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Constant mean curvature n-noids in hyperbolic space

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Using the DPW method, we construct genus zero Alexandrov-embedded constant mean curvature (greater than one) surfaces with any number of Delaunay ends in the hyperbolic space.

Introduction

In [4], Dorfmeister, Pedit and Wu introduced a loop group method (the DPW method) for constructing harmonic maps from a Riemann surface into a symmetric space. As a consequence of the Ruh-Vilms theorem [19], their method provides a Weierstrass-type representation of constant mean curvature surfaces (CMC) in Euclidean space $\mathbb{R}^3$. Many examples have been constructed (see for example [3, 5, 7, 8, 12, 21]). Among them, Traizet [25, 26] showed how the DPW method can construct genus zero n-noids, that is, CMC $H \neq 0$ surfaces conformal to an n-punctured sphere, with Delaunay ends (as Kapouleas did with partial differential equations techniques in [10]) and glue half-Delaunay ends to minimal surfaces (as did Mazzeo and Pacard in [16], also with PDE techniques). Using [18], the embeddedness (or Alexandrov-embeddedness) of these examples can be derived from their Weierstrasss data.

A natural question is whether these constructions can be carried out in $\mathbb{H}^3$. Lawson has shown [15] that to any CMC H surface in $\mathbb{R}^3$ corresponds a CMC H^2+1 surface in $\mathbb{H}^3$, and the Lawson correspondence has been translated in the DPW framework [21]. We thus use the DPW method for CMC $H > 1$ surfaces in $\mathbb{H}^3$ and adapt the techniques used in [18, 25] to construct new examples. Admittedly, the n-noids of [25, 26] already provide equally as many CMC $H > 1$ cousins in $\mathbb{H}^3$. Nevertheless, the Lawson correspondence being only local, these cousin n-noids are immersions of the universal cover of the n-punctured sphere, and they have no reason to descend to a well-defined immersion of the n-punctured sphere itself. Hence, the construction carried out here is not the cousin construction of [25, 26].

The two resulting theorems are as follows:

Theorem 1. *Given a point $p \in \mathbb{H}^3$, $n \geq 3$ distinct unit vectors $u_1, \cdots, u_n$ in the tangent space of $\mathbb{H}^3$ at p and n non-zero real weights $\tau_1, \cdots, \tau_n$ satisfying the balancing condition*

$$\sum_{i=1}^{n} \tau_i u_i = 0 \tag{1}$$

and given $H > 1$, there exists a smooth 1-parameter family of CMC H surfaces $(M_t)_{0<t<T}$ with genus zero, n Delaunay ends and the following properties:

1) *Denoting by $w_{i,t}$ the weight of the i-th Delaunay end,*
$$\lim_{t\to 0} \frac{w_{i,t}}{t} = \tau_i.$$
2) *Denoting by $\Delta_{i,t}$ the axis of the i-th Delaunay end, $\Delta_{i,t}$ converges as t tends to 0 to the oriented geodesic through the point p in the direction of u_i.*
3) *If all the weights τ_i are positive, then M_t is Alexandrov-embeddedd.*

4) *If all the weights* τ_i *are positive and if for all* $i \neq j \in [1, n]$*, the angle* θ_{ij} *between* u_i *and* u_j *satisfies*

$$\left|\sin \frac{\theta_{ij}}{2}\right| > \frac{\sqrt{H^2 - 1}}{2H}, \tag{2}$$

then M_t *is embedded.*

Theorem 2. *Let* $M_0 \subset \mathbb{R}^3$ *be a non-degenerate minimal* n*-noid with* $n \geq 3$ *and let* $H > 1$*. There exists a smooth family of CMC* H *surfaces* $(M_t)_{0<|t|<T}$ *in* $\mathbb{H}^3$ *such that*

1) *The surfaces* M_t *have genus zero and* n *Delaunay ends.*
2) *After a suitable blow-up,* M_t *converges to* M_0 *as* t *tends to* 0*.*
3) *If* M_0 *is Alexandrov-embedded, then all the ends of* M_t *are of unduloidal type if* $t > 0$ *and of nodoidal type if* $t < 0$*. Moreover,* M_t *is Alexandrov-embedded if* $t > 0$*.*

Following the proofs of [25, 26] gives an effective strategy to construct the desired CMC surfaces M_t. This is done in Sections 3 and 4. However, showing that M_t is Alexandrov-embedded requires a precise knowledge of its ends. This is the purpose of our main theorem (Section 2, Theorem 3). We consider a family of holomorphic perturbations of the data giving rise, via the DPW method, to a half-Delaunay embedding $f_0 : \mathbb{D}^* \subset \mathbb{C} \longrightarrow \mathbb{H}^3$ and show that the perturbed induced surfaces $f_t(\mathbb{D}^*)$ are also embedded. Note that the domain on which the perturbed immersions are defined does not depend on the parameter t, which is stronger than f_t having an embedded end, and is critical for showing that the surfaces M_t are Alexandrov-embedded. The essential hypothesis on the perturbations is that they induce a well-defined immersion of the punctured disk. The proof relies on the Frobenius method for linear differential systems with regular singular points. Although this idea has been used in $\mathbb{R}^3$ by Kilian, Rossman, Schmitt [13] and [18], the case of $\mathbb{H}^3$ generates in the differential system two extra resonance points that are unavoidable and make their results inapplicable. Our solution is to extend the Frobenius method to loop-group-valued differential systems. Finally, let us note that there exists a DPW framework for minimal surfaces in $\mathbb{H}^3$ in which this construction has been carried out [2] except for the proof of embeddedness and another one for CMC $H < 1$ surfaces in $\mathbb{H}^3$ [3] in which these techniques should also be efficient.

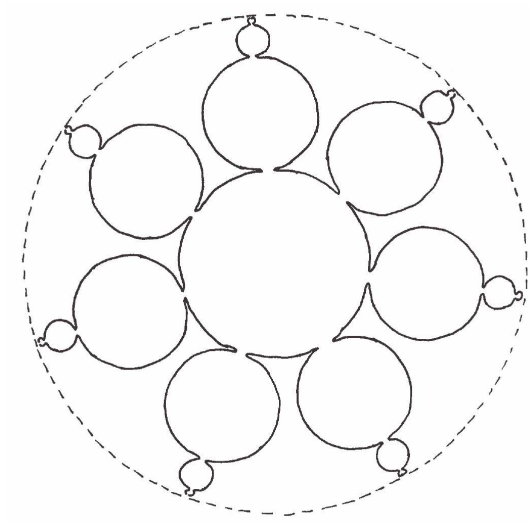

Figure 1: *Theorem 1 ensures the existence of n-noids with small necks. For $H > 1$ small enough ($H \simeq 1.5$ on the picture), there exist coplanar embedded heptanoids.*

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1. Delaunay surfaces in $\mathbb{H}^3$ via the DPW method

We recall the matrix model of $\mathbb{H}^3$ and the DPW expression of the Lawson correspondence for CMC $H > 1$ surfaces in $\mathbb{H}^3$ [21]. We then study Delaunay surfaces and parametrise them by their weight.

1.1. Hyperbolic space

We set the notations for a matrix model of $\mathbb{H}^3$ and give the formulas for rigid motions, geodesics and parallel transportation in this model.

Matrix model. Let $\mathbb{R}^{1,3}$ denote the space $\mathbb{R}^4$ with the Lorentzian metric $\langle x, x\rangle = -x_0^2 + x_1^2 + x_2^2 + x_3^2$. Hyperbolic space is the subset $\mathbb{H}^3$ of vectors $x \in \mathbb{R}^{1,3}$ such that $\langle x, x\rangle = -1$ and $x_0 > 0$, with the metric induced by $\mathbb{R}^{1,3}$. The DPW method constructs CMC immersions into a matrix model of $\mathbb{H}^3$.

Consider the identification

$$x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^{1,3} \simeq X = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in \mathcal{H}_2$$

where $\mathcal{H}_2 := \{M \in \mathcal{M}(2, \mathbb{C}) \mid M^* = M\}$ denotes the Hermitian matrices. In this model, $\langle X, X\rangle = -\det X$ and $\mathbb{H}^3$ is identified with the set $\mathcal{H}_2^{++} \cap \mathrm{SL}(2, \mathbb{C})$ of Hermitian positive definite matrices with determinant 1. This fact leads us to write

$$\mathbb{H}^3 = \{FF^* \mid F \in \mathrm{SL}(2, \mathbb{C})\}.$$

Setting

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

gives us an orthonormal basis $(\sigma_1, \sigma_2, \sigma_3)$ of the tangent space $T_{\mathrm{I}_2}\mathbb{H}^3$ of $\mathbb{H}^3$ at the identity matrix. We choose the orientation of $\mathbb{H}^3$ induced by this basis.

Rigid motions. In the matrix model of $\mathbb{H}^3$, $\mathrm{SL}(2, \mathbb{C})$ acts as rigid motions: for all $p \in \mathbb{H}^3$ and $A \in \mathrm{SL}(2, \mathbb{C})$, this action is denoted by

$$A \cdot p := ApA^* \in \mathbb{H}^3.$$

This action extends to tangent spaces: for all $v \in T_p\mathbb{H}^3$, $A \cdot v := AvA^* \in T_{A\cdot p}\mathbb{H}^3$. The DPW method takes advantage of this fact and contructs immersions in $\mathbb{H}^3$ with the moving frame method.

Geodesics. Let $p \in \mathbb{H}^3$ and $v \in UT_p\mathbb{H}^3$. Define the map

$$\begin{array}{rcccc} \mathrm{geod}(p, v) & : & \mathbb{R} & \longrightarrow & \mathbb{H}^3 \\ & & t & \longmapsto & p\cosh t + v \sinh t. \end{array} \tag{4}$$

Then $\mathrm{geod}(p, v)$ is the unit speed geodesic through p in the direction v. The action of $\mathrm{SL}(2, \mathbb{C})$ extends to oriented geodesics via:

$$A \cdot \mathrm{geod}(p, v) := \mathrm{geod}(A \cdot p, A \cdot v).$$

Parallel transport. Let $p, q \in \mathbb{H}^3$ and $v \in T_p\mathbb{H}^3$. We denote the result of parallel transporting v from p to q along the geodesic of $\mathbb{H}^3$ joining p to q by $\Gamma_p^q v \in T_q\mathbb{H}^3$. The parallel transport of vectors from the identity matrix is easy to compute with Proposition 1.

Proposition 1. *For all $p \in \mathbb{H}^3$ and $v \in T_{\mathrm{I}_2}\mathbb{H}^3$, there exists a unique $S \in \mathcal{H}_2^{++} \cap \mathrm{SL}(2,\mathbb{C})$ such that $p = S \cdot \mathrm{I}_2$. Moreover, $\Gamma_{\mathrm{I}_2}^{p} v = S \cdot v$.*

Proof. The point p is in $\mathbb{H}^3$ identified with $\mathcal{H}_2^{++} \cap \mathrm{SL}(2,\mathbb{C})$. Define S as the unique square root of p in $\mathcal{H}_2^{++} \cap \mathrm{SL}(2,\mathbb{C})$. Then $p = S \cdot \mathrm{I}_2$. Define for $t \in [0,1]$:

$$S(t) := \exp\left(t \log S\right), \qquad \gamma(t) := S(t) \cdot \mathrm{I}_2, \qquad v(t) := S(t) \cdot v.$$

Then $v(t) \in T_{\gamma(t)}\mathbb{H}^3$ because

$$\langle v(t), \gamma(t) \rangle = \langle S(t) \cdot \mathrm{I}_2, S(t) \cdot v \rangle = \langle \mathrm{I}_2, v \rangle = 0$$

and $S \cdot v = v(1) \in T_p\mathbb{H}^3$.

Suppose that S is diagonal. Then

$$S(t) = \begin{pmatrix} e^{\frac{at}{2}} & 0 \\ 0 & e^{\frac{-at}{2}} \end{pmatrix} \qquad (a \in \mathbb{R})$$

and using equations (3) and (4),

$$\gamma(t) = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix} = \mathrm{geod}(\mathrm{I}_2, \sigma_3)(at)$$

is a geodesic curve. Write $v = v^1\sigma_1 + v^2\sigma_2 + v^3\sigma_3$ and compute $S(t) \cdot \sigma_i$ to find

$$v(t) = v^1\sigma_1 + v^2\sigma_2 + v^3 \begin{pmatrix} e^{at} & 0 \\ 0 & -e^{-at} \end{pmatrix}.$$

Compute in $\mathbb{R}^{1,3}$

$$\frac{Dv(t)}{dt} = \left(\frac{dv(t)}{dt}\right)^T = av^3 \left(\gamma(t)\right)^T = 0$$

to see that $v(t)$ is the parallel transport of v along the geodesic γ.

If S is not diagonal, write $S = QDQ^{-1}$ where $Q \in \mathrm{SU}(2)$ and $D \in \mathcal{H}_2^{++} \cap \mathrm{SL}(2,\mathbb{C})$ is diagonal. Then,

$$S \cdot v = Q \cdot \left(D \cdot \left(Q^{-1} \cdot v\right)\right) = Q \cdot \Gamma_{\mathrm{I}_2}^{D\cdot \mathrm{I}_2}(Q^{-1} \cdot v).$$

But for all $A \in \mathrm{SL}(2,\mathbb{C})$, $p, q \in \mathbb{H}^3$ and $v \in T_p\mathbb{H}^3$,

$$A \cdot \Gamma_p^q v = \Gamma_{A\cdot p}^{A\cdot q} A \cdot v$$

and thus

$$S \cdot v = \Gamma^p_{\mathrm{I}_2} v.$$

$\square$

1.2. The DPW method for CMC $H > 1$ surfaces in $\mathbb{H}^3$

We present the loop groups we will need and endow them with a Banach structure. We then recall the ingredients and steps of the DPW method and recall the dressing and gauging action, leading us to the monodromy problem that we will need to solve. We finally study the example of the sphere.

Loop groups. In the DPW method a one-parameter family of surfaces is constructed. The parameter is called "spectral parameter" and will always be in one of the following subsets of $\mathbb{C}$ ($\rho > 1$):

$$\mathbb{S}^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}, \quad \mathbb{A}_\rho = \left\{\lambda \in \mathbb{C} \mid \rho^{-1} < |\lambda| < \rho\right\},$$
$$\mathbb{D}_\rho = \{\lambda \in \mathbb{C} \mid |\lambda| < \rho\}.$$

Any smooth map $f : \mathbb{S}^1 \longrightarrow \mathcal{M}(2, \mathbb{C})$ can be decomposed into its Fourier series

$$f(\lambda) = \sum_{i \in \mathbb{Z}} f_i \lambda^i.$$

Let $|\cdot|$ denote a norm on $\mathcal{M}(2, \mathbb{C})$. Fix some $\rho > 1$ and consider

$$\|f\|_\rho := \sum_{i \in \mathbb{Z}} |f_i| \rho^{|i|}.$$

Let G be a Lie subgroup or subalgebra of $\mathcal{M}(2, \mathbb{C})$. We define

- ΛG as the set of smooth functions $f : \mathbb{S}^1 \longrightarrow G$.
- $\Lambda G_\rho \subset \Lambda G$ as the set of functions f such that $\|f\|_\rho$ is finite. If G is a group (or an algebra) then $(\Lambda G_\rho, \|\cdot\|_\rho)$ is a Banach Lie group (or algebra).
- $\Lambda G_\rho^{\geq 0} \subset \Lambda G_\rho$ as the set of functions f such that $f_i = 0$ for all $i < 0$.
- $\Lambda_+ G_\rho \subset \Lambda G_\rho^{\geq 0}$ as the set of functions such that f_0 is upper-triangular.
- $\Lambda_+^{\mathbb{R}} G_\rho \subset \Lambda_+ G_\rho$ as the set of functions that have positive elements on the diagonal.

Seeing $\mathbb{C}$ as an abelian subalgebra of $\mathcal{M}(2,\mathbb{C})$, the above notations are extended to $\Lambda\mathbb{C}_\rho$ and $\Lambda\mathbb{C}_\rho^{\geq 0}$. Note that every function of ΛG_ρ holomorphically extends to $\mathbb{A}_\rho$ and that every function of $\Lambda G_\rho^{\geq 0}$ holomorphically extends to $\mathbb{D}_\rho$.

In the sequel we will use the Frobenius norm on $\mathcal{M}(2,\mathbb{C})$:

$$|A| := \left(\sum_{i,j} |a_{ij}|^2\right)^{\frac{1}{2}}.$$

Since this norm is sub-multiplicative, the norm $\|\cdot\|_\rho$ is sub-multiplicative. Moreover, for all $A \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$,

$$\left\|A^{-1}\right\|_\rho = \|A\|_\rho$$

and for all $A \in \Lambda\mathcal{M}(2,\mathbb{C})_\rho$ and $\lambda \in \mathbb{A}_\rho$,

$$|A(\lambda)| \leq \|A\|_\rho .$$

The DPW method relies on the Iwasawa decomposition [17], which we state in our context and with the above notations:

Proposition 2. *The multiplication map* $\Lambda\mathrm{SU}(2)_\rho \times \Lambda_+^{\mathbb{R}}\mathrm{SL}(2,\mathbb{C})_\rho \longmapsto \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ *is a smooth diffeomorphism between Banach manifolds. Its inverse map is called "Iwasawa decomposition" and is denoted by*

$$\mathrm{Iwa}(\Phi) = (\mathrm{Uni}(\Phi), \mathrm{Pos}(\Phi))$$

for $\Phi \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$.

Remark 1. *The decomposition of Proposition 2 covers a smaller set than the r-Iwasawa decomposition used in [13, 21] because the loops we consider holomorphically extend to a whole annulus. This allows us to use a stronger strucure than the $\mathcal{C}^k$-topologies on those loops. An elementary proof of Proposition 2 (relying on [17]) can be found in [26].*

The ingredients. Let $H > 1$, $q = \operatorname{arcoth} H > 0$ and $\rho > e^q$. The DPW uses the following input data:

- A Riemann surface Σ.

- A holomorphic 1-form on Σ with values in $\Lambda\mathfrak{sl}(2,\mathbb{C})_\rho$ of the following form:

$$\xi = \begin{pmatrix} \alpha & \lambda^{-1}\beta \\ \gamma & -\alpha \end{pmatrix}$$

  where α, β, γ are holomorphic 1-forms on Σ with values in $\Lambda\mathbb{C}_\rho^{\geq 0}$. The 1-form ξ is called "the potential".

- A base point $z_0 \in \Sigma$.
- An initial condition $\phi \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$.

The recipe. The DPW method consists in the following steps:

1) Let $\widetilde{z}_0$ be any point above z_0 in the universal cover $\widetilde{\Sigma}$ of Σ. Solve on $\widetilde{\Sigma}$ the following Cauchy problem:

$$\begin{cases} d\Phi = \Phi\xi \\ \Phi(\widetilde{z}_0) = \phi. \end{cases} \tag{5}$$

   Then $\Phi : \widetilde{\Sigma} \longrightarrow \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ is called "the holomorphic frame".

2) Compute pointwise on $\widetilde{\Sigma}$ the Iwasawa decomposition of Φ:

$$(F(z), B(z)) := \mathrm{Iwa}\,\Phi(z),$$

   for $z \in \widetilde{\Sigma}$. The unitary part F of this decomposition is called "the unitary frame".

3) Define $f : \widetilde{\Sigma} \longrightarrow \mathbb{H}^3$ via the Sym-Bobenko formula:

$$f(z) = F(z, e^{-q})F(z, e^{-q})^* =: \mathrm{Sym}_q F(z)$$

   where $F(z, \lambda_0) := F(z)(\lambda_0)$.

Then f is a CMC $H > 1$ ($H = \coth q$) conformal immersion from $\widetilde{\Sigma}$ to $\mathbb{H}^3$. Its Gauss map (in the direction of the mean curvature vector) is given by

$$N(z) = F(z, e^{-q})\sigma_3 F(z, e^{-q})^* =: \mathrm{Nor}_q F(z)$$

where σ_3 is defined in (3). The differential of f is given by

$$df(z) = 2\sinh(q) b(z)^2 F(z, e^{-q}) \begin{pmatrix} 0 & \beta(z,0) \\ \overline{\beta(z,0)} & 0 \end{pmatrix} F(z, e^{-q})^* \tag{6}$$

where $b(z) > 0$ is the upper-left entry of $B(z) \mid_{\lambda=0}$. The metric of f is given by

$$ds_f^2(z) = 4\sinh(q)^2 b(z)^4 \left|\beta(z,0)\right|^2$$

and its Hopf differential at $z \in \widetilde{\Sigma}$ reads

$$-2\sinh q\ \beta(z,0) \otimes \gamma(z,0). \tag{7}$$

Remark 2. *The results of this paper hold for any $H > 1$. We thus fix now $H > 1$ and $q = \operatorname{arcoth} H$. Hence,*

$$e^{-q} = \sqrt{\frac{H-1}{H+1}}.$$

Rigid motions. Let $C \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ and define the new holomorphic frame $\widetilde{\Phi} = C\Phi$ with unitary part $\widetilde{F}$ and induced immersion $\widetilde{f} = \mathrm{Sym}_q \widetilde{F}$. If $C \in \Lambda\mathrm{SU}(2)_\rho$, then $\widetilde{F} = CF$ and $\widetilde{\Phi}$ gives rise to the same immersion as Φ up to an isometry of $\mathbb{H}^3$:

$$\widetilde{f}(z) = C(e^{-q}) \cdot f(z).$$

If $C \notin \Lambda\mathrm{SU}(2)_\rho$, this transformation is called a "dressing" and may change the surface.

