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Recent progress in the Zimmer program

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ABSTRACT. This article surveys recent results due to the author and his collaborators on rigidity properties of actions of certain countably infinite groups on compact manifolds. We specifically focus on the results of [12–14, 16, 18]. We primarily focus on groups such as $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ (for $n \geq 3$) and more general lattices Γ in (typically higher-rank) semisimple Lie groups. The actions considered will be either on low-dimensional manifolds (where the dimension is small relative to certain algebraic data associated with the acting group) or actions on tori $\mathbb{T}^d$ and nilmanifolds N/Λ.

CONTENTS

1. Introduction

1.1. Rigidity properties of discrete groups. Consider the family of groups

$$\Gamma = \mathrm{SL}(n, \mathbb{Z})$$

for $n \geq 2$. Relative to various group-theoretic and representation-theoretic properties, there is a stark distinction between the case $n = 2$ and $n \geq 3$. Namely, the group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ is rather "flexible" whereas the group $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ is very "rigid" whenever $n \geq 3$.

As a prototype property where such flexibility versus rigidity phenomena is observed, fix a countably infinite group Γ and consider all normal subgroups Γ' of Γ. When the ambient group Γ is $\mathrm{SL}(2, \mathbb{Z})$, the fundamental group of a compact surface, or the free group F_n on $n \geq 2$ generators, Γ admits many infinite normal subgroups Γ' with infinite index; for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, this can be seen from the fact that $\mathrm{SL}(2, \mathbb{Z})$ contains F_2 as a subgroup of finite index. In contrast, a result of Margulis [70] shows that the only normal subgroups of $\mathrm{SL}(n, \mathbb{Z})$ (and more general irreducible lattices in higher-rank semisimple Lie groups) are either finite or of finite index whenever $n \geq 3$. Similarly, when $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ or when Γ is the fundamental group of a surface, linear representations $\pi \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ are quite flexible. In contrast, for $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, $n \geq 3$, and more general irreducible lattices in higher-rank semisimple Lie groups, linear representation $\pi\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ exhibit a number of well-known rigidity properties.

Fix a compact boundaryless manifold M and denote by $\mathrm{Diff}^r(M)$ the group[1] of C^r diffeomorphisms $f\colon M \to M$. Recall that if $r \geq 1$ is not integral then, writing

$$r = k + \beta \quad \text{for } k \in \mathbb{N} \text{ and } \beta \in (0,1),$$

we say that $f\colon M \to M$ is C^r or is $C^{k+\beta}$ if it is C^k and if the kth derivatives of f are β-Hölder continuous. If vol is a smooth volume form on M, we also write $\mathrm{Diff}^r_{\mathrm{vol}}(M)$ for the group of volume-preserving diffeomorphisms. At times, we also consider $\mathrm{Homeo}(M)$, the group of homeomorphisms of M.

Given a discrete group Γ, a C^r **action of Γ on** M is a homomorphism

$$\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$$

from the group Γ into the group $\mathrm{Diff}^r(M)$; that is, for each $\gamma \in \Gamma$ the image $\alpha(\gamma)$ is a C^r diffeomorphism $\alpha(\gamma)\colon M \to M$ and for $x \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ we have

$$\alpha(\gamma_1\gamma_2)(x) = \alpha(\gamma_1)\big(\alpha(\gamma_2)(x)\big).$$

Fix a discrete, countably infinite group Γ. By analogy, we might ask if various rigidity properties that hold for finite-dimensional linear representations $\pi\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ continue to hold when the vector space $\mathbb{R}^d$ is replaced by a manifold M and the target group $\mathrm{GL}(\mathbb{R}^d)$ is replaced by a group of (possibly volume-preserving) diffeomorphisms $\mathrm{Diff}^r(M)$. That is, for such Γ, we ask if rigidity properties of finite-dimensional linear representations

$$\pi\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$$

continue to hold for "non-linear representations"

$$\alpha\colon \Gamma \to \mathrm{Diff}^r(M).$$

This analogy turns out to be quite fruitful. In the early 1980s, Robert Zimmer established a superrigidity theorem for linear cocycles (see [99] and Theorem 2.14 below). This result extends Margulis's superrigidity theorem (see Theorem 1.3) for linear representations to the setting of linear cocycles of over ergodic measurable actions of higher-rank simple Lie groups and their lattices (see Section 2.3). This, in particular, implies that for groups such as $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, $n \geq 3$, and any volume-preserving action $\alpha\colon \Gamma \to \mathrm{Diff}^1_{\mathrm{vol}}(M)$ on a compact manifold M, the derivative

$$(\gamma, x) \mapsto D_x\alpha(\gamma)$$

coincides with a representation $\pi\colon \Gamma \to \mathrm{GL}(\dim(M),\mathbb{R})$, up to a measurable trivialization of the tangent bundle TM and a compact group.

The cocycle superrigidity theorem, its corollaries, and contemporaneous results of Zimmer's (see [99,100,102–105]) led Zimmer to formulate a number of conjectures and questions concerning (C^∞, volume-preserving) actions of

[1] Equipped with the group operation of composition of maps

$\Gamma = \mathrm{SL}(n,\mathbb{Z})$ for $n \geq 3$ and more general lattices in higher-rank simple Lie groups. These questions and conjectures, as well as a number of more recent extensions, are usually referred to as the **Zimmer program**. Roughly, the Zimmer program aims to establish analogues of rigidity results for linear representations in the setting of smooth actions on compact manifolds.

The recent work [12–14, 16, 18] of the author and his collaborators addressed a number of problems in the Zimmer program; this article surveys the main results of [12, 18]. In the remainder of this introduction, we define the main class of groups we study, namely, lattices in higher-rank (semi-)simple Lie groups, in Section 1.2. We also recall a number of classical rigidity results in Section 1.3 and formulate a version of Margulis' superrigidity theorem in Section 1.4. Finally, we outline a number of topological classification and rigidity results for $\mathbb{Z}$-actions generated by Anosov diffeomorphisms on tori in Section 1.5 that motivate the discussion of group actions on tori in Section 4.

1.2. Semisimple Lie groups and their lattices. Recall that a Lie algebra $\mathfrak{g}$ is **simple** if it is non-abelian and has no non-trivial ideals. A Lie algebra $\mathfrak{g}$ is **semisimple** if it is the direct sum $\mathfrak{g} = \oplus_{i=1}^{\ell}\mathfrak{g}_i$ of simple Lie algebras $\mathfrak{g}_i$; this is equivalent to the fact that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. We say a Lie group G is **simple** (resp. **semisimple**) if its Lie algebra $\mathfrak{g}$ is simple (resp. semisimple). Throughout, we also assume G is connected and has finite center. The primary example discussed in this text is the simple Lie group

$$G = \mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{Mat}_{n\times n} : \det(A) = 1\}$$

with Lie algebra

$$\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathrm{Mat}_{n\times n} : \mathrm{trace}(A) = 0\}.$$

1.2.1. *Lattices in semisimple Lie groups.* Every semisimple Lie group admits a bi-invariant, locally finite Borel measure, called the **Haar measure**, which is unique up to normalization. A **lattice** in G is a discrete subgroup $\Gamma \subset G$ with finite co-volume. That is, if D is a Borel fundamental domain for the right-action of Γ on G, then Γ is a lattice if D has finite volume. If the quotient G/Γ is compact, we say that Γ is a **cocompact** lattice. If G/Γ has finite volume but is not compact we say that Γ is **nonuniform**. The coset space G/Γ is a manifold which admits a left action of G. The Haar measure on G descends to a finite, G-invariant Borel measure on G/Γ which we always normalize to be a probability measure.

Example 1.1. The primary example of a lattice in $G = \mathrm{SL}(n,\mathbb{R})$ is $\Gamma = \mathrm{SL}(n,\mathbb{Z})$. $\mathrm{SL}(n,\mathbb{Z})$ is not cocompact in $\mathrm{SL}(n,\mathbb{R})$. However, $\mathrm{SL}(n,\mathbb{R})$ and more general simple and semisimple Lie groups possess both nonuniform and cocompact lattices. We refer to [96, Sections 6.7, 6.8] for examples and detailed constructions.

Example 1.2. In the case $G = \mathrm{SL}(2,\mathbb{R})$, the fundamental group of any finite-area hyperbolic surface S is a lattice in G. In particular, the fundamental group of a compact hyperbolic surface S is a cocompact lattice in G. This can be seen by identifying the fundamental group of S with the deck group of the hyperbolic plane $\tilde{S} = \mathbb{H} = \mathrm{SO}(2,\mathbb{R})\backslash\mathrm{SL}(2,\mathbb{R})$. For instance, the free group $\Gamma = F_2$ on two generators is a lattice in G as can be seen by giving the punctured torus $S = \mathbb{T}^2 \smallsetminus \{\mathrm{pt}\}$ a hyperbolic metric.

See [96] for further details on constructions and properties of lattices in Lie groups.

1.2.2. *Rank of a semisimple Lie group.* Every semisimple Lie group admits an Iwasawa decomposition

$$G = KAN \tag{1.1}$$

where K is a maximal compact subgroup, A is a simply connected free abelian group of $\mathbb{R}$-diagonalizable (relative to the adjoint representation) elements, and N is unipotent (relative to the adjoint representation). Such a decomposition is unique up to conjugation. See [61] for details.

The (real) **rank** of G is defined to be the dimension of the subgroup $A \simeq \mathbb{R}^{\mathrm{rank}(G)}$. We call such a group A a **maximal split Cartan subgroup.** We often identify A with its Lie algebra $\mathfrak{a}$ via the Lie exponential map and thus obtain an identification of groups $A \simeq \mathfrak{a} \simeq \mathbb{R}^{\mathrm{rank}(G)}$.

In the case of $G = \mathrm{SL}(n,\mathbb{R})$, the standard choice of K, A, and N are

$$K = \mathrm{SO}(n,\mathbb{R}), \qquad A = \{\mathrm{diag}(e^{t_1}, e^{t_2}, \dots, e^{t_n}) : t_1 + \cdots + t_n = 0\}, \tag{1.2}$$

and N, the group of upper-triangular matrices with all diagonal entries equal to 1. Note that, as elements in $\mathrm{SL}(n,\mathbb{R})$ have determinant 1, we have

$$\mathrm{diag}(e^{t_1}, e^{t_2}, \dots, e^{t_n}) \in \mathrm{SL}(n,\mathbb{R})$$

if and only if $t_1 + \cdots + t_n = 0$. The Lie algebra $\mathfrak{a}$ of A is $\mathfrak{a} = \{\mathrm{diag}(t_1, t_2, \dots, t_n) : t_1 + \cdots + t_n = 0\}$. We identify $A \simeq \mathbb{R}^{n-1}$ and have that the rank of $G = \mathrm{SL}(n,\mathbb{R})$ is $\mathrm{rank}(G) = n - 1$.

We say that a simple Lie group G is **higher-rank** if its rank is at least 2. We say that a lattice Γ in a higher-rank simple Lie group G is a **higher-rank lattice.** In particular, $G = \mathrm{SL}(n,\mathbb{R})$ and its lattices are higher-rank whenever $n \geq 3$.

In Example 2.8 below, we give an example of a cocompact lattice Γ in the group $G = \mathrm{SO}(n,n)$ for $n \geq 4$. The group $\mathrm{SO}(n,n)$ has rank n and thus Γ is a higher-rank, cocompact lattice.

For further examples, see Table 1 for calculations of the rank for various matrix groups and see [61, Section VI.4] for examples of Iwasawa decompositions for various matrix groups.

1.3. Rigidity of linear representations. To motivate the results and conjectures discussed below, we briefly review a number of well-known rigidity results in the setting of linear representations of higher-rank lattices.

1.3.1. *Rigidity of finite-dimensional representations.* Let G be a semisimple Lie group and let $\Gamma \subset G$ be a lattice. Considering Γ as an abstract group, one might ask if the inclusion map $\iota\colon \Gamma \to G$ is **locally rigid**; that is, if $\tilde{\pi}\colon \Gamma \to G$ is a homomorphism sufficiently close to ι, are ι and $\tilde{\pi}$ conjugate by an element of G. (Here, the topology on the space of representations is generated by the restriction to a finite generating set of Γ.) Similarly, we may ask if a general linear representation $\pi\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is **locally rigid**: given a sufficiently close representation $\tilde{\pi}\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$, we ask if $\tilde{\pi}$ and π are conjugate.

When $G = \mathrm{SL}(2, \mathbb{R})$, the inclusion $\iota\colon \Gamma \to G$ is not locally rigid. Indeed, when Γ is the fundamental group of a closed orientable surface of genus $g \geq 2$, the space of deformations is the $(6g - g)$-dimensional Teichmuller space.

When Γ is a cocompact lattice in a higher-rank simple Lie group G (or more generally a cocompact lattice in semisimple group G all of whose simple factors are not compact or locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$) then the inclusion $\iota\colon \Gamma \to G$ was shown to be locally rigid; see [88, 90]. In [92], the question of local rigidity of general representations $\pi\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ was reduced to the vanishing of a certain cohomology. The vanishing of this cohomology was studied in [73, 86] and is known to hold for all lattices in higher-rank simple groups.

Beyond local rigidity, Mostow's strong rigidity theorem established for any simple Lie group G that is not locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ and for any two cocompact lattices $\Gamma_1, \Gamma_2 \subset G$, that any isomorphism $h\colon \Gamma_1 \to \Gamma_2$ extends to an isomorphism $h\colon G \to G$. See [75] as well as [69, 81] for extensions to the case of non-uniform lattices.

The strongest rigidity result for finite-dimensional representations is Margulis's superrigidity theorem. In the next section, we formulate in Theorem 1.3 a version for representations of lattices in $\mathrm{SL}(n, \mathbb{R})$. Roughly, Margulis's result establishes for any lattice Γ in a higher-rank simple Lie group G, that any representation $\pi\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ extends to a representation $\overline{\pi}\colon G \to \mathrm{GL}(d, \mathbb{R})$ up to a "compact error" (and possibly passing to finite covers.) This effectively classifies all representations $\Gamma \to \mathrm{GL}(d, \mathbb{R})$ up to conjugacy. See also [24] for extensions to lattices in certain rank-1 groups.

1.3.2. *Rigidity of unitary representations.* Given a Lie group G or a lattice Γ in G, we may consider representations by unitary operators on Hilbert space. For higher-rank simple groups G and their lattices, an important rigidity property was formulated and established by Kazhdan. We say a group G has Kazhdan's **property (T)** if any unitary representation of G with almost invariant vectors has a non-trivial invariant vector. See for instance [96, Chapter 13] and [91] for detailed expositions. Higher-rank simple Lie groups with finite center and their lattices have property (T) but $\mathrm{SL}(2, \mathbb{R})$ and its lattices fail to have property (T).

In Section 5.2 below, we formulate and use a recent strengthening of Kazhdan's property (T), known as **strong property (T)** due to V. Lafforgue. This stronger property continues to hold for higher-rank simple Lie groups and their lattices. See Theorem 5.5 for a precise (re-)formulation we use in our work [12].

1.4. Margulis superrigidity. We formulate a version of Margulis's superrigidity theorem for finite-dimensional linear representations of lattices in $\mathrm{SL}(n,\mathbb{R})$.

THEOREM 1.3 (Margulis superrigidity [71]). *For $n \geq 3$, let Γ be a lattice in $\mathrm{SL}(n,\mathbb{R})$. Given a representation $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ there are*

(1) a linear representation $\hat\rho\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$;
(2) and a compact subgroup $K \subset \mathrm{GL}(d,\mathbb{R})$ that commutes with the image of $\hat\rho$

such that

$$\hat\rho(\gamma)\rho(\gamma)^{-1} \in K$$

for all $\gamma \in \Gamma$.

That is, $\rho = \hat\rho \cdot c$ is the product of the restriction of a representation

$$\hat\rho\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$$

to Γ and a compact-valued representation $c\colon \Gamma \to K$. Moreover, the image of $\hat\rho$ and c commute.

In the case that Γ is nonuniform, one can show that all compact-valued representations $c\colon \Gamma \to K$ have finite image. See also [96, Corollary 16.4.1] for more general criteria which guarantees that the image of c is finite.

For certain cocompact lattices $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$, there exists compact-valued representations $c\colon \Gamma \to \mathrm{SU}(n)$ with infinite image. (See discussion in Example 2.8.) The next theorem, characterizing all homomorphisms from lattices in $\mathrm{SL}(n,\mathbb{R})$ into compact Lie groups, shows that representations into $\mathrm{SU}(n)$ are more-or-less the only such examples. The proof uses the p-adic version of Margulis's superrigidity theorem and some algebra. See [71, Theorem VII.6.5] and [96, Corollary 16.4.2].

THEOREM 1.4. *For $n \geq 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a lattice. Let K be a compact Lie group and $\pi\colon \Gamma \to K$ a homomorphism.*

(1) If Γ is nonuniform then $\pi(\Gamma)$ is finite.
(2) If Γ is cocompact and $\pi(\Gamma)$ is infinite then there is a closed subgroup $K' \subset K$ with

$$\pi(\Gamma) \subset K' \subset K$$

and the Lie algebra of K' is of the form $\mathrm{Lie}(K') = \mathfrak{su}(n) \times \cdots \times \mathfrak{su}(n)$.

The appearance of $\mathfrak{su}(n)$ in (2) of Theorem 1.4 is due to the fact that $\mathfrak{su}(n)$ is the compact real form of $\mathfrak{sl}(n,\mathbb{R})$, the Lie algebra of $\mathrm{SL}(n,\mathbb{R})$. For a cocompact lattice Γ in $\mathrm{SO}(n,n)$ as in Example 2.8 below, the analogue of Theorem 1.4 states that

$$\mathrm{Lie}(K') = \mathfrak{so}(2n) \times \cdots \times \mathfrak{so}(2n).$$

Note that if $d < n$, there is no non-trivial representation

$$\hat\rho\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R});$$

moreover, there is no embedding of $\mathfrak{su}(n)$ in $\mathfrak{sl}(d,\mathbb{R})$. We thus immediately obtain as corollaries of Theorems 1.3 and 1.4 the following.

Corollary 1.5. *For $n \geq 3$, let Γ be a lattice in $G = \mathrm{SL}(n,\mathbb{R})$. Then, for $d < n$, the image of any representation $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ is finite.*

1.5. Rigidity of Anosov diffeomorphisms and hyperbolic maps on tori. We recall a number of facts from smooth dynamics concerning $\mathbb{Z}$-actions on tori generated by Anosov diffeomorphisms and maps that act hyperbolically on homology. These results serve as prototype rigidity results for actions of more general discrete groups and motivate a number of rigidity conjectures and results discussed below.

1.5.1. *Topological rigidity of hyperbolic toral maps.* Given any homeomorphism $f \in \mathrm{Homeo}(\mathbb{T}^d)$ there is a unique matrix $A_f \in \mathrm{GL}(d,\mathbb{Z})$ such that any lift $\tilde f\colon \mathbb{R}^d \to \mathbb{R}^d$ of f is of the form

$$\tilde f(x) = A_f x + \phi(x) \tag{1.3}$$

where $\phi\colon \mathbb{R}^d \to \mathbb{R}^d$ is $\mathbb{Z}^d$-periodic. We call A_f the **linear data** of f. Note that A_f induces a linear map L_{A_f} on the torus $\mathbb{T}^d$ given by

$$L_{A_f}(x + \mathbb{Z}^d) = A_f x + \mathbb{Z}^d.$$

We say that a matrix $A \in \mathrm{GL}(d,\mathbb{Z})$ is **hyperbolic** if no eigenvalue of A has modulus 1. We have the following theorem which characterizes, up to a continuous change of coordinates and possible identification of points, all maps $f\colon \mathbb{T}^d \to \mathbb{T}^d$ whose linear data A_f is hyperbolic.

