

Euler class of taut foliations and Dehn filling

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In this article, we study the Euler class of taut foliations on the Dehn fillings of a \mathbb{Q} -homology solid torus. We give a necessary and sufficient condition for the Euler class of a foliation transverse to the core of the filling solid torus to vanish. We apply this condition to taut foliations on Dehn fillings of hyperbolic fibered manifolds and obtain many new left-orderable Dehn filling slopes on these manifolds. For instance, we show that when X is the exterior of the pretzel knot $P(-2, 3, 2r + 1)$, for $r \geq 3$, $\pi_1(X(\alpha_n))$ is left-orderable for a sequence of positive slopes α_n with $\alpha_0 = 2g - 2$ and $\alpha_n \rightarrow 2g - 1$. Lastly, we prove that given any \mathbb{Q} -homology solid torus, the set of slopes for which the corresponding Dehn fillings admit a taut foliation transverse to the core with zero Euler class is nowhere dense in $\mathbb{R} \cup \{\frac{1}{0}\}$.

1	Introduction	1750
2	The Euler class of tangent plane fields over 3-manifolds	1756
3	Necessary and sufficient conditions for the Euler class to vanish	1758
4	Dehn fillings on fibered manifolds and left-orders	1764
5	Euler class and restrictions of filling slopes	1771
	Acknowledgment	1778
	References	1778

1. Introduction

A co-dimension one foliation on a compact, connected, oriented 3-manifold M is a decomposition of M into a disjoint union of injectively immersed surfaces. These surfaces are called the leaves of the foliation. Throughout the article, we assume that foliations are $C^{\infty,0}$. So all leaves are smoothly immersed, while the transverse structure may only be C^0 . By [Cal01], any topological co-dimension one foliation can be isotoped to be $C^{\infty,0}$. We also assume that foliations are co-orientable (or transversely orientable). Since M is orientable, this assumption is equivalent to requiring that the tangent plane field of the foliation is orientable.

A closed transversal in a foliated 3-manifold is a smooth closed loop that is everywhere transverse to the leaves of the foliation. We call a co-dimension one foliation *taut* if every leaf of the foliation intersects a closed transversal. See [CKR19] for other notions of tautness for $C^{\infty,0}$ foliations.

Conjecture 1.1 (The L-space Conjecture; Conjecture 1 in [BGW13], Conjecture 5 in [Juh15]). Let M be an irreducible \mathbb{Q} -homology sphere. The following statements are equivalent:

- (1) M admits a co-orientable taut foliation.
- (2) $\pi_1(M)$ is left-orderable.
- (3) M is not an L -space.

A nontrivial group G is called *left-orderable* (LO), if there exists a strict total order $<$ on G such that given any elements a, b and c in G , we have $a < b$ if and only if $ca < cb$. And an L -space is a \mathbb{Q} -homology sphere whose Heegaard Floer homology is of the minimal complexity [OS04c, OS04b, OS05, Juh15].

The conjecture has been confirmed for all graph manifolds [BC17, HRRW15, Ras17]. It is also known that (1) \Rightarrow (3) [OS04a, Bow16, KR17]. One motivation of our work is to study the equivalence between (1) and (2).

Given a co-oriented taut foliation $\widehat{\mathcal{F}}$ on an oriented 3-manifold M , the Euler class of the tangent plane field $T\widehat{\mathcal{F}}$, denoted by $e(\widehat{\mathcal{F}})$, is a cohomology class in $H^2(M, \mathbb{Z})$ and the class is zero if and only if the plane field $T\widehat{\mathcal{F}}$ is isomorphic to the product bundle $M \times \mathbb{R}^2 \rightarrow M$ (see §2). Using Thurston's universal circle action associated with taut foliations (see [CD03]), we have the following theorem whose proof can be found in [BH19, §7].

Theorem 1.2 (Thurston; Calegari-Dunfield, Boyer-Hu). *Let M be a \mathbb{Q} -homology 3-sphere. Suppose that M admits a co-oriented taut foliation whose Euler class is zero. Then $\pi_1(M)$ is left-orderable.*

Theorem 1.2 leads us to the natural question: when is the Euler class of a given taut foliation on a \mathbb{Q} -homology sphere zero?

1.1. When is the Euler class zero?

We first note that the Euler class of any oriented tangent plane field over a connected compact 3-manifold is an “even” class (see Proposition 2.1). As a result, if $H^2(M)$ is a direct sum of \mathbb{Z}_2 , then the Euler class of any taut foliation on M must be zero.

Corollary 1.3. *Let M be an oriented 3-manifold satisfying that $H^2(M)$ is isomorphic to a (possibly trivial) direct sum of \mathbb{Z}_2 . If there exists a co-orientable taut foliation on M , then $\pi_1(M)$ is left-orderable.*

In this article, we view \mathbb{Q} -homology spheres as Dehn fillings on \mathbb{Q} -homology solid tori. See §3.1 for the convention of our notations. For simplicity, we will only state the result for Dehn fillings of \mathbb{Z} -homology solid tori below. We refer the readers to Theorem 3.4 and Remark 3.5 for more general statements.

Theorem 1.4. *Let X be a \mathbb{Z} -homology solid torus and F be a properly embedded surface in X representing a generator of $H_2(X, \partial X)$. Suppose that $\widehat{\mathcal{F}}$ is an oriented co-dimension one foliation on $X(p/q)$, $p > 0$ whose restriction to the filling solid torus N is the foliation by meridian disks, and the orientation of the leaves of $\widehat{\mathcal{F}}$ agree with the given orientation of the meridian disk of N . Let $\mathcal{F} = \widehat{\mathcal{F}}|_X$ and σ denote a nowhere vanishing outward pointing section of $T\mathcal{F}$ along ∂X . Then the Euler class $e(T\widehat{\mathcal{F}}) = 0$ in $H^2(X(p/q))$ if and only if $aq \equiv 1 \pmod{p}$, where $e_\sigma(\mathcal{F}) \in H^2(X, \partial X)$ is the relative Euler class of \mathcal{F} associated with the section σ and $a = e_\sigma(\mathcal{F})([F])$.*

The condition that the restriction of $\widehat{\mathcal{F}}$ to the filling solid torus N is the foliation by meridian disks is equivalent to that the core of N is transverse to $\widehat{\mathcal{F}}$ by shrinking N if necessary. Throughout the article, we always assume that $\widehat{\mathcal{F}}|_N$ is the product foliation if $\widehat{\mathcal{F}}$ on $X(\alpha)$ is transverse to the core of N . The existence of taut foliations satisfying this transverseness condition in $X(p/q)$ is also equivalent to that the slope p/q is *strongly CTF-detected* as defined in [BC17, Definition 6.5].

Almost all known taut foliations on Dehn filled manifolds are transverse to the core of the filling solid torus, and hence Theorem 1.4 applies. These include taut foliations constructed in [Rob95, Rob01a, Rob01b, LR14, Kri20, DR19a, DR19b], among which the relative Euler classes $e_\sigma(\mathcal{F})$ in [Rob95, Rob01a, Rob01b, LR14, Kri20] are of Thurston norm one.

1.2. Dehn fillings of fibered manifolds and left-orders

In [Rob01a, Rob01b, Kri20], families of taut foliations are constructed on manifolds obtained from Dehn fillings of the exteriors of fibered knots. In the theorem below, the *canonical meridian* μ_0 and the *degeneracy slope* are defined in §4.1. When X is the exterior of a fibered knot in S^3 , μ_0 is mostly the meridian of the knot [Rob01b, Proposition 7.4]. In particular, this is the case when K is the closure of a positive braid considered in Theorem 1.6.

Theorem 1.5 (Theorem 4.7 in [Rob01b]). *Let X be the exterior of a hyperbolic fibered knot in a closed 3-manifold M with degeneracy slope γ and μ_0 be the canonical meridian of X .*

- (1) *If $\gamma = \mu_0$, then there exists a taut foliation on $X(\alpha)$ transverse to the core of the filling solid torus for any slope $\alpha \neq \mu_0$.*
- (2) *If γ is a positive slope with respect to μ_0 , then there exists a taut foliation on $X(\alpha)$ transverse to the core of the filling solid torus for any $\alpha \in (-\infty, 1)$.*
- (3) *If γ is a negative slope with respect to μ_0 , then there exists a taut foliation on $X(\alpha)$ transverse to the core of the filling solid torus for any $\alpha \in (-1, \infty)$.*

Theorem 1.6 (Theorem 1.2 in [Kri20]). *Let X be the exterior of the closure of a positive 3-braid in S^3 . There exists a taut foliation on $X(\alpha)$ transverse to the core of the filling solid torus for every slope $\alpha < 2g - 1$, where g is the genus of the closed braid.*

Taut foliations in the theorems above are carried by very nice branched surfaces (§4.2), which allows us to compute their Euler classes precisely.

Theorem 1.7. *Suppose that $\widehat{\mathcal{F}}$ is a taut foliation given by Theorem 1.5 or Theorem 1.6. Let $\alpha = p\mu_0 + q\lambda$, where $p > 0$ and μ_0 be the canonical meridian. Then $e(\widehat{\mathcal{F}}) = 0$ in $H^2(X(\alpha))$ if and only if $e(\mathcal{F}) = 0$ in $H^2(X)$*

and

$$(1.1) \quad (2g - 1)|q| \equiv 1 \pmod{p},$$

where g is the genus of the knot and $\mathcal{F} = \widehat{\mathcal{F}}|_X$ is the restriction of $\widehat{\mathcal{F}}$ to the knot exterior X .

For each $g > 0$, we denote the set of slopes that satisfies Equation (1.1) in Theorem 1.7 by

$$\mathcal{L}_g = \{p/q \in \mathbb{Q} \setminus \{0\} : (2g - 1)|q| \equiv 1 \pmod{p}\}.$$

We collect some key properties of the set \mathcal{L}_g below. See §4.4 for more details.

- (1) The set \mathcal{L}_1 contains all nonzero integer slopes. In fact, a slope is in \mathcal{L}_1 if and only if it is a nonzero integer slope with respect to some meridian of X . (Lemma 4.4).
- (2) When $g > 1$, the set \mathcal{L}_g is bounded between $\pm(2g - 1)$. There are two sequences of slopes $\pm\{\alpha_n\}$ in \mathcal{L}_g satisfying $\alpha_0 = 2g - 2$ and $\alpha_n \rightarrow (2g - 1)$ as $n \rightarrow \infty$ (Lemma 4.5).

From Theorem 1.2, we obtain the following results on the left-orderability of 3-manifold groups.

