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A characterization of a hyperplane in two-phase heat conductors

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We consider the Cauchy problem for the heat diffusion equation in the whole Euclidean space consisting of two media with different constant conductivities, where initially one has temperature 0 and the other has temperature 1. Suppose that the interface is connected and uniformly of class C^6. We show that if the interface has a time-invariant constant temperature, then it must be a hyperplane.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a domain with $N \geq 2$. Suppose that $\partial\Omega \neq \emptyset$ and $\partial\Omega$ is connected. Denote by $\sigma = \sigma(x)$ $(x \in \mathbb{R}^N)$ the conductivity distribution of the whole medium given by

$$\sigma = \begin{cases} \sigma_s & \text{in } \Omega, \\ \sigma_m & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where σ_s, σ_m are positive constants with $\sigma_s \neq \sigma_m$.

Let $u = u(x,t)$ be the unique bounded solution of the Cauchy problem for the heat diffusion equation:

$$u_t = \operatorname{div}(\sigma \nabla u) \quad \text{in } \mathbb{R}^N \times (0,+\infty) \ \text{ and } \ u = \mathcal{X}_{\mathbb{R}^N\setminus\Omega} \ \text{ on } \mathbb{R}^N \times \{0\}, \tag{1.2}$$

where $\mathcal{X}_{\mathbb{R}^N\setminus\Omega}$ denotes the characteristic function of the set $\mathbb{R}^N \setminus \Omega$.

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When $\partial\Omega$ is in particular a hyperplane, for instance,

$$\begin{aligned}\Omega &= \{x=(x_1,\dots,x_N)\in\mathbb{R}^N : x_1>0\} \text{ and}\\ \partial\Omega &= \{x=(x_1,\dots,x_N)\in\mathbb{R}^N : x_1=0\},\end{aligned}$$

then we observe that

$$u(x,t)=\frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}} \quad \text{for every } (x,t)\in\partial\Omega\times(0,+\infty). \tag{1.3}$$

Indeed, the uniqueness of the solution of problem (1.2) yields that the solution u does not depend on the variables $x_2,\dots,x_N$. The heat kernel for $N=1$ is explicitly given by [GOO, p. 478]. Denote by $G(x_1,y_1,t)$ the heat kernel written as

$$\begin{aligned}G(x_1,y_1,t) &= \left\{E_-(x_1-y_1,t)+\frac{\sqrt{\sigma_m}-\sqrt{\sigma_s}}{\sqrt{\sigma_m}+\sqrt{\sigma_s}}E_-(x_1+y_1,t)\right\}\mathcal{X}_{\{x_1\le 0,y_1\le 0\}}\\
&+ \frac{2\sqrt{\sigma_m}}{\sqrt{\sigma_m}+\sqrt{\sigma_s}}E_-\left(x_1-\frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}}y_1,t\right)\mathcal{X}_{\{x_1\le 0,y_1>0\}}\\
&+ \left\{E_+(x_1-y_1,t)+\frac{\sqrt{\sigma_s}-\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}}E_+(x_1+y_1,t)\right\}\mathcal{X}_{\{x_1>0,y_1>0\}}\\
&+ \frac{2\sqrt{\sigma_s}}{\sqrt{\sigma_m}+\sqrt{\sigma_s}}E_+\left(x_1-\frac{\sqrt{\sigma_s}}{\sqrt{\sigma_m}}y_1,t\right)\mathcal{X}_{\{x_1>0,y_1\le 0\}},\end{aligned}$$

where $E_\pm(z,t)$ are the Gaussian kernels with conductivities σ_s,σ_m respectively on $\mathbb{R}$ given by

$$E_+(z,t)=(4\pi t\sigma_s)^{-\frac{1}{2}}\exp\left(-\frac{z^2}{4t\sigma_s}\right) \quad\text{and}\quad E_-(z,t)=(4\pi t\sigma_m)^{-\frac{1}{2}}\exp\left(-\frac{z^2}{4t\sigma_m}\right)$$

and each $\mathcal{X}_{\{\cdot\}}$ denotes the characteristic function of the set $\{\cdot\}$. Then the value of u on $\partial\Omega\times(0,+\infty)$ is explicitly given by

$$\begin{aligned}u(0,x_2,\dots,x_N,t) &= \int_{-\infty}^0 G(0,y_1,t)\,dy_1\\
&= \int_{-\infty}^0\left\{E_-(-y_1,t)+\frac{\sqrt{\sigma_m}-\sqrt{\sigma_s}}{\sqrt{\sigma_m}+\sqrt{\sigma_s}}E_-(y_1,t)\right\}dy_1\\
&= \frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}}.\end{aligned}$$

The main purpose of the present paper is to show that the converse also holds true.

Theorem 1.1. *Let u be the solution of problem* (1.2)*. Suppose that $\partial\Omega$ is uniformly of class C^6. If there exists a constant k satisfying*

$$u(x,t) = k \quad \text{for every } (x,t) \in \partial\Omega \times (0,+\infty), \tag{1.4}$$

then $\partial\Omega$ must be a straight line when $N = 2$ and it must be a hyperplane when $N \geq 3$.

Here $\partial\Omega$ is said to be uniformly of class C^6 if each point of $\partial\Omega$ is equipped with the N-dimensional ball of a fixed radius centered at the point in which $\partial\Omega$ is represented as the graph of a function whose C^6 norm is less than a fixed number. We note that if the solution u of problem (1.2) satisfies (1.4) for a constant k, then k must equal $\frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}}$, which is the same as in (1.3), by Proposition 2.2 in section 2.

We mention a remark on the case where $\sigma_s = \sigma_m$. If $\sigma_s = \sigma_m$ and $N \geq 3$, then Theorem 1.1 does not hold. A counterexample is given in [MPS, p. 4824]. Indeed, let $\mathcal{H}$ be a helicoid in $\mathbb{R}^3$. When $\partial\Omega = \mathcal{H} \times \mathbb{R}^{N-3}$ ($\partial\Omega = \mathcal{H}$ for $N = 3$), by the symmetry of $\mathcal{H}$ the solution u satisfies

$$u = \frac{1}{2} \quad \text{on } \partial\Omega \times (0,+\infty). \tag{1.5}$$

For convenience, we give a proof of this fact in subsection A of the Appendices. Moreover, when $\sigma_s = \sigma_m$, without loss of generality when $\sigma_s = \sigma_m = 1$, by using the results of [MPS, N] together with the explicit representation of the solution via Gaussian kernel, we have

Theorem 1.2. *Let u be the unique bounded solution of the following Cauchy problem for the heat equation:*

$$u_t = \Delta u \quad \text{in } \mathbb{R}^N \times (0,+\infty) \quad \text{and} \quad u = \mathcal{X}_{\mathbb{R}^N \setminus \Omega} \quad \text{on } \mathbb{R}^N \times \{0\}. \tag{1.6}$$

Suppose that $\partial\Omega$ is of class C^0. If there exists a constant k satisfying (1.4)*, then $\partial\Omega$ must be a straight line when $N = 2$, it must be either a hyperplane or a helicoid when $N = 3$, and it must be a minimal hypersurface when $N \geq 4$.*

Here $\partial\Omega$ is said to be of class C^0 if each point of $\partial\Omega$ has a neighborhood in $\mathbb{R}^N$ in which $\partial\Omega$ is represented as the graph of a continuous function.