THEOREM 1.6 (Franks [40]). *Suppose a homeomorphism $f\colon \mathbb{T}^d \to \mathbb{T}^d$ has hyperbolic linear data A_f. Then there is a continuous, surjective $h\colon \mathbb{T}^d \to \mathbb{T}^d$ (homotopic to the identity) such that*

$$h \circ f = L_{A_f} \circ h. \tag{1.4}$$

An analogous result holds for homeomorphisms of compact nilmanifolds. We call a continuous surjection h as in (1.4) a **topological semiconjugacy** between f and L_A.

1.5.2. *Local and global rigidity of Anosov diffeomorphisms of tori.* We first recall the definition of an Anosov diffeomorphism.

Definition 1.7. A C^1 diffeomorphism $f\colon M \to M$ of a compact Riemannian manifold M is **Anosov** if there is a Df-invariant splitting of the tangent bundle $TM = E^s \oplus E^u$ and constants $0 < \kappa < 1$ and $C \geq 1$ such that for every $x \in M$ and every $n \in \mathbb{N}$

$$\begin{aligned} \|D_x f^n(v)\| &\leq C\kappa^n \|v\| && \text{for all } v \in E^s(x) \\ \|D_x f^{-n}(w)\| &\leq C\kappa^n \|w\| && \text{for all } w \in E^u(x). \end{aligned}$$

As a primary example, consider a matrix $A \in \mathrm{GL}(n, \mathbb{Z})$ with all eigenvalues of modulus different from 1. Then, with $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ the n-torus, the induced toral automorphism $L_A \colon \mathbb{T}^n \to \mathbb{T}^n$ given by

$$L_A(x + \mathbb{Z}^n) = Ax + \mathbb{Z}^n$$

is a linear Anosov diffeomorphism. More generally, given $v \in \mathbb{T}^n$ we have $f \colon \mathbb{T}^n \to \mathbb{T}^n$ given by

$$f(x) = L_A(x) + v$$

is an affine Anosov diffeomorphism. In dimension 2, a standard example of an Anosov diffeomorphism is given by $L_A \colon \mathbb{T}^2 \to \mathbb{T}^2$ where A is the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

A prototype for *local rigidity* results, it is known (see [1, 74], [54, Corollary 18.2.2]) that Anosov maps are **structurally stable**: if f is Anosov and if g is sufficiently C^1 close to f, then g is also Anosov and there exists a homeomorphism $h \colon \mathbb{T}^n \to \mathbb{T}^n$ such that

$$h \circ g = f \circ h. \tag{1.5}$$

The map h in (1.5) is always Hölder continuous but in general need not be C^1 even when f and g are C^∞. A homeomorphism h as in (1.5) is called a **topological conjugacy** between f and g.

All known examples of Anosov diffeomorphisms occur on finite factors of tori and nilmanifolds. From [40, 68] we have a complete classification—a prototype *global rigidity* result—of Anosov diffeomorphisms on tori (as well as nilmanifolds) up to a continuous change of coordinates. In [39, 67], it was shown that Anosov diffeomorphisms on tori and nilmanifolds always have hyperbolic linear data whence there exists a map h as in Theorem 1.6. Moreover, Manning [68] showed that such a map is necessarily a homeomorphism. This establishes the following.

Theorem 1.8 (Franks–Manning, [40, 68]). *If $f \colon \mathbb{T}^n \to \mathbb{T}^n$ is Anosov, then f is homotopic to L_A for some hyperbolic $A \in \mathrm{GL}(n, \mathbb{Z})$; moreover, there is a homeomorphism $h \colon \mathbb{T}^n \to \mathbb{T}^n$ such that $h \circ f = L_A \circ h$.*

Again, the topological conjugacy h is Hölder continuous but need not be C^1. Conjecturally, all Anosov diffeomorphisms are, up to finite covers, topologically conjugate to affine maps on tori and nilmanifolds.

2. Smooth lattice actions, cocycle superrigidity, and the Zimmer program

To motivate the results discussed in subsequent sections, we outline a number of examples of smooth lattice actions, formulate a version of Zimmer's cocycle superrigidity theorem, and outline a number of general directions within the Zimmer program.

2.1. Standard lattice actions. We give a number "algebraic" or "standard" actions of lattices in Lie groups. We also discuss in Examples 2.9 and 2.10 some modifications of algebraic actions and constructions of more exotic actions.

Example 2.1 (Finite actions)**.** Let Γ' be a finite-index normal subgroup of Γ so that $F := \Gamma/\Gamma'$ is finite. Suppose F acts on a manifold M. As F is a quotient of Γ, we naturally obtain a Γ-action on M.

Definition 2.2. An action $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ is **finite** or **almost trivial** if it factors through the action of a finite group. That is, α is finite if there is a finite-index normal subgroup $\Gamma' \subset \Gamma$ such that $\alpha\restriction_{\Gamma'}$ is the identity transformation.

We remark that by a theorem of Margulis [70], if Γ is a lattice in a higher-rank simple Lie group with finite center, then all normal subgroups of Γ are either finite or of finite-index. Note that an action of a finite group preserves a volume form simply by averaging any volume form over the action.

Example 2.3 (Affine actions on tori)**.** Let $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ (or any finite-index subgroup of $\mathrm{SL}(n,\mathbb{Z})$). Let $M = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. We have a natural action $\alpha\colon \Gamma \to \mathrm{Diff}(\mathbb{T}^n)$ given by

$$\alpha(\gamma)(x + \mathbb{Z}^n) = \gamma \cdot x + \mathbb{Z}^n$$

for any matrix $\gamma \in \mathrm{SL}(n,\mathbb{Z})$.

To generalize to other lattices, let G be a Lie group and let $\Gamma \subset G$ be any lattice. Suppose that Γ admits a representation $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{Z})$. Then we have a natural action $\alpha_\rho\colon \Gamma \to \mathrm{Diff}(\mathbb{T}^d)$ given by

$$\alpha_\rho(\gamma)(x + \mathbb{Z}^d) = \rho(\gamma) \cdot x + \mathbb{Z}^d.$$

Note that these examples preserve a volume form, namely, the Lebesgue measure on $\mathbb{T}^d$.

Remark 2.4 (Genuinely affine actions)**.** Both constructions in Example 2.3 give actions $\alpha\colon \Gamma \to \mathrm{Diff}(\mathbb{T}^d)$ that have global fixed points. That is, the coset of 0 in $\mathbb{T}^d$ is a fixed point of $\alpha(\gamma)$ for every $\gamma \in \Gamma$.

The construction can be modified further to obtain genuinely affine actions without global fixed points. Given a lattice $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ and a representation $\rho\colon \Gamma \to \mathrm{SL}(d,\mathbb{Z})$, there may exist non-trivial elements $c \in H^1_\rho(\Gamma,\mathbb{T}^d)$; that is, $c\colon \Gamma \to \mathbb{T}^d$ is a function with

$$c(\gamma_1\gamma_2) = \rho(\gamma_1)c(\gamma_2) + c(\gamma_1) \tag{2.1}$$

such that there does not exist any $\eta \in \mathbb{T}^d$ with

$$c(\gamma) = \rho(\gamma)\eta - \eta \tag{2.2}$$

for all $\gamma \in \Gamma$. (Equation (2.1) says that c is a cocycle with coefficients in the Γ-module $\mathbb{T}^d$; (2.2) says c is not a coboundary.) We may then define $\tilde{\alpha}\colon \Gamma \to \mathrm{Diff}(\mathbb{T}^d)$ by

$$\tilde{\alpha}(\gamma)(x + \mathbb{Z}^d) = \rho(\gamma) \cdot x + c(\gamma) + \mathbb{Z}^d.$$

Equation (2.1) ensures that $\tilde{\alpha}$ is an action and (2.2) ensures that $\tilde{\alpha}$ is not conjugate to an action by automorphisms.

In the above construction, when Γ is higher-rank, any cocycle $c\colon \Gamma \to \mathbb{T}^d$ is necessarily cohomologous to a torsion-valued ($\mathbb{Q}^d/\mathbb{Z}^d$-valued) cocycle. This follows from the vanishing of $H^1_\rho(\Gamma, \mathbb{R}^d)$ (see for instance [71, Theorem 3 (iii)]) as discussed in [48] (see paragraph preceding Corollary 3) or [57] (see paragraph spanning pages 34 and 35). In particular, $\tilde{\alpha}$ has a global fixed point and is conjugate to an action by automorphisms when restricted to a finite-index subgroup of Γ. See [48] for further details.

Remark 2.5 (Affine Anosov actions)**.** We say that an action $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ of a discrete group Γ is an **Anosov action** if $\alpha(\gamma_0)$ is an Anosov diffeomorphism for some $\gamma_0 \in \Gamma$.

The action $\alpha\colon \mathrm{SL}(n, \mathbb{Z}) \to \mathrm{Diff}(\mathbb{T}^n)$ in Example 2.3 produces an affine Anosov action. Indeed, there exists $\gamma_0 \in \mathrm{SL}(n, \mathbb{Z})$ with no eigenvalues of modulus 1 whence $\alpha(\gamma_0) = L_{\gamma_0}$ is Anosov.

More generally, given a lattice Γ in a Lie group G, the action $\alpha_\rho\colon \Gamma \to \mathrm{Diff}(\mathbb{T}^d)$ discussed in Example 2.3 induced by a representation $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{Z})$ is Anosov if there is some $\gamma_0 \in \Gamma$ such that $\rho(\gamma_0)$ has all eigenvalues of modulus different from 1. This is equivalent to the statement that the representation $\hat{\rho}\colon G \to \mathrm{GL}(d, \mathbb{R})$ extending $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{Z})$ obtained from Margulis superrigidity theorem, Theorem 1.3, has no zero weights (see for instance [18, Lemma 7.1]).

Example 2.6 (Projective actions)**.** Let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be any lattice. Then Γ has a natural linear action on $\mathbb{R}^n$. The linear action of Γ on $\mathbb{R}^n$ induces an action of Γ on the sphere S^{n-1} thought of as the set of unit vectors in $\mathbb{R}^n$: we have $\alpha\colon \Gamma \to \mathrm{Diff}(S^n)$ given by

$$\alpha(\gamma)(x) = \frac{\gamma \cdot x}{\|\gamma \cdot x\|}.$$

Alternatively, we could act on the space of lines in $\mathbb{R}^n$ and obtain an action of Γ on the $(n-1)$-dimensional real projective space $\mathbb{R}P^{n-1}$. This action does not preserve a volume; in fact there is no Borel probability measure preserved by this action. Note also that these actions are not isometric for any Riemannian metric.

Remark 2.7 (Actions on boundaries)**.** Example 2.6 generalizes to actions by lattices Γ in any semisimple Lie group G acting on boundaries (i.e. generalized flag manifolds) of G. Given a semisimple Lie group G with Iwasawa decomposition $G = KAN$, let $M = K \cap C_G(A)$ be the centralizer of A in K. A closed subgroup $Q \subset G$ is **parabolic** if it is conjugate to a group containing MAN. When $G = \mathrm{SL}(n, \mathbb{R})$ we have that M is a finite group and any parabolic subgroup Q is conjugate to a group containing all upper triangular matrices. See [61, Section VII.7] for further discussion on the structure of parabolic subgroups.

Given a semisimple Lie group G, a (finite-index subgroup of a) proper parabolic subgroup $Q \subset G$, and a lattice $\Gamma \subset G$, the coset space $M = G/Q$ is compact and Γ acts on M naturally as

$$\alpha(\gamma)(xQ) = \gamma x Q.$$

When Q is a proper subgroup, these actions never preserve a volume form or any Borel probability measure and are not isometric.

In Example 2.6, the action on the projective space $\mathbb{R}P^{n-1}$ can be seen as the action on $\mathrm{SL}(n, \mathbb{R})/Q$ where Q is the parabolic subgroup

$$Q = \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \right\}.$$

Example 2.8 (Isometric actions)**.** Another important family of algebraic actions are isometric actions obtained from embeddings of cocompact lattices in Lie groups into compact groups.

Isometric actions of cocompact lattices in split orthogonal groups of type D_n*.* For $n \geq 4$, consider the quadratic form in $2n$ variables

$$Q(x_1, \dots, x_n, y_1, \dots, y_n) = x_1^2 + \cdots + x_n^2 - \sqrt{2}(y_1^2 + \cdots + y_n^2).$$

Let

$$B = \mathrm{diag}\left(1, \dots, 1, -\sqrt{2}, \dots, -\sqrt{2}\right) \in \mathrm{GL}(2n, \mathbb{R})$$

be the matrix such that $Q(x) = x^T B x$ for all $x \in \mathbb{R}^{2n}$ and let

$$G = \mathrm{SO}(Q) = \{g \in \mathrm{SL}(2n, \mathbb{R}) \mid g^T B g = B\}$$

be the special orthogonal group associated with Q. We have that

$$\mathrm{SO}(Q) \simeq \mathrm{SO}(n, n)$$

is a Lie group of rank n with restricted root system of type D_n when $n \geq 4$.[2]

Let $\mathbb{K} = \mathbb{Q}[\sqrt{2}]$ and let $\mathbb{Z}[\sqrt{2}]$ be the ring of integers in $\mathbb{K}$. Let

$$\Gamma = \{g \in \mathrm{SL}(2n, \mathbb{Z}[\sqrt{2}]) \mid g^T B g = B\}.$$

Then Γ is a cocompact lattice in G. (See for example Proposition 5.5.8 and Corollary 5.5.10 in [96].)

[2]For $n = 1$, $\mathrm{SO}(1,1)$ is a one-parameter group and for $n = 2$, $\mathrm{SO}(2,2)$ is not simple (it is double covered by $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$). For $n = 3$, $\mathrm{SO}(3,3)$ is double covered by $\mathrm{SL}(4, \mathbb{R})$.

Let $\tau\colon \mathbb{K} \to \mathbb{K}$ be the nontrivial Galois automorphism (so that $\tau(\sqrt{2}) = -\sqrt{2}$) and let τ act coordinate-wise on matrices with entries in $\mathbb{K}$. Given $\gamma \in \Gamma$ we have $\tau(\gamma) = \mathrm{Id}$ if and only if $\gamma = \mathrm{Id}$. Moreover, as $\tau^2 = \mathrm{Id}$ we have

$$\tau(\gamma) \in \mathrm{SO}(\tau(Q)) := \{g \in \mathrm{SL}(2n, \mathbb{R}) \mid g^T \tau(B) g = \tau(B)\} \simeq \mathrm{SO}(2n).$$

In particular, the map $\gamma \mapsto \tau(\gamma)$ gives a representation $\Gamma \to \mathrm{SO}(2n)$ into the compact group $\mathrm{SO}(2n)$ with infinite image.

As $\mathrm{SO}(2n)$ is the isometry group of the sphere $S^{2n-1} = \mathrm{SO}(2n)/\mathrm{SO}(2n-1)$, we obtain an action of Γ by isometries of a manifold of dimension $2n-1$.

Isometric actions of cocompact lattices in $\mathrm{SL}(n, \mathbb{R})$*.* A more complicated construction can be used to build cocompact lattices $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ that possess infinite-image representations $\pi\colon \Gamma \to \mathrm{SU}(n)$ (see discussion in [96, Sections 6.7, 6.8] as well as [96, Warning 16.4.3].) In this case, one obtains isometric actions of certain cocompact lattices Γ in $\mathrm{SL}(n, \mathbb{R})$ on the $(2n-2)$-dimensional homogeneous space

$$\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n-1)).$$

Example 2.9 (Modifications of affine actions)**.** Beyond the "algebraic actions" discussed in Examples 2.3–2.8, it is possible to modify certain algebraic constructions to construct genuinely new actions; these actions might not be conjugate to algebraic actions and may exhibit much weaker rigidity properties. One such construction starts with the standard action of (finite-index subgroups of) $\mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{T}^n$ and creates a non-volume-preserving action by blowing-up fixed points or finite orbits of the action. In [56, Section 4], Katok and Lewis showed this example can be modified to obtain volume-preserving, real-analytic actions of $\mathrm{SL}(n, \mathbb{Z})$ that are not C^0 conjugate to an affine action; moreover, these actions are not locally rigid. In [3, 5, 30], constructions of non-locally rigid, ergodic, volume-preserving actions of any lattice in a simple Lie group are constructed by more general blow-up constructions.

Example 2.10 (Actions factoring over boundaries)**.** A simple construction due to Stuck [89] demonstrates that it is impossible to fully classify all lattice actions in terms of algebraic actions. Let $P \subset \mathrm{SL}(n, \mathbb{R})$ be the group of upper triangular matrices. There is a non-trivial homomorphism $\rho\colon P \to \mathbb{R}$. Now consider any flow (i.e. $\mathbb{R}$-action) on a manifold M and view the flow as a P-action via the image of ρ. Then G acts on the induced space $N = (G \times M)/P$ and the restriction induces a non-volume-preserving, non-finite action of Γ. This example shows—particularly in the non-volume-preserving-case—that care is needed in order to formulate any precise conjectures that assert that every action should be "of an algebraic origin." Note, however, that we obtain a natural map $N \to G/P$ that intertwines Γ-actions; in particular, this action has an "algebraic action" as a factor.

We refer to [31, Sections 9 and 10] for more detailed discussion and references to modifications of algebraic actions and exotic actions.

2.2. Actions of lattices in rank-1 groups. Actions by lattices in higher-rank Lie groups are expected to be rather constrained. Although Example 2.9 and Example 2.10 shows there exists exotic, genuinely "non-algebraic" actions of such groups, these actions are built from modifications of algebraic constructions or factor over algebraic actions. For lattices in rank-one Lie groups such as $\mathrm{SL}(2,\mathbb{R})$, the situation is very different. There exist natural actions that have no algebraic origin and the algebraic actions of such groups seem to exhibit far less rigidity (see especially Example 2.12) than those above.

Example 2.11 (Actions of free groups)**.** Let $G = \mathrm{SL}(2,\mathbb{R})$. The free group $\Gamma = F_2$ is a lattice in G. (For instance, F_2 is the fundamental group of the punctured torus; more explicitly, $\mathrm{SL}(2,\mathbb{Z})$ contains a copy of F_2 as an index 12 subgroup.) Let M be any manifold and let $f, g \in \mathrm{Diff}(M)$. Then f and g generate an action of Γ on M that in general is not of any algebraic origin. In particular, there is no expectation that any rigidity phenomena should hold in general for actions by all lattices in $\mathrm{SL}(2,\mathbb{R})$.

For the next example, recall the definition of an Anosov action from Remark 2.5 (and Definition 1.7 of an Anosov diffeomorphism).

Example 2.12 (Non-standard Anosov actions of $\mathrm{SL}(2,\mathbb{Z})$)**.** Consider the standard action α_0 of $\mathrm{SL}(2,\mathbb{Z})$ on the 2 torus $\mathbb{T}^2$ as constructed in Example 2.3. In [47, Example 7.21], Hurder presents an example of a 1-parameter family of perturbations $\alpha_t \colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Diff}(\mathbb{T}^2)$ of α_0 with the following properties:

(1) Each α_t is a real-analytic, volume-preserving action;
(2) For $t > 0$, α_t is not topologically conjugate to α_0, (even when restricted to a finite-index subgroup of $\mathrm{SL}(2,\mathbb{Z})$.)

Moreover, since α_0 is an Anosov action and since the Anosov property is an open property we have that

(3) each α_t is an Anosov action.

This shows that even affine Anosov actions of $\mathrm{SL}(2,\mathbb{Z})$ fail to exhibit local rigidity properties and that there exist genuinely exotic Anosov actions of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{T}^2$. This is in stark contrast to the case affine Anosov actions of higher-rank lattices which are known to be locally rigid by [60, Theorem 15]. More generally, it is expected that all Anosov actions of higher-rank lattices are smoothly conjugate to affine actions as in Example 2.3 or Remark 2.4 (or analogous constructions in infra-nilmanifolds). See conjecture 4.1 below.

Remark 2.13. There are a number of rank-1 Lie groups whose lattices are known to exhibit some rigidity properties relative to linear representations. For instance, Corlette established superrigidity and arithmeticity of lattices in certain rank-1 simple Lie groups in [24]. In particular, Corlette establishes superrigidity for lattices in $\mathrm{Sp}(n,1)$ and F_4^{-20}, the isometry groups of quaternionic hyperbolic space and the Cayley plane. We note that these groups also exhibit Kazhdan's property (T).