Corollary 1.8. *Let K be a hyperbolic genus one fibered knot in a \mathbb{Q} -homology sphere with the degeneracy slope γ . With respect to the canonical meridian μ_0 , we have*

- (1) *If $\gamma = \mu_0$, then given any $\alpha \in \mathcal{L}_1$, there exists a taut foliation on $X(\alpha)$ of zero Euler class and $\pi_1(X(\alpha))$ is LO. In particular, $\pi_1(X(\alpha))$ is LO for all integer slopes.*
- (2) *If γ is a positive slope, then given any $\alpha \in \mathcal{L}_1 \cap (-\infty, 1)$, there exists a taut foliation on $X(\alpha)$ of zero Euler class and $\pi_1(X(\alpha))$ is LO. In particular, $\pi_1(X(\alpha))$ is LO for all non-positive integer slopes.*
- (3) *If γ is a negative slope, then given any $\alpha \in \mathcal{L}_1 \cap (-1, \infty)$, there exists a taut foliation on $X(\alpha)$ of zero Euler class and $\pi_1(X(\alpha))$ is LO. In particular, $\pi_1(X(\alpha))$ is LO for all non-negative integer slopes.*

Remark 1.9 (Group left-orderability and genus one fibered knots).

Roberts and Shareshian have shown that $\pi_1(X(\alpha))$ is not LO when $\alpha \in [1, \infty)$ in Case (2) [RS10, Theorem 1.3], and therefore the analogous result

holds in Case (3) (see Remark 4.1). Though the L-space Conjecture predicts that $\pi_1(X(\alpha))$ is LO whenever $\alpha \in (-\infty, 1)$ in Case (2) and $\alpha \in (-1, \infty)$ in Case (3), this was previously unknown even for integer slopes.

In Case (1), Fenley first showed that $\pi_1(X(\alpha))$ is LO for any $\alpha \in \mathbb{Z}$ by showing the existence of \mathbb{R} -covered Anosov flows on these manifolds ([Fen94, Theorem D]; also see [IN20, Proposition 3.3]). Recently, Zung has significantly generalized Fenley's approach to pseudo-Anosov flows and non- \mathbb{R} -covered taut foliations. It follows from [Zun20, Theorem 1] that $\pi_1(X(\alpha))$ is LO for any slope $\alpha \neq \mu_0$ when $\gamma = \mu_0$.

Corollary 1.10. *Let X be the exterior of the closure of a positive 3-braid in S^3 of genus $g > 1$. For any slope $\alpha \in \mathcal{L}_g \cap (-\infty, 2g - 1)$, there exists a taut foliation on $X(\alpha)$ of zero Euler class and hence $\pi_1(X(\alpha))$ is LO. In particular, there is a monotone increasing sequence of positive slopes $\{\alpha_n\}$ with $\alpha_0 = 2g - 2$ and $\alpha_n \rightarrow 2g - 1$ as $n \rightarrow \infty$, satisfying $\pi_1(X(\alpha_n))$ is LO for any $n \in \mathbb{N}$.*

Remark 1.11 (Group left-orderability and L-space knots). A positive L-space knot is a knot in S^3 whose exterior admits a positive L-space Dehn filling [OS05]. It is known that the set of finite L-space filling slopes for a genus g positive L-space knot is precisely $[2g - 1, \infty)$ ([OS05, KMOS07], [RR17, Theorem 1.6]). According to the L-space Conjecture, one expects that given X the exterior of a positive L-space knot, the closed manifold $X(\alpha)$ admits taut foliations and $\pi_1(X(\alpha))$ is LO for any slope $\alpha \in (-\infty, 2g - 1)$.

The closures of positive 3-braids are examples of positive L-space knots, among which the pretzel knots $P(-2, 3, 2r + 1)$, $r \geq 3$ are the most studied and have been recognized as the only hyperbolic positive L-space Montesinos knots [LM16, BM18]. In Theorem 1.6, Krishna constructed a taut foliation on $X(\alpha)$ for any $\alpha \in (-\infty, 2g - 1)$. Moreover, Nie has shown that $\pi_1(X(\alpha))$ is not LO for any $\alpha \in [2g - 1, \infty)$ [Nie19, Theorem 2]. So to confirm the L-space conjecture for manifolds obtained from Dehn surgeries along pretzel knots $P(-2, 3, 2r + 1)$, $r \geq 3$, it remains to show that $\pi_1(X(\alpha))$ is LO for any $\alpha \in (-\infty, 2g - 1)$. However, in general, this was confirmed only for slopes α in an open neighborhood of 0 ([Nie19, Theorem 3]; [CD18, HZ19]), except for the case when $r = 3$.

When X is the exterior of the pretzel knot $P(-2, 3, 7)$, Culler and Dunfield proved that $\pi_1(X(\alpha))$ is LO for any $\alpha \in (-\infty, 6)$ [CD18, Figure 3]. The genus of $P(-2, 3, 2r + 1)$ is $r + 2$, so for $r = 3$, the expected LO filling slope

interval is $(-\infty, 9)$. Corollary 1.10 shows that there is an increasing sequence of slopes α_n with $\alpha_0 = 8$ and $\alpha_n \rightarrow 9$ such that $\pi_1(X(\alpha_n))$ are LO.

1.3. Restrictions on the filling slopes

Given a \mathbb{Q} -homology solid torus X , we define \mathcal{S}_X to be the set of slopes satisfying: given any $\alpha \in \mathcal{S}_X$, there exists a taut foliation on $X(\alpha)$ transverse to the core of the Dehn filling solid torus whose Euler class is zero.

By fixing a meridian μ , we identify the set \mathcal{S}_X with a subset of $\mathbb{Q} \cup \{\frac{1}{0}\}$, denoted by $\mathcal{S}_{X,\mu} \subset \mathbb{Q} \cup \{\frac{1}{0}\}$. For simplicity, we state the result when the longitude of X is null-homologous in $H_1(X)$. See Theorem 5.2 for the case when X is an arbitrary \mathbb{Q} -homology solid torus.

Theorem 1.12. *Let X be the exterior of a null-homologous knot of genus $g > 0$ and μ be any fixed meridian.*

- (1) *Outside the interval $(-2g, 2g)$, the set $\mathcal{S}_{X,\mu}$ contains only μ and the integer slopes. That is,*

$$\mathcal{S}_{X,\mu} \setminus (-2g, 2g) \subseteq \mathbb{Z} \cup \{\mu\}.$$

- (2) *The set $\mathcal{S}_{X,\mu}$ is nowhere dense in $\mathbb{R} \cup \{\frac{1}{0}\} \cong \mathbb{R}P^1$. Particularly, it is nowhere dense in $(-2g, 2g)$.*

When we use a different meridian, a slope $\frac{p}{q}$ becomes $\frac{p}{q+np}$ for some $n \in \mathbb{Z}$. Since Theorem 1.12 holds regardless of the choice of the meridian, one can obtain a stronger version of Theorem 1.12(1) (see Corollary 5.3).

Organization of the paper

In §2, we review the definition of the (relative) Euler class and prove Corollary 1.3. Section 3 contains the core of our computations (Lemma 3.1). Theorem 3.4 is also proven in this section. In §4, we apply the results in §3 to studying the Euler classes of taut foliations on Dehn fillings of fibered manifolds. We prove Theorem 1.7 as well as its corollaries on the left-orderability of 3-manifold groups in this section. Finally, Theorem 1.12 and its generalization (Theorem 5.2) are proved in §5.

2. The Euler class of tangent plane fields over 3-manifolds

2.1. The (relative) Euler class

Let $\xi : E \rightarrow B$ be an oriented 2-plane vector bundle over an oriented CW complex B . A 2-plane bundle is called trivial if it is isomorphic to the product bundle $B \times \mathbb{R}^2 \rightarrow B$. It is easy to see that an oriented 2-plane bundle ξ is trivial if and only if there exists a nowhere vanishing section $\sigma : B \rightarrow E$. In fact, given such a section σ , it determines an orientable line bundle γ . Then $\xi \cong \gamma \oplus \xi/\gamma$ is isomorphic to the sum of two orientable line bundles, and hence it is trivial.

We denote the Euler class of an oriented 2-plane bundle $\xi : E \rightarrow B$ by $e(\xi)$. A representative cocycle of the cohomology class $e(\xi)$ can be constructed as follows.

First note that since $\mathbb{R}^2 \setminus \{0\}$ is a $K(\mathbb{Z}, 1)$ space, the only obstruction to the existence of a nowhere vanishing section σ arises when one tries to extend a section over the 1-skeleton $B^{(1)}$ of B to the 2-skeleton $B^{(2)}$. Fixing a nowhere vanishing section $\sigma : B^{(1)} \rightarrow E$ over the 1-skeleton $B^{(1)}$, we construct a cellular 2-cochain $c_\sigma : C_2(B) \rightarrow \mathbb{Z}$ by specifying its value on each 2-cell. Let $\varphi_\alpha : D_\alpha^2 \rightarrow B$ be the characteristic map of a 2-cell e_α and let $E_{D_\alpha^2} \rightarrow D_\alpha^2$ denote the pullback of $E \rightarrow B$ through φ_α . Since D_α^2 is contractible, $E_{D_\alpha^2} \rightarrow D_\alpha^2$ is trivial. We identify $E_{D_\alpha^2}$ with the product $D_\alpha^2 \times \mathbb{R}^2$ by fixing a trivialization. The nowhere vanishing section $\sigma : B^{(1)} \rightarrow E$ determines a nowhere vanishing section of $E_{D_\alpha^2} \rightarrow D_\alpha^2$ along ∂D_α^2 , which we also denote by σ . Since σ is nowhere vanishing, the image of the composition $\partial D_\alpha^2 \xrightarrow{\sigma} E_{D_\alpha^2} \xrightarrow{\cong} D_\alpha^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is contained in $\mathbb{R}^2 \setminus \{0\}$. We set the value of the 2-cochain c_σ on the 2-cell e_α to be the degree of the composite map $\partial D_\alpha^2 \rightarrow \mathbb{R}^2 \setminus \{0\}$. See the diagram below.

$$\begin{array}{ccccc}
 \mathbb{R}^2 \setminus 0 & \longleftarrow & D_\alpha^2 \times \mathbb{R}^2 & \xleftarrow{\cong} & E_{D_\alpha^2} & \longrightarrow & E \\
 & & & \nearrow \sigma & \downarrow & & \downarrow \\
 & & \partial D_\alpha^2 & \longrightarrow & D_\alpha^2 & \xrightarrow{\varphi_\alpha} & B
 \end{array}$$

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The 2-cochain c_σ is a cocycle and the Euler class $e(\xi)$ is defined to be the cohomology class $[c_\sigma]$ in $H^2(B)$. See [MS74, Section 12] or [CC03, Chapter 4] for instance.

Given a subcomplex A of B , suppose that $\xi : E \rightarrow B$ is trivial over A . So there exists a nontrivial section of ξ over A . Then if one starts with a section $\sigma : B^{(1)} \cup A \rightarrow E$, the 2-cocycle c_σ constructed above vanishes over A and hence defines a relative cohomology class $[c_\sigma]$ in $H^2(B, A)$. We call the relative class $[c_\sigma]$ the relative Euler class of ξ associated to the section σ , denoted by $e_\sigma(\xi)$. The relative class vanishes if and only if there exists a section of ξ that extends σ . In general, $e_\sigma(\xi)$ does depend on σ . By construction, the image of $e_\sigma(\xi)$ under the homomorphism $H^2(B, A) \rightarrow H^2(B)$ is the Euler class $e(\xi)$ in $H^2(B)$.

