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The solution of the Kadison-Singer Problem

Adam Marcus and Nikhil Srivastava

Abstract. These lecture notes are meant to accompany two lectures given at the CDM 2016 conference, about the Kadison-Singer Problem. They are meant to complement the survey by the same authors (along with Spielman) which appeared at the last ICM. In the first part of this survey we will introduce the Kadison-Singer problem from two perspectives ($C*$ algebras and spectral graph theory) and present some examples showing where the difficulties in solving it lie. In the second part we will develop the framework of interlacing families of polynomials, and show how it is used to solve the problem. None of the results are new, but we have added annotations and examples which we hope are of pedagogical value.

Contents

1. The Kadison-Singer Problem

Since the Kadison-Singer Problem is a question in C^* algebras, we begin by recalling some basic definitions from that subject. Let $B(\ell_2)$ denote the algebra of bounded operators on the complex Hilbert space $\ell_2(\mathbb{N})$. Such

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operators may be identified with infinite dimensional matrices with bounded operator norm

$$\|M\| := \sup_{x \in \ell_2} \|Mx\|.$$

For our purposes, a C^* *algebra* is a subalgebra $\mathcal{A}$ of $B(\ell_2)$ which is closed in the operator norm topology, closed under taking adjoints (hence the C^*), and contains the identity. The most important example of a C^* algebra in the present context is $D(\ell_2)$, the algebra of bounded diagonal operators on ℓ_2, which may be identified with infinite diagonal matrices with bounded entries. Note that $D(\ell_2)$ is a *maximal abelian subalgebra* of $B(\ell_2)$.

Operator algebras were originally introduced by von Neumann as a rigorous mathematical framework for quantum mechanics, in which bounded self-adjoint operators play the role of physical observables (such as position, momentum, energy). Without making a full digression into quantum theory, we remark that the physical relevance of abelian subalgebras of $B(\ell_2)$ is that they are generated by observables which *commute*, implying that they can be measured simultaneously without being constrained by an uncertainty principle. The physical question that motivated the Kadison-Singer problem is roughly this:

> Given a quantum system (such as an electron in a hydrogen atom) does knowing the outcomes of all measurements with respect to a maximal set of commuting observables (such as the quantum numbers n, ℓ, m, s) *uniquely* determine the outcomes of all possible measurements of all possible observables?

The above is not meant to be mathematically rigorous, and we have left words such as “outcome” deliberately undefined, but we remark that such an assertion, interpreted appropriately, was believed to be true by Dirac .

We need one more notion to arrive at a mathematically precise formulation of the question. A *state* on a C^* algebra $\mathcal{A}$ is a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ with two properties: (a) $\phi(I) = 1$; (2) $\phi(M^*M) \geq 0$ for every $M \in \mathcal{A}$. It is easy to check that the set of states on $\mathcal{A}$ is convex and compact in the w^* topology; let us call this set $\mathcal{S}(A)$. By the Krein-Milman theorem, $\mathcal{S}(\mathcal{A})$ is the convex hull of its extreme points, which are the *pure states* on $\mathcal{A}$. States are supposed to correspond to physical states of a quantum system. The only other facts we will use about states are that they satisfy the Cauchy-Schwartz inequality:

$$|\phi(MN)|^2 \leq \phi(M^*M)\phi(N^*N)$$

and that $\phi(M) \leq \|M\|$.

The most familiar examples of states come from unit vectors: given any $\xi \in \ell_2$ with $\|\xi\| = 1$, it is easy to see that $\rho(M) := \langle \xi, M\xi \rangle$ satisfies (a) and (b). In finite dimensions, one can easily show using elementary linear algebra that these are the only pure states on $B(\mathbb{C}^n)$. This is not at all the case in infinite dimensions.

The Kadison-Singer Problem asks:

> Does every pure state on $D(\ell_2)$ have a unique extension to a state on $B(\ell_2)$?

If one restricts attention to *vector* pure states then the answer to the KSP is easily seen to be yes. The difficulty stems from the fact that the set of all pure states on $D(\ell_2)$ is substantially more complicated than this. For instance, one can take limits of pure states with respect to non-principal ultrafilters to produce pure states which are very different from vector states, and rather inaccessible in concrete terms.[1]

In their original paper Kadison and Singer outlined an elegant approach to proving the conjecture without having to say too much about $\mathcal{S}(D(\ell_2))$. The starting point is to observe that pure states are necessarily very well-behaved on a particular class of operators in $D(\ell_2)$, namely *diagonal projections.*

LEMMA 1.1. *If $P \in D(\ell_2)$ is a diagonal projection and ρ is a pure state on $D(\ell_2)$ then $\rho(P) = 0$ or $\rho(P) = 1$.*

PROOF. Suppose $\rho(P) = \lambda \in (0, 1)$. Observe by linearity that $\rho(I-P) = 1 - \lambda$. Consider the linear functionals $\rho_1, \rho_2 : D(\ell_2) \to \mathbb{C}$ defined by

$$\rho_1(M) := \frac{1}{\lambda}\rho(PM) \qquad \rho_2(M) := \frac{1}{1-\lambda}\rho((I-P)M),$$

and observe that they are both states. But now $\rho = \lambda\rho_1 + (1 - \lambda)\rho_2$, so ρ cannot be a pure state. □

Recall that our goal is to show that for every pure state $\rho : D(\ell_2) \to \mathbb{C}$ there is a unique extension $\hat{\rho} : B(\ell_2) \to \mathbb{C}$. It is clear that at least one canonical extension exists:

$$\hat{\rho}(M) := \rho(\mathrm{diag}(M)),$$

where $\mathrm{diag}(M)$ refers to the diagonal part of M, so to show that it is unique we must show that any extension $\hat{\rho}$ must satisfy

$$\hat{\rho}(M) = \rho(\mathrm{diag}(M)) = \hat{\rho}(\mathrm{diag}(M)),$$

or in other words

$$\hat{\rho}(M - \mathrm{diag}(M)) = 0$$

for every $M \in B(\ell_2)$. This is where diagonal projections and the key notion of a *paving* come in.

DEFINITION 1.2. An $\epsilon-$paving of an operator $M \in B(\ell_2)$ is a finite collection of diagonal projections $P_1, \ldots, P_k$ satisfying $P_1 + \ldots + P_k = I$ and

[1] One could argue that this is physically irrelevant, since non-principal ultrafilters require the axiom of choice and cannot arise in any physical situation, and one would be right. The question above is really a question about a certain mathematical apparatus around quantum mechanics, rather than about the physical world itself.

$$\|P_i M P_i\| \leq \epsilon \|M\|,$$

for every $i = 1, \ldots, k$.

Conjecture 1.3 (Paving Conjecture). *For every $\epsilon > 0$, every zero diagonal $M \in B(\ell_2)$ has an $\epsilon-$paving.*

Theorem 1.4 (Kadison-Singer). *The Paving Conjecture implies a positive solution to the Kadison-Singer Problem.*

Proof. Suppose ρ is a pure state on $D(\ell_2)$ and $\hat{\rho}$ is an extension of it to $B(\ell_2)$. Let $M \in B(\ell_2)$ and $N = M - \text{diag}(M)$, and fix $\epsilon > 0$. Let $P_1, \ldots, P_k$ be an $\epsilon-$paving of N. Observe by linearity that

$$\hat{\rho}(N) = \sum_{i,j \leq k} \hat{\rho}(P_i N P_j). \tag{1}$$

By Lemma 1.1 and linearity, we know that exactly one of the projections, say P_1, satisfies $\hat{\rho}(P_1) = 1$ and for the rest of them we have $\hat{\rho}(P_j) = 0$. By the Cauchy-Schwarz inequality, each term satisfies

$$|\hat{\rho}(P_i N P_j)| \leq \hat{\rho}(P_i^* P_i)\hat{\rho}(P_j^* N^* N P_j) \wedge \hat{\rho}(P_i^* N^* N P_i)\hat{\rho}(P_j^* P_j).$$

Since $P^* P = P$ for any projection P, this implies that only the first term in (1) is nonzero, and we have

$$\hat{\rho}(N) = \hat{\rho}(P_1 N P_1) \leq \|P_1 N P_1\| \leq \epsilon \|N\|,$$

by the paving property. Since $\epsilon > 0$ was arbitrary we conclude that $\hat{\rho}(N) = 0$, as desired. □

The pleasing feature of Conjecture 1.5 is that it makes no mention of pure states. The next major simplification was achieved by Anderson in 1979, who showed that this conjecture is readily implied by a simple to state conjecture about *finite* matrices.

Conjecture 1.5 (Finite Paving Conjecture). *For every $\epsilon > 0$ there is a $k = k(\epsilon)$ such that for every n, every zero diagonal complex $n \times n$ matrix M can be $\epsilon-$paved with k projections.*

Theorem 1.6 (Anderson). *The Finite Paving Conjecture implies the Infinite Paving Conjecture.*

We omit the short proof, which shows that a limit of finite pavings can be used to construct an infinite paving via an Arzela-Ascoli argument. The most important feature of this conjecture is that the number of projections k is allowed to depend only on ϵ and *not on the dimension* n, and this is because any dependence on n precludes a limit. It is also easy to see that it is sufficient to prove the conjecture for a single $\epsilon < 1$ and constant k, since then any smaller ϵ can be achieved by iteration. That said, it is substantially more accessible than the original KSP, and its statement is entirely elementary.

In the decades since Anderson's result, the paving conjecture was shown (using various finite dimensional linear algebra arguments) to be equivalent to several other statements about partitioning matrices or sets of vectors into submatrices or subsets which are "smaller" in some appropriate sense. In particular, the work of Casazza [**CEKP07**] et al. shows that it is equivalent to a number of other conjectures in various fields. A very tangible, combinatorial such statement is the following conjecture of Weaver, which is actually a family of statements indexed by $r \in \mathbb{N}$.

Conjecture 1.7 (Weaver KS_r). *There are universal constants $\epsilon > 0, \delta > 0$ such that the following holds. Suppose $v_1, \ldots, v_m \in \mathbb{C}^n$ are vectors satisfying $\sum_{i=1}^m v_i v_i^* = I$ and $\|v_i\| \leq \delta$. Then there is a partition $[m] = T_1 \cup T_2 \ldots \cup T_r$ such that for every $j = 1, \ldots, r$:*

$$\left\| \sum_{i \in T_j} v_i v_i^* \right\| \leq 1 - \epsilon. \tag{2}$$

The equivalence between this conjecture and paving is obtained by passing from paving zero diagonal matrices to paving Hermitian matrices to paving positive semidefinite matrices (by adding a multiple of the identity) and then to paving projection matrices (via a dilation argument)[2]. Dualizing the statement for projection matrices yields the family of statements KS_r. It was shown in [**Wea04**] that the validity of KS_r for any finite r is equivalent to Kadison-Singer.

