Communications in
Analysis and Geometry
Volume 31, Number 7, 1827–1848, 2023

Singular hyperbolic metrics and negative subharmonic functions

Yu Feng, Yiqian Shi, Jijian Song, and Bin Xu

We propose a conjecture that the monodromy group of a singular hyperbolic metric on a non-hyperbolic Riemann surface is *Zariski dense* in PSL(2, $\mathbb{R}$). By using meromorphic differentials and affine connections, we obtain evidence of the conjecture that the monodromy group of the singular hyperbolic metric cannot be contained in four classes of one-dimensional Lie subgroups of PSL(2, $\mathbb{R}$). Moreover, we confirm the conjecture if the Riemann surface is the once punctured Riemann sphere, the twice punctured Riemann sphere, a once punctured torus or a compact Riemann surface.

1. Introduction

We investigate the monodromy groups of singular hyperbolic metrics on Riemann surfaces, not necessarily compact, in terms of some analytic property of the underlying surfaces. This project lies in the intersection of algebra, analysis and geometry of Riemann surfaces, and establishes a new connection between Differential Geometry and Potential Theory on Riemann surfaces. We also use techniques from some research works on cone spherical metrics during the course of the investigation.

Y.F. is supported in part by China Scholarship Council. Y.S. is supported in part by the National Natural Science Foundation of China (Grant No. 11931009) and Anhui Initiative in Quantum Information Technologies (Grant No. AHY150200). J.S. is partially supported by National Natural Science Foundation of China (Grant Nos. 12001399 and 11831013) and the International Postdoctoral Exchange Fellowship Program 2021 by the Office of China Postdoctoral Council. B.X. is supported in part by the Project of Stable Support for Youth Team in Basic Research Field, CAS (Grant No. YSBR-001) and the National Natural Science Foundation of China (Grant Nos. 11571330, 11971450 and 12071449). Both Y.S. and B.X. are supported in part by the Fundamental Research Funds for the Central Universities (Grant No. WK3470000015).

1.1. Background of singular hyperbolic metrics

There have been many studies on singular hyperbolic metrics. Nitsche [21], Heins [11], Yamada [28], Chou and Wan [2, 3] proved that an isolated singularity of a hyperbolic metric must be either a cone singularity or a cusp one. Actually, if the curvature of a conformal Riemannian metric on a punctured neighbourhood is bounded below and above by two negative constants respectively, then the isolated singularity of the metric must be either a cone singularity or a cusp one (see [11, 13, 20]). We gave the explicit expressions of hyperbolic metrics near isolated singularities in [6, 7] by using Complex Analysis.

Let Σ be a Riemann surface, not necessarily compact, and $\mathrm{D} = \sum_{i=1}^{\infty}(\theta_i - 1)p_i$ an $\mathbb{R}$-divisor on Σ such that $0 \leq \theta_i \neq 1$, where $\{p_i\}_{i=1}^{\infty}$ is a discrete closed subset of Σ. We call $\mathrm{d}s^2$ a *singular hyperbolic metric representing* D if and only if

- $\mathrm{d}s^2$ is a conformal metric of Gaussian curvature -1 on $\Sigma \setminus \operatorname{supp} \mathrm{D}$, where we denote $\{p_i\}_{i=1}^{\infty}$ by $\operatorname{supp} \mathrm{D}$.
- If $\theta_i > 0$, then $\mathrm{d}s^2$ *has a cone singularity at* p_i *with cone angle* $2\pi\theta_i > 0$. That is, in a neighborhood U of p_i, $\mathrm{d}s^2 = e^{2u}|\mathrm{d}z|^2$, where z is a complex coordinate of U with $z(p_i) = 0$ and $u - (\theta_i - 1)\ln|z|$ extends continuously to $z = 0$.
- If $\theta_i = 0$, then $\mathrm{d}s^2$ *has a cusp singularity at* p_i. That is, in a neighborhood V of p_i, $\mathrm{d}s^2 = e^{2u}|\mathrm{d}z|^2$, where z is a complex coordinate of V with $z(p_i) = 0$ and $u + \ln|z| + \ln(-\ln|z|)$ extends continuously to $z = 0$.

It was a classical problem about the existence and uniqueness of a hyperbolic metric with finitely many prescribed singularities on a compact Riemann surface. By the Gauss-Bonnet formula, if $\mathrm{d}s^2$ is a hyperbolic metric representing the divisor $\mathrm{D} = \sum_{i=1}^{n}(\theta_i - 1)p_i$ with $\theta_i \geq 0$ on a compact Riemann surface Σ, then there holds $\chi(\Sigma) + \sum_{i=1}^{n}(\theta_i - 1) < 0$, where $\chi(\Sigma)$ is the Euler number of Σ. More than half a century ago, by using Potential Theory, Heins [11] proved the following.

Theorem 1.1. *There exists a unique hyperbolic metric representing an* $\mathbb{R}$-*divisor* $\mathrm{D} = \sum_{i=1}^{n}(\theta_i - 1)p_i$ *with* $\theta_i \geq 0$ *on a compact Riemann surface* Σ *if and only if* $\chi(\Sigma) + \sum_{i=1}^{n}(\theta_i - 1) < 0$.

Nearly three decades later, by using the PDE method, both McOwen [19] and Troyanov [25] proved the same theorem for hyperbolic metrics with only cone singularities. And McOwen [20, Theorems I and II] proved more general results than Theorem 1.1, which give the existence and uniqueness of conformal metrics with prescribed curvature and singularities on compact Riemann surfaces.

1.2. Developing maps and subharmonic functions

From the viewpoint of a combination of Complex Analysis and G-structure ([8, Theorem 2.12] and [15, Theorem 2.2]), the concept of *developing map* is naturally associated to a singular hyperbolic metric. In this manuscript, we focus on the algebraic property of developing maps of singular hyperbolic metrics on Riemann surfaces, which turns out to be interwoven with the analytic property of underlying surfaces. To give all the details of the story, we need to at first present the existence and the basic properties of developing maps for singular hyperbolic metrics.

Theorem 1.2 ([15, Theorem 2.2]). *Let* $\mathrm{d}s^2$ *be a singular hyperbolic metric representing* D *on a Riemann surface* Σ*. Then there exists a multivalued locally univalent holomorphic map* $f\colon \Sigma\setminus \operatorname{supp}\mathrm{D} \longrightarrow \mathbb{H}=\{z\in\mathbb{C}: \Im z>0\}$ *called a* developing map *of* $\mathrm{d}s^2$ *such that* $\mathrm{d}s^2=f^*\mathrm{d}s_0^2$*, where* $\mathrm{d}s_0^2=|\mathrm{d}z|^2/(\Im z)^2$ *is the hyperbolic metric on the upper half-plane* $\mathbb{H}$ *and the monodromy representation of* f *gives a homomorphism* $\mathfrak{M}_f\colon \pi_1\big(\Sigma\setminus\operatorname{supp}\mathrm{D}\big)\to \mathrm{PSL}(2,\mathbb{R})$*, whose image has a well defined conjugacy class in* $\mathrm{PSL}(2,\mathbb{R})$*.*

By abuse of notation, we just call the conjugacy class of image of $\mathfrak{M}_f$ the *monodromy group* of $\mathrm{d}s^2$. Any two developing maps of $\mathrm{d}s^2$ are related by a fractional linear transformation in $\mathrm{PSL}(2,\mathbb{R})$.

We recall the classification of Riemann surfaces in terms of the existence of a non-constant negative subharmonic function, which is a basic concept in Potential Theory.

Definition 1.3 ([5, p.179]). Let Σ be a Riemann surface. We call Σ *elliptic* if and only if Σ is compact. We call Σ *parabolic* if and only if Σ is not compact and Σ does not carry a negative non-constant subharmonic function. We call Σ *hyperbolic* if and only if Σ carries a negative non-constant subharmonic function. We call Σ *non-hyperbolic* in short if it is either elliptic or parabolic.

For the convenience of readers, we list quite a few examples of parabolic/ hyperbolic Riemann surfaces as follows.

- Each finitely punctured compact Riemann surface is parabolic, since a subharmonic function on such a surface extends to the original compact surface and then it must be constant.
- The maximal abelian cover of a punctured torus is a parabolic Riemann surface by a theorem of Uludağ [27, p.298].
- The unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is a hyperbolic Riemann surface since $\log|z|$ is a negative subharmonic function on it. Hence, subdomains of $\mathbb{D}$ are also hyperbolic.
- The maximal abelian cover of the thrice punctured Riemann sphere is hyperbolic by a result of McKean-Sullivan [18, Section 4]. See Section 3 for the definition of maximal abelian cover.

