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Bergman-Einstein metric on a Stein space with a strongly pseudoconvex boundary

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Let Ω be a Stein space with a compact smooth strongly pseudoconvex boundary. We prove that the boundary is spherical if its Bergman metric over $\mathrm{Reg}(\Omega)$ is Kähler-Einstein.

1. Introduction

For any bounded domain in $D \subset \mathbb{C}^n$, its Bergman metric is a canonical biholomorphically invariant Kähler metric over D. Cheng-Yau [CY80] proved that there exists a complete Kähler-Einstein metric on a bounded pseudoconvex domain in $\mathbb{C}^n$ with a C^2-smooth boundary. A well-known open question initiated from the work of Cheng-Yau [CY80] asks when the Bergman metric on a smoothly bounded domain coincides with its Cheng-Yau Kähler-Einstein metric. Cheng conjectured in [C79] that the Bergman metric of a smoothly bounded strongly pseudoconvex domain is Kähler-Einstein if and only if the domain is biholomorphic to the ball. This conjecture was solved by Fu-Wong [FW97] and Nemirovski-Shafikov [NS06] in the case of complex dimension two and was verified in a recent paper of Huang-Xiao [HX16] for any dimensions. Recently, Ebenfelt-Xiao-Xu [EXX20] introduced a new characterization of the two-dimensional unit ball $\mathbb{B}^2$, more generally, two-dimensional finite ball quotients $\mathbb{B}^2/\Gamma$ in terms of algebracity of the Bergman kernel. There have been also other related studies on versions of the Cheng's conjecture in terms of metrics defined by other important canonical potential functions as in the work of Li [L1, L2, L3].

On a complex space Ω with possible singularities, Kobayashi [Kob] defined the Bergman kernel form on its smooth part $\mathrm{Reg}(\Omega)$ which is naturally identified with the Bergman kernel function in the domain case. The Kobayashi Bergman kernel form can be similarly used to define a Kähler form on $\mathrm{Reg}(\Omega)$ under certain geometric conditions on Ω, which are always

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the case when Ω is a Stein space with a compact smooth strongly pseudoconvex boundary. In this paper, we address the generalized Cheng question of understanding the geometric implication when the Bergman metric of a Stein space with a compact strongly pseudoconvex boundary has the Einstein property.

To state our main theorem, we first introduce a few notations. Let Ω be a Stein space of dimension n with possibly isolated singularity and write $\mathrm{Reg}(\Omega)$ for its regular part. Write $\Lambda^n(\mathrm{Reg}(\Omega))$ for the space of the holomophic $(n,0)$-forms on $\mathrm{Reg}(\Omega)$ and define the Bergman space of Ω as follows:

$$A^2(\Omega) := \left\{ f \in \Lambda^n(\mathrm{Reg}(\Omega)) : (-1)^{\frac{n^2}{2}} \int_{\mathrm{Reg}(\Omega)} f \wedge \overline{f} < \infty \right\}.$$

Then $A^2(\Omega)$ is a Hilbert sapce with the inner product:

$$(f,g) = (-1)^{\frac{n^2}{2}} \int_{\mathrm{Reg}(\Omega)} f \wedge \overline{g}, \text{ for all } f,g \in \Lambda^n(\mathrm{Reg}(\Omega)).$$

We assume that $A^2(\Omega) \neq \{0\}$. Let $\{f_j\}_1^N$ be an orthonormal basis of $A^2(\Omega)$ and define the Bergman kernel to be $K_\Omega = \sum_{j=1}^N f_j \wedge \overline{f}_j$. Here, N is either a natural number or ∞. In a local holomorphic coordinate chart (U,z) on $\mathrm{Reg}(\Omega)$, we have

$$K_\Omega = k_\Omega(z,\overline{z}) dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n \text{ in } U.$$

Assume further that K_Ω is nowhere zero on $\mathrm{Reg}(\Omega)$. We define a Hermitian $(1,1)$-form on $\mathrm{Reg}(\Omega)$ by $\omega_\Omega^B = i\partial\overline{\partial}\log k_\Omega(z,\overline{z})$. We call ω_Ω^B the Bergman metric on Ω if it indeed induces a positive definite metric on $\mathrm{Reg}(\Omega)$.

Notice that if Ω is a Stein space with a compact smooth strongly pseudoconvex boundary then $\overline{\Omega}$ can be compactly embedded into a closed Stein subspace of a certain complex Euclidean space. Then $A^2(\Omega)$ is of infinite dimension and it indeed defines a Bergman metric on $\mathrm{Reg}(\Omega)$.

Our main purpose of this paper is to generalize results obtained in [FW97] and [HX16] to Stein spaces with possible singularities:

Theorem 1.1. *Let Ω be a Stein space with a compact smooth strongly pseudoconvex boundary. If its Bergman metric ω_Ω^B on* $\mathrm{Reg}(\Omega)$ *is Kähler-Einstein then $\partial\Omega$ is spherical.*

2. Proof of Theorem 1.1

In this section, we start with a strongly pseudoconvex complex manifold M with a compact strongly pseudoconvex boundary. We denote by E the exceptional set in M in the sense of Grauert [G62], that is, there exists a blowing down map $\pi : M \to \Omega$ from M to a Stein space Ω with isolated singularities such that $\pi^{-1}(\mathrm{Sing}(\Omega)) = E$ and $\pi : M \setminus E \to \Omega \setminus \mathrm{Sing}(\Omega)$ is a biholomorphic map. Here, we denote by $\mathrm{Sing}(\Omega)$ the set of singularities in Ω and define $\mathrm{Reg}(\Omega) := \Omega \setminus \mathrm{Sing}(\Omega)$. Since the boundary of M is strongly pseudoconvex then by a Theorem of Oshawa [Oh84] and Hill-Nacinovich [HN05, Theorem 3.1] there exists a larger complex manifold $M' \supset \overline{M}$, that contains M as its open subset.

Let $\Omega^{n,0}(\overline{M})$ be the space of smooth $(n, 0)$-forms on M which are smooth up to the boundary. Let $\Omega^{n,0}_c(M)$ be the subspace of $\Omega^{n,0}(\overline{M})$ with elements having compact support in M. We define the L^2 inner product on $\Omega^{n,0}_c(M)$ as following

$$(f, g) = (-1)^{\frac{n^2}{2}} \int_M f \wedge \overline{g} \text{ for all } f, g \in \Omega^{n,0}_c(M).$$

Let $L^2_{(n,0)}(M)$ be the completion of $\Omega^{n,0}_c(M)$ under the above inner product. We denote by $H_s(M), s \in \mathbb{R}$ the Sobolev space of order s on M (see [FK72, Appendix]). Write $\Lambda^n(M)$ for the space of the holomorphic n-forms on M and we define the Bergman space of M to be

$$A^2(M) = \left\{ f \in \Lambda^n(M) : (-1)^{\frac{n^2}{2}} \int_M f \wedge \overline{f} < \infty \right\}.$$

Then $A^2(M)$ is a closed subspace of $L^2_{(n,0)}(M)$.