It seems plausible that lattices in certain rank-1 Lie groups would exhibit some rigidity properties for actions by diffeomorphisms. Currently, there do not seem to be any results in this direction other than results that hold for all property (T) groups.

2.3. Cocycles over group actions and cocycle superrigidity. Consider a standard probability space (X, μ). Let G be a locally compact topological group and let $\alpha\colon G \times X \to X$ be a measurable action of G by μ-preserving transformations. In particular, $\alpha(g)$ is a μ-preserving, measurable transformation of X for each $g \in G$. We will always assume that μ is **ergodic**: the only G-invariant measurable subsets of X are null or conull.

2.3.1. *Linear cocycles over α.* A d-dimensional **measurable linear cocycle** over the action α is a measurable map

$$\mathcal{A}\colon G \times X \to \mathrm{GL}(d, \mathbb{R})$$

satisfying for a.e. $x \in X$ the cocycle condition: for all $g_1, g_2 \in G$

$$\mathcal{A}(g_1 g_2, x) = \mathcal{A}\left(g_1, \alpha(g_2)(x)\right) \mathcal{A}(g_2, x). \tag{2.3}$$

Note if e is the identity element of G, then (2.3) implies that

$$\mathcal{A}(e, x) = \mathcal{A}(e, x)\mathcal{A}(e, x)$$

whence $\mathcal{A}(e, x) = \mathrm{Id}$ for a.e. x

We say two cocycles $\mathcal{A}, \mathcal{B}\colon G \times X \to \mathrm{GL}(d, \mathbb{R})$ are (measurably) **cohomologous** if there is a measurable map $\Phi\colon X \to \mathrm{GL}(d, \mathbb{R})$ such that for a.e. x and every $g \in G$

$$\mathcal{B}(g, x) = \Phi(\alpha(g)(x))^{-1}\mathcal{A}(g, x)\Phi(x). \tag{2.4}$$

We say a cocycle $\mathcal{A}\colon G\times X \to \mathrm{GL}(d, \mathbb{R})$ is **constant** if $\mathcal{A}(g, x)$ is independent of x, that is, if $\mathcal{A}\colon G\times X \to \mathrm{GL}(d, \mathbb{R})$ coincides with a representation $\pi\colon G \to \mathrm{GL}(d, \mathbb{R})$ on a set of full measures.

2.3.2. *Application to smooth volume-preserving actions.* As a primary example of the above abstract setting, let $\alpha\colon G \to \mathrm{Diff}^1_\mu(M)$ be an action of G by C^1 diffeomorphisms of a compact manifold M preserving some Borel probability measure μ. Although the tangent bundle TM may not be a trivial bundle, we may choose a Borel measurable trivialization $\Psi\colon TM \to M \times \mathbb{R}^d$ of the vector-bundle TM where $d = \dim(M)$. We have that Ψ factors over the identity map on M and, writing $\Psi_x\colon T_x \to \mathbb{R}^d$ for the identification of the fiber over x with $\mathbb{R}^d$, we moreover assume that $\|\Psi_x\|$ and $\|\Psi_x^{-1}\|$ are uniformly bounded in x.

Fix such a trivialization Ψ and define $\mathcal{A}$ to be the derivative cocycle relative to this trivialization. To be precise, if $\Psi\colon TM \to M \times \mathbb{R}^d$ is the measurable vector-bundle trivialization then

$$\mathcal{A}(g, x) := \Psi(\alpha(g)(x))D_x\alpha(g)\Psi(x)^{-1}.$$

In this case, the cocycle relation (2.3) is simply the chain rule. Note that if we chose another Borel measurable trivialization $\Psi'\colon TM \to M \times \mathbb{R}^d$ then we obtain a cohomologous cocycle $\mathcal{A}'$. Indeed, we have

$$\mathcal{A}'(g,x) = \Psi'(\alpha(g)(x))\Psi(\alpha(g)(x))^{-1}\mathcal{A}(g,x)\Psi(x)\Psi'(x)^{-1}$$

so we may take $\Phi(x) = \Psi(x)\Psi'(x)^{-1}$ in (2.4).

2.3.3. *Cocycle superrigidity.* We formulate the statement of Zimmer's cocycle superrigidity theorem when G is either $\mathrm{SL}(n,\mathbb{R})$ or a lattice $\mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$. Note that the version formulated by Zimmer (see [101]) had a slightly weaker conclusion. We state the stronger version formulated and proved in [35].

THEOREM 2.14 (Cocycle superrigidity [35, 101]). *For $n \geq 3$, let G be either $\mathrm{SL}(n,\mathbb{R})$ or a lattice in $\mathrm{SL}(n,\mathbb{R})$. Let $\alpha\colon G \to \mathrm{Aut}(X,\mu)$ be an ergodic, measurable action of G by μ-preserving transformations of a standard probability space (X,μ). Let $\mathcal{A}\colon G \times X \to \mathrm{GL}(d,\mathbb{R})$ be a bounded,*[3] *measurable linear cocycle over α.*

Then there exist

1. *a linear representation $\rho\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$;*
2. *a compact subgroup $K \subset \mathrm{GL}(d,\mathbb{R})$ that commutes with the image of ρ;*
3. *a K-valued cocycle $\mathcal{C}\colon G \times X \to K$;*
4. *and a measurable function $\Phi\colon X \to \mathrm{GL}(d,\mathbb{R})$*

such that for a.e. $x \in X$ and every $g \in G$

$$\mathcal{A}(g,x) = \Phi(\alpha(g)(x))^{-1}\rho(g)\mathcal{C}(g,x)\Phi(x). \tag{2.5}$$

Moreover, the images of ρ and $\mathcal{C}$ commute.

In particular, Theorem 2.14 states that any bounded measurable linear cocycle $\mathcal{A}\colon G \times X \to \mathrm{GL}(d,\mathbb{R})$ over the action α is cohomologous to the product of a constant cocycle $\rho\colon G \to \mathrm{SL}(d,\mathbb{R})$ and a compact-valued cocycle $\mathcal{C}\colon G \times X \to K \subset \mathrm{GL}(d,\mathbb{R})$.

When $\mathcal{A}$ is the derivative cocycle associated to a smooth volume-preserving action $\alpha\colon \Gamma \to \mathrm{Diff}^r_{\mathrm{vol}}(M)$, Theorem 2.14 says that the derivative $(\gamma, x) \mapsto D_x\alpha(\gamma)$ coincides—up to a compact group and measurable trivialization of TM—with a representation $\rho\colon G \to \mathrm{SL}(\dim(M),\mathbb{R})$. This, in particular, suggests that non-isometric, volume-preserving actions $\alpha\colon \Gamma \to \mathrm{Diff}^r_{\mathrm{vol}}(M)$ on low-dimensional manifolds should be "derived from" affine actions of Γ. An example of a "derived from" affine action is the example of Katok in Lewis mentioned in Example 2.9. In [57], such a philosophy is carried out for volume-preserving Anosov actions of $\mathrm{SL}(n,\mathbb{Z})$ on $\mathbb{T}^n$.

[3]Here, **bounded** means that for every compact $K \subset G$, the map $K \times X \to \mathrm{GL}(d,\mathbb{R})$ given by $(g,x) \mapsto \mathcal{A}(g,x)$ is bounded. More generally, we may replace the boundedness hypothesis with the hypothesis that the function $x \mapsto \sup_{g\in K} \log \|\mathcal{A}(g,x)\|$ is $L^1(\mu)$. See [35].

2.4. Smooth actions and the Zimmer program. We outline a number of broad directions within the **Zimmer program** inspired by Theorem 2.14 and motivated by analogy with linear representations. More detailed overviews can be found in [29, 31, 32, 62].

2.4.1. *Existence and finiteness of actions in low dimensions.* Given a group Γ and a manifold M, one may ask if there exists a faithful action $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$. More generally, one may ask if there exists an action $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ with infinite image $\alpha(\Gamma)$. From a dynamical perspective, actions of finite groups are rather trivial. Thus, the existence of "non-trivial" actions $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ should aim to find actions with infinite image $\alpha(\Gamma)$. This is the setting for **Zimmer's conjecture** which asserts that when $\dim(M)$ is sufficiently small (relative to certain algebraic data associated with Γ), all actions $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ are finite. See Section 3 for detailed statements of conjectures and further discussion and heuristics.

2.4.2. *Local rigidity.* Local rigidity conjectures and results aim to classify perturbations of actions by showing that all sufficiently small perturbations coincide after a smooth coordinate change. We recall a common definition of local rigidity for C^∞ group actions:

Definition 2.15. An action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ of a finitely generated group Γ is said to be **locally rigid** if, for any action $\tilde\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ sufficiently C^1-close to α, there exists a C^∞ diffeomorphism $h\colon M \to M$ such that

$$h \circ \tilde\alpha(\gamma) \circ h^{-1} = \alpha(\gamma) \qquad \text{for all } \gamma \in \Gamma. \tag{2.6}$$

In Definition 2.15, using that Γ is finitely generated, we define the C^1 distance between α and $\tilde\alpha$ to be

$$\max\{d_{C^1}(\alpha(\gamma), \tilde\alpha(\gamma)) \mid \gamma \in F\}$$

where $F \subset \Gamma$ is a finite, symmetric generating subset.

For isometric actions, local rigidity was shown for cocompact lattices in higher-rank simple Lie groups in [4]. For nonuniform lattices in higher-rank simple groups and more general discrete groups with property (T), local rigidity of isometric actions was shown in [36].

In a non-isometric setting, consider the projective actions discussed in Example 2.6 and Remark 2.7. In [53], the action in Example 2.6 by cocompact lattices was shown to be locally rigid. General projective actions by cocompact lattices as in Remark 2.7 were shown in [60, Theorem 17] to be local rigid.

In a non-isometric, volume-persevering setting, recall the affine Anosov actions on tori (and generalizations to nilmanifolds) discussed in Example 2.3 and Remarks 2.4 and 2.5. For Anosov actions, note that while structural stability (1.5) holds for individual Anosov elements of the action, local rigidity requires that map h in (2.6) intertwines the action of the entire group Γ; moreover, unlike in the case of a single Anosov map where the map h in (1.5) is typically only Hölder continuous, we ask that the map h in (2.6) be smooth in Definition 2.15.

In [47] Hurder proved a number of **deformation rigidity** results for certain standard affine actions; that is, under certain hypotheses, a 1-parameter family of perturbations of an affine action ρ are smoothly conjugate to ρ. A related rigidity phenomenon, the **infinitesimal rigidity**, has been studied for affine Anosov actions in [47, 50, 65, 83].

There are a number of results establishing local rigidity of affine Anosov actions on tori and nilmanifolds including [46, 47, 55, 57, 82, 85]. For the general case of actions by higher-rank lattices on tori and nilmanifolds, the local rigidity problem for affine Anosov actions was settled by Katok and Spatzier in [60].

In [72] Margulis and Qian extended local rigidity to weakly hyperbolic affine actions which allow the representation $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{Z})$ to have zero weights. Fisher and Margulis [37] established local rigidity in full generality for quasi-affine actions by higher-rank lattices which, in particular, includes actions by nilmanifold automorphisms without assuming any hyperbolicity.

We remark that most local rigidity results discussed above require the acting group to have property (T) so that the perturbed action preserves an absolutely continuous invariant measure. In particular, the methods of proof of these results do not hold for irreducible lattices in products of rank-1 Lie groups.

We also remark that the examples due to Katok and Lewis in Example 2.9 are not locally rigid and thus local rigidity of general non-linear actions of higher-rank lattices requires additional hypotheses.

2.4.3. *Global rigidity.* Global rigidity conjectures roughly aim to classify all smooth actions $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ (up to smooth coordinate change) in terms of modifications or extensions of families of standard algebraic examples. Precise global rigidity conjectures often assume a number of strong dynamical hypotheses such the existence of Anosov element and that the action preserve a smooth volume. Furthermore, most global rigidity results and conjectures make assumptions on the underlying manifold M. See Section 4 where we discuss new global rigidity results for actions on tori.

2.4.4. *Actions on the circle.* A direction which is not discussed here concerns actions in dimension 1; namely actions on the circle S^1 or the interval $[0, 1]$. In this setting, there are a number of extra tools that are not available in higher dimensions. We do not attempt to give a complete up to date bibliography. We refer the reader to [77] for an overview of many results concerning group actions on the circle.

3. Finiteness of actions and Zimmer's conjecture

Given a lattice Γ in a higher-rank Lie group G and a manifold M, we consider the existence of non-trivial (i.e. non-finite) actions $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$. Zimmer's conjecture asserts that no such actions should exist when $\dim(M)$ is sufficiently small.

3.1. Zimmer's conjecture for lattices in $\mathrm{SL}(n,\mathbb{R})$**.** First consider the case that $G = \mathrm{SL}(n,\mathbb{R})$ and recall the actions discussed in Example 2.3 and Example 2.6. When $n \geq 3$, Zimmer's conjecture asserts that these are the minimal dimensions in which non-finite actions can occur for lattices in $\mathrm{SL}(n,\mathbb{R})$. We have the following precise formulation.

Conjecture 3.1 (Zimmer's conjecture for lattices in $\mathrm{SL}(n,\mathbb{R})$)**.** *For $n \geq 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a lattice. Let M be a compact manifold.*

(1) If $\dim(M) < n-1$ then any homomorphism $\Gamma \to \mathrm{Diff}(M)$ has finite image.
(2) In addition, if vol is a volume form on M and if $\dim(M) = n-1$ then any homomorphism $\Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$ has finite image.

We are intentionally vague about the regularity of the action in conjecture 3.1 (and conjecture 3.2 below) as it is not clear what is the expected sharp regularity. Zimmer's original conjecture considered only the case of C^∞ volume-preserving actions. See [100, 103, 104]. The non-volume-preserving case first appears in writing in [27] but is motivated by much earlier work on the circle including [19, 45, 95]. Most evidence for the conjecture requires the action to be at least C^1. It is possible the conjecture holds for actions by homeomorphisms; see for instance [8, 94, 95] for a very partial list of results in this directions. Most of the results discussed below require the action to be at least $C^{1+\beta}$ as we use tools from nonuniformly hyperbolic dynamics which typically require some regularity of the derivative. Some of our results can be established for actions by C^1 diffeomorphisms; see in particular Theorem 3.7. However, none of the tools we employ apply to actions by homeomorphisms.

Early results establishing this conjecture in the setting of actions the circle appear in [19, 45, 95] and in the setting of volume-preserving (and more general measure-preserving) actions on surfaces in [41, 42, 80]. See also [44] and [27] for results on real-analytic actions and [20, 22, 23] for results on holomorphic and birational actions. There are also many results (usually in the C^0 setting) for actions of specific lattices on specific manifolds where there are topological obstructions to the group acting; a partial list of such results includes [8, 9, 79, 93, 94, 97, 98, 106].

3.2. Zimmer's conjecture for lattices in other Lie groups. To formulate Zimmer's conjecture for actions of lattices in more general Lie groups, to each simple, non-compact Lie group G we associate 3 positive integers $d_0(G), d_{\mathrm{rep}}(G)$, and $d_{\mathrm{cmt}}(G)$ defined roughly as follows:

(1) $d_0(G)$ is the minimal dimension of G/H as H varies over proper closed subgroups $H \subset G$. We remark that H is necessarily a parabolic subgroup and the quotient is a boundary (or generalized flag manifold) of G. See Remark 2.7.
(2) $d_{\mathrm{rep}}(G)$ is the minimal dimension of a non-trivial linear representation of (the Lie algebra) of G.

(3) $d_{\text{cmt}}(G)$ is the minimal dimension of a non-trivial homogeneous space of a compact real form of G.

See Table 1 where we compute the above numbers for a number of matrix groups, (split) real forms of exceptional Lie algebras, and certain complex matrix groups. We also include another number $r(G)$ which is defined in [12, 16] and arises from certain dynamical arguments; a precise definition which is equivalent to that in [12, 16] is the following:

(4) $r(G)$ is $d_0(G')$ where G' is the largest $\mathbb{R}$-split simple subgroup of G.

This number $r(G)$ gives the bounds appearing in the most general result, Theorem 3.8 below, towards solving Conjecture 3.2. For complete tables of values of $d_{\text{rep}}(G)$, $d_{\text{cmt}}(G)$, and $d_0(G)$, we refer to [21].

Given the examples in Section 2.1 and the integers $d_{\text{rep}}(G)$, $d_{\text{cmt}}(G)$, and $d_0(G)$ defined above, is it natural to conjecture the following full conjecture.

Conjecture 3.2 (Zimmer's conjecture)**.** *Let G be a connected, simple Lie group with finite center. Let $\Gamma \subset G$ be a lattice. Let M be a compact manifold and* vol *a volume form on M. Then*

(1) if $\dim(M) < \min\{d_{\text{rep}}(G), d_{\text{cmt}}(G), d_0(G)\}$ then any homomorphism $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ has finite image;

Table 1. Numerology in appearing in Zimmer's conjecture for various groups. See page 20 for definitions of d_0, d_{vol}, and d_{cmt}. See also [21] for more complete tables. See Theorem 3.8 where the number $r(G)$ appears and [12, 16] or (4), page 20.

G	restricted root system	rank	$d_{\text{rep}}(G)$	$d_{\text{cmt}}(G)$	$d_0(G)$	$r(G)$
$\mathrm{SL}(n,\mathbb{R})$	A_{n-1}	$n-1$	n	$2n-2$	$n-1$	$n-1$
$\mathrm{SO}(n,n+1)$	B_n	n	$2n+1$	$2n$	$2n-1$	$2n-1$
$\mathrm{Sp}(2n,\mathbb{R})$	C_n	n	$2n$	$4n-4$	$2n-1$	$2n-1$
$\mathrm{SO}(n,n)$	D_n	n	$2n$	$2n-1$	$2n-2$	$2n-2$
E_I	E_6	6	27	26	16	16
E_V	E_7	7	56	54	27	27
E_{VIII}	E_8	8	248	112	57	57
F_1	F_4	4	26	16	15	15
G	G_2	2	7	6	5	5
$\mathrm{SL}(n,\mathbb{C})$	A_{n-1}	$n-1$	$2n$	$2n-2$	$2n-2$	$n-1$
$\mathrm{SO}(2n,\mathbb{C})$	D_n	n	$4n$	$2n-1$	$4n-4$	$2n-2$
$\mathrm{SO}(2n+1,\mathbb{C})$	B_n	n	$4n+2$	$2n$	$4n-2$	$2n-1$
$\mathrm{Sp}(2n,\mathbb{C})$	C_n	n	$4n$	$4n-4$	$4n-2$	$2n-1$
$\mathrm{SO}(p,q)$ $p<q$	B_p	p	$p+q$	$p+q-1$	$p+q-2$	$2p-1$

(2) *if* $\dim(M) < \min\{d_{\mathrm{rep}}(G), d_{\mathrm{cmt}}(G)\}$ *then any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$ *has finite image;*
(3) *if* $\dim(M) < \min\{d_0(G), d_{\mathrm{rep}}(G)\}$ *then for any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}(M)$*, the image* $\alpha(\Gamma)$ *preserves a Riemannian metric;*
(4) *if* $\dim(M) < d_{\mathrm{rep}}(G)$ *then for any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$*, the image* $\alpha(\Gamma)$ *preserves a Riemannian metric.*

3.3. Heuristic evidence for Conjecture 3.1. We outline a number of heuristic arguments that support Conjectures 3.1 and 3.2.

3.3.1. *Analogy with linear representations.* Conjecture 3.1 can be seen as a "nonlinear" analogue of Corollary 1.5. Namely, for $d < n$, we replace the vector space $\mathbb{R}^d$ with a d-dimensional manifold M and the linear group $\mathrm{GL}(d, \mathbb{R})$ with the diffeomorphism group $\mathrm{Diff}(M)$ and aim to establish an analogous finiteness result.