1.3. *G*-metrics

Thompson-Wick [24, Section 2] classified all the connected Lie subgroups of $\mathrm{GL}(2,\mathbb{R})$ up to conjugacy. As a consequence, we could obtain all the four classes of connected Lie subgroups of $\mathrm{SL}(2,\mathbb{R})$ as

$$\left\{\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} : t\in\mathbb{R}\right\},\ \left\{\begin{pmatrix}e^t & 0\\ 0 & e^{-t}\end{pmatrix} : t\in\mathbb{R}\right\},$$
$$\left\{\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix} : t\in\mathbb{R}\right\} \quad \text{and} \quad \left\{\begin{pmatrix}e^t & s\\ 0 & e^{-t}\end{pmatrix} : t\in\mathbb{R}, s\in\mathbb{R}\right\}.$$

Moreover, by a direct computation, we also reach the following proposition.

Proposition 1.4. *Up to conjugacy, we can classify all the positive dimensional proper Lie subgroups of* $\mathrm{PSL}(2,\mathbb{R})$ *as the following five classes*:

- *the two-dimensional subgroup* $L = \left\{\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix} : a>0, b\in\mathbb{R}\right\}$;
- *the family* $\{H_{1c} : c>0\}$ *of one-dimensional Lie subgroups* $H_{1c} = \left\{\begin{pmatrix}c^n & t\\ 0 & c^{-n}\end{pmatrix} : n\in\mathbb{Z},\ t\in\mathbb{R}\right\}$;
- *the one-dimensional subgroup* $H_2 = \left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} : a>0\right\}$;
- *the one-dimensional subgroup* $H_2' = H_2 \bigcup \left\{\begin{pmatrix}0 & b\\ -b^{-1} & 0\end{pmatrix} : b>0\right\}$;

- *the one-dimensional subgroup* $H_3 = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} : t \in \mathbb{R} \right\} / \{\pm I_2\}$.

We note that H_2' has two connected components and H_2 is its identity component. Denote by L_0 the normal subgroup $H_{11} = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$ of L.

Definition 1.5. Let $\mathrm{d}s^2$ be a singular hyperbolic metric representing the divisor $\mathrm{D} = \sum_{i=1}^{\infty} (\theta_i - 1)p_i,\ 0 \le \theta_i \ne 1$ on a Riemann surface Σ, and G a positive dimensional proper Lie subgroup of $\mathrm{PSL}(2, \mathbb{R})$. We call $\mathrm{d}s^2$ a *G-metric* if the monodromy group of $\mathrm{d}s^2$ lies in the conjugacy class of G. We call the monodromy group of $\mathrm{d}s^2$ is *Zariski dense* in $\mathrm{PSL}(2,\mathbb{R})$ if it is *not* contained in the conjugacy class of any positive dimensional proper Lie subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

1.4. Main results and the conjecture

We state the main results and the conjecture of this manuscript as follows.

Theorem 1.6. *Let $\mathrm{d}s^2$ be a singular hyperbolic metric on a non-hyperbolic Riemann surface Σ. Then the following two statements hold:*

- $\mathrm{d}s^2$ *is not a G-metric if G is a Lie subgroup of* $\mathrm{PSL}(2, \mathbb{R})$ *which is conjugate to H_2, H_2', H_3 or L_0.*
- *If Σ is a compact Riemann surface, $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$ or a punctured torus, then the monodromy group of* $\mathrm{d}s^2$ *is Zariski dense in* $\mathrm{PSL}(2,\mathbb{R})$.

This theorem stimulates us to propose the following conjecture.

Conjecture 1.7. The monodromy group of a singular hyperbolic metric on a non-hyperbolic Riemann surface is Zariski dense in $\mathrm{PSL}(2,\mathbb{R})$.

Remark 1.8. The compact Riemann surface case of Theorem 1.6 can be thought of as an analogue of [4, Theorem 7], where G. Faltings proved the Zariski dense property in $\mathrm{PSL}(2, \mathbb{C})$ for the monodromy groups of permissible connections belonging to certain uniformization data on a compact Riemann surface. However, we could not mimick the proof of [4, Theorem 7] by Faltings for this case because the cone angles of the singular hyperbolic metric in Theorem 1.6 don't lie in $\{2\pi/n : n \in \mathbb{Z}_{>1}\}$ in general.

We investigate G-metrics as G varies among all the positive dimensional proper Lie subgroups of $\mathrm{PSL}(2,\mathbb{R})$ in the remaining sections of this manuscript. In Section 2, we prove that there exists no H_3-metric on a non-hyperbolic Riemann surface, and construct a family of H_3-metrics with countably many cone singularities on the unit disc. We show the non-existence of either H_2-metric or H_2'-metric on a non-hyperbolic Riemann surface in Sections 3 and 4, respectively. In Section 5, we prove that there exists no L-metric on a compact Riemann surface, $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$ or a punctured torus. We also show that any L-metric on the unit disk is automatically an L_0-metric in this section. We prove Theorem 1.6 as a consequence of the results proved in Sections 2-5 and make a discussion about Conjecture 1.7 in the last section.

2. H_3-metrics

In this section, we use the Poincaré disk model $\left(\mathbb{D}, \frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}\right)$ rather than the upper half plane model $\left(\mathbb{H}, |\mathrm{d}z|^2/(\Im z)^2\right)$ to investigate an H_3-metric $\mathrm{d}s^2$ representing an $\mathbb{R}$-divisor D on a Riemann surface Σ. Hence, there exists a developing map $f\colon \Sigma\setminus \operatorname{supp} \mathrm{D} \longrightarrow \mathbb{D}$ of the metric $\mathrm{d}s^2$ such that $\mathrm{d}s^2 = f^*\left(4|\mathrm{d}z|^2/(1-|z|^2)^2\right)$. Moreover, the monodromy of f lies in

$$\mathrm{U}(1) = \left\{z \mapsto e^{\sqrt{-1}\,t} z \,:\, t\in[0,\,2\pi)\right\}.$$

Hence we may also call the metric $\mathrm{d}s^2$ a U(1)-*metric*. Motivated by [1], we characterize a U(1)-metric in terms of a meromorphic one-form on Σ satisfying some geometric properties (Lemma 2.1). We can also construct a nonconstant bounded subharmonic function by using a developing map of a U(1)-metric on Σ, which implies that Σ must be hyperbolic (Theorem 2.3). In addition, using some meromorphic one-forms, we can construct a family of U(1)-metrics on $\mathbb{D}$ (Proposition 2.4).

Lemma 2.1. *Let* $\mathrm{d}s^2$ *be a* U(1)*-metric representing an* $\mathbb{R}$*-divisor* $\mathrm{D} = \sum_{i=1}^{\infty}(\theta_i - 1)p_i$ *on a Riemann surface* Σ*, and* $f\colon \Sigma\setminus\operatorname{supp}\mathrm{D}\longrightarrow\mathbb{D}$ *a developing map of* $\mathrm{d}s^2$ *with monodromy in* U(1)*. Then the logarithmic differential* $\omega := \mathrm{d}(\log f) = \frac{\mathrm{d}f}{f}$ *of* f *extends to a meromorphic one-form on* Σ *which satisfies the following properties*:

1. *If* $p\in\Sigma\setminus\operatorname{supp}\mathrm{D}$ *is a pole of* ω*, then* p *is a simple pole of* ω *with residue 1.*
2. $\mathrm{d}s^2$ *has no cusp singularity. Moreover, if* $\mathrm{d}s^2$ *has a cone singularity at* $p\in\operatorname{supp}\mathrm{D}$ *with cone angle* $0<2\pi\alpha\notin 2\pi\,\mathbb{Z}$*, then* p *is a simple pole of* ω *with residue* α*.*

(3) *If p is a cone singularity of* $\mathrm{d}s^2$ *with the angle* $2\pi m \in 2\pi\,\mathbb{Z}_{>1}$*, then p is either a zero of ω with order $m-1$ or a simple pole of ω with residue m.*

(4) *The real part* $\Re\omega$ *of ω is exact outside the set of poles of ω:* $2\Re\omega = \mathrm{d}(\log|f|^2)$.

We call ω a *character one-form* of the $\mathrm{U}(1)$-metric $\mathrm{d}s^2$.

Proof. Since the developing map $f\colon \Sigma\setminus \operatorname{supp}\mathrm{D} \to \mathbb{D}$ is a multi-valued holomorphic function with monodromy in $\mathrm{U}(1)$, its logarithmic differential $\omega = \frac{\mathrm{d}f}{f}$ is a (single-valued) meromorphic one-form on $\Sigma\setminus\operatorname{supp}\mathrm{D}$. Then we prove the four properties of ω in what follows, from which ω extends to a meromorphic one-form on Σ.