Let $P : L^2_{(n,0)}(M) \to A^2(M)$ be the orthogonal projection which we call the Bergman projection of M. The reproducing kernel of the Bergman projection is denoted by $K_M(z, w)$. Let $\{f_j\}_{j=1}^{\infty}$ be an orthnormal basis of $A^2(M)$. Let $pr_1 : M \times M \to M$ and $pr_2 : M \times M \to M$ be the natural projection from the product space. Then the reproducing kernel of the Bergman projection P is a (n, n)-form on $M \times M$ which can be written as

$$K_M(z, \overline{w}) = \sum_{j=1}^{\infty} pr_1^* f_j \wedge pr_2^* \overline{f_j} = \sum_{j=1}^{\infty} f_j(z) \wedge \overline{f_j(w)}, \forall (z, w) \in M \times M.$$

Here, $f_j(z)$ and $f_j(w)$ are considered as a $(n, 0)$-forms at (z, w) for each j. Then $K_M(z, \overline{z})$ can be considered as a $2n$-form on M which is called the Bergman kernel form on M. Both $K_M(z,\overline{w})$ and the Bergman kernel $K_M(z,\overline{z})$ are independent of the choice of the orthonormal basis of $A^2(M)$. In a local coordinate chart (U,z) of M with $z=(z_1,\ldots,z_n)$ we have

$$K_M(z,\overline{z}) = k_M(z,\overline{z})dz_1\wedge\cdots\wedge dz_n\wedge d\overline{z_1}\wedge\cdots\wedge d\overline{z_n}, \tag{2.1}$$

where $k_M(z,\overline{z})=\sum_{j=1}^{\infty}|\hat{f}_j(z)|^2$ with $f_j=\hat{f}_j(z)dz_1\wedge\cdots\wedge dz_n$. Then $\omega_M^B=\partial\overline{\partial}\log k_M$ is a well defined Hermitian $(1,1)$-form on M where K_M is nonzero. We call ω_M^B the Bergman metric over the subset where it is positive definite.

Since the Bergman metric over $\mathrm{Reg}(\Omega)$ is well defined, thus ω_M^B is a well defined Bergman metric on $M\setminus E$. Write $g_{\alpha\overline{\beta}}^M=\frac{\partial^2\log k_M}{\partial z_\alpha\partial\overline{z}_\beta}$ and define $G_M(z):=\det(g_{\alpha\overline{\beta}}^M)$. Then the Ricci tensor of the Bergman metric on $M\setminus E$ is given by

$$R_{\alpha\overline{\beta}}^M(z) = -\frac{\partial^2\log G_M(z)}{\partial z_\alpha\partial\overline{z}_\beta}.$$

The Bergman metric on $M\setminus E$ is called Kähler-Einstein when $R_{\alpha\overline{\beta}}^M=cg_{\alpha\overline{\beta}}^M$ for some constant c. It is well-known that the constant c is necessary negative (as we will also see later). Since $\omega_M^B=\pi^*\omega_\Omega^B$ over $M\setminus E$, thus ω_M^B is Kähler-Einstein over $M\setminus E$ if and only if ω_Ω^B is Kähler-Einstein over $\mathrm{Reg}(\Omega)$.

Now, an equivalent version of Theorem 1.1 is as follows:

Theorem 2.1. *Let M be a complex manifold with a compact smoothly stronlgy pseudoconvex boundary. If the Bergman metric on $M\setminus E$ is Kahler-Einstein, then ∂M is spherical.*

With Theorem 2.1 at our disposal and by a similar argument as in the [NS06] and [HX16], we have the following:

Corollary 2.2. *Let M be a Stein manifold with a compact smooth strongly pseudoconvex boundary. If the Bergman metric on M is Kahler-Einstein, then M is biholomorphic to the ball.*

3. Localization of Bergman kernel forms

Assume now that M is a complex manifold with a compact smooth strongly pseudo-convex boundary. Fix $w_0\in M$. Then $K_M(z,w_0)$ is a holomorphic $(n,0)$-form with respect to z and is L^2-integrable.

Let $w=(w_1,\cdots,w_n)$ be coordinates in a neighborhood of w_0. We explain the meaning of L^2-integrablity of $K_M(z,w_0)$: Write $dw=dw_1\wedge\cdots\wedge$

dw_n and $d\overline{w} = d\overline{w}_1 \wedge \cdots d\overline{w}_n$. Write

$$K_M(z, w_0) = \tilde{k}_M(z, w_0) \wedge d\overline{w}|_{w_0}.$$

Then $\tilde{k}_M(z, w_0)$ is a $(n, 0)$-form on M. By saying $K_M(z, w_0)$ is L^2-integrable with respect to z we meant that

$$(-1)^{\frac{n^2}{2}} \int_M \tilde{k}_M(z, w_0) \wedge \overline{\tilde{k}_M(z, w_0)} < \infty.$$

The L^2-integrability of $K(z, w_0)$ does not depend on the choice of coordinates w.

For any $p \in \partial M$, there exists a coordinate chart (U, z) of M' centered at p. Take a smooth strongly pseudocovnex domain $D \subset M \cap U$ such that

$$D \cap B(p, 2\delta) = M \cap B(p, 2\delta) \tag{3.1}$$

where $B(p, 2\delta) = \{q \in U : |z(q)| < 2\delta\}$ with $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ and δ being sufficiently small. We then have the following localization result for which there is no need to assume that the Bergman metric of M is Kähler-Einstein.

Proposition 3.1. *For $p \in \partial M$, let $D \subset M$ be a strongly pseudoconvex domain satisfying (3.1). Let $k_M(z, \overline{z}), k_D(z, \overline{z})$ be given as in (2.1). Then*

$$k_M(z, \overline{z}) = k_D(z, \overline{z}) + \varphi(z), \tag{3.2}$$

where $\varphi(z) \in C^\infty(B(p, \delta) \cap \overline{M})$.

Proof. We will follow the Fefferman [Fe74] localization method developed in the domain case. For clarity, we proceed in two steps.

Step 1. Let (U, w) be a coordinate chart centered at p where $w = (w_1, \cdots, w_n)$ are holomorphic coordinates. Write $d\overline{w}|_w = d\overline{w}_1 \wedge \cdots \wedge d\overline{w}_n|_w, \forall w \in U$. We fix $w \in B(p, r) \cap M$ and set

$$f_w(z) = K_M(z, \overline{w}) - K_D(z, \overline{w})\chi_D(z), z \in M,$$

where χ_D is the characteristic function of D. Write $f_w(z) = \tilde{f}_w(z) \wedge d\overline{w}|_w$ and $\tilde{g}_w(z) = \overline{\partial}\tilde{f}_w$ where $\tilde{f}_w(z)$ is a L^2-integrable $(n, 0)$-form on M, $\tilde{f}_w \perp$

$A^2(M)$ and $\tilde{g}_w$ is a $(n,1)$-form in $H_{-1}(M)$ with

$$\text{supp } \tilde{g}_w \subset \partial D \setminus \partial M.$$

By the smoothing property, there is a sequence of $(n,0)$-form $\{\tilde{f}_w^\varepsilon\}$ on M which are smooth up to $\overline{M}$ such that $\tilde{f}_w^\varepsilon \to \tilde{f}_w$ in the L^2 space. Set $\tilde{g}_w^\varepsilon = \overline{\partial}\tilde{f}_w^\varepsilon$. Since supp $\tilde{g}_w \subset \overline{\partial D \setminus \partial M}$, we can assume that supp $\tilde{g}_w^\varepsilon$ is contained in a ε-neighborhood of $\partial D \setminus \partial M$. Moreover,

$$\tilde{f}_w^\varepsilon \to \tilde{f}_w \text{ in } L^2_{(n,0)}(M),\ \tilde{g}_w^\varepsilon \to \tilde{g}_w \text{ in } H_{-1}(M). \tag{3.3}$$

