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On the moduli space of asymptotically flat manifolds with boundary and the constraint equations

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Carlotto-Li have generalized Marques' path connectedness result for positive scalar curvature $R > 0$ metrics on closed 3-manifolds to the case of compact 3-manifolds with $R > 0$ and mean convex boundary $H > 0$. Using their result, we show that the space of asymptotically flat metrics with nonnegative scalar curvature and mean convex boundary on $\mathbb{R}^3 \backslash B^3$ is path connected. The argument bypasses Cerf's theorem, which was used in Marques' proof but which becomes inapplicable in the presence of a boundary. We also show path connectedness for a class of maximal initial data sets with marginally outer trapped boundary.

1. Introduction

Let $\mathcal{M}_{R>0}$ be the space of positive scalar curvature metrics on X endowed with the smooth topology, and $\mathrm{Diff}(X)$ the group of diffeomorphisms acting on X. Marques [Mar12] proves the following fundamental result.

Theorem 1.1 (Marques [Mar12]). *Let X be a closed 3-manifold admitting a metric of positive scalar curvature. Then $\mathcal{M}_{R>0}/\mathrm{Diff}(X)$ is path connected.*

Marques' beautiful proof combines Perelman's Ricci flow with surgery [Per02], the conformal method, and Gromov-Lawson's gluing [GLJ80].

Based on this and using a deep theorem of Cerf [Cer68] that implies the path connectedness of $\mathrm{Diff}_+(D^3)$, the group of orientation preserving diffeomorphisms of the (closed) 3-disc D^3, Marques proves that the space of asymptotically flat metrics on $\mathbb{R}^3$ with zero scalar curvature is path connected, and furthermore that the space of asymptotically flat vacuum and maximal solutions to the constraint equations of general relativity on $\mathbb{R}^3$ is path connected in a suitable topology. This improves the result of Smith-Weinstein [SW04], who prove a similar result within a more restrictive class of metrics.

Recently, Carlotto-Li [CL19] studied the space of positive scalar curvature metrics with mean convex boundary, $\mathcal{M}_{R>0,H>0}$, on compact 3-manifolds with boundary.[1] After characterizing the topology of such manifolds, they prove the following fundamental generalization of Theorem 1.1.

Theorem 1.2 (Carlotto-Li [CL19]). *Let X be a compact 3-manifold with boundary that admits a metric with positive scalar curvature and mean convex boundary. Then $\mathcal{M}_{R>0,H>0}/\mathrm{Diff}(X)$ is path connected.*

Their delicate argument proceeds in two stages. They first combine Miao's desingularization with the doubling of Gromov-Lawson to double the manifold across its boundary in a controlled way. They then study the resulting double using a sequence of equivariant Ricci flows. Their result extends to $\mathcal{M}_{R>0,H\geq 0}/\mathrm{Diff}(X)$ and $\mathcal{M}_{R\geq 0,H\geq 0}/\mathrm{Diff}(X)$, though the latter behaves differently when X is diffeomorphic to $S^1 \times S^1 \times I$.

In analogy with Marques' second result concerning $\mathbb{R}^3$, we show that Theorem 1.2 implies the following, where B^3 denotes the open topological 3-ball.

Theorem 1.3. *Let $\mathcal{M}^{k,p,\rho}_{R=0,H=0}$ be the space of asymptotically flat metrics on $\mathbb{R}^3 \backslash B^3$ as defined in Definition 2.1. Then $\mathcal{M}^{k,p,\rho}_{R=0,H=0}$ is path connected in the $W^{k,p}_\rho$ topology.*

Remark 1.4. Thus given an asymptotically flat manifold with $R = 0, H = 0$, it is possible to continuously deform it to the Riemannian Schwarzschild manifold (M_S, g_S), i.e., $M_S = \mathbb{R}^3 \backslash B_1(0)$ with $g_S = (1 + \frac{1}{r})^4 g_E$, whilst maintaining $R = 0, H = 0$. Explicit paths of this kind to g_S have been exploited to obtain geometric inequalities as in Bray's proof of the Riemannian Penrose inequality [Bra01].

To prove Theorem 1.3, we construct a path between two arbitrary metrics in $\mathcal{M}^{k,p,\rho}_{R=0,H=0}$, which we denote by g_{-1} and g_2. We apply Lemma 4.2 to g_{-1} and g_2 separately, to obtain a pair of smooth metrics g_0 and g_1 satisfying a list of desirable properties. The rest of the argument lies in connecting g_0 and g_1. This involves using Lemma 4.3 to endow $S^3 \backslash B^3$ (the compact model of $\mathbb{R}^3 \backslash B^3$) with two metrics $\overline{g}_0, \overline{g}_1$ in such a way that $\overline{g}_0, \overline{g}_1$ can be

[1] H is measured pointing out of X.

joined by a continuous path of Yamabe positive metrics by [Esc92] and Theorem 1.2. To finish the proof we need to invert the path on $S^3 \backslash B^3$ to a path in $\mathcal{M}^{k,p,\rho}_{R=0,H=0}$ on $\mathbb{R}^3 \backslash B^3$. This is done in Lemma 4.5, which explicitly constructs the relevant diffeomorphisms.

Carlotto-Li point out that their result implies a statement like Theorem 1.3 by an argument along the lines of [Mar12]. Although this is true, Marques' argument alone does not strictly speaking yield the desired result for $\mathbb{R}^3 \backslash B^3$. Towards the end of his proof, Marques employs the fact that $\mathrm{Diff}_+(D^3)$ is path connected, which follows from [Cer68]. Running the same argument for $\mathbb{R}^3 \backslash B^3$ eventually leads to one consider a possible path between two elements of $\mathrm{Diff}_\partial(A^3)$, the group of boundary fixing diffeomorphisms of the closed 3-annulus $A^3 \simeq D^3_{r>r'} \backslash B^3_{r'}$. But as we show in Section 3, $\mathrm{Diff}_\partial(A^3)$ is not path-connected and so the argument does not close. We get around this issue by finding a more constructive proof that does not invoke the homotopy type of diffeomorphism groups.