2.2. Euler class and \mathbb{Z}_2 -torsion

Proposition 2.1 below states a general fact of the Euler class of an oriented tangent plane field over an oriented 3-manifold. Note that Corollary 1.3 follows immediately from Proposition 2.1 and Theorem 1.2.

Proposition 2.1. *Let M be an oriented closed 3-manifold and $\xi : E \rightarrow M$ be an oriented tangent plane field of M . Then the Euler class $e(\xi)$ lies in the image of the homomorphism $H^2(M, \mathbb{Z}) \xrightarrow{\cdot 2} H^2(M, \mathbb{Z})$ induced by $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$. In particular, if $H^2(M)$ is isomorphic to a direct sum of \mathbb{Z}_2 , then $e(\xi) = 0$.*

Before we prove the proposition, we briefly review the Stiefel-Whitney classes. We refer the reader to [MS74, §4] for the details.

Let $\xi : E \rightarrow B$ denote a vector bundle over a CW complex B and $w_i(\xi) \in H^i(B, \mathbb{Z}_2)$ denote the i^{th} Stiefel-Whitney class of ξ . If ξ is a trivial bundle, then w_i vanishes for all i . The first Stiefel-Whitney class detects the orientability of ξ . That is, $w_1(\xi) = 0$ if and only if ξ is orientable. Suppose that ξ is an oriented vector bundle of dimension n , then the n^{th} Stiefel-Whitney class $w_n \in H^n(B, \mathbb{Z}_2)$ is the mod 2 reduction of the Euler class $e(\xi)$ in $H^n(B, \mathbb{Z})$. When $n = 2$, if we use the notation introduced in §2.1, this means that $w_2(\xi) = [\bar{c}_\sigma]$ with $\bar{c}_\sigma(e_\alpha) = c_\sigma(e_\alpha) \pmod{2}$ for each 2-cell e_α of B . Lastly, if $\xi \cong \xi_1 \oplus \xi_2$ is the Whitney sum of two vector bundles $\xi_1 : E_1 \rightarrow B$ and $\xi_2 : E_2 \rightarrow B$, then we have $w_i(\xi) = \sum_{k=0}^i w_k(\xi_1)w_{i-k}(\xi_2)$.

It is a well-known fact that the tangent bundle TM of a closed orientable 3-manifold M is trivial. To prove Proposition 2.1, we only need $w_2(TM) = 0$, which can be easily verified using Wu's formula [MS74, §11]. Also see [Gei08, Theorem 4.2.1].

Proof of Proposition 2.1. Since both ξ and TM are orientable, the normal line bundle $\eta = TM/\xi$ is orientable and hence trivial. From the decomposition $TM \cong \eta \oplus \xi$, we have

$$w_2(\xi) = w_2(TM) - w_1(\xi)w_1(\eta) - w_2(\eta) = w_2(TM).$$

As $w_2(TM) = 0$, so is $w_2(\xi)$.

Consider the Bockstein sequence associated to the short exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$ as follows:

$$(2.1) \quad \rightarrow H^2(M, \mathbb{Z}) \xrightarrow{\cdot 2} H^2(M, \mathbb{Z}) \xrightarrow{\text{mod } 2} H^2(M, \mathbb{Z}_2) \rightarrow H^3(M, \mathbb{Z}) \rightarrow .$$

Since the Euler class $e(\xi)$ is sent to $w_2(\xi) = 0$ under $H^2(M, \mathbb{Z}) \xrightarrow{\text{mod } 2} H^2(M, \mathbb{Z}_2)$, we have $e(\xi)$ is in the image of $H^2(M, \mathbb{Z}) \xrightarrow{\cdot 2} H^2(M, \mathbb{Z})$ as claimed.

Suppose that $H^2(M, \mathbb{Z}) \cong \oplus \mathbb{Z}_2$ is a direct sum of \mathbb{Z}_2 . Then the homomorphism $H^2(M, \mathbb{Z}) \xrightarrow{\cdot 2} H^2(M, \mathbb{Z})$ is the zero map. Hence, the Euler class $e(\xi) \in \text{Im}(H^2(M, \mathbb{Z}) \xrightarrow{\cdot 2} H^2(M, \mathbb{Z}))$ must be zero. □

3. Necessary and sufficient conditions for the Euler class to vanish

In this section, we prove Theorem 3.4. It gives necessary and sufficient conditions for the Euler class of a foliation on a Dehn filling of a \mathbb{Q} -homology solid torus X to vanish. Theorem 1.4 is the special case of Theorem 3.4 when X is a \mathbb{Z} -homology solid torus.

3.1. Notation and conventions

Let X be a \mathbb{Q} -homology solid torus. That is, X is a compact orientable 3-manifold with $H_*(X; \mathbb{Q}) \cong H_*(D^2 \times S^1; \mathbb{Q})$. It follows that ∂X consists of a single torus. Let F be an oriented connected incompressible, boundary incompressible surface properly embedded in X such that $[F]$ is a generator of $H_2(X, \partial X) \cong \mathbb{Z}$. We may assume that there are $k \geq 1$ connected components of ∂F on ∂X oriented coherently by the induced orientation from F . The *rational longitude* λ of X , or *longitude* for short, is a connected component of ∂F . Let $\iota : \partial X \rightarrow X$ be the inclusion map. Then λ represents the unique primitive integral class up to sign that is in the kernel of $\iota_* : H_1(\partial X, \mathbb{Q}) \rightarrow H_1(X, \mathbb{Q})$. By definition, k is the order of $\iota_*([\lambda])$ in $H_1(X, \mathbb{Z})$. We refer to k as *the order of λ* .

A *meridian* μ is an oriented simple closed curve on ∂X that intersects λ once. Hence, μ and λ form a basis of $H_1(\partial X)$. We orient μ so that the algebraic intersection number of F and μ equals k . In particular, μ intersects both F and λ positively. In Figure 1, we use the exterior of the unknot U in S^3 with the standard orientation (the right hand rule) to illustrate our choice of the orientations on λ , μ and F .

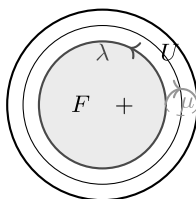


Figure 1.

If X is the exterior of a null-homologous knot in a \mathbb{Q} -homology sphere, then the boundary of a meridional disk of the tubular neighborhood of the knot is a meridional curve, called *the knot meridian*.

A slope α on ∂X is an isotopy class of essential simple closed curves on ∂X . Given μ and λ as above, we write $\alpha = p/q$ in $\mathbb{Q} \cup \{\frac{1}{0}\}$, if $[\alpha] = \pm(p[\mu] + q[\lambda])$ in $H_1(\partial X)$. So $\lambda = 0$ and $\mu = \frac{1}{0}$. If we choose a different meridian μ' of X , then since $[\mu'] = [\mu] + n[\lambda]$ in $H_1(\partial X)$ for some $n \in \mathbb{Z}$, we have α , which equals p/q with respect to μ , now is equal to $p/(q - np)$ with respect to μ' . Given an oriented simple closed curve c on ∂X , to simplify the notation, we often use c to denote both the slope and the homology class in $H_1(\partial X)$ it represents.

We use $X(\alpha)$ to denote the closed 3-manifold obtained by the α -Dehn filling on X . That is, $X(\alpha) = X \cup_f N$ is obtained by attaching a solid torus N to X along ∂X so that the boundary of the meridional disk D of N is identified with α under the gluing map $f : \partial N \rightarrow \partial X$. We orient the disk D so that $f(\partial D) = p\mu + q\lambda$ with $p \geq 0$.

3.2. Cohomology classes in $H^2(X(\alpha))$

In $X(\alpha)$, we identify ∂N with ∂X by the map f . We assume that $\alpha \neq \lambda$, so $X(\alpha)$ is a \mathbb{Q} -homology sphere and we have the following diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \uparrow & & & \\
 0 & \longrightarrow & H^1(\partial X) & \xrightarrow{\delta} & H^2(X(\alpha), \partial X) & \xrightarrow{i^*} & H^2(X(\alpha)) \longrightarrow 0 \\
 & & & \uparrow \cong & & & \\
 & & & H^2(X, \partial X) \oplus H^2(N, \partial N) & & & \\
 & & & \uparrow & & & \\
 & & & 0 & & &
 \end{array}$$

where the vertical isomorphism is from the decomposition

$$(X(\alpha), \partial X) = (X, \partial X) \cup (N, \partial N = f^{-1}(\partial X)).$$

More precisely, given a cohomology class $c \in H^2(X(\alpha), \partial X)$, the corresponding class in $H^2(X, \partial X) \oplus H^2(N, \partial N)$ is given by $(c|_X, c|_N)$, where $c|_X$ and $c|_N$ are restrictions of c to the subspaces X and N respectively. On the other hand, given relative classes c_X and c_N in $H^2(X, \partial X)$ and $H^2(N, \partial N)$, since both of them vanish on the boundaries, we may extend them to the closed manifold $X(\alpha)$ by the zero map. We continue to use c_X and c_N to denote the resulting classes in $H^2(X(\alpha), \partial X)$. Then under the isomorphism $H^2(X, \partial X) \oplus H^2(N, \partial N) \cong H^2(X(\alpha), \partial X)$, the pair (c_X, c_N) is mapped to the sum $c_X + c_N$.

By identifying $H^2(X(\alpha), \partial X)$ and $H^2(X, \partial X) \oplus H^2(N, \partial N)$ as above, we obtain the short exact sequence

$$(3.1) \quad 0 \longrightarrow H^1(\partial X) \xrightarrow{\delta} H^2(X, \partial X) \oplus H^2(N, \partial N) \xrightarrow{i^*} H^2(X(\alpha)) \longrightarrow 0.$$

where $i^*(c_X, c_N) = c_X + c_N$ and $\delta\beta = (\delta\beta, \delta \circ f^*\beta)$ for any $(c_X, c_N) \in H^2(X, \partial X) \oplus H^2(N, \partial N)$ and $\beta \in H^1(\partial X)$. Notice that the short exact sequence (3.1) is the dual of the Mayer-Vietoris sequence of homology groups given by the decomposition $X(\alpha) = X \cup N$.

The lemma below contains the core of the computation in this paper. Later in §3.3, the cohomology class c in the lemma will be the Euler class $e(\widehat{\mathcal{F}})$, and $c'|_X$ and $c'|_N$ will be two relative Euler classes of $\mathcal{F} = \widehat{\mathcal{F}}|_X$ and $\mathcal{D} = \widehat{\mathcal{F}}|_N$ respectively (see Theorem 3.4).

Lemma 3.1. *Fix a meridian μ and let $\alpha = p/q$ with $p > 0$. Assume that $c \in H^2(X(\alpha))$ and c' is a relative class in $H^2(X(\alpha), \partial X)$ satisfying $i^*(c') = c$. Let $c'|_X$ and $c'|_N$ denote the restrictions of c' to $H^2(X, \partial X)$ and $H^2(N, \partial N)$ respectively. Suppose that $c = 0$ in $H^2(X(\alpha))$. Then the following two statements hold:*

- (1) The restriction $c|_X = 0$ in $H^2(X)$;
- (2) Suppose that $a = c'|_X([F])$, $b = c'|_N([D])$ and k is the order of the rational longitude λ . Then a is divisible by k and $(aq)/k \equiv b \pmod{p}$.