Fix a Hermitian metric g on M'. For $0 \le q \le n$, let $L^2_{(n,q)}(M)$ be the space of L^2-integrable (n,q)-forms with respect to g. When $q = 0$, this definition of the space $L^2_{(n,0)}(M)$ is the same as defined in Section 2. We denote by $N^{(q)}$ the $\overline{\partial}$-Neumann operator with respect to $\square^{(q)}$. For convenience, we denote $N^{(q)}$ by N when it dose not cause any confusing. Since M is strongly pseudoconvex, then by the local regularity of N(see [Ke72] and [FK72]) we have

$$\|\xi N\tilde{g}_w^\varepsilon\|_s \le C_s(\|\xi_1\tilde{g}_w^\varepsilon\|_s + \|\tilde{g}_w^\varepsilon\|_{-1}), \forall s \ge 0, \tag{3.4}$$

with $\{C_s\}$ constants independent of w. Here, $\xi(z), \xi_1(z) \in C_0^\infty(B(p, \frac{3}{2}\delta))$ and $\xi_1|_{\text{supp}\xi} \equiv 1$, $\xi|_{B(p,\delta)} \equiv 1$. Since $B(p,2\delta) \cap \partial D \setminus \partial M = \emptyset$, then $\xi_1\tilde{g}_w^\varepsilon \equiv 0$ when ε is sufficiently small. Thus,

$$\|\xi N\tilde{g}_w^\varepsilon\|_s \le C_s\|\tilde{g}_w^\varepsilon\|_{-1}. \tag{3.5}$$

By (3.3) and (3.5), $\{\xi N\tilde{g}_w^\varepsilon\}$ is a Cauchy sequence in $H_s(M)$ for any $s \ge 0$. Assume that $\xi N\tilde{g}_w^\varepsilon \to h$ in $H_s(M)$ for any $s \ge 0$. Then $h \in C^\infty(\overline{M})$. On the other hand, $\tilde{f}_w^\varepsilon - P\tilde{f}_w^\varepsilon = \overline{\partial}^* N\tilde{g}_w^\varepsilon$ where $P : L^2_{(n,0)}(M) \to A^2(M)$ is the Bergman projection. Then

$$\xi(\tilde{f}_w^\varepsilon - P\tilde{f}_w^\varepsilon) = \xi\overline{\partial}^* N\tilde{g}_w^\varepsilon = \overline{\partial}^*(\xi N\tilde{g}_w^\varepsilon) - [\xi, \overline{\partial}^*](\xi_1 N\tilde{g}_w^\varepsilon). \tag{3.6}$$

By (3.5), we have

$$\|\xi(\tilde{f}_w^\varepsilon - P\tilde{f}_w^\varepsilon)\|_s \le C_s\|\tilde{g}_w^\varepsilon\|_{-1}. \tag{3.7}$$

We claim that $\{\|\tilde{g}_w\|_{-1}\}$ has uniform bound with respect to $w \in B(p,\delta) \cap M$. We next give a proof of this Claim as follows:

Choose a real function $\rho \in C^\infty(M')$ such that $\rho \equiv 1$ in a 2σ-neighborhood of $\partial D \setminus \partial M$ denoted by V_σ in M'. Write $K_D(z,w) = \tilde{K}_D(z,w) \wedge d\overline{w}|_w$ for all

$w \in M \cap B(p,\delta)$. Since supp $\tilde{g}_w \subset \partial D \setminus \partial M$, then $\forall \varphi = \sum_{j=1}^n \varphi_j dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_j \in \Omega_c^{(n,1)}(M)$ we have $(\tilde{g}_w, \varphi) = (\tilde{g}_w, \rho\varphi)$ and

$$
\begin{aligned}
(\tilde{g}_w, \rho\varphi) &= (\overline{\partial}\tilde{f}_w, \rho\varphi) = (\overline{\partial}(\tilde{K}_D(z,\overline{w})\chi_D(z)), \rho\varphi) \\
&= (\tilde{K}_D(z,\overline{w})\chi_D(z), \overline{\partial}^*(\rho\varphi)) = \int_D \tilde{K}_D(z,\overline{w}) \wedge \overline{\overline{\partial}^*(\rho\varphi)} \\
&= \int_{V_{2\sigma}} k_D(z,w) dz_1 \wedge \cdots \wedge dz_n \wedge \overline{\overline{\partial}^*(\rho\varphi)},
\end{aligned}
\tag{3.8}
$$

where $\tilde{K}_D(z,w) = k_D(z,w)dz_1 \wedge \cdots \wedge dz_n$. Since $d(V_{2\sigma}, B(p,\delta)) > 0$ when σ, δ are sufficinetly small then by a result of Kerzman [Ke72, Theorem 2] we have

$$
\sup_{z \in V_\sigma} |k_D(z,w)| \leq C, \forall w \in M \cap B(p,\delta)
\tag{3.9}
$$

where C is a constant independent of w. Then from (3.8) and (3.9) we have

$$
|(g_w, \varphi)| \leq C_1 \|\varphi\|_1, \forall w \in B(p,\delta) \cap M,
\tag{3.10}
$$

where the constant C_1 does not depend on $w \in B(p,\delta) \cap M$. Thus, we get the conclusion of the Claim.

On the other hand, $P\tilde{f}_w^\varepsilon \to 0$ in $L^2(M)$ as $\tilde{f}_w \perp A^2(M)$. By (3.6) and the Rellich lemma, we have $\xi(\tilde{f}_w^\varepsilon - P\tilde{f}_w^\varepsilon) \to h_s$ in $H_s(M)$ $\forall s \geq 0$ for a certain h_s. Then by (3.3) we have $h_s = \xi\tilde{f}_w$. Thus, from the above Claim and by taking the limit in (3.7), we have

$$
\|\xi\tilde{f}_w\|_s \leq \tilde{C}_s.
\tag{3.11}
$$

Here, the constant $\tilde{C}_s$ does not depend on $w \in B(p,r) \cap M$.

Step 2. Write $f_w(z) = \tilde{f}_w(z)dw|_w$ and $\tilde{g}_w = \overline{\partial}\tilde{f}_w$. Then $D_w^\alpha \tilde{g}_w = \overline{\partial} D_w^\alpha \tilde{f}_w$ for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$. Here, $\overline{\partial}$ is defined with respect to the z-direction. We still have $D_w^\alpha \tilde{f}_w \perp A^2(M)$ for any $w \in M \cap B(p,\delta)$. Then by a similar argument in Step 1, we have

$$
\|\xi D_w^\alpha \tilde{f}_w\|_s \leq \tilde{C}_s.
\tag{3.12}
$$

Here, constants $\tilde{C}_s$ do not depend on $w \in M \cap B(p,\delta)$. Then by Sobolev embedding theorem, we have that

$$
|\xi D_z^\alpha D_w^\beta \tilde{f}_w(z)| \leq C_{\alpha,\beta}, \forall \alpha, \beta, \forall z \in M, w \in M \cap B(p,\delta),
\tag{3.13}
$$

where $C_{\alpha,\beta}$ are constants. Since $\xi|_{B(p,\delta)} \equiv 1$, thus (3.13) implies that $\tilde{f}_w(z)$ is smooth up to $B(p,\delta) \cap \overline{M} \times B(p,\delta) \cap \overline{M}$. Thus, we get the conclusion of the proposition if we take $z = w \in B(p,\delta) \cap \overline{M}$. □

Remark 3.2. It is an interesting question if we can work directly on the Stein space to get the localization of the Bergman kernel forms. This depends on the regularity of the $\overline{\partial}$-Neumann operator on the Stein space. Whereas the theory of the $\overline{\partial}$-Neumann operator is very well developed on complex manifolds, not much is known about the situation on singular complex spaces. Ruppenthal [Ru11] has proved that the $\overline{\partial}$-Neumann operator $N_{n,1} : L^2_{(n,1)}(\mathrm{Reg}(\Omega)) \to L^2_{(n,1)}(\mathrm{Reg}(\Omega))$ is a compact operator on the Stein space Ω with only isolated singularities and compact strongly pseudoconvex boundary. It is still unknown if $N_{n,1}$ can gain more regularity which is crucial in our proof.