3.3.2. *Invariant measurable metrics.* For $n \geq 3$, let Γ be a lattice in $G = \mathrm{SL}(n, \mathbb{R})$ and consider a volume-preserving action $\alpha\colon \Gamma \to \mathrm{Diff}^1_{\mathrm{vol}}(M)$ where M is a compact manifold of dimension at most $d \leq n-1$. Since there are no representations $\rho\colon \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(d, \mathbb{R})$ for $d < n$, Theorem 2.14 implies that the derivative cocycle is cohomologous to a compact-valued cocycle. In particular, for such an action α we have the following.

Corollary 3.3. *α preserves a 'Lebesgue-measurable Riemannian metric;' that is, there exists a measurable, α-invariant, positive-definite symmetric two-form on TM.*

Indeed, we have that the derivative cocycle is cohomologous to a K-valued cocycle for some compact group $K \subset \mathrm{GL}(d, \mathbb{R})$. One may then pull-back any K-invariant inner product on $\mathbb{R}^d$ to T_xM via the map $\Phi(x)$ in Theorem 2.14 to an $\alpha(\Gamma)$-invariant inner product on M.

Suppose one could show that the Lebesgue-measurable invariant Riemannian metric g in Corollary 3.3 was continuous or C^ℓ. Then α is an action by isometries of g and, as explained in Step 3 of Section 5 below, this combined with Theorem 1.4 implies that the image $\alpha(\Gamma)$ is finite. Thus, Conjecture 3.1(2) follows if one can promote the measurable invariant metric g guaranteed by Corollary 3.3 of Theorem 2.14 to a continuous Riemannian metric.

3.3.3. *Actions with discrete spectrum.* Upgrading the measurable invariant Riemannian metric in Corollary 3.3 to a continuous Riemannian metric in the above heuristic seems quite difficult and is not the approach we take in our proof of Theorem 3.4 below. In [102], Zimmer was able to upgrade the measurable metric to a continuous metric for volume-preserving actions that are very close to isometries. This result now follows from the local rigidity of isometric actions in [4, 36].

Zimmer later established a much stronger result in [105] which provides very strong evidence for the volume-preserving cases in conjecture 3.2. Using the invariant, measurable metric discussed above and that higher-rank lattices have Property (T), Zimmer showed that any volume-preserving action appearing in conjecture 3.2 has discrete spectrum. This, in particular, implies that (the ergodic components of) all volume-preserving actions appearing in conjecture 3.2 are measurably isomorphic to isometric actions.

3.4. Progress on Zimmer's conjecture. Recently the author, together with David Fisher and Sebastian Hurtado, established conjecture 3.1 for actions by cocompact lattices in $\mathrm{SL}(n,\mathbb{R})$ in [12]. We also addressed conjecture 3.2 for actions of cocompact lattices in other Lie groups in [12].

THEOREM 3.4 ([12, Theorem 1.1]). *For $n \geq 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a cocompact lattice. Let M be a compact manifold.*

(1) If $\dim(M) < n-1$ then any homomorphism $\Gamma \to \mathrm{Diff}^{1+\beta}(M)$ has finite image.
(2) In addition, if vol is a volume form on M and if $\dim(M) = n-1$ then any homomorphism $\Gamma \to \mathrm{Diff}^{1+\beta}_{\mathrm{vol}}(M)$ has finite image.

Remark 3.5 (Remarks on Theorem 3.4)**.**

(1) Recently, the authors announced in [13] that the conclusion of Theorem 3.4 holds for actions of $\mathrm{SL}(n,\mathbb{Z})$ for $n \geq 3$. The result for general lattices in $\mathrm{SL}(n,\mathbb{Z})$ as well as analogous results for lattices in other higher-rank simple Lie groups, has been announced [14] by the same authors. See Theorem 3.8.
(2) We stated Theorem 3.4 for actions by C^2 diffeomorphisms in [12] though the proof can be adapted for actions by $C^{1+\beta}$ actions. Our sketch below will assume the action is by C^∞ diffeomorphisms to simplify certain Sobolev space arguments.
(3) Theorem 3.4 and the results in the above remarks solves conjecture 3.1 for actions $C^{1+\beta}$ diffeomorphisms. For actions by C^1 diffeomorphisms, see discussion in Section 3.5.1 below.

3.5. Further results on Zimmer's conjecture.

3.5.1. *Actions in dimensions below the rank and actions by C^1 diffeomorphisms.* Consider a connected, simple Lie group G with finite center. In the excellent article [21], Serge Cantat gave (in French) a proof of Theorem 3.4 by presenting a mostly self-contained proof of the following.

THEOREM 3.6. *Let G be a connected, simple Lie group G with finite center and rank at least 2. Let $\Gamma \subset G$ be a cocompact lattice and let M be a compact manifold.*

(1) If $\dim(M) < \mathrm{rank}(G)$ then any homomorphism $\Gamma \to \mathrm{Diff}^2(M)$ has finite image.
(2) In addition, if vol is a volume form on M and if $\dim(M) \leq \mathrm{rank}(G)$ then any homomorphism $\Gamma \to \mathrm{Diff}^2_{\mathrm{vol}}(M)$ has finite image.

Recently, D. Damjanovich and Z. Zhang observed that certain arguments in the proofs of Theorems 3.4 and 3.6 can be adapted to the setting of actions by C^1-diffeomorphisms. Together with the author, they have announced the following theorem.

THEOREM 3.7 ([11]). *Let $\Gamma \subset G$ be a lattice in a connected, higher-rank simple Lie group G with finite center. Let M be a compact manifold.*

(1) If $\dim(M) < \operatorname{rank}(G)$ then any homomorphism $\Gamma \to \operatorname{Diff}^1(M)$ has finite image.

(2) In addition, if vol is a volume form on M and if $\dim(M) = \operatorname{rank}(G)$ then any homomorphism $\Gamma \to \operatorname{Diff}^1_{\operatorname{vol}}(M)$ has finite image.

For actions by lattices in other higher-rank Lie groups, there is a gap between what is known for C^1 versus $C^{1+\beta}$-actions. Indeed, our number $r(G)$ in Theorem 3.8 below always satisfies $r(G) \geq \operatorname{rank}(G)$ and is a strict inequality unless G has restricted root system of type A_n. The main distinction between the C^1 and $C^{1+\beta}$ setting is in the use of Proposition 6.2 versus Proposition 6.4 discussed in Section 6.4.

3.5.2. *Actions by lattices in general Lie groups.* Theorem 3.6 fails to give the optimal dimension bounds for the analogue of Conjecture 3.1 given in Conjecture 3.2 for actions by lattices in Lie groups other than $\mathrm{SL}(n, \mathbb{R})$. See Table 1 for various conjectured critical dimensions arising in Zimmer's conjecture for other Lie groups.

To state the most general result towards solving Conjecture 3.2, to any simple Lie group G, we associate a non-negative integer $r(G)$ defined in (4), page 20. See also [12, Section 2.2] for equivalent definitions of $r(G)$ and see Table 1 for values of $r(G)$ in various examples of G. For actions of lattices in a general Lie group G, the main result of [12] as well as the announced extension in [14] gives finiteness of actions up to a critical dimension determined by $r(G)$.

THEOREM 3.8 ([12] cocompact case; [14] nonuniform case). *Let $\Gamma \subset G$ be a lattice in a connected, higher-rank simple Lie group G with finite center. Let M be a compact manifold.*

(1) If $\dim(M) < r(G)$ then any homomorphism $\Gamma \to \operatorname{Diff}^{1+\beta}(M)$ has finite image.

(2) In addition, if vol is a volume form on M and if $\dim(M) = r(G)$ then any homomorphism $\Gamma \to \operatorname{Diff}^{1+\beta}_{\operatorname{vol}}(M)$ has finite image.

3.5.3. *Actions by lattices in split real forms.* When G is exceptional or not a split real form, our number $r(G)$ is lower than the conjectured critical dimension in Conjecture 3.2(1) and (2). However, for lattices in all Lie groups that are non-exceptional, split real forms Theorem 3.8 confirms Conjecture 3.2(1) and (2). For $C^{1+\beta}$ actions by lattices in simple Lie groups that are exceptional split real forms, Theorem 3.8 confirms Conjecture 3.2(1).

For instance, for actions by lattices in symplectic groups, we have the following.

Theorem 3.9 ([12, Theorem 1.3] cocompact case; [14] nonuniform case). *For $n \geq 2$, if M is a compact manifold with* $\dim(M) < 2n-1$ *and if* $\Gamma \subset \mathrm{Sp}(2n, \mathbb{R})$ *is a lattice then any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}^2(M)$ *has finite image. In addition, if* $\dim(M) = 2n-1$ *then any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}^2_{\mathrm{vol}}(M)$ *has finite image.*

Similarly, for actions by lattices in split orthogonal groups, we have the following.

Theorem 3.10 ([12, Theorem 1.4] cocompact case; [14] nonuniform case). *Let M be a compact manifold.*

(1) For $n \geq 4$, if $\Gamma \subset \mathrm{SO}(n,n)$ is a lattice and if $\dim(M) < 2n-2$ *then any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}^2(M)$ *has finite image. If* $\dim(M) = 2n-2$ *then any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}^2_{\mathrm{vol}}(M)$ *has finite image.*

(2) For $n \geq 3$, if $\Gamma \subset \mathrm{SO}(n,n+1)$ is a lattice and if $\dim(M) < 2n-1$ *then any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}^2(M)$ *has finite image. If* $\dim(M) = 2n-1$ *then any homomorphism* $\alpha\colon \Gamma \to \mathrm{Diff}^2_{\mathrm{vol}}(M)$ *has finite image.*

For actions by lattices Γ in simple Lie groups that are not split real forms such as $G = \mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(n,m)$ for $m \geq n+2$, or $\mathrm{SU}(n,m)$, Theorem 3.8 above (the main result of [12] for cocompact case, [14] in general) gives finiteness of all actions on manifolds whose dimension is below a certain critical dimension. However, this critical dimension may be below the dimension conjectured by the analogue of Conjecture 3.2 for these groups. See Table 1.

4. Global rigidity of hyperbolic and Anosov actions on tori

We recall that an action $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ is Anosov if $\alpha(\gamma_0)$ is Anosov for some $\gamma_0 \in \Gamma$; see Definition 1.7. The motivation for the results discussed in this section is a desired classification of Anosov actions of higher-rank lattices on compact manifolds. See conjecture 4.1 below. We recall that it is conjectured that the only manifolds M that admit Anosov diffeomorphisms $f\colon M \to M$ are, up to finite covers, homeomorphic to tori and nilmanifolds. If this conjecture is true, Theorem 1.8 would classify all Anosov diffeomorphisms up to a continuous change of coordinates. While this conjecture remains open, it seems plausible that a classification of Anosov actions of higher-rank lattices (or abelian groups) is more tractable. In particular, there is hope that one may show all Anosov actions of higher-rank lattices (or actions of higher-rank Abelian groups satisfying certain irreducibility conditions) occur on tori and nilmanifolds.

We state the following conjecture which is motivated in part by the works of Feres–Labourie [28] and Goetze–Spatzier [46].

Conjecture 4.1 ([31, Conjecture 1.3])**.** *If Γ is a lattice in* $\mathrm{SL}(n, \mathbb{R})$ *where $n \geq 3$, then any C^∞, volume-preserving, Anosov action by Γ on a compact* *manifold is smoothly conjugate to an action by affine automorphisms of an infranilmanifold.*

See also [49, Conjecture 1.1] and [56, Conjecture 1.1] for related conjectures. The assumption in Conjecture 4.1 that the action preserves a volume is a standard assumption though the results outlined here seem to suggest that such a hypothesis may be unnecessary.

The work of Feres-Labourie [28] and Goetze-Spatzier [46] motivating the above conjecture make no assumptions on the topology of M. However, they require strong dynamical hypotheses, particularly that the dimension of M is small relative to G. Outside of these results, results on the global rigidity of Anosov actions typically assume the underlying manifold is a torus or nilmanifold. We take this approach in the following where we consider higher-rank lattices Γ acting on tori $\mathbb{T}^d$. Note that while we specialize to actions on tori, all the results discussed hold in the more general setting of actions on nilmanifolds N/Λ.

4.1. Global topological rigidity. Recall Theorem 1.6 which, for a homeomorphism $f\colon \mathbb{T}^d \to \mathbb{T}^d$ with hyperbolic linear data A_f, produces a continuous, surjective, possibly non-injective change of coordinates $h\colon \mathbb{T}^d \to \mathbb{T}^d$ such that $h \circ f = L_{A_f} \circ h$. In particular, f is semiconjugate to an automorphism of $\mathbb{T}^d$.

Given a discrete group Γ and an action

$$\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$$

by homeomorphisms, we define $\rho(\gamma) \in \mathrm{GL}(d,\mathbb{Z})$ to be the matrix $A_{\alpha(\gamma)}$; that is, for each $\gamma \in \Gamma$, taking any lift $\tilde\alpha(\gamma)\colon \mathbb{R}^d \to \mathbb{R}^d$ of $\alpha(\gamma)$ we have

$$\tilde\alpha(\gamma)(x) = \rho(\gamma)x + \phi_\gamma(x)$$

where $\phi_\gamma\colon \mathbb{R}^d \to \mathbb{R}^d$ is $\mathbb{Z}^d$-periodic. Fixing a basis for homology $H^1(\mathbb{T}^n,\mathbb{Z})$, we may also view $\rho(\gamma)$ as the action $H^1(\mathbb{T}^n,\mathbb{Z})$ induced by α. By the uniqueness of the linear data (1.3), the map $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{Z})$ is a representation.

We call $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{Z})$ the **linear data** of the action $\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$. We abuse notation and also write $\rho\colon \Gamma \to \mathrm{Aut}(\mathbb{T}^d)$ for the action by toral automorphisms induced by ρ:

$$\rho(\gamma)(x + \mathbb{Z}^d) = \rho(\gamma)x + \mathbb{Z}^d.$$

Whether we view $\rho(\gamma)$ as an element of $\mathrm{GL}(d,\mathbb{Z})$ or $\mathrm{Aut}(\mathbb{T}^d)$ should be clear from context.

Below we will always assume that $\rho(\gamma)$ is **hyperbolic** for some $\gamma \in \Gamma$; that is, we assume $\rho(\gamma)$ has no eigenvalue of modulus 1. Then, by Theorem 1.6, there exists a semiconjugacy $h\colon \mathbb{T}^d \to \mathbb{T}^d$ between the maps $\alpha(\gamma)$ and $\rho(\gamma)$ of $\mathbb{T}^d$; global topological rigidity results aim to show such a semiconjugacy h is coherent across the entire action of Γ. That is, assuming that $\rho(\gamma)$ is hyperbolic for some $\gamma \in \Gamma$, we aim to find a continuous, surjective map $h\colon \mathbb{T}^d \to \mathbb{T}^d$ that intertwines the action $\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$ with an affine action whose linear part is $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{Z})$. In the case that Γ is a higher-rank lattice, vanishing of 1st cohomology ensures for such a map h that

$$h \circ \alpha(\gamma) = \rho(\gamma) \circ h \qquad \text{for all } \gamma \in \Gamma' \tag{4.1}$$

where Γ' is a finite-index subgroup of Γ. Note that, as genuinely affine actions exist (see Remark 2.4), one should not expect (4.1) to hold for all $\gamma \in \Gamma$. We remark that any map h satisfying (4.1) will semiconjugate the action of the entire group Γ to an action by affine maps.

For actions by free groups or $\mathrm{SL}(2,\mathbb{Z})$, it is possible to find actions for which no h as in (4.1) exists; moreover, no continuous surjective h intertwines the action with an affine action. See discussion in Example 2.12. However, for actions of lattices Γ in higher-rank simple Lie groups G with hyperbolic linear data ρ (and satisfying a mild lifting hypothesis), the first main result of [18] produces a semiconjugacy h as in (4.1).

For simplicity, we state the following result only for actions on tori though analogous results hold for actions on nilmanifolds.

Theorem 4.2 (c.f. [18, Theorem 1.3]). *Let Γ be a lattice in a connected, higher-rank simple Lie group G with finite center. Let $\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$ be an action by homeomorphisms with linear data $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{Z})$. Suppose that*

(1) the matrix $\rho(\gamma_0)$ is hyperbolic for some $\gamma_0 \in \Gamma$, and
(2) for some finite-index subgroup $\Gamma' \subset \Gamma$, the action $\alpha\colon \Gamma' \to \mathrm{Homeo}(\mathbb{T}^d)$ lifts to an action $\tilde{\alpha}\colon \Gamma' \to \mathrm{Homeo}(\mathbb{R}^d)$.

Then there is a continuous, surjective $h\colon \mathbb{T}^d \to \mathbb{T}^d$ such that

$$h \circ \alpha(\gamma) = \rho(\gamma) \circ h \tag{4.2}$$

for all γ in a finite-index subgroup $\Gamma'' \subset \Gamma$.

In particular, the action $\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$ is semiconjugate to an action by affine maps of $\mathbb{T}^d$.

Previous topological rigidity results for Anosov actions of higher-rank lattices on general nilmanifolds were proven in [72, Theorem 1.3]. More general and topological semiconjugacies are constructed in [38] between actions on general manifolds M whose action on $\pi_1(M)$ factors through an action of a finitely-generated, torsion-free, nilpotent group and affine actions on nilmanifolds. These and related results assume the existence of an invariant probability measure for the non-linear action; the topological rigidity theorem of [18] recovers these results without any assumption on the existence of an invariant measure.

Remark 4.3. The assumption that the action $\alpha\colon \Gamma' \to \mathrm{Homeo}(\mathbb{T}^d)$ lifts to an action $\tilde{\alpha}\colon \Gamma' \to \mathrm{Homeo}(\mathbb{R}^d)$ for some finite-index $\Gamma' \subset \Gamma$ is equivalent to the vanishing of a certain element of $H^2(\Gamma, (\mathbb{R}^d, \rho))$. In particular, the lifting hypotheses can be verified using only knowledge about the linear data ρ.

Sufficient conditions for the vanishing of this cohomological obstruction are given by [43, Theorem 3.1] and [6, Theorem 4.4] and automatically holds in any of the following settings:

(1) $\Gamma = \mathrm{SL}(d, \mathbb{Z})$ acting on $\mathbb{T}^d$, $d \geq 5$.
(2) Γ is a cocompact lattice in G.
(3) α is an action of Γ on a torus $\mathbb{T}^d$ which preserves a probability measure μ.

For more details, see [18, Remark 1.5].

4.2. Global smooth rigidity of Anosov actions. In Section 4.1, we considered continuous actions $\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$ satisfying mild assumptions on the induced action on homology. However, the natural conjectures in the area (see conjecture 4.1) typically concern C^∞ actions $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ containing at least one Anosov element $\alpha(\gamma_0)$. When Γ is a lattice in a higher-rank simple Lie group G, it is then expected that α is smoothly conjugate to an affine action.

Global rigidity for Anosov actions satisfying strong dynamical hypotheses appeared in [47]. Global rigidity for Anosov actions by $\mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{T}^n$, $n \geq 3$, were obtained in [56, 57]. Other global rigidity results appear in [84]. We also remark that Feres-Labourie [28] and Goetze-Spatzier [46] established very strong global rigidity results for Anosov actions, in which no assumptions on the topology of M are made. These results assume the existence of an invariant (smooth or fully supported) probability measure for the non-linear action or additional strong dynamical hypotheses.

In [18], we established the most general global smooth rigidity result for Anosov actions on tori (with analogous results for nilmanifolds).

Theorem 4.4 (c.f. [18, Theorem 1.7]). *Let Γ be a lattice in a connected, higher-rank simple Lie group G with finite center. Let $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ be an action with linear data $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{Z})$. Suppose that*

(1) the element $\alpha(\gamma_0)$ is Anosov for some $\gamma_0 \in \Gamma$, and
(2) for some finite-index subgroup $\Gamma' \subset \Gamma$, the action $\alpha\colon \Gamma' \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ lifts to an action $\tilde{\alpha}\colon \Gamma' \to \mathrm{Diff}^\infty(\mathbb{R}^d)$.