In addition, if $k = 1$, i.e., λ is null-homologous in $H_1(X)$, then the above two conditions are sufficient to conclude that $c = 0$.

Remark 3.2. The statement of Lemma 3.1 doesn't depend on the choice of meridian μ . However, it does depend on the choice of orientations. A different orientation convention than the one we described in §3.1 may result in $(aq)/k \equiv -b \pmod{p}$ in Condition (2).

Proof. The necessity of Condition (1) is obvious. Since the restriction $c|_X$ is the image of c under the inclusion induced map $H^2(X(\alpha)) \rightarrow H^2(X)$, it's zero if c is.

Next we derive Condition (2) by assuming $c = 0$. Since $c = i^*(c') = i^*(c'|_X, c'|_N) = 0$, there exists $\beta \in H^1(\partial X)$ such that $(c'|_X, c'|_N) = (\delta\beta, \delta \circ f^*\beta)$. That is,

$$(3.2) \quad c'|_X = \delta\beta \text{ and } c'|_N = \delta \circ f^*\beta.$$

Hence,

$$b = c'|_N([D]) = \delta \circ f^*\beta([D]) = \beta(f(\partial D)) = q\beta(\lambda) + p\beta(\mu),$$

whereas in the first term

$$(3.3) \quad \beta(\lambda) = \delta\beta([F])/k = c'|_X([F])/k = a/k.$$

Therefore, $a = c'|_X([F]) = k\beta(\lambda)$ is divisible by k and $(aq)/k \equiv b \pmod{p}$ as claimed.

Now assuming that (1) and (2) hold and $k = 1$, we show that $c = 0$ in $H^2(X(\alpha))$. Applying H^2 to the following commutative diagram:

$$\begin{array}{ccc} (X, \partial X) & \longrightarrow & (X(\alpha), \partial X) \\ i \uparrow & & \uparrow i \\ X & \longrightarrow & X(\alpha) \end{array}$$

we have $i^*(c'|_X) = c|_X$ in $H^2(X)$. Since $c|_X = 0$ by assumption, from the exact sequence

$$H^1(X) \xrightarrow{\iota^*} H^1(\partial X) \xrightarrow{\delta} H^2(X, \partial X) \xrightarrow{i^*} H^2(X),$$

there exists $\beta_0 \in H^1(\partial X)$ such that $\delta\beta_0 = c'|_X$. We fix β_0 . Given any $\beta' \in H^1(\partial X)$ with $\delta\beta' = c'|_X$, the difference $\beta_0 - \beta'$ is in the image of ι^* , which is generated by $k\mu^* = \mu^*$ (since $k = 1$). Here μ^* in $H^1(\partial X)$ is the dual of μ . Let $\beta = \beta_0 + n\mu^*$ for some $n \in \mathbb{Z}$. To show that $c = 0$, it remains to show that there exists $n \in \mathbb{Z}$ such that $\delta \circ f^*\beta = c'|_N$ in $H^2(N, \partial N)$.

Since $H^2(N, \partial N) \cong \text{Hom}(H_2(N, \partial N), \mathbb{Z})$, this is equivalent to $\delta \circ f^*\beta([D]) = c'|_N([D])$. By Condition (2), $aq - b$ is divisible by p . Let $n = -\beta_0(\mu) - \frac{aq-b}{p}$ and $\beta = \beta_0 + n\mu^*$. Then

$$\begin{aligned} \delta \circ f^*\beta([D]) &= \beta(f(\partial D)) \\ &= (\beta_0 + n\mu^*)(p\mu + q\lambda) \\ &= p\beta_0(\mu) + q\beta_0(\lambda) + np \\ &= p\beta_0(\mu) + q\beta_0(\lambda) + \left(-\beta_0(\mu) - \frac{aq-b}{p}\right)p \\ &= q\beta_0(\lambda) - aq + b \\ &= q\delta\beta_0([F]) - aq + b \quad (\text{since } k = 1) \\ &= qc'|_X([F]) - aq + b \\ &= b = c'|_N([D]). \end{aligned}$$

Hence we have $\delta\beta = (\delta\beta, \delta \circ f^*\beta) = (c'|_X, c'|_N) = c'$. Therefore, $c = i^*(c') = 0$. □

Remark 3.3. In general, the argument in the proof of Lemma 3.1 shows that a necessary and sufficient condition for $c = 0$ is that both $c|_X = 0$ and $(aq)/k + \beta(\mu)p \equiv b \pmod{kp}$ hold. Here $\beta \in H^1(\partial X)$ satisfies $\delta\beta = c'|_X$ whose existence is guaranteed by the condition $c|_X = 0$. As we have seen in the proof of Lemma 3.1, the value $(aq)/k + \beta(\mu)p \pmod{kp}$ doesn't depend on the choice of β .

3.3. The Euler class of foliations on Dehn filled manifolds

In this section, we apply Lemma 3.1 to computing the Euler class of foliations on $X(p/q)$, $p > 0$. Let $\widehat{\mathcal{F}}$ denote an oriented co-dimension-one foliation

on $X(\alpha)$ such that $\mathcal{D} = \widehat{\mathcal{F}}|_N$ is the foliation by meridional disks. Consequently, $\mathcal{F} = \widehat{\mathcal{F}}|_X$ intersects ∂X transversely in parallel simple closed curves of slope α .

Let $T\widehat{\mathcal{F}}$, $T\mathcal{F}$ and $T\mathcal{D}$ denote the oriented tangent plane fields of $\widehat{\mathcal{F}}$, \mathcal{F} and \mathcal{D} respectively. We use $e(\widehat{\mathcal{F}})$ to denote the Euler class of $T\widehat{\mathcal{F}}$ in $H^2(X(p/q))$. Let σ be a nowhere vanishing section of $T\mathcal{F}|_{\partial X} = T\widehat{\mathcal{F}}|_{\partial X}$ that is transverse to ∂X pointing outwards. Hence, σ is also a nowhere vanishing section of $T\mathcal{D}$ along ∂N pointing into N .

Following §2.1, we obtain three relative Euler classes. They are $e_\sigma(\widehat{\mathcal{F}})$ in $H^2(X(\alpha), \partial X)$, $e_\sigma(\mathcal{F})$ in $H^2(X, \partial X)$ and $e_\sigma(\mathcal{D})$ in $H^2(N, \partial N)$. By definition, $e_\sigma(\widehat{\mathcal{F}})|_X = e_\sigma(\mathcal{F})$ and $e_\sigma(\widehat{\mathcal{F}})|_N = e_\sigma(\mathcal{D})$, where $e_\sigma(\widehat{\mathcal{F}})|_X$ and $e_\sigma(\widehat{\mathcal{F}})|_N$ are restrictions of $e_\sigma(\widehat{\mathcal{F}})$ to X and N respectively. Moreover, $i^*(e_\sigma(\widehat{\mathcal{F}})) = e(\widehat{\mathcal{F}})$ and $i^*(e_\sigma(\mathcal{F})) = e(\mathcal{F})$, where i^* 's are induced by inclusions $(X(p/q), \emptyset) \rightarrow (X(p/q), \partial X)$ and $(X, \emptyset) \rightarrow (X, \partial X)$ respectively.

Theorem 3.4 below follows immediately from Lemma 3.1. Euler classes $e(\widehat{\mathcal{F}})$, $e(\mathcal{F})$, $e_\sigma(\widehat{\mathcal{F}})$, $e_\sigma(\mathcal{F})$ and $e_\sigma(\mathcal{D})$ correspond to cohomology classes c , $c|_X$, c' , $c'|_X$ and $c'|_N$ in the lemma.

Theorem 3.4. *Let X be a \mathbb{Q} -homology solid torus and F be a properly embedded surface in X representing a generator of $H_2(X, \partial X)$. Suppose that $\widehat{\mathcal{F}}$ is an oriented co-dimension one foliation on $X(p/q)$, $p > 0$ whose restriction to the filling solid torus N is the foliation by meridian disks, and the orientation of the leaves of $\widehat{\mathcal{F}}$ agree with the given orientation of the meridian disks of N . Let $\mathcal{F} = \widehat{\mathcal{F}}|_X$ and σ denote a nowhere vanishing outward pointing section of $T\mathcal{F}$ along ∂X . Suppose that the Euler class $e(\widehat{\mathcal{F}}) = 0$ in $H^2(X(p/q))$. Then the following two statements hold:*

- (1) *The Euler class $e(\mathcal{F}) = 0$ in $H^2(X)$;*
- (2) *Let $a = e_\sigma(\mathcal{F})([F])$ and k denote the order of the rational longitude λ . Then a is divisible by k and $(aq)/k \equiv 1 \pmod{p}$.*

In addition, if $k = 1$, i.e., λ is null-homologous in $H_1(X)$, then the above two conditions are sufficient to conclude that $e(\widehat{\mathcal{F}}) = 0$.

Remark 3.5. A necessary and sufficient condition for $e(\widehat{\mathcal{F}}) = 0$ when $k > 1$ in general can be deduced immediately from the condition given in Remark 3.3 with b being replaced by 1.

Intuitively, Condition (1) in Theorem 3.4 says that the plane field $T\widehat{\mathcal{F}}$ is trivial over X , which is obviously necessary for it to be trivial over the entire manifold $X(\alpha)$. Also notice that since $H^2(N) = 0$, we have $T\widehat{\mathcal{F}}|_N$ is

always trivial. Therefore, there exist nowhere vanishing sections of $T\widehat{\mathcal{F}}$ over both X and N . Condition (2) spells out the numerical equation that must be satisfied for the existence of sections over X and N that match along ∂X , so that together they define a global section of $T\widehat{\mathcal{F}}$ over $X(\alpha)$.

Proof of Theorem 3.4. The theorem is Lemma 3.1 with an additional claim that the value of b must equal 1.

By assumption $\mathcal{D} := \widehat{\mathcal{F}}|_N$ is the foliation of N by meridional disks. Let D denote a meridian disk of N . Since the orientation of D inherited from \mathcal{D} agrees with the given orientation on D (see §3.1), by the Poincare-Hopf Theorem, we have $b = e_\sigma(\mathcal{D})([D]) = \chi(D) = 1$. □

Proof of Theorem 1.4. When X is a \mathbb{Z} -homology solid torus, $H^2(X) = 0$ and $k = 1$. Therefore, Theorem 1.4 follows from Theorem 3.4. □

4. Dehn fillings on fibered manifolds and left-orders

In this section, we compute the Euler classes of taut foliations constructed in [Rob01b] (Theorem 1.7) and [Kri20] (Theorem 1.6). These are taut foliations on closed 3-manifolds obtained from Dehn fillings on fibered manifolds.