Let $B_M(z) = G_M(z)/k_M(z,z)$. Then $B_M(z)$ is a globally-defined smooth function on M although $G_M(z)$ and $k_M(z,z)$ are only locally given. The following lemma is a generalization of a result of Diederich [Di70, Theorem 2]:

Lemma 3.3. *$B_M(z) \to \frac{(n+1)^n \pi^n}{n!}$ as $z \to \partial M$.*

Proof. By Lemma 3.1, for any $p \in \partial M$ there exists a strongly pseudocovnex domain $D \subset M$ which satisfies (3.1) such that

$$k_M(z,\overline{z}) = k_D(z,\overline{z}) + \varphi(z) \tag{3.14}$$

where $\varphi(z) \in C^\infty(B(p,\delta) \cap \overline{M})$. Then

$$\log k_M(z,\overline{z}) = \log k_D(z,\overline{z}) + \log\left(1 + \frac{\varphi(z)}{k_D(z,\overline{z})}\right), z \in D \cap B(p,\delta). \tag{3.15}$$

Thus,

$$g^M_{\alpha\overline{\beta}} = g^D_{\alpha\overline{\beta}} + \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta} \log\left(1 + \frac{\varphi(z)}{k_D(z,\overline{z})}\right). \tag{3.16}$$

Since D can be seen as a strongly pseudoconvex domain in $\mathbb{C}^n$ with a smooth boundary, then by Fefferman's asymptotic expansion of Bergman kernels, we have

$$k_D(z,\overline{z}) = \frac{\Phi(z)}{r^{n+1}(z)} + \Psi(z)\log r(z), z \in D. \tag{3.17}$$

where r is a Fefferman defining function for D and $\Phi, \Psi \in C^\infty(\overline{D})$ and $\Phi(z) \neq 0$ for all $z \in \partial D$. Then

$$\log(1 + \frac{\varphi}{k_D(z,\overline{z})}) = \log\left(1 + \frac{\varphi(z) r^{n+1}}{\Phi + \Psi r^{n+1}\log r}\right) = \log(1 + f r^{n+1}) \tag{3.18}$$

where $f = \frac{\varphi(z)}{\Phi + \Psi r^{n+1}\log r}$. Since $n \geq 2$ and $\Phi|_{\partial D} \neq 0$, we have $f \in C^2(B(p,\delta) \cap \overline{M})$. By Taylor's expansion,

$$\log(1 + f r^{n+1}) = f r^{n+1} + O(f^2 r^{2(n+1)}) \text{ as } r \to 0. \tag{3.19}$$

Thus, $[\log(1 + f r^{n+1})]_{\alpha\overline{\beta}} \to 0$ as $z \to B(p,\delta) \cap \partial M$ for $n \geq 2$. Then combining (3.18) and (3.19), one has

$$\frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta} \log\left(1 + \frac{\varphi(z)}{k_D(z,\overline{z})}\right) \to 0.$$

As a consequence,

$$\frac{G_M(z)}{G_D(z)} \to 1 \tag{3.20}$$

as $z \to \partial M \cap B(p,\delta)$. From (3.14) we have

$$\frac{k_M(z,\overline{z})}{G_M(z)} = \frac{k_D(z,\overline{z})}{G_M(z)} + \frac{\varphi(z)}{G_M(z)}. \tag{3.21}$$

Combining (3.20) and (3.21) we have

$$\left| \frac{k_M(z,\overline{z})}{G_M(z)} - \frac{k_D(z,\overline{z})}{G_D(z)} \right| \to 0 \tag{3.22}$$

as $z \to \partial M \cap B(p,\delta)$. By [Di70, Theorem 2], we have

$$\frac{G_D(z)}{k_D(z,\overline{z})} \to \frac{(n+1)^n \pi^n}{n!} \tag{3.23}$$

as $z \to \partial D$. Substituting (3.23) into (3.22) we conclude the proof of the lemma. □

The following proposition is a generalization of a result of Fu-Wong [FW97, Proposition 1.1] which gives a characterization when the Bergman metric on $M \setminus E$ is Kähler-Einstein.

Proposition 3.4. *Let M be a relatively compact strongly pseudoconvex complex manifold with a smooth boundary. The Bergman metric on $M \setminus E$ is Kahler-Einstein if and only if $B_M(z) = \frac{(n+1)^n \pi^n}{n!}$ for all $z \in M \setminus E$.*

Proof. If the Bergman metric on $M \setminus E$ is Kähler-Einstein, then $R^M_{i\bar{j}} = c g^M_{i\bar{j}}$ where c is a constant. By Lemma 3.1 and a direct calculation one has that $R^M_{i\bar{j}} + g^M_{i\bar{j}}$ goes to zero as a tensor with respect to ω^B_M when $z \to \partial M$. Thus, combining the Kähler-Einstein assumption one has $c = -1$ and this implies that $\log B_M(z)$ is a pluriharmonic function on $M \setminus E$. Now, for any holomorphic disk $\phi : \Delta \to M \setminus E$ with ϕ is holomorphic in $\Delta := \{t \in \mathbb{C} : |t| < 1\}$, smooth continuous up to $\overline{\Delta}$ and $\phi(\partial\Delta) \subset \partial M$, we have $\log B_M(\phi(t))$ is harmonic. Since it takes the constant value on the boundary by Lemma 3.3, it takes a constant value $\log \frac{(n+1)^n \pi^n}{n!}$ over Δ. Now, since ∂M is strongly pseudoconvex, the union of such disks fills up an open subset of $M \setminus E$. Since $\log B_M$ is real analytic, we conclude that $B_M \equiv \log \frac{(n+1)^n \pi^n}{n!}$ over $M \setminus E$. If $\log B_M(z)$ takes constant value, then the Bergman metric is obviously Kähler-Einstien. □

Let $D = \{r > 0\}$ be a strongly pseudoconvex domain given in (3.1) where r is a defining for D. Then k_D has following expansion

$$k_D(z, \overline{z}) = \frac{\Phi(z)}{r^{n+1}(z)} + \Psi(z) \log r(z), z \in D \tag{3.24}$$

with $\Phi, \Psi \in C^\infty(\overline{D})$. Then from Proposition 3.4 we have the following

Lemma 3.5. *Let M be a relatively compact strongly pseudoconvex complex manifold with smooth boundary. Assume the Bergman metric on $M \setminus E$ is Kahler-Einstein. Then*

$$\Psi(z) = O(r^k) \text{ on } D \cap B(p, \delta) \tag{3.25}$$

for any $k > 0$.