Gauging. Let $G : \widetilde{\Sigma} \longrightarrow \Lambda_+\mathrm{SL}(2,\mathbb{C})_\rho$ holomorphic and define the new potential:

$$\widehat{\xi} = \xi \cdot G := G^{-1}\xi G + G^{-1}dG.$$

The potential $\widehat{\xi}$ is a DPW potential and this operation is called "gauging". The data (Σ, ξ, z_0, ϕ) and $\left(\Sigma, \widehat{\xi}, z_0, \phi\ G(z_0)\right)$ give rise to the same immersion.

The monodromy problem. Since the immersion f is only defined on the universal cover $\widetilde{\Sigma}$, one might ask for conditions ensuring that it descends to a well-defined immersion on Σ. For any deck transformation $\tau \in \mathrm{Deck}\left(\widetilde{\Sigma}/\Sigma\right)$, define the monodromy of Φ with respect to τ as:

$$\mathcal{M}_\tau(\Phi) := \Phi(\tau(z))\Phi(z)^{-1} \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho.$$

This map is independent of $z \in \widetilde{\Sigma}$ since ξ is well-defined on Σ (see (5)). The standard sufficient conditions for the immersion f to be well-defined on Σ

is the following set of equations, called the monodromy problem in $\mathbb{H}^3$:

$$\forall \tau \in \operatorname{Deck}\left(\widetilde{\Sigma}/\Sigma\right), \quad \left\{ \begin{array}{l} \mathcal{M}_\tau(\Phi) \in \Lambda \mathrm{SU}(2)_\rho, \\ \mathcal{M}_\tau(\Phi)(e^{-q}) = \pm \mathrm{I}_2. \end{array} \right. \tag{8}$$

Use the point $\widetilde{z}_0$ defined in step 1 of the DPW method to identify the fundamental group $\pi_1(\Sigma, z_0)$ with $\operatorname{Deck}(\widetilde{\Sigma}/\Sigma)$. Let $\{\gamma_i\}_{i\in I}$ be a set of generators of $\pi_1(\Sigma, z_0)$. Then the problem (8) is equivalent to

$$\forall i \in I, \quad \left\{ \begin{array}{l} \mathcal{M}_{\gamma_i}(\Phi) \in \Lambda \mathrm{SU}(2)_\rho, \\ \mathcal{M}_{\gamma_i}(\Phi)(e^{-q}) = \pm \mathrm{I}_2. \end{array} \right. \tag{9}$$

Example: the standard sphere. The DPW method can produce spherical immersions of $\Sigma = \mathbb{C} \cup \{\infty\}$ with the potential

$$\xi_S(z, \lambda) = \begin{pmatrix} 0 & \lambda^{-1} dz \\ 0 & 0 \end{pmatrix}$$

and initial condition $\Phi_S(0, \lambda) = \mathrm{I}_2$. The potential is not regular at $z = \infty$ because it has a double pole there. However, the immersion will be regular at this point because ξ_S is gauge-equivalent to a regular potential at $z = \infty$. Indeed, consider on $\mathbb{C}^*$ the gauge

$$G(z, \lambda) = \begin{pmatrix} z & 0 \\ -\lambda & \frac{1}{z} \end{pmatrix}.$$

The gauged potential is then

$$\xi_S \cdot G(z, \lambda) = \begin{pmatrix} 0 & \lambda^{-1} z^{-2} dz \\ 0 & 0 \end{pmatrix}$$

which is regular at $z = \infty$. The holomorphic frame is

$$\Phi_S(z, \lambda) = \begin{pmatrix} 1 & \lambda^{-1} z \\ 0 & 1 \end{pmatrix} \tag{10}$$

and its unitary factor is

$$F_S(z, \lambda) = \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 & \lambda^{-1} z \\ -\lambda \overline{z} & 1 \end{pmatrix}.$$

The induced CMC-H immersion is

$$f_S(z) = \frac{1}{1 + |z|^2} \begin{pmatrix} 1 + e^{2q}|z|^2 & 2z \sinh q \\ 2\overline{z} \sinh q & 1 + e^{-2q}|z|^2 \end{pmatrix}.$$

It is not easy to see that $f_S(\Sigma)$ is a sphere because it is not centered at I_2. To solve this problem, notice that $F_S(z,e^{-q}) = R(q)\widetilde{F}_S(z)R(q)^{-1}$ where

$$R(q) := \begin{pmatrix} e^{\frac{q}{2}} & 0 \\ 0 & e^{\frac{-q}{2}} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \tag{11}$$

and

$$\widetilde{F}_S(z) := \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & z \\ -\overline{z} & 1 \end{pmatrix} \in \mathrm{SU}(2).$$

Apply an isometry by setting

$$\widetilde{f}_S(z) := R(q)^{-1} \cdot f_S(z)$$

and compute

$$\begin{aligned} \widetilde{f}_S(z) &= \frac{1}{1+|z|^2} \begin{pmatrix} e^{-q} + e^q|z|^2 & 2z\sinh q \\ 2\overline{z}\sinh q & e^q + e^{-q}|z|^2 \end{pmatrix} \\ &= (\cosh q)\mathrm{I}_2 + \frac{\sinh q}{1+|z|^2} \begin{pmatrix} |z|^2 - 1 & 2z \\ 2\overline{z} & 1 - |z|^2 \end{pmatrix} \end{aligned}$$

i.e.

$$\widetilde{f}_S(z) = \mathrm{geod}(\mathrm{I}_2, v_S(z))(q) \tag{12}$$

with geod defined in (4) and where in the basis $(\sigma_1,\sigma_2,\sigma_3)$ of $T_{\mathrm{I}_2}\mathbb{H}^3$,

$$v_S(z) := \frac{1}{1+|z|^2}\left(2\,\mathrm{Re}\,z, 2\,\mathrm{Im}\,z, |z|^2 - 1\right) \tag{13}$$

describes a sphere of radius one in the tangent space of $\mathbb{H}^3$ at I_2 (it is the inverse stereographic projection from the north pole). Hence, $\widetilde{f}_S(\Sigma)$ is a sphere centered at I_2 of hyperbolic radius q and $f_S(\Sigma)$ is a sphere of same radius centered at $\mathrm{geod}(\mathrm{I}_2,\sigma_3)(q)$.

One can compute the normal map of f_S:

$$N_S(z) := \mathrm{Nor}_q F_S(z) = R(q) \cdot \widetilde{N}_S(z)$$

where

$$\begin{aligned} \widetilde{N}_S(z) := \mathrm{Nor}_q\left(\widetilde{F}_S(z)\right) &= \frac{1}{1+|z|^2} \begin{pmatrix} e^{-q} - e^q|z|^2 & -2z\cosh q \\ -2\overline{z}\cosh q & e^{-q}|z|^2 - e^q \end{pmatrix} \\ &= -(\sinh q)\mathrm{I}_2 - (\cosh q)v_S(z) = -\dot{\mathrm{geod}}\left(\mathrm{I}_2, v_S(z)\right)(q). \end{aligned}$$

Note that this implies that the normal map Nor_q is oriented by the mean curvature vector.

1.3. Delaunay surfaces

Constant mean curvature $H > 1$ surfaces of revolution in $\mathbb{H}^3$ have been described in the DPW framework in [21]. We recall here the basic facts needed for our purpose and parametrise the data by the weight of the induced surface.

The data. Let $\Sigma = \mathbb{C}^*$, $\xi_{r,s}(z,\lambda) = A_{r,s}(\lambda)z^{-1}dz$ where

$$A_{r,s}(\lambda) := \begin{pmatrix} 0 & r\lambda^{-1}+s \\ r\lambda+s & 0 \end{pmatrix}, \qquad r,s\in\mathbb{R}, \quad \lambda\in\mathbb{S}^1, \tag{14}$$

and initial condition $\Phi_{r,s}(1) = \mathrm{I}_2$. With these data, the holomorphic frame reads

$$\Phi_{r,s}(z) = z^{A_{r,s}}.$$

The unitary frame $F_{r,s}$ can be expressed in terms of elliptic functions (see [21]) and the DPW method states that the map $f_{r,s} = \mathrm{Sym}_q(F_{r,s})$ is a CMC H immersion from the universal cover $\widetilde{\mathbb{C}^*}$ of $\mathbb{C}^*$ into $\mathbb{H}^3$.

Monodromy. Computing the monodromy along $\gamma(\theta) = e^{i\theta}$ for $\theta\in[0,2\pi]$ gives

$$\mathcal{M}\left(\Phi_{r,s}\right) := \mathcal{M}_\gamma\left(\Phi_{r,s}\right) = \exp\left(2i\pi A_{r,s}\right).$$

Recall that $r,s\in\mathbb{R}$ to see that $iA_{r,s}\in\Lambda\mathfrak{su}(2)_\rho$, and thus $\mathcal{M}\left(\Phi_{r,s}\right)\in\Lambda\mathrm{SU}(2)_\rho$: the first equation of (9) is satisfied. To solve the second one, one can determine r and s such that $A_{r,s}(e^{-q})^2 = \frac{1}{4}\mathrm{I}_2$, which will imply that $\mathcal{M}\left(\Phi_{r,s}\right)(e^{-q}) = -\mathrm{I}_2$. This condition is equivalent to

$$r^2+s^2+2rs\cosh q = \frac{1}{4}. \tag{15}$$

Seeing this equation as a polynomial in r and computing its discriminant $(1+4s^2\sinh^2 q > 0)$ ensures the existence of an infinite number of solutions: given a pair $(r,s)\in\mathbb{R}^2$ solving (15), $f_{r,s}$ is a well-defined CMC immersion from $\mathbb{C}^*$ into $\mathbb{H}^3$ with mean curvature $H = \coth q$.

Surface of revolution. Let $(r,s)\in\mathbb{R}^2$ satisfying (15) and let $\theta\in\mathbb{R}$. Then,

$$\Phi_{r,s}\left(e^{i\theta}z\right)=\exp\left(i\theta A_{r,s}\right)\Phi_{r,s}(z).$$

Using $iA_{r,s}\in\Lambda\mathfrak{su}(2)_\rho$ and diagonalising $A_{r,s}(e^{-q})$ gives

$$\begin{aligned}f_{r,s}\left(e^{i\theta}z\right)&=\exp\left(i\theta A_{r,s}(e^{-q})\right)\cdot f_{r,s}(z)\\&=\left(H_{r,s}\exp\left(i\theta D\right)H_{r,s}^{-1}\right)\cdot f_{r,s}(z)\end{aligned}$$

where

$$H_{r,s}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-2\left(re^{q}+s\right)\\2\left(re^{-q}+s\right)&1\end{pmatrix},\qquad D=\begin{pmatrix}\frac{1}{2}&0\\0&\frac{-1}{2}\end{pmatrix}.$$

Noting that $\exp\left(i\theta D\right)$ acts as a rotation of angle θ around the axis $\operatorname{geod}(\mathrm{I}_2,\sigma_3)$ and that $H_{r,s}$ acts as an isometry of $\mathbb{H}^3$ independent of θ shows that $\exp\left(i\theta A_{r,s}(e^{-q})\right)$ acts as a rotation around the axis $H_{r,s}\cdot\operatorname{geod}(\mathrm{I}_2,\sigma_3)$ and that $f_{r,s}$ is CMC $H>1$ immersion of revolution of $\mathbb{C}^*$ into $\mathbb{H}^3$ and by definition (as in [14]) a Delaunay immersion.

The weight as a parameter. For a fixed $H>1$, CMC H Delaunay surfaces in $\mathbb{H}^3$ form a family parametrised by the weight [21]. This weight can be computed in the DPW framework: given a solution (r,s) of (15), the weight w of the Delaunay surface induced by the DPW data $(\mathbb{C}^*,\xi_{r,s},1,\mathrm{I}_2)$ reads

$$w=8\pi rs\sinh q \tag{16}$$

(see [21] or [6] for details).

Lemma 1. *Writing $t:=\frac{w}{2\pi}$ and assuming $t\neq 0$, equations* (15) *and* (16) *imply that*

$$\begin{cases}t\leq T_1,\\ r^2=\frac{1}{8}\left(1-2Ht\pm 2\sqrt{T_1-t}\sqrt{T_2-t}\right),\\ s^2=\frac{1}{8}\left(1-2Ht\pm 2\sqrt{T_1-t}\sqrt{T_2-t}\right)\end{cases} \tag{17}$$

with

$$T_1=\frac{\tanh\frac{q}{2}}{2}<\frac{1}{2\tanh\frac{q}{2}}=T_2.$$

Proof. First, note that (15) and (16) imply

$$r^2 + s^2 = \frac{1}{4}\left(1 - 2t\coth q\right) = \frac{1}{4}(1 - 2Ht)$$

and thus

$$t \le \frac{H}{2} < T_2. \tag{18}$$

If $r = 0$, then $t = 0$. Thus $r \neq 0$ and

$$s = \frac{t}{4r\sinh q}.$$

Equation (15) is then equivalent to

$$r^2 + \frac{t^2}{16r^2\sinh^2 q} + \frac{Ht}{2} = \frac{1}{4} \iff r^4 - \frac{1-2Ht}{4}r^2 + \frac{t^2}{16\sinh^2 q} = 0.$$

Using $\coth q = H$, the discriminant of this quadratic polynomial in r^2 is

$$\Delta(t) = \frac{1}{16}\left(1 - 4Ht + 4t^2\right)$$

which in turn is a quadratic polynomial in t with discriminant

$$\widetilde{\Delta} = \frac{H^2-1}{16} > 0$$

because $H > 1$. Thus

$$\Delta(t) = \frac{(T_1-t)(T_2-t)}{4}$$

because $H = \coth q$. Using (18), $\Delta(t) \ge 0$ if and only if $t \le T_1$ and

$$r^2 = \frac{1}{8}\left(1 - 2Ht \pm 2\sqrt{(T_1-t)(T_2-t)}\right).$$

By symmetry of equations (15) and (16), s^2 is as in (17). $\square$

We consider the two continuous parametrisations of r and s for $t \in (-\infty, T_1)$ such that (r,s) satisfies equations (15) and (16) with $w = 2\pi t$:

$$\begin{cases} r(t) := \frac{\pm 1}{2\sqrt{2}}\left(1 - 2Ht + 2\sqrt{T_1-t}\sqrt{T_2-t}\right)^{\frac{1}{2}}, \\ s(t) := \frac{t}{4r(t)\sinh q}. \end{cases} \tag{19}$$

Choosing the parametrisation satisfying $r > s$ maps the unit circle of $\mathbb{C}^*$ onto a parallel circle of maximal radius, called a “bulge” of the Delaunay surface. As t tends to 0, the immersions tend towards a parametrisation of a sphere on every compact subset of $\mathbb{C}^*$, which is why we call this family of immersions "the spherical family". When $r < s$, the unit circle of $\mathbb{C}^*$ is mapped onto a parallel circle of minimal radius, called a "neck" of the Delaunay surface. As t tends to 0, the immersions degenerate into a point on every compact subset of $\mathbb{C}^*$. Nevertheless, we call this family the "catenoidal family" because applying a blowup to the immersions makes them converge towards a catenoidal immersion of $\mathbb{R}^3$ on every compact subset of $\mathbb{C}^*$ (see Section 4.1 for more details). In both cases, the weight of the induced surfaces is given by $w = 2\pi t$.

2. Perturbed Delaunay immersions

In this section, we study the immersions induced by a perturbation of Delaunay DPW data with small non-vanishing weights in a neighbourhood of $z = 0$. Our results are the same whether we choose the spherical or the catenoidal family of immersions. We thus drop the index r, s in the Delaunay DPW data and replace it by a small value of $t = 4rs\sinh q$ in a neighbourhood of $t = 0$ such that

$$t < T_{\max} := \frac{\tanh\frac{q}{2}}{2} = \frac{1}{2}\left(H - \sqrt{H^2 - 1}\right).$$

For all $\epsilon > 0$, we denote

$$D_\epsilon := \{z \in \mathbb{C} \mid |z| < \epsilon\}, \qquad D_\epsilon^* := D_\epsilon \setminus \{0\}.$$

Definition 1 (Perturbed Delaunay potential). *Let $\rho > e^q$, $0 < T < T_{\max}$ and $\epsilon > 0$. A perturbed Delaunay potential is a continuous one-parameter family $(\xi_t)_{t\in(-T,T)}$ of DPW potentials defined for $(t, z) \in (-T, T) \times D_\epsilon^*$ by*

$$\xi_t(z) = A_t z^{-1} dz + C_t(z) dz$$

where $A_t \in \Lambda\mathfrak{sl}(2,\mathbb{C})_\rho$ is a Delaunay residue as in (14) *satisfying* (19) *and the map $(t, z) \longmapsto C_t(z) \in \Lambda\mathfrak{sl}(2,\mathbb{C})_\rho$ is $\mathcal{C}^1$ (and thus holomorphic with respect to z for all t) and satisfies $C_0(z) = 0$ for all z.*

Theorem 3. *Let $\rho > e^q$, $0 < T < T_{\max}$, $\epsilon > 0$ and ξ_t be a perturbed Delaunay potential $\mathcal{C}^2$ with respect to (t, z). Let Φ_t be a holomorphic frame associated to ξ_t for all t via the DPW method. Suppose that the family of initial* *conditions ϕ_t is $\mathcal{C}^2$ with respect to t, with $\phi_0 = \tilde{z}_0^{A_0}$, and that the monodromy problem* (9) *is solved for all $t \in (-T, T)$. Let $f_t = \mathrm{Sym}_q(\mathrm{Uni}\,\Phi_t)$. Then,*

1) *For all $\delta > 0$, there exist $0 < \epsilon' < \epsilon$, $T' > 0$ and $C > 0$ such that for all $z \in D^*_{\epsilon'}$ and $t \in (-T', T')\backslash\{0\}$,*

$$d_{\mathbb{H}^3}\left(f_t(z), f_t^{\mathcal{D}}(z)\right) \leq C|t||z|^{1-\delta}$$

where $f_t^{\mathcal{D}}$ is a Delaunay immersion of weight $2\pi t$.

2) *There exist $T' > 0$ and $\epsilon' > 0$ such that for all $0 < t < T'$, f_t is an embedding of $D^*_{\epsilon'}$.*

3) *The limit axis as t tends to 0 of the Delaunay immersion $f_t^{\mathcal{D}}$ oriented towards the end at $z = 0$ is given by:*

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -e^q \\ e^{-q} & 1 \end{pmatrix} \cdot \mathrm{geod}\,(\mathrm{I}_2, -\sigma_3) \quad \textit{in the spherical family } (r > s),$$

$$\mathrm{geod}\,(\mathrm{I}_2, -\sigma_1) \quad \textit{in the catenoidal family } (r < s).$$

Let ξ_t and Φ_t as in Theorem 3 with ρ, T and ϵ fixed. This Section is dedicated to the proof of Theorem 3.

The topology used to define the above $\mathcal{C}^k$ regularities is the one induced by the Banach structure of the domain and codomain of the given maps. The $\mathcal{C}^2$-regularity of ξ_t essentially means that $C_t(z)$ is $\mathcal{C}^2$ with respect to (t, z). Together with the $\mathcal{C}^2$-regularity of ϕ_t, it implies that Φ_t is $\mathcal{C}^2$ with respect to (t, z). Thus $\mathcal{M}(\Phi_t)$ is also $\mathcal{C}^2$ with respect to t. These regularities and the fact that there exists a solution Φ_t solving the monodromy problem are used in Section 2.1 to deduce an essential piece of information about the potential ξ_t (Proposition 4). This step then allows us to write in Section 2.2 the holomorphic frame Φ_t in a Mz^AP form given by the Frobenius method (Proposition 5), and to gauge this expression, in order to gain an order of convergence with respect to z (Proposition 6). During this process, the holomorphic frame will loose one order of regularity with respect to t, which is why Theorem 3 asks for a $\mathcal{C}^2$-regularity of the data. Section 2.3 is devoted to the study of dressed Delaunay frames Mz^A in order to ensure that the immersions f_t will converge to Delaunay immersions as t tends to 0, and to estimate the growth of their unitary part around the end at $z = 0$. Section 2.4 proves that these immersions do converge, which is the first point of Theorem 3. Before proving the embeddedness in Section 2.6, Section 2.5

is devoted to the convergence of the normal maps. Finally, we compute the limit axes in Section 2.7.

Note that many estimates in this section resemble the ones given in [13] for perturbed Delaunay ends in $\mathbb{R}^3$. However, as stated in Remark 3.8. of [13], the techniques used in $\mathbb{R}^3$ do not extend directly to $\mathbb{H}^3$ due to the fact that the Sym points $e^{\pm q}$ are no longer on the unit circle. Hence, one interest of this section is to extend these estimates to a uniform annulus.

2.1. A property of the potentials

We begin by diagonalising A_t in a unitary basis (Proposition 3) in order to simplify the computations in Proposition 4, in which we use the Frobenius method for a fixed value of $\lambda = e^{\pm q}$. This will ensure the existence of the $\mathcal{C}^1$ map $P^1 \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ that will be used in Section 2.2 to define the factor P in the Mz^AP form of Φ_t.