Remark 1.5. One wonders whether the arguments of Theorem 1.3 apply to manifolds other than $\mathbb{R}^3 \backslash B^3$, say $(\mathbb{R}^3 \backslash B^3) \# X'$ for X' closed but not diffeomorphic to S^3, or say with multiple boundary components $\mathbb{R}^3 \backslash \bigcup_{i=1} B^3_i$. Without modification however, the methods herein yield at best a result modulo quotients by the relevant diffeomorphism groups.

Remark 1.6. We note that if the initial asymptotically flat metric is smooth with $R = 0, H = 0$, then the path constructed in the proof of Theorem 1.3 produces a continuous path of smooth metrics.

Theorem 1.3 gives the following.

Corollary 1.7. *Let $\mathcal{M}^{k,p,\rho}_{R\geq 0,H\geq 0}$ be the space of asymptotically flat metrics on $\mathbb{R}^3 \backslash B^3$ as defined in Definition 2.1. Then $\mathcal{M}_{R\geq 0,H\geq 0}$ is path connected in the $W^{k,p}_\rho$ topology.*

As in [Mar12], we also consider the constraint equations of general relativity, which are

$$16\pi\mu = R_g + (\mathrm{Tr}_g \sigma)^2 - |\sigma|^2_g, \tag{1.1}$$

$$8\pi J = \mathrm{div}_g (\sigma - (\mathrm{Tr}_g \sigma) g) \tag{1.2}$$

where g is a metric and σ a symmetric two-form on the manifold. We study the set of *asymptotically flat* pairs (g, σ) on $\mathbb{R}^3 \backslash B^3$ solving (1.1), (1.2) with

$\mu = 0, J = 0$ (vaccum) and $\mathrm{Tr}_g \sigma = 0$ (maximal) along with the boundary condition

$$\theta^+ = -H + \mathrm{Tr}_g(\sigma) - \sigma(\nu, \nu) = 0, \quad H \geq 0 \tag{1.3}$$

where ν is the normal to ∂B^3 pointing away from the asymptotically flat end. In general relativistic terms, we say that ∂M is marginally outer trapped when $\theta^+ = 0$.

Corollary 1.8. *Let $\mathcal{M}_{BH}$ be the space of pairs (g, σ) on $\mathbb{R}^3 \backslash B^3$ defined in Definition 2.1, which satisfy* (1.1), (1.2), (1.3) *with $\mu = 0$, $J = 0$, $Tr_g \sigma = 0$. Then $\mathcal{M}_{BH}$ is path connected in the $W^{k,p}_\rho \oplus W^{k-1,p}_{\rho-1}$ topology.*

Corollary 1.8 is relevant to general relativity because $\mathcal{M}_{BH}$ represents a certain class of initial data sets expected to give rise to a black hole spacetimes. The path connectedness of $\mathcal{M}_{BH}$ can be thought of as a necessary condition for the so-called Final State Conjecture, which states that generic black hole initial data sets asymptote to ones that isometrically embed into the Kerr solution. The conjecture may be studied for initial data sets belonging to $\mathcal{M}_{BH}$. Assuming that the subset $K \subsetneq \mathcal{M}_{BH}$ of pairs (g, σ) that isometrically embed into the Kerr spacetime lie in at most one component of $\mathcal{M}_{BH}$, the disconnectedness of $\mathcal{M}_{BH}$ would imply that certain initial data sets describing black holes are unable to approach K in a continuous way, which would spell trouble for the Final State Conjecture. Corollary 1.8 shows that no such tension arises.

2. Definitions and conventions

B^n is the topological open n-ball (not a metric ball), D^n is the closed topological n-disc, and $A^n = D^n_{r > r'} \backslash B^n_{r'}$ is the closed 3-annulus.

If X is a manifold, $\mathrm{Diff}_+(X)$ is the group of orientation preserving diffeomorphisms of X, and if X has a boundary, $\mathrm{Diff}_\partial(X)$ is the group of boundary fixing diffeomorphisms on X.

Denote by g_E the flat Euclidean metric on $\mathbb{R}^3$. For $x \in \mathbb{R}^3$, let $w(x) = (1 + |x|^2)^{1/2}$. Then for any $\rho \in \mathbb{R}$, any $1 < p < \infty$ and any open set $\Omega \subset \mathbb{R}^3$, the weighted Sobolev space $W^{k,p}_\rho(\Omega)$ is the subset of $W^{k,p}_{loc}(\Omega)$ for which the following norm is finite.

$$||u||_{W^{k,p}_{\rho, g_E}(\Omega)} = \sum_{|\beta| \leq k} ||w^{-\rho - \frac{n}{p} + |\beta|} \partial^\beta u||_{L^p(\Omega)} \tag{2.1}$$

Weighted spaces of continuous functions are defined by the norm

$$||u||_{C^k_{\rho,g_E}(\Omega)} = \sum_{|\alpha|\le k} \sup_{x\in\Omega} w(x)^{-\rho+|\alpha|}|\partial^\alpha u(x)| \tag{2.2}$$

and see [Max05, Lemma 2.1] for the standard embedding theorems in this context.

Let $M = \mathbb{R}^3\backslash B^3$ be Euclidean space minus the unit ball. Besides the Euclidean metric g_E, we suppose that M is equipped with another Riemannian, metric $g \in W^{k,p}_{loc}(M)$ such that (M, g) is complete. Here $1/p - k/n < 0$ so that g is continuous. We call (M, g) asymptotically flat of order $\rho < 0$ if $g - g_E \in W^{k,p}_\rho(M)$.

We denote by $W^{k,p}_\rho(M) = W^{k,p}_{\rho,g}(M)$ the corresponding weighted function spaces associated to the metric g. Analogously, we define the weighted function space $L^p_\rho(M)$ and $C^k_\rho(M)$, and $C^\infty_\rho(M) = \cap_{k=0}^\infty C^k_\rho(M)$. These weighted spaces are also serve to define an asymptotically flat initial data set for the constraint equations. The only additional requirement is that σ, the second fundamental form associated with the initial data set, behaves like a first derivative of $g \in W^{k,p}_\rho$ and thus belongs to $W^{k-1,p}_{\rho-1}(M)$.

A metric g on an asymptotically flat manifold M with boundary ∂M is *conformally flat* outside a compact set if there is a compact set containing ∂M outside of which the metric takes the form $u^4 g_E$, where g_E is the flat metric metric.