Proof. By Proposition 3.4, we have the same identities as in [FW97, (1.1)]. Thus,

$$J(k_M) = (-1)^n C_n k_M^{n+2} \text{ on } D \cap B(p, \delta), \tag{3.26}$$

where $C_n = \frac{(n+1)^n \pi^n}{n!}$. On the other hand,

$$k_M = k_D + \varphi(z) \tag{3.27}$$

when $z \in B(p, \delta) \cap D$, where $\varphi \in C^\infty(B(p, \delta) \cap \overline{D})$. Substituting (3.24) and (3.27) into (3.26) and by a similar argument as in the proof of [FW97, Theorem 2.1] we get the conclusion of the lemma. □

Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly pseudocovnex domain with smooth boundary. The following Monge-Ampere type equation on Ω was introduced by Fefferman [Fe76]

$$\begin{aligned} J(u) \equiv (-1)^n \det \begin{pmatrix} u & u_{\overline{\beta}} \\ u_\alpha & u_{\alpha\overline{\beta}} \end{pmatrix} &= 1 \text{ in } \Omega \\ u &= 0 \text{ at } \partial\Omega \end{aligned} \tag{3.28}$$

Fefferman proved that Ω has a smooth defining function r_F which satisfies

$$J(r_F) = 1 + O(r_F^{n+1}).$$

We call r_F a Fefferman's defining function for Ω. Let us recall Fefferman's construction of such defining function. The existence of such an r_F can be established in the following steps: Starting with $\Omega = \{r > 0\}$ and $dr|_{\partial\Omega} \neq 0$, Fefferman defined recursively

$$\begin{aligned} u^1 &= \frac{r}{(J(r))^{1/n+1}}, \\ u^s &= u^{s-1}\left(1 + \frac{1 - J(u^{s-1})}{[n+2-s]s}\right), 2 \leq s \leq n+1. \end{aligned} \tag{3.29}$$

Each u^s satisfies $J(u^s) = 1 + O(r^s)$ and u^{n+1} is what we call Fefferman defining function.

Lemma 3.6. *There exists a Fefferman's defining function r_F for D such that*

$$r_F = \left(\frac{\pi^n}{n!}k_M\right)^{-\frac{1}{n+1}} \text{ on } D\cap B(p,\sigma). \tag{3.30}$$

for some small σ.

Proof. First, by Lemma 3.1 we have $k_M = k_D + \varphi(z)$. Then from the Bergman kernel expansion of k_D we have

$$\begin{aligned} k_M(z,\overline{z}) = k_D + \varphi &= \frac{\Phi(z)}{r^{n+1}} + \Psi(z)\log r + \varphi \\ &= \frac{\Phi + r^{n+1}\Psi\log r + r^{n+1}\varphi}{r^{n+1}} \end{aligned} \tag{3.31}$$

when $z \in D\cap B(p,\delta)$. Since $k_M(z,\overline{z}) > 0$ one has

$$\Phi + r^{n+1}\Psi\log r + r^{n+1}\varphi > 0$$

for all $z\in D\cap B(p,\delta)$. Thus,

$$(k_M)^{-\frac{1}{n+1}}(z) = \frac{r}{(\Phi + r^{n+1}\Psi\log r + r^{n+1}\varphi)^{\frac{1}{n+1}}} \tag{3.32}$$

is well-defined on $D\cap B(p,\delta)$. Moreover, from Lemma 3.5 we have that $(k_M)^{-\frac{1}{n+1}} \in C^\infty(B(p,\delta)\cap\overline{D})$. Then by partition of unity, we can choose a defining funciton r_0 for D such that

$$r_0 = \left(\frac{\pi^n}{n!}k_M\right)^{-\frac{1}{n+1}} \text{ on } D\cap B\left(p,\frac{\delta}{2}\right). \tag{3.33}$$

This idea has been crucially used in Huang-Xiao [HX16] to construct a Fefferman's defining function which satisfy the Monge-Ampere equation.

Let r_F be a Fefferman defining function for D. Then $r_F = hr_0$ for some $h\in C^\infty(\overline{D})$ and $h>0$ on D. Since

$$J(r_F) = h^{n+1}J(r_0) \text{ on } \partial D$$

and $J(r_F)=1$ on ∂D, thus $J(r_0)\neq 0$ on ∂D. Thus, by continuity $J(r_0)\neq 0$ in a neighborhood of ∂D. So the set $K=\{z\in D : J(r_0)=0\}$ is a compact subset of D. Choose a cut-off function χ such that $\chi \equiv 1$ in a neighborhood of ∂D and $\chi \equiv 0$ in a neighborhood of K. Set

$$u^1 = \chi \frac{r_0}{(J(r_0))^{\frac{1}{n+1}}}.$$

Then we still have $J(u^1) = 1$ on ∂D. We notice that the Kahler-Einstein condition of the Bergman metric implies that $J(\frac{\pi^n}{n!}k_M)^{-\frac{1}{n+1}} = 1$ for $z \in D$, so $J(r_0) \equiv 1$ on $D \cap B(p, \frac{\delta}{2})$ by the construction of r_0 in (3.33). Then

$$J(u^1) = 1 \text{ on } D \cap B(p, \sigma), \tag{3.34}$$

for some $\sigma < \frac{\delta}{2}$. Then from Fefferman's construction of Fefferman defining function (3.29) we see that

$$u^1 = u^2 = \cdots = u^{n+1} = r_0 \text{ on } D \cap B(p, \sigma). \tag{3.35}$$

Combing with (3.34) and changing the values of u_{n+1} in a certain compact subset of M if needed, we get the conclusion of the lemma. □

4. Proof of Theorem 2.1

We first recall the Moser normal [CM74] form theory and the notion of Fefferman scalar invariants [Gr85]. Let $X \subset \mathbb{C}^n$ be a real analytic strongly pseudoconvex hypersurface with $p \in X$. There exist coordinates $(z, w) = (z_1, \cdots, z_{n-1}, w)$ such that in this new coordinates $p \leftrightarrow 0$ and X is locally defined by an equation of the form

$$2u = |z|^2 + \sum_{|\alpha| \geq 2, |\beta| \geq 2, v \geq 0} A^l_{\alpha\overline{\beta}} z^\alpha z^{\overline{\beta}} v^l \tag{4.1}$$

where $\alpha = (\alpha_1, \cdots, \alpha_{n-1}), \beta = (\beta_1, \cdots, \beta_{n-1})$ and $A^l_{\alpha\overline{\beta}}$ satisfying

- $A^l_{\alpha\overline{\beta}}$ is symmetric with respect to the permutation of indices in α and β, respectively;
- $\overline{A^l_{\alpha\overline{\beta}}} = A^l_{\beta\overline{\alpha}}$;
- $\mathrm{tr} A^l_{2\overline{2}} = 0, \mathrm{tr}^2 A^l_{3\overline{3}} = 0, \mathrm{tr}^3 A^l_{3\overline{3}} = 0.$

Here, for $p, q \geq 2$, $A^l_{p\overline{q}}$ is the symmetric tensor $[A^l_{\alpha\overline{\beta}}]_{|\alpha|=p,|\beta|=q}$ on $\mathbb{C}^{n-1}$ and the traces are the usual tensorial traces with respect to $\delta_{i\overline{j}}$. Here, (4.1) is called the normal form of X at p and $\{A^l_{\alpha\overline{\beta}}\}$ are called the normal form coefficients. When X is merely smooth, the expansion (4.1) is in the formal sense.

Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with C^∞-smooth boundary with $p = 0 \in \partial D$. Using a Fefferman defining function r in the asymptotic expansion of the Bergman kernel function

$$k_D(z, \overline{z}) = \frac{\phi(z)}{r^{n+1}} + \psi(z)\log r, \tag{4.2}$$

if ∂D is in its normal form at $p = 0$, then any Taylor coefficient at 0 of ϕ of order $\leq n$, and that of ψ of any order is a universal polynomial in the normal coefficients $\{A^l_{\alpha\overline{\beta}}\}$. (See Boutet-Sjostrand [BS75] and a related argument in [Fe79].) In particular, we have the following

Proposition 4.1 ([Ch81], [Gr85]). *Let D be as above and suppose that ∂D is in the Moser normal form up to sufficiently high order. Let r be a Fefferman defining function, and let φ, ψ be as in (4.2). Then $\phi = \frac{n!}{\pi^n} + O(r^2)$. Write $P_2 = \frac{\phi - \frac{n!}{\pi^n}}{r^2}|_{\partial\Omega}$. If $n = 2, P_2 = 0$. If $n \geq 3$, $P_2 = c_n\|A^0_{2\overline{2}}\|^2$ for some universal constant $c_n \neq 0$.*