4.1. Canonical meridian, degeneracy slope and the fractional Dehn twist coefficient

Let X be the exterior of a fibered hyperbolic knot with fiber F in a \mathbb{Q} -homology sphere. So $X = F \times [0, 1]/(x, 1) \sim_h (h(x), 0)$ for an orientation-preserving homeomorphism $h : F \rightarrow F$ with $h|_{\partial F} = \text{id}_{\partial F}$, where h is called the monodromy of the knot. Since the interior of X is hyperbolic, h is freely isotopic to a pseudo-Anosov homeomorphism φ [Thu98]. Note that $\varphi|_{\partial X}$ is never the identity.

The flow lines of the suspension flow of φ on ∂X are path-connected components of the quotient $\sqcup_{p \in \partial F} (p \times [0, 1]) / \sim_\varphi$. The *degeneracy slope* of X , denoted by γ is a closed flow line of the suspension flow of φ , the existence of which is guaranteed by the properties of pseudo-Anosov maps (see [FM12, Part 3] for instance). The concept of degeneracy slope was first considered in [GO89] in the study of essential laminations in Dehn fillings of X .

Instead of the knot meridian $\mu = * \times [0, 1] / \sim_h$, when considering fibered knots in a general \mathbb{Q} -homology sphere, it is often natural to use the so-called *canonical meridian* [Rob01b], which we denote by μ_0 .

To obtain μ_0 , we first follow γ starting at a point in $\gamma \cap \lambda$ until we reach a point in $\gamma \cap \lambda$ again. In general, this is not the same intersection point

as the one we started with. To form a loop, which represents the slope μ_0 , we continue to follow one of the subarcs of the longitude λ back to the initial point. There are two subarcs of λ that can lead us back to the initial intersection point, and we choose the one that intersects γ minimally. If both subarcs have the same intersection number with γ , we follow the convention in [Rob01b, §3] and use the subarc of λ so that γ is a positive slope with respect to μ_0 .

Remark 4.1 (The trace of the monodromy of genus one knots). Let h denote the monodromy of a fibered knot K and $h_* : H_1(F) \rightarrow H_1(F)$ be its induced homology map. When the genus $g(F) = 1$, we have $\gamma = \mu_0$ if and only if the trace $\text{tr}(h_*) > 2$. This is because when $g(F) = 1$, the stable foliation \mathcal{F}_s of the (pseudo-)Anosov representative φ of h has two singular points on the boundary. The orientation of \mathcal{F}_s is preserved by φ when $\text{tr}(h_*) > 2$, so φ must fix both singular points of \mathcal{F}_s on ∂F . Therefore, the degeneracy slope γ intersects λ once and hence equals the canonical meridian μ_0 .

There is another notion closely related to the degeneracy slope, called the *fractional Dehn twist coefficient* $c(h)$ of h . It was first introduced in [HKM08, HKM09] to study the tightness of the contact structure supported by the open book (F, h) . If we write the degeneracy slope $\gamma = p\mu + q\lambda$, where μ is the knot meridian, then the fractional Dehn twist coefficient h , denoted by $c(h)$, equals q/p , the reciprocal of the degeneracy slope [KR13, §2].

Remark 4.2 (The degeneracy slope and the FDTC).

- (1) When $c(h)$ is an integer, the degeneracy slope γ intersects λ once. Therefore, by definition $\mu_0 = \gamma$. When X is the exterior of a fibered knot in S^3 , it is known that $|c(h)| < 1$. Examples of fibered knots in S^3 with $c(h) = 0$ include hyperbolic fibered two-bridge knots [GK90, Theorem 8].
- (2) When $c(h) > 0$, h is called *right-veering*. Fibered knots in S^3 with right-veering monodromies include the positive L-space knots and more generally fibered strongly quasipositive knots [Ni07, Ghi08, HKM09, Hed10]. For these knots, the degeneracy slope is positive with respect to the knot meridian and hence the canonical meridian μ_0 is the same as the knot meridian [Rob01b, Proposition 7.4].

4.2. Branched surfaces

A branched surface B is a topological space locally modeled on one of the pictures in Figure 2. The set of points at which B is not diffeomorphic to \mathbb{R}^2 is called the branch locus. Note that at any point of B there is a well-defined tangent plane. Given a properly embedded branched surface B in a 3-manifold M , let $N(B)$ denote an I -fibered normal neighborhood of B in M , whose local models are depicted in Figure 3.

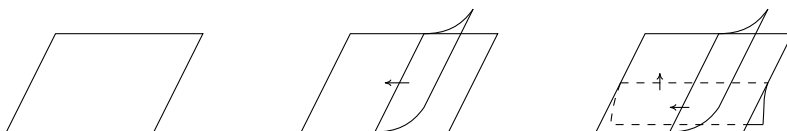


Figure 2: Local models of a branched surface.

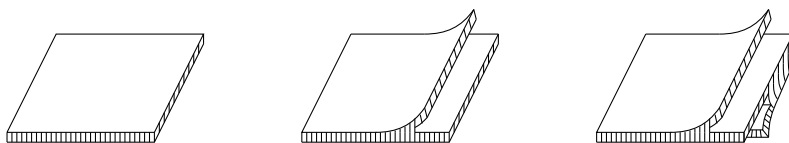


Figure 3: Local models of the normal neighborhood of a branched surface.

Example 4.3. Let X be the exterior of a fibered knot with oriented fiber F and monodromy h . Given a properly embedded arc β in F , one can construct a branched surface $B = \langle F; \mathbb{D} \rangle$ in X by attaching a disk $\mathbb{D} = [0, 1] \times [0, 1]$ to F with the bottom side $[0, 1] \times 0$ of \mathbb{D} attached to the positive side of F along β and the top side $[0, 1] \times 1$ of \mathbb{D} attached to the negative side of F along an arc β' that is freely isotopic to $h(\beta)$ and intersects β minimally. So the remaining two sides of \mathbb{D} are on ∂X . We fix an orientation on \mathbb{D} and tilt the disk \mathbb{D} slightly near F so that the positive normal directions of \mathbb{D} and F agree at the branched locus (Figure 4). If we view $X = F \times [0, 1]/(x, 1) \sim (h(x), 0)$, then the disk \mathbb{D} in X is isotopic to $\beta \times [0, 1]/\sim_h$. A key property of the canonical meridian μ_0 defined in §4.1 is that: up to isotopy, it is disjoint from the disk $\mathbb{D} \cap \partial X$, the green arcs in Figure 4. This follows immediately from the construction of the canonical meridian (see [Rob01b, §4]).

Similarly, given a collection of pairwise disjoint properly embedded arcs $\{\beta_i : i = 1, \dots, m\}$ in F , one can attach m disks $\mathbb{D}_i = [0, 1] \times [0, 1]$ along β_i and $h(\beta_i)$ (up to isotopy) to obtain a branched surface $B = \langle F; \mathbb{D}_1, \dots, \mathbb{D}_m \rangle$.



Figure 4: An example of the branched surface $B = \langle F; \mathbb{D} \rangle$. The branched locus is $\beta \cup \beta'$, where β' is freely isotopic to $h(\beta)$. Since in this example, β and β' are disjoint, locally the branched surface is homeomorphic to one of the first two models in Figure 2. The rightmost figure shows $B \cap \partial X$.

Branched surfaces provide a useful tool for constructing foliations. We say a foliation \mathcal{F} is (*fully*) *carried* by a branched surface B if after possibly splitting a finite number of leaves, leaves of the foliation can be isotoped to lie in $N(B)$ and intersect every I -fiber of $N(B)$ transversely. Intuitively, leaves of the foliation are locally “parallel” to B if it is carried by B .

4.3. Computing the Euler class of taut foliations on Dehn fillings of fibered manifolds

Suppose that $\widehat{\mathcal{F}}$ is a taut foliation on $X(\alpha)$ given in Theorem 1.5 and Theorem 1.6, where X is the exterior of a fibered knot in a \mathbb{Q} -homology sphere, $\alpha = p\mu_0 + q\lambda$ with $p > 0$ and μ_0 is the canonical meridian of X , which equals the knot meridian if X is the exterior of a fibered knot in S^3 .

Since by assumption $\widehat{\mathcal{F}}$ is transverse to the core of the filling solid torus N , we assume that $\mathcal{D} = \widehat{\mathcal{F}}|_N$ is the foliation of N by meridional disks. Let $\mathcal{F} = \widehat{\mathcal{F}}|_X$. We note that \mathcal{F} is carried by a branched surface $B = \langle F; \mathbb{D}_1, \dots, \mathbb{D}_m \rangle$ as described in Example 4.3 ([Rob01b, Theorem 4.1, Corollary 4.3, Corollary 4.4] and [Kri20, §3]).

Proof of Theorem 1.7. As in the proof of Theorem 3.4, we will use Lemma 3.1 with cohomology classes $c, c|_X, c', c'|_X$ and $c'|_N$ in the lemma replaced by Euler classes $e(\widehat{\mathcal{F}}), e(\mathcal{F}), e_\sigma(\widehat{\mathcal{F}}), e_\sigma(\mathcal{F})$ and $e_\sigma(\mathcal{D})$ respectively. Here, σ is again a nowhere vanishing outward pointing section of $T\widehat{\mathcal{F}}$ along ∂X .

Condition (1) of Lemma 3.1 corresponds to the condition that $e(\mathcal{F}) = 0$ in Theorem 1.7. Since X is fibered, we have $k = 1$. Hence, to complete the proof, it remains to verify that Condition (2) in Lemma 3.1 is equivalent to $(2g - 1) \cdot |q| \equiv 1 \pmod{p}$.

Let $B = \langle F; \mathbb{D}_1, \dots, \mathbb{D}_m \rangle$ be a branched surface that carries \mathcal{F} . Up to isotopy we assume that $\mathcal{F}|_{N(B)}$ is transverse to the I -fibers of the normal

neighborhood $N(B)$. Since B contains F as a subsurface, we can orient \mathcal{F} so that $T\mathcal{F}|_F$ is homotopic to the tangent field TF of F . To see this, we first fix a Riemannian metric g on X . Let v_0 and v_1 be the positive normal vector fields of $T\mathcal{F}|_F$ and TF respectively. We assume that v_1 is tangent to the I -fibers of $N(F) = F \times I$, and both v_0 and v_1 are tangent to ∂X . The condition that $T\mathcal{F}|_{N(B)}$ is transverse to the I -fibers of $N(B)$ is equivalent to that $v_0 \cdot v_1 \neq 0$, where the dot product \cdot is taken pointwise. We assume that $v_0 \cdot v_1 > 0$ over F . Otherwise, we reverse the orientation of $\widehat{\mathcal{F}}$ (and hence that of \mathcal{F}) and replace v_0 by $-v_0$. Let $v_t = tv_0 + (1-t)v_1$ for $t \in [0, 1]$. Note that v_t is tangent to ∂X for all t . Moreover, $v_t \cdot v_1 = tv_0 \cdot v_1 + (1-t)\|v_1\|^2$ are positive and hence nonzero. So the desired homotopy ξ_t is given by the plane fields that are normal to v_t for $t \in [0, 1]$.