Let us now define the main spaces of interest.

Definition 2.1. Let k be an integer ≥ 2, let $-1 < \rho < 0$, and $p > 3/k$. Then

(i) $\mathcal{M}^{k,p,\rho}_{R=0,H=0}$ denotes the set of metrics $g \in W^{k,p}_\rho$ on M satisfying $R_g = 0$, $H_g = 0$,

(ii) $\mathcal{M}^{k,p,\rho}_{R\ge0,H\ge0}$ is defined in the same way except that we enlarge the curvature restriction on g from $H_g = 0$ to $H_g \ge 0$ and from $R_g = 0$ to $R_g \ge 0$,

(iii) $\mathcal{M}_{BH}$ is defined as the set of pairs $(g \in W^{k,p}_\rho, \sigma \in W^{k-1,p}_{\rho-1})$ on M that solve (1.1), (1.2), (1.3) with $\mu = 0, J = 0$, $\mathrm{Tr}_g\sigma = 0$.

3. Marques' argument with a boundary

After proving Theorem 1.1 Marques turns to the case of asymptotically flat metrics on $\mathbb{R}^3$ and proves the path connectedness of three spaces,

$\mathcal{M}_{R=0}$, $\mathcal{M}_{R\geq 0}$, and $\mathcal{M}_V$ where $\mathcal{M}_V$ denotes the class of asymptotically flat pairs (g,k) solving the vacuum, maximal constraint equations; Marques uses weighted Hölder spaces rather than Sobolev, but this distinction is immaterial here. The hardest part of the argument concerns $\mathcal{M}_{R=0}$ and proceeds in four stages.

(1) Based on work of Smith-Weinstein [SW04], Marques constructs a continuous path from any metric in $\mathcal{M}_{R=0}$ to a smooth, harmonically flat metric $g \in \mathcal{M}_{R=0}$.

(2) With g in hand, Marques shows the existence of a diffeomorphism $\phi : \mathbb{R}^3 \to S^3\backslash\{p\}$ such that $\phi_*(g) = G^4\bar{g}$ where $\bar{g}$ is a metric on S^3 with positive Yamabe type and G is the Green's function for the conformal Laplacian $\mathcal{L}_{\bar{g}}$, i.e., a function on $S^3\backslash\{p\}$ solving the distributional equation $\mathcal{L}_{\bar{g}}(G) = -4\pi\delta_p$.

(3) Using (2) and a theorem of Palais [Pal59] on extensions of local diffeomorphisms, Marques constructs a continuous family of diffeomorphisms $\phi_\mu : \mathbb{R}^3 \to S^3\backslash\{p\}$ with properties matching those in (2).

(4) The path connectedness of $\mathcal{M}_{R=0}$ results from combining (2) and (3) with Theorem 1.1. The key step is to show that the diffeomorphisms so far constructed can be composed to produce a diffeomorphism $F : \mathbb{R}^3 \to \mathbb{R}^3$ which can be realized by a continuous path of diffeomorphisms $F_\mu : \mathbb{R}^3 \to \mathbb{R}^3$.

To show (4), Marques relies on a theorem of Cerf [Cer68] which implies that $\mathrm{Diff}_+(D^3)$ is path connected. It then remains to show that a diffeomorphism $F : \mathbb{R}^3 \to \mathbb{R}^3$ which is identity outside a compact set can be realized by a continuous path $\mu \in [0,1]$ of diffeomorphisms $F_\mu : \mathbb{R}^3 \to \mathbb{R}^3$, each identity outside a compact set, such that $F_{\mu=0} = \mathrm{id}$ and $F_{\mu=1} = F$. Since each F_μ gives an orientation preserving diffeomorphism of D^3, the path connectedness of $\mathrm{Diff}_+(D^3)$ guarantees the existence of the desired path F_μ.

So what of Marques' argument for $\mathbb{R}^3\backslash B^3$?

At first sight, one can expect (1), (2), (3) to be suitably generalizable. But with regards to (4), one would seek to show that composing the diffeomorphisms constructed produces a diffeomorphism $F : \mathbb{R}^3\backslash B^3 \to \mathbb{R}^3\backslash B^3$ that is identity outside a compact set and identity within a neighborhood of ∂B^3, which permits preserving the mean curvature condition. To find a continuous path of diffeomorphisms playing the role of F_μ, we thus consider $\mathrm{Diff}_\partial(A^3)$. But at this point we run into the issue that $\mathrm{Diff}_\partial(A^3)$ has two connected components – a fact that can be derived from Hatcher's Theorem [Hat83].[2]

Theorem 3.1 (Hatcher's Theorem). *There is a weak homotopic equivalence* $\mathrm{Diff}(S^3) \simeq O(4)$.

Corollary 3.2. $\mathrm{Diff}_\partial(A^3)$ *has two connected components.*

Proof. By Theorem 3.1 and the facts[3] that there is a weak homotopic equivalence $\mathrm{Diff}(S^3) \simeq O(4) \times \mathrm{Diff}_\partial(D^3)$, and that $\mathrm{Diff}_\partial(D^3) \to \mathrm{Diff}(D^3) \to \mathrm{Diff}(S^3)$ is a fibration, it follows that $\mathrm{Diff}_\partial(D^3)$ is weakly contractible. Now consider the action on the standard embedding $D^3_{r'} \hookrightarrow D^3_r$ by $\mathrm{Diff}_\partial(D^3_r)$. Letting $\mathrm{Emb}_+(D^3_{r'}, D^3_r)$ denote the space of orientation preserving embeddings of $D^3_{r'}$ into $D^3_{r>r'}$, this gives a fibration $\mathrm{Diff}_\partial(D^3_r) \to \mathrm{Emb}_+(D^3_{r'}, D^3_r)$ with fiber of the standard embedding given by $\mathrm{Diff}_\partial(A^3)$. Putting these observations together, the long exact sequence of homotopy groups implies that $\pi_0(\mathrm{Diff}_\partial(A^3)) = \pi_1(\mathrm{Emb}_+(D^3_{r'}, D^3_r))$. Since the derivative at the origin gives a projection[4] $\mathrm{Emb}_+(D^3_{r'}, D^3_r) \to SO(3)$, and since $\pi_1(SO(3)) = \mathbb{Z}\backslash 2\mathbb{Z}$, it follows that $\mathrm{Diff}_\partial(A^3)$ has two connected components. $\square$

In sum, the best case seems to be that Marques' argument yields no more than two connected components.