Proof of Theorem 2.1. For any $p \in \partial M$, let D and $B(p, \delta)$ be the sets as chosen in lemma 3.1. Let r_F be the Fefferman defining for D function as chosen in lemma 3.6. By Fefferman's Bergman asymptotic expansion on D, we have

$$k_D(z, z) = \frac{\phi}{r_F^{n+1}} + \psi \log r_F, \tag{4.3}$$

where $\phi, \psi \in C^\infty(\overline{D})$ and $\phi|_{\partial D} \neq 0$. On the other hand, by lemma 3.1,

$$k_M(z, \overline{z}) = k_D(z, \overline{z}) + \varphi(z), z \in B(p, \delta) \cap D$$

where $\varphi \in C^\infty(B(p, \delta) \cap \overline{D})$. Thus,

$$k_M r_F^{n+1} = \phi + \psi r_F^{n+1} \log r_F + \varphi r_F^{n+1} \text{ on } B(p, \delta) \cap D. \tag{4.4}$$

Substituting (3.30) to (4.4) we have

$$\frac{n!}{\pi^n} = \phi + \psi r_F^{n+1} \log r_F + \varphi r_F^{n+1} \text{ on } D \cap B(p, \sigma). \tag{4.5}$$

By [FW97, Lemma 2.2], we have

$$\phi - \varphi r_F^{n+1} - \frac{n!}{\pi^n} = O(r_F^k), \psi = O(r_F^k) \text{ on } D \cap B(p, \sigma), \forall k > 0. \tag{4.6}$$

Thus,

$$\phi - \frac{n!}{\pi^n} = O(r_F^{n+1}) \text{ on } D \cap B(p, \sigma). \tag{4.7}$$

When $n = 2$, $\psi = O(r_F^k)$ on $D \cap B(p, \sigma), \forall k > 0$ implies that $\partial D \cap B(p, \sigma)$ is spherical by a result of Burns-Graham [Gr85, pp.129] (also see [BdM90, pp.23]). When $n \geq 3$, it follows from (4.7) that $P_2 = 0$ on $\partial D \cap B(p, \sigma)$. By Proposition 4.1, $A^0_{2\overline{2}} = 0$ at $q \in D \cap B(p, \sigma)$ if ∂D is in the Moser normal form up to sufficiently high order at q. By a classical result of Chern-Moser, $\partial D \cap B(p, \sigma)$ is spherical. Thus, we get the conclusion of Theorem 2.1. □

Theorem 1.1 is a direct corollary of Theorem 2.1. Huang [H06] proved that a Stein space with possible isolated normal singularities and with a compact strongly pseudoconvex and algebraic boundary is biholomorphic to a ball quotient. Then a direct corollary of Theorem 1.1 and [H06, Theorem 3.1] is the following

Corollary 4.2. *Let Ω be a Stein space with isolated normal singularities and a compact smoothly strongly pseudoconvex boundary $\partial\Omega$. Assume the $\partial\Omega$ is CR equivalent to an algebraic CR manifold in a complex Euclidean space. If the Bergman metric ω_Ω^B on $\mathrm{Reg}(\Omega)$ is Kahler-Einstein then Ω is biholomorphic to a ball quotient $\mathbb{B}^n/\Gamma$ where $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ is finite subgroup with $0 \in \mathbb{B}^n$ the only fixed point of any non-identity element of Γ.*

5. Bergman metric on a ball quotient

Let $\Omega := \mathbb{B}^n/\Gamma$ where Γ is a finite subgroup of $\mathrm{Aut}(\mathbb{B}^n)$ with 0 as the unique fixed point for each non-identity element. Then Ω is a Stein space with only an isolated singularity. Let $\pi : \mathbb{B}^n \to \mathbb{B}^n/\Gamma$ be the standard branched covering map. Write $p = \pi(0)$. Let ω^B be the Bergman metric on Ω. Let $A^2(\Omega)$ be the L^2-integrable holomorphic $(n, 0)$-forms on $\mathrm{Reg}(\Omega)$. Let $\{\alpha_j\}_{j=1}^\infty$ be an orthnormal basis of $A^2(\Omega)$. Locally, write $\alpha_j = a_j dw, j \geq 1$ and $k_\Omega(w, \overline{w}) = \sum_{j=1}^\infty |a_j|^2$. Then $\omega_\Omega^B = i\partial\overline{\partial} \log k_\Omega(w, \overline{w})$. Write $\pi^*\alpha_j = f_j dz$ where $dz = dz_1 \wedge \cdots \wedge dz_n$ and $\{f_j\}$ are holomorphic functions on $\mathbb{B}^n \setminus \{0\}$. By the Hartogs extension theorem, $\{f_j\}$ can be holomorphically extended to $\mathbb{B}^n$. Moreover, f_j satisfies

$$f_j \circ \gamma(z) \det \gamma = f_j(z), \forall \gamma \in \Gamma, \forall z \in \mathbb{B}^n.$$

Set $A^2_\Gamma(\mathbb{B}^n) = \{f \in A^2(\mathbb{B}^n) : f \circ \gamma \det \gamma = f, \forall \gamma \in \Gamma\}$. Then $A^2_\Gamma(\mathbb{B}^n)$ is a closed subspace of $A^2(\mathbb{B}^n)$. Let $P_\Gamma : L^2(\mathbb{B}^n) \to A^2_\Gamma(\mathbb{B}^n)$ be the orthogonal projection. Let $\{f_j\}_{j=1}^\infty$ be an orthonormal basis of $A^2_\Gamma(\mathbb{B}^n)$. Write

$$K_\Gamma(z, \overline{w}) = \sum_{j=1}^{\infty} f_j(z)\overline{f}_j(w), z, w \in \mathbb{B}^n.$$

$K_\Gamma(z, \overline{w})$ is then the Schwarz kernel of P_Γ. That is,

$$P_\Gamma f = \int_{\mathbb{B}^n} K_\Gamma(z, \overline{w}) f(w) dv$$

where dv is the Lebesgue measure on $\mathbb{C}^n$. Define

$$Q_\Gamma f = \int_{\mathbb{B}^n} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} K(\gamma z, \overline{w}) \det \gamma f(w) dv, \forall f \in L^2(\mathbb{B}^n)$$

where $K(z, \overline{w})$ is the Bergman kernel function of the $\mathbb{B}^n$. Then $K(z, \overline{w}) = \frac{n!}{\pi^n} \frac{1}{(1 - z \cdot \overline{w})^{n+1}}$ and $z \cdot \overline{w} = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n$. Then $Q_\Gamma f \in A^2_\Gamma(\mathbb{B}^n)$ for all $f \in L^2(\mathbb{B}^n)$. Moreover,

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} K(\gamma z, \overline{\tau w}) \det \gamma \det \overline{\tau} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} K(\gamma z, \overline{w}) \det \gamma, \forall \tau \in \Gamma. \tag{5.1}$$

In fact, $\overline{\tau^t} = \tau^{-1} \in \Gamma, \forall \tau \in \Gamma$ where τ^t is the transpose matrix of τ, then

$$\begin{aligned}
\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} K(\gamma z, \overline{\tau w}) \det \gamma \det \overline{\tau} &= \frac{c_n}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{1}{(1 - z^t \gamma^t \cdot \overline{\tau w})^{n+1}} \det \gamma \det \overline{\tau} \\
&= \frac{c_n}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{1}{(1 - z^t (\overline{\tau}^t \gamma)^t \overline{w})^{n+1}} \det(\overline{\tau}^t \gamma) \\
&= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} K(\gamma z, \overline{w}) \det \gamma.
\end{aligned}$$

Here, $c_n = \frac{n!}{\pi^n}$.