The main result of [**MSS15b**] is a strong version of KS_r for every r:

Theorem 1.8. *Let $r > 1$ be an integer, and let $u_1, \ldots, u_m \in \mathbb{C}^d$ be vectors such that*

$$\sum_{i=1}^m \mathbb{E}\, u_i u_i^* = I_d \quad \textit{and} \quad \|u_i\|^2 \leq \delta \textit{ for all } i.$$

Then there exists a partition $\{A_1, \ldots A_r\}$ of $[m]$ such that

$$\left\| \sum_{i \in A_j} u_i {u_i}^* \right\| \leq \frac{1}{r} \left(1 + \sqrt{r\delta}\right)^2$$

Since the outer products of the vectors sum to the identity, the best one could hope for is to be able to split the vectors into r groups such that each was *exactly* $(1/r)I$. Hence Theorem 1.8 guarantees that for any vectors v_i, one can get within a factor of $\left(1 + \sqrt{r\delta}\right)^2$ of the best one could get with the best possible u_i.

[2]For an exposition of the (elementary) details of these reductions, the reader is encouraged to consult Tao's blog post https://terrytao.wordpress.com/tag/kadison-singer-problem/.

2. Spectral graph theory

In this section we describe a different, more recent story which leads to the same core problem KS_2, and which is in fact how the present authors were introduced to this problem. The question we consider is:

> Given a finite undirected graph, can it be approximated by a graph with very few edges?

The answer to this question of course depends on what we mean by approximate, and this is where the Laplacian operator comes in. Recall that the discrete Laplacian of a weighted graph $G = (V, E, w)$ may be defined as the following sum of rank one matrices over the edges:

$$L_G = \sum_{(a,b)\in E} w_{(a,b)}(e_a - e_b)(e_a - e_b)^T.$$

In the unweighted d−regular case, it is easy to see that $L = dI - A$, so the eigenvalues of the Laplacian are just d minus the eigenvalues of the adjacency matrix. The Laplacian matrix of a graph always has an eigenvalue of 0; this is a trivial eigenvalue, and the corresponding eigenvectors are the constant vectors.

Following Spielman and Teng, we say that two graphs G and H on the same vertex set V are *spectral approximations* of each other if their Laplacian quadratic forms multiplicatively approximate each other:

$$\kappa_1 \cdot x^T L_H x \le x^T L_G x \le \kappa_2 \cdot x^T L_H x \qquad \forall x \in \mathbb{R}^V,$$

for some approximation factors $\kappa_1, \kappa_2 > 0$. We will write this as

$$\kappa_1 \cdot L_H \preccurlyeq L_G \preccurlyeq \kappa_2 \cdot L_H,$$

where $A \preccurlyeq B$ means that $B - A$ is positive semidefinite, i.e., $x^T(B - A)x \ge 0$ for every x.

The complete graph on n vertices, K_n, is the graph with an edge of weight 1 between every pair of vertices. All of the eigenvalues of L_{K_n} other than 0 are equal to n. If G is a d-regular Ramanujan graph [**LPS88**], then 0 is the trivial eigenvalue of its Laplacian matrix, L_G, and all of the other eigenvalues of L_G are between $d - 2\sqrt{d-1}$ and $d + 2\sqrt{d-1}$. After a simple rescaling, this allows us to conclude that

$$(1 - 2\sqrt{d-1}/d)L_{K_n} \preccurlyeq (n/d)L_G \preccurlyeq (1 + 2\sqrt{d-1}/d)L_{K_n}.$$

So, $(n/d)L_G$ is a good approximation of L_{K_n}.

Batson, Spielman and Srivastava proved that every weighted graph has an approximation that is almost this good.

Theorem 2.1 ([**BSS12**]). *For every $d > 1$ and every weighted graph $G = (V, E, w)$ on n vertices, there exists a weighted graph $H = (V, F, \tilde{w})$ with $\lceil d(n-1)\rceil$ edges that satisfies:*

$$\left(1 - \frac{1}{\sqrt{d}}\right)^2 L_G \preccurlyeq L_H \preccurlyeq \left(1 + \frac{1}{\sqrt{d}}\right)^2 L_G. \tag{3}$$

However, their proof had very little to do with graphs. In fact, they derived their result from the following theorem about sparse weighted approximations of sums of rank one matrices.

THEOREM 2.2 ([**BSS12**]). *Let $v_1, v_2, \ldots, v_m$ be vectors in $\mathbb{R}^n$ with*

$$\sum_i v_i v_i^T = V.$$

For every $\epsilon \in (0,1)$, there exist non-negative real numbers s_i with

$$|\{i : s_i \neq 0\}| \leq \lceil n/\epsilon^2 \rceil$$

so that

$$(1-\epsilon)^2 V \preccurlyeq \sum_i s_i v_i v_i^T \preccurlyeq (1+\epsilon)^2 V. \tag{4}$$

Taking V to be a Laplacian matrix written as a sum of outer products and setting $\epsilon = 1/\sqrt{d}$ immediately yields Theorem 2.1.

Theorem 2.2 is very general and turned out to be useful in a variety of areas including graph theory, numerical linear algebra, and metric geometry (see, for instance, the survey of Naor [**Nao11**]). One of its limitations is that it provides no guarantees on the weights s_i that it produces, which can vary wildly. So it is natural to ask: is there a version of Theorem 2.2 in which all the weights are the same?

This may seem like a minor technical point, but it is actually a fundamental difference. In particular, Gil Kalai observed that the statement of Theorem 2.2 with $V = I$ is similar to Weaver's Conjecture. It turns out that the natural unweighted variant of it is essentially *the same* as Weaver's conjecture.

To make the connection, let us go back to the setting of KS_2 and observe that for any partition of a given set of vectors $v_1, \ldots, v_m$ we have:

$$\sum_{i \in S_1} v_i v_i^* + \sum_{i \in S_2} v_i v_i^* = I,$$

so that condition (2) is equivalent to

$$\epsilon I \preccurlyeq \sum_{i \in S_1} v_i v_i^* \preccurlyeq (1-\epsilon) I.$$

Thus, choosing a subset of the weights s_i to be non-zero in Theorem 2.2 is similar to choosing the set S_1. The essential difference is that Conjecture 1.7 assumes a bound on the lengths of the vectors v_i and in return requires the stronger conclusion that all of the s_i are either 0 or 1. It is easy to see that long vectors are an obstacle to the existence of a good partition; an extreme example is provided by considering an orthonormal basis $e_1, \ldots, e_n$. Weaver's conjecture asserts that this is the only obstacle.

3. Two examples and their expected characteristic polynomials

In this section we discuss two key examples which highlight the difficulties in solving Weaver's problem using familiar combinatorial and random matrix techniques.

Example 1 (Diagonal Case). Let $\delta > 0$ and $m = n/\delta$, and let $v_1, \ldots, v_m$ consist of $1/\delta$ copies each of $\sqrt{\delta}e_1, \sqrt{\delta}e_2, \ldots, \sqrt{\delta}e_n$, where e_i are the standard basis vectors in $\mathbb{C}^n$. Then it is clear that $\|v_i\|^2 = \delta$ for every i and

$$\sum_{i \le m} v_i v_i^* = \frac{1}{\delta} \sum_{i \le n} \delta e_i e_i^* = I.$$

It is not hard to find a balanced partition in this example: for each standard basis vector e_i, simply divide the copies of that vector into subsets of almost equal size. Note that this simple *deterministic* strategy crucially requires knowing that the given vectors can be split up into n groups, each of which is a one-dimensional instance of KS_2. Also note that it would not be as clear how to proceed if one were to (say) add a small amount of noise to each vector — a clustering approach might still work, but would be somewhat nontrivial.

On the other hand, balanced partitions of these vectors are exponentially rare. To see this, consider a uniformly random partition of $v_1, \ldots, v_m$ into $T_1 \cup T_2$. Then T_1 is a random subset of $[m]$, containing each v_i with probability $1/2$. Thus, for any i the probability that all copies of e_i appear in T_1 is $(1/2)^{1/\delta}$, and the probability that this does *not* happen for all $i = 1, \ldots, n$ is:

$$\left(1 - 2^{-1/\delta}\right)^n \approx \exp(-n2^{-1/\delta}),$$

which is exponentially small unless $\delta \le 1/\log(n)$. A similar probability is obtained even if we consider random balanced partitions with $|T_1| = |T_2|$, but we omit the details.

The second example exhibits exactly the opposite kind of behavior and is given by random vectors.

Example 2 (Random Case). Let $\delta > 0$ and $m = n/\delta$ and let $v_1, \ldots, v_m \in \mathbb{R}^n$ be i.i.d. random Gaussian vectors scaled so that

$$\mathbf{E}\|v_i\|^2 = \delta.$$

By standard concentration inequalities (see e.g. [**Bar05**]) we have:

$$\mathbb{P}[\|v_i\|^2 > (1+t)\delta] \le \exp(-t^2 n/4),$$

which implies by a union bound that

$$\max_i \|v_i\|^2 \le (1 + o(1))\delta \qquad \text{with probability at least} \quad 1 - \exp(-cn),$$

as long as $m = \exp(o(n))$. Moreover, by well-known properties of rectangular Gaussian random matrices (e.g., Sections 5.3 and 5.4 of [**Ver10**]) the eigenvalues of

$$V = \sum_{i \leq m} v_i v_i^T$$

are contained in the interval $[(1-\sqrt{\delta}-o(1))^2, (1+\sqrt{\delta}+o(1))^2]$ with exponentially good probability. Thus, the vectors $w_i := V^{-1/2} v_i$ satisfy the conditions of KS_2 with constant at most 5δ whenever (say) $\delta < 1/4$.

It is not clear how to deterministically partition a typical instance of Example 2 — in particular, pairs of vectors are not orthogonal and they do not naturally decompose into lower-dimensional instances of KS_2.

