e_{i+1}).$$

We can now compute the 4-form η_α by using the relation $d\xi_\alpha = 4\eta_\alpha = -4 * \xi_\alpha$. In order to show that the infinitesimal deformation $(\xi_\alpha, \eta_\alpha)$ associated to α is obstructed to second order, we need to compute the quadratic term $Q_3(\eta_\alpha)$ as discussed in equation (4.13) and find an element $\beta \in \mathfrak{su}(3)$ for which the L^2-inner product is non-zero.

To compute $Q_3(\eta_\alpha)$, one can use the algorithm for stable 4-forms on manifolds with G_2 structures as discussed in [10]. Using the fact that $\xi_\alpha = - * \eta_\alpha$, one can easily show that for some non-zero constant c_1, $Q_3(\eta_\alpha) = c_1 * Q_4(\xi_\alpha)$ where $Q_4(\xi_\alpha)$ is the quadratic term associated to ξ_α. Thus, we will instead compute $Q_4(\xi_\alpha)$ and show that the inner product $\langle *Q_4(\xi_\alpha), \xi_\alpha\rangle_{L^2} \neq 0$ to prove obstructedness.

Consider $\varphi_t = \varphi + t\xi_\alpha$ to be a positive 3-form for small t. We will denote the metric and the volume form induced by φ_t by g_t and vol_t respectively. We have a Taylor series expansion

$$g_t = g_0 + tg_1 + t^2 g_2 + O(t^3)).$$

Then one can define the symmetric bi-linear form B_t by

$$(B_t)_{ij} = ((e_i \lrcorner \varphi_t) \wedge (e_j \lrcorner \varphi_t) \wedge \varphi_t)(e_1, \ldots, e_7).$$

The zero order term of B_t, denoted by B_0 is given by $(B_0)_{ij} = ((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi)(e_1, \ldots, e_7) = \delta_{ij}$. Similarly, one can compute the linear term $(B_1)_{ij} = 3((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \xi_\alpha)(e_1, \ldots, e_7)$ and the quadratic term $(B_2)_{ij} =$

$3((e_i \lrcorner \xi_\alpha) \wedge (e_j \lrcorner \xi_\alpha) \wedge \varphi)(e_1, \ldots, e_7)$. The metric is then defined using the relation (see for example, [13])

$$(B_t)_{ij} = 6(g_t)_{ij}\sqrt{\det g_t}.$$

The linear term in vol_t is proportional to $\varphi \wedge \eta_\alpha + \psi \wedge \xi_\alpha$ and thus vanishes since $(\xi_\alpha, \eta_\alpha) \in \Omega^3_{27} \times \Omega^4_{27}$. Using the above formula we get that

$$\mathrm{vol}_t = \sqrt{\det g_t} = 1 + At^2 + O(t^3),$$

where A is a quadratic polynomial in v_1, v_2 and $x_i, i = 1..6$. Using the Taylor series expansion of g_t and $\sqrt{\det g_t}$, we can compute the Taylor series expansion of the Hodge star associated to φ_t, $*_t = *_0 + t *_1 + t^2 *_2 + O(t^3)$. The Hodge star operator $*_t$ can be computed using the formula

$$*_t(e^{i_1 i_2 \ldots i_k}) = \frac{\mathrm{vol}_t}{(7-k)!} g_t^{i_1 j_1} \ldots g_t^{i_k j_k} \epsilon_{j_1 \ldots j_7} e^{j_{k+1} \ldots j_7}.$$

The quadratic term $Q_4(\xi_\alpha)$ is then given by

$$Q_4(\xi_\alpha) = *_2\varphi + *_1\xi_\alpha.$$

In the present case, for a general element $\alpha \in \mathfrak{su}(3)$, the quadratic term turns out to be very complicated and is not very enlightening. We define the cubic polynomial on $X_{1,1}$ by

$$f_\alpha([h]) = \langle *Q_4(\xi_\alpha), \xi_\alpha \rangle_{L^2}.$$

Note that f_α is cubic in α since $Q_4(\xi_\alpha)$ and ξ_α are quadratic and linear in α respectively. This cubic polynomial can be lifted to a polynomial P on the Lie group $\mathrm{SU}(3) \times \mathrm{SU}(2)$ by

$$f_\alpha([h]) = P(h^{-1}\alpha h).$$

This lift enables us to calculate the average of P on $\mathrm{SU}(3) \times \mathrm{SU}(2)$ by using the Peter–Weyl theorem. To express the polynomial P in a compact form, we will set $z_1 = x_2 + ix_1, z_2 = x_4 - ix_3, z_3 = x_6 + ix_5$. Then the cubic polynomial P is given by

$$\begin{aligned} P(h^{-1}\alpha h) = &-\frac{97}{6}(v_1^2v_2+v_2^2v_1)+\frac{25}{9}\mathrm{Re}(z_1z_2z_3)-\frac{29}{6}(v_1^3+v_2^3)\\ &+\frac{5}{3}(v_1+v_2)|z_1|^2+\frac{37}{18}(v_1|z_3|^3+v_2|z_2|^2)\\ &+\frac{31}{9}(v_1|z_2|^3+v_2|z_3|^2) \end{aligned} \tag{5.1}$$