Lemma 5.1.

$$Q_\Gamma = P_\Gamma \text{ on } L^2(\mathbb{B}^n); \quad K_\Gamma(z,\overline{w}) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} K(\gamma z,\overline{w})\det\gamma. \tag{5.2}$$

Proof. For all $f \in L^2(\mathbb{B}^n)$, write $f = f_1 + f_2$ where $f_1 = P_\Gamma f$ and $f_1 \perp f_2$ and $f_2 \perp A^2_\Gamma(\mathbb{B}^n)$. By (5.1), one has

(5.3)

$$\begin{aligned} Q_\Gamma f &= \frac{1}{|\Gamma|}\int_{\mathbb{B}^n}\sum_{\gamma\in\Gamma} K(\gamma z,\overline{w})\det\gamma f_1(w)dv + \frac{1}{|\Gamma|}\int_{\mathbb{B}^n}\sum_{\gamma\in\Gamma} K(\gamma z,\overline{w})\det\gamma f_2(w)dv \\ &= \frac{1}{|\Gamma|}\int_{\mathbb{B}^n}\sum_{\gamma\in\Gamma} K(\gamma z,\overline{w})\det\gamma f_1(w)dv \\ &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\det\gamma f_1(\gamma z) = f_1(z) \\ &= P_\Gamma f. \end{aligned}$$

As a consequence, Q_Γ and P_Γ have the same Schwarz kernel. Thus, we get the conclusion of the second part of the lemma. □

Write $\omega_\Gamma = i\partial\overline{\partial}\log K_\Gamma(z,\overline{z})$. Then we have the following

Lemma 5.2.

$$\pi^*\omega^B_\Omega = \omega_\Gamma. \tag{5.4}$$

Moreover, ω^B_Ω is Kähler-Einstein if and only if ω_Γ is Kähler-Einstein on $\mathbb{B}^n \setminus \{0\}$.

Proof. Let $\{\alpha_j\}$ be an orthnormal basis of $A^2(\Omega)$. Write $\alpha_j = a_j dw$ and $\pi^*\alpha_j = f_j dz$ on $\mathbb{B}^n \setminus \{0\}$. Here $w = (w_1, \cdots, w_n)$ are local coordinates on $\mathrm{Reg}(\Omega)$ and $dw = dw_1 \wedge \cdots \wedge dw_n$. We have $a_j \circ \pi \det \pi' = f_j$. Since $\Gamma \subset$ Aut $(\mathbb{B}^n)$, then $|\det \pi'|^2 = 1$. Thus,

$$|a_j \circ \pi|^2 = |f_j|^2, \forall j. \tag{5.5}$$

(5.6)

$$\frac{1}{i^{n^2}}\int_{\mathbb{B}^n} f_j\overline{f_k}dz\wedge d\overline{z} = \frac{1}{i^{n^2}}\int_{\mathbb{B}^n}\pi^*\alpha_j\wedge\overline{\pi^*\alpha_k} = \frac{1}{i^{n^2}}|\Gamma|\int_\Omega \alpha_j\wedge\overline{\alpha_k} = |\Gamma|\delta_{jk}.$$

For any $f \in A^2_\Gamma(\mathbb{B}^n)$, there exist an $\alpha \in A^2(\Omega)$ such that $\pi^*\alpha = f(z)dz$. Thus, $\{\frac{1}{\sqrt{|\Gamma|}} f_j\}$ is an orthonormal basis of $A^2_\Gamma(\mathbb{B}^n)$. Then combine with (5.5)

$$K_\Gamma(z,\overline{z}) = \frac{1}{|\Gamma|}\sum_{j=1}^\infty |f_j(z)|^2 = \frac{1}{|\Gamma|}|a_j \circ \pi|^2 = \frac{1}{|\Gamma|}\pi^* k_\Omega. \tag{5.7}$$

By taking the $\partial\overline{\partial}\log$ on both sides of the above equation we get the conclusion of the lemma. □

Assume that ω^B_Ω is Kähler-Einstein. Then ω_Γ is Kahler-Einstein on $\mathbb{B}^n \setminus \{0\}$. The Bergman kernel on $\mathbb{B}^n$ is denoted by $K(z,\overline{z})$. Then

$$K(z,\overline{z}) = \frac{n!}{\pi^n}\frac{1}{(1-|z|^2)^{n+1}}.$$

By Lemma 5.1

$$\begin{aligned} K_\Gamma(z,\overline{z}) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} K(\gamma z,\overline{z})\det\gamma = \frac{1}{|\Gamma|}\frac{n!}{\pi^n}\sum_{\gamma\in\Gamma}\frac{1}{(1-\gamma z\cdot\overline{z})^{n+1}}\det\gamma \\ &= \frac{n!}{\pi^n}\frac{1}{|\Gamma|}\left[\frac{1}{(1-|z|^2)^{n+1}} + \Psi(z)\right], \end{aligned} \tag{5.8}$$

where $\Psi = \sum_{\gamma\neq id}\frac{1}{(1-\gamma z\cdot\overline{z})^{n+1}}\det\gamma$. Since $1-\gamma z\cdot\overline{z}\neq 0, \forall z\in\partial B^n$ when $\gamma\neq id$, it follows that $\Psi(z)\in C^\infty(\overline{\mathbb{B}^n})$. Then

$$\omega_\Gamma = i\partial\overline{\partial}\log K_\Gamma = i\partial\overline{\partial}\log\frac{1}{(1-|z|^2)^{n+1}} + i\partial\overline{\partial}\log(1+\tilde{\Psi}) \tag{5.9}$$

where $\tilde{\Psi} = \Psi(z)(1-|z|^2)^{n+1}$. Write $\omega_\Gamma = i\sum_{i,j=1}^n g_{i\overline{j}} dz_i\wedge d\overline{z}_j$. By direct calculation,

$$g_{i\overline{j}} = (n+1)\left\{\frac{\delta_{ij}}{1-|z|^2} + \frac{\overline{z}_i z_j}{(1-|z|^2)^2}\right\} + O((1-|z|^2)^{n-1}). \tag{5.10}$$

Here, $O(f)$ indicates that there exist a constant $C>0$ such that the term can be bounded by $C|f|$ near ∂B^n. Then

$$\begin{aligned} \det g_{i\overline{j}} &= (n+1)^n\frac{1}{(1-|z|^2)^{n+1}} + O((1-|z|^2)^{-n+1}) \\ &= (n+1)^n\frac{1}{(1-|z|^2)^{n+1}}[1+O((1-|z|^2)^2)]. \end{aligned} \tag{5.11}$$

Then the Ricci curvature with respect to ω_Γ is given by

(5.12)
$$\begin{aligned}\Theta_\Gamma &= i\overline{\partial}\partial \log \det g_{i\bar{j}} = -(n+1)i\overline{\partial}\partial \log(1-|z|^2) + \overline{\partial}\partial[O((1-|z|^2)^2)]\\ &= -(n+1)i\overline{\partial}\partial \log(1-|z|^2) + O(1).\end{aligned}$$

Since ω_Γ is Kahler-Einstein on $\mathbb{B}^n \setminus \{0\}$, then $\Theta_\Gamma = c_0\omega_\Gamma$ where c_0 is a constant. From (5.9) and (5.12) we have

$$-(n+1)\overline{\partial}\partial \log(1-|z|^2) + O(1) = c_0[-(n+1)\partial\overline{\partial}\log(1-|z|^2) + \partial\overline{\partial}\log(1+\tilde{\Psi})]. \tag{5.13}$$

Letting $z \to \partial\mathbb{B}^n$, we have $c_0 = -1$.