The proof of Theorem 1.1 consists of two steps. In the first step, we show that the mean curvature of $\partial\Omega$ must vanish with the aid of the barriers for the Laplace-Stieltjes transform of the solution. These barriers are constructed in [CMS, S] under the assumption that $\partial\Omega$ is uniformly of class C^6. Hence, with the aid of the interior estimates for solutions of the minimal surface equation we notice that $\partial\Omega$ is uniformly of class C^ℓ for every $\ell \in \mathbb{N}$. This fact enables us to construct more precise barriers in view of the formal WKB approximation for the Laplace-Stieltjes transform of the solution. The second step is devoted to proving that all the elementary functions of the principal curvatures of $\partial\Omega$ must vanish with the aid of the more precise barriers. Note that we use the fact that $\sigma_s \neq \sigma_m$ only in the second step, that is, even if $\sigma_s = \sigma_m$, we can prove that the mean curvature of $\partial\Omega$ must vanish.

The following sections are organized as follows. In section 2, we quote a lemma from [CMS] and a proposition from [S]. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2 respectively. We also added two Appendices at the end. In subsection A we show how (1.5) follows from the symmetry properites of the helicoid, while in subsection B, we quote a maximum principle for elliptic equations with discontinuous conductivities from [S] and give its proof.

2. Preliminaries

Let us introduce the distance function $\delta = \delta(x)$ of $x \in \mathbb{R}^N$ to $\partial\Omega$ by

$$\delta(x) = \operatorname{dist}(x, \partial\Omega) \quad \text{for} \quad x \in \mathbb{R}^N. \tag{2.1}$$

We quote a lemma concerning the solutions of problem (1.2) from [CMS, Lemma 4.1], which simply comes from the maximum principle and the Gaussian bounds for the fundamental solution of $u_t = \operatorname{div}(\sigma\nabla u)$ due to Aronson [A, Theorem 1, p. 891] (see also [FS, p. 328]). Although [CMS, Lemma 4.1] concerns the case where Ω is bounded, exactly the same proof is applicable even if Ω is unbounded. For $\rho > 0$, we set

$$\Omega_\rho = \{x \in \Omega\ : \delta(x) \geq \rho\} \ \text{ and } \ \Omega^c_\rho = \{x \in \mathbb{R}^N \setminus \Omega\ : \delta(x) \geq \rho\}.$$

Lemma 2.1. *Let u be the solution of problem* (1.2) *with a general conductivity $\sigma = \sigma(x)$ $(x \in \mathbb{R}^N)$ satisfying*

$$0 < \mu \leq \sigma(x) \leq M \quad \textit{for every } x \in \mathbb{R}^N,$$

where μ, M are positive constants. Then the following propositions hold true:

(1) *The solution u satisfies*

$$0 < u < 1 \quad \textit{in } \mathbb{R}^N \times (0, +\infty). \tag{2.2}$$

(2) *For every* $\rho > 0$*, there exist two positive constants* B *and* b *depending only on* $N, \mu, M, \sigma_s, \sigma_m$ *and* ρ *such that*

$$0 < u(x,t) < Be^{-\frac{b}{t}} \quad \text{for every } (x,t) \in \Omega_\rho \times (0,+\infty)$$
$$\text{and} \quad 0 < 1 - u(x,t) < Be^{-\frac{b}{t}} \quad \text{for every } (x,t) \in \Omega_\rho^c \times (0,+\infty).$$

Since a proposition [CMS, Proposition E], where the boundary of the domain is compact, also plays a key role in [CMS], in [S, Proposition 2.3] the proposition was modified in order to deal also with the case where $\partial\Omega$ is unbounded. Denote by $B_r(x)$ an open ball in $\mathbb{R}^N$ with radius $r > 0$ and centered at a point $x \in \mathbb{R}^N$.

Proposition 2.2 ([S]). *Let* Ω *be a possibly unbounded domain in* $\mathbb{R}^N$*, and let* $z_0 \in \partial\Omega$*. Assume that there exists* $\varepsilon > 0$ *such that* $\partial\Omega \cap B_\varepsilon(z_0)$ *is of class* C^2 *and* $\partial\Omega$ *divides* $B_\varepsilon(z_0)$ *into two connected components. Let* $\sigma = \sigma(x)$ $(x \in \mathbb{R}^N)$ *be a general conductivity satisfying*

$$0 < \mu \le \sigma(x) \le M \quad \text{for every } x \in \mathbb{R}^N, \text{ and } \sigma(x) = \begin{cases} \sigma_s & \text{if } x \in B_\varepsilon(z_0) \cap \Omega, \\ \sigma_m & \text{if } x \in B_\varepsilon(z_0) \setminus \Omega, \end{cases}$$

where $\mu, M, \sigma_s,$ *and* σ_m *are positive constants. Let* u *be the bounded solution of problem* (1.2) *for this general conductivity* σ*. Then, as* $t \to +0$*,* u *converges to the number* $\frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}}$ *uniformly on* $\partial\Omega \cap \overline{B_{\frac{1}{2}\varepsilon}(z_0)}$*.*

Proof. For convenience, we mention how to reduce the present case to the case where $\partial\Omega$ is bounded and of class C^2. Since $\partial\Omega \cap B_\varepsilon(z_0)$ is of class C^2, we can find a bounded domain Ω_* with C^2 boundary $\partial\Omega_*$ satisfying

$$\Omega \cap \overline{B_{\frac{2}{3}\varepsilon}(z_0)} \subset \Omega_* \subset \Omega \quad \text{and} \quad \overline{B_{\frac{2}{3}\varepsilon}(z_0)} \cap \partial\Omega \subset \partial\Omega_*.$$

Let us define the conductivity $\sigma_* = \sigma_*(x)$ $(x \in \mathbb{R}^N)$ by

$$\sigma_* = \begin{cases} \sigma_s & \text{in } \Omega_*, \\ \sigma_m & \text{in } \mathbb{R}^N \setminus \Omega_*. \end{cases} \tag{2.3}$$

Let $u_* = u_*(x,t)$ be the bounded solution of problem (1.2) where Ω and σ are replaced with Ω_* and σ_*, respectively. Then, by [CMS, Proposition E], as $t \to +0$, u_* converges to the number $\frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}}$ uniformly on $\partial\Omega \cap \overline{B_{\frac{1}{2}\varepsilon}(z_0)}$.