Let σ be an outward pointing nowhere vanishing vector field that is tangent to $\xi_0 = T\mathcal{F}|_F$ and $\xi_1 = TF$ along ∂F . So σ is tangent to all ξ_t for $t \in [0, 1]$. Therefore, we obtain a relative Euler class of ξ_t as a plane field over $F \times [0, 1]$ in $H^2(F \times [0, 1], \partial F \times [0, 1])$ associated with the section $\sigma \times \text{id}$ over $\partial F \times [0, 1]$, whose projections to $F \times 0$ and $F \times 1$ are $e_\sigma(T\mathcal{F}|_F)$ and $e_\sigma(TF)$ respectively. Hence, $e_\sigma(T\mathcal{F}|_F) = e_\sigma(TF)$. By the Poincare-Hopf Theorem, $e_\sigma(TF)([F]) = \chi(F)$ and we have

$$a = e_\sigma(T\mathcal{F})([F]) = e_\sigma(T\mathcal{F}|_F)([F]) = \chi(F) = 1 - 2g.$$

It remains to determine the value of $b = e_\sigma(\mathcal{D})([D])$. To do so, we need to compare the orientation of the foliation \mathcal{D} and the given orientation on the meridian disk D of N . This is done by comparing their induced orientations on the simple closed curves on ∂X .

Recall that the meridian disk D is oriented so that $f(\partial D) = p\mu_0 + q\lambda$ with $p > 0$, where $f : \partial N \rightarrow \partial X$ is the gluing map (See §3.1). Let α' be a component of $\mathcal{D} \cap \partial X$ contained in $N(B) \cap \partial X$ with the induced boundary orientation from \mathcal{D} . So $\alpha' = \pm(p\mu_0 + q\lambda)$ and $-\alpha'$ is a component of $\mathcal{F} \cap \partial X$ with the induced boundary orientation from \mathcal{F} .

Since α' is contained in $N(B)$ and the canonical meridian μ_0 can be isotoped to be disjoint from $\mathbb{D}_i \cap \partial X$ for $i = 1, \dots, m$ (Example 4.3), we have μ_0 intersects only the portion of α' in $N(\partial F)$, the normal neighborhood of the black line in the rightmost picture of Figure 4). Because we oriented \mathcal{F} so that leaves of \mathcal{F} intersects the I -fibers of $N(F)$ positive, it follows that $-\alpha'$ intersects μ_0 positively.

Hence, $-\alpha' = p\mu_0 + q\lambda$ if $p/q > 0$ and $-\alpha' = -p\mu_0 - q\lambda$ if $p/q < 0$. Therefore,

(1) If $p/q > 0$, then $\alpha' = -(p\mu_0 + q\lambda)$. So \mathcal{D} has the opposite orientation with D and hence $e_\sigma(\mathcal{D})([D]) = -1$. By Lemma 3.1, we have $e(\widehat{\mathcal{F}}) = 0$ if and only if $e(\mathcal{F}) = 0$ and

$$\chi(F) \cdot q \equiv -1 \pmod{p}.$$

Because $p/q > 0$ and $p > 0$ implies that $q > 0$, the above identity is equivalent to $|\chi(F) \cdot q| = 1 \pmod{p}$.

(2) If $p/q < 0$, then $\alpha' = p\mu_0 + q\lambda$. So \mathcal{D} has the same orientation with D and hence $e_\sigma(\mathcal{D})([D]) = 1$. By Lemma 3.1, we have $e(\widehat{\mathcal{F}}) = 0$ if and only if $e(\mathcal{F}) = 0$ and

$$\chi(F) \cdot q \equiv 1 \pmod{p}$$

Because $p/q < 0$ and $p > 0$ implies that $q < 0$, the above identity is again equivalent to $|\chi(F) \cdot q| = 1 \pmod{p}$. □

4.4. Computing the slopes in \mathcal{L}_g

For each $g > 0$, we denote the set of slopes satisfying Equation (1.1) in Theorem 1.7 by

$$\mathcal{L}_g = \{p/q \in \mathbb{Q} \setminus \{0\} : (2g - 1)|q| \equiv 1 \pmod{p}\}.$$

4.4.1. Genus one knots. When $g = 1$, the above equation reduces to

$$(4.1) \quad |q| \equiv 1 \pmod{p},$$

which is equivalent to $|q| = 1 + np$ for some $n \geq 0$. Hence $q = \pm(1 + np)$ with $n \geq 0$ and $p \geq 1$.

Lemma 4.4. *Let*

$$(4.2) \quad \mathcal{L}_1 = \left\{ \pm \frac{p}{1 + np} : p \geq 1 \text{ and } n \geq 0 \right\}.$$

Then \mathcal{L}_1 is the set of slopes for which Equality (1.1) holds and $\mathbb{Z} \setminus 0 \subset \mathcal{L}_1$.

Proof. One obtains all nonzero integer slopes by taking $n = 0$. □

4.4.2. Higher genus knots.

Lemma 4.5. *The set \mathcal{L}_g is bounded between $\pm(2g - 1)$. Let $\alpha_k = (2g - 1) - \frac{1}{k+1}$, $k \in \mathbb{N}$. Then $\pm\alpha_k$ are in \mathcal{L}_g for all $k \in \mathbb{N}$.*

Proof. Given a slope $p/q \in \mathcal{L}_g$, $p > 0$, it satisfies

$$(2g - 1)|q| = 1 + np \text{ for some } n \in \mathbb{N}.$$

Because $g > 1$, n cannot be 0 in the above equation. Hence $p/q \in \mathcal{L}_g$ is equivalent to

$$\frac{p}{|q|} = \frac{1}{n} \left((2g - 1) - \frac{1}{|q|} \right).$$

It follows that the set \mathcal{L}_g is bounded between $\pm(2g - 1)$. One obtains $\pm\alpha_k$ by taking $n = 1$ and $q = \pm(k + 1)$ for $k \in \mathbb{N}$. □

4.5. The proof of results on group left-orderability

Suppose that X is the exterior of a knot in a \mathbb{Z} -homology sphere, then $H^2(X) = 0$ and the condition that $e(\mathcal{F}) = 0$ in Theorem 1.7 always holds. Since Corollary 1.10 only concerns knots in S^3 , it follows from Theorem 1.7 and Lemma 4.5.

Next we prove Corollary 1.8 and an analogous result for fibered knots of higher genus.

Proof of Corollary 1.8. When $\alpha = \lambda$ is the zero slope, the closed 3-manifold is a surface bundle over S^1 . So it is irreducible and $b_1(X(\alpha)) > 0$, which shows that $\pi_1(X(\alpha))$ is LO by [BRW05, Corollary 3.4].

Now suppose that $\alpha \neq \lambda$. Let $\widehat{\mathcal{F}}$ denote a taut foliation on $X(\alpha)$ given in Theorem 1.5. We will show that the Euler class of $e(\widehat{\mathcal{F}})$ is zero and hence by Theorem 1.2, we have $\pi_1(X(\alpha))$ is LO.

Let $\alpha = p\mu_0 + q\lambda \in \mathcal{L}_1$ with $p > 0$. Since $\alpha \in \mathcal{L}_1$, we already have the condition that $(2g - 1)|q| \equiv 1 \pmod{p}$ satisfied and it remains for us to show that $e(\mathcal{F}) = 0$ in $H^2(X)$, where as before $\mathcal{F} = \widehat{\mathcal{F}}|_X$. This is done in [BH19, §10]. We outline the proof here (See the proof of [BH19, Theorem 1.9]).

In the proof of Theorem 1.7, we have shown that the plane field $T\mathcal{F}$ is homotopic to TF over F . In fact, since the taut foliation \mathcal{F} on X is carried by a branched surface of the form $B = \langle F; \mathbb{D} \rangle$, the entire tangent plane field $T\mathcal{F}$ is homotopic to the tangent plane field of the fibration on X , which is the plane field over X tangent to the fibers of X ([HKM08, Lemma 4.4]).

Let $h : F \rightarrow F$ be the monodromy of the fibered knot with $h_{\partial F} = \text{id}$ and $M = (F, h)$ denote the closed manifold given by the open book (F, h) . Let ξ denote the contact structure supported by (F, h) . In [BH19, Theorem 1.9], it is shown that the Euler class of the contact structure $e(\xi) = 0$ in $H^2(M)$.

Because $\tilde{\xi}$ is supported by the open book (F, h) , by definition its restriction $\tilde{\xi}|_X$ is also homotopic to the tangent plane field of the fibration on X . Therefore, $e(\mathcal{F}) = e(\tilde{\xi}|_X) = 0$ in $H^2(X)$. \square

A key fact from the condition that the genus of F is one used implicitly in the above proof is that the mapping class group $\text{MCG}(F)$ is isomorphic to the group of 3-strand braid $B_3 \cong \text{MCG}(D_3)$, where D_3 is the disk with three marked points. The isomorphism from $\text{MCG}(D_3)$ to $\text{MCG}(F)$ is given by lifting elements in $\text{MCG}(D_3)$ to homeomorphisms of F through the twofold branched cover $F \rightarrow D_3$. The fact that h is a lift from a mapping class of D_3 is what allows one to conclude that the Euler class of the contact structure supported by (F, h) is zero in the proof of [BH19, Theorem 1.9].

In general, mapping classes of a surface F of genus g with connected boundary that can be realized as lifts of elements in $\text{MCG}(D_{2g+1})$ are called *symmetric homeomorphisms*. When $g > 1$, they form a proper subgroup of $\text{MCG}(F)$, called the *symmetric mapping class group* ([BH73]; also see [FM12, §9.4]). In [BH19, Theorem 1.9], it was shown that as long as the monodromy $h : F \rightarrow F$ is a symmetric homeomorphism, the Euler class of the contact structure supported by (F, h) is zero. Hence, the argument in the proof of Corollary 1.8 leads us to the following.

Corollary 4.6. *Let K be a hyperbolic fibered knot in a \mathbb{Q} -homology sphere with the degeneracy slope γ . Suppose that the monodromy h of K is a symmetric homeomorphism. Then with respect to the canonical meridian μ_0 , we have*

- (1) *If $\gamma = \mu_0$, then given any $\alpha \in \mathcal{L}_g$, there exists a co-orientable taut foliation on $X(\alpha)$ of zero Euler class and $\pi_1(X(\alpha))$ is LO.*
- (2) *If γ is a positive slope, then given any $\alpha \in \mathcal{L}_g \cap (-\infty, 1)$, there exists a co-orientable taut foliation on $X(\alpha)$ of zero Euler class and hence $\pi_1(X(\alpha))$ is LO.*
- (3) *If γ is a negative slope, then given any $\alpha \in \mathcal{L}_g \cap (-1, \infty)$, there exists a co-orientable taut foliation on $X(\alpha)$ of zero Euler class and hence $\pi_1(X(\alpha))$ is LO.*

5. Euler class and restrictions of filling slopes

Given X a \mathbb{Q} -homology solid torus, let \mathcal{S}_X be the set of slope defined as follows: for any slope $\alpha \in \mathcal{S}_X$, there exists a taut foliation on $X(\alpha)$ transverse to the core of the Dehn filling solid torus whose Euler class is zero.