(1) Suppose that $p \in \Sigma\setminus\operatorname{supp}\mathrm{D}$ is a pole of ω. We choose a function element $\mathfrak{f}$ near p. Since $\mathfrak{f}$ is a univalent holomorphic function near p, there exists a complex coordinate z near p with $z(p) = 0$ such that $\mathfrak{f} = az + b,\ a \neq 0$. Then $\omega = \frac{\mathfrak{f}'(z)}{\mathfrak{f}(z)}\mathrm{d}z = \frac{a}{az+b}\mathrm{d}z$. Since p is a pole of ω, we have $b = 0$ and then $\mathfrak{f} = az,\ \omega = \frac{\mathrm{d}z}{z}$. Hence, p is a simple pole of ω with residue 1.

(2) Since the developing map f of $\mathrm{d}s^2$ has monodromy in $\mathrm{U}(1)$, $\mathrm{d}s^2$ has only cone singularities ([6, §3]). Suppose that p is a cone singularity of $\mathrm{d}s^2$ with angle $0 < 2\pi\alpha \notin 2\pi\,\mathbb{Z}$. By [6, Lemma 2.4], we can choose a function element $\mathfrak{f}$ near p and a complex coordinate z near p such that $\mathfrak{f} = \frac{az^\alpha+b}{cz^\alpha+d}$ with $ad-bc=1$. Since f has monodromy in $\mathrm{U}(1)$, there exists $\theta\in\mathbb{R}$ such that $e^{2\pi\sqrt{-1}\theta}\mathfrak{f} = e^{2\pi\sqrt{-1}\theta}\frac{az^\alpha+b}{cz^\alpha+d} = \frac{ae^{2\pi\sqrt{-1}\alpha}z^\alpha+b}{ce^{2\pi\sqrt{-1}\alpha}z^\alpha+d}$. This is equivalent to the system:

$$\begin{cases} ace^{2\pi\sqrt{-1}\alpha}(1-e^{2\pi\sqrt{-1}\theta}) = 0 \\ (ade^{2\pi\sqrt{-1}\alpha}+bc) - e^{2\pi\sqrt{-1}\theta}(bce^{2\pi\sqrt{-1}\alpha}+ad) = 0\,. \\ bd(1-e^{2\pi\sqrt{-1}\theta}) = 0 \end{cases}$$

Solving it, we find that either $c = b = 0$ or $a = d = 0$. If $a = d = 0$, then $\mathfrak{f} = \frac{b}{cz^\alpha}$, which contradicts that f takes values in $\mathbb{D}$. Thus $c = b = 0$, that is, $\mathfrak{f}(z)$ equals $\mu z^\alpha(\mu\neq 0)$, $\mathfrak{f}(0) = 0$. Hence $\omega = \frac{\alpha}{z}\mathrm{d}z$ and p is a simple pole of ω with residue α.

(3) Suppose that $\mathrm{d}s^2$ has a cone singularity at p with angle $2\pi m \in 2\pi\,\mathbb{Z}_{>1}$. By [6, Lemma 2.4], we can choose a complex coordinate z near p such that $\mathfrak{f} = \frac{az^m+b}{cz^m+d}$ with $ad-bc=1$, and $\omega = \frac{\mathfrak{f}'(z)}{\mathfrak{f}(z)}\mathrm{d}z = \frac{mz^{m-1}}{(az^m+b)(cz^m+d)}\mathrm{d}z$.

If $bd \neq 0$, then p is a zero of ω with order $m-1$, and $\lim_{z\to p} f(z) = \frac{b}{d} \in \mathbb{D}\setminus\{0\}$. If $bd = 0$, then we have $b = 0$ and $d \neq 0$ since $\mathfrak{f}$ takes values in $\mathbb{D}$. Hence, $\mathfrak{f}(z) = \frac{az^m}{cz^m+d}$ and p is a simple pole of ω with residue m.

(4) By a simple computation, we have $2\Re\,\omega = \mathrm{d}\,(\log f) + \mathrm{d}\,\overline{(\log f)} = \mathrm{d}\log|f|^2$. By (1-3), $|f|^2$ is a single-valued smooth function outside the poles of ω, where $\Re\,\omega$ is also exact. □

Remark 2.2. An H_3-metric with non-trivial monodromy has a unique character one-form.

Theorem 2.3. *There exists no H_3-metric on a non-hyperbolic Riemann surface.*

Proof. Let $\mathrm{d}s^2$ be an H_3-metric representing an $\mathbb{R}$-divisor D on a Riemann surface Σ and $f\colon \Sigma\setminus \operatorname{supp}\mathrm{D} \longrightarrow \mathbb{D}$ a developing map with monodromy in $\mathrm{U}(1)$. By the proof of Lemma 2.1, $|f|$ is a single-valued function on $\Sigma\setminus \operatorname{supp}\mathrm{D}$ and extends continuously to Σ. If $p\in\Sigma$ is a cone singularity of $\mathrm{d}s^2$ with angle $0<2\pi\alpha\notin 2\pi\,\mathbb{Z}$, then by (2) of Lemma 2.1, we can choose a neighborhood U of p with complex coordinate z such that $z(p)=0$ and $f(z)=z^\alpha$. Hence, $|f(z)| = |z|^\alpha$ is a subharmonic function on U. If $p\in\Sigma$ is either a smooth point or a cone singularity with angle $2\pi\,m\in 2\pi\,\mathbb{Z}_{>1}$ of $\mathrm{d}s^2$, by (1,3) of Lemma 2.1, we can choose a holomorphic function element $\mathfrak{f}$ in a neighborhood V of p such that $|f(z)| = |\mathfrak{f}(z)|$ is subharmonic on V. Therefore, $|f(z)|$ is a bounded non-constant subharmonic function on Σ, which implies that Σ is a hyperbolic Riemann surface. □

Using certain meromorphic one-forms with simple poles and positive residues on the unit disc $\mathbb{D}$, we construct a family of $\mathrm{U}(1)$-metrics on $\mathbb{D}$ in the following proposition.

Proposition 2.4. *Let $\sum_{j=1}^\infty a_j$ be a convergent series of positive real numbers, $\{z_1, z_2, \ldots\}$ a closed discrete subset of the unit disc $\mathbb{D}$, and $h\colon \mathbb{D}\to\mathbb{C}$ a holomorphic function such that $\Re\int^z h\,\mathrm{d}z$ has an upper bound. Then $\omega := \left(\sum_{j=1}^\infty \frac{a_j}{z-z_j} + h(z)\right)\mathrm{d}z$ is a meromorphic one-form on the unit disc $\mathbb{D}$. And there exists a positive number T and a 1-parameter family of $\mathrm{U}(1)$-metrics on $\mathbb{D}$,*

$$\left\{\mathrm{d}\sigma_\lambda^2 := f_\lambda^*\left(\frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}\right),\ \lambda\in(0,T)\right\} \quad \text{where} \quad f_\lambda(z) = \lambda\cdot\exp\left(\int^z \omega\right),$$

such that ω is the common character one-form of these metrics $\mathrm{d}\sigma_\lambda^2$.

Proof. Since $\sum_{j=1}^{\infty} a_j < \infty$, the series $\sum_{j=1}^{\infty} \frac{a_j}{z-z_j}$ is uniformly convergent in any compact subset K of $\mathbb{D} \setminus \{z_1,\ z_2,\ \ldots\}$. Thus $\sum_{j=1}^{\infty} \frac{a_j}{z-z_j} + h(z)$ is a meromorphic function on $\mathbb{D}$ with simple poles $z_1,\ z_2,\ \ldots$ whose residues are $a_1,\ a_2,\ \ldots$, respectively. By the uniform convergence of $\sum_{j=1}^{\infty} \frac{a_j}{z-z_j}$ on a path in $\mathbb{D} \setminus \{z_1,\ z_2,\ \ldots\}$, we can do the term-by-term integration and obtain

$$\exp\left(\int^z \omega\right) = \prod_{j=1}^{\infty} (z-z_j)^{a_j} \cdot \left(\exp \int^z h \mathrm{d}z\right).$$

Since $\sum_{j=1}^{\infty} a_j < \infty$ and $\Re \int^z h$ has an upper bound, there exists $M > 0$ such that

$$\left|\exp \int^z \omega\right| = \prod_{j=1}^{\infty} |z-z_j|^{a_j} \cdot e^{\Re \int^z h} < 2^{\sum_{j=1}^{\infty} a_j} e^{\Re \int^z h} < M.$$

Solving the equation $\omega = \mathrm{d}(\log f)$ on $\mathbb{D} \setminus \{z_1,\ z_2,\ \ldots\}$, up to a complex multiple with modulus one, we obtain a 1-parameter family of multi-valued locally univalent holomorphic functions $f_\lambda(z) = \lambda \cdot \exp\left(\int^z \omega\right)$, $\lambda \in (0,\ T) := (0,\ 1/M)$, which take values in $\mathbb{D}$ and have monodromy in $\mathrm{U}(1)$. Hence, $\mathrm{d}\sigma_\lambda^2 := f_\lambda^* \left(\frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}\right)$, $\lambda \in (0,\ T)$, form a 1-parameter family of $\mathrm{U}(1)$-metrics with character one-form ω. □

Remark 2.5. The condition $\sum_{j=1}^{\infty} a_j < \infty$ is optimal in Proposition 2.4. In fact, if $\sum_{j=1}^{\infty} a_j = \infty$, then, by taking $z_j = 1 - 1/(j+1)$, we find that the series $\sum_{j=1}^{\infty} \frac{a_j}{z-z_j}$ diverges at $z = 0$.