We observe that the difference $v = u - u_*$ satisfies

$$v_t = \operatorname{div}(\sigma_* \nabla v) \quad \text{in } B_{\frac{2}{3}\varepsilon}(z_0) \times (0, +\infty), \tag{2.4}$$

$$|v| < 1 \quad \text{in } \mathbb{R}^N \times (0, +\infty), \tag{2.5}$$

$$v = 0 \quad \text{on } B_{\frac{2}{3}\varepsilon}(z_0) \times \{0\}. \tag{2.6}$$

Set

$$\mathcal{N} = \left\{ x \in \mathbb{R}^N : \operatorname{dist}(x, \partial B_{\frac{2}{3}\varepsilon}(z_0)) < \frac{1}{100}\varepsilon \right\} \left(= B_{\frac{203}{300}\varepsilon}(z_0) \setminus \overline{B_{\frac{197}{300}\varepsilon}(z_0)} \right).$$

By comparing v with the solutions of the Cauchy problem for the heat diffusion equation with conductivity σ_* and initial data $\pm 2\mathcal{X}_{\mathcal{N}}$ for a short time, with the aid of the Gaussian bounds due to Aronson [A, Theorem 1, p. 891] (see also [FS, p. 328]), we see that there exist two positive constants B and b such that

$$|v(x,t)| \le Be^{-\frac{b}{t}} \quad \text{for every } (x,t) \in \overline{B_{\frac{1}{2}\varepsilon}(z_0)} \times (0, \infty). \tag{2.7}$$

Therefore, since u_* satisfies the conclusion, u also does. □

3. Proof of Theorem 1.1

First of all, Proposition 2.2 yields that the constant k in (1.4) is determined by

$$k = \frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s} + \sqrt{\sigma_m}}. \tag{3.1}$$

Since $\partial\Omega$ is uniformly of class C^6, there exist two positive numbers r and K such that, for every point $p \in \partial\Omega$, there exist an orthogonal coordinate system z and a function $\varphi \in C^6(\mathbb{R}^{N-1})$ such that the z_N coordinate axis lies in the inward normal direction to $\partial\Omega$ at p, the origin is located at p, C^6 norm of φ in $\mathbb{R}^{N-1}$ is less than K, $\varphi(0) = 0$, $\nabla\varphi(0) = 0$ and the set $B_r(p) \cap \Omega$ is written as in the z coordinate system

$$\{z \in B_r(0) : z_N > \varphi(z_1, \dots, z_{N-1})\}.$$

Since $\partial\Omega$ is uniformly of class C^6 as explained above, by choosing a number $\delta_0 > 0$ sufficiently small and setting

(3.2)
$$\mathcal{N}_- = \{x \in \Omega \ : \ 0 < \delta(x) < \delta_0\} \ \text{ and } \ \mathcal{N}_+ = \{x \in \mathbb{R}^N \setminus \overline{\Omega} \ : \ 0 < \delta(x) < \delta_0\},$$

where $\delta(x)$ is the distance function given by (2.1), we see that

$$\sigma = \begin{cases} \sigma_s & \text{in } \mathcal{N}_-, \\ \sigma_m & \text{in } \mathcal{N}_+ \end{cases}, \tag{3.3}$$

$$\delta \in C^6(\overline{\mathcal{N}_\pm}), \ \sup\left\{\left|\frac{\partial^\alpha \delta}{\partial x^\alpha}(x)\right| : x \in \overline{\mathcal{N}_\pm}, |\alpha| \le 6\right\} < +\infty, \tag{3.4}$$

$$\text{for every } x \in \overline{\mathcal{N}_\pm} \text{ there exists a unique } z = z(x) \in \partial\Omega \text{ with } \delta(x) = |x - z|, \tag{3.5}$$

$$z(x) = x - \delta(x)\nabla\delta(x) \ \text{ for all } x \in \overline{\mathcal{N}_\pm}, \tag{3.6}$$

$$\max_{1 \le j \le N-1} |\kappa_j(z)| < \frac{1}{2\delta_0} \ \text{ for every } z \in \partial\Omega, \tag{3.7}$$

where $\kappa_1(z), \dots, \kappa_{N-1}(z)$ denote the principal curvatures of $\partial\Omega$ at a point $z \in \partial\Omega$ with respect to the inward normal direction to $\partial\Omega$. It is shown in [GT, Lemmas 14.16 and 14.17, p. 355] that

$$|\nabla\delta(x)| = 1 \ \text{ and } \ \Delta\delta(x) = \begin{cases} -\sum_{j=1}^{N-1} \frac{\kappa_j(z(x))}{1-\kappa_j(z(x))\delta(x)} & \text{for } x \in \mathcal{N}_-, \\ \sum_{j=1}^{N-1} \frac{\kappa_j(z(x))}{1+\kappa_j(z(x))\delta(x)} & \text{for } x \in \mathcal{N}_+. \end{cases} \tag{3.8}$$

We introduce elementary functions of the principal curvatures at $z \in \partial\Omega$ by

$$H_i(z) = \sum_{j_1 < \cdots < j_i} \kappa_{j_1}(z) \cdots \kappa_{j_i}(z) \ \text{ for } i = 1, \dots, N-1, \tag{3.9}$$

where $\frac{1}{N-1}H_1(z)$ corresponds to the mean curvature of $\partial\Omega$ at $z \in \partial\Omega$ with respect to the inward normal direction to $\partial\Omega$. Then we notice that, for every $i = 1, \dots, N-1$, the composite function $H_i = H_i(z(x))$ satisfies that for $x \in \overline{\mathcal{N}_\pm}$

$$H_i \in C^4(\overline{\mathcal{N}_\pm}), \ \sup\left\{\left|\frac{\partial^\alpha H_i(z(x))}{\partial x^\alpha}\right| : x \in \overline{\mathcal{N}_\pm}, |\alpha| \le 4\right\} < +\infty \tag{3.10}$$

and

$$\nabla\delta(x) \cdot \nabla H_i(z(x)) = 0 \quad \text{for } x \in \overline{\mathcal{N}_\pm}. \tag{3.11}$$

Moreover, as in the proof of [S, Theorem 1.1], by introducing an increasing sequence of bounded subdomains in each of $\mathcal{N}_\pm$ together with an increasing sequence of bounded harmonic functions on each of the subdomains, we can construct a function $\psi = \psi(x)$, as the limit of the sequence, on each of $\mathcal{N}_\pm$ satisfying

(3.12)
$$\Delta\psi = 0 \ \text{ in } \ \mathcal{N}_\pm, \ \psi = 0 \ \text{ on } \partial\Omega, \ \psi = 2 \ \text{ on } \partial\mathcal{N}_\pm \setminus \partial\Omega \text{ and } 0 < \psi < 2 \ \text{ in } \mathcal{N}_\pm,$$

even if each of $\mathcal{N}_\pm$ is unbounded.

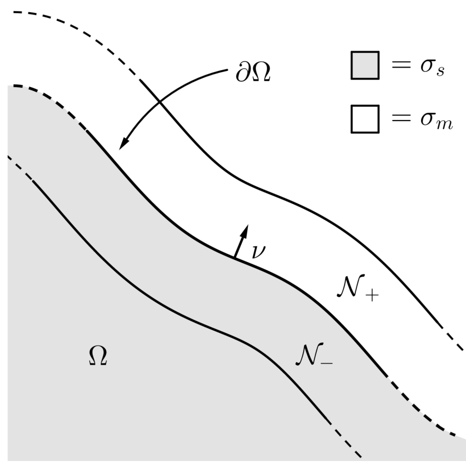

Figure 1: The geometric setting used in the proof.

