

On the averaged Colmez conjecture

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ABSTRACT. This is an expository article on the averaged version of Colmez’s conjecture, relating Faltings heights of CM abelian varieties to Artin L -functions. It is based on the author’s lectures at the Current Developments in Mathematics conference held at Harvard in 2018.

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1. Introduction

The goal of this paper is to give an overview of a proof of the averaged Colmez conjecture, which relates the Faltings heights of CM abelian varieties to logarithmic derivatives of L -functions.

1.1. Statement of the theorem. Suppose E is a CM field, and A is an abelian variety defined over $\mathbb{Q}^{\text{alg}} \subset \mathbb{C}$ that admits complex multiplication by the full ring of integers $\mathcal{O}_E \subset E$. This means that $[E : \mathbb{Q}] = 2 \cdot \dim(A)$, and A admits a ring homomorphism $\mathcal{O}_E \rightarrow \text{End}(A)$.

It is a theorem of Colmez [Col193] that the Faltings height

$$h_{(E, \Phi)}^{\text{Falt}} = h^{\text{Falt}}(A)$$

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depends only on the CM type (E, Φ) of A , and not on the abelian variety A itself. In the same paper in which he proved this theorem, Colmez proposed a conjectural formula for the value of the Faltings height as a linear combination of logarithmic derivatives of Artin L -functions at $s = 0$. The precise statement is recalled here as Conjecture 3.3.1.

Colmez's conjectural formula generalizes both the Chowla-Selberg [CS67] formula for elliptic curves with complex multiplication, and results of Anderson [And82] (following Deligne, Gross, and Shimura) on the periods of A in the special case where E/\mathbb{Q} is Galois with abelian Galois group. See also work of Maillot-Roessler [MR02] and Yoshida [Yos03]. We will say nothing about Anderson's result and the work that preceded it, and instead refer the reader to the expository paper [Gro18]. However, we will take a moment to state the Chowla-Selberg formula, so that the reader may compare it to Theorem A below.

Chowla and Selberg originally stated their theorem as a formula for the values of Ramanujan's discriminant Δ at CM points in the complex upper half plane. It was noted in [Gro80], where the observation is credited to Deligne, that one can use this to obtain a formula for the Faltings heights of elliptic curves with complex multiplication. The connection between the two formulations is explained in detail in §2.

When expressed in terms of Faltings heights, the Chowla-Selberg theorem says that if E is a quadratic imaginary field of discriminant D_E , and if $\Phi \subset \text{Hom}(E, \mathbb{C})$ is either of the two CM types of E , then

$$(1.1.1) \quad h_{(E, \Phi)}^{\text{Falt}} = -\frac{1}{2} \frac{L'(0, \chi)}{L(0, \chi)} - \frac{1}{4} \log |D_E| - \frac{1}{2} \log(2\pi).$$

Here $\chi : \mathbb{A}^\times \rightarrow \{\pm 1\}$ is the quadratic idele class character associated to the extension E/\mathbb{Q} , and $L(s, \chi)$ is the usual Dirichlet L -function (excluding the archimedean Euler factors).

Returning to the case of a general CM field, if one holds E fixed and averages both sides of Colmez's conjectural formula over all CM types $\Phi \subset \text{Hom}(E, \mathbb{C})$, the result is the equality stated below as Theorem A. It is this theorem, which was proved simultaneously by two groups of authors using different methods, that is the subject of this paper.

THEOREM A (Andreatta-Goren-H-Madapusi Pera, Yuan-Zhang). *If E is a CM field with maximal totally real subfield F of degree $d = [F : \mathbb{Q}]$, then*

$$\frac{1}{2^d} \sum_{\Phi} h_{(E, \Phi)}^{\text{Falt}} = -\frac{1}{2} \frac{L'(0, \chi)}{L(0, \chi)} - \frac{1}{4} \log \left| \frac{D_E}{D_F} \right| - \frac{d \log(2\pi)}{2}.$$

Here D_E and D_F are the (absolute) discriminants of E and F ,

$$\chi : \mathbb{A}_F^\times \rightarrow \{\pm 1\}$$

is the idele class character associated with the extension E/F , and the summation on the left is over the 2^d distinct CM types $\Phi \subset \text{Hom}(E, \mathbb{C})$.

For applications of the theorem to the André-Oort conjecture, see work of Tsimmermann [Tsi18].

The proof of Theorem A given in [AGHMP18] is based on the method of Yang [Yan10b, Yan10a], who proved Colmez’s conjecture for some non-Galois quartic CM fields by combining the theory of Borcherds products with some explicit calculations of arithmetic intersection multiplicities on the integral model of a Hilbert modular surface. The arguments of [AGHMP18] proceed along the same general lines, but with the Hilbert modular surface replaced by an orthogonal Shimura variety.

These arithmetic intersection calculations are much in the spirit of theorems and conjectures of Kudla [Kud97b], and variants of it explored in the work of Kudla-Rapoport [KR99, KR00], Bruinier-Yang [BY09], the author [How12, How15], and various subsets of those just named [HY12, BHY15, BKY12, BHK⁺18]. In particular, [How12] and [BHK⁺18] contain similar calculations, but for unitary rather than orthogonal Shimura varieties. The calculations are also reminiscent of those of Gross-Zagier [GZ86] and Gross-Keating [GK93].

If $a, b \in \mathbb{C}$, let us write

$$a \doteq b$$

to mean that $a - b$ is a \mathbb{Q} -linear combination of $\{\log(p) : p \text{ prime}\}$. In this paper we will outline the proof of Theorem A, while also providing a more-or-less complete proof of the weaker relation

$$(1.1.2) \quad \frac{1}{2d} \sum_{\Phi} h_{(E, \Phi)}^{\text{Falt}} \doteq -\frac{1}{2} \frac{L'(0, \chi)}{L(0, \chi)} - \frac{1}{4} \log \left| \frac{D_E}{D_F} \right| - \frac{d \log(2\pi)}{2}.$$

The qualifier “more-or-less” means that the reader must accept as a black box the existence of Borcherds products, and a result of Bruinier-Kudla-Yang on their values at CM points of orthogonal Shimura varieties.

The point is that the arithmetic intersection multiplicities appearing in the proof of Theorem A decompose as sums of local terms, and the weaker relation (1.1.2) only requires computing the archimedean contribution. In particular, one only needs the canonical models over \mathbb{Q} of orthogonal Shimura varieties, not their integral models.

1.2. Outline of the paper. We now give a fairly detailed outline of both the structure of the paper, and of the proof of Theorem A up to \doteq .

In §2 we recall the original Chowla-Selberg formula on the CM values of Ramanujan’s discriminant Δ , and explain how to deduce from it the reformulation (1.1.1) in terms of Faltings heights.

Over the open modular curve \mathcal{M} parametrizing elliptic curves over arbitrary schemes, there is a metrized line bundle of weight one modular forms $\hat{\omega}$. If A is an elliptic curve over a number field \mathbf{k} with everywhere good reduction, the theory of Neron models provides an extension of A to an elliptic curve over $\mathcal{O}_{\mathbf{k}}$, which defines a morphism $\text{Spec}(\mathcal{O}_{\mathbf{k}}) \rightarrow \mathcal{M}$. The Faltings

height of A can be computed by pulling back $\widehat{\omega}$ to a metrized line bundle on $\text{Spec}(\mathcal{O}_{\mathbf{k}})$ and taking its arithmetic degree.

In this way we see that the Faltings heights of elliptic curves are encoded by the metrized line bundle $\widehat{\omega}$. Ramanujan's discriminant enters the picture because it provides a concrete trivialization of the 12th power of this line bundle.

In §3 we recall some basic properties of abelian varieties with complex multiplication, and state Colmez's conjecture in full generality. There is only one new idea here. Given a CM field E and an embedding $E \rightarrow \mathbb{C}$, we construct the *total reflex pair* (E^\sharp, Φ^\sharp) , which satisfies

$$(1.2.1) \quad h_{(E^\sharp, \Phi^\sharp)}^{\text{Falt}} = \frac{1}{2d} \sum_{\Phi} h_{(E, \Phi)}^{\text{Falt}}.$$

Using this, the averaged Colmez conjecture for E can be reduced to the *exact* Colmez conjecture for (E^\sharp, Φ^\sharp) . Admittedly, at the moment this looks less like a reduction and more like retrograde motion.

In §4 we introduce the orthogonal Shimura variety M associated to a rational quadratic space (V, Q) of signature $(n, 2)$ and a choice of maximal lattice $L \subset V$ (*maximal* means that the quadratic form is \mathbb{Z} -valued on L , and that L is maximal among all lattices with this property). It is a smooth n -dimensional Deligne-Mumford stack over \mathbb{Q} .

The Shimura variety M is of Hodge type, but is of PEL type only when $n \in \{0, 1, 2, 3, 4, 6\}$. Thus, while M does not have a simple moduli-theoretic interpretation, it can be embedded into a Siegel moduli space parametrizing polarized abelian varieties. The Kuga-Satake construction provides a particularly natural way to do this, and pulling back the universal polarized abelian variety from the Siegel space yields the *Kuga-Satake abelian scheme*

$$A \rightarrow M$$

of relative dimension $\dim(A) = 2^{n+1}$. The Shimura variety M also carries over it a line bundle of weight one modular forms ω , and a family of special divisors $Z(m, \mu)$ indexed by rational numbers $m > 0$ and cosets $\mu \in L^\vee/L$.

In §4.6 we explain the connection between ω and Faltings heights. If $y \in M(\mathbf{k})$ is a point valued in a number field \mathbf{k} , the Faltings height of the fiber A_y satisfies

$$(1.2.2) \quad \frac{-1}{[\mathbf{k} : \mathbb{Q}]} \sum_{\sigma: \mathbf{k} \rightarrow \mathbb{C}} \log \|v^\sigma\| \doteq \frac{1}{2^n} \cdot h^{\text{Falt}}(A_y) + \log(2\pi)$$

for any nonzero vector $v \in \omega_y$. Here $\|\cdot\|$ is the metric (4.4.3) on ω .

In §5 we define a flat extension of M to a stack \mathcal{M} over \mathbb{Z} . We also extend the Kuga-Satake abelian scheme to $\mathcal{A} \rightarrow \mathcal{M}$, and construct extensions ω and $\mathcal{Z}(m, \mu)$ of the line bundle of modular forms and the family of special divisors to \mathcal{M} . In particular, we obtain a class

$$\widehat{\omega} \in \widehat{\text{Pic}}(\mathcal{M})$$

in the group of metrized line bundles on \mathcal{M} .

The Shimura variety \mathcal{M} , endowed with its line bundle ω and its Kuga-Satake family, is analogous to the modular curve endowed with its line bundle of modular forms and universal elliptic curve. However, on \mathcal{M} there is no natural analogue of the discriminant Δ that can be used to trivialize a power of ω .

This is addressed in §6, where we recall the theory of Borcherds products. The metaplectic double cover of $\text{SL}_2(\mathbb{Z})$ acts via the Weil representation ω_L on the space S_L of \mathbb{C} -valued functions on L^\vee/L . Suppose

$$f(\tau) = \sum_{m \gg -\infty} c(m) \cdot q^m \in M_{1-\frac{n}{2}}^!(\omega_L)$$

is a weakly holomorphic form of weight $1 - n/2$ and representation ω_L . Each Fourier coefficient $c(m) \in S_L$ decomposes as a linear combination

$$c(m) = \sum_{\mu \in L^\vee/L} c(m, \mu) \cdot \varphi_\mu,$$

where $\varphi_\mu \in S_L$ is the characteristic function of μ .

Assuming that all $c(m, \mu)$ are integers, the Borcherds product of f is a rational section $\psi(f)$ of $\omega^{\otimes c(0,0)}$ with divisor

$$\mathcal{Z}(f) = \sum_{\substack{m > 0 \\ \mu \in L^\vee/L}} c(-m, \mu) \cdot \mathcal{Z}(m, \mu).$$

The theory of regularized theta lifts allows one to construct a Green function $\Phi(f)$ for $\mathcal{Z}(f)$, and the resulting class

$$\widehat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(f)) \in \widehat{\text{CH}}^1(\mathcal{M})$$

in the codimension one arithmetic Chow group of \mathcal{M} is essentially equal to $\widehat{\omega}^{\otimes c(0,0)}$ under the isomorphism

$$(1.2.3) \quad \widehat{\text{CH}}^1(\mathcal{M}) \cong \widehat{\text{Pic}}(\mathcal{M}).$$

The precise relation is found in Corollary 6.2.4.

In §7 we choose the quadratic space (V, Q) in a particular way. Let E be a CM field with $[E : \mathbb{Q}] = 2d > 2$. Choose a $\xi \in F^\times$ that is negative at some fixed real embedding $F \rightarrow \mathbb{R}$, and positive at all remaining embeddings. Endow $V = E$, viewed as a vector space over \mathbb{Q} of dimension $2d$, with the quadratic form

$$Q(x) = \text{Trace}_{F/\mathbb{Q}}(\xi x \bar{x})$$

of signature $(2d - 2, 2)$.

The Shimura variety associated to this quadratic space comes equipped with distinguished cycle $\mathcal{Y}_{\mathbb{Z}} \rightarrow \mathcal{M}$, which is regular and finite flat over $\text{Spec}(\mathbb{Z})$. In particular it has dimension 1. We obtain a linear functional

$$[- : \mathcal{Y}_{\mathbb{Z}}] : \widehat{\text{CH}}^1(\mathcal{M}) \rightarrow \mathbb{R}$$

as the composition

$$\widehat{\text{CH}}^1(\mathcal{M}) \rightarrow \widehat{\text{CH}}^1(\mathcal{Y}_{\mathbb{Z}}) \xrightarrow{\widehat{\text{deg}}} \mathbb{R},$$

where the first arrow is pullback of arithmetic divisors, and the second is the arithmetic degree.

The Kuga-Satake abelian scheme $\mathcal{A} \rightarrow \mathcal{M}$ acquires complex multiplication of a particular type when restricted to $\mathcal{Y}_{\mathbb{Z}}$. More precisely, for every point $y \in \mathcal{Y}_{\mathbb{Z}}(\mathbb{C})$ the fiber \mathcal{A}_y is isogenous to a power of an abelian variety with complex multiplication by E^{\sharp} , and CM type a Galois conjugate of Φ^{\sharp} . Using this and the connection between $\widehat{\omega}$ and the Faltings height provided by (1.2.2) we deduce

$$\frac{[\widehat{\omega} : \mathcal{Y}_{\mathbb{Z}}]}{\text{deg}_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} \doteq \frac{1}{2^{d-2}} h_{(E^{\sharp}, \Phi^{\sharp})}^{\text{Falt}} + \log(2\pi),$$

which we rewrite, using (1.2.1), as

$$(1.2.4) \quad \frac{[\widehat{\omega} : \mathcal{Y}_{\mathbb{Z}}]}{\text{deg}_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} \doteq \frac{1}{d \cdot 2^{d-1}} \sum_{\Phi} h_{(E, \Phi)}^{\text{Falt}} + \log(2\pi).$$

See Theorem 7.5.2 for a stronger statement.

In §8 we turn to the calculation of the arithmetic intersection $[\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathbb{Z}}]$. The cycles in question intersect properly, and so this intersection decomposes as the sum

$$[\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathbb{Z}}] = [\mathcal{Z}(f) : \mathcal{Y}_{\mathbb{Z}}]_{\text{fin}} + [\Phi(f) : \mathcal{Y}_{\mathbb{Z}}]_{\infty}$$

of a contribution from finite places and an archimedean term

The archimedean contribution was computed by Bruinier-Kudla-Yang [BKY12], who showed that it can be expressed in terms of the coefficients of the central derivative of a Hilbert modular Eisenstein series of weight one. If one works modulo \doteq , all of the coefficients vanish except for the constant term, and the constant term is essentially the logarithmic derivative of $L(s, \chi)$ at $s = 0$. In particular, the theorem of Bruinier-Kudla-Yang shows that

$$\frac{2d}{c(0, 0)} \cdot \frac{[\Phi(f) : \mathcal{Y}_{\mathbb{Z}}]_{\infty}}{\text{deg}_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} \doteq -2 \frac{L'(0, \chi)}{L(0, \chi)} - \log \left| \frac{D_E}{D_F} \right| + d \cdot \log(4\pi e^{\gamma}).$$

As $[\mathcal{Z}(f) : \mathcal{Y}_{\mathbb{Z}}]_{\text{fin}} \doteq 0$, we deduce

$$(1.2.5) \quad \frac{2d}{c(0, 0)} \cdot \frac{[\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathbb{Z}}]}{\text{deg}_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} \doteq -2 \frac{L'(0, \chi)}{L(0, \chi)} - \log \left| \frac{D_E}{D_F} \right| + d \cdot \log(4\pi e^{\gamma}).$$

See Corollary 8.2.5 for a stronger statement.

Combining (1.2.4), (1.2.5), and the relation

$$\frac{d}{2} \cdot \frac{[\widehat{\omega} : \mathcal{Y}_{\mathbb{Z}}]}{\text{deg}_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} = \frac{d}{2c(0, 0)} \cdot \frac{[\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathbb{Z}}]}{\text{deg}_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} - \frac{d}{4} \cdot \log(4\pi e^{\gamma})$$

obtained by comparing $\widehat{\omega}$ and $\widehat{\mathcal{Z}}(f)$ under the isomorphism (1.2.3), we find that the equality in Theorem A holds up to \doteq .

In order to upgrade from $\stackrel{\circ}{=}$ to actual equality, one must strengthen both (1.2.4) and (1.2.5). It is the latter which is more difficult, and occupies much of [AGHMP18]. To do this one must compute the finite intersection multiplicities $[\mathcal{Z}(m, \mu) : \mathcal{Y}_{\mathbb{Z}}]_{\text{fin}}$ of all special divisors, and compare them with the same Fourier coefficients of Eisenstein series appearing in the work of Bruinier-Kudla-Yang. The precise statement is Theorem 8.2.2.

2. The Chowla-Selberg formula

We recall here the original statement of the Chowla-Selberg formula concerning CM values of Ramanujan's discriminant, and explain how to deduce from it the reformulation in terms of Faltings heights. To some extent this is a formality, and our real purpose is to acquaint the reader some ideas that will appear in our sketch of the proof of Theorem A.

The central idea that we hope to convey here is that Faltings heights of elliptic curves can be computed using the metrized line bundle of weight one modular forms on the modular curve. As Ramanujan's discriminant provides a trivialization of the 12th power of this line bundle, we obtain a connection between discriminants and Faltings heights.

2.1. The analytic formulation. As usual, we let the group $\text{SL}_2(\mathbb{Z})$ act on the complex upper half-plane via

$$g \cdot \tau = \frac{a\tau + b}{c\tau + d},$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By a *weak modular form* of weight k we mean a holomorphic function $f : \mathcal{H} \rightarrow \mathbb{C}$ on the complex upper half-plane satisfying the transformation law

$$f(g\tau) = (c\tau + d)^k \cdot f(\tau)$$

for all $g \in \text{SL}_2(\mathbb{Z})$ as above. The adjective *weak* is added because we are imposing no conditions on the behavior of $f(\tau)$ as $\tau \rightarrow i\infty$.

The most famous example of such a function is Ramanujan's modular discriminant of weight 12, defined by

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

where $q = e^{2\pi i\tau}$. We may view Δ as a function on lattices $L \subset \mathbb{C}$ in the usual way: choose a \mathbb{Z} -basis $\omega_1, \omega_2 \in L$ such that $\omega_1/\omega_2 \in \mathcal{H}$, and set

$$\Delta(L) = \omega_2^{-12} \cdot \Delta(\omega_1/\omega_2).$$

It is clear from the transformation law that this doesn't depend on the choice of basis $\{\omega_1, \omega_2\}$, and of course

$$\Delta(\alpha L) = \alpha^{-12} \Delta(L)$$

for any $\alpha \in \mathbb{C}^\times$.

Fix a quadratic imaginary field $E = \mathbb{Q}(\sqrt{-d})$ of discriminant $-d$. Denote by $\text{CL}(E)$ the ideal class group of E , by $h = |\text{CL}(E)|$ its class number, and by $w = |\mathcal{O}_E^\times|$ the number of number of roots of unity in E . Let

$$\chi : (\mathbb{Z}/d\mathbb{Z})^\times \rightarrow \{\pm 1\}$$

be the corresponding Dirichlet character.

Choose an embedding $E \subset \mathbb{C}$. This allows us to view a fractional ideal $\mathfrak{a} \subset E$ as a lattice in \mathbb{C} , and the product $\Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1})$ depends only on the image of \mathfrak{a} in the ideal class group.