4. Proof

There are three main steps to the proof of Theorem 1.3. From now on M denotes a manifold diffeomorphic to $\mathbb{R}^3\backslash B^3$.

1) **Deformation**. First we show Lemma 4.2, which gives a continuous path $\in \mathcal{M}^{k,p,\rho}_{R=0,H=0}$ from an arbitrary asymptotically flat metric to a metric which is conformally flat outside a compact set.
2) **Compactification**. Then we show Lemma 4.3, which constructs a diffeomorphism from M, now endowed with a metric (conformally flat outside a compact set) with $R = 0$ and $H = 0$, to $(S^3\backslash B^3)\backslash\{p\}$ such that $S^3\backslash B^3$ admits a metric of positive Yamabe type.

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[2] We thank Sander Kupers for showing us that Corollary 3.2 can be shown without Theorem 3.1.
[3] See for instance [Hat83] or more simply [Kup19, II].
[4] See for instance the proof of [Kup19, Lemma 9.2.3].

3) **Interpolation**. In Lemma 4.5 we explicitly construct a continuous family of diffeomorphisms from M to $S^3 \backslash B^3$ that permits combining steps (1) and (2) whilst invoking Theorem 1.2 and [Esc92].

Steps (1), (2), (3) yield a proof of Theorem 1.3, whilst Corollaries 1.7, 1.8 follow relatively straightforwardly from Theorem 1.3.

4.1. Deformation

Before stating and proving Lemma 4.2, we recall the following result of Maxwell. Let $F_{\alpha,\beta}$ denote the following operator $((\Delta - \alpha)|_M, (\partial_\nu + \beta)|_{\partial M}))$, and let n be the dimension of M (3 in our case).

Proposition 4.1 (Maxwell [Max05]). *Let (M, g) be asymptotically flat of class $W^{k,p}_\rho$, $k \geq 2$, $k > n/p$, and suppose $\alpha \in W^{k-2,p}_{\rho-2}$ and $\beta \in W^{k-1-\frac{1}{p},p}$. Then if $2 - n < \rho < 0$ the operator $F_{\alpha,\beta} : W^{k,p}_\rho \to W^{k-2,p}_{\rho-2}(M) \times W^{k-1-\frac{1}{p},p}(\partial M)$ is Fredholm with index 0. Moreover if $\alpha, \beta \geq 0$ then F is an isomorphism.*

We now prove the following.

Lemma 4.2. *Let $g_{-1} \in \mathcal{M}^{k,p,\rho}_{R=0,H=0}$. There is a path $\mu \in [-1, 0] \to g_\mu \in \mathcal{M}^{k,p,\rho}_{R=0,H=0}$ such that g_0 is smooth, conformally flat outside a compact set, minimal boundary, and moreover this path is continuous in the $W^{k,p}_\rho$ topology.*

Proof. Let η be a smooth cut-off function $0 \leq \eta \leq 1$ such that $\eta(t) = 1$ for $t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$. Pick R_0 such that metric ball $B_{R_0}(0)$ contains ∂M. Given the metric $g_{-1} \in \mathcal{M}^{k,p,\rho}_{R=0,H=0}$, set $\eta_R(t) = \eta(t/R)$ for $R > R_0$, and define the new metric

$$g_R = (1 - \eta_R)g_E + \eta_R g_{-1} \tag{4.1}$$

We can now approximate g_R with a smooth g'_R such that $||g_R - g'_R||_{W^{k,p}_\rho}$ is arbitrarily small.

For $\mu \in [-1, 0]$, we define $g_{R,\mu} = (1 + \mu)g'_R - \mu g_{-1}$.

We now observe that by Proposition 4.1, $F_{\frac{1}{8}R_{g_{-1}}, \frac{1}{4}H_{g_{-1}}}$ is an isomorphism, and that consequently, for $||\hat{g} - g_{-1}||_{W^{k,p}_\rho}$ sufficiently small, $F_{\frac{1}{8}R_{\hat{g}}, \frac{1}{4}H_{\hat{g}}}$ is also an isomorphism. So choosing $||g'_R - g_R||$ sufficiently small and R large enough, $F_{\frac{1}{8}R_{g_{R,\mu}}, \frac{1}{4}H_{g_{R,\mu}}}$ is an isomorphism.

We may now consider the unique solution $v_{R,\mu}$ to $F_{\frac{1}{8}R_{g_{R,\mu}},\frac{1}{4}H_{g_{R,\mu}}} = (\frac{1}{8}R_{g_{R,\mu}}, -\frac{1}{4}H_{g_{R,\mu}})$.

Observe that $\Delta v_{R,\mu} = \frac{1}{8}R_{g_{R,\mu}}(1+v_{R_\mu}) = \frac{1}{8}R_{g_{R,\mu}}u$ where $u := 1+v_{R,\mu}$ and that $\partial_\nu v_{R,\mu} = -\frac{1}{4}H_{g_{R,\mu}}(1+v_{R,\mu}) = -\frac{1}{4}H_{g_{R,\mu}}u$. The path in Lemma 4.2 now follows by setting $g_\mu = u^4 g_{R,\mu}$. □

4.2. Compactification

Lemma 4.3. *Let g be a smooth, asymptotically flat, conformally flat metric outside a compact set with $g = u^4 g_E$, on $M \simeq \mathbb{R}^3\backslash B^3$ with $R = 0$ and $H = 0$. Denote by $\overline{M}$ a manifold diffeomorphic $S^3\backslash B^3$ and p a point in the interior of $\overline{M}$. Then there exists a diffeomorphism $\phi : M \to \overline{M}\backslash\{p\}$ and a smooth function $v : M \to \mathbb{R}$ such that*

- *outside a large ball in M we have $\phi = \exp_{\overline{g},p} \circ \mathrm{Inv}$ where $\mathrm{Inv}(x) := \frac{x}{|x|^2}$ is the inversion map,*
- *on $\overline{M}\backslash\{p\}$ the metric $\overline{g} := \phi_*(v^4 g)$ extends smoothly to $\{p\}$,*
- *the mean curvature of $\partial\overline{M}$ satisfies $\overline{H} = 0$,*
- *$(\overline{M}, \overline{g})$ has positive Yamabe type.*

Remark 4.4. This is the boundary version of [Mar12, Theorem 9.3], which shows that an asymptotically flat manifold with positive scalar curvature gives rise to a Yamabe positive metric on S^3.