3. H_2-metrics

Let $\mathrm{d}s^2$ be an H_2-metric on a Riemann surface Σ. In this section, we can obtain a holomorphic one-form on Σ from $\mathrm{d}s^2$ (Lemma 3.4), and prove that Σ must be hyperbolic (Theorem 3.5). To this end, we need to recall some preliminary results.

Proposition 3.1. ([10, Proposition 1.36])

Suppose X is path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \subset \pi_1(X, x_0)$ there is a covering space $p\colon X_H \to X$ such that $p_*(\pi_1(X_H, \widetilde{x_0})) = H$ for a suitably chosen basepoint $\widetilde{x_0} \in X_H$.

Definition 3.2. For a path-connected, locally path-connected, and semilocally simply-connected space X, we call a path-connected covering space $\widetilde{X} \to X$ *abelian* if it is normal and has abelian deck transformation group.

In particular, the commutator subgroup $[\pi_1(X), \pi_1(X)]$ determines a path-connected covering space $X^{\mathrm{Ab}} \xrightarrow{p} X$ by Proposition 3.1. Since the commutator subgroup is normal, X^{Ab} is a normal covering space. And the deck transformation group of the covering $X^{\mathrm{Ab}} \xrightarrow{p} X$ is isomorphic to $\pi_1(X)/[\pi_1(X), \pi_1(X)]$, which is abelian. Hence X^{Ab} is an abelian covering space, which is called the *maximal abelian covering* of X.

Lemma 3.3. ([16, Theorem 2]) *There exists no non-constant bounded harmonic function on any abelian cover of a non-hyperbolic Riemann surface.*

We construct a holomorphic one-form from an H_2-metric on a Riemann surface in the following lemma.

Lemma 3.4. *Let* $\mathrm{d}s^2$ *be an* H_2*-metric representing the divisor* $\mathrm{D} = \sum_{i=1}^{\infty}(\theta_i - 1)p_i$, $0 \le \theta_i \ne 1$ *on a Riemann surface* Σ *and* $f\colon \Sigma \setminus \operatorname{supp}\mathrm{D} \to \mathbb{H}$ *a developing map of* $\mathrm{d}s^2$ *with monodromy in* H_2*. Then the following statements hold.*

(1) *The logarithmic differential* $\omega := \frac{\mathrm{d}f}{f}$ *of* f *is a holomorphic one-form on* $\Sigma \setminus \operatorname{supp}\mathrm{D}$*, which extends to a holomorphic one-form on* Σ*, which we call the* character one-form *of* $\mathrm{d}s^2$*.*

(2) *The singularities of* $\mathrm{d}s^2$ *must be cone singularities with angles lying in* $2\pi\,\mathbb{Z}_{>1}$*. In particular, a cone singularity with angle* $2\pi\, m \in 2\pi\,\mathbb{Z}_{>1}$ *of* $\mathrm{d}s^2$ *is a zero of* ω *with order* $m-1$*.*

(3) *The developing map* $f\colon \Sigma \setminus \operatorname{supp}\mathrm{D} \to \mathbb{H}$ *extends to a multi-valued holomorphic function on* Σ *which also takes values in* $\mathbb{H}$*.*

Proof. (1) Take a point $p \in \Sigma \setminus \operatorname{supp}\mathrm{D}$ and choose a function element $\mathfrak{f}$ of f near p. Since f has monodromy in H_2 and takes values in $\mathbb{H}$, $\omega := \frac{\mathrm{d}\mathfrak{f}}{\mathfrak{f}}$ does not depend on the choice of $\mathfrak{f}$ and $\omega = \frac{\mathrm{d}f}{f}$ is a holomorphic one-form on $\Sigma \setminus \operatorname{supp}\mathrm{D}$. We postpone to (2) the proof that ω extends to a holomorphic one-form on Σ.

(2) Since f has monodromy in H_2, $\mathrm{d}s^2$ has only cone singularities with angles in $2\pi\,\mathbb{Z}_{>1}$ ([6, §3]). Suppose that it has a cone singularity at p with angle $2\pi m \in 2\pi\,\mathbb{Z}_{>1}$. Then, by [6, Lemma 2.4], we can choose a complex coordinate z near p such that $\mathfrak{f} = \frac{az^m+b}{cz^m+d}$ with $ad - bc = 1$, and $\omega = \frac{mz^{m-1}}{(az^m+b)(cz^m+d)}\mathrm{d}z$. Since f takes values in $\mathbb{H}$, we have $bd \ne 0$ which implies that p is a zero of ω with order $m-1$, and $\lim_{z\to p} f(z) = \frac{b}{d} \in \mathbb{H}$. Hence, ω extends to a holomorphic one-form on Σ.

(3) By (2), a function element of f extends to a cone singularity of $\mathrm{d}s^2$ analytically and achieves a value in $\mathbb{H}$. Hence, f extends to a multi-valued holomorphic function which also takes values in $\mathbb{H}$. □

Theorem 3.5. *There exists no H_2-metric on a non-hyperbolic Riemann surface.*

Proof. Let $\mathrm{d}s^2$ be an H_2-metric on a Riemann surface Σ. Then its developing map $f\colon \Sigma\setminus \operatorname{supp}\mathrm{D}\longrightarrow \mathbb{H}$ extends to a multi-valued holomorphic function on Σ which also takes values in $\mathbb{H}$. Consider the maximal abelian covering $\Sigma^{\mathrm{Ab}} \xrightarrow{p} \Sigma$. Then $p_*(\pi_1(\Sigma^{\mathrm{Ab}}, \widetilde{x_0})) = [\pi_1(\Sigma), \pi_1(\Sigma)]$ for a suitably chosen base point $\widetilde{x_0}\in\Sigma^{\mathrm{Ab}}$. Let $[\gamma]$ be the homotopy class of a loop γ based at $\widetilde{x_0}$ and $\mathfrak{f}$ a function element of f near $x_0 = p(\tilde{x_0})$. Then the analytical continuation $\mathfrak{f}\circ p_{[\gamma]}$ of $\mathfrak{f}\circ p$ along γ equals $\mathfrak{f}_{[p\circ\gamma]} = \mathfrak{f}_{[aba^{-1}b^{-1}]} = \mathfrak{f}$ for some two loops a and b based at $x_0 = p(\tilde{x_0})$. Since $f\circ p$ is a non-constant single-valued holomorphic function on Σ^{Ab} taking values in $\mathbb{H}$, both real and imaginary parts of $\frac{f\circ p - i}{f\circ p + i}$ are non-constant bounded harmonic functions on Σ^{Ab}. Thus Σ must be a hyperbolic Riemann surface by Lemma 3.3. □

Remark 3.6. Let $\mathrm{d}s^2$ be an H_2-metric on the unit disc $\mathbb{D}$. Then it must represent an effective divisor D and its developing map f extends to a multi-valued holomorphic function on $\mathbb{D}$. Since $\mathbb{D}$ is simply connected, f is a single-valued holomorphic function on $\mathbb{D}$ and the effective divisor (f) associated to f coincides with D. Hence, the study of H_2-metrics on $\mathbb{D}$ is equivalent to that of the critical sets of analytic self-maps of $\mathbb{D}$. If $\operatorname{supp}\mathrm{D}$ is a finite subset of $\mathbb{D}$, Heins [11, §19] observed that the developing map f for $\operatorname{supp}\mathrm{D}$ are precisely the finite Blaschke products with critical set $\operatorname{supp}\mathrm{D}$. Kraus [14] gave a description of the critical sets of analytic self-maps of $\mathbb{D}$, which are countable in general.

4. H_2'-metrics

In this section, we construct a meromorphic quadratic differential from an H_2' metric on a Riemann surface Σ (Lemma 4.1) and prove that Σ must be hyperbolic (Theorem 4.3).