However, in contrast to the previous example, a random partition works very well here. If we take T_1 to be a random subset of $[m]$ of size $m/2$, then

$$V_1 = \sum_{i \in T_1} v_i v_i^T$$

is itself a Wishart matrix whose expectation is $I/2$. Again by the Bai-Yin theorem, we conclude that the eigenvalues of V_1 are contained in $[\frac{1}{2}(1-\sqrt{2\delta}+o(1))^2, \frac{1}{2}(1+\sqrt{2\delta}-o(1))^2]$ with exponentially high probability. The same is true for $T_2 = [m] \setminus T_1$, so we conclude that a random partition is balanced with high probability.

The difficulty of KS_2 arises from the fact that there are no tools which readily handle both examples and the various possible combinations of them.

The following well-known result in random matrix theory can be used to analyze a random partition, by taking $A_i = v_i v_i^T$ and taking T_1 to be all i such that $\epsilon_i = +1$.

THEOREM 3.1 (Matrix Chernoff [**Tro12**]). *Given random Hermitian matrices $A_1 \ldots A_m \in \mathbb{C}^{n \times n}$ and independent Bernoulli signs $\epsilon_1, \ldots, \epsilon_m$, we have*

$$\mathbb{P}\left[\|\sum_i \epsilon_i A_i\| \geq t\right] \leq n \cdot \exp(-\frac{t^2}{2\|\sum_i A_i^2\|}).$$

Note that for an instance of KS_2 we have $\sum_i A_i^2 = \sum_i \|v_i\|^2 v_i v_i^T \leq \delta I$, so the above probability is less than one when t is a small constant and $\delta \leq c/\log(n)$, yielding a balanced partition. The tightness of this result is witnessed by Example 1, which shows that for larger δ the probability of being unbalanced quickly approaches one. Unfortunately, a result in which δ depends on n is insufficient for Kadison-Singer (since we will take a limit of pavings as $n \to \infty$) and also for various graph theory applications.

There are of course many other results about random matrices which can be used to analyze specific families of instances. However, results which establish dimension-free bounds (without the fatal $\log(n)$ factor introduced by Matrix Chernoff) typically rely on high symmetry assumptions about the vectors (such as i.i.d. entries) or on strong geometric regularity properties (such as log-concavity), which are far too restrictive for the general case. Thus, new ideas are required.

Expected characteristic polynomials. The Method of Interlacing Families is a way of analyzing certain random matrices which is oblivious to the diagonal/random dichotomy above, i.e., it is able to provide a uniform dimension-free bound on both cases and on everything in between. Unlike most results about random matrices, it provides estimates on the eigenvalues which hold with *exponentially small* but nonetheless nonzero probability. As such the method is not really probabilistic in nature, but the language of probability theory provides a convenient notation.

The central idea is to access the distribution of the eigenvalues of a random matrix via its *expected characteristic polynomial*:

$$\mathbf{E}\chi(A) = \mathbf{E}\det(xI - A).$$

Before describing the general approach, let's examine the expected characteristic polynomials corresponding to Examples 1 and 2 above.

Example 3 ($\mathbf{E}\chi$ for the Diagonal Case). Let $v_1, \dots, v_m$ be as in Example 1 with $m = kn$ where $k = 1/\delta$ is an integer. Let

$$A = \sum_{i \le m} b_i v_i v_i^T,$$

where $b_1, \dots, b_m$ are i.i.d. random variables each 0 with probability 1/2 and 1 otherwise, corresponding to a random subset of the vectors. Certainly $\mathbf{E}A = I/2$, and the A is diagonal with independent entries $A_1, \dots, A_n$ indicating the number of times the vector $\sqrt{\delta}e_i$ is chosen. We now have

$$\begin{aligned}\mathbf{E}\det(xI - A) &= \mathbf{E}(x - A_1)(x - A_2)\dots(x - A_n)\\ &= \prod_{i \le n} \mathbf{E}(x - A_i) \quad \text{since the } A_i \text{ are independent}\\ &= (x - 1/2)^n.\end{aligned}$$

There are two interesting things about this calculation. The first is that the expected characteristic polynomial is real-rooted, which is in general not the case since real-rootedness is not necessarily preserved under taking sums. The second is that the roots reflect the behavior of (one half of) the ideal balanced partition: the bad allocations with A_i that are too large or small seem to have "cancelled out".

Example 4 ($\mathbf{E}\chi$ for the Random Case). Let $v_1, \dots, v_m$ be Gaussian random vectors with norm $\mathbf{E}\|v_i\|^2 = \delta$ and let $A = \sum_{i \le m/2} v_i v_i^T$ be the empirical covariance matrix of half of them. Observe that for any matrix M and a single Gaussian random vector v with $\mathbf{E}\|v\|^2 = \delta$ we have:

$$
\begin{aligned}
\mathbf{E}\det(xI - M - vv^T) &= \det(xI - M)\cdot \mathbf{E}(1 - v^T(xI-M)^{-1}v) \\
&= \det(xI-M)(1 - \mathrm{Tr}((xI-M)^{-1} \bullet \mathbf{E}vv^T)) \\
&= \det(xI-M) - \delta \det(xI-M)\cdot \mathrm{Tr}(xI-M)^{-1} \\
&= \left(1 - \delta \frac{d}{dx}\right)\det(xI-M)
\end{aligned}
$$

Thus, adding a random rank one Gaussian outer product corresponds to subtracting off a multiple of the derivative of the characteristic polynomial. Since the v_i in our example are independent, we may apply this fact inductively to conclude that

$$
\mathbf{E}\det(xI - A) = \left(1 - \delta \frac{d}{dx}\right)^{m/2} x^n. \tag{5}
$$

We now observe that if a polynomial $f(x) = \prod_{i\le n}(x - \lambda_i)$ has real roots, then $f(x) - cf'(x)$ also has real roots for every real c; the reason is that $f(x) - cf'(x) = 0$ precisely when

$$
\sum_{i=1}^{n} \frac{1}{x - \lambda_i} = f'(x)/f(x) = 1/c. \tag{6}
$$

By examining the behavior of this rational function between its poles and noting that every multiple root of f is also a root of f' with one less multiplicity, we see that the number of solutions to (6) is equal to the degree of f.

Thus, we conclude that (5) has real roots. But more is true: it turns out that these polynomials are exactly equal to certain orthogonal polynomials known as the *associated Laguerre polynomials*, whose roots have been studied in great detail (see [**MSS14**] Section 3.2 for more details). This connection implies that the roots of $\mathbf{E}\det(xI - A)$ are contained in the interval $[\frac{1}{2}(1-\sqrt{2\delta})^2, \frac{1}{2}(1+\sqrt{2\delta})^2]$, which is precisely what is expected for a random partition in 2.

Thus, in both of the extreme cases, the expected characteristic polynomial is real-rooted and captures the behavior of the kind of partition that we want — greedy in the diagonal case, and random in the random case. It turns out that this is not an accident and for a large class of random matrices the expected characteristic polynomial *always* has real roots, a property which can be used to relate the roots to the distribution of the eigenvalues of the matrix itself. Then, tools from the analytic theory of polynomials can be used to bound the roots, and thereby obtain information about the eigenvalues.

The main theorem produced by this approach is the following:

THEOREM 3.2. *Let $\epsilon > 0$ and let $v_1, \ldots, v_m$ be independent random vectors in $\mathbb{C}^d$ with finite support such that*

$$\sum_{i=1}^{m} \mathbb{E}\, v_i v_i^* = I_d, \tag{7}$$

and

$$\mathbb{E}\, \|v_i\|^2 \leq \epsilon, \textit{for all } i.$$

Then

$$\mathbb{P}\left[\left\|\sum_{i=1}^{m} v_i v_i^*\right\| \leq (1+\sqrt{\epsilon})^2\right] > 0$$

The above theorem compares favorable to what is yielded by the Matrix Chernoff bound, which is that $\| \sum_{i=1}^{m} v_i v_i^* \| \leq C(\epsilon) \cdot \log n$ with high probability. Here we are able to control the deviation at the much smaller scale $(1+\sqrt{\epsilon})^2$, but only with nonzero probability.

In the remainder of this document, we show how to prove this theorem.

4. Interlacing families

Our proofs inherently rely on bounding the largest eigenvalue of various matrices. One key idea in our methods for obtaining such bounds is the use of characteristic polynomials in our analysis. Since the eigenvalues of a matrix A are exactly the roots of its characteristic polynomial $\det(xI - A)$, this does not seem to gain us any leverage. The leverage comes, however, when we replace *random* matrices with *random* characteristic polynomials.

On the surface this may seem like an odd idea. In general, the roots of an average of real rooted polynomials are not necessarily related in any way to the roots of the original collection. However, there are situations where this works quite well.

LEMMA 4.1. *Let $p_1, \ldots, p_k$ be polynomials and $[s, t]$ an interval such that*

- *each $p_i(s)$ has the same sign (or is 0)*
- *each $p_i(t)$ has the same sign (or is 0)*
- *each p_i has exactly one real root in $[s, t]$.*

Then $\sum_i p_i$ has exactly one real root in $[s, t]$ and it lies between the roots of some p_a and p_b.

PROOF. This is illustrated in Figure 1 (the blue line is the average of the red ones). Let $p(x) = \sum_i p_i(x)$ and without loss of generality, assume $p_i(s) \geq 0$ for all i. Then $p(x) \geq 0$ and, since each polynomial switches signs somewhere in the interval $[s, t]$ we have $p(t) \leq 0$. By continuity, there must be a point $r \in [s, t]$ at which $p(r) = 0$.

Now if we look at the value of each p_i at the point r, we know the values must add up to 0. Hence there exist polynomials p_a and p_b such that $p_a(r) \leq 0 \leq p_b(r)$ and these will have roots which are smaller (larger) than r (respectively). □

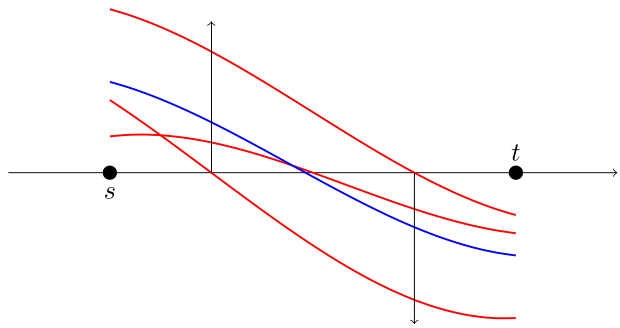

FIGURE 1. Picture of Lemma 4.1.