Then, there is a $\mathbb{C}^\infty$ diffeomorphism $h\colon \mathbb{T}^d \to \mathbb{T}^d$ such that

$$h \circ \alpha(\gamma) = \rho(\gamma) \circ h \tag{4.3}$$

for all γ in a finite-index subgroup $\Gamma'' \subset \Gamma$.

In particular, the action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ is smoothly conjugate to an action by affine maps of $\mathbb{T}^d$.

To establish Theorem 4.4, we first use that $\rho(\gamma_0)$ is hyperbolic if $\alpha(\gamma_0)$ is Anosov. From Theorem 4.2 and Theorem 1.8, we then obtain a homeomorphism $h\colon \mathbb{T}^d \to \mathbb{T}^d$ satisfying (4.3). It remains to show that h is C^∞. This follows by studying the restriction of the action to a higher-rank *abelian* subgroup $\Sigma \subset \Gamma$. Establishing smoothness of a conjugacy h between Anosov actions of higher-rank abelian groups goes back to [56, 58] with more recent results in [33,34]. The most general result which we quote in our work [18] was established in [87].

To apply the results of [87], one must find a higher-rank abelian subgroup $\Sigma \subset \Gamma$ such that (among other technical requirements) $\alpha(\gamma)$ is Anosov for some $\gamma \in \Sigma$. Note that while we assume $\alpha(\gamma_0)$ is Anosov for some $\gamma_0 \in \Gamma$, the element γ_0 need not have a large centralizer in Γ. In [18], the new main new arguments needed to establish Theorem 4.4 promotes the existence of single Anosov element $\alpha(\gamma_0)$ of the action α to the existence of a Zariski dense subsemigroup S of Γ such that $\alpha(\gamma)$ is Anosov for every γ in S.

5. Outline of proof of Theorem 3.4

We outline the proof of Theorem 3.4 in the case of a C^∞ action of a cocompact lattice in $\mathrm{SL}(n,\mathbb{R})$. That is, for $n \geq 3$, we consider a cocompact lattice Γ in $\mathrm{SL}(n,\mathbb{R})$ and show that every homomorphism $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ has finite image whenever

(1) M is a compact manifold of dimension at most $(n-2)$, or
(2) M is a compact manifold of dimension at most $(n-1)$ and α preserves a volume form vol.

The broad outline of the proof consists of 3 steps.

5.1. Step 1: subexponential growth. We present the main technical result from [12] for the case of cocompact lattices in $\mathrm{SL}(n,\mathbb{R})$. In [13,14] we establish an analogous result for actions by nonuniform lattices.

It is a classical fact that all lattices Γ in semisimple Lie groups are finitely generated. Fix a finite symmetric generating set S for Γ. Given $\gamma \in \Gamma$, let $|\gamma| = |\gamma|_S$ denote the **word-length** of γ relative to this generating set; that is,

$$|\gamma| = \min\{k : \gamma = s_k \cdots s_1, s_i \in S\}.$$

Note that if we replace the finite generating set S by another finite generating set S', there is a uniform constant C such that the word-lengths are uniformly distorted:

$$|\gamma|_{S'} \leq C|\gamma|_S.$$

Thus all definitions below will be independent of the choice of S.

Equip TM with a Riemannian metric and corresponding norm.

Definition 5.1. We say that an action $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$ has **uniform subexponential growth of derivatives** if for every $\varepsilon > 0$ there is a $C = C_\varepsilon$ such that for every $\gamma \in \Gamma$,

$$\sup_{x\in M} \|D_x\alpha(\gamma)\| \leq Ce^{\varepsilon|\gamma|}.$$

The following is the main result of [12], formulated here for the case of cocompact lattices in $\mathrm{SL}(n,\mathbb{R})$.

THEOREM 5.2 ([12, Theorem 2.8]). *For $n \geq 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a cocompact lattice. Let $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ be an action. Suppose that either*

(1) $\dim(M) \leq n-2$, *or*
(2) $\dim(M) = n-1$ *and* α *preserves a smooth volume.*

Then α *has uniform subexponential growth of derivatives.*

Remark 5.3. The proof of Theorem 5.2 is the only place in the proof of Theorem 3.4 where cocompactness of Γ is used. It is not required for Steps 2 or 3 below. For $\Gamma = \mathrm{SL}(m, \mathbb{Z})$, the analogue of Theorem 5.2 is announced in [13] and has been announced for more general nonuniform lattices in [14].

5.2. Step 2: strong property (T) and averaging Riemannian metrics. Assume $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ is an action by C^∞ diffeomorphisms.[4] The action α of Γ on M induces an action $\alpha_\#$ of Γ on tensor powers of the cotangent bundle of M by pull-back: Given $\omega \in (T^*M)^{\otimes k}$ write

$$\alpha_\#(\gamma)\omega = \alpha(\gamma^{-1})^*\omega;$$

that is, if $v_1, \dots v_k \in T_xM$ then

$$\alpha_\#(\gamma)\omega(x)(v_1, \dots, v_k) = \omega(x)(D_x\alpha(\gamma^{-1})v_1, \dots, D_x\alpha(\gamma^{-1})v_k).$$

In particular, we obtain an action of Γ on the set of Riemannian metrics which naturally sits as a half-cone inside $S^2(T^*M)$, the vector space of symmetric 2-forms on M. Note that $\alpha_\#$ preserves $C^\ell(S^2(T^*(M)))$, the subspace of all C^ℓ sections of $S^2(T^*M)$ for any $\ell \in \mathbb{N}$.

Fix a volume form vol on M. The norm on TM induced by the background Riemannian metric induces a norm on each fiber of $S^2(T^*M)$. We then obtain a natural notion of measurable and integrable sections of $S^2(T^*M)$ with respect to vol. Let $\mathcal{H}^k = W^{2,k}(S^2(T^*M))$ be the Sobolev space of symmetric 2-forms whose weak derivatives of order ℓ are bounded with respect to the $L^2(\mathrm{vol})$-norm for $0 \leq \ell \leq k$. Then $\mathcal{H}^k$ is a Hilbert space. Let $\|\cdot\|_{\mathcal{H}^k}$ denote the corresponding Sobolev norm on $\mathcal{H}^k$ as well as the induced operator norm on the space $B(\mathcal{H}^k)$ of bounded operators on $\mathcal{H}^k$. Working in local coordinates, the Sobolev embedding theorem implies that

$$\mathcal{H}^k \subset C^\ell(S^2(T^*(M)))$$

as long as

$$\ell < k - \dim(M)/2.$$

In particular, for k sufficiently large, an element ω of $\mathcal{H}^k$ is a C^ℓ section of $S^2(T^*M)$ which will be a C^ℓ Riemannian metric on M if it is positive definite.

The action $\alpha_\#$ is a representation of Γ by bounded operators on $\mathcal{H}^k$. From Theorem 5.2, we obtain strong control on the norm growth of the induced representation $\alpha_\#$. In particular, we obtain that the representation $\alpha_\#\colon \Gamma \to B(\mathcal{H}^k)$ has **subexponential norm growth:**

[4]For $C^{1+\beta}$ actions, one replaces the Hilbert Sobolev spaces $W^{2,k}(S^2(T^*M)))$ below with appropriate Banach Sobolev spaces $W^{p,1}(S^2(T^*M)))$ and verifies such spaces are of the type $\mathcal{E}_{10}$ considered in [26].

Lemma 5.4. *Let* $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ *have uniform subexponential growth of derivatives. Then, for every* $k \in \mathbb{N}$ *and every* $\varepsilon' > 0$ *there is* $C > 0$ *such that*

$$\|\alpha_\#(\gamma)\|_{\mathcal{H}^k} \le Ce^{\varepsilon'|\gamma|}$$

for all $\gamma \in \Gamma$.

The proof of Lemma 5.4 follows from the chain rule, Leibniz rule, and computations that bound the growth of higher-order derivatives by polynomial functions of the growth of the first derivative. See [36, Lemma 6.4] and discussion in [12, Section 6.3].

We use the main result from [26,63]: cocompact lattices Γ in higher-rank simple Lie groups (such as $\mathrm{SL}(n,\mathbb{R})$ for $n \ge 3$) satisfy Lafforgue's **strong Banach property (T)** first introduced in [63]. The result for $\mathrm{SL}(n,\mathbb{R})$ and its cocompact lattices (as well as most other higher-rank simple Lie groups) is established by Lafforgue in Corollary 4.1 and Proposition 4.3 of [63]; for cocompact lattices in certain other higher-rank Lie groups, the results of [26] are needed. See also [25] for the case of nonuniform lattices. Strong Banach property (T) considers representations π of Γ by bounded operators on certain Banach spaces E (of type $\mathcal{E}_{10}$). If such representations have sufficiently slow exponential norm growth, then there exists a sequence of operators p_n converging to a projection p_∞ such that for any vector $v \in E$, the limit $p_\infty(v)$ is π-invariant. In the case that E is a Hilbert space (which we may assume when α is an action by C^∞ diffeomorphisms), we have the following formulation. Note that Lemma 5.4 (which follows from Theorem 5.2) ensures our representation $\alpha_\#$ satisfies the hypotheses of the theorem.

Theorem 5.5 ([25,26,63]). *Let* $\mathcal{H}$ *be a Hilbert space and for* $n \ge 3$, *let* Γ *be a lattice in* $\mathrm{SL}(n,\mathbb{R})$.

There exists $\varepsilon > 0$ *such, that for any representation* $\pi\colon \Gamma \to B(\mathcal{H})$, *if there exists* $C_\varepsilon > 0$ *such that*

$$\|\pi(\gamma)\| \le C_\varepsilon e^{\varepsilon|\gamma|}$$

for all $\gamma \in \Gamma$, *then there exists a sequence of operators* $p_n = \sum w_i \pi(\gamma_i)$ *in* $B(\mathcal{H})$*—where* $w_i \ge 0$, $\sum w_i = 1$, *and* $w_i = 0$ *for every* $\gamma_i \in \Gamma$ *of word-length larger than* n*—such that for any vector* $v \in \mathcal{H}$, *the sequence* $v_n = p_n(v) \in \mathcal{H}$ *converges to an invariant vector* $v^* = p_\infty(v)$.

Moreover, the convergence is exponentially fast: there exist $0 < \lambda < 1$ *and* $C = C_\lambda$ *such that* $\|v_n - v^*\| \le C\lambda^n\|v\|$.

We remark that while we only use convergence of p_n to p_∞ in the strong operator topology, the convergence in Theorem 5.5 actually holds in the norm topology.

Theorem 5.5 as stated in [63] (and its generalization in [26]) requires that Γ be cocompact. The extension to nonuniform lattices is announced in [25]. The exponential convergence in Theorem 5.5 is often not explicitly stated in the definition of strong property (T) or in statements of theorems establishing that the property holds for lattices in higher-rank simple Lie groups; however, the exponential convergence follows from the proofs.

We complete Step 2 with the following computation.

Proposition 5.6. *For $n \geq 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a lattice and let $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ be an action with uniform subexponential growth of derivatives. Then for any ℓ, there is a C^ℓ Riemannian metric g on M invariant under the action of α.*

We note there is no assumption on the dimension of M in the proposition.

Proof. Consider an arbitrary C^∞ Riemannian metric g. For any k, we have $g \in \mathcal{H}^k$. We apply Theorem 5.5 and its notation to the representation $\alpha_\#\colon \Gamma \to B(\mathcal{H}^k)$ with g the initial vector v. We have that $g_n := p_n(g)$ is positive definite and C^∞ for every n. In particular, the limit $g_\infty = p_\infty(g)$ is in the closed cone of positive (possibly indefinite) symmetric 2-tensors in $\mathcal{H}^k$. Having taken k sufficiently large we have that g_∞ is C^ℓ; in particular, g_∞ is continuous, everywhere defined, and positive everywhere. We need only confirm that g_∞ is non-degenerate, i.e. is positive definite on T_xM for every $x \in M$.

Given any $x \in M$ and unit vector $\xi \in T_xM$, for any $\varepsilon > 0$ we have from Definition 5.1 that there is a $C_\varepsilon > 0$ such that

$$\begin{aligned} p_n(g)(\xi,\xi) &= \left(\sum w_i \alpha_\#(\gamma_i) g\right)(\xi,\xi) \\ &= \sum w_i g(D_x\alpha(\gamma_i^{-1})\xi, D_x\alpha(\gamma_i^{-1})\xi) \\ &\geq \frac{1}{C_\varepsilon^2} e^{-2\varepsilon n} \end{aligned}$$

where we use that $w_i > 0$ only when γ_i has word-length at most n.

On the other hand, from the exponential convergence in Theorem 5.5, we have

$$|p_n(g)(\xi,\xi) - p_\infty(g)(\xi,\xi)| \leq C_\lambda \lambda^n.$$

Thus

$$p_\infty(g)(\xi,\xi) \geq \frac{1}{C_\varepsilon^2} e^{-2\varepsilon n} - C_\lambda \lambda^n$$

for all $n \geq 0$. Having taken $\varepsilon > 0$ sufficiently small we can ensure that $C_\varepsilon^2 e^{2\varepsilon n} < \frac{1}{C_\lambda}\lambda^{-n}$ for all sufficiently large n and thus $p_\infty(g)(\xi,\xi) > 0$. ☐

5.3. Step 3: Margulis superrigidity with compact codomain. From Steps 1 and 2 we have that any action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ as in Theorem 3.4 preserves a C^ℓ Riemannian metric g. In the general case of C^2-actions, we have that any action $\alpha\colon \Gamma \to \mathrm{Diff}^2(M)$ preserves a continuous Riemannian metric g. See [12, Theorem 2.7]. We thus have

$$\alpha\colon \Gamma \to \mathrm{Isom}^2_g(M) \subset \mathrm{Diff}^2(M).$$

Let $\dim(M) = m$. The group $\mathrm{Isom}_g(M)$ of isometries of a continuous Riemannian metric is a compact Lie group with

$$\dim(\mathrm{Isom}_g(M)) \leq \frac{m(m+1)}{2}. \tag{5.1}$$

Indeed, the orbit of any point $p \in M$ under $\mathrm{Isom}_g(M)$ has dimension at most m and the dimension of the stabilizer of a point is at most $\frac{m(m-1)}{2}$, the dimension of $\mathrm{SO}(m)$; thus

$$\dim(\mathrm{Isom}_g(M)) \leq m + \frac{m(m-1)}{2}.$$

With $K = \mathrm{Isom}_g(M) \subset \mathrm{Diff}^2(M)$ we thus obtain a compact-valued representation $\alpha\colon \Gamma \to K$. By equation (5.1), if $m < \frac{1}{2}\sqrt{8n^2-7} - \frac{1}{2}$ then $\dim(\mathfrak{su}(n)) = n^2 - 1 > \dim(K)$; by conclusion (2) of Theorem 1.4, $\alpha(\Gamma)$ is thus contained in a 0-dimensional subgroup of K. This holds in particular if $m \leq n-1$. We thus conclude that the image $\alpha(\Gamma) \subset K$ is finite.

6. Suspension space, associated linear functionals, and rigidity

We introduce a technical construction that is common to the proofs of both Theorem 5.2 (and its analogue needed to establish the results in Section 3.5) and Theorem 4.2. This construction takes an action $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ and induces a related G-action on an auxiliary space which we denote by M^α. The auxiliary space M^α has the structure of a fiber bundle over G/Γ.

Properties of the G-action on M^α mimic properties of the Γ-action on M. However, it turns out to be much more powerful to study the G-action on M^α as dynamical properties of the homogeneous G-action on G/Γ can influence related dynamical properties of the G-action on M^α. In particular, dynamical quantities and properties of the G-action on M^α are related to and constrained by algebraic data associated to the G-action on G/Γ.

6.1. Suspension space and induced G-action. Fix G a Lie group and let $\Gamma \subset G$ be a lattice. We need not assume Γ is cocompact. Let M be a compact manifold and let $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ be an action.

On the product $G \times M$ consider the right Γ-action

$$(g, x) \cdot \gamma = (g\gamma, \alpha(\gamma^{-1})(x))$$

and the left G-action

$$a \cdot (g, x) = (ag, x).$$

Define the quotient manifold

$$M^\alpha := (G \times M)/\Gamma.$$

As the G-action on $G \times M$ commutes with the Γ-action, we have an induced left G-action on M^α. When $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$, M^α has a natural C^r structure and the G-action is C^r. For $g \in G$ and $x \in M^\alpha$ we denote this action by $g \cdot x$; if α is C^1, we denote by

$$D_x g \colon T_x M^\alpha \to T_{g\cdot x} M^\alpha$$

the derivative of the diffeomorphism $x \mapsto g \cdot x$ at $x \in M^\alpha$.

We write

$$\pi \colon M^\alpha \to G/\Gamma$$

for the natural projection map. Note that M^α has the structure of a fiber-bundle over G/Γ induced by the map π with fibers diffeomorphic to M. The G-action permutes the M-fibers of M^α.

Equip M^α with a continuous Riemannian metric. For convenience, we may moreover assume the restriction of the metric to G-orbits coincides under push-forward by the projection $\pi \colon M^\alpha \to G/\Gamma$ with the metric on G/Γ induced by a right-invariant (and left K-invariant) metric on G. (We note that if Γ is cocompact, M^α is compact and all metrics are equivalent. In the case that Γ is nonuniform, additional care is needed to ensure the metric is well behaved in the fibers; we will not discuss these technicalities here.)

In the case that $\alpha \colon \Gamma \to \mathrm{Homeo}(M)$, M^α is a topological manifold and the G-action is continuous.

6.2. Dynamics of A on G/Γ. For simplicity, we restrict our discussion to the case $G = \mathrm{SL}(n, \mathbb{R})$. Recall that A denotes the group of positive diagonal matrices (1.2). We have $A \simeq \mathbb{R}^{n-1}$.

Consider the linear functionals

$$\beta^{i,j} \colon A \to \mathbb{R}$$

given as follows: for $i \neq j$,

$$\beta^{i,j}\left(\mathrm{diag}(e^{t_1}, e^{t_2}, \dots, e^{t_n})\right) = t_i - t_j.$$

The linear functionals $\beta^{i,j}$ are the **roots** of G.

In $\mathrm{SL}(n, \mathbb{R})$, associated to each root $\beta^{i,j}$ is a 1-parameter unipotent subgroup $U^{i,j} \subset G$ consisting of matrices whose diagonal entries are 1, (i,j)th entry is an arbitrary real number t, and every other entry is zero. For instance, in $G = \mathrm{SL}(3, \mathbb{R})$ we have the following 1-parameter flows

$$u^{1,2}(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad u^{1,3}(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad u^{2,3}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix},$$

and $U^{i,j}$ is the associated 1-parameter unipotent subgroups of G:

$$U^{i,j} := \{u^{i,j}(t) : t \in \mathbb{R}\}. \tag{6.1}$$

The groups $U^{i,j}$ have the property that conjugation by $a \in A$ dilates their parametrization by $e^{\beta^{i,j}(a)}$:

$$a u^{i,j}(t) a^{-1} = u^{i,j}(e^{\beta^{i,j}(a)} t). \tag{6.2}$$

Consider the left action of A on G/Γ. Given $x = g\Gamma \in G/\Gamma$, consider y in the $U^{i,j}$-orbit of x. Using the parametrization $u^{i,j}$ of $U^{i,j}$, we may write $y = u^{i,j}(t)x$. Then for $a \in A$,

$$ay = u^{i,j}(e^{\beta^{i,j}(a)}t)ax.$$

In particular, the action by $a \in A$ dilates the parametrization of $U^{i,j}$-orbits by exactly $e^{\beta^{i,j}(a)}$.

6.3. Weights of a representation and cohomological rigidity. In the course of proving Theorem 4.2, in Section 7.2 we introduce a twisted dynamical 1-cocycle over the G-action on M^α which is twisted by a representation $\rho\colon G \to \mathrm{GL}(d, \mathbb{R})$. The image $\rho(A)$ of A under this representation consists of commuting matrices that are diagonalizable over $\mathbb{R}$ and hence are jointly diagonalizable. Moreover, the eigenvalues are all positive. Let $\{e_j\}$ be a basis of $\mathbb{R}^d$ relative to which $\rho(A)$ is a group of positive diagonal matrices.