Firstly we recall some basic knowledge of quadratic differentials. A quadratic differential q on Σ is a section of $K_\Sigma\otimes K_\Sigma$, a differential of type (2,0) locally of the form $\phi(z)\mathrm{d}z^2$, where K_Σ is the canonical line bundle of Σ. It is said to be holomorphic (or meromorphic) when $\phi(z)$ is holomorphic (or meromorphic).

Lemma 4.1. *Let* $\mathrm{d}s^2$ *be an* H_2'*-metric on a Riemann surface* Σ*, representing the divisor* $\mathrm{D} = \sum_{i=1}^{\infty}(\theta_i - 1)p_i,\ 0 \le \theta_i \ne 1$*. Let* $f\colon \Sigma \setminus \operatorname{supp}\mathrm{D} \to \mathbb{H}$ *be a developing map of* $\mathrm{d}s^2$*, whose monodromy lies in* H_2'*. Then* $q := \left(\frac{\mathrm{d}f}{f}\right)^{\otimes 2}$ *is a holomorphic quadratic differential on* $\Sigma \setminus \operatorname{supp}\mathrm{D}$ *which extends to a meromorphic quadratic differential on* Σ*, called the* character quadratic differential *of* $\mathrm{d}s^2$*. Moreover, we have the following.*

(1) *The singularities of* $\mathrm{d}s^2$ *must be cone singularities. Suppose that* $\mathrm{d}s^2$ *has a cone singularity at* $p \in \operatorname{supp}\mathrm{D}$ *with the angle* $2\pi\alpha > 0$*, where* α *is a non-integer. Then* $\alpha = \frac{1}{2} + k$*, where* $k \in \mathbb{Z}_{\ge 0}$*. If* $k = 0$*, then* p *is a simple pole of* q*; if* $k > 0$*, then* p *is a zero of* q *with order* $2k - 1$*.*

(2) *If* $\mathrm{d}s^2$ *has a cone singularity at* $p \in \operatorname{supp}\mathrm{D}$ *with angle* $2\pi\, m \in 2\pi\,\mathbb{Z}_{>1}$*, then* p *is a zero of* q *with order* $2m - 2$*.*

(3) *The* $\mathbb{Z}$*-divisor* (q) *associated to* q *is related to* D *by the equation* $(q) = 2 \cdot \mathrm{D}$*.*

Proof. Suppose that $p \in \Sigma \setminus \operatorname{supp}\mathrm{D}$, we choose a function element $\mathfrak{f}$ of f near p. Since f has monodromy in H_2' and takes values in $\mathbb{H}$, $q := \left(\frac{d\mathfrak{f}}{\mathfrak{f}}\right)^{\otimes 2}$ does not depend on the choice of $\mathfrak{f}$ and $q = \left(\frac{\mathrm{d}f}{f}\right)^{\otimes 2}$ is a holomorphic quadratic differential on $\Sigma \setminus \operatorname{supp}\mathrm{D}$. We postpone to (1-2) the proof that q extends to a meromorphic quadratic differential on Σ.

(1) Since the monodromy of f lies in H_2', $\mathrm{d}s^2$ has only cone singularities ([6, §3]). Suppose that $\mathrm{d}s^2$ has a cone singularity at $p \in \operatorname{supp}\mathrm{D}$ with angle $2\pi\alpha > 0$, where α is a non-integer. Then we can choose a function element $\mathfrak{f}$ near p and a complex coordinate z near p such that $\mathfrak{f} = \frac{az^\alpha + b}{cz^\alpha + d}$ with $ad - bc = 1$. Since f has monodromy in H_2', by computation we have $\alpha = \frac{1}{2} + k$ for some $k \in \mathbb{Z}_{\ge 0}$ and $ad = -bc = 1/2$, so $\mathfrak{f}' = \frac{\alpha z^{\alpha-1}}{(cz^\alpha + d)^2}$, $\mathfrak{f}(0) = \frac{b}{d} \in \mathbb{H}$, and $q = \frac{\alpha^2 z^{2\alpha - 2}}{(acz^{2\alpha} + bd)^2}\mathrm{d}z^2$. Therefore, if $k = 0$, then p is a simple pole of φ; if $k > 0$, then p is a zero of φ with order $2k - 1$.

(2) Suppose that $\mathrm{d}s^2$ has a cone singularity at $p \in \operatorname{supp}\mathrm{D}$ with angle $2\pi m \in 2\pi\,\mathbb{Z}_{>1}$. Then we choose a complex coordinate z near p such that $\mathfrak{f} = \frac{az^m + b}{cz^m + d}$ with $ad - bc = 1$ and $q = \frac{m^2 z^{2m-2}}{(az^m + b)^2 (cz^m + d)^2}\mathrm{d}z^2$. Since f takes values in $\mathbb{H}$, we have $bd \ne 0$, which implies that p is a zero of φ with order $2m - 2$, and $\lim_{z \to p} f(z) = \frac{b}{d} \in \mathbb{H}$.

(3) The equation $(q) = 2 \cdot \mathrm{D}$ follows from (1-2).

□

Theorem 4.2. ([5, p.181]). *Let Σ be a Riemann surface and p a point on Σ. There exists the Green function with singularity p on Σ if and only if Σ is hyperbolic.*

Theorem 4.3. *There exists no H_2'-metric on a non-hyperbolic Riemann surface.*

Proof. Let $\mathrm{d}s^2$ be an H_2'-metric on a Riemann surface Σ, $f\colon \Sigma \setminus \mathrm{supp}\, \mathrm{D} \to \mathbb{H}$ a developing map of it with monodromy in H_2' and $q = \left(\frac{\mathrm{d}f}{f}\right)^{\otimes 2}$ the character quadratic differential of $\mathrm{d}s^2$. By Lemma 4.1, we have $(q) = 2\mathrm{D}$, where q has at worst simple poles. As the proof of [23, Lemma 3.1], the quadratic differential q induces the canonical double cover $\pi\colon \widehat{\Sigma} \to \Sigma$, branching over critical points of q whose orders are odd, such that $\pi^*\varphi = \omega^{\otimes 2}$, where ω is a holomorphic one-form on $\widehat{\Sigma}$. Define $\widehat{f} := f \circ \pi$, which is a multi-valued holomorphic function on $\widehat{\Sigma}$. Since $\pi^* q = \pi^*\left(\frac{\mathrm{d}f}{f}^{\otimes 2}\right) = \left(\frac{d\widehat{f}}{\widehat{f}}\right)^{\otimes 2} = \omega^{\otimes 2}$, we have $\omega = \mathrm{d}(\log \widehat{f})$ (up to sign). Hence, $\widehat{f}$ has monodromy in H_2 and is a developing map of the pull-back metric $\pi^*\mathrm{d}s^2$ on $\widehat{\Sigma}$, which is then an H_2-metric. By Theorem 3.5, $\widehat{\Sigma}$ is a hyperbolic Riemann surface.

By Theorem 4.2, we can take the Green function G on $\widehat{\Sigma}$ with singularity $p \in \widehat{\Sigma}$. Then G is a positive harmonic function on $\Sigma \setminus \{p\}$, $G + \log|z|$ is harmonic near p, where z is a complex coordinate centered at p. And G is the minimal function satisfying the aforementioned two properties. Hence, $u := \max\{-1, -G\}$ is a negative subharmonic function on $\widehat{\Sigma}$ and equals constant -1 near p. Moreover, it is neither a constant nor harmonic by the minimal property of G. Note that the double cover π is a branched Galois cover whose deck transformation group is generated by the holomorphic involution τ on $\widehat{\Sigma}$, where τ acts on $\pi^{-1}(x)$ as a swap for all point x outside the critical points of q with odd order. Then

$$F(x) := \frac{u(y) + u \circ \tau(y)}{2}, \quad \text{where} \quad x \in \Sigma \quad \text{and} \quad y \in \pi^{-1}(x),$$

is a negative subharmonic function on Σ. We claim that F is nonconstant. Otherwise, since both u and $u \circ \tau$ are subharmonic and their sum $2F$ is harmonic, we find that u must be harmonic. Contradiction! Hence F is a nonconstant negative subharmonic function on Σ, which implies that Σ is a hyperbolic Riemann surface. $\square$

Remark 4.4. Though Theorem 3.5 is contained in Theorem 4.3, we intentionally spent the preceding section to narrate the former theorem and its preliminary lemma (Lemma 3.4) in detail since they have independent interest.