THEOREM 2.1.1 (Chowla-Selberg [CS67], see also [Wei99]). *We have*

$$(2\pi d)^{12h} \prod_{\mathfrak{a} \in \text{CL}(E)} \Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1}) = \prod_{0 < a < d} \Gamma(a/d)^{6w\chi(a)}.$$

Using Lerch's formula

$$\log(d) + \frac{L'(0, \chi)}{L(0, \chi)} = \frac{w}{2h} \sum_{0 < a < d} \chi(a) \log(\Gamma(a/d))$$

one can rewrite the Chowla-Selberg formula as

$$(2.1.1) \quad \frac{1}{24h} \sum_{\mathfrak{a} \in \text{CL}(E)} \log |\Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1})| = \frac{1}{2} \frac{L'(0, \chi)}{L(0, \chi)} - \frac{1}{2} \log(2\pi).$$

2.2. Arithmetic intersections. We need some rudimentary ideas from arithmetic intersection theory, following [GS90, Sou92, Gil09]. Let \mathcal{M} be a locally integral Deligne-Mumford stack, flat and of finite type over $\text{Spec}(\mathbb{Z})$, and with smooth generic fiber.

DEFINITION 2.2.1. Suppose \mathcal{Z} is a Cartier divisor on \mathcal{M} . A *Green function* for \mathcal{Z} is a smooth real-valued function Φ on the complex orbifold $\mathcal{M}(\mathbb{C}) \setminus \mathcal{Z}(\mathbb{C})$ satisfying the following two properties.

- (1) If f is a meromorphic function on a holomorphic orbifold chart $U \rightarrow \mathcal{M}(\mathbb{C})$ satisfying

$$\text{div}(f) = \mathcal{Z}(\mathbb{C})|_U,$$

then $\Phi|_U + 2 \log |f|$, initially defined on $U \setminus \mathcal{Z}(\mathbb{C})|_U$, extends to a smooth function on U ;

- (2) Pullback by complex conjugation on $\mathcal{M}(\mathbb{C})$ fixes Φ .

DEFINITION 2.2.2. An *arithmetic divisor* on \mathcal{M} is a pair $\widehat{\mathcal{Z}} = (\mathcal{Z}, \Phi)$ consisting of a Cartier divisor \mathcal{Z} on \mathcal{M} and a Green function Φ for \mathcal{Z} . An arithmetic divisor as above is *principal* if it has the form

$$\widehat{\mathcal{Z}} = (\text{div}(f), -2 \log |f|)$$

for some rational function f on \mathcal{M} .

DEFINITION 2.2.3. A *metrized line bundle* on \mathcal{M} is a pair

$$\widehat{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$$

consisting of a line bundle on \mathcal{M} , and a smoothly varying family of Hermitian metrics on its complex fiber. We further require the metrics to be invariant under pullback by complex conjugation on $\mathcal{M}(\mathbb{C})$.

The *codimension one arithmetic Chow group* $\widehat{\text{CH}}^1(\mathcal{M})$ is the quotient of the group of all arithmetic divisors on \mathcal{M} by the subgroup of principal arithmetic divisors. If we denote by $\widehat{\text{Pic}}(\mathcal{M})$ the group of metrized line bundles (under tensor product), there is a canonical isomorphism

$$(2.2.1) \quad \widehat{\text{Pic}}(\mathcal{M}) \cong \widehat{\text{CH}}^1(\mathcal{M})$$

sending a metrized line bundle $\widehat{\mathcal{L}}$ to the arithmetic divisor

$$\widehat{\text{div}}(s) = (\text{div}(s), -2 \log \|s\|)$$

for any nonzero rational section s of \mathcal{L} .

Now suppose that \mathcal{Y} is a regular Deligne-Mumford stack, finite and flat over $\text{Spec}(\mathbb{Z})$. In particular $\dim(\mathcal{Y}) = 1$, and the assumption of regularity means that we need not distinguish between Cartier divisors and Weil divisors. There is a linear functional

$$\widehat{\text{deg}} : \widehat{\text{CH}}^1(\mathcal{Y}) \rightarrow \mathbb{R}$$

called the *arithmetic degree*, defined as follows. As any divisor on \mathcal{Y} has empty generic fiber, any arithmetic divisor on \mathcal{Y} decomposes uniquely as a finite sum

$$(0, \Phi) + \sum_i m_i \cdot (\mathcal{Z}_i, 0)$$

in which $m_i \in \mathbb{Z}$, each \mathcal{Z}_i is an irreducible effective divisor on \mathcal{Y} supported in a single nonzero characteristic, and Φ is a complex conjugation invariant function on the 0-dimensional orbifold $\mathcal{Y}(\mathbb{C})$. Thus it suffices to define

$$\widehat{\text{deg}}(\mathcal{Z}, 0) = \sum_{y \in \mathcal{Z}(\mathbb{F}_p^{\text{alg}})} \frac{\log(p)}{|\text{Aut}(y)|}$$

when \mathcal{Z} is irreducible and supported in characteristic p , and define

$$\widehat{\text{deg}}(0, \Phi) = \frac{1}{2} \sum_{y \in \mathcal{Y}(\mathbb{C})} \frac{\Phi(y)}{|\text{Aut}(y)|}.$$

Keeping \mathcal{Y} and \mathcal{M} as above, suppose we are given a morphism

$$\mathcal{Y} \rightarrow \mathcal{M}.$$

Composing the arithmetic degree with pullback of metrized line bundles

$$\widehat{\text{CH}}^1(\mathcal{M}) \cong \widehat{\text{Pic}}(\mathcal{M}) \rightarrow \widehat{\text{Pic}}(\mathcal{Y}) \cong \widehat{\text{CH}}^1(\mathcal{Y})$$

defines a linear functional

$$[- : \mathcal{Y}] : \widehat{\text{CH}}^1(\mathcal{M}) \rightarrow \mathbb{R}$$

called *arithmetic intersection against* \mathcal{Y} .

Suppose $\widehat{\mathcal{Z}} = (\mathcal{Z}, \Phi)$ is an arithmetic divisor on \mathcal{M} , and suppose further that the underlying Cartier divisor \mathcal{Z} is effective and meets \mathcal{Y} properly in the sense that

$$\mathcal{Z} \cap \mathcal{Y} \stackrel{\text{def}}{=} \mathcal{Z} \times_{\mathcal{M}} \mathcal{Y}$$

has dimension 0. In this simple case there is a decomposition

$$[\widehat{\mathcal{Z}} : \mathcal{Y}] = [\mathcal{Z} : \mathcal{Y}]_{\text{fin}} + [\Phi : \mathcal{Y}]_{\infty}$$

where the finite contribution is

$$(2.2.2) \quad [\mathcal{Z} : \mathcal{Y}]_{\text{fin}} = \sum_p \log(p) \left(\sum_{y \in (\mathcal{Z} \cap \mathcal{Y})(\mathbb{F}_p^{\text{alg}})} \frac{\text{length}(\mathcal{O}_{\mathcal{Z} \cap \mathcal{Y}, y}^{\text{et}})}{|\text{Aut}(y)|} \right),$$

and the archimedean contribution is

$$[\Phi : \mathcal{Y}]_{\infty} = \frac{1}{2} \sum_{y \in \mathcal{Y}(\mathbb{C})} \frac{\Phi(y)}{|\text{Aut}(y)|}.$$

2.3. The Faltings height. Suppose A is an elliptic curve defined over a number field \mathbf{k} , and that A has everywhere good reduction. Denote in the same way its Néron model

$$(2.3.1) \quad A \rightarrow \text{Spec}(\mathcal{O}_{\mathbf{k}}).$$

The space of global 1-forms on A is a projective $\mathcal{O}_{\mathbf{k}}$ -module of rank one. After choosing a nonzero element

$$\eta \in H^0(A, \Omega_{A/\mathcal{O}_{\mathbf{k}}}),$$

define the archimedean part of the Faltings height by

$$h_{\infty}^{\text{Falt}}(A, \eta) = \frac{-1}{2[\mathbf{k} : \mathbb{Q}]} \sum_{\sigma : \mathcal{O}_{\mathbf{k}} \rightarrow \mathbb{C}} \log \left| \int_{A^{\sigma}(\mathbb{C})} \eta^{\sigma} \wedge \overline{\eta^{\sigma}} \right|,$$

where $A^{\sigma} \rightarrow \text{Spec}(\mathbb{C})$ is the base change of A by σ , and similarly for η^{σ} . Define the finite part of the Faltings height by

$$h_f^{\text{Falt}}(A, \eta) = \frac{1}{[\mathbf{k} : \mathbb{Q}]} \log |H^0(A, \Omega_{A/\mathcal{O}_{\mathbf{k}}})/\mathcal{O}_{\mathbf{k}}\eta|.$$

The *Faltings height*

$$h^{\text{Falt}}(A) = h_{\infty}^{\text{Falt}}(A, \eta) + h_f^{\text{Falt}}(A, \eta)$$

is independent of the choice of η , and is unchanged if we enlarge \mathbf{k} .

Let \mathcal{M} be the Deligne-Mumford stack over \mathbb{Z} classifying elliptic curves, and let $\pi : \mathcal{A} \rightarrow \mathcal{M}$ be the universal elliptic curve. The *line bundle of weight one modular forms* on \mathcal{M} is

$$\omega = \pi_* \Omega_{\mathcal{A}/\mathcal{M}} \cong \text{Lie}(\mathcal{A})^{-1}.$$

We endow ω with the Faltings metric, which at a complex point $y \in \mathcal{M}(\mathbb{C})$ assigns the norm

$$\|\eta\|_y^2 = \left| \int_{\mathcal{A}_y(\mathbb{C})} \eta \wedge \bar{\eta} \right|$$

to a global 1-form $\eta \in \omega_y \cong H^0(\mathcal{A}_y, \Omega_{\mathcal{A}_y/\mathbb{C}})$, and denote by

$$\widehat{\omega} \in \widehat{\text{Pic}}(\mathcal{M})$$

the resulting metrized line bundle on \mathcal{M} .

The Néron model (2.3.1) is an elliptic curve over $\mathcal{O}_{\mathbf{k}}$, and so determines a morphism

$$\mathcal{Y} \stackrel{\text{def}}{=} \text{Spec}(\mathcal{O}_{\mathbf{k}}) \rightarrow \mathcal{M}.$$

Unwinding the definitions, we find that

$$(2.3.2) \quad h^{\text{Falt}}(A) = \frac{[\widehat{\omega} : \mathcal{Y}]}{[\mathbf{k} : \mathbb{Q}]}.$$

2.4. Chowla-Selberg revisited. For each integer k there is an isomorphism of \mathbb{C} -vector spaces

$$\begin{array}{c} \{\text{weak modular forms of weight } k\} \\ \downarrow f \mapsto \underline{f} \\ \{\text{global holomorphic sections of } \omega^{\otimes k}\}. \end{array}$$

To make this isomorphism explicit, let the group $\text{SL}_2(\mathbb{Z})$ act on $\mathcal{H} \times \mathbb{C}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)^k \cdot z \right).$$

There is a commutative diagram

$$\begin{array}{ccc} \text{SL}_2(\mathbb{Z}) \backslash (\mathcal{H} \times \mathbb{C}) & \longrightarrow & \text{SL}_2(\mathbb{Z}) \backslash \mathcal{H} \\ \cong \downarrow & & \cong \downarrow \\ \omega^{\otimes k}(\mathbb{C}) & \longrightarrow & \mathcal{M}(\mathbb{C}) \end{array}$$

in which the isomorphism on the left sends $(\tau, z) \in \mathcal{H} \times \mathbb{C}$ to the complex elliptic curve

$$A_\tau(\mathbb{C}) = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$$

together with the vector $z \in \mathbb{C} \cong \text{Lie}(A_\tau)^{\otimes -k}$ in the fiber of $\omega^{\otimes k}$ at this point. Here the Lie algebra $\text{Lie}(A_\tau)$, and hence also its tensor powers, has been trivialized in the obvious way. If f is a weak modular form of weight k , then

$$\underline{f}(\tau) = (\tau, (2\pi i)^k f(\tau))$$

defines a section to the top horizontal arrow, which we interpret as a global holomorphic section of $\omega^{\otimes k}$.

Using the q -expansion principle, one can show that the holomorphic global section $\underline{\Delta}$ on $\mathcal{M}(\mathbb{C})$ determined by Δ is algebraic and descends to the \mathbb{Q} -stack $\mathcal{M}_{\mathbb{Q}}$. In fact, it extends (necessarily uniquely) to a global section

$$\underline{\Delta} \in H^0(\mathcal{M}, \omega^{\otimes 12}),$$

which trivializes the line bundle $\omega^{\otimes 12}$.

PROPOSITION 2.4.1. *The isomorphism (2.2.1) sends the metrized line bundle $\widehat{\omega}^{\otimes 12}$ to the arithmetic divisor*

$$(0, \Phi) \in \widehat{\text{CH}}^1(\mathcal{M}),$$

where, for any $s \in \mathcal{M}(\mathbb{C})$ we choose a lattice $L_s \subset \mathbb{C}$ satisfying

$$\mathcal{A}_s(\mathbb{C}) \cong \mathbb{C}/L_s,$$

and define

$$\Phi(s) = -12 \cdot \log \left| 4\pi^2 \Delta(L_s)^{1/6} \int_{\mathbb{C}/L_s} dz \wedge d\bar{z} \right|.$$

PROOF. As $\underline{\Delta}$ defines a nowhere vanishing section of $\omega^{\otimes 12}$, the isomorphism (2.2.1) sends

$$\widehat{\omega}^{\otimes 12} \mapsto (0, -2 \log \|\underline{\Delta}\|).$$

At any point $s \in \mathcal{M}(\mathbb{C})$ the fiber

$$\underline{\Delta}_s \in \omega_s^{\otimes 12} \cong H^0(\mathcal{A}_s(\mathbb{C}), \Omega_{\mathcal{A}_s/\mathbb{C}}^{\otimes 12})$$

is given explicitly by $\underline{\Delta}_s = \Delta(L_s) \cdot (2\pi i dz)^{\otimes 12}$, and so

$$\|\underline{\Delta}\|_s = (2\pi)^{12} \cdot |\Delta(L_s)| \cdot \left| \int_{\mathbb{C}/L_s} dz \wedge d\bar{z} \right|^6.$$

The claim follows immediately. □

We can use the above proposition to give a formula for the Faltings height of any elliptic curve A over a number field $\mathbf{k} \subset \mathbb{C}$ with everywhere good reduction. Let $s \in \mathcal{M}(\mathbf{k})$ be the corresponding point on the modular curve. The assumption of good reduction implies that the corresponding morphism $\text{Spec}(\mathbf{k}) \rightarrow \mathcal{M}$ extends to

$$\mathcal{Y} \stackrel{\text{def}}{=} \text{Spec}(\mathcal{O}_{\mathbf{k}}) \rightarrow \mathcal{M},$$

and combining (2.3.2) with Proposition 2.4.1 gives

$$\begin{aligned} h^{\text{Falt}}(A) &= \frac{1}{12} \cdot \frac{[\widehat{\omega}^{\otimes 12} : \mathcal{Y}]}{[\mathbf{k} : \mathbb{Q}]} \\ &= \frac{1}{24[\mathbf{k} : \mathbb{Q}]} \sum_{y \in \mathcal{Y}(\mathbb{C})} \frac{\Phi(y)}{|\text{Aut}(y)|} \\ (2.4.1) \quad &= \frac{-1}{2[\mathbf{k} : \mathbb{Q}]} \sum_{\sigma: \mathbf{k} \rightarrow \mathbb{C}} \log \left| 4\pi^2 \Delta(L_{s^\sigma})^{1/6} \int_{\mathbb{C}/L_{s^\sigma}} dz \wedge d\bar{z} \right|. \end{aligned}$$

In the case where A has complex multiplication, this formula simplifies even further.

PROPOSITION 2.4.2. *Let A be an elliptic curve defined over a number field \mathbf{k} , and having everywhere good reduction. If A admits complex multiplication by \mathcal{O}_E , then*

$$\frac{1}{24h} \sum_{\mathfrak{a} \in \text{CL}(E)} \log |\Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1})| = -h^{\text{Falt}}(A) - \frac{1}{4} \log(d) - \log(2\pi).$$

PROOF. After enlarging \mathbf{k} we are free to assume that \mathbf{k}/\mathbb{Q} is Galois. A choice of embedding $\mathbf{k} \hookrightarrow \mathbb{C}$ then identifies

$$\text{Gal}(\mathbf{k}/\mathbb{Q}) \cong \{\text{embeddings } \mathbf{k} \rightarrow \mathbb{C}\}.$$

It follows from the theory of complex multiplication that \mathbf{k} contains the Hilbert class field H of E , and that $A \cong A^\sigma$ for every $\sigma \in \text{Gal}(\mathbf{k}/H)$.

Every $\sigma \in \text{Gal}(\mathbf{k}/\mathbb{Q})$ determines a class $\mathfrak{a}_\sigma \in \text{CL}(E)$ characterized by

$$A^\sigma(\mathbb{C}) \cong \mathbb{C}/\mathfrak{a}_\sigma.$$

The theory of complex multiplication implies that the resulting function $\text{Gal}(\mathbf{k}/\mathbb{Q}) \rightarrow \text{CL}(E)$ factors through a surjective two-to-one function

$$\text{Gal}(H/\mathbb{Q}) \rightarrow \text{CL}(E).$$

If $c \in \text{Gal}(H/\mathbb{Q})$ is complex conjugation, then $\mathfrak{a}_{c\sigma}$ is the complex conjugate of \mathfrak{a}_σ in the ideal class group.

We now fix, for every $\sigma \in \text{Gal}(H/\mathbb{Q})$ a fractional ideal $\mathfrak{a}_\sigma \subset E$ such that $A^\sigma(\mathbb{C}) \cong \mathbb{C}/\mathfrak{a}_\sigma$, and do this in such a way that $\mathfrak{a}_{c\sigma}$ is the complex conjugate of \mathfrak{a}_σ in E . This implies

$$\Delta(\mathfrak{a}_{c\sigma}) = N(\mathfrak{a}_\sigma)^{-12} \Delta(\mathfrak{a}_\sigma^{-1}),$$

and the formula (2.4.1) simplifies to

$$\begin{aligned} h^{\text{Falt}}(A) &= \frac{-1}{2[H : \mathbb{Q}]} \sum_{\sigma \in \text{Gal}(H/\mathbb{Q})} \log \left| 4\pi^2 \Delta(\mathfrak{a}_\sigma)^{\frac{1}{6}} \int_{\mathbb{C}/\mathfrak{a}_\sigma} dz \wedge d\bar{z} \right| \\ &= \frac{-1}{2[H : \mathbb{Q}]} \sum_{\sigma \in \text{Gal}(H/\mathbb{Q})} \log \left| 4\pi^2 d^{\frac{1}{2}} \Delta(\mathfrak{a}_\sigma)^{\frac{1}{6}} N(\mathfrak{a}_\sigma) \right| \\ &= \frac{-1}{[H : \mathbb{Q}]} \sum_{\sigma \in \text{Gal}(H/\mathbf{k})} \log \left| 4\pi^2 d^{\frac{1}{2}} \Delta(\mathfrak{a}_\sigma)^{\frac{1}{12}} \Delta(\mathfrak{a}_{c\sigma})^{\frac{1}{12}} N(\mathfrak{a}_\sigma) \right| \\ &= \frac{-1}{2h} \sum_{\mathfrak{a} \in \text{CL}(E)} \log \left| 4\pi^2 d^{\frac{1}{2}} \Delta(\mathfrak{a})^{\frac{1}{12}} \Delta(\mathfrak{a}^{-1})^{\frac{1}{12}} \right|. \end{aligned}$$

This is equivalent to the stated formula. □

Combining the preceding proposition with (2.1.1) yields the Chowla-Selberg formula in the form (1.1.1).

3. Colmez's conjecture

This section contains a general discussion of CM algebras and CM abelian varieties, including the statement of Colmez's conjecture in full generality.

There is only one new idea. We will show in Corollary 3.4.3 that the averaged Faltings height appearing in Theorem A can be rewritten as the Faltings height of a *single* CM abelian variety with a very particular CM type (E^\sharp, Φ^\sharp) . Such abelian varieties will later appear in the Kuga-Satake family over an orthogonal Shimura variety.

3.1. CM fields. Before discussing abelian varieties with complex multiplication, we recall some basic facts about CM fields and CM types. We also introduce the notions of total reflex algebra and total reflex type of a CM field.

DEFINITION 3.1.1. The following terminology is standard:

- (1) A *CM field* is a totally imaginary quadratic extension of a totally real number field.
- (2) A *CM algebra* is a finite product of CM fields.
- (3) A *CM type* of a CM algebra E is a set

$$\Phi \subset \text{Hom}(E, \mathbb{C})$$

of \mathbb{Q} -algebra maps such that $\Phi \sqcup \bar{\Phi} = \text{Hom}(E, \mathbb{C})$.

- (4) A *CM pair* is a pair (E, Φ) consisting of a CM algebra E and a CM type $\Phi \subset \text{Hom}(E, \mathbb{C})$.