Lemma 4.1 asserts that, as long as our collection of polynomials has its roots bunched together inside disjoint intervals, then the sum of the polynomials is real rooted and one can compare the roots of the sum to the individual polynomials. To characterize the collections of polynomials for which this holds, we recall the definition of interlacing polynomials.

DEFINITION 4.2. We say that a real rooted polynomial $g(x) = \alpha_0 \prod_{i=1}^{n-1}(x-\alpha_i)$ *interlaces* a real rooted polynomial $f(x) = \beta_0 \prod_{i=1}^{n}(x-\beta_i)$ if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n$$

We say that $g(x)$ *strictly interlaces* $f(x)$ if all of these inequalities are strict. We say that polynomials $f_1(x), \ldots, f_k(x)$ have a *common interlacing* if there is a polynomial $g(x)$ so that $g(x)$ interlaces $f_i(x)$ for each i.

In the event that a collection of polynomials $f_1, \ldots f_k$ has a common interlacer g, the roots of g separate the roots of the f_i in exactly the way necessary for Lemma 4.1 to hold. This leads to the following corollary:

COROLLARY 4.3. *Let $f_1, \ldots, f_k$ be polynomials of the same degree that are real-rooted and have positive leading coefficients. Define*

$$f_\emptyset = \sum_{i=1}^{k} f_i.$$

If $f_1, \ldots, f_k$ have a common interlacing, then there exists an i so that the largest root of f_i is at most the largest root of $f_\emptyset$.

The hope would be to apply Lemma 4.3 to the collection of polynomials defined in Section 5. These polynomials, however, do not have a common interlacing. Instead, we will need to use Lemma 4.3 inductively on subcollections of these polynomials that do have a common interlacing. This inspires the following definition from [**MSS15a**]:

DEFINITION 4.4. Let $S_1, \ldots, S_m$ be finite sets and for every assignment $s_1, \ldots, s_m \in S_1 \times \cdots \times S_m$ let $f_{s_1,\ldots,s_m}(x)$ be a real-rooted degree n polynomial

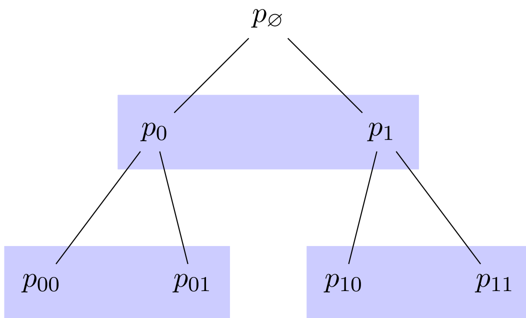

FIGURE 2. A tree of partial assignment polynomials. The purple blocks denote subsets of polynomials that have a common interlacer.

with positive leading coefficient. For a partial assignment $s_1, \ldots, s_k \in S_1 \times \ldots \times S_k$ with $k < m$, define

$$f_{s_1,\ldots,s_k} \stackrel{\text{def}}{=} \sum_{s_{k+1} \in S_{k+1}, \ldots, s_m \in S_m} f_{s_1,\ldots,s_k,s_{k+1},\ldots,s_m},$$

as well as

$$f_\emptyset \stackrel{\text{def}}{=} \sum_{s_1 \in S_1, \ldots, s_m \in S_m} f_{s_1,\ldots,s_m}.$$

We say that the polynomials $\{f_{s_1,\ldots,s_m}\}$ form an *interlacing family* if for all $k = 0, \ldots, m-1$, and all $s_1, \ldots, s_k \in S_1 \times \cdots \times S_k$, the polynomials

$$\{f_{s_1,\ldots,s_k,t}\}_{t \in S_{k+1}}$$

have a common interlacing.

Given an interlacing family, one can form a tree of partial assignment polynomials with $f_\emptyset$ at the top (we avoid saying "root" since we already use it as a synonym for "zeroes") and where each polynomial $f_{s_1,\ldots,s_k}$ will have the collection of polynomials $\{f_{s_1,\ldots,s_k,t}\}_{t \in S_{k+1}}$ as its children. The idea, then, will be to apply Lemma 4.3 iteratively as one walks down the tree (see Figure 2).

THEOREM 4.5. *Let $S_1, \ldots, S_m$ be finite sets and let $\{f_{s_1,\ldots,s_m}\}$ be an interlacing family of polynomials. Then, there exists some $s_1, \ldots, s_m \in S_1 \times \cdots \times S_m$ so that the largest root of $f_{s_1,\ldots,s_m}$ is less than the largest root of $f_\emptyset$.*

PROOF. From the definition of an interlacing family, we know that the polynomials $\{f_t\}$ for $t \in S_1$ have a common interlacing and that their sum is $f_\emptyset$. So, Lemma 4.3 tells us that one of the polynomials has largest root at most the largest root of $f_\emptyset$. We now proceed inductively. For any $s_1, \ldots, s_k$, we know that the polynomials $\{f_{s_1,\ldots,s_k,t}\}$ for $t \in S_{k+1}$ have a common interlacing and that their sum is $f_{s_1,\ldots,s_k}$. So, for some choice of t (say s_{k+1}) the largest root of the polynomial $f_{s_1,\ldots,s_{k+1}}$ is at most the largest root of $f_{s_1,\ldots,s_k}$. □

Our first goal will be to prove that the characteristic polynomials of sums of independent rank one random matrices form an interlacing family. According to Definition 4.4, this requires establishing the existence of certain common interlacings. We will do this using the fact that common interlacings are equivalent to real-rootedness statements, a result which seems to have been discovered a number of times. The following appears as Theorem 2.1 of Dedieu [**Ded92**], (essentially) as Theorem $2'$ of Fell [**Fel80**], and as (a special case of) Theorem 3.6 of Chudnovsky and Seymour [**CS07**].

LEMMA 4.6. *Let $f_1, \ldots, f_k$ be (univariate) polynomials of the same degree with positive leading coefficients. Then $f_1, \ldots, f_k$ have a common interlacing if and only if $\sum_{i=1}^{k} \lambda_i f_i$ is real rooted for all convex combinations $\lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1$.*

We establish the necessary real-rootedness statements in Section 5.

Our second goal will be to bound the largest root of the corresponding $f_\emptyset$ and then use Theorem 4.5 to assert the existance of some polynomial in the original collection that has largest root smaller than $f_\emptyset$.

The analysis will benefit from a slight generalization of interlacing that allows for polynomials to have the same degree. Let f and g be real rooted polynomials and let r_f and r_g be their smallest roots (respectively).

DEFINITION 4.7. We say that f *subinterlaces* g (written $f \ll g$) if either

1. f interlaces g, or
2. $r_f \leq r_g$ and $f(x)/(x - r_f)$ interlaces g.

In particular, we will use the following theorem of Hermite, Kakeya, and Obreschkoff:

THEOREM 4.8. *Hermite-Kakeya-Obreschkoff Let f and g be real rooted polynomials. Then $af + bg$ is real rooted for all $a, b \in \mathbb{R}$ if and only if $f \ll g$ or $g \ll f$.*

5. The mixed characteristic polynomial

In this section, we obtain a useful formula for the expected characteristic polynomials which are relevant to the proof of Theorem 3.2, and show that these polynomials are always real-rooted, which is crucial to the interlacing method.

We begin by recording some well-known facts from linear algebra. For a Hermitian matrix $M \in \mathbb{C}^{d \times d}$ we write the characteristic polynomial of M in a variable x as

$$\chi[M](x) = \det(xI - M).$$

The following lemma is sometimes known as the *matrix determinant lemma* or *rank-1 update formula.*

LEMMA 5.1. *If A is an invertible matrix and u, v are vectors, then*

$$\det(A + uv^*) = \det(A)(1 + v^* A^{-1} u)$$

We will utilize Jacobi's formula for the derivative of the determinant of a matrix.

THEOREM 5.2. *If A and B are matrices of the same dimensions and A is invertible, then*

$$\partial_t \det(A + tB)\Big|_{t=0} = \det(A) \operatorname{Tr}\left(A^{-1}B\right).$$

Using the previous two results, we have the following easy corollary.

COROLLARY 5.3. *For an invertible matrix A and random vector v, we have*

$$\mathbb{E}\det(A - vv^*) = (1 - \partial_t) \det\left(A + t\,\mathbb{E}\,vv^*\right)\Big|_{t=0}$$

PROOF. By Lemma 5.1, we have

$$\begin{aligned}
\mathbb{E}\det(A - vv^*) &= \mathbb{E}\det(A)\left(1 - v^*A^{-1}v\right) \\
&= \mathbb{E}\det(A)\left(1 - \operatorname{Tr}\left(A^{-1}vv^*\right)\right) \\
&= \det(A) - \det(A)\,\mathbb{E}\operatorname{Tr}\left(A^{-1}vv^*\right) \qquad (8)
\end{aligned}$$

On the other hand, by Theorem 5.2, we have

$$\begin{aligned}
(1 - \partial_t)\det\left(A + t\,\mathbb{E}\,vv^*\right)\Big|_{t=0} &= \det\left(A + t\,\mathbb{E}\,vv^*\right)\Big|_{t=0} \\
&\quad - \det(A)\operatorname{Tr}\left(A^{-1}\,\mathbb{E}\,vv^*\right) \\
&= \det(A) - \det(A)\operatorname{Tr}\left(A^{-1}\,\mathbb{E}\,vv^*\right)
\end{aligned}$$

which is the same as (8) by switching the order of summation in the expectation/trace. □

Let $v_1, \ldots, v_m$ be independent random column vectors in $\mathbb{C}^d$ with finite support. For each i, let $A_i = \mathbb{E}\,v_iv_i^*$. Then,

THEOREM 5.4.

$$\mathbb{E}\,\chi\left[\sum_{i=1}^{m} v_iv_i^*\right](x) = \left(\prod_{i=1}^{m} 1 - \partial_{z_i}\right)\det\left(xI + \sum_{i=1}^{m} z_iA_i\right)\Bigg|_{z_1=\cdots=z_m=0}. \qquad (9)$$