Proposition 3. *There exist $e^q < R < \rho$ and $0 < T' < T$ such that for all $t \in (-T', T')$, $A_t = H_t D_t H_t^{-1}$ with $H_t \in \Lambda\mathrm{SU}(2)_R$ and $iD_t \in \Lambda\mathfrak{su}(2)_R$. Moreover, H_t and D_t are smooth with respect to t.*

Proof. For all $\lambda \in \mathbb{S}^1$,

$$\begin{aligned} -\det A_t(\lambda) &= \frac{1}{4} + \frac{t\lambda^{-1}(\lambda - e^q)(\lambda - e^{-q})}{4\sinh q} \\ &= \frac{1}{4} + \frac{t}{4\sinh q}\left(\lambda + \lambda^{-1} - 2\cosh q\right) \in \mathbb{R}. \end{aligned} \tag{20}$$

Extending this determinant as a holomorphic function on $\mathbb{A}_\rho$, there exists $T' > 0$ such that

$$\left| -\det A_t(\lambda) - \frac{1}{4} \right| < \frac{1}{4} \quad \forall (t,\lambda) \in (-T', T') \times \mathbb{A}_\rho \tag{21}$$

With this choice of T', the function $\mu_t : \mathbb{A}_\rho \longrightarrow \mathbb{C}$ defined as the positive-real-part square root of $(-\det A_t)$ is holomorphic on $\mathbb{A}_\rho$ and real-valued on $\mathbb{S}^1$. Note that μ_t is also the positive-real-part eigenvalue of A_t and thus $A_t = H_t D_t H_t^{-1}$ with

$$H_t(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{-(r\lambda^{-1}+s)}{\mu_t(\lambda)} \\ \frac{r\lambda+s}{\mu_t(\lambda)} & 1 \end{pmatrix}, \qquad D_t(\lambda) = \begin{pmatrix} \mu_t(\lambda) & 0 \\ 0 & -\mu_t(\lambda) \end{pmatrix}. \tag{22}$$

Let $e^q < R < \rho$. For all $t \in (-T', T')$, $\mu_t \in \Lambda\mathbb{C}_R$ and the map $t \mapsto \mu_t$ is smooth on $(-T', T')$. Moreover, $H_t \in \Lambda\mathrm{SU}(2)_R$, $iD_t \in \Lambda\mathfrak{su}(2)_R$ and these functions are smooth with respect to t. □

Remark 3. *The bound $t < T'$ ensures that that $4\det A_t(\lambda)$ is an integer only for $t = 0$ and $\lambda = e^{\pm q}$. These points make the Frobenius system resonant, but they are precisely the points that bear an extra piece of information due to the hypotheses on $\mathcal{M}(\Phi_t)(e^q)$ and Φ_0. Allowing the parameter t to leave the interval $(-T', T')$ would bring other resonance points and make Section 2.2 invalid. This is why Theorem 3 does not state that the end of the immersion f_t is a Delaunay end for all t.*

Remark 4. *At $t = 0$, the change of basis H_t in the diagonalisation of A_t takes different values whether $r > s$ (spherical family) or $r < s$ (catenoidal family). One has:*

$$H_0(\lambda) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix} \qquad \text{in the spherical case,} \tag{23}$$

$$H_0(\lambda) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \text{in the catenoidal case.} \tag{24}$$

In both cases, $\mu_0 = \frac{1}{2}$, and thus D_0 is the same.

A basis of $\boldsymbol{\Lambda\mathcal{M}(2,\mathbb{C})_\rho}$. Let R and T' given by Proposition 3. Identify $\Lambda\mathcal{M}(2,\mathbb{C})_\rho$ with the free $\Lambda\mathbb{C}_\rho$-module $\mathcal{M}(2, \Lambda\mathbb{C}_\rho)$ and define for all $t \in (-T', T')$ the basis

$$\mathcal{B}_t = H_t\,(E_1, E_2, E_3, E_4)\,H_t^{-1} =: (X_{t,1}, X_{t,2}, X_{t,3}, X_{t,4})$$

where

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For all $t \in (-T', T')$, write

$$C_t(0) = \begin{pmatrix} tc_1(t) & \lambda^{-1}tc_2(t) \\ tc_3(t) & -tc1(t) \end{pmatrix} = \sum_{j=1}^{4} t\widehat{c}_j(t) X_{t,j}. \tag{25}$$

The functions $c_j, \widehat{c}_j$ are $\mathcal{C}^1$ with respect to $t \in (-T', T')$ and take values in $\Lambda\mathbb{C}_R$. Moreover, the functions $c_i(t)$ holomorphically extend to $\mathbb{D}_\rho$.

Proposition 4. *There exists a continuous function* $\widetilde{c}_3 : (-T', T') \longrightarrow \Lambda\mathbb{C}_R$ *such that for all* $\lambda \in \mathbb{S}^1$ *and* $t \in (-T', T')$,

$$\widehat{c}_3(t) = t(\lambda - e^q)(\lambda - e^{-q})\widetilde{c}_3(t).$$

Proof. It suffices to show that $\widehat{c}_3(0) = 0$ and that the holomorphic extension of $\widehat{c}_3(t)$ satisfies $\widehat{c}_3(t, e^{\pm q}) = 0$ for all t.

To show that $\widehat{c}_3(0) = 0$, recall that the monodromy problem (9) is solved for all t and note that $\mathcal{M}(\Phi_0) = -\mathrm{I}_2$, which implies that, as a function of t, the derivative of $\mathcal{M}(\Phi_t)$ at $t = 0$ is in $\Lambda\mathfrak{su}(2)_\rho$. On the other hand, using the proof of Theorem 5.1.2. in [11] (or Proposition 8 in [25]),

$$\frac{d\mathcal{M}(\Phi_t)}{dt} \mid_{t=0} = \left(\int_\gamma \Phi_0 \frac{d\xi_t}{dt} \mid_{t=0} \Phi_0^{-1} \right) \mathcal{M}(\Phi_0)$$

where γ is a generator of $\pi_1(D_\epsilon^*, z_0)$. Expanding the right-hand side gives

$$-\int_\gamma z^{A_0} \frac{dA_t}{dt} \mid_{t=0} z^{-A_0} z^{-1} dz - \int_\gamma z^{A_0} \frac{dC_t(z)}{dt} \mid_{t=0} z^{-A_0} dz \in \Lambda\mathfrak{su}(2)_\rho.$$

Using [11] once again, note that the first term is the derivative of $\mathcal{M}(z^{A_t})$ at $t = 0$, which is in $\Lambda\mathfrak{su}(2)_\rho$ because $\mathcal{M}(z^{A_t}) \in \Lambda\mathrm{SU}(2)_\rho$ and $\mathcal{M}(z^{A_0}) = -\mathrm{I}_2$. Therefore, the second term is also in $\Lambda\mathfrak{su}(2)_\rho$. Diagonalising A_0 with Proposition 3 and using $H_0 \in \Lambda\mathrm{SU}(2)_R$ gives

$$2i\pi \operatorname*{Res}_{z=0} \left(z^{D_0} H_0^{-1} \frac{d}{dt} C_t(z) \mid_{t=0} H_0 z^{-D_0} \right) \in \Lambda\mathfrak{su}(2)_R.$$

But using Equation (25),

$$\begin{aligned} z^{D_0} H_0^{-1} \frac{d}{dt} C_t(z) \mid_{t=0} H_0 z^{-D_0} &= z^{D_0} H_0^{-1} \left(\sum_{j=1}^{4} \widehat{c}_j(0) X_{0,j} \right) H_0 z^{-D_0} \\ &= \sum_{j=1}^{4} \widehat{c}_j(0) z^{D_0} E_j z^{-D_0} \\ &= \begin{pmatrix} \widehat{c}_1(0) & z\widehat{c}_2(0) \\ z^{-1}\widehat{c}_3(0) & \widehat{c}_4(0) \end{pmatrix}. \end{aligned}$$

Thus

$$2i\pi \begin{pmatrix} 0 & 0 \\ \widehat{c}_3(0) & 0 \end{pmatrix} \in \Lambda\mathfrak{su}(2)_R$$

which gives $\widehat{c}_3(0) = 0$.

Let $\lambda_0 \in \{e^q, e^{-q}\}$ and $t \neq 0$. Using the Frobenius method (Theorem 4.11 of [23] and Lemma 11.4 of [22]) at the resonant point λ_0 ensures the existence of $\epsilon' > 0$, $B, M \in \mathcal{M}(2, \mathbb{C})$ and a holomorphic map $P : D_{\epsilon'} \longrightarrow \mathcal{M}(2, \mathbb{C})$ such that for all $z \in D^*_{\epsilon'}$,

$$\begin{cases} \Phi_t(z, \lambda_0) = M z^B z^{A_t(\lambda_0)} P(z), \\ B^2 = 0, \\ P(0) = \mathrm{I}_2, \\ [A_t(\lambda_0), d_z P(0)] + d_z P(0) = C_t(0, \lambda_0) - B. \end{cases}$$

Compute the monodromy of Φ_t at $\lambda = \lambda_0$:

$$\begin{aligned} \mathcal{M}(\Phi_t)(\lambda_0) &= M \exp(2i\pi B) z^B \exp(2i\pi A_t(\lambda_0)) z^{-B} M^{-1} \\ &= -M \exp(2i\pi B) M^{-1}. \end{aligned}$$

Since the monodromy problem (9) is solved, this quantity equals $-\mathrm{I}_2$. Use $B^2 = 0$ to show that $B = 0$ and thus

$$\begin{cases} \Phi_t(z, \lambda_0) = M z^{A_t(\lambda_0)} P(z), \\ P(0) = \mathrm{I}_2, \\ [A_t(\lambda_0), d_z P(0)] + d_z P(0) = C_t(0, \lambda_0). \end{cases}$$

Diagonalise $A_t(\lambda_0)$ with Proposition 3 and write $d_z P(0) = \sum p_j X_{t,j}$ to get for all $1 \leq j \leq 4$

$$p_j \left([D_t(\lambda_0), E_j] + E_j \right) = t\widehat{c}_j(t, \lambda_0) E_j.$$

In particular, using $\mu_t(\lambda_0) = 1/2$,

$$t\widehat{c}_3(t, \lambda_0) = p_3 \left([D_t(\lambda_0), E_3] + E_3 \right) = 0.$$

$\square$

Note that with the help of equations (25) and (23) or (24), and one can compute the series expansion of $\widehat{c}_3(0)$:

$$\widehat{c}_3(0) = \frac{-\lambda^{-1}}{2} c_2(0,0) + \mathcal{O}(\lambda^0) \quad \text{if} \quad r < s,$$

$$\widehat{c}_3(0) = \frac{1}{2} c_3(0,0) + \mathcal{O}(\lambda) \quad \text{if} \quad r > s.$$

Hence,

$$sc_2(t,0) + rc_3(t,0) \underset{t\to 0}{\longrightarrow} 0. \tag{26}$$

The following map will be useful in the next section:

$$\begin{aligned} t \in (-T', T') \longmapsto P^1(t) := t\widehat{c}_1(t)X_{t,1} + \frac{t\widehat{c}_2(t)}{1+2\mu_t}X_{t,2} \\ + \frac{t\widehat{c}_3(t)}{1-2\mu_t}X_{t,3} + t\widehat{c}_4(t)X_{t,4}. \end{aligned} \tag{27}$$

For all t, Proposition 4 ensures that the map $P^1(t,\lambda)$ holomorphically extends to $\mathbb{A}_R$. Taking a smaller value of R if necessary, $P^1(t) \in \Lambda\mathcal{M}(2,\mathbb{C})_R$ for all t. Moreover,

$$\operatorname{tr} P^1(t) = t\widehat{c}_1(t) + t\widehat{c}_4(t) = \operatorname{tr} C_t(0) = 0.$$

Thus $P^1 \in \mathcal{C}^1((-T',T'), \Lambda\mathfrak{sl}(2,\mathbb{C})_R)$.

2.2. The $z^A P$ form of Φ_t

The map P^1 defined above allows us to use the Frobenius method in a loop group framework and in the non-resonant case, that is, for all t (Proposition 5). The techniques used in [18] will then apply in order to gauge the Mz^AP form and gain an order on z (Proposition 6).

Proposition 5. *There exists $\epsilon' > 0$ such that for all $t \in (-T',T')$ there exist $M_t \in \Lambda\mathrm{SL}(2,\mathbb{C})_R$ and a holomorphic map $P_t : D_{\epsilon'} \longrightarrow \Lambda\mathrm{SL}(2,\mathbb{C})_R$ such that for all $z \in D^*_{\epsilon'}$,*

$$\Phi_t(z) = M_t z^{A_t} P_t(z).$$

Moreover, M_t is $\mathcal{C}^1$ with respect to t, $M_0 = \mathrm{I}_2$, $P_t(z)$ is $\mathcal{C}^1$ with respect to (t,z) (and hence holomorphic in z for all t), $P_0(z) = \mathrm{I}_2$ for all z and $P_t(0) = \mathrm{I}_2$ for all t.

Proof. For all $k \in \mathbb{N}^*$ and $t \in (-T',T')$, define the linear map

$$\begin{array}{rcccc} \mathcal{L}_{t,k} & : & \Lambda\mathcal{M}(2,\mathbb{C})_\rho & \longrightarrow & \Lambda\mathcal{M}(2,\mathbb{C})_\rho \\ & & X & \longmapsto & [A_t, X] + kX. \end{array}$$

Use the bases $\mathcal{B}_t$ and restrict $\mathcal{L}_{t,k}$ to $\Lambda\mathcal{M}(2,\mathbb{C})_R$ to get

$$\mathrm{Mat}_{\mathcal{B}_t}\mathcal{L}_{t,k} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k+2\mu_t & 0 & 0 \\ 0 & 0 & k-2\mu_t & 0 \\ 0 & 0 & 0 & k \end{pmatrix}.$$

Note that

$$\det \mathcal{L}_{t,k} = k^2(k^2 - 4\mu_t^2).$$

Equation (21) implies that $|\mu_t|^2 < 1$ and thus, for all $k \geq 2$, $\det \mathcal{L}_{t,k}$ is an invertible element of $\Lambda\mathbb{C}_R$ which implies that $\mathcal{L}_{t,k}$ is invertible for all $t \in (-T', T')$ and $k \geq 2$.

Write

$$C_t(z) = \sum_{k\in\mathbb{N}} C_{t,k} z^k.$$

With $P^0 := \mathrm{I}_2$ and P^1 as in Equation (27), define for all $k \geq 1$:

$$P^{k+1}(t) := \mathcal{L}_{t,k+1}^{-1}\left(\sum_{i+j=k} P^i(t) C_{t,j}\right).$$

so that the sequence $(P^k)_{k\in\mathbb{N}} \subset \mathcal{C}^1\left((-T',T'), \Lambda\mathfrak{sl}(2,\mathbb{C})_\rho\right)$ satisfies the following recursive system for all $t \in (-T', T')$:

$$\begin{cases} P^0(t) = \mathrm{I}_2, \\ \mathcal{L}_{t,k+1}(P^{k+1}(t)) = \sum_{i+j=k} P^i(t) C_{t,j}. \end{cases}$$

With $P_t(z) := \sum P^k(t) z^k$, the Frobenius method ensures convergence for all t (see [23]). Restricting to a compact interval in $(-T', T')$ if necessary, there exists $\epsilon' > 0$ such that for all $z \in D_{\epsilon'}^*$ and $t \in (-T', T')$,

$$\Phi_t(z,\lambda) = M_t z^{A_t} P_t(z)$$

where $M_t \in \Lambda\mathrm{SL}(2,\mathbb{C})_R$ is $\mathcal{C}^1$ with respect to t, $P_t(z)$ is $\mathcal{C}^1$ with respect to t and z, and for all t, $P_t : D_{\epsilon'} \longrightarrow \Lambda\mathrm{SL}(2,\mathbb{C})_R$ is holomorphic and satisfies $P_t(0) = \mathrm{I}_2$. Moreover, the map P^1 defined in (27) vanishes at $t = 0$ and thus $P_0(z) = \mathrm{I}_2$ for all $z \in D_{\epsilon'}$, which implies that $M_0 = \mathrm{I}_2$. □

Proposition 6. *There exists $\epsilon' > 0$ such that for all $t \in (-T', T')$ there exist an admissible gauge $G_t : D_\epsilon \longrightarrow \Lambda_+\mathrm{SL}(2,\mathbb{C})_R$, a change of coordinates $h_t : D_{\epsilon'} \longrightarrow D_\epsilon$, a holomorphic map $\widetilde{P}_t : D_{\epsilon'} \longrightarrow \Lambda\mathrm{SL}(2,\mathbb{C})_R$ and $\widetilde{M}_t \in$*

$\Lambda\mathrm{SL}(2,\mathbb{C})_R$ *such that for all* $z \in D_{\epsilon'}^*$,

$$h_t^*\left(\Phi_t G_t\right)(z) = \widetilde{M}_t z^{A_t} \widetilde{P}_t(z).$$

Moreover, $\widetilde{M}_t$ *is* $\mathcal{C}^1$ *with respect to* t, $\widetilde{M}_0 = \mathrm{I}_2$ *and there exists a uniform* $C > 0$ *such that for all* t *and* z,

$$\left\|\widetilde{P}_t(z) - \mathrm{I}_2\right\|_\rho \leq C|t||z|^2.$$

Proof. The proof goes as in Section 3.3 of [18]. Expand $P^1(t)$ given by Equation (27) as a series to get (this is a tedious but simple computation):

$$P^1(t,\lambda) = \begin{pmatrix} 0 & \frac{stc_2(t,0)+rtc_3(t,0)}{2s}\lambda^{-1} \\ \frac{stc_2(t,0)+rtc_3(t,0)}{2r} & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\lambda^0) & \mathcal{O}(\lambda^0) \\ \mathcal{O}(\lambda) & \mathcal{O}(\lambda^0) \end{pmatrix}.$$

Define

$$p_t := 2\sinh q(sc_2(t,0) + rc_3(t,0))$$

so that

$$g_t := p_t A_t - P^1(t) \in \Lambda_+\mathfrak{sl}(2,\mathbb{C})_R$$

and recall Equation (26) together with $P_0 = \mathrm{I}_2$ to show that $g_0 = 0$. Thus

$$G_t := \exp\left(g_t z\right) \in \Lambda_+\mathrm{SL}(2,\mathbb{C})_R$$

is an admissible gauge. Let $\epsilon' < |p_t|^{-1}$ for all $t \in (-T', T')$. Define

$$\begin{array}{rcccc} h_t & : & D_{\epsilon'} & \longrightarrow & D_\epsilon \\ & & z & \longmapsto & \frac{z}{1+p_t z}. \end{array}$$

Then,

$$\widetilde{\xi}_t := h_t^*\left(\xi_t \cdot G_t\right) = A_t z^{-1} dz + \widetilde{C}_t(z) dz$$

is a perturbed Delaunay potential as in Definition 1 such that $\widetilde{C}_t(0) = 0$ for all $t \in (-T', T')$. The holomorphic frame

$$\widetilde{\Phi}_t := h_t^*\left(\Phi_t G_t\right)$$

satisfies $d\widetilde{\Phi}_t = \widetilde{\Phi}_t \widetilde{\xi}_t$. With $\widetilde{C}_t(0) = 0$, one can apply the Frobenius method on $\widetilde{\xi}_t$ to get

$$\widetilde{\Phi}_t(z) = \widetilde{M}_t z^{A_t} \widetilde{P}_t(z)$$

with $\widetilde{M}_0 = \mathrm{I}_2$ and

$$\left\|\widetilde{P}_t(z) - \mathrm{I}_2\right\|_R \leq C|t||z|^2.$$

$\square$

Conclusion. The new frame $\widetilde{\Phi}_t$ is associated to a perturbed Delaunay potential $(\widetilde{\xi}_t)_{t\in(-T',T')}$, defined for $z \in D^*_{\epsilon'}$, with values in $\Lambda\mathfrak{sl}(2,\mathbb{C})_R$ and of the form

$$\widetilde{\xi}_t(z) = A_t z^{-1}dz + \widehat{C}_t(z)zdz.$$

Note that $\widetilde{C}_t(z) \in \Lambda\mathfrak{sl}(2,\mathbb{C})_R$ is now $\mathcal{C}^1$ with respect to (t,z). The monodromy problem (9) is solved for $\widetilde{\Phi}_t$ and for any $\widetilde{z}_0$ in the universal cover $\widetilde{D}^*_{\epsilon'}$, $\widetilde{\Phi}_0(\widetilde{z}_0) = \widetilde{z}_0^{A_0}$. Moreover, writing $\widetilde{f}_t := \mathrm{Sym}_q(\mathrm{Uni}\,\widetilde{\Phi}_t)$ and $f_t := \mathrm{Sym}_q(\mathrm{Uni}\,\Phi_t)$, then $\widetilde{f}_t = h_t^* f_t$ with $h_0(z) = z$. Hence in order to prove Theorem 3 it suffices to prove the following proposition.