REMARK 3.1.2. Suppose $E = E_1 \times \cdots \times E_r$ is a CM algebra, with each factor E_i a CM field. There is a canonical identification

$$\text{Hom}(E, \mathbb{C}) = \text{Hom}(E_1, \mathbb{C}) \sqcup \cdots \sqcup \text{Hom}(E_r, \mathbb{C}),$$

and every CM type of E has the form

$$\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_r$$

for Φ_i a CM type of E_i .

Let $\mathbb{Q}^{\text{alg}} \subset \mathbb{C}$ be the algebraic closure of \mathbb{Q} in \mathbb{C} , and set $G_{\mathbb{Q}} = \text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$. The following is an exercise in Galois theory.

PROPOSITION 3.1.3. *The construction*

$$B \mapsto \text{Hom}(B, \mathbb{C})$$

establishes a bijection between the set of isomorphism classes of finite étale \mathbb{Q} -algebras, and the set of isomorphism classes of finite sets with a continuous action of $G_{\mathbb{Q}}$.

Suppose E is a CM field, and choose an embedding $\iota_0 : E \rightarrow \mathbb{C}$. To this data we can associate a *total reflex pair* (E^\sharp, Φ^\sharp) as follows. First note that the group $G_{\mathbb{Q}}$ acts on the set of all CM types of E by

$$g \circ \Phi = \{g \circ \varphi : \varphi \in \Phi\}.$$

By Proposition 3.1.3 there is a finite étale \mathbb{Q} -algebra E^\sharp characterized by

$$\{\text{CM types of } E\} \cong \text{Hom}(E^\sharp, \mathbb{C})$$

as finite sets with $G_{\mathbb{Q}}$ -actions. The embedding ι_0 determines a subset

$$\Phi^\sharp = \{\text{CM types of } E \text{ containing } \iota_0\} \subset \{\text{CM types of } E\} \cong \text{Hom}(E^\sharp, \mathbb{C}).$$

We call E^\sharp and Φ^\sharp the *total reflex algebra* and *total reflex type*, respectively.

PROPOSITION 3.1.4. *Fix a CM field E and an embedding $\iota_0 : E \rightarrow \mathbb{C}$.*

- (1) *The total reflex pair (E^\sharp, Φ^\sharp) is CM pair.*
- (2) *There are natural homomorphisms*

$$\text{Nm}^\sharp : E^\times \rightarrow (E^\sharp)^\times, \quad \text{Tr}^\sharp : E \rightarrow E^\sharp,$$

called the total reflex norm and total reflex trace, respectively.

- (3) *If we hold E fixed but change ι_0 , the CM algebra E^\sharp is unchanged, and Φ^\sharp is replaced by a CM type in the same $G_{\mathbb{Q}}$ -orbit.*

PROOF. The third claim is elementary, and left to the reader. To prove the first two claims, we relate our total reflex pair to the classical notions of reflex field and reflex type.

Fix $G_{\mathbb{Q}}$ -orbit representatives

$$\Phi_1, \dots, \Phi_r \in \{\text{CM types of } E\},$$

and let $\text{Stab}(\Phi_i) \subset G_{\mathbb{Q}}$ be the stabilizer of Φ_i . For each CM pair (E, Φ_i) let (E'_i, Φ'_i) be the classical reflex CM pair, defined by

$$E'_i = \{x \in \mathbb{Q}^{\text{alg}} : \sigma(x) = x, \forall \sigma \in \text{Stab}(\Phi_i)\},$$

and

$$\Phi'_i = \{\sigma^{-1}|_{E'_i} : \sigma \in G_{\mathbb{Q}}, \sigma \circ \iota_0 \in \Phi_i\} \subset \text{Hom}(E'_i, \mathbb{C}).$$

It is an exercise in Galois theory to show that there is an isomorphism of \mathbb{Q} -algebras

$$E^\sharp \cong E'_1 \times \dots \times E'_r$$

such that the natural bijection

$$\text{Hom}(E^\sharp, \mathbb{C}) \cong \text{Hom}(E'_1, \mathbb{C}) \sqcup \dots \sqcup \text{Hom}(E'_r, \mathbb{C})$$

identifies $\Phi^\sharp = \Phi'_1 \sqcup \dots \sqcup \Phi'_r$.

It is known from the classical theory that each (E', Φ') is a CM type, and hence so is (E^\sharp, Φ^\sharp) . Moreover, the classical theory provides homomorphisms

$$\begin{aligned} \mathrm{Nm}_{\Phi_i} &= \prod_{\varphi \in \Phi_i} \varphi : E^\times \rightarrow (E'_i)^\times \\ \mathrm{Tr}_{\Phi_i} &= \sum_{\varphi \in \Phi_i} \varphi : E \rightarrow E'_i, \end{aligned}$$

and the total reflex norm and total reflex trace are constructed from these in the obvious way using the isomorphism $E^\sharp \cong E'_1 \times \cdots \times E'_r$. \square

3.2. CM abelian varieties. Suppose A is an abelian variety defined over a number field \mathbf{k} , and assume that A extends to a semi-abelian scheme

$$A \rightarrow \mathrm{Spec}(\mathcal{O}_{\mathbf{k}})$$

in the sense of [BLR90]. For example, if A has everywhere good reduction then its Néron model provides such an extension.

The Faltings height of A is defined exactly as in the case of an elliptic curve. Pick a nonzero rational section η of the line bundle $\pi_* \Omega_{A/\mathcal{O}_{\mathbf{k}}}^{\dim(A)}$ on $\mathrm{Spec}(\mathcal{O}_{\mathbf{k}})$. Define the *archimedean part of the Faltings height*

$$h_\infty^{\mathrm{Falt}}(A, \eta) = \frac{-1}{2[\mathbf{k} : \mathbb{Q}]} \sum_{\sigma : \mathbf{k} \rightarrow \mathbb{C}} \log \left| \int_{A^\sigma(\mathbb{C})} \eta^\sigma \wedge \overline{\eta^\sigma} \right|,$$

where A^σ is the base change of A via $\sigma : \mathbf{k} \rightarrow \mathbb{C}$, and similarly for η^σ . Define the *finite part of the Faltings height*

$$h_f^{\mathrm{Falt}}(A, \eta) = \frac{1}{[\mathbf{k} : \mathbb{Q}]} \sum_{\mathfrak{p} \subset \mathcal{O}_{\mathbf{k}}} \mathrm{ord}_{\mathfrak{p}}(\eta) \cdot \log(N(\mathfrak{p})),$$

where $\mathrm{ord}_{\mathfrak{p}}$ is defined by choosing an isomorphism of $\mathcal{O}_{\mathbf{k}, \mathfrak{p}}$ -modules

$$H^0(\mathrm{Spec}(\mathcal{O}_{\mathbf{k}}), \pi_* \Omega_{A/\mathcal{O}_{\mathbf{k}}}^{\dim(A)}) \otimes_{\mathcal{O}_{\mathbf{k}}} \mathcal{O}_{\mathbf{k}, \mathfrak{p}} \cong \mathcal{O}_{\mathbf{k}, \mathfrak{p}}.$$

The *Faltings height*

$$(3.2.1) \quad h^{\mathrm{Falt}}(A) = h_f^{\mathrm{Falt}}(A, \eta) + h_\infty^{\mathrm{Falt}}(A, \eta)$$

is independent of the choice of η , and is unchanged if we enlarge the number field \mathbf{k} . This allows us to define the Faltings height of any abelian variety over $\mathbb{Q}^{\mathrm{alg}}$, by choosing a model over a number field with everywhere semi-abelian reduction.

Now fix a CM algebra E of dimension $2d$, and assume that A is an abelian variety of dimension d defined over \mathbb{C} , equipped with an injective ring homomorphism

$$(3.2.2) \quad E \rightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Such a homomorphism endows the homology group $H_1(A(\mathbb{C}), \mathbb{Q})$ with the structure of a free E -module of rank one. Using this and the Hodge decomposition

$$H_1(A(\mathbb{C}), \mathbb{C}) \cong \mathrm{Lie}(A) \oplus \overline{\mathrm{Lie}(A)},$$

it is easy to see that there is a unique CM type Φ such that

$$\mathrm{Lie}(A) \cong \prod_{\varphi \in \Phi} \mathbb{C}_\varphi$$

as $E \otimes_{\mathbb{Q}} \mathbb{C}$ modules, where $\mathbb{C}_\varphi = \mathbb{C}$ with E acting through $\varphi : E \rightarrow \mathbb{C}$.

In the situation above, we say that A has *complex multiplication of type* (E, Φ) . If (3.2.2) restricts to a homomorphism $\mathcal{O}_E \rightarrow \mathrm{End}(A)$, we say that A has *complex multiplication of type* (\mathcal{O}_E, Φ) .

The following proposition is standard. See for example the statement and proof of Proposition 1.1 in Chapter 5 of [Lan83].

PROPOSITION 3.2.1. *Suppose A is an abelian variety over \mathbb{C} with complex multiplication of type (E, Φ) .*

- (1) *There is a model of A defined over a number field, and this model may be chosen to have everywhere good reduction.*
- (2) *Any other abelian variety over \mathbb{C} with complex multiplication of type (E, Φ) is E -linearly isogenous to A .*

THEOREM 3.2.2 (Colmez [Col93]). *Suppose A is an abelian variety over $\mathbb{Q}^{\mathrm{alg}}$ with complex multiplication of type (\mathcal{O}_E, Φ) . The Faltings height*

$$h_{(E, \Phi)}^{\mathrm{Falt}} = h^{\mathrm{Falt}}(A)$$

depends only on the pair (E, Φ) , and not on A itself. It is unchanged if Φ is replaced by a CM type in the same $G_{\mathbb{Q}}$ -orbit.

3.3. Colmez’s conjecture. Fix a CM pair (E, Φ) . Theorem 3.2.2 says that the Faltings height $h_{(E, \Phi)}^{\mathrm{Falt}}$ depends only on the Galois-theoretic data (E, Φ) . It is natural to ask if there is some formula for it that makes this more transparent. This is precisely what Colmez’s conjecture provides: an expression for $h_{(E, \Phi)}^{\mathrm{Falt}}$ in terms of Artin L -functions, that makes no mention of abelian varieties.

Define a complex-valued function on $G_{\mathbb{Q}}$ by

$$A_{(E, \Phi)}(g) = |\Phi \cap g \circ \Phi|.$$

By averaging over the $G_{\mathbb{Q}}$ -orbit of Φ , we obtain a function

$$A_{(E, \Phi)}^0 = \frac{1}{[G_{\mathbb{Q}} : \mathrm{Stab}(\Phi)]} \sum_{g \in G_{\mathbb{Q}}/\mathrm{Stab}(\Phi)} A_{(E, g \circ \Phi)}$$

on $G_{\mathbb{Q}}$ that is locally constant, and constant on conjugacy classes. As such there is a unique decomposition

$$A_{(E, \Phi)}^0 = \sum_{\chi} m_{(E, \Phi)}(\chi) \cdot \chi$$

as a linear combination of Artin characters. That is, each $\chi : G_{\mathbb{Q}} \rightarrow \mathbb{C}$ is the character of a continuous representation $\rho_{\chi} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(W_{\chi})$ on a finite

dimensional complex vector space. For each such Artin character χ let

$$L(s, \chi) = \prod_p \frac{1}{\det(1 - p^{-s} \cdot \rho_\chi(\text{Fr}_p))}$$

be the usual Artin L -function, where the product is over all primes $p < \infty$ of \mathbb{Q} , and

$$\rho_\chi(\text{Fr}_p) : W^{I_p} \rightarrow W^{I_p}$$

is Frobenius acting on the subspace $W^{I_p} \subset W$ of vectors fixed by the inertia subgroup $I_p \subset G_{\mathbb{Q}}$ of some chosen place of \mathbb{Q}^{alg} above p .

Let $c \in G_{\mathbb{Q}}$ be complex conjugation. Using the observation that

$$A_{(E, \Phi)}^0(g) + A_{(E, \Phi)}^0(c \circ g) = |\Phi|$$

is independent of g , one can check that any nontrivial Artin character with $m_{(E, \Phi)}(\chi) \neq 0$ must satisfy $\chi(c) = -\chi(\text{id})$. Hence, by Proposition 3.4 in Chapitre I of [Tat84], the L -function $L(s, \chi)$ has neither a zero nor a pole at $s = 0$. Define

$$Z_{(E, \Phi)} = \sum_{\chi} m_{(E, \Phi)}(\chi) \frac{L'(0, \chi)}{L(0, \chi)}$$

and

$$\mu_{(E, \Phi)} = \sum_{\chi} m_{(E, \Phi)}(\chi) \log(f_\chi),$$

where the sums are over all Artin characters, and f_χ is the Artin conductor of χ . A good reference for Artin representations, including the definition of the Artin conductor, is [MM97]. The *Colmez height* of the CM pair (E, Φ) is defined by

$$h_{(E, \Phi)}^{\text{Col}} = -Z_{(E, \Phi)} - \frac{\mu_{(E, \Phi)}}{2}.$$

The following is our restatement of Conjecture 0.4 of [Col93].

CONJECTURE 3.3.1 (Colmez). *For any CM pair (E, Φ) we have*

$$h_{(E, \Phi)}^{\text{Falt}} = h_{(E, \Phi)}^{\text{Col}}.$$

Following ideas of Gross [Gro78] and Anderson [And82], Colmez was able to prove the abelian case of Conjecture 3.3.1 up to a rational multiple of $\log(2)$. The $\log(2)$ error term was later removed by Obus.

THEOREM 3.3.2 (Colmez [Col93], Obus [Obu13]). *If E/\mathbb{Q} is Galois with abelian Galois group, then*

$$h_{(E, \Phi)}^{\text{Falt}} = h_{(E, \Phi)}^{\text{Col}}$$

holds for every CM type $\Phi \subset \text{Hom}(E, \mathbb{C})$.

3.4. The averaged version. Let E be a CM field of degree $2d$. Let $F \subset E$ be its maximal totally real subfield, and let

$$\chi : \mathbb{A}_F^\times \rightarrow \{\pm 1\}$$

be the associated quadratic character. Denote by $L(s, \chi)$ the usual L -function, and by

$$\Lambda(s, \chi) = |D_E/D_F|^{\frac{s}{2}} \cdot \Gamma_{\mathbb{R}}(s+1)^d \cdot L(s, \chi)$$

the completed L -function. Here

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2),$$

and D_E and D_F are the (absolute) discriminants of E and F . The completed L -function satisfies the function equation

$$\Lambda(1-s, \chi) = \Lambda(s, \chi),$$

and

$$\frac{\Lambda'(0, \chi)}{\Lambda(0, \chi)} = \frac{L'(0, \chi)}{L(0, \chi)} + \frac{1}{2} \log \left| \frac{D_E}{D_F} \right| - \frac{d \log(4\pi e^\gamma)}{2}.$$

As Colmez himself noted, the right hand side of the identity of Conjecture 3.3.1 simplifies considerably if one averages over all CM types of E . The following is an elementary exercise. See the proof of [AGHMP18, Proposition 9.3.1] for details.

PROPOSITION 3.4.1. *For a fixed CM field E of degree $2d$,*

$$\frac{1}{2d} \sum_{\Phi} h_{(E, \Phi)}^{\text{Col}} = -\frac{1}{2} \cdot \frac{L'(0, \chi)}{L(0, \chi)} - \frac{1}{4} \cdot \log \left| \frac{D_E}{D_F} \right| - \frac{d \log(2\pi)}{2}$$

where the summation on the left is over all CM types $\Phi \subset \text{Hom}(E, \mathbb{C})$.

Proposition 3.4.1 is the justification for calling Theorem A the averaged Colmez conjecture. Somewhat oddly, nowhere in our proof of Theorem A does there ever appear an abelian variety with complex multiplication by E . What will appear are abelian varieties with complex multiplication by the total reflex algebra E^\sharp defined in §3.1. Recall that the total reflex type Φ^\sharp depends on the additional choice of an embedding $\iota_0 : E \rightarrow \mathbb{C}$, but the $G_{\mathbb{Q}}$ -orbit of Φ^\sharp does not. It follows that $A_{(E^\sharp, \Phi^\sharp)}^0$, $h_{(E^\sharp, \Phi^\sharp)}^{\text{Falt}}$, and $h_{(E^\sharp, \Phi^\sharp)}^{\text{Col}}$ depend only on E , and not on ι_0 .

The connection between abelian varieties with CM by E and CM by E^\sharp is proved by the following proposition. The proof is elementary Galois theory, and we refer the reader to [AGHMP18, Proposition 9.3.2] for the proof.

PROPOSITION 3.4.2. *For any CM field E we have*

$$A_{(E^\sharp, \Phi^\sharp)}^0 = \frac{1}{2d} \sum_{\Phi} A_{(E, \Phi)}^0,$$

where the summation on the right is over all CM types of E .

COROLLARY 3.4.3. *For any CM field E we have*

$$h_{(E^\sharp, \Phi^\sharp)}^{\text{Falt}} = \frac{1}{2d} \sum_{\Phi} h_{(E, \Phi)}^{\text{Falt}}, \quad h_{(E^\sharp, \Phi^\sharp)}^{\text{Col}} = \frac{1}{2d} \sum_{\Phi} h_{(E, \Phi)}^{\text{Col}}$$

where the summations are over all CM types of E .

PROOF. The Colmez height of (E, Φ) depends only on the function $A_{(E, \Phi)}^0$, and the dependence is linear. The same is true of the Faltings height, but this is a nontrivial theorem of Colmez [Col93, Théorème 0.3]. Both equalities therefore follow from Proposition 3.4.2. \square

4. Orthogonal Shimura varieties

Let (V, Q) be a quadratic space over \mathbb{Q} of signature $(n, 2)$, with $n \geq 1$. The associated bilinear form is denoted

$$(4.0.1) \quad [x_1, x_2] = Q(x_1 + x_2) - Q(x_1) - Q(x_2).$$

Fix a maximal lattice $L \subset V$.

In this section we explain how to attach to this data a Shimura variety M over \mathbb{Q} . This Shimura variety is not of PEL type, so does not have any simple interpretation as a moduli space of polarized abelian varieties. It is, however, of Hodge type, which means that it can be embedded into a Siegel moduli space parametrizing polarized abelian varieties.

By pulling back the universal family we obtain the *Kuga-Satake abelian scheme* $A \rightarrow M$. The Shimura variety carries over it a *line bundle of weight one modular forms* ω , endowed with a Petersson metric. We will see that this line bundle can be used to compute the Faltings heights of the fibers of $A \rightarrow M$.

4.1. Spinor similitudes. We recall some basic definitions in the theory of Clifford algebras and spinor groups. More details may be found in [Shi10].

The Clifford algebra $C = C(V)$ is the quotient of the full tensor algebra of V by the two-sided ideal generated by all elements $x \otimes x - Q(x)$ with $x \in V$. It carries a $\mathbb{Z}/2\mathbb{Z}$ -grading

$$C = C^+ \oplus C^-$$

inherited from the usual grading on the tensor algebra. The natural \mathbb{Q} -linear map $V \rightarrow C$ is injective, and we use it to regard $V \subset C^-$ as a subspace. The Clifford algebra has dimension $2^{\dim(V)}$ as a \mathbb{Q} -vector space, and is generated as a \mathbb{Q} -algebra by $V \subset C^-$.

The quadratic space (V, Q) has an associated *spinor similitude group*¹

$$G = \text{GSpin}(V).$$

It is an algebraic group over \mathbb{Q} with rational points

$$G(\mathbb{Q}) = \{g \in (C^+)^{\times} : gVg^{-1} = V\}.$$

¹In the terminology of [Shi10], this is the *even Clifford group*.

The group G acts on V via the *standard representation* $g \bullet v = gvg^{-1}$, and this action determines a short exact sequence of algebraic groups

$$1 \rightarrow \mathbb{G}_m \rightarrow G \xrightarrow{g \mapsto g \bullet} \mathrm{SO}(V) \rightarrow 1.$$

There is a unique \mathbb{Q} -linear involution $c \mapsto c^*$ on C characterized by

$$(v_1 \cdots v_r)^* = v_r^* \cdots v_1^*$$

for $v_1, \dots, v_r \in V$. One can show that every $g \in G(\mathbb{Q})$ satisfies $g^*g \in \mathbb{Q}^\times$, and the *spinor similitude*

$$\nu : G \rightarrow \mathbb{G}_m$$

is defined by $\nu(g) = g^*g$. Its kernel is the usual spin double cover of $\mathrm{SO}(V)$, and its restriction to the central \mathbb{G}_m sends $z \mapsto z^2$.

4.2. The orthogonal Shimura variety. We construct a Shimura variety from the data (V, Q) and the choice of maximal lattice $L \subset V$.