PROOF. For a positive definite matrix M, set

$$a_k(M) = \mathbb{E}\det\left(M - \sum_{i=1}^{k} v_iv_i^*\right)$$

and

$$b_k(M) = \left(\prod_{i=1}^{k} 1 - \partial_{z_i}\right)\det\left(M + \sum_{i=1}^{k} z_iA_i\right)\Bigg|_{z_1=\cdots=z_k=0}.$$

We will prove by induction (on k) that $a_k(M) = b_k(M)$. As the base case, we have

$$a_0(x) = \mathbb{E}\det(M) = \det(M) = b_0(x)$$

so we may assume $a_i(M) = b_i(M)$ for all $i < k$. Now Corollary 5.3 implies

$$\begin{aligned} a_k(M) &= \mathbb{E} \det \left(M - \sum_{i=1}^{k} v_i v_i^* \right) \\ &= \mathop{\mathbb{E}}_{v_1,\dots,v_{k-1}} \mathop{\mathbb{E}}_{v_k} \det \left(M - \sum_{i=1}^{k-1} v_i v_i^* - v_k v_k^* \right) \\ &= \mathop{\mathbb{E}}_{v_1,\dots,v_{k-1}} (1 - \partial_{z_k}) \det \left(M - \sum_{i=1}^{k-1} v_i v_i^* + z_k A_k \right) \Big|_{z_k=0} \end{aligned}$$

and so by the inductive hypothesis (with $M' = M + z_k A_k$, which for z_k sufficiently close to 0 is still positive definite), we get

$$\begin{aligned} a_k(M) &= (1 - \partial_{z_k}) \left(\prod_{i=1}^{k-1} 1 - \partial_{z_i} \right) \det \left(M' + \sum_{i=1}^{k-1} z_i A_i \right) \Big|_{z_1=\dots=z_{k-1}=0} \Big|_{z_k=0} \\ &= \left(\prod_{i=1}^{k} 1 - \partial_{z_i} \right) \det \left(M + \sum_{i=1}^{k} z_i A_i \right) \Big|_{z_1=\dots=z_k=0} \\ &= b_k(M). \end{aligned}$$

Hence $a_k(M) = b_k(M)$ for all positive definite M. In particular, $a_m(xI) = b_m(xI)$ for $x > 0$. But $a_m(xI)$ and $b_m(xI)$ are finite degree polynomials, so equality on any interval implies equality everywhere. □

In particular, the above formula shows that the expected characteristic polynomial is a function of the covariance matrices A_i. We call this polynomial the *mixed characteristic polynomial* of $A_1, \dots, A_m$, and denote it by $\mu [A_1, \dots, A_m] (x)$.

To see that these polynomials are real-rooted, we draw on the theory of multivariate real stable polynomials.

Stable polynomials. For a complex number z, let $\mathsf{Im}(z)$ denote its imaginary part. We recall that a polynomial $p(z_1, \dots, z_m) \in \mathbb{C}[z_1, \dots, z_m]$ is *stable* if whenever $\mathsf{Im}(z_i) > 0$ for all i, $p(z_1, \dots, z_m) \neq 0$. A polynomial p is *real stable* if it is stable and all of its coefficients are real. A univariate polynomial is real stable if and only if it is real rooted (as defined at the beginning of Section 4).

One of the classical theorems in this area gives a direct link between interlacing and real stability.

THEOREM 5.5 (Hermite-Biehler). *Let f and g be polynomials with real coefficients. Then $g + if$ is stable if and only if f and g have all real roots and $f \ll g$.*

To prove that the polynomials we construct in this paper are real stable, we begin with an observation of Borcea and Brändén [**BB08**, Proposition 2.4].

PROPOSITION 5.6. *If $A_1, \ldots, A_m$ are positive semidefinite Hermitian matrices, then the polynomial*

$$\det\left(\sum_i z_i A_i\right)$$

is real stable.

We will generate new real stable polynomials from the one above by applying operators of the form $(1 - \partial_{z_i})$. One can use general results, such as Theorem 1.3 of [**BB10**] or Proposition 2.2 of [**LS81**], to prove that these operators preserve real stability. It is also easy to prove it directly using the fact that the analogous operator on univariate polynomials preserves stability of polynomials with complex coefficients. For example, the following theorem appears as Corollary 18.2a in Marden [**Mar85**], and is similar to Corollary 5.4.1 of Rahman and Schmeisser [**RS02**].

THEOREM 5.7. *If all the zeros of a degree d polynomial $q(z)$ lie in a (closed) circular region A, then for $\lambda \in \mathbb{C}$, all the zeros of*

$$q(z) - \lambda q'(z)$$

lie in the convex region swept out by translating A in the magnitude and direction of the vector $d\lambda$.

COROLLARY 5.8. *If $p \in \mathbb{R}[z_1, \ldots, z_m]$ is real stable, then so is*

$$(1 - \lambda\partial_{z_1})p(z_1, \ldots, z_m).$$

for any $\lambda \in \mathbb{R}$.

PROOF. Let $x_2, \ldots, x_m$ be numbers with positive imaginary part. Then, the univariate polynomial

$$q(z_1) = p(z_1, z_2, \ldots, z_m)\big|_{z_2=x_2,\ldots,z_m=x_m}$$

is stable. That is, all of its zeros lie in the circular region consisting of numbers with non-positive imaginary part. As this region is invariant under translation by d, $(1 - \lambda\partial_{z_1})q(z)$ is stable. This implies that $(1 - \lambda\partial_{z_1})p$ has no roots in which all of the variables have positive imaginary part. □

We will also use the fact that real stability is preserved under setting variables to real numbers (see, for instance, [**Wag11**, Lemma 2.4(d)]).

PROPOSITION 5.9. *If $p \in \mathbb{R}[z_1, \ldots, z_m]$ is real stable and $a \in \mathbb{R}$, then $p|_{z_1=a} = p(a, z_2, \ldots, z_m) \in \mathbb{R}[z_2, \ldots, z_m]$ is real stable.*

Now it is immediate from Proposition 5.6 and Corollary 5.8 that the mixed characteristic polynomial is real rooted.

COROLLARY 5.10. *The mixed characteristic polynomial of positive semidefinite Hermitian matrices is real rooted.*

PROOF. Proposition 5.6 tells us that

$$\det\left(xI + \sum_{i=1}^{m} z_i A_i\right)$$

is real stable. Corollary 5.8 tells us that

$$\left(\prod_{i=1}^{m} 1 - \partial_{z_i}\right) \det\left(xI + \sum_{i=1}^{m} z_i A_i\right)$$

is real stable as well. Finally, Proposition 5.9 shows that setting all of the z_i to zero preserves real stability. As the resulting polynomial is univariate, it is real rooted. □

Finally, we use the real rootedness of mixed characteristic polynomials to show that every sequence of independent finitely supported random vectors $v_1, \ldots, v_m$ defines an interlacing family. For $i \in [m]$, let ℓ_i be the size of the support of v_i, and let

$$\mathbb{P}\left[v_i = w_{i,j}\right] = p_{i,j}$$

for $j = 1, \ldots, \ell_i$.

For a vector $s \in [l_1] \times \cdots \times [l_m]$, we define

$$q_s(x) = \left(\prod_{i=1}^{m} p_{i,s_i}\right) \chi\left[\sum_{i=1}^{m} w_{i,s_i} w_{i,s_i}^*\right](x).$$

THEOREM 5.11. *The polynomials q_s form an interlacing family.*

PROOF. For $0 \leq k < m$ and $t \in [l_1] \times \cdots \times [l_k]$, we will write the conditionally expected polynomials

$$q_t(x) = \left(\prod_{i=1}^{k} p_{i,t_i}\right) \mathop{\mathbb{E}}_{v_{k+1},\ldots,v_m} \chi\left[\sum_{i=1}^{k} w_{i,t_i} w_{i,t_i}^* + \sum_{j=k+1}^{m} v_j v_j^*\right](x).$$

In particular,

$$q_\emptyset(x) = \mathop{\mathbb{E}}_{v_1,\ldots,v_m} \chi\left[\sum_{j=1}^{m} v_j v_j^*\right](x).$$

is the expected characteristic polynomial of the random matrix appearing in Theorem3.2. For a given $t = t_1, \ldots, t_k$ and a given $r \in \ell_{k+1}$ let (t, r) denote the vector $t_1, \ldots, t_k, r$. In this language we need to prove that for every t, the polynomials $\{q_{(t,r)}(x) : r \in \ell_{k+1}\}$ have a common interlacing. By Lemma 4.6, it suffices to prove that for any choice of real numbers $\{\alpha_r\}$ with $0 \leq \alpha_r \leq 1$ and $\sum_r \alpha_r = 1$, the polynomial

$$\sum_{r \in \ell_{k+1}} \alpha_r q_{(t,r)}(x) \tag{10}$$

is real-rooted. But notice that

$$\begin{aligned} q_t(x) &= \left(\prod_{i=1}^{k} p_{i,t_i}\right) \mathop{\mathbb{E}}_{v_{k+1},\dots,v_m} \chi\left[\sum_{i=1}^{k} w_{i,t_i} w_{i,t_i}^* + \sum_{j=k+1}^{m} v_j v_j^*\right](x). \\ &= \sum_{r\in\ell_{k+1}} \left(\prod_{i=1}^{k} p_{i,t_i}\right) p_{k+1,r} \\ &\quad \times \mathop{\mathbb{E}}_{v_{k+2},\dots,v_m} \chi\left[\sum_{i=1}^{k-1} w_{i,j} w_{i,j}^* + w_{k+1,r} w_{k+1,r}^* + \sum_{i=k+2}^{m} v_i v_i^*\right](x) \\ &= \sum_{r\in\ell_{k+1}} p_{k+1,r}\, q_{(t,r)}(x) \end{aligned}$$

which (if we set $\alpha_r = p_{k+1,r}$) is precisely the polynomial in (10). Thus it suffices to show that q_t is real-rooted (independent of the values of $p_{k+1,r}$). Denoting $\mathbb{E}\, v_i v_i^* = A_i$, we have that for $t = t_1, \dots, t_k$,

$$q_t(x) = \left(\prod_{i=1}^{k} p_{i,t_i}\right) \mu\left[w_{1,t_1} w_{1,t_1}^*, \dots, w_{k,t_k} w_{k,t_k}^*, A_{k+1}, \dots, A_m\right](x),$$

a multiple of a mixed characteristic polynomial. But by Corollary 5.10, such a polynomial is real-rooted regardless of what $A_{k+1} = \mathbb{E}\, v_{k+1} v_{k+1}^*$ is, and therefore is real-rooted independent of the distribution on v_k, as needed. □

6. The multivariate barrier argument

Our goal in this section is to prove an upper bound on the roots of the mixed characteristic polynomial $\mu[A_1, \dots, A_m](x)$ as a function of the A_i, in the case of interest $\sum_{i=1}^{m} A_i = I$. Our main theorem is:

THEOREM 6.1. *Let $A_1, \dots, A_m$ be positive semidefinite Hermitian matrices satisfying*

$$\sum_{i=1}^{m} A_i = I \quad \textit{and} \quad \mathrm{Tr}\,(A_i) \le \epsilon$$

for all i. Then the largest root of $\mu[A_1, \dots, A_m](x)$ is at most $(1 + \sqrt{\epsilon})^2$.