Proposition 7. *Let $\rho > e^q$, $0 < T < T_{\max}$, $\epsilon > 0$ and ξ_t be a perturbed Delaunay potential as in Definition 1. Let Φ_t be a holomorphic frame associated to ξ_t for all t via the DPW method. Suppose that the monodromy problem* (9) *is solved for all $t \in (-T,T)$ and that*

$$\Phi_t(z) = M_t z^{A_t} P_t(z)$$

where $M_t \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ is $\mathcal{C}^1$ with respect to t, satisfies $M_0 = \mathrm{I}_2$, and $P_t : D_\epsilon \longrightarrow \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ is a holomorphic map such that for all t and z,

$$\|P_t(z) - \mathrm{I}_2\|_\rho \leq C|t||z|^2$$

where $C > 0$ is a uniform constant. Let $f_t = \mathrm{Sym}_q(\mathrm{Uni}\,\Phi_t)$. Then the three points of Theorem 3 hold for f_t.

We now reset the values of ρ, T and ϵ and suppose that we are given a perturbed Delaunay frame ξ_t and a holomorphic frame Φ_t associated to it and satisfying the hypotheses of Proposition 7.

2.3. Dressed Delaunay frames

In this section we study dressed Delaunay frames arising from the DPW data $(\widetilde{\mathbb{C}}^*, \xi_t^{\mathcal{D}}, 1, M_t)$, where $\widetilde{\mathbb{C}}^*$ is the universal cover of $\mathbb{C}^*$ and

$$\xi_t^{\mathcal{D}}(z) := A_t z^{-1}dz$$

with A_t as in (14) satisfying (19), and M_t as in Proposition 6. The induced holomorphic frame is

$$\Phi_t^{\mathcal{D}}(z) = M_t z^{A_t}.$$

Note that the fact that the monodromy problem (9) is solved for Φ_t implies that it is solved for $\Phi_t^{\mathcal{D}}$ because P_t is holomorphic on D_ϵ. Let $\widetilde{D}_1^*$ be the universal cover of D_1^* and let

$$F_t^{\mathcal{D}} := \operatorname{Uni}\Phi_t^{\mathcal{D}}, \qquad B_t^{\mathcal{D}} := \operatorname{Pos}\Phi_t^{\mathcal{D}}, \qquad f_t^{\mathcal{D}} := \operatorname{Sym}_q F_t^{\mathcal{D}}.$$

In this section, our goal is to prove the following proposition.

Proposition 8. *The immersion $f_t^{\mathcal{D}}$ is a CMC H Delaunay immersion of weight $2\pi t$ for $|t|$ small enough. Moreover, for all $\delta > 0$ and $e^q < R < \rho$ there exists $C, T' > 0$ such that*

$$\left\|F_t^{\mathcal{D}}(z)\right\|_R \leq C|z|^{-\delta}.$$

for all $(t,z) \in (-T', T') \times \widetilde{D}_1^*$.

Delaunay immersion. We will need the following lemma, inspired by [20].

Lemma 2. *Let $M \in \mathrm{SL}(2,\mathbb{C})$ and $\mathcal{A} \in \mathfrak{su}(2)$ such that*

$$M \exp(\mathcal{A}) M^{-1} \in \mathrm{SU}(2). \tag{28}$$

Then there exist $U \in \mathrm{SU}(2)$ and $K \in \mathrm{SL}(2,\mathbb{C})$ such that $M = UK$ and $[K, \mathcal{A}] = 0$.

Proof. The matrix M^*M is Hermitian and positive-definite. Thus, there exists $Q \in \mathrm{SU}(2)$ and D diagonal such that $M^*M = QD^2Q^{-1}$ with

$$D = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \quad x > 0.$$

Equation (28) together with $\mathcal{A} \in \mathfrak{su}(2)$ imply $[M^*M, \mathcal{A}] = 0$, and hence $[D^2, Q^{-1}\mathcal{A}Q] = 0$. But $Q^{-1}\mathcal{A}Q \in \mathfrak{su}(2)$, therefore $D = \mathrm{I}_2$ or $Q^{-1}\mathcal{A}Q$ is diagonal. In any case, $[D, Q^{-1}\mathcal{A}Q] = 0$. Defining K as the square root of M^*M, this implies $[K, \mathcal{A}] = 0$. Setting $U = MK^{-1}$ and checking that $U^*U = \mathrm{I}_2$ ends the proof. $\square$

Corollary 1. *There exists $T' > 0$ such that for all $t \in (-T', T')$,*

$$\Phi_t(z) = U_t z^{A_t} K_t$$

where $U_t \in \Lambda\mathrm{SU}(2)_R$ and $K_t \in \Lambda\mathrm{SL}(2,\mathbb{C})_R$ for any $e^q < R < \rho$.

Proof. Write $\mathcal{M}\left(\Phi_t^{\mathcal{D}}\right) = M_t \exp\left(\mathcal{A}_t\right) M_t^{-1}$ with $\mathcal{A}_t := 2i\pi A_t \in \Lambda\mathfrak{su}(2)_\rho$ continuous on $(-T, T)$. The map

$$M \longmapsto \sqrt{M^* M} = \exp\left(\frac{1}{2}\log M^* M\right)$$

is a diffeomorphism from a neighbourhood of $\mathrm{I}_2 \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ to another neighbourhood of I_2. Using the convergence of M_t towards I_2 as t tends to 0, this allows us to use Lemma 2 pointwise on $\mathbb{A}_\rho$ and thus construct $K_t := \sqrt{M_t^* M_t} \in \Lambda\mathrm{SL}(2,\mathbb{C})_R$ for all $t \in (-T', T')$ and any $e^q < R < \rho$. Let $U_t := M_t K_t^{-1} \in \Lambda\mathrm{SL}(2,\mathbb{C})_R$ and compute $U_t U_t^*$ to show that $U_t \in \Lambda\mathrm{SU}(2)_R$. Use Lemma 2 to show that $[K_t(\lambda), \mathcal{A}_t(\lambda)] = 0$ for all $\lambda \in \mathbb{S}^1$. Hence $[K_t, \mathcal{A}_t] = 0$ and thus $\Phi_t^{\mathcal{D}} = U_t z^{A_t} K_t$. $\square$

Returning to the proof of Proposition 8, let $\theta \in \mathbb{R}$, $z \in \mathbb{C}^*$ and $e^q < R < \rho$. Apply Corollary 1 to get

$$\Phi_t^{\mathcal{D}}(e^{i\theta} z) = U_t \exp(i\theta A_t) U_t^{-1} \Phi_t^{\mathcal{D}}(z)$$

and note that $U_t \in \Lambda\mathrm{SU}(2)_R$, $iA_t \in \Lambda\mathfrak{su}(2)_R$ imply

$$R_t(\theta) := U_t \exp(i\theta A_t) U_t^{-1} \in \Lambda\mathrm{SU}(2)_R. \tag{29}$$

Hence

$$F_t^{\mathcal{D}}(e^{i\theta z}) = R_t(\theta) F_t^{\mathcal{D}}(z)$$

and

$$f_t^{\mathcal{D}}(e^{i\theta} z) = R_t(\theta, e^{-q}) \cdot f_t^{\mathcal{D}}(z).$$

Use Section 1.3 and note that U_t does not depend on θ to see that $f_t^{\mathcal{D}}$ is a CMC immersion of revolution and hence a Delaunay immersion. Its weight can be read from its Hopf diffferential, which in turn can be read from the potential $\xi_t^{\mathcal{D}}$ (see Equation (7)). Thus $f_t^{\mathcal{D}}$ is a CMC H Delaunay immersion of weight $2\pi t$, which proves the first part of Proposition 8.

Restricting to a meridian. Note that for all $t \in (-T', T')$ and $z \in \mathbb{C}^*$,

$$\left\|F_t^{\mathcal{D}}(z)\right\|_R \leq C \left\|F_t^{\mathcal{D}}(|z|)\right\|_R$$

where

$$C = \sup\left\{\|R_t(\theta)\|_R \mid (t,\theta) \in (-T',T') \times [0, 2\pi]\right\}$$

depends only on R. We thus restrict $F_t^{\mathcal{D}}$ to $\mathbb{R}_+^*$ with $\widehat{F}_t^{\mathcal{D}}(x) := F_t^{\mathcal{D}}(|z|)$ ($x = |z|$).

Grönwall over a period. Recalling the Lax Pair associated to $F_t^{\mathcal{D}}$ (see Appendix C in [18]), the restricted map $\widehat{F}_t^{\mathcal{D}}$ satisfies $d\widehat{F}_t^{\mathcal{D}} = \widehat{F}_t^{\mathcal{D}}\widehat{W}_t dx$ with

$$\widehat{W}_t(x,\lambda) = \frac{1}{x}\begin{pmatrix} 0 & \lambda^{-1} r b^2(x) - s b^{-2}(x) \\ s b^{-2}(x) - \lambda r b^2(x) & 0 \end{pmatrix}$$

where $b(x)$ is the upper-left entry of $B_t^{\mathcal{D}}(x)\mid_{\lambda=0}$. Recall Section 1.2 and define

$$g_t(x) = 2\sinh q|r|b(x)^2 x^{-1}$$

so that the metric of $f_t^{\mathcal{D}}$ reads $g_t(x)^2|dz|^2$. Let $\widetilde{f}_t^{\mathcal{D}} := \exp^* f_t^{\mathcal{D}}$. Then the metric of $\widetilde{f}_t^{\mathcal{D}}$ satisfies

$$d\widetilde{s}^2 = 4r^2(\sinh q)^2 b^4(e^u)(du^2 + d\theta^2)$$

at the point $u + i\theta = \log z$. Using Proposition 13 of Section A gives

$$\int_0^{S_t} 2|r|b^2(e^u)du = \pi \quad \text{and} \quad \int_0^{S_t} \frac{du}{2\sinh q|r|b^2(e^u)} = \frac{\pi}{|t|}$$

where $S_t > 0$ is the period of the profile curve of $\widetilde{f}_t$. Thus

$$\int_1^{e^{S_t}} |rb^2(x)x^{-1}|dx = \frac{\pi}{2} = \int_1^{e^{S_t}} |sb^{-2}(x)x^{-1}|dx.$$

Using

$$\left\|\widehat{W}_t(x)\right\|_R = \sqrt{2}\left|sb^{-2}(x)x^{-1}\right| + 2R\left|rb^2(x)x^{-1}\right|,$$

we deduce

$$\int_1^{e^{S_t}} \left\|\widehat{W}_t(x)\right\|_R dx = \frac{\pi}{2}(2R + \sqrt{2}) < C \tag{30}$$

where $C > 0$ is a constant depending only on ρ and T. Applying Grönwall's lemma to the inequality

$$\left\|\widehat{F}_t^{\mathcal{D}}(x)\right\|_R \leq \left\|\widehat{F}_t^{\mathcal{D}}(1)\right\|_R + \int_1^x \left\|\widehat{F}_t^{\mathcal{D}}(u)\right\|_R \left\|\widehat{W}_t(u)\right\|_R |du|$$

gives

$$\left\|\widehat{F}_t^{\mathcal{D}}(x)\right\|_R \leq \left\|\widehat{F}_t^{\mathcal{D}}(1)\right\|_R \exp\left(\int_1^x \left\|\widehat{W}_t(u)\right\|_R |du|\right).$$

Use Equation (30) together with the fact that $\widetilde{F}_0^{\mathcal{D}}(0) = F_0^{\mathcal{D}}(1) = \mathrm{I}_2$ and the continuity of Iwasawa decomposition to get $C, T > 0$ such that for all $t \in (-T', T')$ and $x \in [1, e^{S_t}]$

$$\left\|\widehat{F}_t^{\mathcal{D}}(x)\right\|_R \leq C. \tag{31}$$

Control over the periodicity matrix. Let $t \in (-T', T')$ and $\Gamma_t := \widehat{F}_t^{\mathcal{D}}(xe^{S_t})\widehat{F}_t^{\mathcal{D}}(x)^{-1} \in \Lambda\mathrm{SU}(2)_R$ for all $x > 0$. The periodicity matrix Γ_t does not depend on x because $\widehat{W}_t(xe^{S_t}) = \widehat{W}_t(x)$ (by periodicity of the metric in the log coordinate). Moreover,

$$\|\Gamma_t\|_R = \left\|\widehat{F}_t^{\mathcal{D}}(e^{S_t})\widehat{F}_t^{\mathcal{D}}(1)^{-1}\right\|_R \leq \left\|\widehat{F}_t^{\mathcal{D}}(e^{S_t})\right\|_R \left\|\widehat{F}_t^{\mathcal{D}}(1)\right\|_R,$$

and using Equation (31),

$$\|\Gamma_t\|_R \leq C. \tag{32}$$

Conclusion. Let $x < 1$. Then there exist $k \in \mathbb{N}^*$ and $\zeta \in \left[1, e^{S_t}\right)$ such that $x = \zeta e^{-kS_t}$. Thus using equations (31) and (32),

$$\left\|\widehat{F}_t^{\mathcal{D}}(x)\right\|_R \leq \left\|\Gamma_t^{-k}\right\|_R \left\|\widehat{F}_t^{\mathcal{D}}(\zeta)\right\|_R \leq C^{k+1}.$$

Writing

$$k = \frac{\log\zeta}{S_t} - \frac{\log x}{S_t},$$

one gets

$$C^k = \exp\left(\frac{\log\zeta}{S_t}\log C\right)\exp\left(\frac{-\log C}{S_t}\log x\right) \leq Cx^{-\delta_t}$$

where $\delta_t = \frac{\log C}{S_t}$ does not depend on x and tends to 0 as t tends to 0 (because $S_t \underset{t\to 0}{\longrightarrow} +\infty$). Returning back to $F_t^{\mathcal{D}}$, we showed that for all $\delta > 0$ there exist

$T' > 0$ and $C > 0$ such that for all $t \in (-T', T')$ and $0 < |z| < 1$,

$$\left\| F_t^{\mathcal{D}}(z) \right\|_R \leq C|z|^{-\delta}$$

and Proposition 8 is proved.

2.4. Convergence of the immersions

In this section, we prove the first part of Theorem 3: the convergence of the immersions f_t towards the immersions $f_t^{\mathcal{D}}$. Our proof relies on the Iwasawa decomposition being a diffeomorphism in a neighbourhood of I_2.

Behaviour of the Delaunay positive part. Let $z \in \widetilde{D}_1^*$. The Delaunay positive part satisfies

$$\left\| B_t^{\mathcal{D}}(z) \right\|_\rho = \left\| F_t^{\mathcal{D}}(z)^{-1} M_t z^{A_t} \right\|_\rho \leq \left\| F_t^{\mathcal{D}}(z) \right\|_\rho \|M_t\|_\rho \left\| z^{A_t} \right\|_\rho .$$

Diagonalise $A_t = H_t D_t H_t^{-1}$ as in Proposition 3. By continuity of H_t and M_t, and according to Proposition 8, there exists $C, T' > 0$ such that for all $t \in (-T', T')$

$$\left\| B_t^{\mathcal{D}}(z) \right\|_R \leq C|z|^{-\delta} \left\| z^{-\mu_t} \right\|_R .$$

Recall Equation (20) and extend $\mu_t^2 = -\det A_t$ to $\mathbb{A}_{\widetilde{R}}$ with $\rho > \widetilde{R} > R$. One can thus assume that for $t \in (-T', T')$ and $\lambda \in \mathbb{A}_{\widetilde{R}}$,

$$|\mu_t(\lambda)| < \frac{1}{2} + \delta,$$

which implies that

$$\left| z^{-\mu_t(\lambda)} \right| \leq |z|^{\frac{-1}{2} - \delta}.$$

This gives us the following estimate in the $\Lambda\mathbb{C}_R$ norm (using Cauchy formula and $\widetilde{R} > R$):

$$\left\| z^{-\mu_t} \right\|_R \leq C|z|^{\frac{-1}{2} - \delta}$$

and

$$\left\| B_t^{\mathcal{D}}(z) \right\|_R \leq C|z|^{\frac{-1}{2} - 2\delta}. \tag{33}$$

Behaviour of a holomorphic frame. Let

$$\widetilde{\Phi}_t := B_t^{\mathcal{D}} \left(\Phi_t^{\mathcal{D}}\right)^{-1} \Phi_t \left(B_t^{\mathcal{D}}\right)^{-1}.$$

Recall Proposition 6 and use Equation (33) to get for all $t \in (-T', T')$ and $z \in D_\epsilon^*$:

$$\left\|\widetilde{\Phi}_t(z) - \mathrm{I}_2\right\|_R = \left\|B_t^{\mathcal{D}}(z)\left(P_t(z) - \mathrm{I}_2\right) B_t^{\mathcal{D}}(z)^{-1}\right\|_R \leq C|t||z|^{1-4\delta}.$$

Behaviour of the perturbed immersion. Note that

$$\begin{aligned}\widetilde{\Phi}_t &= B_t^{\mathcal{D}} \left(\Phi_t^{\mathcal{D}}\right)^{-1} \Phi_t \left(B_t^{\mathcal{D}}\right)^{-1}\\ &= \left(F_t^{\mathcal{D}}\right)^{-1} F_t \times B_t \left(B_t^{\mathcal{D}}\right)^{-1}\end{aligned}$$

and recall that the Iwasawa decomposition is differentiable at the identity to get

$$\left\|F_t^{\mathcal{D}}(z)^{-1}F_t(z) - \mathrm{I}_2\right\|_R = \left\|\mathrm{Uni}\,\widetilde{\Phi}_t(z) - \mathrm{Uni}\,\mathrm{I}_2\right\|_R \leq C|t||z|^{1-4\delta}.$$

The map

$$\widetilde{F}_t(z) := F_t^{\mathcal{D}}(z, e^{-q})^{-1}F_t(z, e^{-q}) \in \mathrm{SL}(2,\mathbb{C})$$

satisfies

$$\left|\widetilde{F}_t(z) - \mathrm{I}_2\right| \leq \left\|F_t^{\mathcal{D}}(z)^{-1}F_t(z) - \mathrm{I}_2\right\|_R \leq C|t||z|^{1-4\delta}. \tag{34}$$

Lemma 3. *There exists a neighbourhood $V \subset \mathrm{SL}(2,\mathbb{C})$ of I_2 and $C > 0$ such that for all $A \in \mathrm{SL}(2,\mathbb{C})$,*

$$A \in V \implies \left|\mathrm{tr}\left(AA^*\right) - 2\right| \leq C\left|A - \mathrm{I}_2\right|^2.$$

Proof. Consider $\exp : U \subset \mathfrak{sl}(2,\mathbb{C}) \longrightarrow V \subset \mathrm{SL}(2,\mathbb{C})$ as a local chart of $\mathrm{SL}(2,\mathbb{C})$ around I_2. Let $A \in V$. Write

$$\begin{array}{rccc} f & : & \mathrm{SL}(2,\mathbb{C}) & \longrightarrow & \mathbb{R}\\ & & X & \longmapsto & \mathrm{tr}\left(XX^*\right)\end{array}$$

and $a = \log A \in U$ to get

$$|f(A) - f(\mathrm{I}_2)| \leq df(\mathrm{I}_2)\cdot a + C|a|^2.$$

Notice that for all $a \in \mathfrak{sl}(2,\mathbb{C})$,

$$df(\mathrm{I}_2) \cdot a = \mathrm{tr}(a + a^*) = 0$$

to end the proof. □

Corollary 2. *There exists a neighbourhood $V \subset \mathrm{SL}(2,\mathbb{C})$ of I_2 and $C > 0$ such that for all $F_1, F_2 \in \mathrm{SL}(2,\mathbb{C})$,*

$$F_2^{-1}F_1 \in V \implies d_{\mathbb{H}^3}(f_1, f_2) < C\left|F_2^{-1}F_1 - \mathrm{I}_2\right|,$$

where $f_i = F_i \cdot \mathrm{I}_2 \in \mathbb{H}^3$.

Proof. Just note that

$$d_{\mathbb{H}^3}(f_1, f_2) = \cosh^{-1}\left(-\langle f_1, f_2\rangle\right) = \cosh^{-1}\left(\frac{1}{2}\,\mathrm{tr}(f_2^{-1}f_1)\right)$$

and that

$$\mathrm{tr}\left(f_2^{-1}f_1\right) = \mathrm{tr}\left(\left(F_2{F_2}^*\right)^{-1}F_1{F_1}^*\right) = \mathrm{tr}\,\widetilde{F}\widetilde{F}^*$$

where $\widetilde{F} = F_2^{-1}F_1$. Apply Lemma 3 and use $\cosh^{-1}(1+x) \sim \sqrt{2x}$ as $x \to 0$ to end the proof. □

Without loss of generality, we can suppose from (34) that $C|t||z|^{1-4\delta}$ is small enough for $\widetilde{F}_t(z)$ to be in V for all t and z. Apply Corollary 2 to end the proof of the first point in Theorem 3:

$$d_{\mathbb{H}^3}\left(f_t(z), f_t^{\mathcal{D}}(z)\right) \le C|t||z|^{1-4\delta}.$$

2.5. Convergence of the normal maps

Before starting the proof of the second part of Theorem 3, we will need to compare the normal maps of our immersions. Let $N_t := \mathrm{Nor}_q F_t$ and $N_t^{\mathcal{D}} := \mathrm{Nor}_q F_t^{\mathcal{D}}$ be the normal maps associated to the immersions f_t and $f_t^{\mathcal{D}}$. This section is devoted to the proof of the following proposition.