Given $a \in A$, let $\lambda_j(a)$ denote the eigenvalues of $\rho(a)$ associated to the eigenvector e_j. We have that $\lambda_j\colon A \to \mathbb{R}_+$ is multiplicative: for $a, b \in A$

$$\lambda_j(ab) = \lambda_j(a)\lambda_j(b).$$

Let $\chi_j\colon A \to \mathbb{R}$ be the linear functional

$$\chi_j(a) = \log \lambda_j(a).$$

The linear functionals $\chi_j\colon A \to \mathbb{R}$ are the **weights** of the representation ρ. For any semisimple Lie group G, the roots $\{\beta\}$ of G are the non-zero weights of the adjoint representation $G \to \mathrm{GL}(\mathfrak{g})$. See Examples 7.5 and 7.6 below for explicit examples.

In Section 7.2 below, we construct ρ-twisted 1-cocycle $c\colon G \times M^\alpha \to \mathbb{R}^d$. Proving Theorem 4.2 reduces to showing the cocycle is a ρ-twisted coboundary: there is a function $\eta\colon M^\alpha \to \mathbb{R}^d$ such that for all $g \in G$,

$$c(g, x) = \rho(g)\eta(x) - \eta(g \cdot x). \tag{6.3}$$

(For the time being, we are intentionally vague about the regularity of c and η.)

Following the proof of of Theorem 1.6, it is relatively straightforward to build a function $\eta\colon M^\alpha \to \mathbb{R}^d$ such that (6.3) holds for all $g \in A$. To show that (6.3) holds for *all* $g \in G$, we work along each unipotent root subgroup U^β and observe the following: if $\beta\colon A \to \mathbb{R}$ is a root such that no weight $\chi\colon A \to \mathbb{R}$ is proportional to β, then (6.3) holds for all g in the associated root subgroup U^β. See Sections 7.3.3 and 7.3.4 for details. See especially Proposition 7.7 which guarantees in the setting of Theorem 4.2 that there are sufficiently many roots β that are not proportional to any weight χ of ρ so that (6.3) holds for all $g \in G$.

6.4. Fiberwise Lyapunov exponents and measurable rigidity. We now consider the case that M^α is the suspension of an action $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$ that is at least C^1. We again restrict the G-action on M^α to an A-action on M^α. Suppose μ is an A-invariant Borel probability measure on M^α. Here, invariance means that

$$\mu(B) = \mu(a \cdot B)$$

for all $a \in A$ and Borel sets $B \subset M^\alpha$. We also assume that μ is **ergodic**: every A-invariant Borel set is null or conull.

Associated to each root β of G is a unipotent root subgroup $U^\beta \subset G$ whose Lie algebra is $\mathfrak{g}^\beta$ (or $\mathfrak{g}^\beta \oplus \mathfrak{g}^{2\beta}$ if 2β is also a root). The goal of this section is to formulate sufficient conditions under which an ergodic, A-invariant Borel probability measure μ on M^α is invariant under the subgroup $U^\beta \subset G$ for a root β of G.

Write $\pi_*\mu$ for the image of μ under the projection $M^\alpha \to G/\Gamma$:

$$\pi_*\mu(B) = \mu(\pi^{-1}(B)).$$

Then $\pi_*\mu$ is an A-invariant Borel probability measure on G/Γ. Note that the (normalized) Haar measure on G/Γ is an A-invariant Borel probability measure on G/Γ; however, there may exist other A-invariant Borel probability measures on G/Γ. Below we will always assume that the measure $\pi_*\mu$ has exponentially small mass in the cusps of G/Γ; that is, if $B(x_0, R)$ denotes a metric ball in G/Γ centered at x_0 of radius R then there is some $C > 0$ and $\kappa > 0$ such that

$$(\pi_*\mu)(G/\Gamma \smallsetminus B(x_0, R)) \le Ce^{-\kappa R}.$$

This always holds whenever Γ is cocompact (so that M^α is compact) or whenever $\pi_*\mu$ is the Haar measure on G/Γ.

6.4.1. *Fiberwise Lyapunov exponents.* Recall that $\pi\colon M^\alpha \to G/\Gamma$ denotes the canonical projection. We let $F = \ker(D\pi)$ be the **fiberwise tangent bundle:** for $x \in M^\alpha$, $F(x) \subset T_xM^\alpha$ is the $\dim(M)$-dimensional subspace tangent to the fiber through x. The G-action (and hence the A-action) on M^α permutes the fibers of the fiber-bundle M^α. In particular, the derivatives of the G- and A-actions preserve the fiberwise tangent subbundle $F \subset TM^\alpha$.

Equip $A \simeq \mathbb{R}^{\mathrm{rank}(G)}$ with a norm $|\cdot|$. We may apply the higher-rank Oseledec's theorem (see [15, 78]) to the fiberwise-derivative cocycle

$$(g, x) \mapsto D_xg\restriction_F$$

to obtain the following: There are

(1) a measurable set $\Lambda \subset M^\alpha$ with $\mu(\Lambda) = 1$;
(2) linear functionals $\lambda_1^F, \lambda_2^F, \dots, \lambda_p^F\colon A \to \mathbb{R}$;
(3) a μ-measurable, A-invariant splitting $F(x) = \bigoplus_{i=1}^p E^i(x)$ of the subbundle $F \subset TM$ defined for $x \in \Lambda$

such that for every $x \in \Lambda$ and every $v \in E^i(x) \smallsetminus \{0\}$

$$\lim_{|a|\to\infty} \frac{\log \|D_x a(v)\| - \lambda_i^F(a)}{|a|} = 0. \tag{6.4}$$

Note that 6.4 implies convergence along rays: for $a \in A$ and $v \in E^i(x) \smallsetminus \{0\}$

$$\lim_{k\to\infty} \frac{1}{k} \log \|D_x a^k(v)\| = \lambda_i^F(a). \tag{6.5}$$

The convergence in 6.4 is taken along *any* sequence $n \to \infty$; this is stronger than (6.5) and is typically needed in applications.

The linear functionals $\lambda_i^F \colon A \to \mathbb{R}$ are called the **fiberwise Lyapunov exponent functionals** or simply **fiberwise Lyapunov exponents** for the A-action on (M^α, μ).

6.4.2. *Resonant and nonresonant roots.* We consider the possibility of positive proportionality between the roots $\beta \colon A \to \mathbb{R}$ of G with the fiberwise Lyapunov exponents $\lambda_j^F \colon A \to \mathbb{R}$ of the action of A on (M^α, μ).

Definition 6.1. Let μ be an ergodic, A-invariant Borel probability measure on M^α as above.

(1) We say a root β of G is **resonant with** μ if there is a fiberwise Lyapunov exponent λ_i^F that is positively proportional to β: there is $c > 0$ such that for all $a \in A$

$$\beta(a) = c\lambda_i^F(a).$$

(2) If β is not resonant with μ, we say β is **nonresonant**.

6.4.3. *Measurable rigidity: invariance principle.* We have the following criteria under which an A-invariant measure is automatically U^β-invariant. Note this holds for C^1-actions.

Proposition 6.2. *Let G be a higher-rank simple Lie group. Let $\Gamma \subset G$ be a lattice, and let M be a compact manifold. Let $\alpha \colon \Gamma \to \mathrm{Diff}^1(M)$ and let μ be an ergodic, A-invariant Borel probability measure on M^α such that $\pi_*\mu$ is the normalized Haar measure on G/Γ.*

Given a root β, if there exists $a_0 \in A$ such that

(1) $\beta(a_0) > 0$, and
(2) $\lambda_i^F(a_0) = 0$ for every fiberwise Lyapunov exponent $\lambda_i^F \colon A \to \mathbb{R}$,

then μ is U^β-invariant.

Proposition 6.2 follows directly from the invariance principle of Avila and Viana [2]; this result generalizes (to non-linear cocycles) an earlier result of Ledrappier for projective cocycles [64].

When G is a connected simple Lie group with Lie algebra $\mathfrak{g}$, we obtain the following as an immediate corollary of Proposition 6.2.

Corollary 6.3. *Let G be a connected, higher-rank simple Lie group. Let $\Gamma \subset G$ be a lattice, and let M be a compact manifold. Let $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$ be an action and let μ be an ergodic, A-invariant Borel probability measure on M^α such that $\pi_*\mu$ is the normalized Haar measure on G/Γ. Then if either*

(1) $\dim(M) < \mathrm{rank}(G)$, *or*
(2) $\dim(M) \leq \mathrm{rank}(G)$ *and α preserves a volume form* vol

then μ is G-invariant.

Proof sketch. In case (1) where $\dim(M) < \mathrm{rank}(G)$, we have $\bigcap \ker \lambda_i^F$ is a vector subspace of dimension at least 1. If α preserves a volume form vol, then there is a linear relation

$$\sum \lambda_i^F = 0$$

on the exponents and hence in case (2), we still have that $\bigcap \ker \lambda_i^F$ has dimension at least 1. Then, there exists an $a_0 \in A$ such that

(a) $a_0 \neq e$, and
(b) $\lambda_i^F(a_0) = 0$ for every fiberwise Lyapunov exponent λ_i^F.

Let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{g}$ generated by all $\mathfrak{g}^\beta$ such that $\beta(a_0) \neq 0$. Then μ is invariant under the connected Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$. However, conditions (a) and (b) imply that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$; as $a_0 \neq e$, we have $\mathfrak{h} \neq \{0\}$ and as $\mathfrak{g}$ is simple, it follows that $\mathfrak{h} = \mathfrak{g}$ and thus $H = G$. $\square$

6.4.4. *Measurable rigidity: nonresonant invariance principle.* We have the following proposition whose conclusion is stronger than the conclusion of Proposition 6.2. This is the main technical proposition from [16]. Note, however, that the following proposition requires the action to be least $C^{1+\beta}$ ($\beta > 0$).

Proposition 6.4 ([16, Proposition 5.1])**.** *Let G be a higher-rank simple Lie group. Let $\Gamma \subset G$ be a lattice, and let M be a compact manifold. Let $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ be an action and let μ be an ergodic, A-invariant Borel probability measure on the space M^α such that $\pi_*\mu$ is the normalized Haar measure on G/Γ.*

Then, given a nonresonant root β of G, the measure μ is U^β-invariant.

The proof of Proposition 6.4 is found in [16] and uses the "entropy product structure" of measures μ invariant under higher-rank abelian groups formulated in [17].

When G is a connected simple Lie group with Lie algebra $\mathfrak{g}$, we obtain the following as an immediate corollary of Proposition 6.2. Recall (see Section 3.2) we define a number $r(G)$ to be $d_0(G')$ where G' is a maximal $\mathbb{R}$-split simple subgroup of G.

Corollary 6.5. *Let G be a connected, higher-rank simple Lie group. Let $\Gamma \subset G$ be a lattice, and let M be a compact manifold. Let $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$*

be an action and let μ be an ergodic, A-invariant Borel probability measure on M^α such that $\pi_*\mu$ is the normalized Haar measure on G/Γ. Then if either

(1) $\dim(M) < r(G)$, *or*

(2) $\dim(M) \leq r(G)$ *and α preserves a smooth volume form* vol *on M*

then μ is G-invariant.

Proof sketch. Let $H \subset G$ denote the subgroup under which μ is invariant. Then $A \subset H$ and $U^\beta \subset H$ for every nonresonant β. By dimension counting, under either hypothesis the Lie algebra $\mathfrak{h}$ of H contains all-but-at-most-$(r(G)-1)$ root spaces. From the definition of $r(g)$, the only such subalgebra is $\mathfrak{h} = \mathfrak{g}$ whence $H = G$. $\square$

7. Discussion of the proof of Theorem 4.2

Let G be a higher-rank simple Lie group with finite center. We may take $G = \mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$. Let $\Gamma \subset G$ be a lattice. Let $\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$ be an action by homeomorphisms and let $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{Z})$ denote its linear data (see Section 4.1). For simplicity we assume the following:

(1) $\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$ lifts to an action $\tilde\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{R}^d)$;

(2) $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{Z})$ extends to a representation $\rho\colon G \to \mathrm{GL}(d,\mathbb{R})$.

By Margulis's superrigidity theorem, Theorem 1.3, the linear data $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{Z})$ always satisfies a condition close to (2). We assume (2) to simplify the following exposition though it is not strictly required.

Recall that we also assume that

(3) $\rho(\gamma_0)$ is hyperbolic for some $\gamma_0 \in \Gamma$.

It follows (see for instance [18, Lemma 7.1]) that $\rho(g_0)$ is hyperbolic for some $g_0 \in A$.

Under the above assumptions, we claim as in Theorem 1.3 that there exists a continuous surjection $h\colon \mathbb{T}^d \to \mathbb{T}^d$ such that

$$h \circ \alpha(\gamma) = \rho(\gamma) \circ h \tag{7.1}$$

for all $\gamma \in \Gamma$.

7.1. Reformulation in suspension spaces.

Let

$$M^\alpha = (G \times \mathbb{T}^d)/\Gamma, \qquad M^\rho = (G \times \mathbb{T}^d)/\Gamma$$

denote the suspension spaces whose construction was given in Section 6. Note the right Γ quotient is determined by the corresponding action on $\mathbb{T}^d$. A priori, M^α and M^ρ need not be homeomorphic.

We also define covering spaces $\tilde M^\alpha$ and $\tilde M^\rho$ of M^α and M^ρ as follows: let $\tilde M^\alpha = (G \times \mathbb{R}^d)/\Gamma$ where Γ acts as

$$(g,x)\cdot\gamma = (g\gamma, \tilde\alpha(\gamma)^{-1}(x))$$

and similarly let $\tilde M^\rho = (G \times \mathbb{R}^d)/\Gamma$ where Γ acts as

$$(g,x)\cdot\gamma = (g\gamma, \rho(\gamma)^{-1}x).$$

Instead of constructing a continuous, surjective $h\colon \mathbb{T}^d \to \mathbb{T}^d$ satisfying (7.1), it is sufficient (and equivalent) to construct a continuous, surjective map $H\colon M^\alpha \to M^\rho$ such that the following hold:

(1) H intertwines the G-actions on M^α and M^ρ: for all $g \in G$ and $y \in M^\alpha$
$$H(g \cdot y) = g \cdot H(y).$$

(2) H factors over the identity: if $\pi^\alpha\colon M^\alpha \to G/\Gamma$ and $\pi^\rho\colon M^\rho \to G/\Gamma$ are the canonical fiber-bundle projections then
$$\pi^\rho \circ H = \pi^\alpha.$$

In criterion (1), the G-action on the left-hand side is the G-action on $\tilde{M}^\alpha$ and the G-action on the right-hand side is the G-action on $\tilde{M}^\rho$.

7.1.1. *Measurable parametrization and reformulation.* Let $D \subset G$ be a Borel, Dirichlet fundamental domain for the right Γ-action on G. That is,

(1) D is a Borel subset of D;
(2) $\bigcup_{\gamma\in\Gamma} D \cdot \gamma = G$;
(3) $D \cdot \gamma \cap D \neq \varnothing$ if and only if $\gamma = e$;
(4) $D \subset \{x \in G \mid d(x, \Gamma) = d(x, e)\}$.

We use D to define measurable parametrizations of M^α and $\tilde{M}^\alpha$:

$$M^\alpha = D \times \mathbb{T}^d, \qquad \tilde{M}^\alpha = D \times \mathbb{R}^d \tag{7.2}$$

where we identify $(\overline{g}, x) \in D \times \mathbb{T}^d$ with $(\overline{g}, x)\Gamma \in M^\alpha$. We similarly measurably parametrize

$$M^\rho = D \times \mathbb{T}^d, \qquad \tilde{M}^\rho = D \times \mathbb{R}^d. \tag{7.3}$$

In the proof of Theorem 4.2, it is not necessary to require a priori that H be continuous. We actually produce a *measurable* function $H\colon M^\alpha \to M^\rho$ with the following properties:

(1) relative to the parametrizations (7.2) and (7.3), $H\colon D \times \mathbb{T}^d \to D \times \mathbb{T}^d$ has the form
$$H(\overline{g}, x) = (\overline{g}, h_{\overline{g}}(x));$$

(2) $h_{\overline{g}}\colon \mathbb{T}^d \to \mathbb{T}^d$ is defined, continuous, and homotopic to the identity for (Haar) almost every $\overline{g} \in D$;
(3) the assignment $\overline{g} \mapsto h_{\overline{g}}$ is measurable;
(4) H intertwines the G-actions on M^α and M^ρ: for all $g \in G$ and $y \in M^\alpha$
$$H(g \cdot y) = g \cdot H(y).$$

From degree arguments, we have that $h_{\overline{g}}$ is surjective for almost every $\overline{g}$. A priori, (especially in the case that Γ is nonuniform) the map $h_{\overline{g}}\colon \mathbb{T}^d \to \mathbb{T}^d$ is only defined for almost every $\overline{g} \in D$. However, since the G-actions on M^α and M^ρ are continuous and since G acts transitively on the base G/Γ, it follows a posteriori that $h_{\overline{g}}$ is defined for *every* $\overline{g} \in D$ and thus that $H\colon M^\alpha \to M^\rho$ is a continuous surjection.

7.1.2. *Measurable reformulation on $\tilde{M}^\alpha$ and $\tilde{M}^\rho$.* To actually prove Theorem 4.2, we work on the (measurable parametrizations of the) covering spaces $\tilde{M}^\alpha$ and $\tilde{M}^\rho$ and find a measurable function $\tilde{H}\colon \tilde{M}^\alpha \to \tilde{M}^\rho$ with the following properties:

(1) relative to the parametrizations (7.2) and (7.3), $\tilde{H}\colon D \times \mathbb{R}^d \to \mathbb{R}^d$ has the form
$$\tilde{H}(\overline{g}, x) = (\overline{g}, \tilde{h}_{\overline{g}}(x));$$
(2) $\tilde{h}_{\overline{g}}\colon \mathbb{R}^d \to \mathbb{R}^d$ is defined, $\mathbb{Z}^d$-equivariant, and continuous for almost every $\overline{g} \in D$;
(3) the assignment $\overline{g} \mapsto \tilde{h}_{\overline{g}}$ is measurable;
(4) for $\overline{g} \in D$ and $g \in G$, writing $g\overline{g} = g'\gamma$ for $g' \in D$ and $\gamma \in \Gamma$, we have

$$\rho(\gamma)\tilde{h}_{\overline{g}} = \tilde{h}_{g'}(\tilde{\alpha}(\gamma)(x)). \tag{7.4}$$

In (2), the $\mathbb{Z}^d$-equivariance asserts that
$$\tilde{h}_{\overline{g}}(x+n) = \tilde{h}_{\overline{g}}(x) + n$$
for all $n \in \mathbb{Z}^d$ so that $\tilde{h}_{\overline{g}}\colon \mathbb{R}^d \to \mathbb{R}^d$ descends to a map $h_{\overline{g}}\colon \mathbb{T}^d \to \mathbb{T}^d$ homotopic to the identity. Property (4) simply asserts that $\tilde{H}$ intertwines the G-actions on $\tilde{M}^\alpha$ and $\tilde{M}^\rho$: given $(\overline{g}, x) \in D \times \mathbb{R}^d$ and $g \in G$, writing $g\overline{g} = g'\gamma$ with $g' \in D$, we have
$$\begin{aligned}
\tilde{H}\left(g \cdot (\overline{g}, x)\right) &= \tilde{H}\left(g\overline{g}, x\right) = \tilde{H}\left(g'\gamma, x\right) \\
&= \tilde{H}\left(g', \tilde{\alpha}(\gamma)(x)\right) \\
&= \left(g', \tilde{h}_{g'}(\tilde{\alpha}(\gamma)(x))\right) \\
&= \left(g', \rho(\gamma)\tilde{h}_{\overline{g}}(x)\right) \\
&= \left(g'\gamma, \tilde{h}_{\overline{g}}(x)\right) = \left(g\overline{g}, \tilde{h}_{\overline{g}}(x)\right) \\
&= g \cdot \tilde{H}\left(\overline{g}, x\right)
\end{aligned}$$

7.2. Twisted cocycle reformulation. Note that the identity map $D \times \mathbb{R}^d \to D \times \mathbb{R}^d$ satisfies all the requirements of a map $\tilde{H}$ from Section 7.1.2 except for property (4). To construct a family of functions $\overline{g} \mapsto \tilde{h}_{\overline{g}}$ as in Section 7.1.2, we define a twisted 1-cocycle which measures the defect in the identity map satisfying (7.4). If this twisted 1-cocycle is a twisted coboundary, we are then able to modify the identity map to produce the desired family of maps $\{\tilde{h}_{\overline{g}}, \overline{g} \in D\}$ satisfying the criteria of Section 7.1.2.