Using the standard representation $G \rightarrow \mathrm{SO}(V)$, the group of real points $G(\mathbb{R})$ acts on the n -dimensional hermitian symmetric domain

$$(4.2.1) \quad \mathcal{D} = \{z \in V_{\mathbb{C}} : [z, z] = 0, [z, \bar{z}] < 0\} / \mathbb{C}^\times \subset \mathbb{P}(V_{\mathbb{C}}).$$

There are two connected components $\mathcal{D} = \mathcal{D}^+ \sqcup \mathcal{D}^-$, interchanged by the action of any $\gamma \in G(\mathbb{R})$ with $\nu(\gamma) < 0$.

The pair (G, \mathcal{D}) is a Shimura datum with reflex field \mathbb{Q} . More precisely, we can realize

$$\mathcal{D} \subset \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$$

as a $G(\mathbb{R})$ -conjugacy class as follows. Given a point $z \in \mathcal{D}$, write $z = x + iy$ for vectors $x, y \in V_{\mathbb{R}}$. The span $\mathbb{R}x + \mathbb{R}y \subset V_{\mathbb{R}}$ is a negative definite plane, with its own Clifford algebra $C_z \subset C_{\mathbb{R}}$. If we identify $\mathbb{C} \cong C_z^+$ as \mathbb{R} -algebras using

$$i \mapsto \frac{xy}{\sqrt{Q(x)Q(y)}},$$

we obtain a homomorphism $\mathbb{C} \rightarrow C_{\mathbb{R}}^+$, which restricts to a homomorphism $h_z : \mathbb{C}^\times \rightarrow G(\mathbb{R})$. This defines an element $h_z \in \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$.

The choice of maximal lattice $L \subset V$ determines a compact open subgroup of $G(\mathbb{A}_f)$ as follows. Let

$$(4.2.2) \quad C_{\widehat{\mathbb{Z}}} \subset C_{\mathbb{A}_f}$$

be the $\widehat{\mathbb{Z}}$ -subalgebra generated by the profinite completion $\widehat{L} \subset V_{\mathbb{A}_f}$, and form the intersection

$$K = C_{\widehat{\mathbb{Z}}}^\times \cap G(\mathbb{A}_f)$$

inside of $C_{\mathbb{A}_f}^\times$. This compact open subgroup of $G(\mathbb{A}_f)$ stabilizes \widehat{L} , and acts trivially on the discriminant group

$$(4.2.3) \quad \widehat{L}^\vee / \widehat{L} \cong L^\vee / L.$$

Here L^\vee is the dual lattice of L relative to the bilinear form (4.0.1).

By Deligne’s theory of canonical models of Shimura varieties, the orbifold

$$M(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f) / K$$

is the space of complex points of a Deligne-Mumford stack M over \mathbb{Q} . Using the complex uniformization, one can check that M has dimension $n = \dim(V) - 2$.

4.3. The Kuga-Satake abelian scheme. There is a \mathbb{Q} -linear *reduced trace*

$$\text{Trd} : C \rightarrow \mathbb{Q}.$$

If $\dim(V) = 2m$ is even, the Clifford algebra C is a central simple \mathbb{Q} -algebra, and the \mathbb{Q}^{alg} -linear extension

$$M_{2m}(\mathbb{Q}^{\text{alg}}) \cong C \otimes_{\mathbb{Q}} \mathbb{Q}^{\text{alg}} \xrightarrow{\text{Trd}} \mathbb{Q}^{\text{alg}}$$

of the reduced trace is the usual trace on matrices. If $\dim(V) = 2m + 1$ is odd, the center of C is a quadratic étale \mathbb{Q} -algebra (so either $\mathbb{Q} \oplus \mathbb{Q}$ or a quadratic field extension), and C is a central simple algebra over its center. The \mathbb{Q}^{alg} -linear extension

$$M_{2m}(\mathbb{Q}^{\text{alg}}) \oplus M_{2m}(\mathbb{Q}^{\text{alg}}) \cong C \otimes_{\mathbb{Q}} \mathbb{Q}^{\text{alg}} \xrightarrow{\text{Trd}} \mathbb{Q}^{\text{alg}}$$

of the reduced trace is the sum of the usual traces on matrices.

Let $H = C$, viewed as a vector space over \mathbb{Q} . Choose vectors $e, f \in V$ such that $[e, f] = 0$, and

$$Q(e) < 0, \quad Q(f) < 0.$$

The element $\delta = ef \in C^\times$ satisfies $\delta^* = -\delta$, which allows us to define a symplectic form

$$\psi_\delta : H \otimes H \rightarrow \mathbb{Q}$$

by $\psi_\delta(h_1, h_2) = \text{Trd}(h_1 \delta h_2^*)$. We may rescale e and f in order to assume that ψ_δ is $\widehat{\mathbb{Z}}$ -valued on the $\widehat{\mathbb{Z}}$ -submodule

$$(4.3.1) \quad H_{\widehat{\mathbb{Z}}} \subset H_{\mathbb{A}_f}$$

determined by (4.2.2).

Letting $G(\mathbb{Q}) \subset C^\times$ act on H by left multiplication, we obtain a closed immersion of algebraic groups

$$G \rightarrow \text{GSp}(H),$$

under which the symplectic similitude on $\text{GSp}(H)$ restricts to the spinor similitude on G , and the action of $K \subset G(\mathbb{A}_f)$ stabilizes the $\widehat{\mathbb{Z}}$ -lattice (4.3.1). This closed immersion determines a morphism from (G, \mathcal{D}) to the Siegel Shimura datum determined by (H, ψ_δ) , and hence determines a morphism

$$M \rightarrow X$$

to the \mathbb{Q} -stack X parametrizing abelian varieties of dimension

$$2^{n+1} = \frac{\dim(H)}{2}$$

equipped with a polarization of some fixed degree (depending on ψ_δ). Pulling back the universal object over X , we obtain the *Kuga-Satake abelian scheme*

$$A \rightarrow M.$$

It carries a polarization that depends on the choice of ψ_δ , but the underlying abelian scheme is independent of the choice.

4.4. Automorphic vector bundles. Like any Shimura variety, M has a theory of automorphic vector bundles [Har84, Har85, Har86, Mil90].

For us this means the following: any representation $G \rightarrow \mathrm{GL}(N)$ on a finite dimensional \mathbb{Q} -vector space has a *de Rham* realization, which is vector bundle \mathbf{N}_{dR} on M endowed with a decreasing filtration $F^\bullet \mathbf{N}_{dR}$ by local direct summands. At a complex point $[z, g] \in M(\mathbb{C})$ the fiber of \mathbf{N}_{dR} is just the vector space $N_{\mathbb{C}}$ endowed with the Hodge filtration determined by

$$\mathbb{S} \xrightarrow{h_z} G(\mathbb{R}) \rightarrow \mathrm{GL}(N_{\mathbb{R}}).$$

The representation $G \rightarrow \mathrm{GL}(N)$ also has a *Betti realization*

$$\mathbf{N}_B = G(\mathbb{Q}) \backslash \mathcal{D} \times N \times G(\mathbb{A}_f) / K.$$

This is a local system of \mathbb{Q} -vector spaces on the complex fiber $M(\mathbb{C})$, which is related to the de Rham realization by a canonical isomorphism

$$\mathbf{N}_{dR}(\mathbb{C}) \cong \mathbf{N}_B \otimes_{\mathbb{Q}} \mathcal{O}_{M(\mathbb{C})}$$

of holomorphic vector bundles on the complex fiber. A choice of K -stable $\widehat{\mathbb{Z}}$ -lattice $N_{\widehat{\mathbb{Z}}} \subset N_{\mathbb{A}_f}$ determines a local subsystem of \mathbb{Z} -modules

$$\mathbf{N}_{B, \mathbb{Z}} \subset \mathbf{N}_B,$$

whose fiber at a complex point $[z, g] \in M(\mathbb{C})$ is the \mathbb{Z} -lattice $gN_{\widehat{\mathbb{Z}}} \cap N$ in the \mathbb{Q} vector space N . Of course the intersection here takes place in $N_{\mathbb{A}_f}$.

We apply these constructions to the representations

$$(4.4.1) \quad G \rightarrow \mathrm{SO}(V), \quad G \rightarrow \mathrm{GSp}(H)$$

to obtain vector bundles \mathbf{V}_{dR} and \mathbf{H}_{dR} on M , along with filtrations

$$0 = F^2 \mathbf{V}_{dR} \subsetneq F^1 \mathbf{V}_{dR} \subsetneq F^0 \mathbf{V}_{dR} \subsetneq F^{-1} \mathbf{V}_{dR} = \mathbf{V}_{dR}$$

and

$$0 = F^1 \mathbf{H}_{dR} \subsetneq F^0 \mathbf{H}_{dR} \subsetneq F^{-1} \mathbf{H}_{dR} = \mathbf{H}_{dR}.$$

The symmetric bilinear pairing on V induces a symmetric bilinear pairing

$$[\cdot, \cdot] : \mathbf{V}_{dR} \otimes \mathbf{V}_{dR} \rightarrow \mathcal{O}_M$$

under which $F^1 \mathbf{V}_{dR}$ is an isotropic line, and $F^0 \mathbf{V}_{dR} = (F^1 \mathbf{V}_{dR})^\perp$. Similarly, the symplectic form on H induces an alternating bilinear pairing

$$\psi_\delta : \mathbf{H}_{dR} \otimes \mathbf{H}_{dR} \rightarrow \mathcal{O}_M$$

under which $F^0 \mathbf{H}_{dR}$ is maximal isotropic. Among these vector bundles, the line bundle $F^1 \mathbf{V}_{dR}$ will play a distinguished role.

DEFINITION 4.4.1. The *line bundle of weight one modular forms* is

$$\omega = F^1 \mathbf{V}_{dR}.$$

The vector bundles \mathbf{V}_{dR} and \mathbf{H}_{dR} are closely related. In fact, we can identify

$$(4.4.2) \quad V \subset \text{End}(H)$$

as a G -stable subspace using the left multiplication action of $V \subset C$ on $C = H$. There is a corresponding inclusion

$$\mathbf{V}_{dR} \subset \underline{\text{End}}(\mathbf{H}_{dR})$$

as a local direct summand, respecting the natural filtrations. If x_1 and x_2 are local sections of \mathbf{V}_{dR} , viewed as endomorphisms of \mathbf{H}_{dR} , then

$$[x_1, x_2] = x_1 \circ x_2 + x_2 \circ x_1$$

as local sections of $\mathcal{O}_M \subset \underline{\text{End}}(\mathbf{H}_{dR})$.

At a complex point $s = [z, g] \in M(\mathbb{C})$, all of this can be made completely explicit in terms of the isotropic vector $z \in V_{\mathbb{C}}$. The filtration on $\mathbf{V}_{dR,s} = V_{\mathbb{C}}$ is given by

$$F^2 \mathbf{V}_{dR,s} = 0, \quad F^1 \mathbf{V}_{dR,s} = \mathbb{C}z, \quad F^0 \mathbf{V}_{dR,s} = (\mathbb{C}z)^\perp, \quad F^{-1} \mathbf{V}_{dR,s} = V_{\mathbb{C}},$$

while the filtration on $\mathbf{H}_{dR,s} = H_{\mathbb{C}}$ is given by

$$F^1 \mathbf{H}_{dR,s} = 0, \quad F^0 \mathbf{H}_{dR,s} = zH_{\mathbb{C}}, \quad F^{-1} \mathbf{H}_{dR,s} = H_{\mathbb{C}}.$$

In particular, suppose we pull back the line bundle of weight one modular forms $\omega = F^1 \mathbf{H}_{dR}$ via the complex uniformization

$$\mathcal{D} \xrightarrow{z \rightarrow [z,g]} M(\mathbb{C})$$

for some fixed $g \in G(\mathbb{A}_f)$. The pullback is just the restriction to $\mathcal{D} \subset \mathbb{P}(V_{\mathbb{C}})$ of the tautological bundle on projective space. In other words, the fiber of the line bundle ω at a complex point $[z, g] \in M(\mathbb{C})$ is the isotropic line

$$\omega_{[z,g]} = \mathbb{C}z \subset V_{\mathbb{C}}.$$

We can use this identification to define the *Petersson metric* on $\omega_{[z,g]}$ by

$$(4.4.3) \quad \|z\|^2 = -[z, \bar{z}].$$

4.5. Special divisors. We now define a family of divisors on M , following Kudla [Kud97a, Kud04]. These divisors also play a prominent role in work of Borcherds [Bor98] and Bruinier [Bru02].

For $x \in V$ of positive length, define an analytic divisor on (4.2.1) by

$$\mathcal{D}_x = \{z \in \mathcal{D} : z \perp x\}.$$

For every positive $m \in \mathbb{Q}$ and every $\mu \in L^\vee/L$ we define a complex orbifold

$$(4.5.1) \quad Z(m, \mu)(\mathbb{C}) = \bigsqcup_{g \in G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K} \Gamma_g \backslash \left(\bigsqcup_{\substack{x \in \mu_g + L_g \\ Q(x)=m}} \mathcal{D}_x \right).$$

Here we have set

$$\Gamma_g = G(\mathbb{Q}) \cap gKg^{-1}.$$

Recalling that the action $G \rightarrow \mathrm{SO}(V)$ is denoted $g \mapsto g\bullet$, the \mathbb{Z} -lattice $L_g \subset V$ is defined by

$$(4.5.2) \quad \widehat{L}_g = g\bullet\widehat{L},$$

and we have set

$$\mu_g = g\bullet\mu \in L_g^\vee/L_g.$$

We now use the Kuga-Satake abelian scheme $A \rightarrow M$ of §4.3 to give a more moduli-theoretic interpretation of (4.5.1). As explained in §4.4, the $\widehat{\mathbb{Z}}$ -lattice (4.3.1) determines a local system of \mathbb{Z} -modules $\mathbf{H}_{B,\mathbb{Z}}$ on $M(\mathbb{C})$, along with an isomorphism

$$\mathbf{H}_{B,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{M(\mathbb{C})} \cong \mathbf{H}_{dR}(\mathbb{C})$$

of holomorphic vector bundles on $M(\mathbb{C})$. The Kuga-Satake abelian scheme over $M(\mathbb{C})$ can be identified with the analytic family of complex tori

$$(4.5.3) \quad A(\mathbb{C}) = \mathbf{H}_{B,\mathbb{Z}} \backslash \mathbf{H}_{dR}(\mathbb{C}) / F^0 \mathbf{H}_{dR}(\mathbb{C}).$$

Similarly, the K -stable $\widehat{\mathbb{Z}}$ -lattices $\widehat{L} \subset \widehat{L}^\vee$ in $V_{\mathbb{A}_f}$ determine local systems of \mathbb{Z} -modules

$$\mathbf{L}_B \subset \mathbf{L}_B^\vee$$

inside the local system of \mathbb{Q} -modules \mathbf{V}_B determined by $G \rightarrow \mathrm{SO}(V)$. Using the fact that K acts trivially on the discriminant group (4.2.3), one obtains a canonical isomorphism of local systems of abelian groups

$$(4.5.4) \quad \mathbf{L}_B^\vee/\mathbf{L}_B \cong (L^\vee/L) \otimes_{\mathbb{Z}} \underline{\mathbb{Z}}.$$

Suppose we have a complex point $s \in M(\mathbb{C})$, and an quasi-endomorphism

$$x \in \mathrm{End}(A_s)_{\mathbb{Q}}$$

of the fiber of the Kuga-Satake abelian scheme. Using (4.5.3), one obtains an induced endomorphism $x_B \in \mathrm{End}_{\mathbb{Q}}(\mathbf{H}_{B,s})$, called the *Betti realization* of x . We say that x is *special* if its Betti realization lies in the \mathbb{Q} -subspace

$$\mathbf{V}_{B,s} \subset \mathrm{End}_{\mathbb{Q}}(\mathbf{H}_{B,s}).$$

The space

$$V(A_s) = \{x \in \mathrm{End}(A_s) : x \text{ is special}\}$$

of all special endomorphisms of A_s is a free \mathbb{Z} -module of finite rank, endowed with a positive definite \mathbb{Z} -valued quadratic form characterized by the equality $Q(x) = x \circ x$ in $\mathrm{End}(A_s)$.

REMARK 4.5.1. The rank of $V(A_s)$ behaves erratically as $s \in M(\mathbb{C})$ varies. It can be as small as 0, or as large as $\dim(V) - 1 = n + 1$.

More generally, for any $\mu \in L^\vee/L$, denote by

$$V_\mu(A_s) \subset \text{End}(A_s)_\mathbb{Q}$$

the set of special quasi-endomorphisms x whose Betti realizations $x_B \in \mathbf{V}_{B,s}$ lie in the \mathbb{Z} -submodule $\mathbf{L}_{B,s}^\vee$, and which are sent to μ under the isomorphism

$$\mathbf{L}_{B,s}^\vee/\mathbf{L}_{B,s} \cong L^\vee/L$$

induced by (4.5.4). In particular

$$V_\mu(A_s) \subset V(A_s)_\mathbb{Q}.$$

Taking $\mu = 0$ recovers $V(A_s)$.

The proof of the following proposition is a straightforward exercise.

PROPOSITION 4.5.2. *For any positive rational number m and any $\mu \in L^\vee/L$, the set of (isomorphism classes of) complex points $Z(m, \mu)(\mathbb{C})$ is in bijection with the set of pairs (s, x) consisting of a point $s \in M(\mathbb{C})$, and a special quasi-endomorphism $x \in V_\mu(A_s)$ such that $Q(x) = m$.*

The orbifold (4.5.1) can be realized as the space of complex points of a smooth Deligne-Mumford stack over \mathbb{Q} of dimension $n - 1$, endowed with a finite and unramified morphism

$$Z(m, \mu) \rightarrow M.$$

This is a special case of the more general constructions of §5.4.

Even though this morphism is not a closed immersion, one can construct from it a Cartier divisor on M as follows. Every geometric point of M admits an étale neighborhood $U \rightarrow M$ small enough that the pullback $Z(m, \mu)_U \rightarrow M_U$ restricts to a closed immersion on every connected component of its domain, and each such closed immersion is defined locally by a single nonzero equation. Thus every connected component of $Z(m, \mu)_U$ determines an effective Cartier divisor on M_U . Summing over all connected components, and then glueing over an étale cover, defines an effective Cartier divisor on M . We will refer to both the stacks $Z(m, \mu)$ and their associated Cartier divisors as *special divisors* on M .

4.6. The connection to Faltings heights. We want to explain how the line bundle ω , endowed with its Petersson metric (4.4.3), can be used to compute Faltings heights of fibers of the Kuga-Satake abelian scheme

$$A \rightarrow M,$$

in much the same way that the line bundle of modular forms on the usual modular curve computes Faltings heights of elliptic curves (2.3.2).

There is a short exact sequence

$$(4.6.1) \quad 0 \rightarrow F^0 \mathbf{H}_{dR} \rightarrow \mathbf{H}_{dR} \rightarrow \text{Lie}(A) \rightarrow 0$$

of vector bundles on M , whose fiber at every complex point $y \in M(\mathbb{C})$ is canonically identified with the \mathbb{C} -linear dual of the Hodge short exact sequence

$$0 \rightarrow H^0(A_y, \Omega_{A_y/\mathbb{C}}) \rightarrow H_{dR}^1(A_y/\mathbb{C}) \rightarrow H^1(A_y, \mathcal{O}_{A_y}) \rightarrow 0.$$

We will use this to relate ω to the determinant of the dual Lie algebra

$$\mathrm{Lie}(A)^\vee \cong \pi_* \Omega_{A/M}.$$

PROPOSITION 4.6.1. *There is an isomorphism of line bundles*

$$\omega^{\otimes \dim(A)} \cong \det(\pi_* \Omega_{A/M})^{\otimes 2} \otimes \det(\mathbf{H}_{dR}).$$

PROOF. Recalling that

$$\omega = F^1 \mathbf{V}_{dR} \subset F^1 \underline{\mathrm{End}}(\mathbf{H}_{dR}) \cong \underline{\mathrm{Hom}}(\mathbf{H}_{dR}/F^0 \mathbf{H}_{dR}, F^0 \mathbf{H}_{dR}),$$

there is a short exact sequence

$$0 \rightarrow \omega \otimes F^0 \mathbf{H}_{dR} \rightarrow \omega \otimes \mathbf{H}_{dR} \xrightarrow{x \otimes a \rightarrow xa} F^0 \mathbf{H}_{dR} \rightarrow 0$$

of vector bundles on M . Taking determinants yields an isomorphism

$$(4.6.2) \quad \omega^{\otimes \dim(A)} \otimes \det(\mathbf{H}_{dR}) \cong \det(F^0 \mathbf{H}_{dR})^{\otimes 2}.$$

Explicitly, if we set $r = 2^{n+1}$, let a_1, \dots, a_r be a local basis of $F^0 \mathbf{H}_{dR}$, and extend it to a local basis $a_1, \dots, a_r, b_1, \dots, b_r$ of \mathbf{H}_{dR} , then

$$x^{\otimes \dim(A)} \otimes (a_1 \wedge \dots \wedge a_r \wedge b_1 \wedge \dots \wedge b_r) \mapsto (a_1 \wedge \dots \wedge a_r) \otimes (xb_1 \wedge \dots \wedge xb_r)$$

for any local generator x of $\omega \subset \mathbf{V}_{dR} \subset \underline{\mathrm{End}}(\mathbf{H}_{dR})$.