Proposition 9. *For all $\delta > 0$ there exist $\epsilon', T', C > 0$ such that for all $t \in (-T', T')$ and $z \in D_\epsilon^*$,*

$$\left\|\Gamma_{f_t(z)}^{f_t^{\mathcal{D}}(z)} N_t(z) - N_t^{\mathcal{D}}(z)\right\|_{T\mathbb{H}^3} \le C|t||z|^{1-\delta}.$$

The following lemma measures the deviation from Euclidean geometry in the parallel transportation of unitary vectors.

Lemma 4. *Let $a, b, c \in \mathbb{H}^3$, $v_a \in T_a\mathbb{H}^3$ and $v_b \in T_b\mathbb{H}^3$ both unitary. Let $\mathcal{A}$ be the hyperbolic area of the triangle (a, b, c). Then*

$$\left\|\Gamma_a^b v_a - v_b\right\|_{T_b\mathbb{H}^3} \leq \mathcal{A} + \|\Gamma_a^c v_a - \Gamma_b^c v_b\|_{T_c\mathbb{H}^3}.$$

Proof. Just use the triangular inequality and Gauss-Bonnet formula in $\mathbb{H}^2$ to write:

$$\begin{aligned}\left\|\Gamma_a^b v_a - v_b\right\|_{T_b\mathbb{H}^3} &= \left\|\Gamma_c^a\Gamma_b^c\Gamma_a^b v_a - \Gamma_c^a\Gamma_b^c v_b\right\|_{T_a\mathbb{H}^3} \\ &\leq \left\|\Gamma_c^a\Gamma_b^c\Gamma_a^b v_a - v_a\right\|_{T_a\mathbb{H}^3} + \|v_a - \Gamma_c^a\Gamma_b^c v_b\|_{T_a\mathbb{H}^3} \\ &\leq \mathcal{A} + \|\Gamma_a^c v_a - \Gamma_b^c v_b\|_{T_c\mathbb{H}^3}.\end{aligned}$$

□

Lemma 5 below clarifies how the unitary frame encodes the immersion and the normal map.

Lemma 5. *Let $f = \mathrm{Sym}_q F$ and $N = \mathrm{Nor}_q F$. Denoting by $(S(z), Q(z)) \in \mathcal{H}_2^{++} \cap \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SU}(2)$ the polar decomposition of $F(z, e^{-q})$, then*

$$f = S^2 \quad \textit{and} \quad N = \Gamma_{\mathrm{I}_2}^f (Q \cdot \sigma_3).$$

Proof. The formula for f is straightforward after noticing that $QQ^* = \mathrm{I}_2$ and $S^* = S$. The formula for N is a direct consequence of Proposition 1. □

Proof of Proposition 9. Let $\delta > 0$, $t \in (-T', T')$ and $z \in D_{\epsilon'}^*$. Using Lemma 4,

$$\left\|\Gamma_{f_t(z)}^{f_t^{\mathcal{D}}(z)} N_t(z) - N_t^{\mathcal{D}}(z)\right\| \leq \mathcal{A} + \left\|\Gamma_{f_t(z)}^{\mathrm{I}_2} N_t(z) - \Gamma_{f_t^{\mathcal{D}}(z)}^{\mathrm{I}_2} N_t^{\mathcal{D}}(z)\right\|$$

where $\mathcal{A}$ is the hyperbolic area of the triangle $\left(\mathrm{I}_2, f_t(z), f_t^{\mathcal{D}}(z)\right)$. Using Heron's formula in $\mathbb{H}^2$ (see [1], p.66), Proposition 8 and the first part of Theorem 3,

$$\mathcal{A} \leq d_{\mathbb{H}^3}\left(f_t(z), f_t^{\mathcal{D}}(z)\right) \times d_{\mathbb{H}^3}\left(\mathrm{I}_2, f_t^{\mathcal{D}}(z)\right) \leq C|t||z|^{1-\delta}.$$

Moreover, denoting by Q_t and $Q_t^{\mathcal{D}}$ the unitary parts of $F_t(e^q)$ and $F_t^{\mathcal{D}}(e^q)$ in their polar decomposition and using Lemma 5 together with Corollary 3 and Equation (34),

$$\begin{aligned}\left\|\Gamma^{\mathrm{I}_2}_{f_t(z)} N_t(z) - \Gamma^{\mathrm{I}_2}_{f_t^{\mathcal{D}}(z)} N_t^{\mathcal{D}}(z)\right\| &= \left\|Q_t(z)\cdot\sigma_3 - Q_t^{\mathcal{D}}(z)\cdot\sigma_3\right\|_{T_{\mathrm{I}_2}\mathbb{H}^3}\\ &\le C\left\|F_t^{\mathcal{D}}(z)\right\|_R^2 \left\|F_t^{\mathcal{D}}(z)^{-1}F_t^{\mathcal{D}}(z) - \mathrm{I}_2\right\|_R\\ &\le C|t||z|^{1-3\delta}.\end{aligned}$$

2.6. Embeddedness

In this section, we prove the second part of Theorem 3. We thus assume that $t > 0$. We suppose that $C, \epsilon, T, \delta > 0$ satisfy Proposition 9 and the first part of Theorem 3.

Lemma 6. *Let $r_t > 0$ such that the tubular neighbourhood of $f_t^{\mathcal{D}}(\mathbb{C}^*)$ with hyperbolic radius r_t is embedded. There exists $T > 0$ and $0 < \epsilon' < \epsilon$ such that for all $0 < t < T$, $x \in \partial D_\epsilon$ and $y \in D^*_{\epsilon'}$,*

$$d_{\mathbb{H}^3}\left(f_t^{\mathcal{D}}(x), f_t^{\mathcal{D}}(y)\right) > 4r_t.$$

Proof. The convergence of $f_t^{\mathcal{D}}(\mathbb{C}^*)$ towards a chain of spheres implies that r_t tends to 0 as t tends to 0. If $f_t^{\mathcal{D}}$ does not degenerate into a point, then it converges towards the parametrisation of a sphere, and for all $0 < \epsilon' < \epsilon$ there exists $T > 0$ satisfying the inequality. If $f_t^{\mathcal{D}}$ does degenerate into a point, then a suitable blow-up makes it converge towards a catenoidal immersion on the punctured disk D^*_ϵ (see Section 4.1). This implies that for $\epsilon' > 0$ small enough, there exists $T > 0$ satisfying the inequality. □

We can now prove embeddedness with the same method than [18]. Let $\mathcal{D}_t := f_t^{\mathcal{D}}(\mathbb{C}^*) \subset \mathbb{H}^3$ be the image Delaunay surface of $f_t^{\mathcal{D}}$. We denote by $\eta_t^{\mathcal{D}} : \mathcal{D}_t \longrightarrow T\mathbb{H}^3$ the Gauss map of $\mathcal{D}_t$. We also write $\mathcal{M}_t = f_t(D^*_\epsilon)$ and $\eta_t : \mathcal{M}_t \longrightarrow T\mathbb{H}^3$. Let r_t be the maximal value of α such that the following map is a diffeomorphism:

$$\begin{array}{cccc}\mathcal{T} & : & (-\alpha,\alpha)\times\mathcal{D}_t & \longrightarrow & \operatorname{Tub}_\alpha \mathcal{D}_t \subset \mathbb{H}^3\\ & & (s,p) & \longmapsto & \operatorname{geod}\left(p, \eta_t^{\mathcal{D}}(p)\right)(s).\end{array}$$

According to Lemma 11, the maximal tubular radius satisfies $r_t \sim t$ as t tends to 0. Using the first part of Theorem 3, we thus assume that on D^*_ϵ,

$$d_{\mathbb{H}^3}\left(f_t(z), f_t^{\mathcal{D}}(z)\right) \le \alpha r_t$$

where $\alpha < 1$ is given by Lemma 12 of Section A.

Let π_t be the projection from $\mathrm{Tub}_{r_t}\mathcal{D}_t$ to $\mathcal{D}_t$. Then the map

$$\begin{array}{rccc} \varphi_t & : & D_\epsilon^* & \longrightarrow & \mathcal{D}_t \\ & & z & \longmapsto & \pi_t \circ f_t(z) \end{array}$$

is well-defined and satisfies

$$d_{\mathbb{H}^3}\left(\varphi_t(z), f_t^{\mathcal{D}}(z)\right) \leq 2\alpha r_t \tag{35}$$

because of the triangular inequality.

Lemma 7. *For $t > 0$ small enough, φ_t is a local diffeomorphism on D_ϵ^*.*

Proof. It suffices to show that for all $z \in D_\epsilon^*$,

$$\left\|\Gamma_{\varphi_t(z)}^{f_t(z)} \eta_t^{\mathcal{D}}(\varphi_t(z)) - N_t(z)\right\| < 1. \tag{36}$$

Using Lemma 4 (we drop the variable z to ease the notation),

$$\left\|\Gamma_{\varphi_t}^{f_t} \eta_t^{\mathcal{D}}(\varphi_t) - N_t\right\| \leq \mathcal{A} + \left\|\Gamma_{\varphi_t}^{f_t^{\mathcal{D}}} \eta_t^{\mathcal{D}}(\varphi_t) - \Gamma_{f_t}^{f_t^{\mathcal{D}}} N_t\right\|$$

where $\mathcal{A}$ is the area of the triangle $\left(f_t, f_t^{\mathcal{D}}, \varphi_t\right)$. Recall the isoperimetric inequality in $\mathbb{H}^2$ (see [24]):

$$\mathcal{P}^2 \geq 4\pi\mathcal{A} + \mathcal{A}^2$$

from which we deduce

$$\mathcal{A} \leq \mathcal{P}^2 \leq \left(2d_{\mathbb{H}^3}\left(f_t, f_t^{\mathcal{D}}\right) + 2d_{\mathbb{H}^3}\left(\varphi_t, f_t^{\mathcal{D}}\right)\right)^2 \leq (6\alpha r_t)^2$$

which uniformly tends to 0 as t tends to 0. Using the triangular inequality and Proposition 9,

$$\left\|\Gamma_{\varphi_t}^{f_t^{\mathcal{D}}} \eta_t^{\mathcal{D}}(\varphi_t) - \Gamma_{f_t}^{f_t^{\mathcal{D}}} N_t\right\| \leq \left\|\Gamma_{\varphi_t}^{f_t^{\mathcal{D}}} \eta_t^{\mathcal{D}}(\varphi_t) - N_t^{\mathcal{D}}\right\| + C|t||z|^{1-\delta}$$

and the second term of the right-hand side uniformly tends to 0 as t tends to 0. Because α satisfies Lemma 12 in Section A,

$$\left\|\Gamma_{\varphi_t}^{f_t^{\mathcal{D}}} \eta_t^{\mathcal{D}}(\varphi_t) - N_t^{\mathcal{D}}\right\| < 1$$

which implies Equation (36). □

Let $\epsilon' > 0$ given by Lemma 6. The restriction

$$\begin{array}{rcccc}\widetilde{\varphi}_t & : & \varphi_t^{-1}\left(\varphi_t(D^*_{\epsilon'})\right) \cap D^*_\epsilon & \longrightarrow & \varphi_t\left(D^*_{\epsilon'}\right) \\ & & z & \longmapsto & \varphi_t(z)\end{array}$$

is a covering map because it is a proper local diffeomorphism between locally compact spaces. To show this, proceed by contradiction as in $\mathbb{R}^3$ (see [18]): let $(x_i)_{i\in\mathbb{N}} \subset \varphi_t^{-1}\left(\varphi_t(D^*_{\epsilon'})\right) \cap D^*_\epsilon$ such that $\left(\widetilde{\varphi}_t(x_i)\right)_{i\in\mathbb{N}}$ converges to $p \in \varphi_t\left(D^*_{\epsilon'}\right)$. Then $(x_i)_i$ converges to $x \in \overline{D}_\epsilon$. Using Equation (35) and the fact that $f_t^{\mathcal{D}}$ has an end at 0, $x \neq 0$. If $x \in \partial D_\epsilon$, denoting $\widetilde{x} \in D^*_{\epsilon'}$ such that $\widetilde{\varphi}_t(\widetilde{x}) = p$, one has

$$d_{\mathbb{H}^3}\left(f_t^{\mathcal{D}}(x), f_t^{\mathcal{D}}(\widetilde{x})\right) < d_{\mathbb{H}^3}\left(f_t^{\mathcal{D}}(x), p\right) + d_{\mathbb{H}^3}\left(f_t^{\mathcal{D}}(\widetilde{x}), \widetilde{\varphi}_t(\widetilde{x})\right) < 4\alpha r_t < 4r_t$$

which contradicts the definition of ϵ'.

Let us now prove as in [18] that $\widetilde{\varphi}_t$ is a one-sheeted covering map. Let $\gamma : [0,1] \longrightarrow D^*_{\epsilon'}$ be a loop of winding number 1 around 0, $\Gamma = f_t^{\mathcal{D}}(\gamma)$ and $\widetilde{\Gamma} = \widetilde{\varphi}_t(\gamma) \subset \mathcal{D}_t$ and let us construct a homotopy between Γ and $\widetilde{\Gamma}$. For all $s \in [0,1]$, let $\sigma_s : [0,1] \longrightarrow \mathbb{H}^3$ be a geodesic arc joining $\sigma_s(0) = \widetilde{\Gamma}(s)$ to $\sigma_s(1) = \Gamma(s)$. For all $s, r \in [0,1]$, $d_{\mathbb{H}^3}\left(\sigma_s(r), \Gamma(s)\right) \leq \alpha r_t$ which implies that $\sigma_s(r) \in \operatorname{Tub}_{r_t} \mathcal{D}_t$ because $\mathcal{D}_t$ is complete. One can thus define the following homotopy between Γ and $\widetilde{\Gamma}$

$$\begin{array}{rcccc}H & : & [0,1]^2 & \longrightarrow & \mathcal{D}_t \\ & & (r,s) & \longmapsto & \pi_t \circ \sigma_s(r)\end{array}$$

where π_t is the projection from $\operatorname{Tub}_{r_t} \mathcal{D}_t$ to $\mathcal{D}_t$. Using the fact that $f_t^{\mathcal{D}}$ is an embedding, the degree of Γ is one, and the degree of $\widetilde{\Gamma}$ is also one. Hence, $\widetilde{\varphi}_t$ is one-sheeted.

Finally, the map $\widetilde{\varphi}_t$ is a one-sheeted covering map and hence a diffeomorphism, so $f_t\left(D^*_{\epsilon'}\right)$ is a normal graph over $\mathcal{D}_t$ contained in its embedded tubular neighbourhood and $f_t\left(D^*_{\epsilon'}\right)$ is thus embedded, which proves the second part of Theorem 3.

2.7. Limit axis

In this section, we prove the third part of Theorem 3 and compute the limit axis of $f_t^{\mathcal{D}}$ as t tends to 0. Recall that $f_t^{\mathcal{D}} = \operatorname{Sym}_q\left(\operatorname{Uni}\left(M_t z^{A_t}\right)\right)$ where M_t tends to I_2 as t tends to 0. Hence, the limit axis of $f_t^{\mathcal{D}}$ and $\widetilde{f}_t^{\mathcal{D}} := \operatorname{Sym}_q\left(\operatorname{Uni}\left(z^{A_t}\right)\right)$ are the same. Two cases can occur, whether $r > s$ or $r < s$.

Spherical family. At $t = 0$, $r = \frac{1}{2}$ and $s = 0$. The limit potential is thus

$$\xi_0(z,\lambda) = \begin{pmatrix} 0 & \frac{\lambda^{-1}}{2} \\ \frac{\lambda}{2} & 0 \end{pmatrix} z^{-1}dz.$$

Consider the gauge

$$G(z,\lambda) = \frac{1}{\sqrt{2z}} \begin{pmatrix} 1 & 0 \\ \lambda & 2z \end{pmatrix}.$$

The gauged potential is then

$$\xi_0 \cdot G(z,\lambda) = \begin{pmatrix} 0 & \lambda^{-1}dz \\ 0 & 0 \end{pmatrix} = \xi_S(z,\lambda)$$

where ξ_S is the spherical potential as in Section 1.2. Let $\widetilde{\Phi} := z^{A_0}G$ be the gauged holomorphic frame and compute

$$\begin{aligned} \widetilde{\Phi}(1,\lambda) &= G(1,\lambda) \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ \lambda & 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix} \\ &= H_0(\lambda)\Phi_S(1,\lambda) \end{aligned}$$

where Φ_S and H_0 are defined in (10) and (23). This means that $\widetilde{\Phi} = H_0\Phi_S$, $\operatorname{Uni}\widetilde{\Phi} = H_0F_S$ and $\operatorname{Sym}_q(\operatorname{Uni}\widetilde{\Phi}) = H_0(e^{-q}) \cdot f_S$ because $H_0 \in \Lambda\mathrm{SU}(2)_R$. Thus using equations (12) and (13),

$$\begin{aligned} \widetilde{f_0^{\mathcal{D}}}(\infty) &= \operatorname{Sym}_q(\operatorname{Uni}\widetilde{\Phi})(\infty) = H_0(e^{-q}) \cdot f_S(\infty) \\ &= \big(H_0(e^{-q})R(q)\big) \cdot \operatorname{geod}(\mathrm{I}_2, \sigma_3)(q) \\ &= H_0(e^{-q}) \cdot \operatorname{geod}(\mathrm{I}_2, \sigma_3)(2q). \end{aligned}$$

And with the same method,

$$\widetilde{f_0^{\mathcal{D}}}(0) = H_0(e^{-q}) \cdot \operatorname{geod}(\mathrm{I}_2, \sigma_3)(0).$$

This means that the limit axis of $\widetilde{f_t^{\mathcal{D}}}$ as $t \to 0$, oriented from $z = \infty$ to $z = 0$ is given in the spherical family by

$$H_0(e^{-q}) \cdot \operatorname{geod}(\mathrm{I}_2, -\sigma_3).$$

Catenoidal family. We cannot use the same method as above, as the immersion $\widetilde{f}_t^{\mathcal{D}}$ degenerates into the point I_2. Use Proposition 12 of Section 4.1 to get

$$\widehat{f} := \lim_{t\to 0} \frac{1}{t}\left(f_t - \mathrm{I}_2\right) = \psi \subset T_{\mathrm{I}_2}\mathbb{H}^3$$

where ψ is the immersion of a catenoid of axis oriented by $-\sigma_1$ as $z \to 0$. This suffices to show that the limit axis oriented from the end at ∞ to the end at 0 of the catenoidal family $\widetilde{f}_t^{\mathcal{D}}$ converges as t tends to 0 to the oriented geodesic $\operatorname{geod}(\mathrm{I}_2, -\sigma_1)$.

3. Gluing Delaunay ends to hyperbolic spheres

In this section, we follow step by step the method Martin Traizet used in $\mathbb{R}^3$ ([25]) to construct CMC $H > 1$ n-noids in $\mathbb{H}^3$ and prove Theorem 1. This method relies on the Implicit Function Theorem and aims to find a pair (ξ_t, Φ_t) satisfying the hypotheses of Theorem 3 around each pole of an n-punctured sphere. More precisely, the Implicit Function Theorem is used to solve the monodromy problem around each pole and to ensure that the potential is regular at $z = \infty$. The set of equations characterising this problem at $t = 0$ is the same as in [25], and the partial derivative with respect to the parameters is the same as in [25] at $t = 0$. Therefore, the Implicit Function Theorem can be used exactly as in [25] and we do not repeat it here. Showing that the surface has Delaunay ends involves slightly different computations, but the method is the same as in [25], namely, find a suitable gauge and change of coordinates around each pole of the potential in order to retrieve a perturbed Delaunay potential as in Definition 1. One can then apply Theorem 3. Finally, we show that the surface is Alexandrov-embedded (and embedded in some cases) by adapting the arguments of [26] to the case of $\mathbb{H}^3$.

3.1. The DPW data

Let $H > 1$, $q = \operatorname{arcoth} H$ and $\rho > e^q$. Let $n \geq 3$, $u_1, \cdots, u_n$ unitary vectors of $T_{\mathrm{I}_2}\mathbb{H}^3$ and $\tau_1, \cdots, \tau_n$ non-zero real weights. Suppose, by applying a rotation, that $u_i \neq \pm\sigma_3$ for all $i \in [1, n]$. Let $v_S : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{S}^2$ defined as in Equation (13) and $\pi_i := v_S^{-1}(u_i) \in \mathbb{C}^*$. Consider $3n$ parameters a_i, b_i, $p_i \in \Lambda\mathbb{C}_\rho^{\geq 0}$ assembled into a vector $\mathbf{x} \in \left(\Lambda\mathbb{C}_\rho^{\geq 0}\right)^{3n}$ which stands in a neighbourhood of a central value $\mathbf{x}_0 \in \left(\Lambda\mathbb{C}_\rho^{\geq 0}\right)^{3n}$ so that the central values of a_i and p_i are τ_i and π_i. The central value of b_i will be denoted ν_i. Introduce a real parameter

t in a neighbourhood of 0 and define

$$\beta_t(\lambda) := t\left(\lambda - e^q\right)\left(\lambda - e^{-q}\right).$$

The potential we use is

$$\xi_{t,\mathbf{x}}(z,\lambda) := \begin{pmatrix} 0 & \lambda^{-1}dz \\ \beta_t(\lambda)\omega_{\mathbf{x}}(z,\lambda) & 0 \end{pmatrix}$$

where

$$\omega_{\mathbf{x}}(z,\lambda) := \sum_{i=1}^{n}\left(\frac{a_i(\lambda)}{(z-p_i(\lambda))^2} + \frac{b_i(\lambda)}{z-p_i(\lambda)}\right)dz.$$

The initial condition is the identity matrix, taken at the point $z_0 = 0 \in \Omega$ where

$$\Omega = \{z \in \mathbb{C} \mid \forall i \in [1,n], |z - \pi_i| > \epsilon\}$$

and $\epsilon > 0$ is a fixed constant such that the disks $D(\pi_i, 2\epsilon) \subset \mathbb{C}$ are disjoint and do not contain 0. Although the poles $p_1, \dots, p_n$ of the potential $\xi_{t,\mathbf{x}}$ are functions of λ, $\xi_{t,\mathbf{x}}$ is well-defined on Ω for $\mathbf{x}$ sufficiently close to $\mathbf{x}_0$. We thus define $\Phi_{t,\mathbf{x}}$ as the solution to the Cauchy problem (5) with data $(\Omega, \xi_{t,\mathbf{x}}, 0, \mathrm{I}_2)$.