As in the proofs of [CMS, Theorem 1.5 in section 5], we introduce the function $w = w(x, \lambda)$ by the Laplace-Stieltjes transform of $u(x, \cdot)$ restricted on the semiaxis of real positive numbers

$$w(x, \lambda) = \lambda \int_0^\infty e^{-\lambda t} u(x, t)\, dt \quad \text{for } (x, \lambda) \in \mathbb{R}^N \times (0, +\infty).$$

Observe from (1.1), (1.2), (1.4) and (3.1) that for every $\lambda > 0$

$$\sigma_s \Delta w - \lambda w = 0 \quad \text{in } \Omega, \tag{3.13}$$
$$\sigma_m \Delta(1-w) - \lambda(1-w) = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \tag{3.14}$$
$$0 < w < 1 \quad \text{in } \mathbb{R}^N, \tag{3.15}$$
$$w = \frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}} \ \text{ and } \ \sigma_s \frac{\partial w}{\partial \nu}\Big|_- = \sigma_m \frac{\partial w}{\partial \nu}\Big|_+ \quad \text{on } \partial\Omega, \tag{3.16}$$

where ν denotes the outward unit normal vector to $\partial\Omega$, $+$ denotes the limit from outside of Ω and $-$ that from inside of Ω. Moreover, it follows from (2) of Lemma 2.1 that there exist two positive constants $\tilde{B}$ and $\tilde{b}$ satisfying:

$$0 < w(x,\lambda) \le \tilde{B} e^{-\tilde{b}\sqrt{\lambda}} \quad \text{for every } (x,\lambda) \in (\partial\mathcal{N}_- \setminus \partial\Omega) \times (0,+\infty), \tag{3.17}$$
$$0 < 1 - w(x,\lambda) \le \tilde{B} e^{-\tilde{b}\sqrt{\lambda}} \quad \text{for every } (x,\lambda) \in (\partial\mathcal{N}_+ \setminus \partial\Omega) \times (0,+\infty). \tag{3.18}$$

3.1. Proving that the mean curvature of $\partial\Omega$ vanishes

Let us first consider w on $\mathcal{N}_-$. Since w satisfies (3.13) and the first equality of (3.16), in view of the formal WKB approximation of w for sufficiently large $\tau = \frac{\lambda}{\sigma_s}$

$$w(x,\lambda) \sim \frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}} e^{-\sqrt{\tau}\delta(x)} \sum_{j=0}^{\infty} A_j(x) \tau^{-\frac{j}{2}} \ \text{ with some coefficients } \{A_j(x)\},$$

we introduce two functions $f_{1,\pm} = f_{1,\pm}(x,\lambda)$ defined for $(x,\lambda) \in \overline{\mathcal{N}_-} \times (0,+\infty)$ by

$$f_{1,\pm}(x,\lambda) = \frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}} e^{-\frac{\sqrt{\lambda}}{\sqrt{\sigma_s}}\delta(x)} \left[A_0(x) + \frac{\sqrt{\sigma_s}}{\sqrt{\lambda}} A_{1,\pm}(x) \right],$$

where

(3.19)

$$A_0(x) = \left\{ \prod_{j=1}^{N-1} \Big[1 - \kappa_j(z(x))\delta(x) \Big] \right\}^{-\frac{1}{2}},$$

$$A_{1,\pm}(x) = \int_0^{\delta(x)} \left[\frac{1}{2} \Delta A_0(x(\tau)) \pm 1 \right] \exp\left(-\frac{1}{2} \int_\tau^{\delta(x)} \Delta\delta(x(\tau'))d\tau' \right) d\tau,$$

with $x(\tau) = z(x) - \tau\,\nu(z(x))$ for $0 < \tau < \delta(x)$. We observe that for $x \in \overline{\mathcal{N}_-}$

$$\prod_{j=1}^{N-1}\Big[1-\kappa_j(z(x))\delta(x)\Big] = 1 + \sum_{i=1}^{N-1}(-1)^i H_i(z(x))(\delta(x))^i. \tag{3.20}$$

With (3.8), (3.11) and (3.20) at hand, by straightforward computations we obtain that

$$\nabla\delta\cdot\nabla A_0 = -\frac{1}{2}(\Delta\delta)A_0, \quad \nabla\delta\cdot\nabla A_{1,\pm} = -\frac{1}{2}(\Delta\delta)A_{1,\pm} + \frac{1}{2}\Delta A_0 \pm 1 \ \text{ in } \overline{\mathcal{N}_-}, \tag{3.21}$$

$$\sigma_s\Delta f_{1,\pm} - \lambda f_{1,\pm} = \frac{\sigma_s\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}}e^{-\frac{\sqrt{\lambda}}{\sqrt{\sigma_s}}\delta(x)}\left(\mp 2 + \frac{\sqrt{\sigma_s}}{\sqrt{\lambda}}\Delta A_{1,\pm}\right) \ \text{ in } \overline{\mathcal{N}_-}, \tag{3.22}$$

and

$$A_0 = 1,\ A_{1,\pm} = 0, \quad f_{1,\pm} = \frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}} \ \text{ on } \partial\Omega, \tag{3.23}$$

for every $\lambda > 0$. Moreover, (3.4), (3.7), (3.10) and (3.20) yield that

$$|\Delta A_{1,\pm}| \le c_1 \ \text{ in } \overline{\mathcal{N}_-} \tag{3.24}$$

for some positive constant c_1. Therefore, it follows from (3.22), (3.24), (3.17) and the definition of $f_{1,\pm}$ that there exist two positive constants λ_1 and η_1 such that

$$\sigma_s\Delta f_{1,+} - \lambda f_{1,+} < 0 < \sigma_s\Delta f_{1,-} - \lambda f_{1,-} \ \text{ in } \overline{\mathcal{N}_-}, \tag{3.25}$$

$$\max\{|f_{1,+}|, |f_{1,-}|, w\} \le e^{-\eta_1\sqrt{\lambda}} \ \text{ on } \partial\mathcal{N}_- \setminus \partial\Omega, \tag{3.26}$$

for every $\lambda \ge \lambda_1$.

For every $(x,\lambda) \in \overline{\mathcal{N}_-} \times (0, +\infty)$, we define the two functions $w_{1,\pm} = w_{1,\pm}(x,\lambda)$ by

$$w_{1,\pm}(x,\lambda) = f_{1,\pm}(x,\lambda) \pm \psi(x)e^{-\eta_1\sqrt{\lambda}}, \tag{3.27}$$

where $\psi(x)$ is given by (3.12). Then, in view of (3.13), (3.16), (3.23), (3.25) and (3.26), we notice that

$$\begin{aligned}
&\sigma_s \Delta w_{1,+} - \lambda w_{1,+} < 0 = \sigma_s \Delta w - \lambda w < \sigma_s \Delta w_{1,-} - \lambda w_{1,-} && \text{in } \mathcal{N}_-, \\
&w_{1,+} = w = w_{1,-} = \tfrac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}} && \text{on } \partial\Omega, \\
&w_{1,-} < w < w_{1,+} && \text{on } \partial\mathcal{N}_- \setminus \partial\Omega,
\end{aligned} \tag{3.28}$$

for every $\lambda \geq \lambda_1$, and hence we get that

$$w_{1,-} < w < w_{1,+} \quad \text{in } \ \mathcal{N}_-, \tag{3.29}$$

for every $\lambda \geq \lambda_1$, by the comparison principle (see Proposition B.1 in Appendix). Thus, combining (3.29) with (3.28) yields that

$$\frac{\partial w_{1,+}}{\partial \nu} \leq \frac{\partial w}{\partial \nu}\Big|_- \leq \frac{\partial w_{1,-}}{\partial \nu} \quad \text{on } \ \partial\Omega, \tag{3.30}$$

for every $\lambda \geq \lambda_1$.