7.2.1. *The twisted cocycle.* Given $g \in G$, $\overline{g} \in D$, and $x \in \mathbb{R}^d$, write $g\overline{g} = g'\gamma$ for $g' \in D$ and $\gamma \in \Gamma$. Define

$$c\left(g, (\overline{g}, x)\right) = \rho(g')\tilde{\alpha}(\gamma)(x) - \rho(g)\rho(\overline{g})(x). \tag{7.5}$$

For notational simplicity, given any $\overline{g} \in G$, write $\overline{g} = g''\gamma''$ where $g'' \in D$ and $\gamma'' \in D$ and set

$$c(g, (\overline{g}, x)) := c\left(g, (g'', \tilde{\alpha}(\gamma'')(x))\right).$$

In this way, we view c as a function $c\colon G \times \tilde{M}^\alpha \to \mathbb{R}^d$ where $c(g, y)$ is defined independently of the representative of $y = (\overline{g}, x)$ in $G \times \mathbb{R}^d$. In particular, given $g_0, g \in G, \overline{g} \in D$, and $x \in \mathbb{R}^d$ we will write equivalently

$$c(g_0, g \cdot (\overline{g}, x)) = c(g_0, (g\overline{g}, x)) = c(g_0, (g', \tilde{\alpha}(\gamma)x)) \tag{7.6}$$

where $g\overline{g} = g'\gamma$ for $g' \in D$ and $\gamma \in \Gamma$. Note that the action in the left-most term is the action in $\tilde{M}^\alpha$.

We have the following properties of the function $c\colon G \times D \times \mathbb{R}^d \to \mathbb{R}^d$.

Claim 7.1. *The function $c\colon G \times D \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following for every $g \in G$, $\overline{g} \in D$, and $x \in \mathbb{R}^d$:*

(1) the function $\mathbb{R}^d \to \mathbb{R}^d$ given by $x \mapsto c\left(g, (\overline{g}, x)\right)$ is continuous;
(2) $c\left(g, (\overline{g}, x + n)\right) = c\left(g, (\overline{g}, x)\right)$ for every $n \in \mathbb{Z}^d$;
(3) for every $g_1, g_2 \in G$ we have

$$c\left(g_1 g_2, (\overline{g}, x)\right) = \rho(g_1) c\left(g_2, (\overline{g}, x)\right) + c\left(g_1, g_2 \cdot (\overline{g}, x)\right) \tag{7.7}$$

Using the convention (7.6) and writing $g_2\overline{g} = g''\gamma''$ for $g'' \in D$ and $\gamma'' \in \Gamma$, we may rewrite the second term on the right-hand side of equation (7.7) as

$$c\left(g_1, g_2 \cdot (\overline{g}, x)\right) = c\left(g_1, (g_2\overline{g}, x)\right) = c\left(g_1, (g'', \tilde{\alpha}(\gamma'')(x))\right).$$

From Claim 7.1, the function $c\colon G \times D \times \mathbb{R}^d \to \mathbb{R}^d$ defines a **ρ-twisted 1-cocycle** over the G-action on $\tilde{M}^\alpha$ which descends to a ρ-twisted 1-cocycle over the G-action on M^α.

Claim 7.1(1) and (2) are clear. We give the computation for (3).

Proof of Claim 7.1(3). Write $g_2\overline{g} = g''\gamma''$ and $g_1 g_2 \overline{g} = g'\gamma'''$ where $g'', g' \in D$. Let $\gamma' = \gamma'''(\gamma'')^{-1}$. Then $g_1 g'' = g'\gamma'$. We have

$$\begin{aligned}
c\left(g_1 g_2, (\overline{g}, x)\right) &= \rho(g')\tilde{\alpha}(\gamma''')(x) - \rho(g_1 g_2)\rho(\overline{g})(x) \\
&= \rho(g')\tilde{\alpha}(\gamma')(\tilde{\alpha}(\gamma'')(x)) - \rho(g_1 g_2)\rho(\overline{g})x \\
&\qquad + \rho(g_1)\rho(g'')\tilde{\alpha}(\gamma'')(x) - \rho(g_1)\rho(g'')\tilde{\alpha}(\gamma'')(x) \\
&= \rho(g')\tilde{\alpha}(\gamma')(\tilde{\alpha}(\gamma'')(x)) - \rho(g_1)\rho(g'')\tilde{\alpha}(\gamma'')(x) \\
&\qquad + \rho(g_1)\left[\rho(g'')\tilde{\alpha}(\gamma'')(x) - \rho(g_2)\rho(\overline{g})x\right] \\
&= c(g_1, (g'', \tilde{\alpha}(\gamma'')(x))) + \rho(g_1)c(g_2, (\overline{g}, x)) \\
&= c(g_1, g_2 \cdot (\overline{g}, x)) + \rho(g_1)c(g_2, (\overline{g}, x)).
\end{aligned}$$

□

7.2.2. *The twisted cocycle equation.* Suppose there exists a measurable family of functions $\{\eta_{\overline{g}}\colon \mathbb{R}^d \to \mathbb{R}^d, \overline{g} \in D\}$ such that for almost every $\overline{g} \in D$

(1) $\eta_{\overline{g}}$ is continuous;
(2) $\eta_{\overline{g}}$ is $\mathbb{Z}^d$-invariant: $\eta_{\overline{g}}(x+n) = \eta_{\overline{g}}(x)$ for all $n \in \mathbb{Z}^d$;
(3) for every $g \in G$ and $x \in \mathbb{R}^d$, writing $g\overline{g} = g'\gamma$ with $g' \in D$, we have

$$c(g,(\overline{g},x)) = \rho(g)\eta_{\overline{g}}(x) - \eta_{g'}(\tilde{\alpha}(\gamma)(x)). \tag{7.8}$$

Let $\eta\colon \tilde{M}^\alpha \to \mathbb{R}^d$ be given by $\eta(\overline{g},x) = \eta_{\overline{g}}(x)$ via the parametrization $\tilde{M}^\alpha \simeq D \times \mathbb{R}^d$. Then for $g \in G$ and $(\overline{g},x) \in D \times \mathbb{R}^d \simeq M^\alpha$, writing $g\overline{g} = g'\gamma$ for $g' \in D$ we have

$$\eta(g \cdot (\overline{g},x)) = \eta_{g'}(\tilde{\alpha}(\gamma)(x)),$$

whence we have

$$c(g,(\overline{g},x)) = \rho(g)\eta(\overline{g},x) - \eta(g \cdot (\overline{g},x)). \tag{7.9}$$

Equation (7.9) is a **twisted cocycle equation**. A function η solving (7.9) (or family $\{\eta_{\overline{g}}, \overline{g} \in D\}$ solving (7.8)) is said to be a solution to the cocycle equation. The existence of such a function η implies that the ρ-twisted cocycle $c\colon G \times \tilde{M}^\alpha \to \mathbb{R}^d$ is a **ρ-twisted coboundary**.

7.2.3. *Constructing a semiconjugacy from a solution to the cocycle equation.* From measurable family of functions $\{\eta_{\overline{g}}\}$ as in Section 7.2.2 solving (7.8), we construct a family of maps $\tilde{h}_{\overline{g}}$ as in Section 7.1.2. For $\overline{g} \in D$ and $x \in \mathbb{R}^d$, define

$$\tilde{h}_{\overline{g}}(x) = x + \rho(\overline{g})^{-1}\eta_{\overline{g}}(x).$$

We clearly have that

(1) $\tilde{h}_{\overline{g}}\colon \mathbb{R}^d \to \mathbb{R}^d$ is defined, $\mathbb{Z}^d$-equivariant, and continuous for almost every $\overline{g} \in D$;
(2) the map $\overline{g} \mapsto \tilde{h}_{\overline{g}}$ is measurable.

Moreover, given $\overline{g} \in D$ and $g \in G$, writing $g\overline{g} = g'\gamma$ for $g' \in D$ and $\gamma \in \Gamma$, we check

$$\begin{aligned}
\rho(\gamma)\tilde{h}_{\overline{g}}(x) &= \rho(\gamma)x + \rho(\gamma)\rho(\overline{g})^{-1}\eta_{\overline{g}}(x) \\
&= \rho(\gamma)x + \rho(g')^{-1}\rho(g)\eta_{\overline{g}}(x) \\
&= \rho(\gamma)x + \rho(g')^{-1}\left[c(g,(\overline{g},x)) + \eta_{g'}(\tilde{\alpha}(\gamma)(x))\right] \\
&= \rho(\gamma)x + \rho(g')^{-1}\left[\rho(g')\tilde{\alpha}(\gamma)(x) - \rho(g)\rho(\overline{g})(x) + \eta_{g'}(\tilde{\alpha}(\gamma)(x))\right] \\
&= \tilde{\alpha}(\gamma)(x) + \rho(g')^{-1}\eta_{g'}(\tilde{\alpha}(\gamma)(x)) \\
&= \tilde{h}_{g'}(\tilde{\alpha}(\gamma)(x)).
\end{aligned}$$

It follows that the family $\overline{g} \mapsto \tilde{h}_{\overline{g}}$ satisfies the properties of Section 7.1.2.

7.3. Solving the cocycle equation. It remains to show there exists a measurable family $\{\eta_{\overline{g}} : \overline{g} \in D\}$ of functions satisfying the conditions of Section 7.2.2. In particular, we need to solve (7.8), the twisted cocycle equation.

We introduce some notation to simplify certain expressions below. Let $\{\eta_{\overline{g}} : \overline{g} \in D\}$ be a measurable family of functions $\eta_{\overline{g}} \colon \mathbb{R}^d \to \mathbb{R}^d$ parameterized by $\overline{g} \in D$. Given $\overline{g} \in G \smallsetminus D$, write $\overline{g} = g'\gamma'$ with $g' \in D$ and declare

$$\eta_{\overline{g}}(x) = \eta_{g'}(\tilde{\alpha}(\gamma')(x)).$$

With this notation, we reformulate that a family $\{\eta_{\overline{g}}\}$ parameterized over $\overline{g} \in D$ solves (7.8) if the family parameterized over $\overline{g} \in G$ satisfies

$$c(g,(\overline{g},x)) = \rho(g)\eta_{\overline{g}}(x) - \eta_{g\overline{g}}(x). \tag{7.10}$$

7.3.1. *Solving the cocycle equation for generic $g_0 \in A$.* Recall that $A \subset G$ denotes a maximal split Cartan subgroup obtained by a choice of Iwasawa decomposition (1.1). In the case $G = \mathrm{SL}(n,\mathbb{R})$, A is the group of positive diagonal matrices (1.2).

Recall that we assume $\rho(\gamma_0)$ is hyperbolic for some $\gamma_0 \in \Gamma$. It then follows that $\rho(g_0)$ is a hyperbolic matrix for some choice of $g_0 \in A$. We first build a family $\{\eta_{\overline{g}}\}$ such that (7.8) (or its reformulation (7.10)) holds for this fixed $g = g_0 \in G$. Using the cohomological rigidity outlined in Section 6.3, it will then follow that (7.8) holds for all $g \in G$.

Lemma 7.2. *Fix $g_0 \in A$ with $\rho(g_0)$ hyperbolic. Then there exists a measurable family of functions $\{\eta_{\overline{g}} \colon \mathbb{R}^d \to \mathbb{R}^d\}$ parameterized by $\overline{g} \in D$ such that for almost every $\overline{g} \in D$*

(1) $\eta_{\overline{g}}$ is continuous;
(2) $\eta_{\overline{g}}$ is $\mathbb{Z}^d$-invariant;
(3) for every $x \in \mathbb{R}^d$, writing $g_0\overline{g} = g'\gamma$, we have

$$c(g_0,(\overline{g},x)) = \rho(g_0)\eta_{\overline{g}}(x) - \eta_{g'}(\tilde{\alpha}(\gamma)(x)) = \rho(g_0)\eta_{\overline{g}}(x) - \eta_{g_0\overline{g}}(x). \tag{7.11}$$

We remark that the proof of Lemma 7.2 is, up to minor notational differences and certain technical difficulties when Γ is nonuniform, nearly identical to the proof of Theorem 1.6.

Proof sketch. Let $\{\lambda_j\}$ denote the set of the eigenvalues of $\rho(g_0)$. We have that

$$\{\lambda_j\} \subset (0,1) \cup (1,\infty).$$

For each j, let $E_j \subset \mathbb{R}^d$ denote the eigenspace corresponding to λ_j. Write

$$E = \bigoplus_{0<\lambda_j<1} E_j, \qquad F = \bigoplus_{\lambda_j>1} E_j.$$

Then $\mathbb{R}^d$ decomposes as a direct sum $\mathbb{R}^d = E \oplus F$. Given a vector $v \in \mathbb{R}^d$, decompose $v = v^E \oplus v^F$ according to this decomposition. In particular, for each $c(g,(\overline{g},x)) \in \mathbb{R}^d$, we write

$$c(g,(\overline{g},x)) = c^E(g,(\overline{g},x)) + c^F(g,(\overline{g},x)).$$

Given $\overline{g} \in D$ and $x \in \mathbb{R}^d$ formally define $\eta_{\overline{g}}(x)$ by

$$\eta_{\overline{g}}(x) := -\sum_{k=1}^{\infty} \rho(g_0)^{k-1}\Big(c^E(g_0, g_0^{-k} \cdot (\overline{g}, x))\Big) + \sum_{k=0}^{\infty} \rho(g_0)^{-k-1}\Big(c^F(g_0, g_0^{k} \cdot (\overline{g}, x))\Big). \tag{7.12}$$

Note that the $\mathbb{Z}^d$-invariance of $c(g, (\overline{g}, \cdot))$ formally passes to $\mathbb{Z}^d$-invariance of $\eta_{\overline{g}}(\cdot)$. Writing $g_0\overline{g} = g'\gamma$ we formally check that (7.11) holds. Indeed, with $k' = k+1$ and $k'' = k-1$, we have

$$\begin{aligned}
\rho(g_0)\eta_{\overline{g}}(x) &= -\sum_{k=1}^{\infty} \rho(g_0)^{k}\Big(c^E(g_0, g_0^{-k} \cdot (\overline{g}, x))\Big) \\
&\qquad + \sum_{k=0}^{\infty} \rho(g_0)^{-k}\Big(c^F(g_0, g_0^{k} \cdot (\overline{g}, x))\Big) \\
&= -\sum_{k=0}^{\infty} \rho(g_0)^{k}\Big(c^E(g_0, g_0^{-k} \cdot (\overline{g}, x))\Big) + c^E(g_0, (\overline{g}, x)) \\
&\qquad + \sum_{k=1}^{\infty} \rho(g_0)^{-k}\Big(c^F(g_0, g_0^{k} \cdot (\overline{g}, x))\Big) + c^F(g_0, (\overline{g}, x)) \\
&= -\sum_{k'=1}^{\infty} \rho(g_0^{k'-1})\Big(c^E(g_0, g_0^{-k'} \cdot (g_0 \cdot (\overline{g}, x)))\Big) + c^E(g_0, (\overline{g}, x)) \\
&\qquad + \sum_{k''=0}^{\infty} \rho(g_0^{-k''-1})\Big(c^F(g_0, g_0^{k''} \cdot (g_0 \cdot (\overline{g}, x)))\Big) + c^F(g_0, (\overline{g}, x)) \\
&= \eta_{g_0 \cdot \overline{g}}(x) + c(g_0, (\overline{g}, x)) = \eta_{g'}(\tilde{\alpha}(\gamma)(x)) + c(g_0, (\overline{g}, x)).
\end{aligned}$$

It remains to argue for almost every $\overline{g} \in D$ that $\eta_{\overline{g}}(x)$ is well-defined and continuous. In the case that Γ is cocompact, we have

$$\sup\{\|c(g_0, (\overline{g}, x))\| : x \in \mathbb{R}^d, \overline{g} \in D\} < \infty$$

and hence the series in (7.12) converge geometrically for every $\overline{g} \in D$ and continuity follows.

In the case that Γ is nonuniform, the main result of [66] and our choice of D as a Dirichlet domain imply for almost every $\overline{g} \in D$ that the sequences

$$k \mapsto \sup_{x \in \mathbb{R}^d} \|c^E(g_0, g_0^{-k} \cdot (\overline{g}, x))\|, \qquad k \mapsto \sup_{x \in \mathbb{R}^d} \|c^F(g_0, g_0^{k} \cdot (\overline{g}, x))\|$$

grow at most subexponentially in k. Then the series in (7.12) converge geometrically to a continuous function $\eta_{\overline{g}}$ for almost every $\overline{g} \in D$. □

7.3.2. *Solving the cocycle equation on the centralizer of* g_0. Using that $\rho(g_0)$ is hyperbolic, one can show that any family of functions $\{\eta_{\overline{g}}, \overline{g} \in D\}$ satisfying the conclusion of Lemma 7.2 is essentially unique. That is, if $\{\overline{\eta}_{\overline{g}}, \overline{g} \in D\}$ is another family satisfying the conclusion of Lemma 7.2, then $\eta_{\overline{g}} = \overline{\eta}_{\overline{g}}$ for almost every $\overline{g} \in D$. Using this uniqueness, it is easy to show the family $\{\eta_{\overline{g}}, \overline{g} \in D\}$ obtained from Lemma 7.2 solves (7.10) for any g contained in the centralizer of g_0.

Lemma 7.3. *Let g be an element of the centralizer of g_0 in G. Then for almost every $\overline{g} \in D$ and every $x \in \mathbb{R}^d$*

$$c(g, (\overline{g}, x)) = \rho(g)\eta_{\overline{g}}(x) - \eta_{g\overline{g}}(x)).$$

PROOF SKETCH. Fix g commuting with g_0. Given $\overline{g} \in D$, define $\overline{\eta}_{\overline{g}} \colon \mathbb{R}^d \to \mathbb{R}^d$ by

$$\overline{\eta}_{\overline{g}}(x) = \rho(g)^{-1}[c(g, (\overline{g}, x)) + \eta_{g\overline{g}}(x)].$$

We verify that $c(g_0, (\overline{g}, x)) = \rho(g_0)\overline{\eta}_{\overline{g}}(x) - \overline{\eta}_{g_0\overline{g}}(x)$. Indeed

$$\begin{aligned}
\rho(g_0)&\overline{\eta}_{\overline{g}}(x) - \overline{\eta}_{g_0\overline{g}}(x) \\
&= \rho(g)^{-1}[\rho(g_0)c(g, (\overline{g}, x)) + \rho(g_0)\eta_{g\overline{g}}(x)] \\
&\qquad - \rho(g)^{-1}[c(g, (g_0\overline{g}, x) + \eta_{g_0 g\overline{g}}(x)] \\
&= \rho(g)^{-1}[\rho(g_0)c(g, (\overline{g}, x)) + c(g_0, (g\overline{g}, x)) + \eta_{g_0 g\overline{g}}(x)] \\
&\qquad - \rho(g)^{-1}[-\rho(g)c(g_0, (\overline{g}, x)) + c(gg_0, (\overline{g}, x)) + \eta_{g_0 g\overline{g}}(x)] \\
&= c(g_0, (\overline{g}, x)) + \rho(g)^{-1}[\rho(g_0)c(g, (\overline{g}, x)) + c(g_0, (g\overline{g}, x)) - c(g_0 g, (\overline{g}, x))] \\
&= c(g_0, (\overline{g}, x)),
\end{aligned}$$

where we use that c is a twisted cocycle (see (7.7)) in the second and fourth equalities. By the essential uniqueness of solutions to (7.11), we have $\overline{\eta}_{\overline{g}}(x) = \eta_{\overline{g}}(x)$ for almost every $\overline{g}$ and every x. $\square$

7.3.3. *Resonant and nonresonant roots.* We make precise the ideas outlined in Section 6.3. Recall that we identify A (via the Lie exponential map) with $\mathbb{R}^{\mathrm{rank}(G)}$. Then the roots of G are a family of (nonzero) linear functionals $\beta \colon A \to \mathbb{R}$. Given a linear representation $\rho \colon G \to \mathrm{SL}(d, \mathbb{R})$, the matrices

$$\{\rho(a) : a \in A\}$$

are jointly diagonalizable over $\mathbb{R}$ with positive eigenvalues. The weights $\{\chi\}$ of ρ are linear functionals $\chi \colon A \to \mathbb{R}$ given by the logarithms of the joint eigenvalues of eigenvalues of $\{\rho(a) : a \in A\}$. See Section 6.3 and Examples 7.5 and 7.6 below.