The main properties of this potential are the same as in [25], namely: it is a perturbation of the spherical potential $\xi_{0,\mathbf{x}}$ and the factor $(\lambda - e^{-q})$ in β_t ensures that the second equation of the monodromy problem (8) is solved.

Let $\{\gamma_1, \cdots, \gamma_{n-1}\}$ be a set of generators of the fundamental group $\pi_1(\Omega, 0)$ and define for all $i \in [1, n-1]$

$$M_i(t,\mathbf{x}) := \mathcal{M}_{\gamma_i}(\Phi_{t,\mathbf{x}}).$$

Noting that

$$\lambda \in \mathbb{S}^1 \implies \lambda^{-1}\left(\lambda - e^q\right)\left(\lambda - e^{-q}\right) = -2\left(\cosh q - \operatorname{Re}\lambda\right) \in \mathbb{R},$$

the unitarity of the monodromy is equivalent to

$$\widetilde{M}_i(t,\mathbf{x})(\lambda) := \frac{\lambda}{\beta_t(\lambda)} \log M_i(t,\mathbf{x})(\lambda) \in \Lambda\mathfrak{su}(2)_\rho.$$

Note that at $t = 0$, the expression above takes the same value as in [25], and so does the regularity conditions. One can thus apply Propositions 2 and 3 of [25] which we recall in Proposition 10 below.

Proposition 10. *For t in a neighbourhood of 0, there exists a unique smooth map $t \mapsto \mathbf{x}(t) = (a_{i,t}, b_{i,t}, p_{i,t})_{1 \leq i \leq n} \in (\Lambda \mathbb{C}_\rho^{\geq 0})^{3n}$ such that $\mathbf{x}(0) = \mathbf{x}_0$, the monodromy problem and the regularity problem are solved at $(t, \mathbf{x}(t))$ and the following normalisations hold:*

$$\forall i \in [1, n-1], \qquad \mathrm{Re}(a_{i,t}) \mid_{\lambda=0} = \tau_i \quad \textit{and} \quad p_{i,t} \mid_{\lambda=0} = \pi_i.$$

Moreover, at $t = 0$, $\mathbf{x}_0$ is a constant of $\mathbb{C}^{3n}$ such that

$$\nu_i = \frac{-2\tau_i \overline{\pi_i}}{1 + |\pi_i|^2} \quad \textit{and} \quad \sum_{i=1}^{n} \tau_i u_i = 0.$$

Now write $\omega_t := \omega_{\mathbf{x}(t)}$, $\xi_t := \xi_{t,\mathbf{x}(t)}$ and apply the DPW method to define the holomorphic frame Φ_t associated to ξ_t on the universal cover $\widetilde{\Omega}$ of Ω with initial condition $\Phi_t(0) = \mathrm{I}_2$. Let $F_t := \mathrm{Uni}\, \Phi_t$ and $f_t := \mathrm{Sym}_q F_t$. The monodromy problem for Φ_t being solved, f_t descends to a well-defined CMC H immersion on Ω. Use Theorem 3 and Corollary 1 of [25] to extend f_t to $\Sigma_t := \mathbb{C} \cup \{\infty\} \setminus \{p_{1,t}(0), \ldots, p_{n,t}(0)\}$ and define $M_t = f_t(\Sigma_t)$. Moreover, with the same proof as in [25] (Proposition 4, point (2)), $a_{i,t}$ is a real constant with respect to λ for all i and t.

3.2. Delaunay ends

Perturbed Delaunay potential. Let $i \in [1, n]$. We are going to gauge ξ_t around its pole $p_{i,t}(0)$ and show that the gauged potential is a perturbed Delaunay potential as in Definition 1. Let $(r, s) : (-T, T) \longrightarrow \mathbb{R}^2$ be the continuous solution to (see Section 1.3 for details)

$$\begin{cases} rs = t a_{i,t}, \\ r^2 + s^2 + 2rs \cosh q = \frac{1}{4}, \\ r > s. \end{cases}$$

For all t and λ, define $\psi_{i,t,\lambda}(z) := z + p_{i,t}(\lambda)$ and

$$G_t(z, \lambda) := \begin{pmatrix} \frac{\sqrt{z}}{\sqrt{r + s\lambda}} & 0 \\ \frac{-\lambda}{2\sqrt{z}\sqrt{r+s\lambda}} & \frac{\sqrt{r+s\lambda}}{\sqrt{z}} \end{pmatrix}.$$

For T small enough, one can thus define on a uniform neighbourhood of 0 the potential

$$\begin{aligned}\widetilde{\xi}_{i,t}(z,\lambda) &:= ((\psi^*_{i,t,\lambda}\xi_t)\cdot G_t)(z,\lambda)\\ &= \begin{pmatrix} 0 & r\lambda^{-1}+s \\ \frac{\beta_t(\lambda)}{r+s\lambda}(\psi^*_{i,t,\lambda}\omega_t(z))z^2+\frac{\lambda}{4(r+s\lambda)} & 0\end{pmatrix} z^{-1}dz.\end{aligned}$$

Note that by definition of r, s and β_t,

$$(r+s\lambda)(r\lambda+s)=\frac{\lambda}{4}+\beta_t(\lambda)a_{i,t}$$

and thus

$$\widetilde{\xi}_{i,t}(z,\lambda)=A_t(\lambda)z^{-1}dz+C_t(z,\lambda)dz$$

with A_t as in Equation (14) satisfies Equation (19) and C_t as in Definition 1. The potential $\widetilde{\xi}_{i,t}$ is thus a perturbed Delaunay potential as in Definition 1. Moreover, using Theorem 3 of [25], the induced immersion $\widetilde{f}_{i,t}$ satisfies

$$\widetilde{f}_{i,t}=\psi^*_{i,t,0}f_t.$$

Applying Theorem 3. The holomorphic frame $\widetilde{\Phi}_{i,t}:=\Phi_t G_{i,t}$ associated to $\widetilde{\xi}_{i,t}$ satisfies the regularity and monodromy hypotheses of Theorem 3, but at $t=0$ and $z=1$,

$$\begin{aligned}\widetilde{\Phi}_{i,0}(1,\lambda) &= \begin{pmatrix}1 & (1+\pi_i)\lambda^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\ \frac{-\lambda}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1-\pi_i & (1+\pi_i)\lambda^{-1}\\ -\lambda & 1\end{pmatrix} =: M_i(\lambda),\end{aligned}$$

and thus $\widetilde{\Phi}_{i,0}(z)=M_i z^{A_0}$. Recall (23) and let $Q_i:=\operatorname{Uni}(M_iH_0)$. Using Lemma 2.1. in [18], Q_i can be made explicit and one can find a change of coordinates h and a gauge G such that $\widehat{\Phi}_{i,t}:=H_0Q_i^{-1}(h^*\widetilde{\Phi}_{i,t})G$ solves $d\widehat{\Phi}_{i,t}=\widehat{\Phi}_{i,t}\widehat{\xi}_{i,t}$ where $\widehat{\xi}_{i,t}$ is a perturbed Delaunay potential and $\widehat{\Phi}_{i,0}(z)=z^{A_0}$. Explicitely,

$$Q_i(\lambda)=\frac{1}{\sqrt{1+|\pi_i|^2}}\begin{pmatrix}1 & \lambda^{-1}\pi_i\\ -\lambda\overline{\pi}_i & 1\end{pmatrix}$$

and

$$h(z)=\frac{(1+|\pi_i|^2)z}{1-\overline{\pi}_i z},\qquad G(z,\lambda)=\frac{1}{\sqrt{1-\overline{\pi}_i z}}\begin{pmatrix}1 & 0\\ -\lambda\overline{\pi}_i z & 1-\overline{\pi}_i z\end{pmatrix}.$$

One can thus apply Theorem 3 on $\widehat{\xi}_{i,t}$ and $\widehat{\Phi}_{i,t}$, which proves the existence of the family $(M_t)_{0<t<T}$ of CMC H surfaces of genus zero and n Delaunay ends, each of weight (according to Equation (16))

$$w_{i,t} = 8\pi r s \sinh q = \frac{8\pi t a_{i,t}}{\sqrt{H^2-1}},$$

which proves the first part of Theorem 1 (after a normalisation on t). Let $\widehat{f}_{i,t} := \mathrm{Sym}_q(\mathrm{Uni}\,\widehat{\Phi}_{i,t})$ and $\widehat{f}^{\mathcal{D}}_{i,t}$ the Delaunay immersion given by Theorem 3.

Limit axis. In order to compute the limit axis of f_t at the end around $p_{i,t}$, let $\widehat{\Delta}_{i,t}$ be the oriented axis of $\widehat{f}^{\mathcal{D}}_{i,t}$ at $w=0$. Then, using Theorem 3,

$$\widehat{\Delta}_{i,0} = H_0(e^{-q}) \cdot \mathrm{geod}\left(\mathrm{I}_2, -\sigma_3\right).$$

And using $\widehat{f}_{i,t}(w) = H_0(e^{-q})Q_i(e^{-q})^{-1} \cdot (h^* f_t(z))$,

$$\widehat{\Delta}_{i,0} = H_0(e^{-q})Q_i(e^{-q})^{-1} \cdot \Delta_{i,0}$$

and thus

$$\Delta_{i,0} = Q_i(e^{-q}) \cdot \mathrm{geod}(\mathrm{I}_2, -\sigma_3).$$

Computing $M_i H_0 = \Phi_S(\pi_i)$ as in (10), one has $Q_i = F_S(\pi_i)$. Hence

$$\Delta_{i,0} = \mathrm{geod}\left(f_S(\pi_i), -N_S(\pi_i)\right)$$

where N_S is the normal map associated to Φ_S. Using Equation (11), $f_S(z) = R(q) \cdot \widetilde{f}_S(z)$ and $N_S(z) = R(q) \cdot \widetilde{N}_S(z)$ where $\widetilde{N}_S$ is the normal map of $\widetilde{f}_S$. Using Equation (12) and the fact that $\widetilde{f}_S$ is a spherical immersion gives

$$\widetilde{N}_S(z) = \Gamma_{\mathrm{I}_2}^{\widetilde{f}_S(z)}\left(-v_S(z)\right)$$

and thus

$$\begin{aligned}\Delta_{i,0} &= \mathrm{geod}\left(R(q)\cdot \widetilde{f}_S(\pi_i), -R(q)\cdot \widetilde{N}_S(\pi_i)\right)\\ &= R(q)\cdot \mathrm{geod}\left(\widetilde{f}_S(\pi_i), \Gamma_{\mathrm{I}_2}^{\widetilde{f}_S(\pi_i)} v_S(\pi_i)\right)\\ &= R(q)\cdot \mathrm{geod}\left(\mathrm{I}_2, u_i\right).\end{aligned}$$

Apply the isometry given by $R(q)^{-1}$ and note that $R(q)$ does not depend on i to prove point 2 of Theorem 1.

3.3. Embeddedness

We suppose that $t > 0$ and that all the weights τ_i are positive, so that the ends of f_t are embedded. Recall the definition of Alexandrov-embeddedness (as stated in [26]):

Definition 2. *A surface $M^2 \subset \mathcal{M}^3$ of finite topology is Alexandrov-embedded if M is properly immersed, if each end of M is embedded, and if there exists a compact 3-manifold $\overline{W}$ with boundary $\partial\overline{W} = \overline{S}$, n points $p_1, \cdots, p_n \in \overline{S}$ and a proper immersion $F : W = \overline{W}\backslash\{p_1, \cdots, p_n\} \longrightarrow \mathcal{M}$ whose restriction to $S = \overline{S}\backslash\{p_1, \cdots, p_n\}$ parametrises M.*

The following lemma is proved in [26] in $\mathbb{R}^3$ and for surfaces with catenoidal ends, but the proof is the same in $\mathbb{H}^3$ for surfaces with any type of embedded ends. For any oriented surface M with Gauss map N and any $r > 0$, the tubular map of M with radius r is defined by

$$\begin{array}{cccccc} \mathcal{T} & : & (-r, r) \times M & \longrightarrow & \operatorname{Tub}_r M \\ & & (s, p) & \longmapsto & \operatorname{geod}(p, N(p))(s). \end{array}$$

Lemma 8. *Let M be an oriented Alexandrov-embedded surface of $\mathbb{H}^3$ with n embedded ends. Let $r > 0$ and suppose that the tubular map of M with radius r is a local diffeomorphism. With the notations of Definition 2, there exist a hyperbolic 3-manifold W' containing W and a local isometry $F' : W' \longrightarrow \mathbb{H}^3$ extending F such that the tubular neighbourhood $\operatorname{Tub}_r S$ is embedded in W'.*

In order to show that M_t is embedded, we will use the techniques of [26]. We thus begin by lifting M_t to $\mathbb{R}^3$ with the exponential map at the identity $\exp_{\mathrm{I}_2} : \mathbb{R}^3 \longrightarrow \mathbb{H}^3$. This map is a diffeomorphism, so M_t is Alexandrov-embedded if and only if its lift $\widehat{M_t}$ to $\mathbb{R}^3$ given by the immersion

$$\widehat{f_t} := \exp_{\mathrm{I}_2}^{-1} \circ f_t : \Sigma_t \longrightarrow \mathbb{R}^3$$

is Alexandrov-embedded.

Let $T, \epsilon > 0$ such that f_t (and hence $\widehat{f_t}$) is an embedding of $D^*(p_{i,t}, \epsilon)$ for all $i \in [1, n]$ and let $f_{i,t}^{\mathcal{D}} : \mathbb{C}\backslash\{p_{i,t}\} \longrightarrow \mathbb{H}^3$ be the Delaunay immersion approximating f_t in $D^*(p_{i,t}, \epsilon)$. Let $\widehat{f_{i,t}^{\mathcal{D}}} := \exp_{\mathrm{I}_2}^{-1} \circ f_{i,t}^{\mathcal{D}}$. Apply an isometry of $\mathbb{H}^3$ so that the limit immersion f_0 maps Σ_0 to a n-punctured geodesic sphere of hyperbolic radius q centered at I_2. Then $\widehat{f_0}(\Sigma_0)$ is a Euclidean sphere of radius q centered at the origin. Define

$$\begin{array}{rcccc} \widehat{N}_t & : & \Sigma_t & \longrightarrow & \mathbb{S}^2 \\ & & z & \longmapsto & d(\exp_{\mathrm{I}_2}^{-1})(f_t(z))N_t(z). \end{array}$$

At $t = 0$, $\widehat{N}_0$ is the normal map of $\widehat{f}_0$ (by Gauss Lemma), but not for $t > 0$ because the Euclidean metric of $\mathbb{R}^3$ is not the metric induced by $\exp_{\mathrm{I}_2}$.

Let

$$\begin{array}{rcccc} h_i & : & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & \left\langle x, -\widehat{N}_0(p_{i,0}) \right\rangle \end{array}$$

be the height function in the direction of the limit axis.

As in [26], one can show that

Claim 1. *There exist $\delta < \delta'$ and $0 < \epsilon' < \epsilon$ such that for all $i \in [1, n]$ and $0 < t < T$,*

$$\max_{C(p_{i,t},\epsilon)} h_i \circ \widehat{f}_t < \delta < \min_{C(p_{i,t},\epsilon')} h_i \circ \widehat{f}_t \leq \max_{C(p_{i,t},\epsilon')} h_i \circ \widehat{f}_t < \delta'.$$

Define for all i and t:

$$\gamma_{i,t} := \left\{ z \in D^*_{p_{i,t},\epsilon} \mid h_i \circ \widehat{f}_t(z) = \delta \right\}, \quad \gamma'_{i,t} := \left\{ z \in D^*_{p_{i,t},\epsilon'} \mid h_i \circ \widehat{f}_t(z) = \delta' \right\}.$$

From their convergence as t tends to 0,

Claim 2. *The regular curves $\gamma_{i,t}$ and $\gamma'_{i,t}$ are topological circles around $p_{i,t}$.*

Define $D_{i,t}, D'_{i,t}$ as the topological disks bounded by $\gamma_{i,t}, \gamma'_{i,t}$, and $\Delta_{i,t}, \Delta'_{i,t}$ as the topological disks bounded by $\widehat{f}_t(\gamma_{i,t}), \widehat{f}_t(\gamma'_{i,t})$. Let $\mathcal{A}_{i,t} := D_{i,t} \backslash D'_{i,t}$. Then $\widehat{f}_t(\mathcal{A}_{i,t})$ is a graph over the plane $\{h_i(x) = \delta\}$. Moreover, for all $z \in D^*_{i,t}$, $h_i \circ \widehat{f}_t(z) \geq \delta' > \delta$. Thus

Claim 3. *The intersection $\widehat{f}_t(D^*_{i,t}) \cap \Delta_{i,t}$ is empty.*

Define a sequence $(R_{i,t,k})$ such that $\widehat{f}_t(D^*_{i,t})$ transversally intersects the planes $\{h_i(x) = R_{i,t,k}\}$. Define

$$\gamma_{i,t,k} := \left\{ z \in D^*_{i,t} \mid h_i \circ \widehat{f}_{i,t}(z) = R_{i,t,k} \right\},$$

and the topological disks $\Delta_{i,t,k} \subset \{h_i(x) = R_{i,t,k}\}$ bounded by $\widehat{f}_t(\gamma_{i,t,k})$. Define $\mathcal{A}_{i,t,k}$ as the annuli bounded by $\gamma_{i,t}$ and $\gamma_{i,t,k}$. Define $W_{i,t,k} \subset \mathbb{R}^3$ as the interior of $\widehat{f}_t(\mathcal{A}_{i,t,k}) \cup \Delta_{i,t} \cup \Delta_{i,t,k}$ and

$$W_{i,t} := \bigcup_{k \in \mathbb{N}} W_{i,t,k}.$$

Hence,

Claim 4. *The union $\widehat{f}_t(D^*_{i,t}) \cup \Delta_{i,t}$ is the boundary of a topological punctured ball $W_{i,t} \subset \mathbb{R}^3$.*

The union

$$\widehat{f}_t\left(\Sigma_t \backslash \left(D_{1,t} \cup \cdots D_{n,t}\right)\right) \cup \Delta_{1,t} \cup \cdots \cup \Delta_{n,t}$$

is the boundary of a topological ball $W_{0,t} \subset \mathbb{R}^3$. Take

$$W_t := W_{0,t} \cup W_{1,t} \cup \cdots \cup W_{n,t}$$

to show that $\widehat{M_t}$, and hence M_t is Alexandrov-embedded for $t > 0$ small enough.

Lemma 9. *Let $S \subset \mathbb{H}^3$ be a sphere of hyperbolic radius q centered at $p \in \mathbb{H}^3$. Let $n \geq 2$ and $\{u_i\}_{i \in [1,n]} \subset T_p\mathbb{H}^3$. Let $\{p_i\}_{i \in [1,n]}$ defined by $p_i = S \cap \operatorname{geod}(p, u_i)(\mathbb{R}_+)$. For all $i \in [1,n]$, let $S_i \subset \mathbb{H}^3$ be the sphere of hyperbolic radius q such that $S \cap S_i = \{p_i\}$. For all $(i,j) \in [1,n]^2$, let θ_{ij} be the angle between u_i and u_j.*

If for all $i \neq j$,

$$\theta_{ij} > 2 \arcsin\left(\frac{1}{2\cosh q}\right)$$

then $S_i \cap S_j = \emptyset$ for all $i \neq j$.