On the other hand, taking determinants in (4.6.1) yields an isomorphism

$$\det(\mathbf{H}_{dR}) \cong \det(F^0 \mathbf{H}_{dR}) \otimes \det(\mathrm{Lie}(A)),$$

which we rewrite as

$$\det(F^0 \mathbf{H}_{dR}) \cong \det(\mathbf{H}_{dR}) \otimes \det(\pi_* \Omega_{A/M}).$$

Substituting this expression into the right hand side of (4.6.2) gives the desired isomorphism. \square

As in the introduction, $a \doteq b$ means that $a - b$ is a \mathbb{Q} -linear combination of $\{\log(p) : p \text{ prime}\}$.

THEOREM 4.6.2. *Suppose \mathbf{k} is a number field, and $y \in M(\mathbf{k})$. If $v \in \omega_y$ is any nonzero vector, then*

$$\frac{-1}{[\mathbf{k} : \mathbb{Q}]} \sum_{\sigma: \mathbf{k} \rightarrow \mathbb{C}} \log \|v^\sigma\| \doteq 2^{-n} h^{\mathrm{Falt}}(A_y) + \log(2\pi).$$

PROOF. For any line bundle \mathcal{L} on M endowed with a metric $\|\cdot\|$ on the complex fiber $M(\mathbb{C})$, and any complex point $y \in M(\mathbf{k})$, abbreviate

$$\log \|\mathcal{L}_y\| = \frac{1}{[\mathbf{k} : \mathbb{Q}]} \sum_{\sigma: \mathbf{k} \rightarrow \mathbb{C}} \log \|v^\sigma\|$$

for any nonzero vector $v \in \mathcal{L}_y$. Of course this depends on the choice of v , but different choices change the value by a \mathbb{Q} -linear combination of $\log(p)$, yielding a well-defined element

$$\log \|\mathcal{L}_y\| \in \mathbb{R}/\mathbb{Z} \doteq.$$

For a complex point $y \in M(\mathbb{C})$, the *Faltings metric* on the fiber at y of $\det(\pi_*\Omega_{A/M})$ is defined by

$$\|\eta\|^2 = \left| \int_{A_y(\mathbb{C})} \eta \wedge \bar{\eta} \right|$$

for any top degree global holomorphic form $\eta \in \det(\pi_*\Omega_{A_y/\mathbb{C}})$ on $A_y(\mathbb{C})$. This makes $\det(\pi_*\Omega_{A/M})$ into a metrized line bundle.

The local system

$$\mathbf{H}_{B,\mathbb{Z}} \subset \mathbf{H}_{B,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{M(\mathbb{C})} \cong \mathbf{H}_{dR,M(\mathbb{C})}$$

of \mathbb{Z} -modules determines a rank 1 local system

$$\det(\mathbf{H}_{B,\mathbb{Z}}) \subset \det(\mathbf{H}_{dR,M(\mathbb{C})}),$$

and we define the *volume metric* on $\det(\mathbf{H}_{dR})$ by declaring that $\|e\|^2 = 1$ for any local generator e of $\det(\mathbf{H}_{B,\mathbb{Z}})$. At a complex point $y \in M(\mathbb{C})$ the dual volume metric on

$$\det(\mathbf{H}_{dR,y}^\vee) \cong \det(H_{dR}^1(A_y/\mathbb{C})) \cong H_{dR}^{2\dim(A_y)}(A_y/\mathbb{C})$$

is just integration of top degree forms:

$$\|u\|^2 = \left| \int_{A_y(\mathbb{C})} u \right|^2.$$

LEMMA 4.6.3. *If \mathbf{k} is a number field and $y \in M(\mathbf{k})$, then*

$$\log \|\det(\mathbf{H}_{dR})_y\| \doteq -\dim(A) \cdot \log(2\pi)$$

and

$$\log \|\det(\pi_*\Omega_{A/M})_y\| \doteq -h^{\text{Falt}}(A_y).$$

PROOF. For the first claim, suppose we have a nonzero vector

$$u \in \det(\mathbf{H}_{dR,y}^\vee) \cong \det(H_{dR}^1(A_y/\mathbf{k})).$$

As in [DMOS82, page 22], there is a scalar $\text{Tr}_{dR}(u) \in \mathbf{k}$ such that

$$\frac{1}{(2\pi i)^{\dim(A)}} \int_{A_{y,\sigma}(\mathbb{C})} u^\sigma = \sigma(\text{Tr}_{dR}(u)),$$

for every $\sigma : \mathbf{k} \rightarrow \mathbb{C}$. This implies that

$$\begin{aligned} \log \|\det(\mathbf{H}_{dR,y}^\vee)\| &= \frac{1}{[\mathbf{k} : \mathbb{Q}]} \sum_{\sigma : \mathbf{k} \rightarrow \mathbb{C}} \log \|u^\sigma\| \\ &= \dim(A) \cdot \log(2\pi) - \frac{1}{[\mathbf{k} : \mathbb{Q}]} \log |\text{Nm}_{\mathbf{k}/\mathbb{Q}}(\text{Tr}_{dR}(u))| \\ &\doteq \dim(A) \cdot \log(2\pi), \end{aligned}$$

and the first claim follows.

For the second claim, directly from the definition (3.2.1), we compute

$$\begin{aligned} \log \|\det(\pi_*\Omega_{A/M})_y\| &= \frac{1}{2[\mathbf{k}:\mathbb{Q}]} \sum_{\sigma:\mathbf{k}\rightarrow\mathbb{C}} \log \left| \int_{A^\sigma(\mathbb{C})} \eta^\sigma \wedge \overline{\eta^\sigma} \right| \\ &= -h_\infty^{\text{Falt}}(A, \eta) \end{aligned}$$

for $\eta \in \det(\pi_*\Omega_{A_y/\mathbf{k}})$ any nonzero vector. Now use $h_f^{\text{Falt}}(A, \eta) \stackrel{\circ}{=} 0$. □

We can now complete the proof of Theorem 4.6.2. It is an exercise in linear algebra to show that the isomorphism of Proposition 4.6.1 respects the metrics, and hence

$$\dim(A) \cdot \log \|\omega_y\| \stackrel{\circ}{=} 2 \cdot \log \|\det(\pi_*\Omega_{A/M})_y\| + \log \|\det(\mathbf{H}_{dR})_y\|$$

Combining this with the lemmas shows that

$$\dim(A) \cdot \log \|\omega_y\| \stackrel{\circ}{=} -2 \cdot h^{\text{Falt}}(A_y) - \dim(A) \cdot \log(2\pi).$$

To complete the proof of the theorem, recall that $\dim(A) = 2^{n+1}$. □

5. Integral models

As in §4, let (V, Q) be a quadratic space over \mathbb{Q} of signature $(n, 2)$ with $n \geq 1$, and let $L \subset V$ be a maximal lattice. Let M be the associated orthogonal Shimura variety of §4.2.

5.1. Integral models in the self-dual case. Assume that $L^\vee = L$.

Recall that the constructions of §4.3 provide us with a morphism

$$M \rightarrow X,$$

where X is the Deligne-Mumford stack over \mathbb{Q} parametrizing abelian schemes of dimension 2^{n+1} endowed with a polarization of some fixed degree (depending on the choice of ψ_δ). By virtue of its definition as a moduli space, it has a natural extension to a stack \mathcal{X} over \mathbb{Z} , and we define \mathcal{M} to be the normalization² of \mathcal{X} in M . By definition, \mathcal{M} is a normal Deligne-Mumford stack sitting in a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathcal{X}. \end{array}$$

The Kuga-Satake abelian scheme of §4.3 extends to an abelian scheme

$$\mathcal{A} \rightarrow \mathcal{M}$$

in an obvious way: simply pull back the universal object over \mathcal{X} .

THEOREM 5.1.1 (Kisin [Kis10], Kim-Madapusi Pera [KM16]). *The integral model \mathcal{M} is smooth over \mathbb{Z} , and is independent of the choice of ψ_δ . Moreover, there is a theory of automorphic vector bundles on \mathcal{M} , extending the theory in the generic fiber described in §4.4.*

²See the Stacks project, §28.51 on relative normalization.

In particular, the final claim of the theorem provides us with filtered vector bundles \mathbf{V}_{dR} and \mathbf{H}_{dR} on \mathcal{M} extending those already constructed in the generic fiber, along with an inclusion

$$\mathbf{V}_{dR} \subset \underline{\text{End}}(\mathbf{H}_{dR}).$$

A distinguished role will be played by the *line bundle of weight one modular forms*

$$\omega = F^1 \mathbf{V}_{dR}.$$

REMARK 5.1.2. Theorem 5.1.1 remains true even if the assumption $L^\vee = L$ is relaxed slightly. The important thing is that the compact open subgroup $K = \prod K_p$ be hyperspecial at every prime p .

- When $p > 2$, K_p is hyperspecial if and only if $L_p^\vee = L_p$.
- When n is even, K_2 is hyperspecial if and only if $L_2^\vee = L_2$.
- When n is odd the inclusion $L_2 \subset L_2^\vee$ is always proper, and K_2 is hyperspecial if and only if $[L_2^\vee : L_2] = 2$.

Thus our hypothesis $L^\vee = L$ implies that n is even, and could have been weakened to

$$[L^\vee : L] = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

We won't ever need this stronger version of Theorem 5.1.1.

5.2. Special divisors in the self-dual case. Keep $L = L^\vee$ as above.

Suppose S is any scheme equipped with a morphism $S \rightarrow \mathcal{M}$, and let $\mathcal{A}_S \rightarrow S$ be the pullback of the Kuga-Satake abelian scheme. As in §4.5, one can define a \mathbb{Z} -submodule

$$V(\mathcal{A}_S) \subset \text{End}(\mathcal{A}_S)$$

of *special endomorphisms*, endowed with a positive definite \mathbb{Z} -valued quadratic form Q satisfying $x \circ x = Q(x) \cdot \text{id}$. The definition of a special endomorphism is now more subtle, as the Betti realization x_B used in §4.5 to define $V(\mathcal{A}_S)$ when $S = \text{Spec}(\mathbb{C})$ is not available for arbitrary S . Instead, one can use de Rham, ℓ -adic, and crystalline cohomology in conjunction.

Briefly, the vector bundle $\mathbf{H}_{dR,S}$ on S can be identified with the \mathcal{O}_S -linear dual of the first relative de Rham cohomology of \mathcal{A}_S , and so there is a canonical homomorphism

$$\text{End}(\mathcal{A}_S) \xrightarrow{x \mapsto x_{dR}} \text{End}(\mathbf{H}_{dR,S})$$

called *de Rham realization*. As we have already noted, (4.4.2) induces an inclusion of vector bundles

$$\mathbf{V}_{dR,S} \subset \underline{\text{End}}(\mathbf{H}_{dR,S}).$$

We say that $x \in \text{End}(\mathcal{A}_S)$ is *de Rham special* if its de Rham realization lies in the submodule

$$H^0(S, \mathbf{V}_{dR,S}) \subset \text{End}(\mathbf{H}_{dR,S}).$$

Now fix a prime ℓ . The representations (4.4.1) determine lisse étale sheaves $\mathbf{V}_{\ell,S[1/\ell]}$ and $\mathbf{H}_{\ell,S[1/\ell]}$ of \mathbb{Q}_ℓ -modules over

$$S[1/\ell] = S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[1/\ell]),$$

related by an inclusion

$$\mathbf{V}_{\ell,S[1/\ell]} \subset \underline{\mathrm{End}}(\mathbf{H}_{\ell,S[1/\ell]}).$$

The sheaf $\mathbf{H}_{\ell,S[1/\ell]}$ is just the ℓ -adic Tate module of $\mathcal{A}_{S[1/\ell]}$, and so there is an ℓ -adic realization map

$$\mathrm{End}(\mathcal{A}_S) \xrightarrow{x \mapsto x_\ell} \mathrm{End}(\mathbf{H}_{\ell,S[1/\ell]}).$$

We say that $x \in \mathrm{End}(\mathcal{A}_S)$ is ℓ -special if its ℓ -adic realization x_ℓ lies in the submodule

$$H^0(S, \mathbf{V}_{\ell,S[1/\ell]}) \subset \mathrm{End}(\mathbf{H}_{\ell,S[1/\ell]}).$$

Now fix a prime p and set $S_p = S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p)$. From the representations (4.4.1) one can construct crystals of \mathcal{O}_{S_p} -modules $\mathbf{V}_{\mathrm{crys},S_p}$ and $\mathbf{H}_{\mathrm{crys},S_p}$, along with an inclusion

$$\mathbf{V}_{\mathrm{crys},S_p} \subset \underline{\mathrm{End}}(\mathbf{H}_{\mathrm{crys},S_p})$$

and a crystalline realization map

$$\mathrm{End}(\mathcal{A}_S) \xrightarrow{x \mapsto x_{\mathrm{crys}}} \mathrm{End}(\mathbf{H}_{\mathrm{crys},S_p}).$$

We then define $x \in \mathrm{End}(\mathcal{A}_S)$ to be *crystalline special* if its crystalline realization lies in the submodule

$$H^0(S, \mathbf{V}_{\mathrm{crys},S_p}) \subset \mathrm{End}(\mathbf{H}_{\mathrm{crys},S_p})$$

for every prime p .

Finally, one defines $x \in \mathrm{End}(\mathcal{A}_S)$ to be *special* if it is de Rham special, ℓ -special for every prime ℓ , and crystalline special. This definition may seem unwieldy, but it is simplified by the fact that specialness is an extremely rigid property. If S is connected, and if there exists a geometric point $s \rightarrow S$ such that the restriction of $x : \mathcal{A}_S \rightarrow \mathcal{A}_S$ to the fiber $x_s : \mathcal{A}_s \rightarrow \mathcal{A}_s$ is special, then x itself is special. See [AGHMP18, §4.3].

DEFINITION 5.2.1. For any positive integer m , the *special divisor*

$$\mathcal{Z}(m) \rightarrow \mathcal{M}$$

is the \mathcal{M} -stack with functor of points

$$\mathcal{Z}(m)(S) = \{x \in V(\mathcal{A}_S) : Q(x) = m\}$$

for any \mathcal{M} -scheme $S \rightarrow \mathcal{M}$.

5.3. Integral models in the maximal case. We now drop the assumption $L^\vee = L$, and return to the general case of a maximal lattice $L \subset V$.

Choose an isometric embedding of V into a quadratic space V^\diamond of signature $(n^\diamond, 2)$, and do this in such a way that L is contained in a self-dual lattice $L^\diamond \subset V^\diamond$. The isometric embedding $V \rightarrow V^\diamond$ induces a homomorphism of Clifford algebras, which restricts to a morphism on the associated groups of spinor similitudes. We obtain a morphism of the associated Shimura data, and hence a (finite and unramified) morphism $M \rightarrow M^\diamond$ of the associated Shimura varieties over \mathbb{Q} .

The construction of §5.1 provides us with a smooth integral model \mathcal{M}^\diamond of M^\diamond , and a Kuga-Satake abelian scheme $\mathcal{A}^\diamond \rightarrow \mathcal{M}^\diamond$. If we now define \mathcal{M} as the normalization of \mathcal{M}^\diamond in M , then \mathcal{M} is a normal Deligne-Mumford stack, flat over \mathbb{Z} with generic fiber M , sitting in a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & M^\diamond \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathcal{M}^\diamond. \end{array}$$

The Kuga-Satake abelian scheme $A \rightarrow M$ extends uniquely to an abelian scheme

$$\mathcal{A} \rightarrow \mathcal{M}.$$

Having already constructed a line bundle ω^\diamond of weight one modular forms on \mathcal{M}^\diamond , we can pull it back to \mathcal{M} to obtain a line bundle ω on \mathcal{M} . There is no reason to expect that \mathcal{M} admits any reasonable theory of automorphic vector bundles, extending the theory in the generic fiber, but the line bundle ω will suffice for our purposes. By endowing ω with the Petersson metric (4.4.3) in the complex fiber, we obtain a metrized line bundle

$$(5.3.1) \quad \widehat{\omega} \in \widehat{\text{Pic}}(\mathcal{M}),$$

as in §2.2.

For a proof of the following, see [AGHMP18, Proposition 4.4.1].

PROPOSITION 5.3.1. *The integral model \mathcal{M} and its line bundle ω do not depend on the choice of L^\diamond used in their construction, and the Kuga-Satake abelian scheme $A \rightarrow M$ extends uniquely to an abelian scheme $\mathcal{A} \rightarrow \mathcal{M}$.*

5.4. Special divisors in the maximal case. Fix a rational number $m > 0$, a coset $\mu \in L^\vee/L$, and an \mathcal{M} -scheme $S \rightarrow \mathcal{M}$.

Consider the Kuga-Satake abelian scheme

$$\mathcal{A}^\diamond \rightarrow \mathcal{M}^\diamond$$

associated with the self-dual lattice L^\diamond used in the construction of \mathcal{M} . Viewing S as an \mathcal{M}^\diamond -scheme using $\mathcal{M} \rightarrow \mathcal{M}^\diamond$, we have already defined a quadratic space of special endomorphisms $V(\mathcal{A}_S^\diamond)$ in §5.2. According to [AGHM17, Proposition 2.5.1] there is a canonical isometric embedding

$$\Lambda \rightarrow V(\mathcal{A}_S^\diamond),$$

where $\Lambda = \{\lambda \in L^\diamond : \lambda \perp L\}$. This allows us to define

$$(5.4.1) \quad V(\mathcal{A}_S) = \{x \in V(\mathcal{A}_S^\diamond) : x \perp \Lambda\}.$$

Of course $V(\mathcal{A}_S)$ inherits from $V(\mathcal{A}_S^\diamond)$ a positive definite quadratic form Q .

In fact, one can realize

$$V(\mathcal{A}_S) \subset \text{End}(\mathcal{A}_S)$$

in such way that $x \circ x = Q(x) \cdot \text{id}$. One first shows that \mathcal{A} comes equipped with a natural right action of the integral Clifford algebra $C(L)$, and similarly \mathcal{A}^\diamond comes with a right action of $C(L^\diamond)$. The isometric embedding $L \rightarrow L^\diamond$ induces a ring homomorphism $C(L) \rightarrow C(L^\diamond)$, and there is a $C(L^\diamond)$ -linear isomorphism

$$(5.4.2) \quad \mathcal{A}_S \otimes_{C(L)} C(L^\diamond) \cong \mathcal{A}_S^\diamond,$$

where the left hand side is Serre’s tensor construction. One can then identify (5.4.1) with the \mathbb{Z} -module of all $C(L)$ -linear endomorphisms of \mathcal{A}_S such that the induced endomorphism $x \otimes \text{id}$ of (5.4.2) lies in $V(\mathcal{A}_S^\diamond)$. For all of this see [AGHM17, §2.5].

Using the self-duality of L^\diamond , one can easily check that the projections to the two factors of $V \oplus \Lambda_{\mathbb{Q}} \cong V^\diamond$ induce isomorphisms

$$\begin{array}{ccc} & L^\diamond / (L \oplus \Lambda) & \\ & \swarrow & \searrow \\ L^\vee / L & & \Lambda^\vee / \Lambda, \end{array}$$

which allow us to view $\mu \in \Lambda^\vee / \Lambda$. Using this and

$$V(\mathcal{A}_S)_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}} \cong V(\mathcal{A}_S^\diamond)_{\mathbb{Q}},$$

we define

$$V_\mu(\mathcal{A}_S) = \{x \in V(\mathcal{A}_S)_{\mathbb{Q}} : x + \mu \in V(\mathcal{A}_S^\diamond)_{\mathbb{Q}}\}.$$

Taking $\mu = 0$ recovers $V(\mathcal{A}_S)$.

DEFINITION 5.4.1. For any rational number $m > 0$ and any $\mu \in L^\vee / L$, the *special divisor*

$$(5.4.3) \quad \mathcal{Z}(m, \mu) \rightarrow \mathcal{M}$$

is the \mathcal{M} -stack with functor of points

$$\mathcal{Z}(m, \mu)(S) = \{x \in V_\mu(\mathcal{A}_S) : Q(x) = m\}.$$

The stack $\mathcal{Z}(m, \mu)$ has dimension $\dim(\mathcal{M}) - 1$, and the map (5.4.3) is finite and unramified. Its complex points can be identified with (4.5.1) in a natural way. Although (5.4.3) is not a closed immersion, one can construct from it an effective Cartier divisor on \mathcal{M} , as explained in the discussion following Proposition 4.5.2. See also [AGHM17, §2.7] for more details.