We begin by performing a simple but useful change of variables that will allow us to reason separately about the effect of each A_i on the roots of $\mu[A_1, \dots, A_m](x)$.

LEMMA 6.2. *Let $A_1, \dots, A_m$ be Hermitian positive semidefinite matrices. If $\sum_i A_i = I$, then*

$$\mu[A_1, \dots, A_m](x) = \left(\prod_{i=1}^{m} 1 - \partial_{y_i}\right) \det\left(\sum_{i=1}^{m} y_i A_i\right)\Big|_{y_1=\dots=y_m=x}. \tag{11}$$

PROOF. For any differentiable function f, we have

$$\partial_{y_i}(f(y_i))\big|_{y_i=z_i+x} = \partial_{z_i} f(z_i + x).$$

So, the lemma follows by substituting $y_i = z_i + x$ into the expression (11), and observing that it produces the expression on the right hand side of (9). $\square$

Let us write

$$\mu\left[A_1, \ldots, A_m\right](x) = Q(x, x, \ldots, x), \tag{12}$$

where $Q(y_1, \ldots, y_m)$ is the multivariate polynomial on the right hand side of (11). The bound on the roots of $\mu\left[A_1, \ldots, A_m\right](x)$ will follow from a "multivariate upper bound" on the roots of Q, defined as follows.

DEFINITION 6.3. Let $p(z_1, \ldots, z_m)$ be a multivariate polynomial. We say that $z \in \mathbb{R}^m$ is *above* the roots of p if

$$p(z + t) > 0 \qquad \text{for all} \qquad t = (t_1, \ldots, t_m) \in \mathbb{R}^m, t_i \geq 0,$$

i.e., if p is positive on the nonnegative orthant with origin at z.

We will denote the set of points which are above the roots of p by Ab_p (for convenience, we will say that $\mathsf{Ab}_0 = \mathbb{R}^m$). A simple lemma we will find useful is that the region above the roots of p never shrinks under the operation of partial differentiation.

LEMMA 6.4. *For any real stable polynomial p, $\mathsf{Ab}_p \subseteq \mathsf{Ab}_{p_{z_i}}$.*

To prove Theorem 6.1, it is sufficient by (12) to show that $(1+\sqrt{\epsilon})^2 \cdot \mathbf{1} \in \mathsf{Ab}_Q$, where $\mathbf{1}$ is the all-ones vector. We will achieve this by an inductive "barrier function" argument. In particular, we will construct Q iteratively via a sequence of operations of the form $(1 - \partial_{y_i})$, and we will track the locations of the roots of the polynomials that arise in this process by studying the evolution of the functions defined below.

Our procedure will be to transform the real stable polynomial

$$p(z_1, \ldots, z_n) = \det\left(z_i \sum_i A_i\right)$$

into

$$p_n(x) = \prod_i (1 - \partial_i) \det\left(z_i \sum_i A_i\right)$$

iteratively, keeping track of what happens to the points above the roots of p. At the beginning, the region above the roots will be the positive orthant (since the A_i are all positive semidefinite). At step k, we will performing the $(1 - \partial_k)$ operation to the polynomial, which (for lack of a better analogy) one can think of as a hitting a metal cast of the "above the roots" region with a hammer. In particular, it will do two things:

1. Shift the entire region in the z_k direction
2. Cause the region to flatten inwards (in all directions)

Both effects will cause the zeros of the polynomial to move away from the origin, but the goal will be to bound the amount of movement. Our method of obtaining such a bound uses a collection of measurements that tell us how convex the polynomial is at a given point, in a given variable. We call these measurements "barrier functions" and we will need one such measurement for each variable in p.

DEFINITION 6.5. Given a real stable polynomial p and a point $z = (z_1, \dots, z_m) \in \mathsf{Ab}_p$, the *barrier function of* p *in direction* i *at* z is defined as

$$\Phi_p^i(z) = \frac{\partial_{z_i} p(z)}{p(z)}.$$

Equivalently, we may define Φ_p^i as

$$\Phi_p^i(z_1, \dots, z_m) := \frac{q'_{z,i}(z_i)}{q_{z,i}(z_i)} = \sum_{j=1}^{r} \frac{1}{z_i - \lambda_j}, \tag{13}$$

where the univariate restriction

$$q_{z,i}(t) := p(z_1, \dots, z_{i-1}, t, z_{i+1}, \dots, z_m) \tag{14}$$

has roots $\lambda_1, \dots, \lambda_r$ (which are all real, by Proposition 5.9).

Note that the barrier functions are (for general points) not particularly well behaved, but we will only be considering them on the set of points that are above the roots, where they have a number of nice properties that we will exploit. Of course applying the $(1 - \partial_k)$ operation will have an effect on the values of the barrier functions, and so we will need to mindful of this change as well. One observation that will simplify this is that the effect of applying $(1 - \partial_k)$ to the barrier function Φ^j can be calculated with all of the other variables (not z_k or z_j) fixed. Hence it suffices to understand the effects on bivariate polynomials, an example of which is shown in Figure 3.

Our proof of these will use an observation of Terry Tao that uses a characterization of interlacing polynomials that appears in [**Wag11**].

LEMMA 6.6. *Let f and g be real rooted polynomials with leading coefficient having the same sign such that $f \ll g$. Then*

$$(-1)^k \frac{\partial^k}{(\partial x)^k} \frac{f(x)}{g(x)} \bigg|_{x=y} \geq 0$$

for all $y \in \mathsf{Ab}_g$.

PROOF. Let $\beta_1, \dots \beta_n$ be the roots of g. Note that the equation

$$f(x) = g(x) \left(s + \sum_i \frac{t_i}{x - \beta_i} \right)$$

defines $n+1$ linearly independent equations in $n+1$ variables (one for each coefficient) and therefore has a solution. Furthermore, one can check that each t_i is nonnegative by noting that $f(\beta_i) = t_i g'(\beta_i)$ and using the fact that both $f(\beta_i)$ and $g'(\beta_i)$ alternate between nonpositive and nonnegative values (the

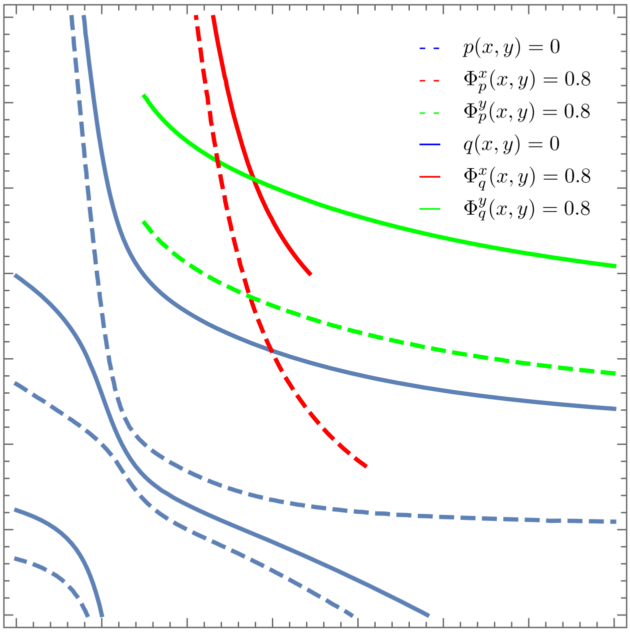

FIGURE 3. The effect of the $(1-\partial_y)$ operator on the bivariate real stable polynomial $p(x,y) = 4+12x+8x^2+17y+29xy+8x^2y+14y^2+13xy^2+y^3$. Note $q(x,y) = (1-\partial_y)p(x,y)$.

first is due to the interlacing, the second is always true). The result then follows by taking the derivatives and using the fact that $y-\beta_i > 0$ for all i. □

Using this, we can show the two analytic properties of barrier functions that we need: at any point above the roots of a real stable polynomial, the barrier functions are nonincreasing and convex in every coordinate.

LEMMA 6.7. *Suppose p is real stable and $z \in \mathsf{Ab}_p$. Then for all $i, j \le m$ and $\delta \ge 0$,*

$$\Phi_p^i(z+\delta e_j) \le \Phi_p^i(z), \text{ and} \qquad \text{(monotonicity)} \tag{15}$$

$$\Phi_p^i(z+\delta e_j) \le \Phi_p^i(z) + \delta \cdot \partial_{z_j}\Phi_p^i(z+\delta e_j) \qquad \text{(convexity).} \tag{16}$$

PROOF. If $i = j$ then consider the real-rooted univariate restriction $q(x_i) = \prod_{k=1}^r (x_i - \lambda_k)$ defined in (14). Since $z \in \mathsf{Ab}_p$ we know that $z_i > \lambda_k$ for all k. Monotonicity follows immediately by considering each term in (13), and convexity is easily established by computing

$$\partial_{x_i}^2 \left(\frac{1}{x_i - \lambda_k}\right)\bigg|_{x=z} \frac{2}{(z_i-\lambda_k)^3}$$

which is positive since (for $z \in \mathsf{Ab}_p$) $z_i > \lambda_k$. For the case when $i \ne j$, we fix all variables other than x_i and x_j and consider the bivariate restriction

$$q(x_i, x_j) := p(z_1, \ldots, x_i, \ldots, x_j, \ldots, z_m).$$

Using Lemma 6.6, both monotonicity and convexity would follow by showing that $f \ll g$ where

$$f(x_j) := \partial_{x_i} q(z_i, x_j) \quad \text{and} \quad g(x_j) := q(z_i, x_j)$$

By Corollary 5.8, $(1 + \lambda \partial_{x_i})q$ is real stable (and so $g + \lambda f$ is real rooted) for all λ. Hence by Theorem 4.8 either $f \ll g$ or $g \ll f$.