Proof. Without loss of generality, we assume that $p = \mathrm{I}_2$. We use the ball model of $\mathbb{H}^3$ equipped with its metric

$$ds^2_{\mathbb{B}}(x) = \frac{4 ds_E^2}{\left(1 - \|x\|_E^2\right)^2}$$

where ds_E is the euclidean metric and $\|x\|_E$ is the euclidean norm. In this model, the sphere S is centered at the origin and has euclidean radius $R =$

$\tanh \frac{q}{2}$. For all $i \in [1, n]$, the sphere S_i has euclidean radius

$$r = \frac{1}{2}\left(\tanh \frac{3q}{2} - \tanh \frac{q}{2}\right) = \frac{\tanh \frac{q}{2}}{2\cosh q - 1}.$$

Let $j \neq i$. In order to have $S_i \cap S_j = \emptyset$, one must solve

$$(R + r) \sin \frac{\theta_{ij}}{2} \geq r,$$

which gives the expected result. □

In order to prove the last part of Theorem 1, just note that

$$H = \coth q \implies \frac{1}{2\cosh q} = \frac{\sqrt{H^2 - 1}}{2H}.$$

Suppose that the angle θ_{ij} between u_i and u_j satisfies Equation (2) for all $i \neq j$. Then for $t > 0$ small enough, the proper immersion F_t given by Definition 2 is injective (because of the convergence towards a chain of spheres) and hence M_t is embedded.

Remark 5. *This means for example that in hyperbolic space, one can construct embedded CMC n-noids with seven coplanar ends or more.*

4. Gluing Delaunay ends to minimal n-noids

Again, this section is an adaptation of Traizet's work in [26] applied to the proof of Theorem 2. We first give in Section 4.1 a blow-up result for CMC $H > 1$ surfaces in Hyperbolic space. We then introduce in Section 4.2 the DPW data giving rise to the surface M_t of Theorem 2 and prove the convergence towards the minimal n-noid. Finally, using the same arguments as in [26], we prove Alexandrov-embeddedness in Section 4.3.

4.1. A blow-up result

As in $\mathbb{R}^3$ (see [26]), the DPW method accounts for the convergence of CMC $H > 1$ surfaces in $\mathbb{H}^3$ towards minimal surfaces of $\mathbb{R}^3$ (after a suitable blow-up). We work with the following Weierstrass parametrisation:

$$W(z) = W(z_0) + \operatorname{Re} \int_{z_0}^{z} \left(\frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega\right) \tag{37}$$

Proposition 11. *Let Σ be a Riemann surface, $(\xi_t)_{t\in I}$ a family of DPW potentials on Σ and $(\Phi_t)_{t\in I}$ a family of solutions to $d\Phi_t = \Phi_t\xi_t$ on the universal cover $\widetilde{\Sigma}$ of Σ, where $I \subset \mathbb{R}$ is a neighbourhood of 0. Fix a base point $z_0 \in \widetilde{\Sigma}$ and $\rho > e^q > 1$. Assume that*

1) *$(t,z) \mapsto \xi_t(z)$ and $t \mapsto \Phi_t(z_0)$ are $\mathcal{C}^1$ maps into $\Omega^1(\Sigma, \Lambda\mathfrak{sl}(2,\mathbb{C})_\rho)$ and $\Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ respectively.*
2) *For all $t \in I$, Φ_t solves the monodromy problem (8).*
3) *$\Phi_0(z,\lambda)$ is independent of λ:*

$$\Phi_0(z,\lambda) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}.$$

Let $f_t = \mathrm{Sym}_q(\mathrm{Uni}(\Phi_t)) : \Sigma \longrightarrow \mathbb{H}^3$ be the CMC $H = \coth q$ immersion given by the DPW method. Then, identifying $T_{\mathrm{I}_2}\mathbb{H}^3$ with $\mathbb{R}^3$ via the basis $(\sigma_1, \sigma_2, \sigma_3)$ defined in (3),

$$\lim_{t\to 0} \frac{1}{t}(f_t - \mathrm{I}_2) = W$$

where W is a (possibly branched) minimal immersion with the following Weierstrass data:

$$g(z) = \frac{a(z)}{c(z)}, \quad \omega(z) = -4(\sinh q)c(z)^2 \frac{\partial \xi_{t,12}^{(-1)}(z)}{\partial t} |_{t=0}.$$

The limit is for the uniform $\mathcal{C}^1$ convergence on compact subsets of Σ.

Proof. With the same arguments as in [26], $(t,z) \mapsto \Phi_t(z)$, $(t,z) \mapsto F_t(z)$ and $(t,z) \mapsto B_t(z)$ are $\mathcal{C}^1$ maps into $\Lambda\mathrm{SL}(2,\mathbb{C})_\rho$, $\Lambda\mathrm{SU}(2)_\rho$ and $\Lambda_+^{\mathbb{R}}\mathrm{SL}(2,\mathbb{C})_\rho$ respectively. At $t = 0$, Φ_0 is constant. Thus F_0 and B_0 are constant with respect to λ:

$$F_0 = \frac{1}{\sqrt{|a|^2+|c|^2}} \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix}, \quad B_0 = \frac{1}{\sqrt{|a|^2+|c|^2}} \begin{pmatrix} |a|^2+|c|^2 & \bar{a}b + \bar{c}d \\ 0 & 1 \end{pmatrix}.$$

Thus $F_0(z, e^{-q}) \in \mathrm{SU}(2)$ and $f_0(z)$ degenerates into the identity matrix. Let $b_t := B_{t,11}|_{\lambda=0}$ and β_t the upper-right residue at $\lambda = 0$ of the potential ξ_t.

Recalling Equation (6),

$$df_t(z) = 2b_t(z)^2 \sinh q F_t(z, e^{-q}) \begin{pmatrix} 0 & \beta_t(z) \\ \overline{\beta_t(z)} & 0 \end{pmatrix} F_t(z, e^{-q})^*.$$

Hence $(t, z) \mapsto df_t(z)$ is a $\mathcal{C}^1$ map. At $t = 0$, $\xi_0 = \Phi_0^{-1} d\Phi_0$ is constant with respect to λ, so $\beta_0 = 0$ and $df_0(z) = 0$. Define $\widetilde{f}_t(z) := \frac{1}{t}\left(f_t(z) - \mathrm{I}_2\right)$ for $t \neq 0$. Then $d\widetilde{f}_t(z)$ extends at $t = 0$, as a continuous function of (t, z) by

$$\begin{aligned} d\widetilde{f}_0 = \frac{d}{dt} df_t \mid_{t=0} &= 2 \sinh q \begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix} \begin{pmatrix} 0 & \beta' \\ \overline{\beta'} & 0 \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ -c & a \end{pmatrix} \\ &= 2 \sinh q \begin{pmatrix} -ac\beta' - \overline{ac\beta'} & a^2\beta' - \overline{c^2\beta'} \\ \overline{a^2\beta'} - c^2\beta' & ac\beta' + \overline{ac\beta'} \end{pmatrix} \end{aligned}$$

where $\beta' = \frac{d}{dt}\beta_t \mid_{t=0}$. In $T_{\mathrm{I}_2}\mathbb{H}^3$, this gives

$$d\widetilde{f}_0 = 4 \sinh q \operatorname{Re}\left(\frac{1}{2}\beta'(a^2 - c^2), \frac{-i}{2}\beta'(a^2 + c^2), -ac\beta'\right).$$

Writing $g = \frac{a}{c}$ and $\omega = -4c^2\beta' \sinh q$ gives:

$$\widetilde{f}_0(z) = \widetilde{f}_0(z_0) + \operatorname{Re} \int_{z_0}^{z} \left(\frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega\right).$$

□

As a useful example for Proposition 11, one can show the convergence of Delaunay surfaces in $\mathbb{H}^3$ towards a minimal catenoid.

Proposition 12. *Let $q > 0$, $A_t = A_{r,s}$ as in* (14) *with $r \leq s$ satisfying* (19). *Let $\Phi_t(z) := z^{A_t}$ and $f_t := \operatorname{Sym}_q(\operatorname{Uni}\Phi_t)$. Then*

$$\widetilde{f} := \lim_{t \to 0} \frac{1}{t}\left(f_t - \mathrm{I}_2\right) = \psi$$

where $\psi : \mathbb{C}^* \longrightarrow \mathbb{R}^3$ is the immersion of a catenoid centered at $(0, 0, 1)$, of neck radius 1 and of axis orientd by the positive x-axis in the direction from $z = 0$ to $z = \infty$.

Proof. Compute

$$\Phi_0(z,\lambda) = \begin{pmatrix} \cosh\left(\frac{\log z}{2}\right) & \sinh\left(\frac{\log z}{2}\right) \\ \sinh\left(\frac{\log z}{2}\right) & \cosh\left(\frac{\log z}{2}\right) \end{pmatrix}$$

and

$$\frac{\partial \xi_{t,12}^{(-1)}(z)}{\partial t}\,|_{t=0} = \frac{z^{-1}dz}{2\sinh q}$$

in order to apply Proposition 11 and get

$$\widetilde{f}(z) = \widetilde{f}(1) + \mathrm{Re}\int_1^z \left(\frac{1}{2}(1-g^2)\omega, \frac{i}{2}(1+g^2)\omega, g\omega\right)$$

where

$$g(z) = \frac{z+1}{z-1} \qquad \text{and} \qquad \omega(z) = \frac{-1}{2}\left(\frac{z-1}{z}\right)^2 dz.$$

Note that $\Phi_t(1) = \mathrm{I}_2$ for all t to show that $\widetilde{f}(1) = 0$ and get

$$\widetilde{f}(z) = \mathrm{Re}\int_1^z \left(w^{-1}dw, \frac{-i}{2}(1+w^2)w^{-2}dw, \frac{1}{2}(1-w^2)w^{-2}dw\right).$$

Integrating gives for $(x,y) \in \mathbb{R}\times[0,2\pi]$:

$$\widetilde{f}(e^{x+iy}) = \psi(x,y)$$

where

$$\begin{array}{cccc} \psi : & \mathbb{R}\times[0,2\pi] & \longrightarrow & \mathbb{R}^3 \\ & (x,y) & \longmapsto & (x, \cosh(x)\sin(y), 1-\cosh(x)\cos(y)) \end{array}$$

and hence the result. □

4.2. The DPW data

In this Section, we introduce the DPW data inducing the surface M_t of Theorem 2. The method is very similar to Section 3 and to [26], which is why we omit the details.

The data. Let (g,ω) be the Weierstrass data (for the parametrisation defined in (37)) of the minimal n-noid $M_0 \subset \mathbb{R}^3$. If necessary, apply a Möbius transformation so that $g(\infty) \notin \{0,\infty\}$, and write

$$g(z) = \frac{A(z)}{B(z)}, \qquad \omega(z) = \frac{B(z)^2 dz}{\prod_{i=1}^n (z-p_{i,0})^2}.$$

Let $H > 1$, $q > 0$ so that $H = \coth q$ and $\rho > e^q$. Consider $3n$ parameters $a_i, b_i, p_i \in \Lambda\mathbb{C}_\rho^{\geq 0}$ ($i \in [1,n]$) assembled into a vector $\mathbf{x}$. Let

$$A_{\mathbf{x}}(z,\lambda) = \sum_{i=1}^n a_i(\lambda) z^{n-1}, \qquad B_{\mathbf{x}}(z,\lambda) = \sum_{i=1}^n b_i(\lambda) z^{n-1}$$

and

$$g_{\mathbf{x}}(z,\lambda) = \frac{A_{\mathbf{x}}(z,\lambda)}{B_{\mathbf{x}}(z,\lambda)}, \qquad \omega_{\mathbf{x}}(z,\lambda) = \frac{B_{\mathbf{x}}(z,\lambda)^2 dz}{\prod_{i=1}^n (z-p_i(\lambda))^2}.$$

The vector $\mathbf{x}$ is chosen in a neighbourhood of a central value $\mathbf{x}_0 \in \mathbb{C}^{3n}$ so that $A_{\mathbf{x}_0} = A$, $B_{\mathbf{x}_0} = B$ and $\omega_{\mathbf{x}_0} = \omega$. Let $p_{i,0}$ denote the central value of p_i. Introduce a real parameter t in a neighbourhood of 0 and write

$$\beta_t(\lambda) := \frac{t(\lambda - e^q)(\lambda - e^{-q})}{4\sinh q}.$$

The potential we use is

$$\xi_{t,\mathbf{x}}(z,\lambda) = \begin{pmatrix} 0 & \lambda^{-1}\beta_t(\lambda)\omega_{\mathbf{x}}(z,\lambda) \\ d_z g_{\mathbf{x}}(z,\lambda) & 0 \end{pmatrix}$$

defined for $(t,\mathbf{x})$ sufficiently close to $(0,\mathbf{x}_0)$ on

$$\Omega = \{z \in \mathbb{C} \mid \forall i \in [1,n], |z - p_{i,0}| > \epsilon\} \cup \{\infty\}$$

where $\epsilon > 0$ is a fixed constant such that the disks $D(p_{i,0}, 2\epsilon)$ are disjoint. The initial condition is

$$\phi(\lambda) = \begin{pmatrix} i g_{\mathbf{x}}(z_0,\lambda) & i \\ i & 0 \end{pmatrix}$$

taken at $z_0 \in \Omega$ away from the poles and zeros of g and ω. Let $\Phi_{t,\mathbf{x}}$ be the holomorphic frame arising from the data $(\Omega, \xi_{t,\mathbf{x}}, z_0, \phi)$ via the DPW method and $f_{t,\mathbf{x}} := \mathrm{Sym}_q(\mathrm{Uni}\,\Phi_{t,\mathbf{x}})$.

Follow Section 6 of [26] to show that the potential $\xi_{t,\mathbf{x}}$ is regular at the zeros of $B_{\mathbf{x}}$ and to solve the monodromy problem around the poles at $p_{i,0}$ for

$i \in [1, n-1]$. The Implicit Function Theorem allows us to define $\mathbf{x} = \mathbf{x}(t)$ in a small neighbourhood $(-T, T)$ of $t = 0$ satisfying $\mathbf{x}(0) = \mathbf{x}_0$ and such that the monodromy problem is solved for all t. We can thus drop from now on the index $\mathbf{x}$ in our data. As in [26], f_t descends to Ω and analytically extends to $\mathbb{C} \cup \{\infty\} \setminus \{p_{1,0}, \ldots, p_{n,0}\}$. This defines a smooth family $(M_t)_{-T<t<T}$ of CMC H surfaces of genus zero with n ends in $\mathbb{H}^3$.

The convergence of $\frac{1}{t}\left(M_t - \mathrm{I}_2\right)$ towards the minimal n-noid M_0 (point 2 of Theorem 2) is a straightforward application of Proposition 11 together with

$$\frac{\Phi_{0,11}(z)}{\Phi_{0,21}(z)} = g(z), \quad -4\left(\sinh q\right)\left(\Phi_{0,21}(z)\right)^2 \frac{\partial \xi_{t,12}^{(-1)}(z)}{\partial t} = \omega(z).$$

Delaunay residue. To show that ξ_t is a perturbed Delaunay potential around each of its poles, let $i \in [1, n]$ and follow Section 3.2 with

$$\psi_{i,t,\lambda}(z) = g_t^{-1}\left(z + g_t(p_{i,t}(\lambda))\right).$$

Define

$$\widetilde{\omega}_{i,t}(z,\lambda) := \psi_{i,t,\lambda}^* \omega_t(z)$$

and

$$\alpha_{i,t}(\lambda) := \operatorname*{Res}_{z=0}(z\widetilde{\omega}_{i,t}(z,\lambda)).$$

Use Proposition 6, Claim 1 of [26] to show that for T small enough, $\alpha_{i,t}$ is real and does not depend on λ. Set

$$\begin{cases} rs = \frac{t\alpha_{i,t}}{4\sinh q}, \\ r^2 + s^2 + 2rs\cosh q = \frac{1}{4}, \\ r < s \end{cases}$$

and

$$G_t(z,\lambda) = \begin{pmatrix} \frac{\sqrt{r\lambda+s}}{\sqrt{z}} & \frac{-1}{2\sqrt{r\lambda+s}\sqrt{z}} \\ 0 & \frac{\sqrt{z}}{\sqrt{r\lambda+s}} \end{pmatrix}.$$

Define the gauged potential

$$\widetilde{\xi}_{i,t}(z,\lambda) := \left((\psi_{i,t,\lambda}^* \xi_t) \cdot G_t\right)(z,\lambda)$$

and compute its residue to show that it is a perturbed Delaunay potential as in Definition 1.

Applying Theorem 3. At $t = 0$ and $z = 1$, writing $\pi_i := g(p_{i,0})$ to ease the notation,

$$\widetilde{\Phi}_{i,0}(1,\lambda) = \begin{pmatrix} i\,(1+\pi_i) & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} 1+\pi_i & 1-\pi_i \\ 1 & -1 \end{pmatrix} =: M_i,$$

and thus $\widetilde{\xi}_{i,0}(z) = M_i z^{A_0}$. Recall (24) and let $Q_i := \mathrm{Uni}\left(M_i H_0^{-1}\right)$. Using Lemma 2 of [26], Q_i can be made explicit and one can find a change of coordinates h and a gauge G such that $\widehat{\Phi}_{i,t} := (Q_i H_0)^{-1}\left(h^* \widetilde{\Phi}_{i,t}\right) G$ solves $d\widehat{\Phi}_{i,t} = \widehat{\Phi}_{i,t}\widehat{\xi}_{i,t}$ where $\widehat{\xi}_{i,t}$ is a perturbed Delaunay potential and $\widehat{\Phi}_{i,0}(z) = z^{A_0}$. One can thus apply Theorem 3 on $\widehat{\xi}_{i,t}$ and $\widehat{\Phi}_{i,t}$, which proves the existence of the family $(M_t)_{-T<t<T}$ of CMC H surfaces of genus zero and n Delaunay ends, each of weight (according to Equation (16))

$$w_{i,t} = 8\pi r s \sinh q = 2\pi t \alpha_{i,t},$$

which proves the first part of Theorem 2. Let $\widehat{f}_{i,t} := \mathrm{Sym}_q\left(\mathrm{Uni}\,\widehat{\Phi}_{i,t}\right)$ and let $\widehat{f}^{\mathcal{D}}_{i,t}$ be the Delaunay immersion given by Theorem 3.

Limit axis. In order to compute the limit axis of f_t at the end around $p_{i,t}$, let $\widehat{\Delta}_{i,t}$ be the oriented axis of $\widehat{f}^{\mathcal{D}}_{i,t}$ at $z = 0$. Then, using Theorem 3,

$$\widehat{\Delta}_{i,0} = \mathrm{geod}\left(\mathrm{I}_2, -\sigma_1\right).$$

And using $\widehat{f}_{i,t}(z) = (Q_i H_0)^{-1} \cdot (h^* f_t(z))$,

$$\widehat{\Delta}_{i,0} = (Q_i H_0)^{-1} \cdot \Delta_{i,0},$$

and thus

$$\Delta_{i,0} = (Q H_0) \cdot \mathrm{geod}(\mathrm{I}_2, -\sigma_1).$$

Compute $H_0 \cdot (-\sigma_1) = \sigma_3$ and note that $M_i H_0^{-1} = \Phi_0(\pi_i)$ to get

$$\Delta_{i,0} = \mathrm{geod}\left(\mathrm{I}_2, N_0(p_{i,0})\right)$$

where N_0 is the normal map of the minimal immersion.

Type of the ends. Suppose that t is positive. Then the end at $p_{i,t}$ is unduloidal if and only if its weight is positive; that is, $\alpha_{i,t}$ is positive. Use Proposition 6 of [26] to show that if the normal map N_0 of M_0 points toward the inside, then $\alpha_{i,0} = \tau_i$ where $2\pi\tau_i N_0(p_{i,0})$ is the flux of M_0 around the end at $p_{i,0}$ ($\alpha_{i,0} = -\tau_i$ for the other orientation). Thus if M_0 is Alexandrov-embedded, then the ends of M_t are of unduloidal type for $t > 0$ and of nodoidal type for $t < 0$.

4.3. Alexandrov-embeddedness

In order to show that M_t is Alexandrov-embedded for $t > 0$ small enough, one can follow the proof of Proposition 7 in [26]. Note that this proposition does not use the fact that M_t is CMC H, but relies on the fact that the ambient space is $\mathbb{R}^3$. This leads us to lift f_t to $\mathbb{R}^3$ via the exponential map at the identity, hence defining an immersion $\widehat{f}_t : \Sigma_t \longrightarrow \mathbb{R}^3$ which is not CMC anymore, but is Alexandrov-embedded if and only if f_t is Alexandrov-embedded. Let $\psi : \Sigma_0 \longrightarrow M_0 \subset \mathbb{R}^3$ be the limit minimal immersion. In order to adapt the proof of [26] and show that M_t is Alexandrov-embedded, one will need the following Lemma.