6. Regularized theta lifts and Borcherds products

Keep the notation of §4. Thus (V, Q) is a quadratic space of signature $(n, 2)$ with $n \geq 1$, and \mathcal{M} is the orthogonal Shimura variety over \mathbb{Z} determined by a maximal lattice $L \subset V$.

6.1. Regularized theta lifts. One can construct Green functions for certain linear combinations of special divisors $\mathcal{Z}(m, \mu)$ on \mathcal{M} using the theory of regularized theta lifts. The construction is due to the physicists Harvey and Moore, and was used by Borcherds in [Bor98] to simplify and extend the construction of Borcherds products first introduced in [Bor95]. The theory was then further extended by Bruinier [Bru02].

Recall that the *metaplectic double cover*

$$\widetilde{\mathrm{SL}}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z})$$

is the group of all pairs (g, ϕ) in which

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and $\phi : \mathcal{H} \rightarrow \mathbb{C}$ is a holomorphic function satisfying $\phi(\tau)^2 = c\tau + d$. Multiplication is given by

$$(g_1, \phi_1(\tau)) \cdot (g_2, \phi_2(\tau)) = (g_1 g_2, \phi_1(g_2 \tau) \phi_2(\tau)).$$

The finite dimensional vector space $S_L = \mathbb{C}[L^\vee/L]$ of \mathbb{C} -valued functions on L^\vee/L admits an action

$$(6.1.1) \quad \omega_L : \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \rightarrow \mathrm{Aut}(S_L)$$

of the metaplectic double cover $\mathrm{SL}_2(\mathbb{Z})$, called the *Weil representation*. As in [AGHM17, Remark 3.1.1], our ω_L is the complex conjugate of the representation ρ_L used by Borcherds [Bor98]. If n is even, as it will be in our application to Colmez's conjecture, then ω_L factors through $\mathrm{SL}_2(\mathbb{Z})$.

Suppose $f : \mathcal{H} \rightarrow S_L$ is a holomorphic function satisfying

$$f(g\tau) = \phi(\tau)^{2-n} \omega_L(g, \phi) f(\tau)$$

for every $(g, \phi) \in \widetilde{\mathrm{SL}}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$. Any such f admits a Fourier expansion

$$f(\tau) = \sum_{m \in \mathbb{Q}} c(m) q^{-m},$$

with coefficients $c(m) \in S_L$, and each coefficient can be expanded uniquely as a linear combination

$$c(m) = \sum_{\mu \in L^\vee/L} c(m, \mu) \cdot \varphi_\mu,$$

where $\varphi_\mu \in S_L$ is the characteristic function of μ .

If $f(\tau)$ is meromorphic at ∞ , in the sense that $c(m)$ vanishes for all sufficiently negative m , we call $f(\tau)$ a *weakly holomorphic modular form* of weight $1 - \frac{n}{2}$ and representation ω_L . The infinite dimensional \mathbb{C} -vector

space space of all weakly holomorphic modular forms of weight $1 - \frac{n}{2}$ and representation ω_L is denoted $M_{1-\frac{n}{2}}^1(\omega_L)$.

It is a theorem of McGraw [McG03] that $M_{1-\frac{n}{2}}^1(\omega_L)$ admits a \mathbb{C} -basis of forms for which all $c(m, \mu)$ are integers, and we fix one such form

$$(6.1.2) \quad f(\tau) = \sum_{m \gg -\infty} c(m)q^{-m} \in M_{1-\frac{n}{2}}^1(\omega_L).$$

If $c(m, \mu) \neq 0$ then $m + Q(\mu) \in \mathbb{Z}$, and in particular

$$m \in \frac{1}{[L^\vee : L]} \cdot \mathbb{Z}.$$

It follows that the \mathbb{Z} -linear combination of special divisors

$$\mathcal{Z}(f) = \sum_{m > 0} \sum_{\mu \in L^\vee / L} c(-m, \mu) \cdot \mathcal{Z}(m, \mu)$$

is actually a finite sum, and so defines a Cartier divisor on \mathcal{M} . We will use the theory of regularized theta lifts to construct a Green function $\Phi(f)$ for this divisor.

Suppose $z \in \mathcal{D}$, so that $z \in V_{\mathbb{C}}$ is an isotropic vector with $[z, \bar{z}] < 0$. If we decompose $z = x + iy$ with $x, y \in V_{\mathbb{R}}$, then $x \perp y$ and $Q(x) = Q(y)$ is negative. Let $P_z \subset V_{\mathbb{R}}$ be the oriented negative definite plane spanned by the ordered basis x, y . It is an easy exercise to check that $z \mapsto P_z$ defines an isomorphism

$$\mathcal{D} \cong \{\text{oriented negative definite planes in } V_{\mathbb{R}}\}$$

of smooth manifolds. Define

$$\text{pr}_z : V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}, \quad \text{pr}_z^\perp : V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$$

to be the orthogonal projections to P_z and P_z^\perp , respectively.

For every $g \in G(\mathbb{A}_F)$ the associated lattice L_g of (4.5.2) comes with an isomorphism

$$L^\vee / L \xrightarrow{g^\bullet} L_g^\vee / L_g,$$

and hence every $\varphi \in S_L$ determines a function $\varphi_g : L_g^\vee / L_g \rightarrow \mathbb{C}$. Define the *theta kernel*

$$\theta_L : \mathcal{H} \times \mathcal{D} \times G(\mathbb{A}_f) \rightarrow S_L^\vee$$

by

$$\theta_L(\tau, z, g, \varphi) = v \sum_{x \in L_g^\vee} \varphi_g(x) \cdot e^{2\pi i \tau \cdot Q(\text{pr}_z^\perp(x))} \cdot e^{2\pi i \bar{\tau} \cdot Q(\text{pr}_z(x))}.$$

The theta kernel descends to a function

$$\theta_L : \mathcal{H} \times \mathcal{M}(\mathbb{C}) \rightarrow S_L^\vee,$$

which transforms in the variable $\tau \in \mathcal{H}$ like a modular form of weight $\frac{n}{2} - 1$ for the representation contragredient to (6.1.1). Using the tautological pairing

$S_L \otimes S_L^\vee \rightarrow \mathbb{C}$, any weakly modular form (6.1.2) therefore defines a scalar-valued function

$$f \cdot \theta_L : \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H} \times \mathcal{M}(\mathbb{C}) \rightarrow \mathbb{C},$$

and we may attempt to define a function

$$\Phi(f, y) = \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}} f(\tau) \cdot \theta_L(\tau, y) \cdot \frac{du dv}{v^2}$$

of the variable $y \in \mathcal{M}(\mathbb{C})$ by integrating over the variable $\tau = u + iv$ in the upper half-plane.

The above integral diverges due to the pole of $f(\tau)$ at the cusp ∞ , but can be regularized as follows. First define

$$\Phi(f, y, s) = \lim_{T \rightarrow \infty} \int_{F_T} f(\tau) \cdot \theta_L(\tau, y) \cdot \frac{du dv}{v^{s+2}},$$

where

$$F_T = \{\tau \in \mathcal{H} : u^2 + v^2 \geq 1, |u| \leq \frac{1}{2}, v \leq T\}$$

is the usual fundamental domain for $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$ truncated at height T . The limit exists for $\mathrm{Re}(s) \gg 0$, and admits meromorphic continuation to all s .

DEFINITION 6.1.1. The *regularized theta lift* of f is the function on $\mathcal{M}(\mathbb{C})$ defined by

$$\Phi(f, y) = \text{constant term of } \Phi(f, y, s) \text{ at } s = 0.$$

The regularized theta lift $\Phi(f)$ is defined at every point of $\mathcal{M}(\mathbb{C})$, but is discontinuous at points of the divisor $\mathcal{Z}(f)(\mathbb{C})$. Its values along $\mathcal{Z}(f)(\mathbb{C})$, while interesting, play no role in the present work. They do play a fundamental role in the results of [AGHM17] and [BY09].

THEOREM 6.1.2 (Borcherds [Bor98]). *The regularized theta lift $\Phi(f)$ is a Green function for the divisor $\mathcal{Z}(f)$ on \mathcal{M} , and hence determines a class*

$$\widehat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(f)) \in \widehat{\mathrm{CH}}^1(\mathcal{M}).$$

We will state a stronger result in Theorem 6.2.1 below.

6.2. Borcherds products. We need to produce rational sections of powers of the line bundle ω on \mathcal{M} with known Petersson norms (4.4.3). Such sections were systematically constructed by Borcherds [Bor95, Bor98, Bor99].

Fix an

$$f \in M_{1-\frac{n}{2}}^!(\omega_L),$$

and assume that all coefficients $c(m, \mu)$ in the Fourier expansion (6.1.2) are integers. After possible replacing f by a positive integer multiple, there is an associated *Borcherds product* $\psi(f)$. This was initially constructed by Borcherds as a meromorphic function on the hermitian symmetric domain \mathcal{D} . It is easy to construct from this function a meromorphic section of $\omega^{\otimes c(0,0)}$ on the complex fiber $\mathcal{M}(\mathbb{C})$, and a calculation of Borcherds shows that the

divisor of the meromorphic section is none other than the analytic divisor $\mathcal{Z}(f)(\mathbb{C})$ on $\mathcal{M}(\mathbb{C})$.

If the quadratic space V contains an isotropic vector, the Shimura variety $\mathcal{M}(\mathbb{C})$ is noncompact, and sections of ω and its powers have q -expansions. Using the explicit q -expansion computed by Borcherds, one can show that $\psi(f)$ is algebraic and defined over \mathbb{Q} , and that its divisor, when viewed as a rational section of $\omega^{\otimes c(0,0)}$ on the integral model, is $\mathcal{Z}(f)$. In fact, these same results are true even when V contains no isotropic vector. Of course the proofs are now more complicated, as $\mathcal{M}(\mathbb{C})$ is compact and no theory of q -expansions is available.

THEOREM 6.2.1 (Borcherds [Bor98], H.-Madapusi Pera [HM18]). *After possibly replacing f by a positive integer multiple, there is a rational section $\psi(f)$ of the line bundle $\omega^{\otimes c(0,0)}$ on \mathcal{M} such that*

$$\operatorname{div}(\psi(f)) = \mathcal{Z}(f),$$

and such that

$$\Phi(f) = -2 \log \|\psi(f)\| + c(0,0) \cdot \log(4\pi e^\gamma)$$

on $\mathcal{M}(\mathbb{C}) \setminus \mathcal{Z}(f)(\mathbb{C})$.

REMARK 6.2.2. Our $\psi(f)$ is equal to $(2\pi i)^{c(0,0)} \Psi(f)^{\otimes 2}$, where $\Psi(f)$ is the meromorphic section more commonly referred to as the Borcherds product.

REMARK 6.2.3. When V contains an isotropic vector and $L^\vee = L$, Theorem 6.2.1 is contained in the work of Hörmann [Hör10] and [Hör14].

COROLLARY 6.2.4. *Inside the codimension one arithmetic Chow group*

$$\widehat{\operatorname{Pic}}(\mathcal{M}) \cong \widehat{\operatorname{CH}}^1(\mathcal{M})$$

we have the equality

$$c(0,0) \cdot \widehat{\omega} = \widehat{\mathcal{Z}}(f) - c(0,0) \cdot (0, \log(4\pi e^\gamma)),$$

where $(0, \log(4\pi e^\gamma))$ means the trivial divisor on \mathcal{M} endowed with the constant Green function $\log(4\pi e^\gamma)$ on the complex fiber.

7. The big CM cycle

We are now going to switch the point of view slightly from previous sections. Rather than starting with a quadratic space as our initial data, we will start with a CM field E of degree $2d > 2$, and build from it a quadratic space (V, Q) of signature $(2d-2, 2)$ in such a way that the resulting $G = \operatorname{GSpin}(V)$ comes with a distinguished maximal torus $T \subset G$.

Geometrically, this maximal torus corresponds to a 0-cycle $Y \rightarrow M$ of points at which the Kuga-Satake abelian scheme is isogenous to a power of an abelian variety with complex multiplication by the total reflex algebra E^\sharp defined in §3.1.

7.1. The initial data. Let E be a CM field of degree $2d > 2$, and let F be its maximal totally real subfield. Denote by

$$\iota_0, \dots, \iota_{d-1} : F \rightarrow \mathbb{R}$$

the real embeddings of F , and choose a $\xi \in F^\times$ such that $\iota_0(\xi)$ is negative, while $\iota_1(\xi), \dots, \iota_{d-1}(\xi)$ are positive. The rank two quadratic space

$$(\mathcal{V}, \mathcal{Q}) = (E, \xi \cdot \text{Norm}_{E/F})$$

over F has signature

$$\text{sig}(\mathcal{V}, \mathcal{Q}) = ((0, 2), (2, 0), \dots, (2, 0)).$$

That is, the quadratic form is negative definite at ι_0 , and positive definite at the remaining real places.

Define a quadratic space V over \mathbb{Q} by

$$(7.1.1) \quad (V, Q) = (\mathcal{V}, \text{Trace}_{F/\mathbb{Q}} \circ \mathcal{Q})$$

of signature $(2d - 2, 2)$. As in §4.1, let $C = C^+ \oplus C^-$ be its Clifford algebra.

One can similarly form the Clifford algebra of the F -quadratic space \mathcal{V} . It is a $\mathbb{Z}/2\mathbb{Z}$ -graded quaternion algebra over F . The even part is isomorphic to E , while the odd part is simply \mathcal{V} itself. Obviously the relation (7.1.1) should imply some relation between the Clifford algebras of V and \mathcal{V} . The relation is slightly subtle, and involves the total reflex algebra E^\sharp of §3.1.

PROPOSITION 7.1.1. *There is a canonical injection of \mathbb{Q} -algebras $E^\sharp \rightarrow C^+$, which makes C into a free E^\sharp -module of rank 2^d .*

PROOF. Choose a totally real number field \mathbf{k} that is Galois over \mathbb{Q} , and for any \mathbb{Q} -vector space W set $W_{\mathbf{k}} = W \otimes_{\mathbb{Q}} \mathbf{k}$. We may choose \mathbf{k} in such a way that

$$F_{\mathbf{k}} \cong \underbrace{\mathbf{k} \oplus \dots \oplus \mathbf{k}}_{d \text{ times}}.$$

This implies that $E_{\mathbf{k}} \cong \mathbf{k}_1 \oplus \dots \oplus \mathbf{k}_d$ for CM fields $\mathbf{k}_1, \dots, \mathbf{k}_d$, each having \mathbf{k} as its maximal totally real subfield. An exercise in Galois theory shows that

$$E_{\mathbf{k}}^\sharp \cong \mathbf{k}_1 \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} \mathbf{k}_d$$

as \mathbf{k} -algebras.

The action of F on $V = \mathcal{V}$ induces an orthogonal decomposition of \mathbf{k} -quadratic spaces $V_{\mathbf{k}} \cong W_1 \oplus \dots \oplus W_d$, in which each W_i is a 1-dimensional vector space over \mathbf{k}_i , and a corresponding isomorphism of \mathbf{k} -algebras

$$C_{\mathbf{k}} \cong D_1 \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} D_d,$$

in which $D_i \cong \mathbf{k}_i \oplus W_i$ is the Clifford algebra of the 2-dimensional quadratic space W_i over \mathbf{k} .

In particular, we have a natural inclusion $E_{\mathbf{k}}^\sharp \hookrightarrow C_{\mathbf{k}}$. One can check that this is compatible with the $\text{Gal}(\mathbf{k}/\mathbb{Q})$ action on source and target, and descends to an injection $E^\sharp \rightarrow C$ with the desired properties. \square

7.2. A maximal torus. The relation (7.1.1) endows the spinor similitude group $G = \mathrm{GSpin}(V)$ with extra structure, namely a distinguished maximal torus $T \subset G$, which we now describe.

First define tori over \mathbb{Q} by

$$T_F = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m, \quad T_E = \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$$

and

$$T_{F^\sharp} = \mathrm{Res}_{F^\sharp/\mathbb{Q}} \mathbb{G}_m, \quad T_{E^\sharp} = \mathrm{Res}_{E^\sharp/\mathbb{Q}} \mathbb{G}_m,$$

where E^\sharp is the total reflex algebra of E , and F^\sharp is its maximal totally real subalgebra. Define a quotient torus

$$T = T_E / \ker(\mathrm{Norm}_{F/\mathbb{Q}} : T_F \rightarrow \mathbb{G}_m),$$

and a subtorus $T^\sharp \subset T_{E^\sharp}$ by the cartesian diagram

$$\begin{array}{ccc} T^\sharp & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow \\ T_{E^\sharp} & \xrightarrow{\mathrm{Norm}_{E^\sharp/F^\sharp}} & T_{F^\sharp} \end{array}$$

where the vertical arrow on the right is induced by the inclusion $\mathbb{Q} \subset F^\sharp$.

The total reflex norm of Proposition 3.1.4 induces a morphism

$$\mathrm{Nm}^\sharp : T_E \rightarrow T_{E^\sharp},$$

which takes values in the subgroup T^\sharp . Moreover, this morphism restricts to $\mathrm{Norm}_{F/\mathbb{Q}}$ on the subgroup $T_F \subset T_E$, and so factors through a morphism

$$(7.2.1) \quad \mathrm{Nm}^\sharp : T \rightarrow T^\sharp.$$

Let ψ_δ be a symplectic form on the \mathbb{Q} -vector space $H = C$, of the type constructed in §4.3. Using Proposition 7.1.1, we can view H as a left module over $E^\sharp \subset C$, and hence we obtain a faithful representation

$$T_{E^\sharp} \rightarrow \mathrm{GL}(H).$$

It is not hard to see that this restricts to a morphism

$$(7.2.2) \quad T^\sharp \rightarrow \mathrm{GSp}(H),$$

and there is a cartesian diagram of algebraic groups

$$(7.2.1) \quad \begin{array}{ccc} T & \longrightarrow & G \\ \downarrow & & \downarrow \\ T^\sharp & \xrightarrow{(7.2.2)} & \mathrm{GSp}(H) \end{array}$$

over \mathbb{Q} . Here the vertical arrow on the right is the symplectic representation of §4.3, and the top horizontal arrow is uniquely determined by the commutativity of the diagram.

7.3. Shimura data. Now we attach a 0-dimensional Shimura variety to the maximal torus $T \subset G$ defined in §7.2.

Recall that we have a distinguished real embedding $\iota_0 : F \rightarrow \mathbb{R}$, characterized as the unique real place of F at which the quadratic space \mathcal{V} is negative definite. In other words, by the condition that $\mathcal{V} \otimes_{F, \iota_0} \mathbb{R}$ is a real quadratic space of signature $(0, 2)$. Fix an extension to

$$\iota_0 : E \rightarrow \mathbb{C}.$$

Recall that $V = \mathcal{V}$ has the structure of a 1-dimension vector space over E . This structure determines a decomposition

$$V_{\mathbb{C}} = \bigoplus_{\iota : E \rightarrow \mathbb{C}} V(\iota)$$

into complex lines, where

$$V(\iota) = \{x \in V_{\mathbb{C}} : \forall \beta \in E, \beta x = \iota(\beta)x\}.$$

In particular, we obtain a distinguished line $V(\iota_0) \subset V_{\mathbb{C}}$. It is an easy exercise in linear algebra to check that this line is isotropic, and defines a point

$$z_0 \in \mathcal{D} \subset \text{Hom}(\mathbb{S}, G_{\mathbb{R}})$$

in the hermitian symmetric domain (4.2.1). Moreover, the corresponding morphism $\mathbb{S} \rightarrow G_{\mathbb{R}}$ takes values in $T_{\mathbb{R}}$.

The pair $(T, \{z_0\})$ is a Shimura datum with reflex field $\iota_0(E) \subset \mathbb{C}$, and the inclusion $T \subset G$ defines a morphism of Shimura data

$$(7.3.1) \quad (T, \{z_0\}) \rightarrow (G, \mathcal{D}).$$

7.4. The big CM cycle. Fix a maximal lattice $L \subset V$. As in §4.2, this choice determines a compact open subgroup $K \subset G(\mathbb{A}_f)$, and an orthogonal Shimura variety M over \mathbb{Q} . Let \mathcal{M} be the integral model defined in §5.3.

Fix any compact open subgroup $K_T \subset T(\mathbb{A}_f)$ small enough that

$$K_T \subset K \cap T(\mathbb{A}_f).$$

The canonical model Y of the 0-dimensional Shimura variety

$$(7.4.1) \quad Y(\mathbb{C}) = T(\mathbb{Q}) \backslash \{z_0\} \times T(\mathbb{A}_f) / K_T$$

is a reduced Deligne-Mumford stack Y , finite over the reflex field $\iota_0(E)$. We can define an integral model \mathcal{Y} simply by taking the normalization of $\text{Spec}(\iota_0(\mathcal{O}_E))$ in Y . Thus \mathcal{Y} is a regular Deligne-Mumford stack, finite and flat over $\iota_0(\mathcal{O}_E)$ with generic fiber Y .