To show that, in fact, $f \ll g$, we will consider the sum of the roots. That is, it suffices to show that the sum of the roots of f is at most the sum of the roots of g. Write

$$q(x, y) = a(y)x^n + b(y)x^{n-1} + \dots$$

Since taking partial derivatives preserves real stability,

$$\partial_x^{n-1} q(x, y) = (n-1)!(nxa(y) + b(y))$$

is real stable. Hence $b(y) + ia(y)$ is stable and so by Theorem 5.5, we have $a \ll b$. Using Lemma 6.6 again, this implies $a(y)b'(y) - b(y)a'(y) \geq 0$.

Now since (x, y) is above the roots of q, it is also above the roots of $\partial_x^n q(x, y) = n!a(y)$ and so $a'(y)$ and $a(y)$ have the same sign. Hence

$$\frac{b'(y)}{a'(y)} \geq \frac{b(y)}{a(y)} \tag{17}$$

Note that the sum of the roots of g is $-b/a$ and the sum of the roots of f is $-b'/a'$. Thus (17) is asserting that the sum of the roots of f is at most the sum of the roots of g (as needed). □

Remark 6.8. Our original proof of monotonicity and convexity used a powerful characterization of bivariate real stable polynomials due to Helton and Vinnikov [**HV07**] and Lewis, Parrilo and Ramana [**LPR05**]. While this characterization is extremely useful, it (incorrectly) gave the impression that such a powerful result was required to prove Lemma 6.7. James Renegar, in particular, pointed out that the lemma follows directly from well-known properties of hyperbolic polynomials. We chose the proof given here since it has the benefit of remaining in the domain of real stable polynomials.

Our first observation is that when a point above the roots has a small enough boundary function in a given direction, it remains above the roots after applying an operator in that direction. Pictorially, this asserts that the dotted green line in Figure 3 will always be contained inside the solid blue line.

Lemma 6.9. *Let p be a real stable polynomial, and let z be a point above the roots of p which satisfies $\Phi_p^i(z) < 1$. Then z is also above the roots of $p - \partial_{z_i} p$.*

Proof. Let t be a nonnegative vector. As Φ is nonincreasing in each coordinate we have $\Phi_p^i(z + t) < 1$, whence

$$\partial_{z_i} p(z + t) < p(z + t) \implies (p - \partial_{z_i} p)(x + t) > 0,$$

as desired. □

While Lemma 6.9 proves what we need for a single iteration, it is not strong enough for an inductive argument because the application of a $(1 - \partial_{z_k})$ operator will typically cause all of the barrier functions to increase. As previously mentioned, the effect of the $(1 - \partial_{z_k})$ operator will be to shift in the z_k direction and flatten away from the origin. To remedy this, we will translate our upper bounds in the z_k direction as well (see Figure 4). Certainly this will compensate for the shift in the z_k direction, but we will need to move extra in order to compensate for the flattening of the region. How much extra will be determined by the value of the barrier function in that direction. In particular, by exploiting the convexity properties of the Φ_p^i, we arrive at the following useful strengthening of Lemma 6.9.

LEMMA 6.10. *Suppose $p(z_1, \ldots, z_m)$ is real stable with $z \in \mathsf{Ab}_p$, and $\delta > 0$ satisfies*

$$\Phi_p^j(z) \le 1 - \frac{1}{\delta}. \tag{18}$$

Then for all i,

$$\Phi^i_{p - \partial_{z_j} p}(z + \delta e_j) \le \Phi_p^i(z).$$

The proof follows directly from property (16) of Lemma 6.7. We refer the reader to [**MSS15b**] for the details. The effect of Lemma 6.10 can be seen in Figure 4. By moving far enough in the y direction, the given point is able to move back inside the regions defined by the Φ^x and Φ^y functions.

It should now be clear how the proof proceeds — at each step, we will apply the operator $(1 - \partial_k)$ and then move our upper bound in that direction. We will bound the amount we move using the barrier function in that direction, while also taking care that we have moved far enough to cause the barrier functions in all of the other directions to go down (so that they will still be small when the time comes to use them).

PROOF OF THEOREM 6.1. Let

$$P(y_1, \ldots, y_m) = \det\left(\sum_{i=1}^m y_i A_i\right).$$

Set $t = \sqrt{\epsilon} + \epsilon$. As all of the matrices A_i are positive semidefinite and

$$\det\left(t \sum_i A_i\right) = \det(tI) > 0,$$

the vector $t\mathbf{1}$ is above the roots of P.

By Theorem 5.2,

$$\Phi_P^i(y_1, \ldots, y_m) = \frac{\partial_i P(y_1, \ldots, y_m)}{P(y_1, \ldots, y_m)} = \mathrm{Tr}\left(\left(\sum_{i=1}^m y_i A_i\right)^{-1} A_i\right).$$

So,

$$\Phi_P^i(t\mathbf{1}) = \mathrm{Tr}\,(A_i)\,/t \le \epsilon/t = \epsilon/(\epsilon + \sqrt{\epsilon}),$$

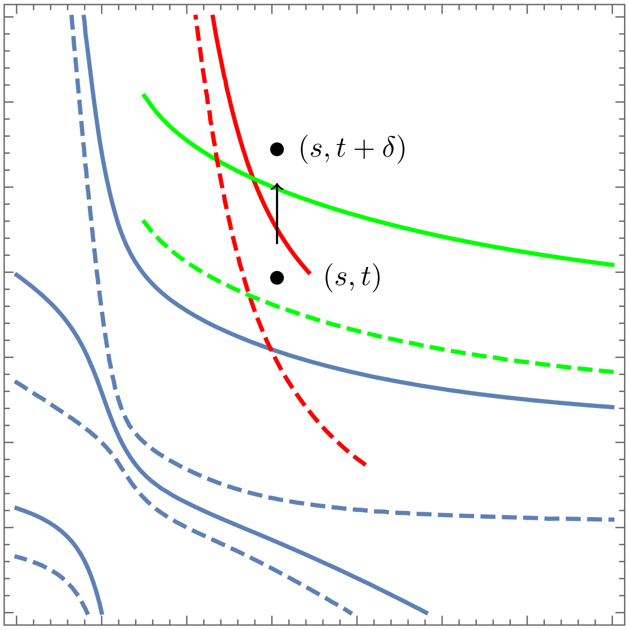

FIGURE 4. Moving $\delta = \frac{1}{1-\Phi_p^y(s,t)}$ in the y direction moves the point back inside the region defined by the two barrier functions.

which we define to be ϕ. Set

$$\delta = 1/(1-\phi) = 1+\sqrt{\epsilon}.$$

For $k \in [m]$, define

$$P_k(y_1, \dots, y_m) = \left(\prod_{i=1}^{k} 1 - \partial_{y_i}\right) P(y_1, \dots, y_m).$$

Note that $P_m = Q$.

Set x^0 to be the all-t vector, and for $k \in [m]$ define x^k to be the vector that is $t+\delta$ in the first k coordinates and t in the rest. By inductively applying Lemmas 6.9 and 6.10, we prove that x^k is above the roots of P_k, and that for all i

$$\Phi^i_{P_k}(x^k) \le \phi.$$

It follows that the largest root of

$$\mu\left[A_1, \dots, A_m\right](x) = P_m(x, \dots, x)$$

is at most

$$t+\delta = 1+\sqrt{\epsilon}+\sqrt{\epsilon}+\epsilon = (1+\sqrt{\epsilon})^2. \qquad \square$$

7. Theorems

We now combine the results in the previous sections to prove Conjecture 1.7, thereby proving Conjecture 1.5 and showing that the Kadison-Singer Problem has a positive solution.

We first complete the proof of Theorem 3.2, restated here for convenience.

THEOREM 7.1 (Theorem 3.2). *Let $\epsilon > 0$ and let $v_1, \dots, v_m$ be independent random vectors in $\mathbb{C}^d$ with finite support such that*

$$\sum_{i=1}^{m} \mathbb{E}\, v_i v_i^* = I_d, \tag{19}$$

and

$$\mathbb{E}\, \|v_i\|^2 \leq \epsilon, \text{for all } i.$$

Then

$$\mathbb{P}\left[\left\|\sum_{i=1}^{m} v_i v_i^*\right\| \leq (1+\sqrt{\epsilon})^2\right] > 0$$

PROOF. Let $A_i = \mathbb{E}\, v_i v_i^*$. We have

$$\mathrm{Tr}\,(A_i) = \mathbb{E}\,\mathrm{Tr}\,(v_i v_i^*) = \mathbb{E}\, v_i^* v_i = \mathbb{E}\,\|v_i\|^2 \leq \epsilon,$$

for all i.

The expected characteristic polynomial of the $\sum_i v_i v_i^*$ is the mixed characteristic polynomial $\mu\,[A_1, \dots, A_m]\,(x)$. Theorem 6.1 implies that the largest root of this polynomial is at most $(1+\sqrt{\epsilon})^2$.

For $i \in [m]$, let l_i be the size of the support of the random vector v_i, and let v_i take the values $w_{i,1}, \dots, w_{i,l_i}$ with probabilities $p_{i,1}, \dots, p_{i,l_i}$. Theorem 5.11 tells us that the polynomials $q_{j_1,\dots,j_m}$ are an interlacing family. So, Theorem 4.5 implies that there exist $j_1, \dots, j_m$ so that the largest root of the characteristic polynomial of

$$\sum_{i=1}^{m} w_{i,j_i} w_{i,j_i}^*$$

is at most $(1+\sqrt{\epsilon})^2$. □

It is worth noting here that the bound $(1+\sqrt{\epsilon})^2$ is asymptotically tight, as can be seen by picking random vectors so that

$$\mathbb{E}\, v_i v_i^* = \frac{1}{n} I$$

for all i. The resulting polynomial is an associated Laguerre polynomial whose largest root is (asymptotically) exactly this bound. We believe this polynomial is actually the extremal polynomial for this problem, but are unable to prove it.

Since the outer products of the vectors sum to the identity, the best one could hope for is to be able to split the vectors into r groups such that each was *exactly* $(1/r)I$. Hence Theorem 1.8 guarantees that for any vectors v_i, one can get within a factor of $\left(1+\sqrt{r\delta}\right)^2$ of the best one could get with the best possible v_i.