Proof. Let $\phi : \mathbb{R}^3 \to S^3\backslash\{p\}$ be the inverse stereographic projection which we restrict to M. We define $v : M \to \mathbb{R}$ by

$$\begin{cases} v(x) := \frac{1}{u(x)|x|} & \text{for } x \text{ near } \infty, \\ v(x) := 1 & \text{for } x \text{ in a neighborhood of } \Sigma. \end{cases} \tag{4.2}$$

where Σ denotes the boundary of $\mathbb{R}^3\backslash B^3$. Clearly we can find $v(x)$ which smoothly interpolates between these two regions.

By construction $\overline{g}$ can be extended smoothly from $\overline{M}\backslash\{p\}$ to $\overline{M}$ and we have $\overline{H} = H$ due to $v(x) = 1$ in a neighborhood of Σ.

Moreover, it is easy to see that ϕ can be constructed such that $\phi = \exp \circ \mathrm{Inv}$ outside a large ball.

Thus it remains to show that $(\overline{M}, \overline{g})$ has positive Yamabe type.

For this, we begin with showing that $G := \phi_*(\frac{1}{v})$ is a conformal Green's function on $(\overline{M}, \overline{g})$, that is

$$\mathcal{L}_{\overline{g}} G = -4\pi\delta_p \qquad \text{and} \qquad \partial_\nu G = -\frac{\bar{H}}{4} G = 0. \tag{4.3}$$

Here $\mathcal{L}$ is again the conformal Laplacian $\mathcal{L}G = \Delta G - \frac{1}{8} R_{\overline{g}} G$. Observe that $\partial_\nu u = 0$ and $\bar{H} = 0$, so the second condition of being a conformal Green's function is satisfied. Since the formula for scalar curvature under conformal transformation yields

$$0 = R = -G^5 \mathcal{L}_{\overline{g}} G, \tag{4.4}$$

we obtain $\mathcal{L}_{\overline{g}} G = 0$ outside $\{p\}$. Near $\{p\}$ we have by construction $G(y) = \frac{1}{|y|} + \mathcal{O}(1)$ where $|y|$ denotes the distance from p to y with respect to $\overline{g}$.

To show (4.3), we first let f be a smooth test function on $\overline{M}$. In that case, there exists, for every $\epsilon > 0$, a $\delta \in (0, \epsilon]$ such that $|f(x) - f(p)| \leq \epsilon$ for $x \in B_\delta(\infty)$. Thus, we have

$$\int_{B_\delta(p)} f\Delta G = (f(p) + \mathcal{O}(\epsilon)) \int_{S_\delta(p)} \partial_\nu G \tag{4.5}$$

$$= (f(p) + \mathcal{O}(\epsilon))(-4\pi + \mathcal{O}(\delta^2)) \tag{4.6}$$

$$= -4\pi f(p) + \mathcal{O}(\epsilon) \tag{4.7}$$

which shows (4.3).

Escobar showed in [Esc92] that for some $c \in \mathbb{R}$ there exists a solution to the Yamabe equation with boundary, i.e. there is a $\zeta : \overline{M} \to \mathbb{R}^+$ such that

$$\begin{cases} \frac{R}{8}\zeta - \Delta\zeta = c\zeta^5 & \text{for } x \in \overline{M}, \\ \partial_\nu \zeta = -\frac{H}{4}\phi & \text{for } x \in \overline{\Sigma}. \end{cases} \tag{4.8}$$

The Yamabe type is then characterized by the sign of c. Thus, to show that $\overline{M}$ has positive Yamabe type it suffices to show that $c > 0$. We compute

$$c \int_{\overline{M}} G\zeta^5 = \int_{\overline{M}} (-G\Delta\zeta + 8GR\zeta) \tag{4.9}$$

$$= -\int_{\overline{\Sigma}} G\partial_\nu \zeta + \int_{\overline{M}} \left(\langle \nabla\zeta, \nabla G \rangle + \frac{1}{8} GR\zeta \right) \tag{4.10}$$

$$= \int_{\overline{\Sigma}} \frac{1}{4} \bar{H} G \zeta - \int_{\overline{M}} \zeta \mathcal{L}_g G + \int_{\Sigma} \zeta \partial_\nu G \tag{4.11}$$

$$= 4\pi\zeta(p) > 0. \tag{4.12}$$

This finishes the proof. □

4.3. Interpolation

It remains to show that the path $\mu \in [0,1] \to \overline{g}_\mu$ gives rise to a continuous path from g_0 to g_1 in $\mathcal{M}^{k,p,\rho}_{R=0,H=0}$. To do this, one needs to 'invert' Lemma 4.3 in a suitable way. This will be done using the usual blow up of $\overline{g}_\mu$ by its conformal Green's function G_μ. This blow up gives a new metric on $\overline{M}\backslash\{p\}$, which in turn can be pulled back onto M by a continuous family of diffeomorphisms $\phi_\mu : M \to \overline{M}\backslash\{p\}$. The challenge is now to construct $\{\phi_\mu\}$.

Lemma 4.5. *Let $\overline{g}_0$ and $\overline{g}_1$ be two metrics coming from Lemma 4.3, and let $\overline{g}_\mu$ be a C^∞-continuous path between these metrics. Moreover, let $G_\mu : \overline{M} \to \mathbb{R}$ be a continuous family of functions which satisfy in a small neighborhood of p*

$$G_\mu(y) = \frac{1}{|y|_{\overline{g}_\mu}} + A_\mu + \mathcal{O}_k(|y|_{\overline{g}_\mu}) \tag{4.13}$$

where $|y|_{\overline{g}_\mu}$ denotes the distance of y to p with respect to $\overline{g}_\mu$. Then there exists a continuous path of diffeomorphisms ϕ_μ such that $\mu \in [0,1] \to g_\mu \equiv \phi_\mu^*(G_\mu^4 \overline{g}_\mu)$ is a continuous path of asymptotically flat metrics $g_\mu \in C^\infty(M) \cap W^{k,p}_\rho(M)$ on M.