The natural morphism $Y(\mathbb{C}) \rightarrow M(\mathbb{C})$ induced by (7.3.1) is algebraic, and descends to the subfield $\iota_0(E) \subset \mathbb{C}$. One can show that it extends uniquely to a morphism

$$(7.4.2) \quad \mathcal{Y} \rightarrow \mathcal{M} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\iota_0(\mathcal{O}_E))$$

of integral models over $\iota_0(\mathcal{O}_E)$.

The notation $\mathcal{Y}(\mathbb{C})$ is potentially ambiguous, as it could refer either to the complex points of \mathcal{Y} viewed as a stack over \mathbb{Z} , or the complex points

of \mathcal{Y} viewed as a stack over $\iota_0(\mathcal{O}_E)$. To disambiguate, we understand that $\mathcal{Y}(\mathbb{C})$ always means the latter. In other words $\mathcal{Y}(\mathbb{C})$ is the same as (7.4.1).

We define $\mathcal{Y}_{\mathbb{Z}} = \mathcal{Y}$, but viewed as a stack over \mathbb{Z} rather than $\iota_0(\mathcal{O}_E)$. Thus the stacks $\mathcal{Y}_{\mathbb{Z}}$ and \mathcal{Y} are the same, but have different complex points. More precisely,

$$\mathcal{Y}_{\mathbb{Z}}(\mathbb{C}) \cong \bigsqcup_{\sigma: \iota_0(\mathcal{O}_E) \rightarrow \mathbb{C}} \mathcal{Y}^{\sigma}(\mathbb{C}).$$

Composing (7.4.2) with the projection

$$\mathcal{M} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\iota_0(\mathcal{O}_E)) \rightarrow \mathcal{M},$$

we obtain a morphism of \mathbb{Z} -stacks

$$\mathcal{Y}_{\mathbb{Z}} \rightarrow \mathcal{M}$$

called the *big CM cycle*. As in §2.2, there is an induced homomorphism

$$(7.4.3) \quad [- : \mathcal{Y}_{\mathbb{Z}}] : \widehat{\text{CH}}^1(\mathcal{M}) \rightarrow \mathbb{R}$$

called *arithmetic intersection against $\mathcal{Y}_{\mathbb{Z}}$* .

This linear functional depends on the choice of compact open subgroup $K_T \subset T(\mathbb{A}_f)$ used to define $\mathcal{Y}_{\mathbb{Z}}$, but if we set

$$\text{deg}_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}}) = \sum_{y \in \mathcal{Y}_{\mathbb{Z}}(\mathbb{C})} \frac{1}{|\text{Aut}(y)|},$$

then the rescaled linear functional

$$\widehat{\mathcal{Z}} \mapsto \frac{[\widehat{\mathcal{Z}} : \mathcal{Y}_{\mathbb{Z}}]}{\text{deg}_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})}$$

is independent of the choice.

At a point in the image of $\mathcal{Y}_{\mathbb{Z}}(\mathbb{C}) \rightarrow \mathcal{M}(\mathbb{C})$ the fiber of the Kuga-Satake abelian scheme is isogenous to a power of an abelian variety with complex multiplication. More precisely, we have the following proposition.

PROPOSITION 7.4.1. *The pullback of the Kuga-Satake abelian scheme*

$$\mathcal{A} \rightarrow \mathcal{M}$$

to any complex point $y \in \mathcal{Y}_{\mathbb{Z}}(\mathbb{C})$ is isogenous to 2^d copies of an abelian variety over \mathbb{C} with complex multiplication by E^{\sharp} , and CM type a Galois conjugate of Φ^{\sharp} .

PROOF. First suppose $y \in \mathcal{Y}(\mathbb{C}) \subset \mathcal{Y}_{\mathbb{Z}}(\mathbb{C})$, so that $y = [z_0, g]$ for some $g \in T(\mathbb{A}_f) \subset G(\mathbb{A}_f)$. The fiber of the Kuga-Satake abelian scheme at y is then given by

$$\mathcal{A}_y(\mathbb{C}) \cong (gH_{\mathbb{Z}} \cap H) \backslash H_{\mathbb{C}} / z_0 H_{\mathbb{C}},$$

where we are identifying $z_0 \in V_{\mathbb{C}} \subset \text{End}(H_{\mathbb{C}})$ using the left multiplication action of $V \subset C$ on $C = H$.

We know from Proposition 7.1.1 that left multiplication by the subalgebra $E^{\sharp} \subset C^+$ makes H into a free E^{\sharp} -module of rank 2^d . It is an exercise

in linear algebra to check that the subspace $z_0H_{\mathbb{C}} \subset H_{\mathbb{C}}$ is stable under the action of E^{\sharp} . This subspace is obviously stable under the right multiplication action of C , and hence we obtain an action

$$M_{2d}(E^{\sharp}) \cong E^{\sharp} \otimes_{\mathbb{Q}} C \rightarrow \text{End}(\mathcal{A}_y)_{\mathbb{Q}}.$$

One can then use this action to decompose \mathcal{A}_y as a product, up to isogeny, of 2^d factors. Each factor admits complex multiplication by E^{\sharp} , and all factors are E^{\sharp} -linearly isogenous to one another. Another exercise in linear algebra shows that the eigenspace of a \mathbb{Q} -algebra map $\varphi \in \text{Hom}(E^{\sharp}, \mathbb{C})$ in

$$\text{Lie}(\mathcal{A}_y) \cong H_{\mathbb{C}}/z_0H_{\mathbb{C}}$$

has dimension 2^d if $\varphi \in \Phi^{\sharp}$, and has dimension 0 otherwise. This implies that each of the isogeny factors of \mathcal{A}_y has complex multiplication of type $(E^{\sharp}, \Phi^{\sharp})$.

A general point $y \in \mathcal{Y}_{\mathbb{Z}}(\mathbb{C})$ is always $\text{Aut}(\mathbb{C}/\mathbb{Q})$ -conjugate to a point of $\mathcal{Y}(\mathbb{C})$, and hence decomposes, up to isogeny, as a product of abelian varieties with CM by E^{\sharp} and CM type a Galois conjugate of Φ^{\sharp} . \square

7.5. The average Faltings height. We now relate $[\widehat{\omega} : \mathcal{Y}_{\mathbb{Z}}]$ to the average Faltings height appearing in Theorem A. Before stating the result, we need a definition. If p is any rational prime, the action of E on $V = \mathcal{V}$ induces a decomposition

$$V_{\mathfrak{p}} = \bigoplus_{\mathfrak{p}|p} \mathcal{V}_{\mathfrak{p}}$$

over the primes $\mathfrak{p} \subset \mathcal{O}_F$ above p . For any such \mathfrak{p} set

$$L_{\mathfrak{p}} = L_p \cap \mathcal{V}_{\mathfrak{p}},$$

where the intersection is taken inside of V_p . The quadratic form Q on L_p restricts to a \mathbb{Z}_p -valued quadratic form on $L_{\mathfrak{p}}$.

DEFINITION 7.5.1. We say that a rational prime p is *good* if it satisfies the following properties.

- (1) For every prime $\mathfrak{p} \subset \mathcal{O}_F$ above p that is unramified in E , the \mathbb{Z}_p -lattice $L_{\mathfrak{p}} \subset \mathcal{V}_{\mathfrak{p}}$ is self-dual with respect to the \mathbb{Z}_p -valued quadratic form, and is stable under the action of $\mathcal{O}_{E,\mathfrak{p}}$.
- (2) For every prime $\mathfrak{p} \subset \mathcal{O}_F$ above p that is ramified in E , the \mathbb{Z}_p -lattice $L_{\mathfrak{p}} \subset \mathcal{V}_{\mathfrak{p}}$ is maximal with respect to the \mathbb{Z}_p -valued quadratic form, and there exists an $\mathcal{O}_{E,\mathfrak{q}}$ -stable lattice $\Lambda_{\mathfrak{p}} \subset \mathcal{V}_{\mathfrak{p}}$ such that

$$\Lambda_{\mathfrak{p}} \subset L_{\mathfrak{p}} \subset \mathfrak{d}_{E_{\mathfrak{q}}/F_{\mathfrak{p}}}^{-1} \Lambda_{\mathfrak{p}}.$$

Here $\mathfrak{q} \subset \mathcal{O}_E$ is the unique prime above \mathfrak{p} , and $\mathfrak{d}_{E_{\mathfrak{q}}/F_{\mathfrak{p}}}^{-1} \subset \mathcal{O}_{E,\mathfrak{q}}$ is the inverse different of $E_{\mathfrak{q}}/F_{\mathfrak{p}}$.

Denote by Σ_{bad} the (finite) set of rational primes that are not good.

THEOREM 7.5.2. *The equality*

$$(7.5.1) \quad \frac{[\widehat{\omega} : \mathcal{Y}_{\mathbb{Z}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} = \frac{1}{d \cdot 2^{d-1}} \sum_{\Phi} h_{(E, \Phi)}^{\text{Falt}} + \log(2\pi)$$

holds up to a \mathbb{Q} -linear combination of $\{\log(p) : p \in \Sigma_{\text{bad}}\}$.

PROOF. We will not give a complete proof of this result. Instead, we will first prove that (7.5.1) holds up to a \mathbb{Q} -linear combination of $\{\log(p) : p \text{ prime}\}$, and then explain how to extract the stronger statement from [AGHMP18].

Recalling the notation $\overset{\circ}{=}$ of the introduction, if we choose any global section s of $\omega|_{\mathcal{Y}_{\mathbb{Z}}}$ then, directly from the definition of (7.4.3),

$$\frac{[\widehat{\omega} : \mathcal{Y}_{\mathbb{Z}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} \overset{\circ}{=} \frac{-1}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} \sum_{y \in \mathcal{Y}_{\mathbb{Z}}(\mathbb{C})} \frac{\log \|s_y\|}{|\text{Aut}(y)|}.$$

Using Theorem 4.6.2, one can rewrite this as

$$\begin{aligned} \frac{[\widehat{\omega} : \mathcal{Y}_{\mathbb{Z}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} &\overset{\circ}{=} \frac{-1}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} \sum_{y \in \mathcal{Y}_{\mathbb{Z}}(\mathbf{k})} \frac{1}{|\text{Aut}(y)|} \sum_{\sigma: \mathbf{k} \rightarrow \mathbb{C}} \frac{\log \|s_{y\sigma}\|}{[\mathbf{k} : \mathbb{Q}]} \\ &\overset{\circ}{=} \frac{1}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} \sum_{y \in \mathcal{Y}_{\mathbb{Z}}(\mathbf{k})} \frac{h^{\text{Falt}}(A_y)}{2^{2d-2} |\text{Aut}(y)|} + \log(2\pi) \end{aligned}$$

for any number field $\mathbf{k} \subset \mathbb{C}$ Galois over \mathbb{Q} and large enough that $\mathcal{Y}_{\mathbb{Z}}(\mathbf{k}) = \mathcal{Y}_{\mathbb{Z}}(\mathbb{C})$. Now combine this with Proposition 7.4.1 and Corollary 3.4.3, which imply

$$\frac{1}{2^{2d-2}} \cdot h^{\text{Falt}}(A_y) \overset{\circ}{=} \frac{1}{2^{d-2}} \cdot h_{(E^{\sharp}, \Phi^{\sharp})}^{\text{Falt}} = \frac{1}{d \cdot 2^{d-1}} \cdot \sum_{\Phi} h_{(E, \Phi)}^{\text{Falt}}.$$

We explain how to extract Theorem 7.5.2 from the results of [AGHMP18]. In [AGHMP18, §9] there is another stack \mathcal{Y}_0 , regular and finite flat over $\iota(\mathcal{O}_E)$, equipped with a finite morphism $\mathcal{Y} \rightarrow \mathcal{Y}_0$. On this stack there is a metrized line bundle $\widehat{\omega}_0$. Theorem 9.4.2 of [AGHMP18] asserts that

$$\frac{\widehat{\deg}(\widehat{\omega}_0)}{\deg_{\mathbb{C}}(\mathcal{Y}_{0\mathbb{Z}})} + \frac{1}{2d} \cdot \log |D_F| = \frac{1}{d \cdot 2^{d-1}} \sum_{\Phi} h_{(E, \Phi)}^{\text{Falt}} + \log(2\pi)$$

where $\mathcal{Y}_{0\mathbb{Z}}$ is \mathcal{Y}_0 regarded as a stack over \mathbb{Z} , and

$$\widehat{\deg} : \widehat{\text{CH}}^1(\mathcal{Y}_{0\mathbb{Z}}) \rightarrow \mathbb{R}$$

is the arithmetic degree of §2.2. It is shown in the proof of Proposition 9.5.1 of [AGHMP18] that

$$\frac{2d \cdot [\widehat{\omega} : \mathcal{Y}_{\mathbb{Z}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} = \frac{2d \cdot \widehat{\deg}(\widehat{\omega}_0)}{\deg_{\mathbb{C}}(\mathcal{Y}_{0\mathbb{Z}})} + \log |D_F|$$

holds up to a \mathbb{Q} -linear combination of $\{\log(p) : p \in \Sigma_{\text{bad}}\}$. Note that the factors $2d/\deg_{\mathbb{C}}(\cdot)$ were mistakenly omitted from equation (9.5.1) of [AGHMP18]. \square

8. The arithmetic intersection formula

We keep the data of §7. In particular, E is a CM field of degree $2d > 2$ with maximal totally real subfield F .

Recall that we have fixed an embedding $\iota_0 : F \rightarrow \mathbb{R}$, and an extension of it to a complex embedding of E . We chose an element $\xi \in F^\times$ that is negative at ι_0 , and positive at all other archimedean places, and constructed from it a quadratic space (V, Q) of signature $(2d-2, 2)$. A choice of maximal lattice $L \subset V$ then determined an orthogonal Shimura variety \mathcal{M} endowed with a metrized line bundle $\widehat{\omega}$ and a big CM cycle

$$\mathcal{Y}_{\mathbb{Z}} \rightarrow \mathcal{M}.$$

We saw (Theorem 7.5.2) that the arithmetic intersection $[\widehat{\omega} : \mathcal{Y}_{\mathbb{Z}}]$ is essentially the averaged Faltings height appearing in Theorem A. On the other hand, we have seen (Corollary 6.2.4) that the theory of Borcherds products relates the metrized line bundle $\widehat{\omega}$ to the arithmetic divisor $\widehat{\mathcal{Z}}(f)$, where f is a suitable (vector-valued) weakly holomorphic form on the complex upper half-plane.

To prove Theorem A, it remains to relate the arithmetic intersection $[\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathbb{Z}}]$ to the logarithmic derivative at $s = 0$ of $L(s, \chi)$. This will be done by relating both quantities to the constant term of an Eisenstein series. These calculations are the technical core of [AGHMP18].

At this point in the exposition we abandon any pretense of providing complete proofs. We hope only to highlight to the reader some of the main ideas, and provide references to the appropriate places in [AGHMP18] where proofs can be found.

8.1. An incoherent Eisenstein series. We now recall from the work of Bruinier-Kudla-Yang [BKY12] a very particular Eisenstein series on the adelic group $\text{SL}_2(\mathbb{A}_F)$. We will follow the exposition of §6.1 and §6.2 of [AGHMP18], although everything we say can be found in [Kud97b] in greater generality.

Recall that our initial data in the construction of the big CM cycle

$$\mathcal{Y}_{\mathbb{Z}} \rightarrow \mathcal{M}$$

is a quadratic space $(\mathcal{V}, \mathcal{Q})$ over F of rank two, which is negative definite at $\iota_0 : F \rightarrow \mathbb{R}$, but positive definite at all other archimedean places. Define a rank two quadratic space $(\mathcal{C}, \mathcal{Q})$ over the adèle ring \mathbb{A}_F by declaring that \mathcal{C} is positive definite at *every* archimedean place of F , and that $\mathcal{C}_{\mathfrak{p}} \cong \mathcal{V}_{\mathfrak{p}}$ as $F_{\mathfrak{p}}$ quadratic spaces for every finite place \mathfrak{p} of F . Thus \mathcal{C} and $\mathcal{V} \otimes_F \mathbb{A}_F$ are isomorphic everywhere locally except at the archimedean place ι_0 .

REMARK 8.1.1. In the terminology of [Kud97b], the quadratic space \mathcal{C} over \mathbb{A}_F is *incoherent* in the sense that it does not arise as the adelization of any quadratic space over F .

Let \mathcal{C}_∞ be the product of the archimedean components of \mathcal{C} , and identify the finite part of \mathcal{C} with $\widehat{\mathcal{V}} = \mathcal{V} \otimes_F \widehat{F}$, so that

$$\mathcal{C} \cong \mathcal{C}_\infty \times \widehat{\mathcal{V}}.$$

For each $\mu \in L^\vee/L$ the space

$$S(\mathcal{C}) \cong S(\mathcal{C}_\infty) \otimes S(\widehat{\mathcal{V}})$$

of Schwartz functions on \mathcal{C} contains a distinguished element $\varphi_\infty^1 \otimes \varphi_\mu$. The archimedean component $\varphi_\infty^1 \in S(\mathcal{C}_\infty)$ is defined as the product over all archimedean places v of F of the Gaussian distributions

$$\varphi_v^1(x) = e^{-2\pi\mathcal{Q}_v(x)} \in S(\mathcal{C}_v),$$

where \mathcal{Q}_v is the quadratic form on \mathcal{C}_v . The finite component $\varphi_\mu \in S(\widehat{\mathcal{V}})$ is the characteristic function of the compact open subset $\mu + \widehat{L} \subset \widehat{\mathcal{V}}$.

The group $\mathrm{SL}_2(\mathbb{A}_F)$ acts³ on $S(\mathcal{C})$ via the Weil representation $\omega_{\mathcal{C}}$. Using this representation to translate the Schwartz function $\varphi_\infty^1 \otimes \varphi_\mu$, we define

$$(8.1.1) \quad \Phi(g, s, \varphi_\mu) = \omega_{\mathcal{C}}(g)(\varphi_\infty^1 \otimes \varphi_\mu)(0) \cdot |a|^s$$

for $g \in \mathrm{SL}_2(\mathbb{A}_F)$, $s \in \mathbb{C}$, and $\mu \in L^\vee/L$. Here $|a|^s$ is defined by factoring

$$g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k$$

with $k \in \mathrm{SL}_2(\widehat{\mathcal{O}}_F)$.

The function (8.1.1) satisfies the transformation law

$$\Phi\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} g, s, \varphi_\mu\right) = \chi(a)|a|^{s+1}\Phi(g, s, \varphi_\mu)$$

for all $a \in \mathbb{A}_F^\times$ and $b \in \mathbb{A}_F$. In other words, it lies in the space $I(s, \chi)$ of the representation obtained by inducing the character $\chi|\cdot|^s$ from the standard Borel subgroup of $\mathrm{SL}_2(\mathbb{A}_F)$. Thus we may form the Eisenstein series

$$E(g, s, \varphi_\mu) = \sum_{\gamma \in B(F) \backslash \mathrm{SL}_2(F)} \Phi(\gamma g, s, \varphi_\mu),$$

where $B \subset \mathrm{SL}_2$ is the group of upper triangular matrices.

Our ordering of the real embeddings $\iota_0, \dots, \iota_{d-1} : F \rightarrow \mathbb{R}$ fixes an isomorphism $F \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^d$, and we define

$$\mathcal{H}_F \subset F \otimes_{\mathbb{Q}} \mathbb{C}$$

to be the preimage of $\mathcal{H}^d \subset \mathbb{C}^d$ under this isomorphism.

³There is no need to work with the metaplectic group, as $\mathrm{rank}_{\mathbb{A}_F}(\mathcal{C}) = 2$ is even.

We now de-adelize the automorphic form $E(g, s, \varphi_\mu)$ to obtain a classical Hilbert modular Eisenstein series in the variable $\vec{\tau} \in \mathcal{H}_F$. Writing $\vec{\tau} = \vec{u} + i\vec{v}$ with $\vec{u}, \vec{v} \in F \otimes_{\mathbb{Q}} \mathbb{R}$ and \vec{v} totally positive, we define

$$g_{\vec{\tau}} = \begin{pmatrix} 1 & \vec{u} \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\vec{v}} & \\ & 1/\sqrt{\vec{v}} \end{pmatrix} \in \mathrm{SL}_2(F \otimes_{\mathbb{Q}} \mathbb{R}),$$

and view $g_{\vec{\tau}} \in \mathrm{SL}_2(\mathbb{A}_F)$ with trivial non-archimedean components. The function

$$E(\vec{\tau}, s, \varphi_\mu) = \frac{1}{\sqrt{\mathrm{Norm}_{F/\mathbb{Q}}(\vec{v})}} \cdot E(g_{\vec{\tau}}, s, \varphi_\mu)$$

is a nonholomorphic Hilbert modular form of parallel weight 1, and satisfies a functional equation in $s \mapsto -s$. Moreover, this Eisenstein series vanishes identically at the center $s = 0$ of the functional equation.