5. *L*-metrics

In this section, we investigate an L-metric $\mathrm{d}s^2$ on a Riemann surface Σ. In the first subsection, we find that $\mathrm{d}s^2$ induces an affine connection on Σ (Lemma 5.3), by which we prove that there exists no L-metric on a compact Riemann surface (Corollary 5.4). In the second one, we prove that there exists no L-metric on $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$ or a punctured torus. We pay special attention to L_0-metrics in the last subsection and prove the non-existence of them on a non-hyperbolic Riemann surface.

5.1. There exists no *L*-metric on a compact Riemann surface

In Differential Geometry, an affine connection is a geometric concept on a smooth manifold which connects nearby tangent spaces so that it allows us to differentiate tangent vector fields ([26, Section 6]). However, an *affine connection* in the following definition is a different one in Complex Analysis which was introduced by Gunning [9] and Mandelbaum [17, p.264]. We will use the latter in this subsection.

Definition 5.1 ([9], [17, p.264])**.** Let Σ be a Riemann surface, $\{U_\alpha, z_\alpha\}$ a complex atlas on Σ, and $\psi_{\alpha\beta} = z_\alpha \circ z_\beta^{-1}\colon z_\beta(U_\alpha\cap U_\beta)\to z_\alpha(U_\alpha\cap U_\beta)$ the coordinate transition functions. We call a family of meromorphic functions $\{h_\alpha\colon U_\alpha \to \overline{\mathbb{C}} = \mathbb{C}\cup\infty\}$ an *affine connection* on Σ if for all $p\in U_\alpha\cap U_\beta$ there holds $h_\beta\big(z_\beta(p)\big) = h_\alpha\big(z_\alpha(p)\big)\cdot\left(\frac{\mathrm{d}z_\beta}{\mathrm{d}z_\alpha}\right)^{-1} + \theta_1\big(\psi_{\alpha\beta}\big(z_\beta(p)\big)\big)$, where $\theta_1\psi(z) = \frac{\psi''(z)}{\psi'(z)}$.

Let $h=\{h_\alpha\}$ be an affine connection on Σ. For each point p in U_α, we choose a small loop Γ winding around p on counterclockwise and define the *residue* of h at p by $\mathrm{Res}(h,\, p) = \frac{1}{2\pi i}\int_\Gamma h_\alpha \mathrm{d}z_\alpha$, and the *residue* of h by $\mathrm{Res}(h) = \sum_{p\in\Sigma}\mathrm{Res}(h,p)$ provided that there exist only finitely many nonzero summands. Note that $\mathrm{Res}(h,p)$ is independent of the choice of representatives for h ([17, p.270]).

Lemma 5.2. [17, Lemma 2] *Let $h=\{h_\alpha\}$ be an affine connection on a compact Riemann surface X. Then* $\mathrm{Res}(h) = -\chi(X)$, *where $\chi(X)$ is the Euler number of X.*

Lemma 5.3. *Let* $\mathrm{d}s^2$ *be an* L*-metric on a Riemann surface* Σ*, representing the divisor* $\mathrm{D} = \sum_{i=1}^{\infty}(\theta_i - 1)p_i,\ 0 \le \theta_i \ne 1$*. Let* $f\colon \Sigma \setminus \operatorname{supp}\mathrm{D} \longrightarrow \mathbb{H}$ *be a developing map of* $\mathrm{d}s^2$*, whose monodromy lies in* L*. Let* $\{U_\alpha, z_\alpha\}$ *be a complex atlas on* Σ *and* $\psi_{\alpha\beta} = z_\alpha \circ z_\beta^{-1}\colon z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$ *the coordinate transition functions. Then we have the following.*

1. $h := \{h_\alpha := f''/f'\colon U_\alpha \to \overline{\mathbb{C}}\}$ *is an affine connection on* $\Sigma \setminus \operatorname{supp}\mathrm{D}$*, which extends to an affine connection on* Σ*.*
2. *A singularity of* $\mathrm{d}s^2$ *is either a cusp singularity or a cone one with angle* $2\pi m \in 2\pi\,\mathbb{Z}_{>1}$*. Moreover, both a cusp singularity and a cone one with angle* $2\pi m$ *are simple poles of* f''/f'*, where the residues of* h *are* -1 *and* $m-1$*, respectively.*

Proof. (1) We choose a point p in $\big(U_\alpha \cap U_\beta\big) \setminus \operatorname{supp}\mathrm{D}$ and take a function element $\mathfrak{f}$ of f in a neighborhood of p in $U_\alpha \cap U_\beta$, where f''/f' is a single-valued function since f has monodromy in L. Since

$$\mathfrak{f}'(z_\beta(p)) = \mathfrak{f}'\big(z_\alpha(p)\big)\psi'_{\alpha\beta}\big(z_\beta(p)\big)$$

and

$$\mathfrak{f}''\big(z_\beta(p)\big) = \mathfrak{f}'\big(z_\alpha(p)\big)\psi''_{\alpha\beta}\big(z_\beta(p)\big) + \mathfrak{f}''\big(z_\alpha(p)\big) \times \Big(\psi'_{\alpha\beta}\big(z_\beta(p)\big)\Big)^2,$$

we have

$$\begin{aligned} h_\beta\big(z_\beta(p)\big) &= \frac{\mathfrak{f}''\big(z_\beta(p)\big)}{\mathfrak{f}'\big(z_\beta(p)\big)} = \frac{\psi''_{\alpha\beta}\big(z_\beta(p)\big)}{\psi'_{\alpha\beta}\big(z_\beta(p)\big)} + \frac{\mathfrak{f}''\big(z_\alpha(p)\big)}{\mathfrak{f}'\big(z_\alpha(p)\big)}\psi'_{\alpha\beta}\big(z_\beta(p)\big) \\ &= \theta_1\Big(\psi_{\alpha\beta}\big(z_\beta(p)\big)\Big) + h_\alpha\big(z_\alpha(p)\big)\left(\frac{\mathrm{d}z_\alpha}{\mathrm{d}z_\beta}\right)(p). \end{aligned}$$

Hence, f''/f' defines an affine connection on $\Sigma \setminus \operatorname{supp}\mathrm{D}$. We postpone to (2) the proof that it extends to Σ.

(2) Since the monodromy of f lies in L, a singularity of $\mathrm{d}s^2$ is either a cusp singularity or a cone one with angle $2\pi m \in 2\pi\,\mathbb{Z}_{>1}$ ([6, §3]). If $p \in \operatorname{supp}\mathrm{D}$ is a cusp singularity of $\mathrm{d}s^2$, by [6, Lemma 2.4], we can choose a function element $\mathfrak{f}$ near p and a complex coordinate z near p such that $\mathfrak{f} = \frac{a\log z + b}{c\log z + d}$ with $ad - bc = 1$. Since the monodromy of f belongs to L, then by computation we have $c = 0$, $ad = 1$ and $\mathfrak{f} = a^2\log z + ab$. Since $\Re\log z < 0$ as $|z| << 1$ and $\mathfrak{f}$ takes values in $\mathbb{H}$, we obtain $a^2 = -\sqrt{-1}r$

for some $r > 0$. Hence, $h_\alpha = \frac{\mathfrak{f}''}{\mathfrak{f}'} = -\frac{1}{z}$ in a neighborhood u_α of p and $\mathrm{Res}(h, p) = -1$.

Suppose that $\mathrm{d}s^2$ has a cone singularity at p with angle $2\pi m \in 2\pi\,\mathbb{Z}_{>1}$. Then we can choose a function element $\mathfrak{f}$ near p and a complex coordinate z near p such that $\mathfrak{f} = \frac{az^m+b}{cz^m+d}$ with $ad - bc = 1$. Since $\mathfrak{f}$ is an open map, we have $\lim_{z\to p}\mathfrak{f}(z) = \frac{b}{d} \in \mathbb{H}$ and $bd \neq 0$. Hence, we obtain $\mathfrak{f}'(z) = \frac{mz^{m-1}}{(cz^m+d)^2}$, $\frac{\mathfrak{f}''}{\mathfrak{f}'} = \frac{m-1}{z} - \frac{2cmz^{m-1}}{cz^m+d}(d \neq 0)$, and $\mathrm{Res}(h,p) = m-1$.