Therefore, by recalling the definition of $w_{1,\pm}$, it follows from (3.21), (3.23) and (3.8) that, for every $\lambda \geq \lambda_1$, we have the following chain of inequalities on $\partial\Omega$:

$$\begin{aligned}
&\frac{\sigma_s\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}}\left\{-\frac{1}{2}\sum_{j=1}^{N-1}\kappa_j - \frac{\sqrt{\sigma_s}}{\sqrt{\lambda}}\left(\frac{1}{2}\Delta A_0 + 1\right)\right\} + \sigma_s\frac{\partial\psi}{\partial\nu}e^{-\eta_1\sqrt{\lambda}} \\
&\leq \sigma_s \frac{\partial w}{\partial \nu}\Big|_- - \frac{\sqrt{\sigma_s}\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}}\sqrt{\lambda} \\
&\leq \frac{\sigma_s\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}}\left\{-\frac{1}{2}\sum_{j=1}^{N-1}\kappa_j - \frac{\sqrt{\sigma_s}}{\sqrt{\lambda}}\left(\frac{1}{2}\Delta A_0 - 1\right)\right\} - \sigma_s\frac{\partial\psi}{\partial\nu}e^{-\eta_1\sqrt{\lambda}}.
\end{aligned} \tag{3.31}$$

This implies that on $\partial\Omega$

$$-\frac{\sigma_s\sqrt{\sigma_m}}{2(\sqrt{\sigma_s}+\sqrt{\sigma_m})}\sum_{j=1}^{N-1}\kappa_j = \sigma_s\frac{\partial w}{\partial\nu}\Big|_- - \frac{\sqrt{\sigma_s}\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}}\sqrt{\lambda} + O(1/\sqrt{\lambda}) \text{ as } \lambda \to +\infty. \tag{3.32}$$

Next, we consider $1 - w$ on $\mathcal{N}_+$. By the similar arguments as above, since

$$1 - w = \frac{\sqrt{\sigma_s}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}} \quad \text{on } \partial\Omega,$$

we can construct barriers for $1-w$ on $\mathcal{N}_+$ with the aid of (3.18) by replacing σ_s with σ_m. Thus, proceeding similarly yields that on $\partial\Omega$

(3.33)
$$\frac{\sigma_m\sqrt{\sigma_s}}{2(\sqrt{\sigma_s}+\sqrt{\sigma_m})}\sum_{j=1}^{N-1}\kappa_j=\sigma_m\frac{\partial w}{\partial \nu}\Big|_+-\frac{\sqrt{\sigma_s}\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}}\sqrt{\lambda}+O(1/\sqrt{\lambda}) \text{ as } \lambda\to+\infty,$$

where we have taken into account both the sign of the mean curvature with (3.8) and the normal direction to $\partial\Omega$. Therefore, by combining (3.32) and (3.33) with the second equality of (3.16) we conclude that on $\partial\Omega$

$$H_1=\sum_{j=1}^{N-1}\kappa_j=O(1/\sqrt{\lambda}) \text{ as } \lambda\to+\infty,$$

and hence the mean curvature of $\partial\Omega$ must vanish, that is, $\partial\Omega$ is a minimal hypersurface properly embedded in $\mathbb{R}^N$ (see (3.9) for H_1). In particular when $N=2$, the curvature of the curve $\partial\Omega$ vanishes and the conclusion of Theorem 1.1 holds.

Note that in this subsection 3.1 we did not use the fact that $\sigma_s\neq\sigma_m$.

3.2. Proving that all the principal curvaures of $\partial\Omega$ vanish and $\partial\Omega$ must be a hyperplane

We may consider the case where $N\geq 3$. It suffices to show that $H_i=0$ on $\partial\Omega$ for every $i=1,\dots,N-1$. Since we already know in subsection 3.1 that $H_1=0$ on $\partial\Omega$, we start induction with supposing that there exists a number $p\in\{2,\dots,N-1\}$ satisfying

$$H_1=\cdots=H_{p-1}=0 \text{ on } \partial\Omega. \tag{3.34}$$

Then we will prove that $H_p=0$ on $\partial\Omega$. By subsection 3.1, $\partial\Omega$ must be real analytic and moreover, by the interior estimates for solutions of the minimal surface equation (see [GT, Corollary 16.7, p. 407]), we see that $\partial\Omega$ is uniformly of class C^ℓ for every $\ell\in\mathbb{N}$, and hence (3.4) and (3.10) are improved as follows: For every $\ell\in\mathbb{N}$,

$$\sup\left\{\left|\frac{\partial^\alpha\delta}{\partial x^\alpha}(x)\right| : x\in\overline{\mathcal{N}_\pm}, |\alpha|\leq\ell\right\}<+\infty, \tag{3.35}$$

and

$$\text{(3.36)} \qquad \sup\left\{\left|\frac{\partial^\alpha H_i(z(x))}{\partial x^\alpha}\right| : 1 \le i \le N-1,\ x \in \overline{\mathcal{N}_\pm}, |\alpha| \le \ell\right\} < +\infty.$$

Therefore we can introduce the following more precise barriers $f_{n,\pm} = f_{n,\pm}(x,\lambda)$ for w on $\mathcal{N}_-$ such that for $(x,\lambda) \in \overline{\mathcal{N}_-} \times (0,+\infty)$ and for every $n \ge 2$

$$f_{n,\pm}(x,\lambda) = \frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}} e^{-\frac{\sqrt{\lambda}}{\sqrt{\sigma_s}}\delta(x)} \left[A_0(x) + \sum_{j=1}^{n-1}\left(\frac{\sqrt{\sigma_s}}{\sqrt{\lambda}}\right)^j A_j(x) + \left(\frac{\sqrt{\sigma_s}}{\sqrt{\lambda}}\right)^n A_{n,\pm}(x)\right],$$

where A_0 is given by (3.19) and for $j = 1, \cdots, n-1$,

(3.37)

$$A_j(x) = \int_0^{\delta(x)} \left[\frac{1}{2}\,\Delta A_{j-1}(x(\tau))\right] \exp\left(-\frac{1}{2}\int_\tau^{\delta(x)} \Delta\delta(x(\tau'))d\tau'\right) d\tau,$$

$$A_{n,\pm}(x) = \int_0^{\delta(x)} \left[\frac{1}{2}\,\Delta A_{n-1}(x(\tau)) \pm 1\right] \exp\left(-\frac{1}{2}\int_\tau^{\delta(x)} \Delta\delta(x(\tau'))d\tau'\right) d\tau$$

with $x(\tau) = z(x) - \tau\,\nu(z(x))$ for $0 < \tau < \delta(x)$.