Recall that associated to each root β is a root subspace $\mathfrak{g}^\beta \subset \mathfrak{g}$ which has the property that

$$\mathrm{Ad}_a(X) = e^{\beta(a)}X$$

for $a \in A$ and $X \in \mathfrak{g}^\beta$.

Definition 7.4. Let G be a semisimple Lie group and let $\rho\colon G \to \mathrm{GL}(d,\mathbb{R})$ be a representation

(1) We say a root β of G is **resonant with** ρ if there exists a (non-zero) weight χ of ρ that is positively proportional to β:
$$\beta(a) = c\chi(a)$$
for some $c > 0$ and all $a \in A$.

(2) If β is not resonant with ρ, we say β is **nonresonant** with ρ.

(3) We say $\rho\colon G \to \mathrm{SL}(d,\mathbb{R})$ is **strongly nonresonant** if every root of G is nonresonant with ρ.

(4) We say $\rho\colon G \to \mathrm{SL}(d,\mathbb{R})$ is **weakly nonresonant** if the Lie subalgebra generated by $\mathfrak{g}^\beta$ for all nonresonant roots β of G coincides with $\mathfrak{g}$.

From the structure theory of $\mathrm{SL}(2,\mathbb{R})$ representations, it follows that a root β is resonant if and only if $-\beta$ is resonant. In particular, if β is a nonresonant root then β is also not negatively proportional to any weight of ρ.

We demonstrate the above definitions with the following examples.

Example 7.5. If $G = \mathrm{SL}(n,\mathbb{R})$ then the roots are given by $\{\beta^{i,j}\}$ where
$$\beta^{i,j}(\mathrm{diag}(e^{t_1},\dots,e^{t_n})) = t_i - t_j.$$

Let $\rho\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$ be the standard representation on $\mathbb{R}^n$. Then the weights are $\{\chi_i\}$ where
$$\chi_i(\mathrm{diag}(e^{t_1},\dots,e^{t_n})) = t_i.$$
In this case, ρ has no zero weights and every root $\beta^{i,j}$ is nonresonant with ρ. Here, ρ is a strongly nonresonant representation.

Let $\mathrm{Ad}\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}(\mathfrak{sl}(n,\mathbb{R}))$ be the adjoint representation
$$\mathrm{Ad}_g(X) = gXg^{-1}.$$
Then the non-zero weights coincide with the roots of G; in particular, every root $\beta^{i,j}$ is resonant with Ad. Thus Ad is neither strongly nonresonant nor weakly nonresonant. However, note that the adjoint representation has zero weights whose corresponding eigenspace is the Lie algebra of A and so this representation is not relevant to Theorem 4.2.

Example 7.6. If $G = \mathrm{Sp}(2n,\mathbb{R})$ we may find an Iwasawa decomposition $G = KAN$ such that
$$A = \{\mathrm{diag}(e^{t_1},\dots,e^{t_n},e^{-t_1},\dots,e^{-t_n})\}.$$
The roots of G are the linear functionals
$$\mathrm{diag}(e^{t_1},\dots,e^{t_n},e^{-t_1},\dots,e^{-t_n}) \mapsto \pm t_i \pm t_j, \quad i<j$$
and
$$\mathrm{diag}(e^{t_1},\dots,e^{t_n},e^{-t_1},\dots,e^{-t_n}) \mapsto \pm 2t_i.$$

Let $\rho\colon \mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{SL}(2n,\mathbb{R})$ be the standard representation on $\mathbb{R}^{2n}$. Then the weights of ρ are

$$\mathrm{diag}(e^{t_1},\dots,e^{t_n},e^{-t_1},\dots,e^{-t_n}) \mapsto \pm t_i.$$

In this case, ρ has no zero weights. Roots of the form

$$\mathrm{diag}(e^{t_1},\dots,e^{t_n},e^{-t_1},\dots,e^{-t_n}) \mapsto \pm 2t_i$$

are positively proportional to weights of the form

$$\mathrm{diag}(e^{t_1},\dots,e^{t_n},e^{-t_1},\dots,e^{-t_n}) \mapsto \pm t_i.$$

However, roots of the form

$$\mathrm{diag}(e^{t_1},\dots,e^{t_n},e^{-t_1},\dots,e^{-t_n}) \mapsto \pm t_i \pm t_j, \quad i < j$$

are nonresonant with the representation ρ; moreover, the root spaces corresponding to nonresonant roots generate the entire Lie algebra. It follows that ρ is a weakly nonresonant representation.

The following proposition is a key technical result from [18]. This allows us to conclude that the family $\{\eta_{\overline{g}}, \overline{g} \in D\}$ constructed in Lemma 7.2 solves (7.10) for all $g \in G$.

Proposition 7.7 ([18, Lemma 2.7])**.** *Suppose that G is a higher-rank simple Lie group. Let $\psi\colon G \to \mathrm{SL}(d,\mathbb{R})$ be a finite-dimensional real representation of G. Suppose that all weights χ of ρ are non-zero. Then ρ is weakly nonresonant.*

Moreover, if the restricted root system of G is not of type C_ℓ, then ψ is strongly nonresonant.

Recall that the assumption that $\rho(\gamma_0)$ is hyperbolic for some $\gamma_0 \in \Gamma$ implied that $\rho(g_0)$ was hyperbolic for some $g_0 \in A$. From this, we conclude from Proposition 7.7 that any representations ρ as in Theorem 4.2 has all weights non-zero and is hence is weakly nonresonant.

7.3.4. *Solving the cocycle equation along nonresonant root subgroups.* Associated to each root β of G is a unipotent root subgroup $U^\beta \subset G$ whose Lie algebra is $\mathfrak{g}^\beta$ or $\mathfrak{g}^\beta \oplus \mathfrak{g}^{2\beta}$ if 2β is also a root. When $G = \mathrm{SL}(n,\mathbb{R})$ we have that $U^{\beta^{i,j}}$ is the group of matrices whose diagonal entries are 1, (i,j)th entry is an arbitrary real number t, and every other entry is zero.

Lemma 7.8. *Let β be a nonresonant root. Then for all $u \in U^\beta$, almost every $\overline{g} \in D$, and every $x \in \mathbb{R}^d$ we have*

$$c(u,(\overline{g},x)) = \rho(u)\eta_{\overline{g}}(x) - \eta_{u\overline{g}}(x)).$$

Proof sketch. Recall our fixed $g_0 \in A$ with $\rho(g_0)$ hyperbolic. Using that β is nonresonant with ρ and that $\rho(g_0)$ is hyperbolic we may find $g_\beta \in A$ such that

(1) $\beta(g_\beta) = 0$;
(2) $\rho(g_\beta)$ is hyperbolic.

From (1), we have that g_β and u commute for all $u \in U^\beta$. From Lemma 7.3 we have

$$c(g_\beta, (\overline{g}, x)) = \rho(g_\beta)\eta_{\overline{g}}(x) - \eta_{g_\beta \overline{g}}(x).$$

Fix $u \in U^\beta$ and define $\hat\eta_{\overline{g}}\colon \mathbb{R}^d \to \mathbb{R}^d$ by

$$\hat\eta_{\overline{g}}(x) = \rho(u)^{-1}[c(u, (\overline{g}, x)) + \eta_{u\overline{g}}(x)].$$

Using that u and g_β commute, as in the proof of Lemma 7.3 we have that

$$c(g_\beta(\overline{g}, x)) = \rho(g_\beta)\hat\eta_{\overline{g}}(x) - \hat\eta_{g_\beta \overline{g}}(x) \tag{7.13}$$

for almost every $\overline{g} \in D$. As $\rho(g_\beta)$ is hyperbolic, we again have that any measurable family $\{\hat\eta_{\overline{g}}\}$ of continuous, $\mathbb{Z}^d$-invariant functions satisfying (7.13) is unique up to null sets of D. Thus $\hat\eta_{\overline{g}}(x) = \eta_{\overline{g}}(x)$ for almost every $\overline{g} \in D$ and the conclusion follows. $\square$

7.3.5. *Completion of the proof.* Using that the representation $\rho\colon G \to \mathrm{GL}(d, \mathbb{R})$ is weakly nonresonant, we have that $\{\eta_{\overline{g}}, \overline{g} \in D\}$ constructed in Lemma 7.2 solves (7.10) for all $g \in G$. Indeed, as the root spaces $\mathfrak{g}^\beta$ corresponding to nonresonant roots β generate all of $\mathfrak{g}$, for any $g \in G$, almost every $\overline{g} \in D$ and every $x \in \mathbb{R}^d$, we conclude that (7.10) holds.

After possibly modifying the family $\{\eta_{\overline{g}}\}$ on a null set of $\overline{g} \in D$, we may reverse the order of quantifiers and conclude: for almost every $\overline{g} \in D$, every $x \in \mathbb{R}^d$, and every $g \in G$ equation (7.10) holds. Thus, the family $\{\eta_{\overline{g}}\}$ satisfies the conditions of Section 7.2.2 and, as discussed in Section 7.1.2 and Section 7.2.3, Theorem 4.2 follows.

8. Discussion of the proof of Theorem 5.2

The main theorem of the paper [12] is Theorem 5.2; we then deduce Theorem 3.4 using strong property (T) and Margulis's superrigidity theorem. Analogues of Theorem 5.2 for nonuniform lattices are the main results of [13, 14]. The main technical result in [12] is stated as Proposition 8.1 below which we state without proof. For cocompact lattices in $\mathrm{SL}(n, \mathbb{R})$, self-contained proofs of Proposition 8.1 can be found in [21] and [10]. We state the key proposition, Proposition 8.1, and outline the proof of Theorem 5.2.

8.1. Key proposition. Recall that given an action $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$, the suspension space M^α is a fiber bundle with induced G-action. We write $\pi\colon M^\alpha \to G/\Gamma$ for the canonical projection map and let $F := \ker D\pi$ denote the fiberwise tangent bundle. Then F is a G-invariant subbundle.

The majority of [12] is devoted to establishing (in the case of cocompact $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$) the following proposition. An analogous proposition for nonuniform lattices is established in [13, 14]. Recall that for $G = \mathrm{SL}(n, \mathbb{R})$, $A \subset G$ denotes the subgroup of positive diagonal matrices.

Proposition 8.1. *For $n \geq 3$, let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a cocompact lattice. Let M be a compact manifold and let $\alpha\colon \Gamma \to \mathrm{Diff}^1(M)$ be an action that fails to have uniform subexponential growth of derivatives. Then there exists an ergodic, A-invariant, Borel probability measure μ on M^α such that*

(1) the image $\pi_*\mu$ of μ under $\pi\colon M^\alpha \to G/\Gamma$ is the normalized Haar measure on G/Γ, and
(2) there is an $a \in A$ such that

$$\lim_{n\to\infty} \frac{1}{n} \int \log \|D_x a^n \restriction_F\| \, d\mu(x) = \lambda > 0. \tag{8.1}$$

Note that the limit in (8.1) exists since the a-invariance of μ implies that sequence

$$n \mapsto \int \log \|D_x a^n \restriction_F\| \, d\mu(x)$$

is subadditive; in particular, the limit in (8.1) is actually

$$\inf_n \frac{1}{n} \int \log \|D_x a^n \restriction_F\| \, d\mu(x).$$

We remark that the statement and proof of Proposition 8.1 is independent of the dimension of M.

8.2. Proof of Theorem 5.2 assuming Proposition 8.1 and Corollary 6.5. Assuming the conclusion of Theorem 5.2 is false, we take μ to be the A-invariant measure guaranteed by Proposition 8.1. From Corollary 6.5 (or Corollary 6.3) and the dimension bounds on M, we conclude that the A-invariant measure μ is in fact G-invariant. We then obtain a contradiction with Zimmer's cocycle superrigidity theorem, Theorem 2.14.

Proof of Theorem 5.2. Let $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ be as in Theorem 5.2. For the sake of contradiction, assume that

$$\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$$

fails to have uniform subexponential growth of derivatives. Let μ be the measure guaranteed by Proposition 8.1. In either case considered in Theorem 5.2, it follows from Corollary 6.5 (or Corollary 6.3) that μ is G-invariant.

The fiberwise tangent $F = \ker D\pi$ is G-invariant. We may then apply Zimmer's cocycle superrigidity theorem, Theorem 2.14, to the fiberwise derivative cocycle $\mathcal{A}(g, x) = D_x g\restriction_{F(x)}$ over the $\mathrm{SL}(n, \mathbb{R})$-action on (M^α, μ). Since the fibers have dimension at most $n - 1$ and since there are no nontrivial representations $\rho\colon \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(d, \mathbb{R})$ for $d < n$, it follows from Theorem 2.14 that the fiberwise derivative cocycle $\mathcal{A}(g, x) = D_x g\restriction_{F(x)}$ is cohomologous to a compact-valued cocycle: there is a compact group $K \subset \mathrm{SL}(d, \mathbb{R})$ and measurable $\Phi\colon M^\alpha \to \mathrm{GL}(d, \mathbb{R})$ such that

$$\Phi(g \cdot x) D_x g\restriction_{F(x)} \Phi(x)^{-1} \in K.$$

By Poincaré recurrence to sets on which the norm and conorm of Φ are bounded, it follows for any $g \in G$ and $\varepsilon > 0$ that the set of $x \in M^\alpha$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \log \|D_x g^n \restriction_{F(x)}\| \geq \varepsilon$$

has μ-measure zero. This contradiction with (8.1) completes the proof of Theorem 5.2. □

9. Work in progress: actions in the critical dimension

For $n \geq 3$, consider a lattice $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$. Let M be a closed connected manifold of dimension $(n-1)$ and consider an action $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$. We aim to classify all possible actions α.

9.1. Case 1: α has subexponential growth. If the action $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ has uniform subexponential growth of derivatives then it follows from Step 2 of Section 5 that the image $\alpha(\Gamma)$ is contained in the isometry group $\mathrm{Isom}_g(M)$ for some continuous Riemannian metric on M. As in Step 3 of Section 5, we have that that the image $\alpha(\Gamma)$ is finite since

$$\dim\left(\mathrm{Isom}^{1+\beta}_g(M)\right) \leq \frac{1}{2}(n^2-n) < n^2-1 = \dim(\mathfrak{su}(n)).$$

9.2. Case 2: α has exponential growth. If the action $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ fails to have uniform subexponential growth of derivatives, Proposition 8.1 gives an A-invariant Borel probability measure on M^α satisfying (8.1). However, in this case μ cannot be G-invariant. Indeed, exactly as in the proof in Section 8.2, if μ were G-invariant then equation (8.1) and the dimension bound $\dim(M) < n$ would yield a contradiction with Zimmer's cocycle superrigidity.

Let $H \subset G$ denote the subgroup preserving μ. Then H is a proper subgroup of G. Using Proposition 6.4 and the structure theory of $\mathrm{SL}(n, \mathbb{R})$ it follows that

(1) H is, up to conjugation, the group

$$H = \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \right\};$$

(2) there are $n-1$ distinct fiberwise Lyapunov exponents $\{\lambda_i^F\}$ and each Lyapunov exponent is positively proportional to a root of the form $\beta^{j,1}$ for $2 \leq j \leq n$;

(3) for some $k \in \mathbb{N}$, there is an H-invariant subset $S \subset M^\alpha$ with $\mu(S) = 1$ such that S meets almost every fiber of the fiber-bundle M^α in exactly k points.

9.2.1. *Measurable classification of* α. Using that μ is not $U^{\beta^{j,1}}$-invariant for any $2 \leq j \leq n$, in [16, Theorem 1.7] we showed that the action $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ has the projective action on $\mathbb{R}P^{n-1}$ equipped with a smooth volume as a measurable factor. Precisely, if $\dim(M) = n-1$ and if the action $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ fails to preserve any probability measure (and so in particular is not an action by isometries) then

(1) there exists a Borel probability measure $\overline{\mu}$ on M such that $\alpha(\gamma)_*\overline{\mu}$ is in the same measure class as $\overline{\mu}$ for every $\gamma \in \Gamma$;
(2) a measurable map $\psi\colon M \to \mathbb{R}P^{n-1}$;
(3) $k \in N$ such that ψ is k-to-1 off a $\overline{\mu}$-null set;

such that

(a) $\psi_*\overline{\mu}$ is a smooth volume on $\mathbb{R}P^{n-1}$ and
(b) $\psi \circ \alpha(\gamma)(x) = \psi(x) \cdot \gamma^{-1}$ for $\overline{\mu}$-almost every x.

To build the measure $\overline{\mu}$, first lift the probability measure μ on M^α to a Γ-invariant, H-invariant, locally finite measure $\tilde{\mu}$ on $G \times M$. Since μ is H-invariant, $\tilde{\mu}$ induces a measure class $\hat{\mu}$ on

$$H\backslash(G \times M) = H\backslash G \times M = \mathbb{R}P^{n-1} \times M$$

which projects to a smooth measure on $\mathbb{R}P^{n-1} = H\backslash G$. Moreover, we show that the natural map

$$\mathbb{R}P^{n-1} \times M \to M$$

is injective on a $\hat{\mu}$ full measure set. We take $\overline{\mu}$ to be the image of $\hat{\mu}$ under the natural map $\mathbb{R}P^{n-1} \times M \to M$. Then, the map $\mathbb{R}P^{n-1} \times M \to M$ has a measurable inverse $\hat{\psi}\colon M \to \mathbb{R}P^{n-1} \times M$ which, post-composed with the projection $\mathbb{R}P^{n-1} \times M \to \mathbb{R}P^{n-1}$ gives the map ψ with the desired properties.

9.2.2. *Topological and smooth classification.* Applying tools from the theory of measure rigidity for non-uniformly hyperbolic actions of abelian groups (especially techniques developed in [51, 52, 59]) we can say much stronger properties about the measures μ and $\tilde{\mu}$ above. In particular, the measure $\tilde{\mu}$ on $G \times M$ above is locally supported on finitely many graphs of $C^{1+\beta}$, H-invariant functions $\Phi_j\colon G \to M$, $1 \leq j \leq k$.

Using the functions Φ_j, we fully recover the topology of M (as a covering space of $\mathbb{R}P^{n-1}$) and the dynamics of the action α. In particular, this shows the following result, which is current work in progress with Federico Rodriguez Hertz and Zhiren Wang.

THEOREM 9.1. *For $n \geq 3$, let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a lattice. Let M be a closed, connected $(n-1)$-dimensional manifold, and let $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ be an action with infinite image. Then, there is either*

(1) a $C^{1+\beta}$ diffeomorphism $h\colon M \to S^{n-1}$, or
(2) a $C^{1+\beta}$ diffeomorphism $h\colon M \to \mathbb{R}P^{n-1}$.

Moreover, for every $x \in M$ and $\gamma \in \Gamma$ we have

$$h(\alpha(\gamma)(x)) = \gamma \cdot h(x)$$

where the action of γ is as in Example 2.6.

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