REMARK 8.1.2. The claim about the weight of $E(\vec{\tau}, s, \varphi_\mu)$ follows from the particular choice of Schwartz function $\varphi_\infty^{\mathbf{1}}$, and is the reason for the otherwise mysterious superscript $\mathbf{1}$. The claim about the vanishing at $s = 0$ follows from the incoherence of the quadratic space \mathcal{L} .

Consider the central derivative

$$E'(\vec{\tau}, 0, \varphi_\mu) = \frac{d}{ds} E(\vec{\tau}, s, \varphi_\mu) \Big|_{s=0}.$$

Like any Hilbert modular form it admits a q -expansion, but because it is nonholomorphic the coefficients depend on the imaginary part

$$\vec{v} \in F \otimes_{\mathbb{Q}} \mathbb{R}$$

of $\vec{\tau}$. We will apply what one might call the lazy man's holomorphic projection to this q -expansion, and simply throw away those parts of the coefficients that depend on \vec{v} .

To make this more precise, we first expand

$$E'(\vec{\tau}, 0, \varphi_\mu) = \frac{1}{\Lambda(0, \chi)} \sum_{\alpha \in F} a_F(\alpha, \vec{v}, \mu) \cdot q^\alpha,$$

where

$$q^\alpha = e^{2\pi i \mathrm{Trace}_{F/\mathbb{Q}}(\alpha \vec{\tau})}$$

and $\mathrm{Trace}_{F/\mathbb{Q}} : F \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \mathbb{C}$ is the usual trace.

For any totally positive $\alpha \in F$, the set of places of F at which \mathcal{V} does not represent α is finite of even cardinality. As \mathcal{V} is negative definite at one archimedean place, and positive definite at the others, it follows that the set of nonarchimedean places

$$(8.1.2) \quad \mathrm{Diff}(\alpha) = \{\text{primes } \mathfrak{p} \subset \mathcal{O}_F : \mathcal{V}_{\mathfrak{p}} \text{ does not represent } \alpha\}$$

is finite with odd cardinality. In particular it is nonempty. Moreover, it is not hard to see that every $\mathfrak{p} \in \mathrm{Diff}(\alpha)$ is either inert or ramified in E .

PROPOSITION 8.1.3. *Let $F_+ \subset F$ be the subset of totally positive elements. If $\alpha \in F_+$ then*

$$a_F(\alpha, \mu) \stackrel{\text{def}}{=} a_F(\alpha, \vec{v}, \varphi_\mu)$$

is independent of \vec{v} . Moreover,

- (1) *if $|\text{Diff}(\alpha)| > 1$ then $a_F(\alpha, \mu) = 0$,*
- (2) *if $\text{Diff}(\alpha) = \{\mathfrak{p}\}$ then*

$$\frac{a_F(\alpha, \mu)}{\Lambda(\chi, 0)} \in \mathbb{Q} \cdot \log(N(\mathfrak{p})).$$

As for the $\alpha = 0$ term, there is a constant $a_F(0, \mu) \in \mathbb{C}$ such that

$$\frac{a_F(0, \vec{v}, \mu)}{\Lambda(0, \chi)} = \frac{a_F(0, \mu)}{\Lambda(0, \chi)} + \varphi_\mu(0) \log(\text{Norm}_{F/\mathbb{Q}}(\vec{v})).$$

This constant satisfies

$$\frac{a_F(0, \mu)}{\Lambda(0, \chi)} = \begin{cases} -2 \frac{L'(0, \chi)}{L(0, \chi)} - \log \left| \frac{D_E}{D_F} \right| + d \cdot \log(4\pi e^\gamma) & \text{if } \mu = 0 \\ 0 & \text{if } \mu \neq 0 \end{cases}$$

up to a \mathbb{Q} -linear combination of $\{\log(p) : p \in \Sigma_{\text{bad}}\}$, where Σ_{bad} is the set of bad primes of Definition 7.5.1.

PROOF. The claims concerning $a_F(\alpha, \mu)$ with $\alpha \in F_+$ follow from [BKY12, Proposition 4.6]. The claims about the constant term follow from [AGHMP18, Proposition 6.2.3] and [AGHMP18, Proposition 7.8.2]. \square

Now form the formal q -expansion

$$\mathcal{E}_F(\vec{\tau}, \mu) = a_F(0, \mu) + \sum_{\alpha \in F_+} a_F(\alpha, \mu) \cdot q^\alpha.$$

This formal q -expansion is the lazy man’s holomorphic projection indicated above. We stress that this is not a modular form, and we have no interest in issues of convergence. If we formally restrict $\mathcal{E}_F(\vec{\tau}, \mu)$ to the diagonally embedded upper half-plane $\mathcal{H} \subset \mathcal{H}_F$, the result is a formal q -expansion

$$\mathcal{E}(\tau, \mu) = \sum_{m \geq 0} a(m, \mu) \cdot q^m$$

where $a(0, \mu) = a_F(0, \mu)$, and

$$(8.1.3) \quad a(m, \mu) = \sum_{\substack{\alpha \in F_+ \\ \text{Trace}_{F/\mathbb{Q}}(\alpha) = m}} a_F(\alpha, \mu)$$

for all rational numbers $m > 0$.

8.2. The arithmetic intersection formula. As in §6.1 (with $n = 2d - 2$), suppose that

$$(8.2.1) \quad f(\tau) = \sum_{m \gg -\infty} c(m)q^{-m} \in M_{2-d}^!(\omega_L)$$

has all $c(m, \mu) \in \mathbb{Z}$, and let

$$\widehat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(f)) \in \widehat{\text{CH}}^1(\mathcal{M})$$

be the arithmetic divisor of Theorem 6.1.2. We wish to compute its image under the arithmetic intersection

$$[- : \mathcal{Y}_{\mathbb{Z}}] : \widehat{\text{CH}}^1(\mathcal{M}) \rightarrow \mathbb{R}.$$

LEMMA 8.2.1. *For any rational number $m > 0$ and any $\mu \in L^\vee/L$, the fiber product*

$$\mathcal{Z}(m, \mu) \cap \mathcal{Y}_{\mathbb{Z}} \stackrel{\text{def}}{=} \mathcal{Z}(m, \mu) \times_{\mathcal{M}} \mathcal{Y}_{\mathbb{Z}}$$

has dimension 0, and is supported in finitely many nonzero characteristics. In other words, $\mathcal{Y}_{\mathbb{Z}}$ intersects all special divisors properly.

PROOF. This follows from [AGHMP18, Proposition 7.6.1], which also gives information about which characteristics can appear in the fiber product. \square

The significance of the lemma is that, as per the discussion of §2.2, there is a decomposition

$$(8.2.2) \quad [\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathbb{Z}}] = [\mathcal{Z}(f) : \mathcal{Y}_{\mathbb{Z}}]_{\text{fin}} + [\Phi(f) : \mathcal{Y}_{\mathbb{Z}}]_{\infty}$$

as the sum of a finite part

$$[\mathcal{Z}(f) : \mathcal{Y}_{\mathbb{Z}}]_{\text{fin}} = \sum_{\substack{m > 0 \\ \mu \in L^\vee/L}} c(-m, \mu) \cdot [\mathcal{Z}(m, \mu) : \mathcal{Y}_{\mathbb{Z}}]_{\text{fin}}$$

and an archimedean part

$$[\Phi(f) : \mathcal{Y}_{\mathbb{Z}}]_{\infty} = \frac{1}{2} \sum_{y \in \mathcal{Y}_{\mathbb{Z}}(\mathbb{C})} \frac{\Phi(f, y)}{|\text{Aut}(y)|}.$$

The following theorem is the most technically difficult part of [AGHMP18].

THEOREM 8.2.2. *For any rational number $m > 0$ and any $\mu \in L^\vee/L$, the equality*

$$\frac{2d}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} \cdot [\mathcal{Z}(m, \mu) : \mathcal{Y}_{\mathbb{Z}}]_{\text{fin}} = -\frac{a(m, \mu)}{\Lambda(0, \chi)}$$

holds up to a \mathbb{Q} -linear combination of $\{\log(p) : p \in \Sigma_{\text{bad}}\}$. Here Σ_{bad} is the set of bad primes of Definition 7.5.1.

PROOF. We only give the barest hint of the idea.

To any scheme S equipped with a morphism $S \rightarrow \mathcal{M}$, we associated in §5.4 a positive definite quadratic space $V(\mathcal{A}_S)_{\mathbb{Q}}$ over \mathbb{Q} , along with a subset

$$V_{\mu}(\mathcal{A}_S) \subset V(\mathcal{A}_S)_{\mathbb{Q}}.$$

Let Q denote the quadratic form on $V(\mathcal{A}_S)_{\mathbb{Q}}$. If the morphism $S \rightarrow \mathcal{M}$ factors through $\mathcal{Y}_{\mathbb{Z}} \rightarrow \mathcal{M}$, then $V(\mathcal{A}_S)_{\mathbb{Q}}$ carries more structure: it is an F -vector space (in fact, an E -vector space), and carries an F -valued quadratic form \mathcal{Q} satisfying

$$Q = \text{Trace}_{F/\mathbb{Q}} \circ \mathcal{Q}.$$

See [AGHMP18, Corollary 5.4.6].

Suppose $\alpha \in F_+$. Recalling that $\mathcal{Y}_{\mathbb{Z}}$ is just the stack \mathcal{Y} of §7.4, but viewed as a stack over \mathbb{Z} rather than $\iota_0(\mathcal{O}_E) \cong \mathcal{O}_E$, we define a stack over \mathcal{Y} whose functor of points assigns to any morphism $S \rightarrow \mathcal{Y}$ the set

$$\mathcal{Z}_F(\alpha, \mu)(S) = \{x \in V_{\mu}(\mathcal{A}_S) : \mathcal{Q}(x) = \alpha\}.$$

By comparing with the definition (5.4.3) of $\mathcal{Z}(m, \mu)$, we obtain a canonical decomposition

$$\mathcal{Z}(m, \mu) \times_{\mathcal{M}} \mathcal{Y} \cong \bigsqcup_{\substack{\alpha \in F_+ \\ \text{Trace}_{F/\mathbb{Q}}(\alpha) = m}} \mathcal{Z}_F(\alpha, \mu)$$

of stacks over \mathcal{O}_E , intended to mirror to the relation (8.1.3). Using (2.2.2), the finite part of the arithmetic intersection becomes

$$\begin{aligned} [\mathcal{Z}(m, \mu) : \mathcal{Y}_{\mathbb{Z}}]_{\text{fin}} &= \sum_p \log(p) \left(\sum_{y \in (\mathcal{Z}(m, \mu) \cap \mathcal{Y}_{\mathbb{Z}})(\mathbb{F}_p^{\text{alg}})} \frac{\text{length}(\mathcal{O}_{(\mathcal{Z}(m, \mu) \cap \mathcal{Y}_{\mathbb{Z}}), y}^{\text{et}})}{|\text{Aut}(y)|} \right) \\ &= \sum_{\alpha \in F_+} \sum_{\substack{\mathfrak{p} \subset \mathcal{O}_F \\ y \in \mathcal{Z}_F(\alpha, \mu)(\mathbb{F}_p^{\text{alg}})}} \frac{\text{length}(\mathcal{O}_{\mathcal{Z}_F(\alpha, \mu), y}^{\text{et}})}{|\text{Aut}(y)|} \cdot \log(N(\mathfrak{p})), \end{aligned}$$

where $\mathbb{F}_p^{\text{alg}}$ is an algebraic closure of $\mathcal{O}_F/\mathfrak{p}$.

Recall the set $\text{Diff}(\alpha)$ primes of \mathcal{O}_F defined by (8.1.2). If $|\text{Diff}(\alpha)| > 1$ then $\mathcal{Z}_F(\alpha, \mu) = \emptyset$. On the other hand, if $\text{Diff}(\alpha) = \{\mathfrak{p}\}$ contains a single prime $\mathfrak{p} \subset \mathcal{O}_F$, then $\mathcal{Z}_F(\alpha, \mu)$ has dimension 0, and is supported at the unique prime of \mathcal{O}_E above \mathfrak{p} . See [AGHMP18, Proposition 7.6.1].

Now suppose that $\text{Diff}(\alpha) = \{\mathfrak{p}\}$ where \mathfrak{p} lies above a rational prime $p \notin \Sigma_{\text{bad}}$. Every étale local ring of $\mathcal{Z}_F(\alpha, \mu)$ has the same length

$$\ell_{\mathfrak{p}}(\alpha) = \text{length}(\mathcal{O}_{\mathcal{Z}_F(\alpha, \mu), y}^{\text{et}}),$$

which is given by an explicit formula in terms of $\text{ord}_{\mathfrak{p}}(\alpha)$, and satisfies

$$\ell_{\mathfrak{p}}(\alpha) \log(N(\mathfrak{p})) \sum_{y \in \mathcal{Z}_F(\alpha, \mu)(\mathbb{F}_p^{\text{alg}})} \frac{1}{|\text{Aut}(y)|} = -\frac{\text{deg}_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})}{2d} \cdot \frac{a_F(\alpha, \mu)}{\Lambda(0, \chi)}.$$

This calculation is carried out in [AGHMP18, §7], and lies at the core of the proof of the average Colmez conjecture. See [AGHMP18, Theorem 7.7.4] and [AGHMP18, Theorem 7.8.1]. The lengths of the local rings are determined using integral p -adic Hodge theory to compute formal deformation spaces of p -divisible groups with extra structure. The summation on the left, which counts geometric points on the stack $\mathcal{Z}_F(\alpha, \mu)$, can be rewritten so that it counts vectors of norm α in lattices contained in a *nearby* positive definite F -quadratic space ${}^{\mathfrak{p}}\mathcal{V}$ obtained by modifying the quadratic space \mathcal{V} at \mathfrak{p} and the unique archimedean place at which it is negative definite. This essentially realizes the summation as a theta series coefficient, and the connection with the Eisenstein series coefficient $a_F(\alpha, \mu)$ comes via the Siegel-Weil formula. This follows the method for counting points used in [HY12] and [How12] for Hilbert modular surfaces and unitary Shimura varieties, respectively.

Combining the above with Proposition 8.1.3 shows that

$$[\mathcal{Z}(m, \mu) : \mathcal{Y}_{\mathbb{Z}}]_{\text{fin}} = \sum_{\substack{\alpha \in F_+ \\ \mathfrak{p} \subset \mathcal{O}_F \\ y \in \mathcal{Z}_F(\alpha, \mu)(\mathbb{F}_{\mathfrak{p}}^{\text{alg}})}} \frac{\ell_{\mathfrak{p}}(\alpha) \cdot \log(N(\mathfrak{p}))}{|\text{Aut}(y)|} = -\frac{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})}{2d} \cdot \frac{a_F(\alpha, \mu)}{\Lambda(0, \chi)}$$

up to a linear combination of $\{\log(p) : p \in \Sigma_{\text{bad}}\}$, as desired. □

The archimedean part of (8.2.2) was computed by Bruinier-Kudla-Yang, following methods of [BY09] and [Sch09]. Similar formulas on Hilbert modular surfaces and modular curves appeared earlier in work of Bruinier-Yang [BY06] and Gross-Zagier [GZ86], respectively, but those calculations used different methods.

THEOREM 8.2.3 (Bruinier-Kudla-Yang [BKY12]). *For any f as in (8.2.1), the regularized theta lift of §6.1 satisfies*

$$\frac{d}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} \sum_{y \in \mathcal{Y}_{\mathbb{Z}}(\mathbb{C})} \frac{\Phi(f, y)}{\text{Aut}(y)} = \sum_{\substack{\mu \in L^{\vee}/L \\ m \geq 0}} \frac{a(m, \mu) \cdot c(-m, \mu)}{\Lambda(0, \chi)}.$$

REMARK 8.2.4. The actual result proved in [BKY12] is stronger than Theorem 8.2.3. Whereas we only need regularized theta lifts of weakly holomorphic forms, the general result proved in [loc. cit.] includes regularized theta lifts of harmonic weak Maass forms. This greater generality is not needed for the proof of the averaged Colmez conjecture.

The weakly holomorphic form (8.2.1) may be chosen so that $c(0, 0) \neq 0$. Putting together the finite and archimedean intersection calculations above yields the following arithmetic intersection formula.

COROLLARY 8.2.5. *For any f as in (8.2.1), the equality*

$$\frac{2d}{c(0, 0)} \cdot \frac{[\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathbb{Z}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} = -2 \frac{L'(0, \chi)}{L(0, \chi)} - \log \left| \frac{D_E}{D_F} \right| + d \cdot \log(4\pi e^{\gamma})$$

holds up to a \mathbb{Q} -linear combination of $\{\log(p) : p \in \Sigma_{\text{bad}}\}$.

PROOF. Combining Theorems 8.2.2 and 8.2.3 with the decomposition (8.2.2) shows that

$$2d \cdot \frac{[\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathbb{Z}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} = \sum_{\mu \in L^{\vee}/L} \frac{a(0, \mu) \cdot c(0, \mu)}{\Lambda(0, \chi)}$$

holds up to a \mathbb{Q} -linear combination of $\{\log(p) : p \in \Sigma_{\text{bad}}\}$. Combining this with Proposition 8.1.3 completes the proof. \square

8.3. Putting everything together. We can now prove Theorem A of the introduction.

THEOREM 8.3.1. *For any CM field E with totally real subfield $F \subset E$ of degree $[F : \mathbb{Q}] = d$, we have the equality*

$$(8.3.1) \quad \frac{1}{2^d} \sum_{\Phi} h_{(E, \Phi)}^{\text{Falt}} = -\frac{1}{2} \cdot \frac{L'(0, \chi)}{L(0, \chi)} - \frac{1}{4} \cdot \log \left| \frac{D_E}{D_F} \right| - \frac{d \log(2\pi)}{2},$$

where the summation is over all CM types $\Phi \subset \text{Hom}(E, \mathbb{C})$.

PROOF. Recall the finite set of bad primes Σ_{bad} of Definition 7.5.1. Theorem 7.5.2 and Corollary 6.2.4 give us the equalities

$$\frac{1}{2^d} \sum_{\Phi} h_{(E, \Phi)}^{\text{Falt}} = \frac{d}{2} \cdot \frac{[\widehat{\omega} : \mathcal{Y}_{\mathbb{Z}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} - \frac{d}{2} \cdot \log(2\pi)$$

and

$$\frac{d}{2} \cdot \frac{[\widehat{\omega} : \mathcal{Y}_{\mathbb{Z}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} = \frac{d}{2c(0, 0)} \cdot \frac{[\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathbb{Z}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathbb{Z}})} - \frac{d}{4} \cdot \log(4\pi e^{\gamma}),$$

up to a \mathbb{Q} -linear combination of $\{\log(p) : p \in \Sigma_{\text{bad}}\}$. Comparing these with Corollary 8.2.5 shows that the desired equality (8.3.1) holds up to a \mathbb{Q} -linear combination of $\{\log(p) : p \in \Sigma_{\text{bad}}\}$.

We now write

$$\frac{1}{2^d} \sum_{\Phi} h_{(E, \Phi)}^{\text{Falt}} = -\frac{1}{2} \cdot \frac{L'(0, \chi)}{L(0, \chi)} - \frac{1}{4} \cdot \log \left| \frac{D_E}{D_F} \right| - \frac{d \log(2\pi)}{2} + \sum_p b_E(p) \log(p)$$

for constants $b_E(p) \in \mathbb{Q}$ satisfying $b_E(p) = 0$ whenever $p \notin \Sigma_{\text{bad}}$. By the \mathbb{Q} -linear independence of $\{\log(p)\}$, the constants $b_E(p)$ are uniquely determined by this relation, and each depends only on E and p .

In particular $b_E(p)$ does not depend on the choice of $\xi \in F^{\times}$ used to define the quadratic space (V, Q) , or on the choice of maximal lattice $L \subset V$. The set Σ_{bad} , however, depends very much on these choices. According to Proposition 9.5.2 of [AGHMP18], for any given prime p one can choose the auxiliary data in such a way that $p \notin \Sigma_{\text{bad}}$, and hence $b_E(p) = 0$. \square

The proof above is somewhat simpler than what is presented in §9.5 of [AGHMP18], where some calculations are carried out after enlarging the Shimura variety \mathcal{M} to some \mathcal{M}^{\diamond} as in §5.3. The point is that at the time [AGHMP18] was written, the results of [HM18] were not yet available,

and so the equality of divisors of Theorem 6.2.1 was only known (by work of Hörmann, as in Remark 6.2.3) up to a sum of divisors supported in characteristics dividing $[L^\vee : L]$. The arguments of [AGHMP18] were written in such a way that they only require this weaker result.

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