Proof. Let $\phi : M \to \overline{M}\backslash\{p\}$ be the diffeomorphism of Lemma 4.3. We choose ϵ so small such that

- $\mathbb{R}^3 \backslash B_{\frac{1}{\epsilon}}(0)$ is in the conformally flat regime of g_0 and g_1 so that $\overline{g}_0 = \overline{g}_1 = \psi_* g_E$ in a small neighborhood around $\{p\}$,
- $\phi = \psi \circ \mathrm{Inv}$ outside $B_{\frac{1}{\epsilon}}(0)$ where $\psi = \exp_{\overline{g}_1,p} = \exp_{\overline{g}_0,p} = \exp_{g_E,p}$ in a small neighborhood around $\{p\}$,
- $(\mathbb{R}^3 \backslash B_{\frac{1}{\epsilon}}(0)) \cap \Sigma = \emptyset$.

Our continuous family of diffeomorphisms ϕ_μ can now be defined as follows

$$\phi_\mu = \begin{cases} \phi & \text{inside } B_{\frac{1}{\epsilon}}(0) \\ \psi_\mu \circ \mathrm{Inv} & \text{outside } B_{\frac{1}{\epsilon}}(0) \end{cases} \tag{4.14}$$

where $\psi_\mu : B_\epsilon(0) \hookrightarrow S^3$ is a continuous path of maps, each diffeomorphic onto their image, such that $\psi_0 = \psi_1 = \psi$ on $B_\epsilon(0)$. The task is now to show that $g_\mu = \phi_\mu^*(G_\mu^4 \overline{g}_\mu)$ is an asymptotically flat metric on M. To show this we must construct ψ_μ suitably. To do so we set $\psi_\mu \equiv \psi \circ f_\mu$ where $f_\mu : B_\epsilon(0) \to B_\epsilon(0)$ is a diffeomorphism onto its image with $f_0 = f_1 = \mathrm{Id}$. It now remains to specify f_μ.

On $T_pS^3 = \mathbb{R}^3$ we have both the standard metric $\psi_* g_E$ and the metric $\overline{g}_\mu(p)$. We choose an orthonormal basis $\{e_1, e_2, e_3\}_\mu$ for g_E and rotate it such a way that $\overline{g}_\mu(p)$ is diagonal, i.e. $\overline{g}_\mu(p) = \mathrm{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2)_\mu$ for some numbers $\lambda_i > 0$. Note that the choice of orthonormal basis is continuous with respect to μ. Although $\{e_1, e_2, e_3\}_\mu$ and $(\lambda_1, \lambda_2, \lambda_3)_\mu$ depend on μ, we hereby suppress this to declutter the notation.

We now define $f_\mu : B_\epsilon(0) \to B_\epsilon(0)$

$$f_\mu(x_1, x_2, x_3) = \left(\frac{x_1}{\hat{\lambda}_1(r)}, \frac{x_2}{\hat{\lambda}_2(r)}, \frac{x_3}{\hat{\lambda}_3(r)} \right) \tag{4.15}$$

and where $\hat{\lambda}_j$, $j = 1, 2, 3$, are smooth, μ-dependent and satisfy

$$\hat{\lambda}_j(r) = \lambda_j \tag{4.16}$$

for $r = \sqrt{x_1^2 + x_2^2 + x_3^2} \le \xi$ where $0 < \xi < \epsilon$ is chosen below. Moreover, we impose that

$$\lambda_j(r) = 1 \tag{4.17}$$

for $\frac{\epsilon}{2} \le x \le \epsilon$ and that $\frac{r}{\hat{\lambda}_j(r)}$ is strictly monotone increasing. We can do this by choosing $\frac{\xi}{2 \min\{\lambda_1, \lambda_2, \lambda_3\}} < \frac{\epsilon}{2}$. Since $[0, 1]$ is compact, we can choose ξ uniform in μ. Moreover, $f_\mu : B_\epsilon(0) \to B_\epsilon(0)$ is a diffeomorphism that depends continuously on μ, and ψ_μ is a diffeomorphism onto its image.

It now remains to show that the resulting metric g_μ is asymptotically flat. Note that this follows not immediately since $\psi_\mu(x) \neq \exp_{\overline{g}_\mu, p}(x)$.

For y in a neighborhood of p, define a metric $\hat{g}$ by $\hat{g}(y) = \psi_*(\overline{g}_\mu(p))$, where we have interpreted ψ as map $\psi : B_\rho(0) \subset T_pS^3 \to S^3$ for some sufficiently small ρ.

We now compare $\overline{g}_\mu$ and $\hat{g}$. For this purpose we take our previous orthogonal basis with respect to $\overline{g}_\mu(p)$, and orthonormal with respect to g_E of T_pS^3 and scale it to be orthonormal with respect to $\overline{g}_\mu(p)$; that is, we set $\overline{e}_i = \frac{1}{\lambda_i} e_i$, $i = 1, 2, 3$, and exponentiate it onto S^3 to obtain normal coordinates on S^3. By construction we have in these coordinate $\hat{g}_{ij} = \delta_{ij}$ and the normal coordinate formula yields

$$(\overline{g}_\mu)_{ij}(y) = \delta_{ij} + \mathcal{O}_2(|y|^2_{\overline{g}_\mu}) \tag{4.18}$$

in a small neighborhood around $\{p\}$. Here the $\mathcal{O}_2$ denotes the higher order estimates

$$\partial_k(\overline{g}_\mu)_{ij} = \mathcal{O}(|y|), \quad \partial_k\partial_l(\overline{g}_\mu)_{ij} = \mathcal{O}(1). \tag{4.19}$$