THEOREM 7.2. *Let $r > 0$ be an integer, and let $u_1, \ldots, u_m \in \mathbb{C}^d$ be vectors such that*

$$\sum_{i=1}^{m} \mathbb{E}\, u_i u_i^* = I_d \quad \textit{and} \quad \|u_i\|^2 \leq \delta \textit{ for all } i.$$

Then there exists a partition $\{A_1, \ldots A_r\}$ of $[m]$ such that

$$\left\| \sum_{i \in A_j} u_i {u_i}^* \right\| \leq \frac{1}{r}\left(1+\sqrt{r\delta}\right)^2$$

PROOF. For each $i \in [m]$ and $k \in [r]$, define $w_{i,k} \in \mathbb{C}^{rd}$ to be the direct sum of r vectors from $\mathbb{C}^d$, all of which are 0^d (the 0-vector in $\mathbb{C}^d$) except for the k^{th} one (which is a copy of u_i). Now let $v_1, \ldots, v_m$ be independent random vectors such that v_i takes the values $\{\sqrt{r} w_{i,k}\}_{k=1}^{r}$ each with probability $1/r$.

These vectors satisfy

$$\mathbb{E}\, v_i v_i^* = \begin{pmatrix} u_i u_i^* & 0_{d\times d} & \ldots & 0_{d\times d} \\ 0_{d\times d} & u_i u_i^* & \ldots & 0_{d\times d} \\ \vdots & & \ddots & \vdots \\ 0_{d\times d} & 0_{d\times d} & \ldots & u_i u_i^*, \end{pmatrix} \quad \text{and} \quad \|v_i\|^2 = r\,\|u_i\|^2 \leq r\delta.$$

So,

$$\sum_{i=1}^{m} \mathbb{E}\, v_i v_i^* = I_{rd}$$

and we can apply Theorem 3.2 with $\epsilon = r\delta$ to show that there exists an assignment of each v_i so that

$$\left\| \sum_{k=1}^{r} \sum_{i : v_i = w_{i,k}} \left(\sqrt{r} w_{i,k}\right)\left(\sqrt{r} w_{i,k}\right)^* \right\| \leq (1+\sqrt{r\delta})^2.$$

Setting $A_k = \{i : v_i = w_{i,k}\}$ implies that

$$\left\| \sum_{i \in A_k} u_i {u_i}^* \right\| = \left\| \sum_{i \in A_k} w_{i,k} {w_{i,k}}^* \right\| \leq \frac{1}{r}\left\| \sum_{k=1}^{r} \sum_{i : v_i = w_{i,k}} \left(\sqrt{r} w_{i,k}\right)\left(\sqrt{r} w_{i,k}\right)^* \right\|$$
$$\leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta}\right)^2.$$

and this is true for all k. □

Note that the bound in Theorem 1.8 is actually quite good. Since the outer products of the vectors sum to the identity, the best one could hope for is to be able to split the vectors into r groups such that each was *exactly* $(1/r)I$. Hence Theorem 1.8 guarantees that for any vectors v_i, one can get within a factor of $\left(1+\sqrt{r\delta}\right)^2$ of the best one could get with the best possible v_i.

8. Extensions

There are multiple issues that occur when attempting to apply Theorem 3.2 in the context of more general matrices. Two of these issues have since been resolved, and we list the resulting theorems here without proof. The first is a result due to Michael Cohen [**Coh**] that allows one to get bounds in situations where the original matrices are not rank 1 [**Coh**].

THEOREM 8.1. *Let $\epsilon > 0$ and let $A_1, \ldots, A_m$ be independent random positive semidefinite matrices in $\mathbb{C}^{d\ timesd}$ with finite support such that*

$$\sum_{i=1}^{m} \mathbb{E}\, A_i = I_d \quad \textit{and} \quad \mathbb{E}\,\mathrm{Tr}\,(A_i) \leq \epsilon, \textit{for all } i.$$

Then

$$\mathbb{P}\left[\left\|\sum_{i=1}^{m} A_i\right\| \leq (1+\sqrt{\epsilon})^2\right] > 0.$$

This then leads to a similar generalization of Theorem 1.8. The issue with trying to apply the original proof in the high rank setting is that the expected characteristic polynomial $\mathbf{E}\det(xI - \sum_i A_i)$ need not be real-rooted for random high-rank matrices A — the simplest example is just to take 2×2 A_i equal to I and $-I$ with equal probability, yielding the expected polynomial:

$$(x-1)^2 + (x+1)^2,$$

which is strictly positive on the real line. Cohen's idea is to replace the characteristic polynomial $\chi(\sum_i A_i)$ by the *mixed characteristic polynomial*:

$$\mu(A_1, \ldots, A_m) := \mathbf{E}\det(\sum_{i=1}^{m} B_i)$$

where the B_i are any rank one random matrices such that $\mathbf{E}B_i (= A_i)$. This polynomial is necessarily real-rooted by Corollary 5.10, and by construction multilinear in the A_i, so the same argument as in the rank-1 case shows that there exist $(A_1, \ldots, A_m$ such that the largest root of $\mu(A_1, \ldots, A_m)$ is at most the largest root of $\mathbf{E}\mu(A_1, \ldots, A_m)$. The latter polynomial is just $\mathbf{E}\chi(xI - \sum_i B_i)$, so its largest root is at most $(1+\sqrt{\epsilon})^2$ where $\epsilon = \mathbf{E}\mathrm{Tr}(B_i) =$

$\mathbf{E}\mathrm{Tr}(A_i)$. The main technical content of Cohen's result, which is a kind of convexity result, is that

$$\lambda_{max}\chi(\sum_{i\le m} A_i) \le \lambda_{max}\mu(A_1,\dots,A_m)$$

for any positive semidefinite A_i. Combining this with the previous result gives Theorem 8.1.

It should be noted that this extension only works when one wishes to find a polynomial whose *largest root* is *small*. In general, the method of interlacing polynomials will supply a bound on either side of any chosen root (for example, to find a polynomial whose smallest root is large), but we currently can only get such a bound in the rank 1 case. Fortunately, the majority of applications seem to involve the largest root, so for many of the known results, the extension can typically be applied without issue.

One reaction to the high rank extension is to think that now one might be able to find a partition of the type in Theorem 1.8 with some number of added constraints (that one can impose combinatorially using added dimensions orthogonal to the original vectors). This, unfortunately, does not appear to be as useful as one might think, since the addition will cause the expected trace of the matrices to go up, thereby decreasing the accuracy of the bound. This suggests that a similar extension would not hold if one was to find an analogue of Theorem 3.2 for indefinite self adjoint matrices (which would be interesting in its own right). This does suggest as an interesting open question whether one can find a bound analogous to Theorem 3.2 when the restriction on the matrices is expressed in some other (for example, Schatten) norm.

The other issue is with the constraint (19). This has been dealt with in a paper of Akemann and Weaver [**AW14**] that shows (among other things) the following extension of Theorem 1.8.

THEOREM 8.2. *Let $r > 0$ be an integer, and let $u_1,\dots,u_m \in \mathbb{C}^d$ be vectors such that*

$$\sum_{i=1}^m \mathbb{E}\, u_i u_i^* \le I_d \quad \textit{and} \quad \|u_i\|^2 \le \delta \textit{ for all } i.$$

Now let $t_1,\dots,t_n$ satisfy $0 \le t_i \le 1$. Then there exists a partition $\{A_1,\dots A_r\}$ of $[m]$ such that

$$\left\| \sum_{i\in A_j} u_i {u_i}^* - \sum_i t_i u_i {u_i}^* \right\| \le O(\delta^{1/8}).$$

Their method of proof proceeds by proving a weighted version of Theorem 3.2 and then giving a series of successive approximations (see [**AW14**] for details).

Lastly we mention that even though Theorem 3.2 is asymptotically tight, further restrictions on the random vectors can lead to improved bounds. This leads to slight improvements in the guarantee of Theorem 1.8 in cases where the partition size is small. Looking at the proof of Theorem 1.8, there is a direct correspondence between the number of partitions and the rank of the expected matrix that is constructed. This in turn corresponds to the degree of the variable z_i in the polynomial

$$\det\left(xI - \sum_i z_i A_i\right).$$

This degree restriction can sometimes lead to tighter bounds on the convexity of the barrier functions. For example, the following improvement of Lemma 6.10 was shown in [**BCMS16**]:

LEMMA 8.3. *Assume $p(x, y)$ is quadratic in x and let*

$$\Phi_p^x \leq \left(1 - \frac{1}{\delta}\right)\frac{1}{2-\delta}$$

*for some $\delta \in (1, 2)$. Now let $q(x, y) = (1 - \partial_x)p(x + \delta, y)$ and assume that (x_0, y_0) is above the roots of **both** p and q. Then*

$$\Phi_q^y \leq \Phi_p^y.$$

Using this, the authors then go on to show an improvement of Theorem 1.8 when $r = 2$ and $\delta < 1/2$:

THEOREM 8.4. *Let $u_1, \dots, u_m \in \mathbb{C}^d$ be vectors such that*

$$\sum_{i=1}^m \mathbb{E}\, u_i u_i^* = I_d \quad \textit{and} \quad \|u_i\|^2 \leq \delta \leq \frac{1}{2} \textit{ for all } i.$$

Then there exists a partition $\{A_1, A_2\}$ of $[m]$ such that

$$\left\|\sum_{i \in A_j} u_i {u_i}^*\right\| \leq \frac{1}{2} + \sqrt{\delta}\sqrt{1-\delta}.$$

The argument in [**BCMS16**] follows the same general pattern as the one used in our proof Theorem 6.1, but has a number of added issues. One of the nicer occurrences in the proof of Theorem 6.1 is that the conditions necessary for Lemma 6.10 were strictly stronger than the conditions necessary for Lemma 6.9. As a result, the constraint provided by Lemma 6.10 is the only relevant one in determining the optimal values of δ and t. With the improved version of Lemma 8.3, this is no longer the case and so one must balance two competing constraints. In the end, Theorem 8.4 only gives a slight improvement over the value

$$\frac{1}{2} + \sqrt{2\delta} + \delta$$

that one gets directly from Theorem 1.8.

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Princeton University
E-mail address: adam.marcus@princeton.edu

UC Berkeley
E-mail address: nikhil@math.berkeley.edu