Therefore, $h = \{h_\alpha = \frac{f''}{f'} : U_\alpha \to \overline{\mathbb{C}}\}$ extends to an affine connection on Σ. □

Corollary 5.4. *There exists no L-metric on a compact Riemann surface X.*

Proof. Suppose that $\mathrm{d}s^2$ is an L-metric representing an $\mathbb{R}$-divisor D on X. Then, by Lemma 5.3, it has either cusp singularities or cone singularities with angles lying in $2\pi\,\mathbb{Z}_{>1}$. Hence, the divisor D has form $\mathrm{D} = \sum_{i=1}^{j}(m_i - 1)p_i + \sum_{i=j+1}^{n}(-1)p_i$, where $m_i \in \mathbb{Z}_{>1}$. By Lemma 5.3, we have $\mathrm{Res}(h, p_i) = m_i - 1$ as $1 \le i \le j$, and $\mathrm{Res}(h, p_i) = -1$ as $i > j$. By Lemma 5.2, we obtain $-\chi(X) = \sum_{i=1}^{j}(m_i - 1) - (n - j)$ and $\chi(X) + \sum_{i=1}^{n}(\theta_i - 1) = \chi(X) + \sum_{i=1}^{j}(m_i - 1) - (n - j) = 0$, which contradicts Theorem 1.1. □

5.2. Non-existence of L-metrics on $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$ or a punctured torus

To prove this, we need the following.

Theorem 5.5. ([27, p.298]) *Let Σ be a Riemann surface such that none of its abelian covers is hyperbolic. Then Σ is the Riemann sphere, $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$, a torus or a punctured torus.*

Theorem 5.6. *There exists no L-metric on $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$ or a punctured torus.*

Proof. Let $\mathrm{d}s^2$ be an L-metric on a Riemann surface Σ and $f : \Sigma \setminus \mathrm{supp}\,\mathrm{D} \to \mathbb{H}$ a developing map of it with monodromy in L. By Lemma 5.3, the singularity of $\mathrm{d}s^2$ is either a cusp singularity or a cone one with angle $2\pi m \in \mathbb{Z}_{>1}$. By the proof of Lemma 5.3, if $p \in \mathrm{supp}\,\mathrm{D}$ is a cusp singularity of $\mathrm{d}s^2$, then we can choose a function element $\mathfrak{f}$ near p and a complex coordinate z near p such that $\mathfrak{f} = -\sqrt{-1}r\log z + s$, where $r > 0$ and $s \in \mathbb{R}$. Since $-\Im\,\mathfrak{f} = r\log|z|$

extends to a negative subharmonic function in a neighborhood of p, $-\Im f$ forms a multi-valued negative subharmonic function on Σ with monodromy in H_2. Taking the maximal abelian covering $\Sigma^{\mathrm{Ab}} \xrightarrow{\pi} \Sigma$, we obtain a negative non-constant subharmonic function $(-\Im f) \circ \pi$ on Σ^{Ab}, which implies that Σ^{Ab} is a hyperbolic Riemann surface. By Theorem 5.5, the Riemann surface Σ is not isomorphic to $\overline{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$, a torus, or a punctured torus. □

5.3. L_0-metric

In this subsection, we obtain a meromorphic one-form satisfying some geometric properties from an L_0 metric on a Riemann surface (Lemma 5.8), by which we show the non-existence of L_0-metrics on a non-hyperbolic Riemann surface (Theorem 5.9). To this end, we need a preliminary lemma as follows.

Lemma 5.7. ([22, Theorem 3.6.1]) *Let U be an open subset of $\mathbb{C}$, E a closed polar set, and u a subharmonic function on $U \setminus E$. Suppose that each point of $U \setminus E$ has a neighbourhood N such that u is bounded above on $N \setminus E$. Then u extends uniquely to a subharmonic function U.*

Lemma 5.8. *Let $\mathrm{d}s^2$ be an L_0-metric on a Riemann surface Σ, representing the divisor $\mathrm{D} = \sum_{i=1}^{\infty}(\theta_i - 1)p_i$, and $f\colon \Sigma \setminus \operatorname{supp} \mathrm{D} \to \mathbb{H}$ a developing map of $\mathrm{d}s^2$ with monodromy in L_0. Then $\omega := \mathrm{d}f$ is a holomorphic one-form on $\Sigma \setminus \operatorname{supp} \mathrm{D}$ and extends to a meromorphic one-form on Σ, which we call the* character one-form *of $\mathrm{d}s^2$. Moreover, we have the following.*

(1) *If $\mathrm{d}s^2$ has a cusp singularity at $p \in \operatorname{supp} \mathrm{D}$, then p is a simple pole of ω with residue $-\sqrt{-1}r$, where $r > 0$.*

(2) *If $\mathrm{d}s^2$ has a cone singularity at $p \in \operatorname{supp} \mathrm{D}$ with angle $2\pi m \in 2\pi\,\mathbb{Z}_{>1}$, then p is a zero of ω with order $m - 1$.*

Proof. Take a point p on $\Sigma \setminus \operatorname{supp} \mathrm{D}$ and choose a function element $\mathfrak{f}$ of f near p. Since f has monodromy in L_0, $\omega = \mathrm{d}\mathfrak{f}$ does not depend on the choice of the function element $\mathfrak{f}$. Hence $\omega = \mathrm{d}f$ is a holomorphic one-form on $\Sigma \setminus \operatorname{supp} \mathrm{D}$. We postpone to (1-2) the proof that it extends to a meromorphic one-form on Σ.

(1) By Lemma 5.3, $\mathrm{d}s^2$ has cusp singularities or cone ones with angles lying in $2\pi\,\mathbb{Z}_{>1}$. Let $p \in \operatorname{supp} \mathrm{D}$ be a cusp singularity of $\mathrm{d}s^2$. By the proof of Lemma 5.3, we can choose a function element $\mathfrak{f}$ near p and a complex

coordinate z near p such that $\mathfrak{f} = -\sqrt{-1}r\log z + s$, where $r > 0$. So $\omega = \mathrm{d}\mathfrak{f} = \frac{-\sqrt{-1}r}{z}\mathrm{d}z$ has a simple pole at p with residue $-\sqrt{-1}r$.

(2) Suppose that $\mathrm{d}s^2$ has a cone singularity at p with angle $2\pi m \in 2\pi\,\mathbb{Z}_{>1}$. We can choose a function element $\mathfrak{f}$ near p and a complex coordinate z near p such that $\mathfrak{f} = \frac{az^m+b}{cz^m+d}$ with $ad - bc = 1$, and $\omega = \mathrm{d}\mathfrak{f} = \frac{mz^{m-1}}{(cz^m+d)^2}\mathrm{d}z$. Since $\mathfrak{f}$ takes values in $\mathbb{H}$, we can see that $d \neq 0$, p is a zero of ω with order $m - 1$, and $\lim_{z\to p} \mathfrak{f}(z) = \frac{b}{d} \in \mathbb{H}$. □

Theorem 5.9. *There exists no L_0-metric on a non-hyperbolic Riemann surface.*

Proof. Let $\mathrm{d}s^2$ be an L_0-metric $\mathrm{d}s^2$ on a Riemann surface Σ, and $f\colon \Sigma \setminus \operatorname{supp}\mathrm{D} \to \mathbb{H}$ a developing map of $\mathrm{d}s^2$ with monodromy in L_0. Then $-\Im f$ is a negative non-constant harmonic function on $\Sigma \setminus \operatorname{supp}\mathrm{D}$ and then it is also subharmonic. Since an isolated point is polar, by Lemma 5.7, $-\Im f$ extends to a negative non-constant subharmonic function on Σ. Hence, Σ is a hyperbolic Riemann surface. □

Proposition 5.10. *An L-metric on $\mathbb{D}$ actually has monodromy in L_0.*

Proof. Let $\mathrm{d}s^2$ be an L-metric representing an $\mathbb{R}$-divisor D on the unit disc $\mathbb{D}$. Let $f\colon \mathbb{D} \setminus \operatorname{supp}\mathrm{D} \to \mathbb{H}$ a developing map of $\mathrm{d}s^2$ with monodromy in L. By Lemma 5.3, f''/f' is a meromorphic function on $\mathbb{D}$ whose residues are all integers. Hence, taking $z_0 \in \mathbb{D} \setminus \operatorname{supp}\mathrm{D}$, we find that $f'(z) = \exp\left(\int_{z_0}^{z} \frac{f''}{f'}\mathrm{d}z\right) + f'(z_0)$ is a meromorphic function on $\mathbb{D}$, and then the monodromy of f lies in L_0. □

Example 5.11 ([15, Example 2.1])**.** Let $\sum_{j=1}^{\infty} a_j$ be a convergent series of positive numbers and $\{z_j\}_{j=1}^{\infty} \subset \mathbb{D}$ a closed discrete subset. Then $h(z) := \sum_{j=1}^{\infty} \frac{a_j}{z - z_j}$ is a meromorphic function on the unit disc $\mathbb{D}$ and there exists a real number λ_0 and a 1-parameter family of L_0-metrics representing the same $\mathbb{Z}$-divisor $\mathrm{D} = (h)$ on $\mathbb{D}$. Hence, these metrics have cusp singularities at z_j's and a cone singularity of angle $2\pi\big(1 + \operatorname{ord}_w(h)\big)$ at a zero w of h.