With (3.8), (3.11) and (3.20) at hand, by straightforward computations we obtain that, in $\overline{\mathcal{N}_-}$ (compare with (3.21)–(3.24)):

$$\text{(3.38)} \qquad \nabla\delta \cdot \nabla A_0 = -\frac{1}{2}(\Delta\delta)A_0,$$

$$\text{(3.39)} \qquad \nabla\delta \cdot \nabla A_j = -\frac{1}{2}(\Delta\delta)A_j + \frac{1}{2}\Delta A_{j-1} \quad \text{for } j = 1, \dots, n-1,$$

$$\text{(3.40)} \qquad \nabla\delta \cdot \nabla A_{n,\pm} = -\frac{1}{2}(\Delta\delta)A_{n,\pm} + \frac{1}{2}\Delta A_{n-1} \pm 1,$$

(3.41)

$$\sigma_s\Delta f_{n,\pm} - \lambda f_{n,\pm} = \frac{\sigma_s\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}}\left(\frac{\sqrt{\sigma_s}}{\sqrt{\lambda}}\right)^{n-1} e^{-\frac{\sqrt{\lambda}}{\sqrt{\sigma_s}}\delta(x)}\left(\mp 2 + \frac{\sqrt{\sigma_s}}{\sqrt{\lambda}}\Delta A_{n,\pm}\right),$$

and on $\partial\Omega$

$$\text{(3.42)} \qquad A_0 = 1,\ A_1 = \cdots = A_{n-1} = A_{n,\pm} = 0, \quad f_{n,\pm} = \frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}},$$

for every $\lambda > 0$. Moreover, (3.35), (3.36), (3.7) and (3.20) yield that

$$|\Delta A_{n,\pm}| \le c_n \quad \text{in } \overline{\mathcal{N}_-} \tag{3.43}$$

for some positive constant c_n. Then, by replacing $f_{1,\pm}$ with $f_{n,\pm}$, we can use the same comparison arguments as in (3.25) – (3.30) of subsection 3.1 to conclude that there exist two positive constants λ_n and η_n satisfying

$$\frac{\partial w_{n,+}}{\partial \nu} \le \frac{\partial w}{\partial \nu}\Big|_- \le \frac{\partial w_{n,-}}{\partial \nu} \quad \text{on } \partial\Omega \tag{3.44}$$

for every $\lambda \ge \lambda_n$, where

$$w_{n,\pm}(x,\lambda) = f_{n,\pm}(x,\lambda) \pm \psi(x) e^{-\eta_n\sqrt{\lambda}} \tag{3.45}$$

with $\psi(x)$ given by (3.12). Since $\Delta\delta = 0$ on $\partial\Omega$, it follows from (3.8), (3.38)–(3.40) and (3.42) that on $\partial\Omega$

$$\begin{aligned} \frac{\partial w_{n,\pm}}{\partial \nu} &= -\nabla\delta \cdot \nabla w_{n,\pm} \\ &= \frac{\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}} \left\{ \frac{\sqrt{\lambda}}{\sqrt{\sigma_s}} - \frac{1}{2}\sum_{j=1}^{n-1} \left(\frac{\sqrt{\sigma_s}}{\sqrt{\lambda}}\right)^j \Delta A_{j-1} \right. \\ &\quad \left. - \frac{1}{2}\left(\frac{\sqrt{\sigma_s}}{\sqrt{\lambda}}\right)^n (\Delta A_{n,\pm} \pm 2) \right\} \\ &\quad \pm \frac{\partial \psi}{\partial \nu} e^{-\eta_n\sqrt{\lambda}}. \end{aligned} \tag{3.46}$$

It follows from (3.34) that for $x \in \overline{\mathcal{N}_-}$

$$\prod_{j=1}^{N-1} \Big[1 - \kappa_j(z(x))\delta(x)\Big] = 1 + \sum_{i=p}^{N-1} (-1)^i H_i(z(x))(\delta(x))^i. \tag{3.47}$$

We choose, for instance, $n = N-1$. Let us show that for every $s \in \{0, \dots, p-2\}$ as $\delta(x) \to 0$

$$\begin{aligned} \Delta A_s(x) = &-2^{-(s+1)}(-1)^p (s+2)! \binom{p}{s+2} H_p(z(x))\,(\delta(x))^{p-2-s} \\ &+ O\left((\delta(x))^{p-1-s}\right). \end{aligned} \tag{3.48}$$

By (3.47) and (3.19), we have that as $\delta(x) \to 0$

$$A_0(x) = 1 - \frac{1}{2}(-1)^p H_p(z(x))(\delta(x))^p + O\left((\delta(x))^{p+1}\right).$$

Then, it follows from the first equality of (3.8) that as $\delta(x) \to 0$

$$\Delta A_0(x) = -\frac{1}{2}(-1)^p H_p(z(x))p(p-1)(\delta(x))^{p-2} + O\left((\delta(x))^{p-1}\right),$$

which means that (3.48) holds for $s = 0$. Suppose that (3.48) holds for $s = q - 1 \in \{0, \dots, p-2\}$. Then we have from (3.37) that

$$\begin{aligned}
A_q(x) = & \int_0^{\delta(x)} \left[-2^{-(q+1)}(-1)^p(q+1)! \binom{p}{q+1} H_p(z(x))\tau^{p-1-q} + O\left((\tau)^{p-q}\right) \right] \\
& \times \exp\left(-\frac{1}{2} \int_\tau^{\delta(x)} \Delta\delta(x(\tau'))d\tau' \right) d\tau \\
& = -2^{-(q+1)}(-1)^p(q+1)! \binom{p}{q+1} H_p(z(x)) \int_0^{\delta(x)} \tau^{p-1-q} d\tau \\
& \quad + O\left((\delta(x))^{p-q+1}\right) \\
& = -2^{-(q+1)}(-1)^p q! \binom{p}{q} H_p(z(x))\delta(x)^{p-q} + O\left((\delta(x))^{p-q+1}\right).
\end{aligned}$$

Thus it follows from the first equality of (3.8) that as $\delta(x) \to 0$

$$\begin{aligned}
\Delta A_q(x) &= -2^{-(q+1)}(-1)^p q! \binom{p}{q} H_p(z(x))(p-q)(p-q-1)\delta(x)^{p-q-2} \\
&\quad + O\left((\delta(x))^{p-q-1}\right) \\
&= -2^{-(q+1)}(-1)^p (q+2)! \binom{p}{q+2} H_p(z(x))\delta(x)^{p-q-2} \\
&\quad + O\left((\delta(x))^{p-q-1}\right),
\end{aligned}$$

which means that (3.48) holds for $s = q$. Hence formula (3.48) holds true for every $s \in \{0, \dots, p-2\}$.

Formula (3.48) implies that on $\partial\Omega$

$$\Delta A_s = 0 \quad \text{for } s < p-2 \text{ and } \Delta A_{p-2} = -2^{-(p-1)}(-1)^p p! H_p,$$

and hence it follows from (3.44) and (3.46) that on $\partial\Omega$ as $\lambda \to \infty$

(3.49)
$$\sigma_s \frac{\partial w}{\partial \nu}\Big|_- = \frac{\sqrt{\sigma_s}\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}} \left\{\sqrt{\lambda} + p! 2^{-p}(-1)^p (\sigma_s)^{\frac{p}{2}} H_p \lambda^{-\frac{p-1}{2}}\right\} + O\left(\lambda^{-\frac{p}{2}}\right).$$

Next, as in the end of subsection 3.1, we proceed to consider $1-w$ on $\mathcal{N}_+$. By replacing $w,\ \sigma_s$ with $1-w,\ \sigma_m$, respectively and taking into account both the sign of H_p and the normal direction to $\partial\Omega$, by the same arguments we infer that on $\partial\Omega$ as $\lambda \to \infty$

$$(3.50)\quad \sigma_m \frac{\partial w}{\partial \nu}\Big|_+ = \frac{\sqrt{\sigma_s}\sqrt{\sigma_m}}{\sqrt{\sigma_s}+\sqrt{\sigma_m}} \left\{\sqrt{\lambda} + p! 2^{-p} (\sigma_m)^{\frac{p}{2}} H_p \lambda^{-\frac{p-1}{2}}\right\} + O\left(\lambda^{-\frac{p}{2}}\right).$$

Here we used the fact that, corresponding to the choice of the normal direction to $\partial\Omega$, the sign of H_p changes if p is odd and it does not change if p is even. Since $\sigma_s \neq \sigma_m$, by combining (3.49) and (3.50) with the second equality of (3.16) we conclude that on $\partial\Omega$

$$H_p = O(1/\sqrt{\lambda}) \ \text{ as } \lambda \to \infty,$$

and hence H_p must vanish on $\partial\Omega$. Therefore we obtain that $H_i = 0$ on $\partial\Omega$ for every $i = 1, \dots, N-1$. This means that all the principal curvatures of $\partial\Omega$ vanish and thus $\partial\Omega$ must be a hyperplane.