The next step in proving obstructedness is to show that the average value of P on $\mathrm{SU}(3)\times\mathrm{SU}(2)$ is non-zero. For this, we appeal to the Peter–Weyl theorem. The Peter–Weyl theorem states that for any compact Lie group G, we have

$$L^2(G)=\bigoplus_{V_\gamma\in G_{irr}}\mathrm{Hom}(V_\gamma,G)\otimes V_\gamma$$

where G_{irr} denotes the set of all non-isomorphic irreducible representations of G.

The cubic polynomial P lies in the $\mathrm{SU}(3)\times\mathrm{SU}(2)$ representation $\mathrm{Sym}^3\mathfrak{su}(3)$. The average value of the function $P(g^{-1}\xi g)$ on $\mathrm{SU}(3)\times\mathrm{SU}(2)$ is the same as the average value of $R(h^{-1}\alpha h)$ where R is the projection of P to the invariant polynomials. This is because $(P-R)(h^{-1}\alpha h)$ lies in the non-trivial part of the Peter–Weyl decomposition and has an average value of zero. The unique trivial sub-representation of $\mathrm{Sym}^3\mathfrak{su}(3)$ is generated by the determinant polynomial $i\det$ on $\mathfrak{su}(3)$ which is given by

$$\begin{aligned} i\det(g^{-1}\alpha g) = &-(v_1v_2^2+v_2v_1^2)+(v_1+v_2)|z_1|^2\\ &-(v_1|z_3|^2+v_2|z_2|^2)+2\mathrm{Re}(z_1z_2z_3). \end{aligned}$$

The average value of the polynomial P can be computed by computing the inner product of P with $i\det$. On $\mathfrak{su}(3)$, since the Killing form B is non-degenarate, $g=-\frac{1}{12}B$ defines an inner product on $\mathfrak{su}(3)$. The inner product g induces an inner product on $\mathrm{Sym}^3\mathfrak{su}(3)$ in the natural way. All the computations that follow are done using g.

If E_{ij} denotes the matrix with 1 as the (i,j)-th entry and zero elsewhere, then the subspace of $\mathfrak{su}(3)$ generated by $\{E_{ij}-E_{ji}+i(E_{ij}+E_{ji})\mid i,j=1,2,3,i\neq j\}$ is orthogonal to $\mathrm{Span}\{E_{11}-iE_{33},E_{22}-iE_{33}\}$. Moreover $E_{ij}-E_{ji}+i(E_{ij}+E_{ji}),\ i,j=1,2,3,i\neq j$ are also orthogonal to each other. Thus the only non-trivial terms occurring in the inner product of P and $i\det$ are,

$$\begin{aligned}
&\|v_1^2v_2+v_2^2v_1\|^2=\frac{1}{3}, \quad \|\mathrm{Re}(z_1z_2z_3)\|^2=\frac{2}{3},\\
&\langle v_1^3+v_2^3, v_1^2v_2+v_2^2v_1\rangle=-\frac{1}{4}, \quad \|(v_1+v_2)|z_1|^2\|^2=1,\\
&\|v_1|z_3|^2+v_2|z_2|^2\|^2=\frac{4}{3}, \quad \langle v_1|z_2|^2+v_2|z_3|^2, v_1|z_3|^2+v_2|z_2|^2\rangle=-\frac{1}{3}.
\end{aligned}$$

From (5.1) and the above computations we have that

$$\begin{aligned}
\langle P, i\det\rangle &= \frac{97}{6}\left(\frac{1}{3}\right)+\frac{50}{9}\left(\frac{2}{3}\right)+\frac{29}{6}\left(-\frac{1}{4}\right)+\frac{5}{3}(1)-\frac{37}{18}\left(\frac{4}{3}\right)-\frac{31}{9}\left(-\frac{1}{3}\right)\\
&=\frac{191}{24}\neq 0.
\end{aligned}$$

Thus we get the following theorem.

Theorem 5.1. *The infinitesimal deformations of the homogeneous nearly G_2 structure on the Aloff–Wallach space $X_{1,1} \cong \frac{SU(3)\times SU(2)}{SU(2)\times U(1)}$ are all obstructed.*

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