- (1) By Theorem 1.2, given any $\alpha \in \mathcal{S}_X$, we have $\pi_1(X(\alpha))$ is left-orderable.
- (2) Given a taut foliation $\widehat{\mathcal{F}}$ on $X(\alpha)$ for which the core of the filling solid torus is a closed transversal, if $\alpha \notin \mathcal{S}_X$, then the Euler class $e(\widehat{\mathcal{F}}) \neq 0$.

We fix a meridian μ and identify the set \mathcal{S}_X with a subset of $\mathbb{Q} \cup \{\frac{1}{0}\}$. We denote it by $\mathcal{S}_{X,\mu} \subset \mathbb{Q} \cup \{\frac{1}{0}\}$. The aim of this section is to gain a better understanding of the distribution of the set $\mathcal{S}_{X,\mu}$ in $\mathbb{R} \cup \{\frac{1}{0}\}$. We prove Theorem 1.12 in this section.

5.1. The Thurston norm of the relative Euler class of a taut foliation

Let $\widehat{\mathcal{F}}$ be a taut foliation on $X(\alpha)$ transverse to the core of the filling solid torus and such that $\mathcal{F} = \widehat{\mathcal{F}}|_X$ intersects ∂X in simple closed curves of slope α . Since \mathcal{F} is taut, it is a well-known result that the Thurston dual norm of the relative class $e_\sigma(\mathcal{F}) \in H^2(X, \partial X)$ is at most 1 [Thu86, §3, Corollary 1]. This follows from the fact that any properly embedded connected incompressible and boundary incompressible surface in a tautly foliated 3-manifold can be isotoped to be either a leaf of the foliation or transverse to the foliation everywhere except for at a finite number of saddle points [Thu86, §3, Theorem 4] (also see [Gab00, Theorem 3.7]). We state Thurston's result in the form that is the most convenient for us in Theorem 5.1 below.

Theorem 5.1 (Thurston). *Let X be an oriented \mathbb{Q} -homology solid torus and \mathcal{F} be a co-oriented taut foliation on X that is transverse to ∂X . Let F be an oriented connected incompressible surface properly embedded in X such that $[F]$ represents a generator of $H_2(X, \partial X)$. Assume that each boundary component of F is either tangent or transverse to $\mathcal{F}|_{\partial X}$. Then the relative Euler class $e_\sigma(\mathcal{F}) \in H^2(X, \partial X)$, associated to a nowhere vanishing outward-pointing (or inward-pointing) section $\sigma : \partial X \rightarrow T\mathcal{F}|_{\partial X}$ satisfies:*

- (i) $|e_\sigma(\mathcal{F})([F])| \leq |\chi(F)|$.
- (ii) $e_\sigma(\mathcal{F})([F]) \equiv \chi(F) \pmod{2}$.

Proof. We may assume that σ is tangent to F along ∂F . If F is isotopic to a leaf of the foliation \mathcal{F} , then up to isotopy, $T\mathcal{F}|_F = \pm TF$. By the Poincaré-Hopf Theorem, we have $e_\sigma([F]) = \pm\chi(F)$, where the sign is $+$ if the orientations on F and \mathcal{F} are the same and $-$ otherwise.

Now suppose that F cannot be isotoped to a leaf. Since \mathcal{F} is taut, X is irreducible [Nov65, Ros68]. So the incompressibility of F implies that F

is also boundary incompressible [Hat, Lemma 1.10] and hence F can be isotoped to be transverse to \mathcal{F} except at a finite number of saddle (index -1) tangential points in the interior of F [Thu86, Gab00]. Let $\mathcal{L} = \mathcal{F} \cap F$ be the induced singular foliation on F . At each singular point p of \mathcal{L} , we have $T_p\mathcal{F} = \pm T_pF$. Let e_+ (resp. e_-) be the total number of singular points on \mathcal{L} , where the orientation of TF is the same with (resp. opposite to) the orientation on $T\mathcal{F}$. Hence $\chi(F) = -e_+ - e_-$ and $e_\sigma([F]) = -e_+ + e_-$ by the Poincaré-Hopf Theorem and the definition of Euler class. It follows that $|e_\sigma([F])| \leq |\chi(F)|$ and that $e_\sigma([F])$ has the same parity with $\chi(F)$. \square

5.2. Restrictions on the Dehn filling slopes

Let \times be the Thurston norm ([Thu86]) of a generator c of $H_2(X, \partial X) \cong \mathbb{Z}$, which is defined to be

$$\min \{ \max\{0, -\chi(F)\} \mid F \text{ is oriented properly embedded in } X \text{ and } [F] = c \}.$$

Theorem 1.12 is the special case of Theorem 5.2 below with $k = 1$ and $\times = 2g - 1, g > 0$.

Theorem 5.2. *Let X be a \mathbb{Q} -homology solid torus, $k \geq 1$ the order of the longitude of X and \times the Thurston norm of a generator of $H_2(X, \partial X)$. Assume that $\times \neq 0$. Then given any meridian μ , we have the following:*

- (1) *Outside $(-\frac{\times}{k} - 1, \frac{\times}{k} + 1)$, the set $\mathcal{S}_{X,\mu}$ only contains μ and the integer slopes. That is,*

$$\mathcal{S}_{X,\mu} \setminus \left(-\frac{\times}{k} - 1, \frac{\times}{k} + 1\right) \subseteq \mathbb{Z} \cup \left\{\frac{1}{0}\right\}.$$

- (2) *The set $\mathcal{S}_{X,\mu}$ is nowhere dense in $\mathbb{R} \cup \{\frac{1}{0}\} \cong \mathbb{R}P^1$. Particularly, it is nowhere dense within $(-\frac{\times}{k} - 1, \frac{\times}{k} + 1)$.*

There is a useful corollary that follows immediately from Theorem 5.2. It gives a simple criterion for a slope to be in \mathcal{S}_X .

Corollary 5.3. *Let X be a \mathbb{Q} -homology solid torus, $k \geq 1$ be the order of the rational longitude and \times the Thurston norm of a generator of $H_2(X, \partial X)$. Fix a meridian μ . Given a co-oriented taut foliation $\widehat{\mathcal{F}}$ on $X(p/q)$ that is transverse to the core of the filling solid torus, if $|\frac{p}{p+nq}| > \frac{\times}{k} + 1$ for some $n \in \mathbb{Z}$ and $\frac{p}{p+nq} \notin \mathbb{Z} \cup \{\frac{1}{0}\}$, then the Euler class $e(\widehat{\mathcal{F}}) \neq 0$.*

Proof. Given any $n \in \mathbb{Z}$, let $\mu' = \mu - n\lambda$ be another meridian. Then

$$\alpha = p\mu + q\lambda = p\mu' + (np + q)\lambda.$$

Hence $\alpha = \frac{p}{q+np}$ with respect to μ' . By Theorem 5.2(1), since $|\frac{p}{p+nq}| > \frac{x}{k} + 1$ and $\frac{p}{p+nq} \notin \mathbb{Z} \cup \{\frac{1}{0}\}$, we have $\frac{p}{q+np} \notin \mathcal{S}_{X,\mu'}$. Therefore, $e(\widehat{\mathcal{F}}) \neq 0$. \square

For instance, let X be the exterior of a genus one knot in an \mathbb{Z} -homology sphere. So we have $k = 1$ and $x = 1$. Consider slope $\frac{5}{12}$. Since

$$\frac{5}{12 - 2 \times 5} = \frac{5}{2} > |\chi(F)| + 1 = 2,$$

by Corollary 5.3 we can conclude that $\frac{5}{12}$ is not in \mathcal{S}_X . Consequently, the Euler class of any taut foliation on $X(\frac{5}{12})$ that is transverse to the core of the filling solid torus is nonzero.

Remark 5.4. By the definition of the Thurston norm, both Theorem 5.2 and Corollary 5.3 hold if we replace x by $-\chi(F)$, where F is any connected, oriented, properly embedded incompressible surface in X that represents a generator of $H_2(X, \partial X)$. Namely, F doesn't need to be norm-minimizing, though one achieves the best bound if it is.

Proof of Theorem 5.2(1). Let $\alpha = p\mu + q\lambda$ in \mathcal{S}_X with $p \geq 0$. Since $p/q \in \mathcal{S}_{X,\mu}$ there exists a co-orientable taut foliation, denoted by $\widehat{\mathcal{F}}$ on $X(p/q) = X \cup_f N$ that is transverse to the core of N and $e(\widehat{\mathcal{F}}) = 0$. Shrinking N if necessary, we may assume that $\mathcal{F} = \widehat{\mathcal{F}}|_X$ intersects ∂X transversely in simple closed curves of slope p/q and $\mathcal{D} = \widehat{\mathcal{F}}|_N$ is the foliation of N by meridional disks. We show that if $p/q \notin \mathbb{Z} \cup \{\frac{1}{0}\}$, then $|p/q| \leq \frac{x}{k} + 1$. This proves Part (1) of the theorem.

Let F denote a connected, oriented, properly embedded norm-minimizing surface which represents a generator in $H_2(X, \partial X)$. Since $p/q \notin \mathbb{Z}$, we have $p \neq 0$. We orient $\widehat{\mathcal{F}}$ as in Theorem 3.4. Then by Theorem 3.4(2) we have $e(\widehat{\mathcal{F}}) = 0$ implies that $(aq)/k \equiv 1 \pmod{p}$, where $a = e_\sigma(\mathcal{F})([F])$ is an integer divisible by k . We let $a' = a/k$. So the equation $(aq)/k \equiv 1 \pmod{p}$ is equivalent to $a'q = 1 + sp$ for some $s \in \mathbb{Z}$. Also because $p/q \notin \mathbb{Z} \cup \{\frac{1}{0}\}$, we have $s \neq 0$ and $q \neq 0$. Hence, $a'q = 1 + sp$ is also equivalent to

$$(5.1) \quad \frac{p}{q} = \frac{1}{s} \left(a' - \frac{1}{q} \right).$$

By Theorem 5.1, $a' = \frac{a}{k}$ is between $\pm\chi(F)/k$. Therefore,

$$\left| \frac{p}{q} \right| \leq \left| \frac{1}{s} \left(a' - \frac{1}{q} \right) \right| \leq |a'| + 1 \leq |\chi(F)|/k + 1 = \frac{x}{k} + 1.$$

This completes the proof of Theorem 5.2 Part (1). □

To prove Theorem 5.2(2), we construct a slightly larger set denoted by \mathcal{M}_g if $k = 1$, and \mathcal{M}_x in general. This set is highly symmetric and easy to visualize, which helps us gain a better intuition of the set \mathcal{S}_X . We will show that the set \mathcal{M}_x is nowhere dense in $\mathbb{R} \cup \{\frac{1}{0}\}$. Since \mathcal{S}_X is a subset of it, it is also nowhere dense.