Note that in this subsection 3.2 we used the fact that $\sigma_s \neq \sigma_m$.

4. Proof of Theorem 1.2

Let u be the solution of problem (1.6). From (1.4) we see that $\partial\Omega$ is a stationary isothermic surface of u. Thus by [MPS, Theorem 2.2, p. 4825] $\partial\Omega$ must be a real analytic hypersurface embedded in $\mathbb{R}^N$. Hence Proposition 2.2 yields that $k = \frac{1}{2}$. Let $x \in \partial\Omega$. Then it follows from the explicit representation of u via Gaussian kernel that for every $t > 0$

$$\begin{aligned}
\frac{1}{2} &= u(x,t) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \mathcal{X}_{\Omega^c}(\xi) e^{-\frac{|x-\xi|^2}{4t}} d\xi \\
&= (4\pi t)^{-\frac{N}{2}} \int_0^\infty e^{-\frac{r^2}{4t}} \left(\int_{\partial B_r(x)} \mathcal{X}_{\Omega^c}(\xi) dS_\xi \right) dr \\
&= (4\pi t)^{-\frac{N}{2}} \int_0^\infty e^{-\frac{r^2}{4t}} |\Omega^c \cap \partial B_r(x)| dr,
\end{aligned}$$

where $\Omega^c = \mathbb{R}^N \setminus \Omega$, dS_ξ indicates the area element of the sphere $\partial B_r(x)$ and $|\Omega^c \cap \partial B_r(x)|$ does the $(N-1)$-dimensional Hausdorff measure of the set $\Omega^c \cap \partial B_r(x)$. Thus we infer that

$$\int_0^\infty e^{-\frac{r^2}{4t}} \left(|\Omega^c \cap \partial B_r(x)| - \frac{1}{2}|\partial B_r(x)| \right) dr = 0 \text{ for every } t > 0.$$

Since the Laplace transform is injective, we conclude that for each point $x \in \partial\Omega$

$$|\Omega^c \cap \partial B_r(x)| - \frac{1}{2}|\partial B_r(x)| = 0 \text{ for almost every } r > 0. \tag{4.1}$$

Then the following formula also holds true:

$$\frac{|\Omega^c \cap B_r(x)|}{|B_r(x)|} = \frac{1}{2} \text{ for every } r > 0 \text{ and } x \in \partial\Omega, \tag{4.2}$$

where the same symbol $|\cdot|$ indicates the N-dimensional Lebesgue measure of sets.

When $N \geq 2$, by [MPS, Theorem 1.2, p. 4823] (4.2) yields that $\partial\Omega$ must have zero mean curvature. Hence, when $N = 2$, $\partial\Omega$ must be a straight line, and when $N \geq 3$, $\partial\Omega$ must be a minimal hypersurface embedded in $\mathbb{R}^N$.

In view of the sufficient regularity of $\partial\Omega$, it follows from (4.1) that for every point $p \in \partial\Omega$, there exist numbers $\delta_p > 0$ and $r_p > 0$ satisfying

(4.3)
$$|\Omega^c \cap \partial B_r(x)| - \frac{1}{2}|\partial B_r(x)| = 0 \text{ for every } 0 < r < r_p \text{ and } x \in B_\delta(p) \cap \partial\Omega.$$

When $N = 3$, by [N, Theorem, p. 234], (4.3) yields that $\partial\Omega$ must be either a hyperplane or a helicoid. This completes the proof of Theorem 1.2. □

Appendices

First of all, let us give a proof of (1.5).

Appendix A. Proof of (1.5)

Let $\mathcal{H} \subset \mathbb{R}^3$ be the helicoid given by

$$\left\{ (x_1, x_2, x_3) = (\rho \cos s, \rho \sin s, s) \ : \ (\rho, s) \in \mathbb{R}^2 \right\}.$$

(See [CMII, pp. 8–9] for the helicoid). Notice that $\mathcal{H}$ is the boundary of the following unbounded domain:

$$\Omega = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;:\; x_2 \cos x_3 - x_1 \sin x_3 > 0\right\}. \tag{A.1}$$

We now introduce two isometries that are deeply related to the symmetries of $\mathcal{H}$. For $\alpha \in \mathbb{R}$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we set:

$$\begin{aligned} k_\alpha(x) &= (x_1 \cos\alpha - x_2 \sin\alpha,\ x_1 \sin\alpha + x_2 \cos\alpha,\ x_3 + \alpha), \\ g(x) &= (x_1, -x_2, -x_3). \end{aligned} \tag{A.2}$$

Here k_α is a *screwing motion* obtained by rotation of angle α in the x_1-x_2 plane, followed by a translation of length α in the x_3 direction. Notice that Ω and $\mathbb{R}^3 \setminus \overline{\Omega}$ are preserved by the action of k_α, while they get switched by that of g:

$$\begin{aligned} k_\alpha(\Omega) = \Omega, \quad k_\alpha(\mathbb{R}^3 \setminus \overline{\Omega}) = \mathbb{R}^3 \setminus \overline{\Omega}, \\ g(\Omega) = \mathbb{R}^3 \setminus \overline{\Omega}, \quad g(\mathbb{R}^3 \setminus \overline{\Omega}) = \Omega. \end{aligned} \tag{A.3}$$

Finally, since $x_2 \cos x_3 - x_1 \sin x_3 = 0$ for $x \in \mathcal{H}$, the restrictions of g and k_α to $\mathcal{H}$ are related by the following formula:

(A.4)
$$g(x_1, x_2, x_3) = (x_1, -x_2, -x_3) = k_{-2x_3}(x_1, x_2, x_3) \quad \text{for all } (x_1, x_2, x_3) \in \mathcal{H}.$$

Let $u = u(x, t)$ be the unique bounded solution of the following Cauchy problem for the heat diffusion equation:

$$u_t = \Delta u \quad \text{in } \mathbb{R}^3 \times (0, +\infty) \ \text{ and } \ u = \mathcal{X}_{\mathbb{R}^3 \setminus \Omega} \ \text{ on } \mathbb{R}^3 \times \{0\}, \tag{A.5}$$

where Ω is the unbounded domain defined in (A.1). Moreover, for arbitray real α, define the following functions:

$$v_\alpha(x,t) = u\left(k_\alpha(x), t\right) \quad \text{and} \quad w(x,t) = u\left(g(x), t\right) \quad \text{for} \quad (x,t) \in \mathbb{R}^3 \times (0, \infty).$$

Since both k_α and g are isometries, by (A.3) we deduce that v_α and w are bounded solutions of the following Cauchy problems.