6. Proof of Theorem 1.6 and some discussions

Proof of Theorem 1.6. Let $\mathrm{d}s^2$ be a singular hyperbolic metric on a non-hyperbolic Riemann surface Σ. The first statement follows from Theorems 4.3, 2.3 and 5.9. The second one follows from the first one, Corollary 5.4, Theorem 5.6, and the classification of positive dimensional proper Lie subgroups of $\mathrm{PSL}(2, \mathbb{R})$ given in the introduction. □

We would like to make a discussion about Conjecture 1.7, which claims that *a singular hyperbolic metric on a non-hyperbolic Riemann surface has Zariski dense monodromy in* $\mathrm{PSL}(2, \mathbb{R})$. By Theorems 4.3, 2.3 and 5.9, the conjecture is reduced to proving that *the monodromy of a singular hyperbolic metric on a non-hyperbolic Riemann surface does not lie in L*. To show its subtlety, let us consider a special parabolic Riemann surface – the thrice punctured sphere $\mathbb{C} \setminus \{0, 1\}$, whose maximal Abelian cover $\big(\mathbb{C} \setminus \{0, 1\}\big)^{\mathrm{Ab}}$ is a hyperbolic Riemann surface by a result of McKean-Sullivan [18, Section 4]. Hence, the argument for the non-existence of L-metrics in the proof of Theorem 5.9 becomes invalid on $\mathbb{C} \setminus \{0, 1\}$.

Both Proposition 2.4 and Example 5.11 show that the uniqueness of hyperbolic metric representing a given $\mathbb{R}$-divisor fails on $\mathbb{D}$. However, the hyperbolic metrics there are not complete in general even after adding their cone singularities. We come up with the following.

Question 6.1. Let $\mathrm{D} = \sum_{n=1}^{\infty} (\theta_j - 1)p_j$ be an $\mathbb{R}$-divisor on $\mathbb{D}$ such that $0 \leq \theta_j \neq 1$ and $\{p_n\}$ is a discrete closed subset of $\mathbb{D}$. When does there exist a singular hyperbolic metric $\mathrm{d}s^2$ representing D such that it extends to a complete metric on $\mathbb{D} \setminus \{p_n : \theta_n = 0\}$? If yes, would it be unique? Hulin-Troyanov [12, Theorem 8.2] answered these two questions affirmatively if $\mathrm{supp}\, \mathrm{D}$ is a finite subset of $\mathbb{D}$.

Acknowledgement

B.X. would like to thank Andriy Haydys, Rafe Mazzeo and Song Sun for many insightful discussions, and their hospitality during his visit to Freiburg University in Winter 2018 and Summer 2019, Stanford University and UC Berkeley in Spring 2019. B.X. expresses his gratitude to Martin de Borbon, Alexandre Eremenko, Qiongling Li, Dmitri Panov, Yi Qi, Mao Sheng and Ming Xu for helpful suggestions and comments. Part of the work was done while B.X. and Y.F. were visiting ShanghaiTech University in Spring 2019 and UC Berkeley in 2020, respectively. Y.F. appreciates her hospitality during his visit in Summer 2021 to Institute of Geometry and Physics, USTC, where he and the other three authors finalized this manuscript.

References

[1] Q. Chen, W. Wang, Y. Y. Wu and B. Xu, *Conformal metrics with constant curvature one and finitely many conical singularities on compact Riemann surface.* Pacific Journal of Mathematics **273** (2015), 75–100.

[2] K. S. Chou and T. Wan, *Asymptotic radial symmetry for solutions of* $\Delta u + e^u = 0$ *in a punctured disc.* Pacific Journal of Mathematics **163** (1994), 269–276.

[3] K. S. Chou and T. Wan, *Correction to "Asymptotic radial symmetry for solutions of* $\Delta u + e^u = 0$ *in a punctured disc".* Pacific Journal of Mathematics **171** (1995), 589–590.

[4] G. Faltings, *Real projective structures on Riemann surfaces.* Compositio Math. **48**:2 (1982), 223–269.

[5] H. M. Farkas and I. Kra, *Riemann surfaces (Second edition).* Graduate Texts in Mathematics, 71. Springer-Verlag, New York, 1992.

[6] Y. Feng, Y. Q. Shi and B. Xu, *Isolated singularities of conformal hyperbolic metrics.* Chinese Journal of Contemporary Mathematics. **40** (2019), 15–26.

[7] Y. Feng, Y. Q. Shi and B. Xu, *Characterizing isolated singularities of conformal hyperbolic metrics.* J. Univ. Sci. Technol. China. **50** (2020), 87–93.

[8] S. Gong, H. Liu and B. Xu, *Locally removable singularities for Kähler metrics with constant holomorphic sectional curvature.* Proc. Amer. Math. Soc. **148**:5 (2020), 2179–2191.

[9] R. C. Gunning, *Special coordinate coverings of Riemann surfaces.* Math. Ann. **170** (1967), 67–86.

[10] A. Hatcher, *Algebraic Topology.* Cambridge University Press, 2002.

[11] M. Heins, *On a class of conformal metrics.* Nagoya Math. J **21** (1962), 1–60.

[12] D. Hulin and M. Troyanov, *Prescribing curvature on open surfaces.* Math. Ann. **293** (1992), 277–315.

[13] D. Kraus and O. Roth, *The behaviour of solutions of the Gaussian curvature equation near an isolated boundary point.* Math. Proc. Cambr. Phil. Soc **145** (2008), 643–667.

[14] D. Kraus, *Critical sets of bounded analytic functions, zero sets of Bergman spaces and nonpositive curvature.* Proc. London Math. Soc **273** (2012), 1–26.

[15] B. Li, L. Li and B. Xu, *Hyperbolic metrics and bounded projective functions with isolated singularities.* Ann. Acad. Sci. Fenn-M. **45** (2020), 687–698.

[16] T. Lyons and D. Sullivan, *Function theory, random paths and covering spaces.* J. Differential Geom **19** (1984), 299–323.

[17] R. Mandelbaum, *Branched structures on Riemann surfaces.* Trans. Amer. Math. Soc. **163** (1972), 261–275.

[18] H. P. McKean and D. Sullivan, *Brownian motion and harmonic functions on the class surface of the thrice puntured sphere*, Adv. Math, **51** (1984), 203–211.

[19] R. C. McOwen, *Point singularities and conformal metrics on Riemann surfaces.* Proc. AMS **103** (1988), 222–224.

[20] R. C. McOwen, *Prescribed curvature and singularities of conformal metrics on Riemann surfaces.* J. Math. Anal. Appl **177** (1993), 287–298.

[21] J. Nitsche, *Über die isolierten Singularitäten der Lösungen von $\Delta u = e^u$.* Math. Z **68** (1957), 316–324.

[22] T. Ransford, *Potential Theory in the Complex Plane.* Cambridge University Press, 1995.

[23] J. Song, Y. Cheng, B. Li and B. Xu, *Drawing Cone Spherical Metrics via Strebel Differential.* Int. Math. Res. Not. **11** (2020), 3341–3363.

[24] G. Thompson and Z. Wick, *Subalgebras of* $\mathfrak{gl}(3,\mathbb{R})$. Extracta Mathematica, **27** (2012), 201–230.

[25] M. Troyanov, *Prescribing curvature on compact surfaces with conical singularities.* Trans. Amer. Math. Soc **324** (1991), 793–821.

[26] L. W. Tu, *Differential Geometry. Connections, Curvature, and Characteristic Classes.* Graduate Texts in Mathematics, **275**. Springer, 2017.

[27] A. Muhammed Uludağ, *Existence of Green function and bounded harmonic functions on Galois covers of Riemannian manifolds.* Osaka J. Math **38** (2001), 295–301.

[28] A. Yamada, *Bounded analytic functions and metrics of constant curvature of Riemann surfaces.* Kodai Math. J **11** (1988), 317–324.

CAS Wu Wen-Tsun Key Laboratory of Mathematics and School of Mathematical Sciences
University of Science and Technology of China
Hefei 230026, China
E-mail address: yuf@mail.ustc.edu.cn

CAS Wu Wen-Tsun Key Laboratory of Mathematics and School of Mathematical Sciences
University of Science and Technology of China
Hefei 230026, China
E-mail address: yqshi@ustc.edu.cn

Center for Applied Mathematics, School of Mathematics
Tianjin University, Tianjin, 300350, China
E-mail address: smath@tju.edu.cn

CAS Wu Wen-Tsun Key Laboratory of Mathematics and School of Mathematical Sciences
University of Science and Technology of China
Hefei 230026, China
E-mail address: bxu@ustc.edu.cn

Received September 25, 2020
Accepted September 2, 2021