We first consider the case when $k = 1$. This is equivalent to X being the exterior of a null-homologous knot K in a \mathbb{Q} -homology sphere.

Proposition 5.5. *Let X be the exterior of a null-homologous knot of genus $g > 0$ in an oriented \mathbb{Q} -homology sphere. Let \mathcal{M}_g be the subset of \mathbb{Q} that is uniquely determined by the following properties:*

- (1) $\mathcal{M}_g = \mathbb{Z} \cup (\cup_{n \geq 1} \mathcal{M}_{g,n})$.
- (2) $\mathcal{M}_{g,1} = \{m + \frac{1}{s} \mid s \in \mathbb{Z} \setminus \{0\}, \text{ and } m \text{ is an odd number between } \pm(2g - 1)\}$.
- (3) $\mathcal{M}_{g,n} = \frac{1}{n}\mathcal{M}_{g,1}$ for $n \geq 1$.

Then $\mathcal{S}_{X,\mu}$ is a proper subset of $\mathcal{M}_g \cup \{\frac{1}{0}\}$ for any given meridian μ .

An example of \mathcal{M}_g when $g = 2$ is depicted in the Figure 5 below.

Proof of Proposition 5.5. Let μ be a meridian. Suppose that p/q is a slope in $\mathcal{S}_{X,\mu}$. Then by the definition of $\mathcal{S}_{X,\mu}$, there exists a co-orientable taut foliation $\widehat{\mathcal{F}}$ with Euler class zero on $X(p/q)$ that is transverse to the core of the filling solid torus. Shrinking N if necessary, we assume that $\mathcal{D} = \widehat{\mathcal{F}}|_N$ is the foliation of N by meridional disks. We also orient $\widehat{\mathcal{F}}$ as in Theorem 3.4. We want to show that if $p/q \neq \frac{1}{0}$ (so $q \neq 0$), then p/q is in \mathcal{M}_g .

Let F be an oriented norm-minimizing Seifert surface. Since the Euler class $e(\widehat{\mathcal{F}}) = 0$, by Theorem 3.4, we have $aq = 1 \pmod{p}$, where $a = e_\sigma(\mathcal{F})([F])$. So there exists an integer $n_0 \in \mathbb{Z}$, such that $aq = 1 + n_0p$. If

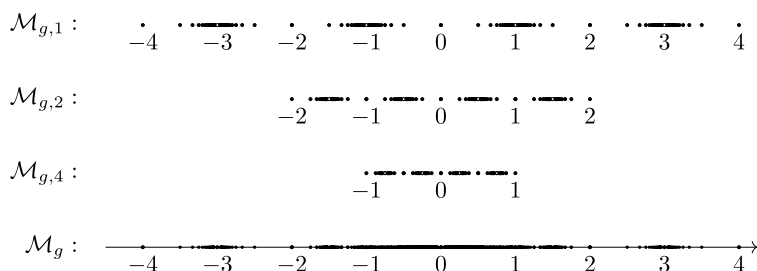


Figure 5: Points indicate slopes in $\mathcal{M}_g = \mathbb{Z} \cup (\cup_{n \geq 1} \mathcal{M}_{g,n})$ as defined in Proposition 5.5 with $g = 2$. We plotted values until points around limit points became indistinguishable. One can see that $\mathcal{M}_{g,1}$ is contained in the interval $[-2g, 2g] = [-4, 4]$ and consists of 8 sequences: $\{m \pm \frac{1}{k} : m = \pm 1, \pm 3 \text{ and } k \in \mathbb{N}\}$ as described in Proposition 5.5(2). By Proposition 5.5(3), each $\mathcal{M}_{g,n}$ for $n > 1$ can be obtained by shrinking $\mathcal{M}_{g,1}$ proportionally. Though the points become denser and denser as they get closer to 0, the set \mathcal{M}_g is in fact nowhere dense by Proposition 5.6.

$n_0 = 0$, then $aq = 1$, which implies that $q = \pm 1$. Hence $p/q \in \mathbb{Z} \subset \mathcal{M}_g$. Suppose that $n_0 \neq 0$. Since $q \neq 0$, we have $aq = 1 + n_0p$ is equivalent to

$$\begin{aligned} \frac{p}{q} &= \frac{1}{n_0} \left(a - \frac{1}{q} \right) = \pm \frac{1}{|n_0|} \left(a - \frac{1}{q} \right) \\ &= \frac{1}{|n_0|} \left(\pm a - \frac{\pm 1}{q} \right). \end{aligned}$$

By Theorem 5.1, $a = e_\sigma(\mathcal{F})([F])$ is an odd integer bounded between $\pm(2g - 1)$. Therefore, the slope $\frac{p}{q}$ lies in $\frac{1}{n} \mathcal{M}_{g,1} = \mathcal{M}_{g,n} \subset \mathcal{M}_g$, where $n = |n_0|$. \square

Proposition 5.6. \mathcal{M}_g is nowhere dense in \mathbb{R} .

Proof. We show that the closure of \mathcal{M}_g in \mathbb{R} , denoted by $\overline{\mathcal{M}_g}$ has empty interior.

Let (a, b) be an open interval in \mathbb{R} . We claim that $(a, b) \not\subseteq \overline{\mathcal{M}_g}$ (i.e. $(a, b) \cap \overline{\mathcal{M}_g} \neq (a, b)$). By taking a sub-interval of (a, b) if necessary, we may assume that $0 \notin [a, b]$. We consider the case that $(a, b) \subset (-\infty, 0)$. When $(a, b) \subset (0, \infty)$, the argument is similar.

Note that since $\mathcal{M}_{g,1}$ is bounded between $\pm 2g$, the set $\mathcal{M}_{g,n}$ is bounded between $\pm \frac{2g}{n}$ for $n \geq 1$. Let $n_0 > 0$ be an integer sufficiently large so that

the closure of $\cup_{n \geq n_0} \mathcal{M}_{g,n}$ is disjoint from (a, b) . Hence

$$(a, b) \cap \overline{\mathcal{M}_g} = (a, b) \cap \overline{\cup_{n=1}^{n_0} \mathcal{M}_{g,n}}.$$

It is easy to see that $\mathcal{M}_{g,1}$ is nowhere dense in \mathbb{R} and hence each $\mathcal{M}_{g,n} = \frac{1}{n} \mathcal{M}_{g,1}$ is nowhere dense. A finite union of nowhere dense sets is again nowhere dense, so we have $\cup_{n=1}^{n_0} \mathcal{M}_{g,n}$ is nowhere dense. Therefore, $(a, b) \cap \overline{\cup_{n=1}^{n_0} \mathcal{M}_{g,n}} \neq (a, b)$. \square

Now we are ready to finish the proof of Theorem 5.2.

Proof of Theorem 5.2(2). Let X be a \mathbb{Q} -homology solid torus, $k \geq 1$ the order of the longitude of X . Given any meridian μ , we will show that $\mathcal{S}_{X,\mu} \setminus \{\frac{1}{0}\}$ is nowhere dense in \mathbb{R} which is equivalent to $\mathcal{S}_{X,\mu}$ is nowhere dense in $\mathbb{R} \cup \{\frac{1}{0}\}$.

When $k = 1$, we have shown that $\mathcal{S}_{X,\mu} \setminus \{\frac{1}{0}\}$ is a subset of \mathcal{M}_g in Proposition 5.5. Since \mathcal{M}_g is nowhere dense by Proposition 5.6, we have $\mathcal{S}_{X,\mu}$ is nowhere dense.

In the case that $k > 1$. Let x denote the Thurston norm of a generator of $H_2(X, \partial X)$. We define a set \mathcal{M}_x , which is analogous to \mathcal{M}_g as follows:

- (1) $\mathcal{M}_x = \mathbb{Z} \cup (\cup_{n \geq 1} \mathcal{M}_{x,n})$.
- (2) $\mathcal{M}_{x,1} = \{m + \frac{1}{s} \mid s \in \mathbb{Z} \setminus \{0\}, \text{ and } m \text{ is an integer between } \pm \frac{x}{k}\}$.
- (3) $\mathcal{M}_{x,n} = \frac{1}{n} \mathcal{M}_{x,1}$ for $n \geq 1$.

Notice that given any slope $p/q \in \mathcal{S}_{X,\mu} \setminus \{\frac{1}{0}\}$ and a properly co-oriented taut foliation $\widehat{\mathcal{F}}$ on $X(p/q)$ that is transverse to the core of the filling solid torus with zero Euler class, by Theorem 3.4 and Theorem 5.1, we have $a'q \equiv 1 \pmod{p}$, where $a' = \frac{a}{k}$ is an integer between $\pm \frac{x}{k}$. Then the same argument as in Proposition 5.5 shows that $\mathcal{S}_{X,\mu} \setminus \{\frac{1}{0}\} \subset \mathcal{M}_x$. One can also show that \mathcal{M}_x is nowhere dense in \mathbb{R} using the exact argument in Proposition 5.6. The conclusion follows. \square

5.3. A remark on integral Dehn fillings

When proving the left-orderability of the fundamental group of a toroidal 3-manifold using certain gluing criteria (see [CLW13, Theorem 2.7] and [GL14, Lemma 2.9]), it is often useful to have a sequence of slopes $\{\alpha_k\}$ converging to μ satisfying that $\pi_1(X(\alpha_k))$ is left-orderable. One approach of obtaining such a sequence is by constructing taut foliations on $X(\alpha_k)$ with zero Euler classes and then applying Theorem 1.2. If the taut foliations are transverse

to the core of the filling solid torus, then Theorem 5.2 says that α_k must be integer slopes when k becomes sufficiently large.

This motivates us to state a sufficient condition for the Euler class to be zero when the slope $\alpha \in \mathbb{Z}$.

Proposition 5.7. *Assume that X is the exterior of a knot of genus $g > 0$ in an oriented \mathbb{Z} -homology sphere. Suppose that $\widehat{\mathcal{F}}$ is an oriented foliation on $X(m)$, $m \in \mathbb{Z} \setminus 0$ whose restriction to the filling solid torus N is the foliation by meridian disks and the orientations of the leaves of $\widehat{\mathcal{F}}$ agree with the given orientations of the meridian disks of N . Let $\mathcal{F} = \widehat{\mathcal{F}}|_X$ and σ denote a nowhere vanishing outward pointing section of $T\mathcal{F}$ along ∂X . Then $e(\widehat{\mathcal{F}}) = 0$ in $H^2(X(m))$ if*

- (1) $e_\sigma(\mathcal{F})([F]) = 1$ when slope $m > 0$,
- (2) $e_\sigma(\mathcal{F})([F]) = -1$ when slope $m < 0$.

Here F is any oriented surface in X that represents a generator of $H_2(X, \partial X)$.

Proof. By Theorem 1.4, the Euler class $e(\widehat{\mathcal{F}}) = 0$ if and only if $aq \equiv 1 \pmod{p}$, where $a = e_\sigma(\mathcal{F})([F])$ and $q = 1$ (resp. $q = -1$) for $p/q = m > 0$ (resp. $p/q = m < 0$). Then the proposition follows. \square

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