Furthermore, $\psi_\mu = \exp_{\hat{g}_\mu,p}$ in a small neighborhood of the origin. Next, we pull back the normal coordinates under Df_μ to obtain an orthonormal basis in $T_y\mathbb{R}^3$ for $|y|_{\overline{g}_\mu}$ very small. After scaling and inverting, this gives then an orthonormal basis for $T_x\mathbb{R}^3$, $|x|$ very large, which we denote by $\{\tilde{e}_i\}$. We compute

$$g_\mu(\tilde{e}_i, \tilde{e}_j) =((\psi \circ f_\mu \circ \mathrm{Inv})^*(G_\mu^4 \overline{g}_\mu))(x)(\tilde{e}_i, \tilde{e}_j) \tag{4.20}$$

$$=\frac{1}{|x|^4_{g_\mu}} \left(G_\mu\left(\psi \circ f_\mu\left(\frac{x}{|x|^2_{g_\mu}}\right)\right)\right)^4 \overline{g}_\mu\left(\frac{x}{|x|^2_{g_\mu}}\right)(\overline{e}_i, \overline{e}_j) \tag{4.21}$$

Our assumptions on G together with equation (4.18) yield in particular

$$G_\mu(y) = \frac{1}{|y|_{\hat{g}_\mu}} + \mathcal{O}(1) \tag{4.22}$$

and

$$|\nabla G_\mu(y)|_{\overline{g}_\mu} = \mathcal{O}(|y|^{-2}_{\hat{g}_\mu}), \quad |\nabla^2 G_\mu(y)|_{\overline{g}_\mu} = \mathcal{O}(|y|^{-3}_{\hat{g}_\mu}). \tag{4.23}$$

Combining this with the equation $|\psi \circ f_\mu(\frac{x}{|x|^2_{g_\mu}})|_{\hat{g}_\mu} = \frac{1}{|x|_{g_\mu}}$ implies

$$\frac{1}{|x|^4_{g_\mu}} \left(G_\mu\left(\psi \circ f_\mu\left(\frac{x}{|x|^2_{g_\mu}}\right)\right)\right)^4 = 1 + \mathcal{O}_2(|x|^{-1}_{g_\mu}) \tag{4.24}$$

and thus

$$g_\mu(\tilde{e}_i, \tilde{e}_j) = (1 + \mathcal{O}_2(|x|^{-1}_{g_\mu}))(\delta_{ij} + \mathcal{O}_2(|x|^{-2}_{g_\mu})) = \delta_{ij} + \mathcal{O}_2(|x|^{-1}_{g_\mu}), \tag{4.25}$$

i.e. $g_\mu \in C^\infty(M) \cap C^2_{-1}(M)$ is asymptotically flat, which in turn implies $g_\mu \in W^{2,p}_\rho(M)$. Higher k proceeds identically, and thus $g_\mu \in W^{k,p}_\rho(M)$. $\square$

4.4. Proof of main theorems

Proof of Theorem 1.3. Take any two metrics g_{-1} and g_2 in $\mathcal{M}^{k,p,\rho}_{R=0,H=0}$ on M. Our goal is to find a path g_μ, $\mu \in [-1,2]$ in $\mathcal{M}^{k,p,\rho}_{R=0,H=0}$ connecting g_{-1} and g_2.

Lemma 4.2 gives us a continuous path connecting g_{-1} to g_0 and g_2 to g_1 respectively, with g_0, g_1 smooth, $R = 0$, $H = 0$, conformally flat outside a compact set containing ∂M.

Lemma 4.3 gives us a diffeomorphism from (M, g_0) and (M, g_1) onto $(S^3 \backslash B^3) \backslash \{p\}$ that produces the manifold $(S^3 \backslash B^3, \overline{g}_0)$ and $(S^3 \backslash B^3, \overline{g}_1)$ where $\overline{g}_0$ and $\overline{g}_1$ are of positive Yamabe type with minimal boundary.

Escobar's work on the Yamabe problem with boundary [Esc92] guarantees that we can find a C^∞-continuous path $\overline{g}_\mu$, $\mu \in [0, 1/3]$ within the conformal class of $\overline{g}_0$ to a metric with positive scalar curvature metric and minimal boundary. Equally, we have a path from $\overline{g}_1$ to $\overline{g}_{2/3}$ with $\overline{g}_{2/3}$ having positive scalar curvature and minimal boundary.

Theorem 1.2 of Carlotto-Li gives us a path $\overline{g}_\mu$, $\mu \in [1/3, 2/3]$ between these metrics. Next, we find as in [Esc92] conformal Green's functions G_μ, $\mu \in [0,1]$, i.e., functions satisfying $\partial_\nu G_\mu - \frac{1}{4} H_{\overline{g}_\mu} G_\mu = 0$ and $\Delta_{\overline{g}_\mu} G_\mu + \frac{1}{8} R_{\overline{g}_\mu} G_\mu = 4\pi\delta_p$ where δ_p is the Dirac δ function. This places us in the setting of Lemma 4.5, which allows us to lift the path g_μ, $\mu \in [0,1]$ to a path g_μ of asymptotically flat metrics of zero scalar curvature and minimal boundary. Hence, we have constructed a continuous path g_μ between g_{-1} and g_2. □

Proof of Corollary 1.7. Consider the following PDE

$$\begin{cases} \Delta\phi - \frac{1}{8} R\phi = 0 & \text{on } \mathbb{R}^3 \setminus B^3, \\ \partial_\nu \phi + \frac{1}{4} H\phi = 0 & \text{on } \partial B^3 \end{cases} \tag{4.26}$$

where $\phi \to 1$ at ∞. As in the proof of Lemma 4.2, we can use Proposition 4.1 to solve this equation. Thus we can conformally deform both the scalar curvature and mean curvature to zero. This places us in the setting of Theorem 1.3, which finishes the proof. □

For Corollary 1.8, we start with the relevant space of interest, $\mathcal{M}_{BH}$, defined to be the set of all doubles (g, σ) satisfying the vacuum, maximal