Theorem 5.3. *Set $u = \log K_\Gamma$. Then the Bergman metric on* Reg(Ω) *is Kähler-Einstein with $n \geq 2$ if u satisfies the following complex Monge-Ampere equation*

$$\det(u_{i\bar{j}}) = ce^u \ \text{on}\ \mathbb{B}^n \setminus \{0\},\ u|_{\partial\mathbb{B}_n} = \infty. \tag{5.14}$$

where $c = \frac{(n+1)^n\pi^n|\Gamma|}{n!}$. Conversely, if u satisfies (5.14), then the Bergman metric on Reg(Ω) *is Kähler-Einstein.*

Proof. We only need to prove the necessary part. The proof is similar to that for $B_M = const$. From $\Theta_\Gamma = -\omega_\Gamma$, we have that $\log(\det u_{i\bar{j}}) - u$ is a pluriharmonic function on $\mathbb{B}^n \setminus \{0\}$. Write $v = \log(\det u_{i\bar{j}}) - u$. Since $n \geq 2$, then v can be smoothly extended to $\mathbb{B}^n$ which is still denoted by v. Then v is a pluriharmonic function on $\mathbb{B}^n$. Thus, $u = \log K_\Gamma$ satisfies the following

$$\det u_{i\bar{j}} = e^v e^u. \tag{5.15}$$

Substituting (5.11) to (5.15) we have

(5.16)
$$\frac{(n+1)^n}{(1-|z|^2)^{n+1}}[1+O((1-|z|^2)^2)] = e^v\frac{n!}{\pi^n}\frac{1}{|\Gamma|}\left[\frac{1}{(1-|z|^2)^{n+1}} + \Psi(z)\right].$$

Letting $z \to \partial\mathbb{B}^n$, we have

$$e^v \to \frac{(n+1)^n\pi^n|\Gamma|}{n!}.$$

Since v is pluriharmonic on $\mathbb{B}^n$, then

$$e^v \equiv \frac{(n+1)^n \pi^n |\Gamma|}{n!}, \forall z \in \mathbb{B}^n.$$

Thus, $u = \log K_\Gamma$ satisfies the following Monge-Ampere equation

$$\det u_{i\bar{j}} = ce^u, \tag{5.17}$$

where $c = \frac{(n+1)^n \pi^n |\Gamma|}{n!}$. □

We notice that if $u = \log K_\Gamma$ satisfies (5.17) with $K_\Gamma(0,0) \neq 0$, then by continuity ω_Γ is a well-defined complete Kahler-Einstein metric over $\mathbb{B}^n$. Hence, by the uniqueness of the Cheng-Yau metric [CY80], ω_Γ is a hyperbolic metric and thus by the uniformization theorem, we see that $\Gamma = \{\mathrm{id}\}$ and thus Ω is biholomorphic to the ball. Namely, we have the following:

Corollary 5.4. *Let $\Gamma \subset Aut_0(\mathbb{B}^n)$ with $n \geq 2$ be a non-trivial finite subgroup with 0 as the only fixed point for each non-identity element of Γ. Let K_Γ be the function defined in (5.1). If $K_\Gamma(0,0) \neq 0$, then the Bergman metric of $\mathrm{Reg}(\mathbb{B}^n/\Gamma)$ is not Kähler-Einstein.*

Example 5.5. Suppose $\Omega = \mathbb{B}^3/\Gamma$, where $\Gamma = \{\gamma_1, \gamma_2\}$ and $\gamma_1 = id, \gamma_2 = \mathrm{diag}(-1,-1,-1)$.

$$K_\Gamma = \frac{3}{\pi^3}\left[\frac{1}{(1-|z|^2)^4} - \frac{1}{(1+|z|^2)^4}\right] = \frac{4!}{\pi^3}\frac{|z|^2(1+|z|^4)}{(1-|z|^4)^4}. \tag{5.18}$$

Thus,

$$K_\Gamma(0,0) = 0.$$

Set $u = \log K_\Gamma$. Then

$$\begin{aligned} u &= \log\frac{4!}{\pi^3} + \log|z|^2 + \log(1+|z|^4) - \log(1-|z|^4)^4 \\ &= \log\frac{4!}{\pi^3} + \log|z|^2 + 5|z|^4 + O(|z|^8). \end{aligned} \tag{5.19}$$

By direct calculation,

(5.20)

$$u_{1\bar{1}} = \frac{|z_2|^2}{|z|^4} + 10|z|^2 + 10|z_1|^2 + O(|z|^6),\ u_{1\bar{2}} = -\frac{1}{|z|^4}\bar{z}_1 z_2 + 10\bar{z}_1 z_2 + O(|z|^6)$$

$$u_{2\bar{1}} = -\frac{1}{|z|^4} z_1\bar{z}_2 + 10 z_1\bar{z}_2 + O(|z|^6),\ u_{2\bar{2}} = \frac{|z_1|^2}{|z|^4} + 10|z|^2 + 10|z_2|^2 + O(|z|^6).$$

Then $\det u_{i\bar{j}}(0) = 20$, but $K_\Gamma(0,0) = 0$. Thus, it follows that $u = \log K_\Gamma$ does not satisfy the Monge-Ampere equation (5.14). Hence, the Bergman metric on Ω is not Kähler-Einstein.

When $n = 1$ and for any finite subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{B}^1)$, assume $|\Gamma| = r, 1 \leq r < \infty$. It is well known that $\Gamma = \{1, e^{2\pi i \frac{1}{r}}, \cdots, e^{2\pi i \frac{r-1}{r}}\}$. Thus, on $\mathbb{B}^1$

(5.21)
$$K_\Gamma(z, \bar{z}) = \frac{1}{\pi|\Gamma|} \sum_{\gamma \in \Gamma} \frac{1}{(1 - \gamma z \cdot \bar{z})^2} \det \gamma = \frac{1}{\pi r} \sum_{j=1}^{r} \frac{1}{[1 - e^{2\pi i \frac{j}{r}} |z|^2]^2} e^{2\pi i \frac{j}{r}}.$$

By Taylor's expansion,

(5.22)
$$K_\Gamma = \frac{1}{\pi} \sum_{j=1}^{r} \sum_{k=0}^{\infty} (k+1) e^{2\pi i \frac{j}{r}(k+1)} |z|^{2k} = \frac{r}{\pi} \sum_{k=1}^{\infty} k |z|^{2(kr-1)} = \frac{r}{\pi} \frac{|z|^{2(r-1)}}{(1 - |z|^{2r})^2}.$$

Set $u = \log K_\Gamma$. Then $u_{1\bar{1}} = 2r^2 \frac{|z|^{2(r-1)}}{(1-|z|^{2r})^2}$. Since $c = 2\pi r$, then one sees immediately that

$$u_{1\bar{1}} = ce^u \text{ on } \mathbb{B}^1 \setminus \{0\}. \tag{5.23}$$

Notice that the sufficient part of Theorem 5.3 holds even for $n = 1$. We have the following:

Proposition 5.6. *For any finite subgroup* $\Gamma \subset \mathrm{Aut}_0(\mathbb{B}^1)$, *its Bergman metric on* $\mathrm{Reg}(\mathbb{B}^1/\Gamma)$ *is Kähler-Einstein.*

We finish off this paper by recalling the following generalized Cheng conjecture formulated in [HX20]:

Conjecture 5.7. *Let* Ω *be a normal Stein space with a compact spherical boundary of complex dimension* $n \geq 2$. *If the Bergman metric over* $\mathrm{Reg}(\Omega)$ *is Kähler-Einstein, then* Ω *is biholomorphic to* $\mathbb{B}^n$.

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