Lemma 10. *Let $\widetilde{f}_t := \frac{1}{t}\widehat{f}_t$. Then $\widetilde{f}_t$ converges to ψ on compact subsets of Σ_0.*

Proof. For all z,

$$\exp_{\mathrm{I}_2}(\widehat{f}_0(z)) = f_0(z) = \mathrm{I}_2,$$

so $\widehat{f}_0(z) = 0$. Thus

$$\lim_{t\to 0} \widetilde{f}_t(z) = \frac{d}{dt}\widehat{f}_t(z) \mid_{t=0} .$$

Therefore, using Proposition 12,

$$\begin{aligned}
\psi(z) &= \lim_{t\to 0} \frac{1}{t}\left(f_t(z) - \mathrm{I}_2\right) \\
&= \lim_{t\to 0} \frac{1}{t}\Big(\exp_{\mathrm{I}_2}(\widehat{f}_t(z)) - \exp_{\mathrm{I}_2}(\widehat{f}_0(z))\Big) \\
&= \frac{d}{dt}\exp_{\mathrm{I}_2}(\widehat{f}_t(z)) \mid_{t=0} \\
&= d\exp_{\mathrm{I}_2}(0)\cdot \frac{d}{dt}\widehat{f}_t(z) \mid_{t=0} \\
&= \lim_{t\to 0} \widetilde{f}_t(z).
\end{aligned}$$

$\square$

Appendix A. CMC surfaces of revolution in $\mathbb{H}^3$

Following Sections 2.2 and 2.3 of [9],

Proposition 13. *Let $X : \mathbb{R} \times [0, 2\pi] \longrightarrow \mathbb{H}^3$ be a conformal immersion of revolution with metric $g^2(s)\left(ds^2 + d\theta^2\right)$. If X is CMC $H > 1$, then g is periodic and denoting by S its period,*

$$\sqrt{H^2-1}\int_0^S g(s)ds = \pi \quad \textit{and} \quad \int_0^S \frac{ds}{g(s)} = \frac{2\pi^2}{|w|}$$

where w is the weight of X, as defined in [14].

Proof. According to Equation (11) in [9], writing $\tau = \frac{\sqrt{|w|}}{\sqrt{2\pi}}$ and $g = \tau e^\sigma$,

$$(\sigma')^2 = 1 - \tau^2\left(\left(He^\sigma + \iota e^{-\sigma}\right)^2 - e^{2\sigma}\right) \tag{A.1}$$

where $\iota \in \{\pm 1\}$ is the sign of w. The solutions σ are periodic with period $S > 0$. Apply an isometry and a change of the variable $s \in \mathbb{R}$ so that

$$\sigma'(0) = 0 \quad \text{and} \quad \sigma(0) = \min_{s\in\mathbb{R}} \sigma(s).$$

By symmetry of Equation (A.1), one can thus define

$$a := e^{2\sigma(0)} = \min_{s\in\mathbb{R}} e^{2\sigma(s)} \quad \text{and} \quad b := e^{2\sigma\left(\frac{S}{2}\right)} = \max_{s\in\mathbb{R}} e^{2\sigma(s)}.$$

With these notations, Equation (A.1) can be written in a factorised form as

$$(\sigma')^2 = \tau^2(H^2-1)e^{-2\sigma}\left(b - e^{2\sigma}\right)\left(e^{2\sigma} - a\right) \tag{A.2}$$

with

$$a = \frac{1 - 2\iota\tau^2 H - \sqrt{1 - 4\tau^2(\iota H - \tau^2)}}{2\tau^2(H^2-1)} \tag{A.3}$$

and

$$b = \frac{1 - 2\iota\tau^2 H + \sqrt{1 - 4\tau^2(\iota H - \tau^2)}}{2\tau^2(H^2-1)}.$$

In order to compute the first integral, change variables $v = e^\sigma$, $y = \sqrt{b - v^2}$ and $x = \frac{y}{\sqrt{b-a}}$ and use Equation (A.2) to get

$$\begin{aligned}
\sqrt{H^2-1}\int_0^S \tau e^{\sigma(s)}ds &= 2\sqrt{H^2-1}\int_{\sqrt{a}}^{\sqrt{b}} \frac{\tau v dv}{\tau\sqrt{H^2-1}\sqrt{b-v^2}\sqrt{v^2-a}} \\
&= -2\int_{\sqrt{b-a}}^0 \frac{dy}{\sqrt{b-a-y^2}} \\
&= 2\int_0^1 \frac{dx}{\sqrt{1-x^2}} \\
&= \pi.
\end{aligned}$$

In the same manner with the changes of variables $v = e^{-\sigma}$, $y = \sqrt{a^{-1} - v^2}$ and $x = \frac{y}{\sqrt{a^{-1}-b^{-1}}}$,

$$\begin{aligned}
\int_0^S \frac{ds}{\tau e^{\sigma(s)}} &= \frac{-2}{\tau\sqrt{H^2-1}}\int_{a^{-1/2}}^{b^{-1/2}} \frac{dv}{v\sqrt{b-v^{-2}}\sqrt{v^{-2}-a}} \\
&= \frac{2}{\tau^2\sqrt{H^2-1}}\int_0^{\sqrt{a^{-1}-b^{-1}}} \frac{dy}{\sqrt{b-a-aby^2}} \\
&= \frac{2}{\tau^2\sqrt{H^2-1}\sqrt{ab}}\int_0^1 \frac{dx}{\sqrt{1-x^2}} \\
&= \frac{\pi}{\tau^2}
\end{aligned}$$

because $ab = \frac{1}{H^2-1}$. □

Lemma 11. *Let $\mathcal{D}_t$ be a Delaunay surface in $\mathbb{H}^3$ of constant mean curvature $H > 1$ and weight $2\pi t > 0$ with Gauss map η_t. Let r_t be the maximal value of R such that the map*

$$\begin{array}{rccc}
T & : & (-R,R)\times\mathcal{D}_t & \longrightarrow & \mathrm{Tub}_{r_t}\subset\mathbb{H}^3 \\
& & (r,p) & \longmapsto & \mathrm{geod}(p,\eta_t(p))(r)
\end{array}$$

is a diffeomorphism. Then $r_t \sim t$ as t tends to 0.

Proof. The quantity r_t is the inverse of the maximal geodesic curvature of the surface. This maximal curvature is attained for small values of t on the points of minimal distance between the profile curve and the axis. Checking the direction of the mean curvature vector at this point, the maximal curvature curve is not the profile curve but the parallel curve. Hence r_t is the minimal hyperbolic distance between the profile curve and the axis. A study of the profile curve's equation as in Proposition 13 shows that

$$r_t = \sinh^{-1}\left(\tau \exp\left(\sigma_{\min}\right)\right) = \sinh^{-1}\left(\tau\sqrt{a(\tau)}\right).$$

But using Equation (A.3), as τ tends to 0, $a \sim \tau^2 = |t|$, which gives the expected result. $\square$

Lemma 12. *Let $\mathcal{D}_t$ be a Delaunay surface in $\mathbb{H}^3$ of weight $2\pi t > 0$ with Gauss map η_t and maximal tubular radius r_t. There exist $T > 0$ and $\alpha < 1$ such that for all $0 < t < T$ and $p, q \in \mathcal{D}_t$ satisfying $d_{\mathbb{H}^3}(p,q) < \alpha r_t$,*

$$\left\|\Gamma_p^q \eta_t(p) - \eta_t(q)\right\| < 1.$$

Proof. Let $t > 0$. Then for all $p, q \in \mathcal{D}_t$,

$$\left\|\Gamma_p^q \eta_t(p) - \eta_t(q)\right\| \leq \sup_{s\in\gamma_t} \|II_t(s)\| \times \ell(\gamma_t)$$

where II_t is the second fundamental form of $\mathcal{D}_t$, $\gamma_t \subset \mathcal{D}_t$ is any path joining p to q and $\ell(\gamma_t)$ is the hyperbolic length of γ_t. Using the fact that the maximal geodesic curvature κ_t of $\mathcal{D}_t$ satisfies $\kappa_t \sim \coth r_t$ as t tends to zero, there exists a uniform constant $C > 0$ such that

$$\sup_{s\in\mathcal{D}_t} \|II_t(s)\| < C \coth r_t.$$

Let $0 < \alpha < (1+C)^{-1} < 1$ and suppose that $d_{\mathbb{H}^3}(p,q) < \alpha r_t$. Let $\sigma_t : [0,1] \to \mathbb{H}^3$ be the geodesic curve of $\mathbb{H}^3$ joining p to q. Then $\sigma_t([0,1]) \subset \mathrm{Tub}_{\alpha r_t}$ and thus the projection $\pi_t : \sigma_t([0,1]) \to \mathcal{D}_t$ is well-defined. Let $\gamma_t := \pi_t \circ \sigma_t$. Then

$$\begin{aligned}
\left\|\Gamma_p^q \eta_t(p) - \eta_t(q)\right\| &\leq C \coth r_t \times \sup_{s\in\sigma_t} \|d\pi_t(s)\| \\
&\leq C \coth r_t \times \sup_{s\in\mathrm{Tub}_{\alpha r_t}} \|d\pi_t(s)\| \times d_{\mathbb{H}^3}(p,q) \\
&\leq C \coth r_t \times \frac{\tanh r_t}{\tanh r_t - \tanh(\alpha r_t)} \times \alpha r_t \\
&\leq \frac{C\alpha r_t}{\tanh r_t - \tanh(\alpha r_t)} \sim \frac{C\alpha}{1-\alpha} < 1
\end{aligned}$$

as t tends to zero. $\square$

Appendix B. Remarks on the polar decomposition

Let $\mathrm{SL}(2,\mathbb{C})^{++}$ be the subset of $\mathrm{SL}(2,\mathbb{C})$ whose elements are hermitian positive definite. Let

$$\begin{array}{rcccc} \mathrm{Pol} & : & \mathrm{SL}(2,\mathbb{C}) & \longrightarrow & \mathrm{SL}(2,\mathbb{C})^{++}\times \mathrm{SU}(2) \\ & & A & \longmapsto & (\mathrm{Pol}_1(A),\mathrm{Pol}_2(A)) \end{array}$$

be the polar decomposition on $\mathrm{SL}(2,\mathbb{C})$. This map is differentiable and satisfies the following proposition.

Proposition 14. *For all $A\in \mathrm{SL}(2,\mathbb{C})$, $\|d\mathrm{Pol}_2\,(A)\| \leq |A|$.*

Proof. We first write the differential of Pol_2 at the identity in an explicit form. Writing

$$\begin{array}{rcccc} d\mathrm{Pol}_2(\mathrm{I}_2) & : & \mathfrak{sl}(2,\mathbb{C}) & \longrightarrow & \mathfrak{su}(2) \\ & & M & \longmapsto & \mathrm{pol}_2(M) \end{array}$$

gives

$$\mathrm{pol}_2\begin{pmatrix} a & b \\ c & -a\end{pmatrix} = \begin{pmatrix} i\,\mathrm{Im}\,a & \frac{b-\bar{c}}{2} \\ \frac{c-\bar{b}}{2} & -i\,\mathrm{Im}\,a\end{pmatrix}.$$

Note that for all $M\in\mathfrak{sl}(2\mathbb{C})$,

$$\begin{aligned} |\mathrm{pol}_2(M)|^2 &= 2\,(\mathrm{Im}\,a)^2 + \frac{1}{4}\left(|b-\bar{c}|^2 + \left|c-\bar{b}\right|^2\right) \\ &\leq |M|^2 - \frac{1}{2}\,|b+\bar{c}|^2 \\ &\leq |M|^2\,. \end{aligned}$$

We then compute the differential of Pol_2 at any point of $\mathrm{SL}(2,\mathbb{C})$. Let $(S_0,Q_0)\in \mathrm{SL}(2,\mathbb{C})^{++}\times \mathrm{SU}(2)$. Consider the differentiable maps

$$\begin{array}{rcccc} \phi & : & \mathrm{SL}(2,\mathbb{C}) & \longrightarrow & \mathrm{SL}(2,\mathbb{C}) \\ & & A & \longmapsto & S_0AQ_0 \end{array} \quad \text{and} \quad \begin{array}{rcccc} \psi & : & \mathrm{SU}(2) & \longrightarrow & \mathrm{SU}(2) \\ & & Q & \longmapsto & QQ_0. \end{array}$$

Then $\psi\circ\mathrm{Pol}_2\circ\phi^{-1} = \mathrm{Pol}_2$ and for all $M\in T_{S_0Q_0}\mathrm{SL}(2,\mathbb{C})$,

$$d\mathrm{Pol}_2\,(S_0Q_0)\cdot M = \mathrm{pol}_2\left(S_0^{-1}MQ_0^{-1}\right)Q_0.$$

Finally, let $A \in \mathrm{SL}(2,\mathbb{C})$ with polar decomposition $\mathrm{Pol}(A) = (S, Q)$. Then for all $M \in T_A\mathrm{SL}(2,\mathbb{C})$,

$$|d\mathrm{Pol}_2(A) \cdot M| = \left|\mathrm{pol}_2\left(S^{-1}MQ^{-1}\right)Q\right| \leq |S| \times |M|$$

and thus using

$$S = \exp\left(\frac{1}{2}\log\left(AA^*\right)\right)$$

gives

$$\|d\mathrm{Pol}_2(A)\| \leq |S| \leq |A|\,.$$

$\square$

Corollary 3. *Let $0 < q < \log\rho$ and $F_1, F_2 \in \Lambda\mathrm{SU}(2)_\rho$ with unitary parts $Q_i = \mathrm{Pol}_2(F_i(e^{-q}))$. Let $\epsilon > 0$ such that*

$$\left\|F_2^{-1}F_1 - \mathrm{I}_2\right\|_\rho < \epsilon.$$

If ϵ is small enough, then there exists a uniform $C > 0$ such that for all $v \in T_{\mathrm{I}_2}\mathbb{H}^3$,

$$\|Q_2 \cdot v - Q_1 \cdot v\|_{T_{\mathrm{I}_2}\mathbb{H}^3} \leq C\,\|F_2\|_\rho^2\,\epsilon.$$

Proof. Let $v \in T_{\mathrm{I}_2}\mathbb{H}^3$ and consider the following differentiable map

$$\begin{array}{cccc} \phi & : & \mathrm{SU}(2) & \longrightarrow & T_{\mathrm{I}_2}\mathbb{H}^3 \\ & & Q & \longmapsto & Q \cdot v. \end{array}$$

Then

$$\begin{aligned} \|Q_2 \cdot v - Q_1 \cdot v\|_{T_{\mathrm{I}_2}\mathbb{H}^3} &= \|\phi(Q_2) - \phi(Q_1)\|_{T_{\mathrm{I}_2}\mathbb{H}^3} \\ &\leq \sup_{t\in[0,1]} \|d\phi(\gamma(t))\| \times \int_0^1 |\dot\gamma(t)|\,dt \end{aligned}$$

where $\gamma : [0,1] \longrightarrow \mathrm{SU}(2)$ is a path joining Q_2 to Q_1. Recalling that $\mathrm{SU}(2)$ is compact gives

$$\|Q_2 \cdot v - Q_1 \cdot v\|_{T_{\mathrm{I}_2}\mathbb{H}^3} \leq C\,|Q_2 - Q_1| \tag{B.4}$$

where $C > 0$ is a uniform constant. But writing $A_i = F_i(e^{-q}) \in \mathrm{SL}(2,\mathbb{C})$,

$$|Q_2 - Q_1| = |\mathrm{Pol}_2(A_2) - \mathrm{Pol}_2(A_1)| \\ \leq \sup_{t\in[0,1]} \|d\mathrm{Pol}_2(\gamma(t))\| \times \int_0^1 |\dot\gamma(t)|\, dt$$

where $\gamma : [0,1] \longrightarrow \mathrm{SL}(2,\mathbb{C})$ is a path joining A_2 to A_1. Take for example

$$\gamma(t) := A_2 \ \exp\left(t \log\left(A_2^{-1} A_1\right)\right).$$

Suppose now that ϵ is small enough for log to be a diffeomorphism from $D(\mathrm{I}_2, \epsilon) \cap \mathrm{SL}(2,\mathbb{C})$ to $D(0,\epsilon') \cap \mathfrak{sl}(2,\mathbb{C})$. Then

$$\left\|A_2^{-1}A_1 - \mathrm{I}_2\right\| \leq \left\|F_2^{-1}F_1 - \mathrm{I}_2\right\|_\rho < \epsilon$$

implies

$$|\gamma(t)| \leq \widetilde{C}\, |A_2| \quad \text{and} \quad |\dot\gamma(t)| \leq \widetilde{C}\widehat{C}\, |A_2|\, \epsilon$$

where $\widetilde{C}, \widehat{C} > 0$ are uniform constants. Using Proposition 14 gives

$$|Q_2 - Q_1| \leq \widehat{C}\widetilde{C}^2\, |A_2|^2\, \epsilon$$

and inserting this inequality into (B.4) gives

$$\|Q_2 \cdot v - Q_1 \cdot v\|_{T_{\mathrm{I}_2}\mathbb{H}^3} \leq C\widehat{C}\widetilde{C}^2\, |A_2|^2\, \epsilon \leq C\widehat{C}\widetilde{C}^2\, \|F_2\|_\rho^2\, \epsilon.$$

□

References

[1] D. V. Alekseevskij, E. B. Vinberg, and A. S. Solodovnikov, *Geometry of Spaces of Constant Curvature*, in E. B. Vinberg, editor, Geometry II: Spaces of Constant Curvature, Encyclopaedia of Mathematical Sciences, 1–138, Springer, Berlin, Heidelberg (1993), ISBN 978-3-662-02901-5.

[2] A. I. Bobenko, S. Heller, and N. Schmitt, *Minimal n-noids in hyperbolic and anti-de Sitter 3-space*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **475** (2019), no. 2227, 20190173. Publisher: Royal Society.

[3] J. Dorfmeister, J.-I. Inoguchi, and S. Kobayashi, *Constant mean curvature surfaces in hyperbolic 3-space via loop groups*, Journal für die reine und angewandte Mathematik **2014** (2014) 1.

[4] J. Dorfmeister, F. Pedit, and H.-H. Wu, *Weierstrass type representation of harmonic maps into symmetric spaces*, Communications in Analysis and Geometry **6** (1998), no. 4, 633–668. Publisher: International Press of Boston, Somerville, MA.

[5] J. Dorfmeister and H. Wu, *Construction of constant mean curvature n-noids from holomorphic potentials*, Mathematische Zeitschrift **258** (2008), no. 4, 773–803.

[6] S. Fujimori, S. Kobayashi, and W. Rossman, *Loop group methods for constant mean curvature surfaces*, `arXiv:math/0602570`, (2009).

[7] S. Heller, *Higher genus minimal surfaces in S3 and stable bundles*, Journal für die reine und angewandte Mathematik **2013** (2013), no. 685, 105–122. Publisher: De Gruyter Section: Journal für die reine und angewandte Mathematik.

[8] ———, *Lawson's genus two surface and meromorphic connections*, Mathematische Zeitschrift **274** (2013), no. 3, 745–760.

[9] M. Jleli and R. López, *Bifurcating Nodoids in Hyperbolic Space*, Advanced Nonlinear Studies **15** (2015), no. 4, 849–865.

[10] N. Kapouleas, *Complete Constant Mean Curvature Surfaces in Euclidean Three-Space*, Annals of Mathematics **131** (1990), no. 2, 239–330. Publisher: Annals of Mathematics.

[11] M. Kilian, Constant mean curvature cylinders, Ph.D. thesis, University of Massachusetts Ahmerst (2000).

[12] M. Kilian, I. McIntosh, and N. Schmitt, *New constant mean curvature surfaces*, Experimental Mathematics **9** (2000), no. 4, 595–611. Publisher: A K Peters, Ltd.

[13] M. Kilian, W. Rossman, and N. Schmitt, *Delaunay ends of constant mean curvature surfaces*, Compositio Mathematica **144** (2008), no. 1, 186–220. Publisher: London Mathematical Society.

[14] N. J. Korevaar, R. Kusner, W. H. Meeks, and B. Solomon, *Constant Mean Curvature Surfaces in Hyperbolic Space*, American Journal of Mathematics **114** (1992), no. 1, 1–43. Publisher: Johns Hopkins University Press.

[15] H. B. Lawson, *Complete Minimal Surfaces in S3*, Annals of Mathematics **92** (1970), no. 3, 335–374. Publisher: Annals of Mathematics.

[16] R. Mazzeo and F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, Comm. Anal. Geom. **9** (2001), no. 1, 169–237.

[17] A. Pressley and G. Segal, Loop Groups, Oxford Mathematical Monographs, Oxford University Press, Oxford, New York (1988), ISBN 978-0-19-853561-4.

[18] T. Raujouan, *On Delaunay ends in the DPW method*, Indiana Univ. Math. J. **69** (2020), 2365–2393.

[19] E. A. Ruh and J. Vilms, *The tension field of the Gauss map*, Transactions of the American Mathematical Society **149** (1970), no. 2, 569–573.

[20] N. Schmitt, *Constant mean curvature trinoids*, arXiv:math/0403036, (2004).

[21] N. Schmitt, M. Kilian, S.-P. Kobayashi, and W. Rossman, *Unitarization of monodromy representations and constant mean curvature trinoids in 3-dimensional space forms*, Journal of the London Mathematical Society **75** (2007), no. 3, 563–581. Publisher: Oxford Academic.

[22] M. E. Taylor, Introduction to Differential Equations, American Mathematical Soc. (2011), ISBN 978-0-8218-5271-2.

[23] G. Teschl, Ordinary Differential Equations and Dynamical Systems, American Mathematical Soc. (2012), ISBN 978-0-8218-8328-0.

[24] E. Teufel, *A generalization of the isoperimetric inequality in the hyperbolic plane*, Archiv der Mathematik **57** (1991), no. 5, 508–513.

[25] M. Traizet, *Construction of constant mean curvature n-noids using the DPW method*, J. Reine Angew. Math. **763** (2020), 223–249.

[26] ———, *Gluing Delaunay ends to minimal n-noids using the DPW method*, Math. Ann. **377** (2020), no. 3-4, 1481–1508.

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