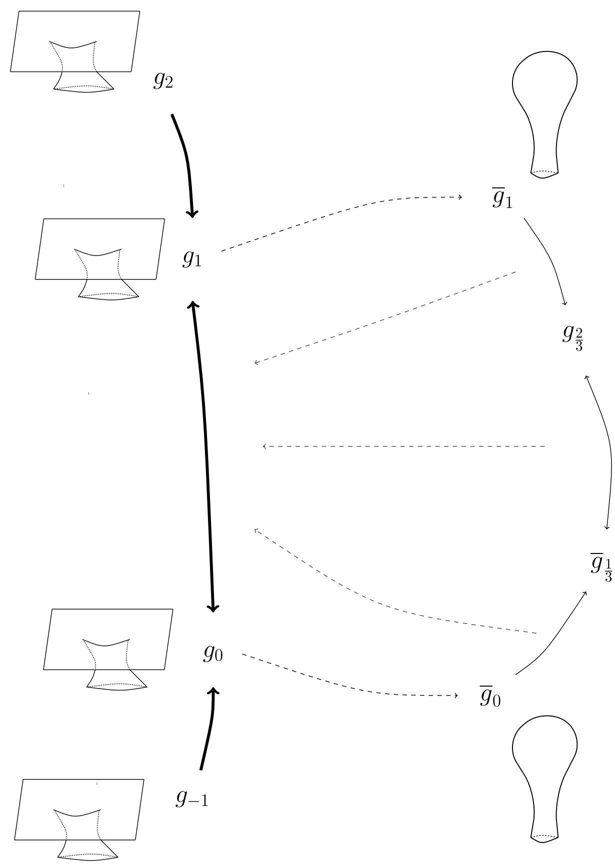

Figure 1: Lemma 4.2 gives $g_2 \to g_1$ and $g_{-1} \to g_0$, and Lemma 4.3 gives $g_1 \to \overline{g}_1$ and $g_0 \to \overline{g}_0$. The path from $\overline{g}_1$ to $\overline{g}_0$ follows from [Esc92] and Theorem 1.2, and finally Lemma 4.5 gives the blue dashed line which permits lifting the path of metrics on $S^3 \backslash B^3$ to a path of metrics on $M = \mathbb{R}^3 \backslash B^3$.

constraint equations

$$(4.27) \qquad R = |\sigma|_g^2, \qquad \operatorname{tr}_g \sigma = 0, \qquad \nabla_i \sigma^i_j = 0$$

such that $g \in W^{k,p}_\rho$ is asymptotically flat, $\sigma \in W^{k-1,p}_{\rho-1}(M)$, and the surface $\Sigma \equiv \partial M$ satisfies $H + \sigma(\nu,\nu) \leq 0$ and $H \geq 0$ where ν is the normal to ∂B^3 pointing away from the asymptotically flat end..

Proof of Corollary 1.8. Our task is to show that $\mathcal{M}_{BH}$ is path connected in $W^{k,p}_\rho(M) \oplus W^{k-1,p}_{\rho-1}(M)$. To do so we consider the larger set $\tilde{\mathcal{M}}_{BH}$ given by replacing $R = |\sigma|_g^2$ with $R \geq |\sigma|_g^2$ and $H \geq -\sigma(\nu,\nu) \geq 0$. Consider now the deformation

$$(g,\sigma) \to (g,(1-\mu)\sigma), \quad \mu \in [0,1] \tag{4.28}$$

Since $H \geq -\sigma(\nu,\nu) \geq 0$ and $R \geq |\sigma|_g^2$ this deformation take place in $\tilde{\mathcal{M}}_{BH}$, from which it follows by Corollary 1.7 that $\tilde{\mathcal{M}}_{BH}$ is path connected. We consider the conformal deformations $\hat{g} = u^4 g$ and $\hat{\sigma} = u^{-2}\sigma$, which in turn implies $|\hat{\sigma}|_{\hat{g}}^2 = u^{-12}|\sigma|_g^2$. We would like to preserve the mean curvature condition $H \geq -\sigma(\nu,\nu) \geq 0$ and the scalar curvature condition $R = |\sigma|_g^2$ under this conformal deformations. Recalling the formula for the change in mean curvature

$$\hat{H} = u^{-2}H + 4u^{-3}\partial_\nu u \tag{4.29}$$

and scalar curvature

$$\hat{R} = u^{-4}R - 8u^{-5}\Delta u \tag{4.30}$$

we are led to the Lichnerowicz equation

$$\begin{cases} \Delta u - \frac{1}{8}Ru + \frac{1}{8}|\sigma|^2 u^{-7} = 0 & \text{on } \mathbb{R}^3 \setminus B^3, \\ \partial_\nu u + \frac{1}{4}Hu - \frac{1}{4}\sigma(\nu,\nu)u^{-3} = 0 & \text{on } \partial B^3 \end{cases} \tag{4.31}$$

where $u \to 1$ at ∞.

To solve this equation we use the method of sub and super solutions from [Max05, Proposition 3.5]. We note that $u = 1$ is a supersolution. Next, as in the proof of Lemma 4.2, we solve the equation

$$\begin{cases} -\Delta\psi + \frac{1}{8}R\psi = -\frac{1}{8}R, \\ \partial_\nu\psi + \frac{1}{4}H\psi = -\frac{1}{4}H \end{cases} \tag{4.32}$$

where $\psi \to 0$ at ∞ and $\psi \in W_\rho^{k,p}$. Denoting $\underline{u} = 1 + \psi$ we have $0 < \underline{u} \leq 1$ and

$$-\Delta\underline{u} + \frac{1}{8}R\underline{u} - \frac{1}{8}|\sigma|^2\underline{u}^{-7} = -\frac{1}{8}|\sigma|^2\underline{u}^{-7} \leq 0. \tag{4.33}$$

Moreover,

$$\partial_\nu\psi + \frac{1}{4}H(1+\psi) - \frac{1}{4}H(1+\psi)^{-3} = -\frac{1}{4}H(1+\psi)^{-3} \leq 0. \tag{4.34}$$

and so $\underline{u}$ is a subsolution, and thus there exists a solution $u \in 1 + W^{k,p}_\rho$ with $0 < u \leq 1$.

Uniqueness now follows in a standard way. Namely, if u_1 and u_2 are solutions, then by the Lichnerowicz equation, $u_1 - u_2$ satisfies

(4.35)
$$\left(\left(\Delta - \frac{1}{8} \left(R - |\sigma|^2 \frac{u_1^{-7} - u_2^{-7}}{u_1 - u_2} \right) \right), \left(\partial_\nu + \frac{1}{4} H \left(1 - \frac{u_1^{-3} - u_2^{-3}}{u_1 - u_2} \right) \right) \right) \times (u_1 - u_2) = 0$$

By the isomorphism of Proposition 4.1, it follows that $u_1 = u_2$. □

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