$$(v_\alpha)_t = \Delta v_\alpha \quad \text{in } \mathbb{R}^3 \times (0, +\infty) \ \text{ and } \ v_\alpha = \mathcal{X}_{\mathbb{R}^3 \setminus \Omega} \ \text{ on } \mathbb{R}^3 \times \{0\}, \tag{A.6}$$
$$w_t = \Delta w \quad \text{in } \mathbb{R}^3 \times (0, +\infty) \ \text{ and } \ w = \mathcal{X}_{\Omega} \ \text{ on } \mathbb{R}^3 \times \{0\}. \tag{A.7}$$

In particular, unique solvability of the Cauchy problems above yields

$$\text{(A.8)} \qquad v_\alpha = u \quad \text{and} \quad u + w = 1 \quad \text{in } \mathbb{R}^3 \times (0, \infty), \quad \text{for all } \alpha \in \mathbb{R}.$$

Fix now an arbitrary pair $(x, t) \in \mathcal{H} \times (0, \infty)$ and choose $\alpha = -2x_3$. By combining both identities in (A.8) with (A.4) we get the following chain of equalities.

$$1 = u(x,t) + u(g(x), t) = u(x,t) + u\left(k_{-2x_3}(x), t\right) = 2u(x,t).$$

That is, $u(x, t) = 1/2$ for all $(x, t) \in \mathcal{H} \times (0, \infty)$. We have therefore proved (1.5) when $N = 3$. The case $N \geq 4$ follows by separation of variables. □

Appendix B. A maximum principle for unbounded domains

For convenience, we quote a maximum principle together with its proof for an elliptic equation in unbounded domains in $\mathbb{R}^N$ from [S, Proposition A.3].

Proposition B.1. *Let $D \subset \mathbb{R}^N$ be an unbounded domain, and let $\sigma = \sigma(x)$ $(x \in D)$ be a general conductivity satisfying*

$$0 < \mu \leq \sigma(x) \leq M \quad \textit{for every } x \in \mathbb{R}^N,$$

where μ, M are positive constants. Assume that $w \in H^1_{loc}(D) \cap L^\infty(D) \cap C^0(\overline{D})$ satisfies

$$-\mathrm{div}(\sigma \nabla w) + \lambda w \geq 0 \quad \textit{in } D \quad \textit{and} \quad w \geq 0 \quad \textit{on } \partial D$$

for some constant $\lambda > 0$. Then $w \geq 0$ in D, and moreover, either $w > 0$ in D or $w \equiv 0$ in D.

Remark B.2. *When D is bounded, this proposition is well known and holds true for every $\lambda \geq 0$. However, when D is unbounded, this proposition is not true for $\lambda = 0$. Indeed, a counterexample is given in* [ABR, p. 37], *where $N \geq 3,\ D = \{x \in \mathbb{R}^N : |x| > 1\},\ \sigma(x) \equiv 1$ and $w(x) = |x|^{2-N} - 1$.*

Proof of Proposition B.1. Define $v = v(x)$ by

$$v(x) = e^{-\delta|x|} w(x) \quad \text{for } x \in \overline{D},$$

where $\delta > 0$ is a constant which will be chosen later. Then $v \in H^1_{loc}(D) \cap L^\infty(D) \cap C^0(\overline{D})$ and moreover

$$\lim_{|x|\to\infty} v(x) = 0, \tag{B.1}$$

since $w \in L^\infty(D)$. For every $\varepsilon > 0$, we consider a nonnegative function

$$\varphi(x) = \max\{-\varepsilon - v(x), 0\} \quad \text{for } x \in \overline{D}.$$

Since $v \in H^1_{loc}(D) \cap L^\infty(D) \cap C^0(\overline{D})$ and $v \geq 0$ on ∂D, it follows from (B.1) that φ is compactly supported in D and $\varphi \in H^1_0(D)$, and hence $e^{-2\delta|\cdot|}\varphi(\cdot) \in H^1_0(D)$. Therefore we obtain

$$\begin{aligned} 0 &\leq \int_D \left\{\sigma(x)\nabla w(x)\cdot\nabla\left(\varphi(x)e^{-2\delta|x|}\right) + \lambda w(x)\varphi(x)e^{-2\delta|x|}\right\} dx \\ &= \int_{D\cap\{v<-\varepsilon\}} \sigma e^{-\delta|x|}\left\{\left(\delta v\frac{x}{|x|} + \nabla v\right)\cdot\left(\nabla\varphi - 2\delta\varphi\frac{x}{|x|}\right) + \frac{\lambda}{\sigma}v\varphi\right\} dx. \end{aligned} \tag{B.2}$$

Notice that

$$\varphi(x) = \begin{cases} -\varepsilon - v(x) & \text{if } v(x) < -\varepsilon, \\ 0 & \text{if } v(x) \geq -\varepsilon, \end{cases} \quad \text{and}$$
$$\nabla\varphi(x) = \begin{cases} -\nabla v(x) & \text{if } v(x) < -\varepsilon, \\ 0 & \text{if } v(x) \geq -\varepsilon. \end{cases}$$

By setting

$$I = \sigma^{-1}e^{\delta|x|} \times \text{ the integrand of the integral (B.2)},$$

we have

$$\begin{aligned} I &= -|\nabla v|^2 - \frac{\lambda}{\sigma}v^2 + 2\delta^2 v^2 + \delta v\frac{x}{|x|}\cdot\nabla v + \varepsilon\left(2\delta^2 v + 2\delta\frac{x}{|x|}\cdot\nabla v - \frac{\lambda}{\sigma}v\right) \\ &\leq -\left\{1 - \delta\left(\frac{1}{2} + \varepsilon\right)\right\}|\nabla v|^2 - \left\{\frac{\lambda}{\sigma}\left(1 - \frac{\varepsilon}{2}\right) - \left(2\delta^2 + \frac{\delta}{2}\right)\right\}v^2 + \varepsilon\left(\frac{\lambda}{2\sigma} + \delta\right). \end{aligned}$$

Here we have used Cauchy's inequality $2ab \leq a^2 + b^2$ and the fact that $v < 0$ in the integrand of (B.2). Therefore, since $0 < \mu \leq \sigma(x) \leq M$, we can choose

$\delta > 0$ sufficiently small to obtain that if $0 < \varepsilon < 1$ then

$$I \le -\frac{1}{4}\left(|\nabla v|^2 + \frac{\lambda}{M}v^2\right) + \varepsilon\left(\frac{\lambda}{2\mu} + \delta\right)$$

and hence

$$\mu \int\limits_{D\cap\{v<-\varepsilon\}} e^{-\delta|x|}\left(|\nabla v|^2 + \frac{\lambda}{M}v^2\right)dx \le M\varepsilon\left(\frac{2\lambda}{\mu} + 4\delta\right)\int\limits_{D} e^{-\delta|x|}dx.$$

By choosing a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ as $n \to \infty$ and letting $n \to \infty$, we conclude that

$$\int\limits_{D\cap\{v<0\}} e^{-\delta|x|}\left(|\nabla v|^2 + \frac{\lambda}{M}v^2\right)dx = 0$$

and hence $v \ge 0$ in D. Therefore $w \ge 0$ in D. Once this is shown, the last part follows from the strong maximum principle (see [GT, Theorem 8.19, pp. 198–199]). $\square$

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