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# BCOV's Feynman rule of quintic 3-folds

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Abstract. We briefly outline the proof given by Chang-Guo-Li of the Yamaguchi-Yau polynomiality and the Bershadsky-Cecotti-Ooguri-Vafa Feynman rule conjectures for the Gromov-Witten invariants of the quintic Calabi-Yau threefolds.

The last two decades is the period when *Mirror Symmetry* has grown to a fully developed subject in mathematics. Gromov-Witten invariants of Calabi-Yau threefolds, a mathematical part of *Mirror Symmetry*, has been progressing steadily during this period, like on the Strominger-Yau-Zaslow conjecture and the homological mirror symmetry conjecture. For the sake of the length of this note, and for the limited knowledge of the author, in this note, we will restrict ourselves to the new development in understanding the Yamaguchi-Yau polynomiality and the Bershadsky-Cecotti-Ooguri-Vafa Feynman rule conjecture in algebraic geometry.

0.0.1. *A brief history.* The theory of Gromov-Witten invariants (GW invariants) is a mathematical foundation to the type II Super-String theory. In terms of algebraic geometry, it is a "virtual" counting of holomorphic maps from algebraic curves to a projective variety. GW-invariants of Calabi-Yau threefolds (CY threefolds) are the most challenging ones, and among CY threefolds, the quintic CY threefolds are the focus of numerous study. In this note, we will briefly ourline the recent progress in understanding the GW-invariants of quintic CY threefolds.

The investigation of GW-invariants went through several periods of development. Inspired by the $\sigma$-model in Super-String theory [**Wit90, Wit92**], Ruan-Tian [**RT95**] constructed the GW-invariants of semi-positive, including the Calabi-Yau, symplectic manifolds. Later, Li-Tian and Behrend-Fantechi [**LT98, BF97**] constructed the GW-invariants of all smooth projective varieties via algebraic geometry.

By the time the mathematical construction of the GW-inariants of CY threefolds was settled, several groups had already made progresses in uncovering the structure of genus zero GW invariants of CY threefolds. A milestone was set, in 1989, by Super-String theorists Candelas-Ossa-Green-

Parkes [**CdOGP91**], when they derived a closed formula for the generating function of the genus zero GW invariants of quintic CY threefolds.

In 1996, based on the work of Kontsevich [**Kon93**] relating the genus zero GW invariants of a smooth quintic Calabi-Yau to that of $\mathbb{P}^4$, the formula of Candelas et al. was proved by Givental and Lian-Liu-Yau [**Giv96, LLY97**].

In 1991, Bershadsky-Cecotti-Ooguri-Vafa (BCOV) derived a closed formula for the generating function $F_1$ of the genus one GW invariants of a quintic CY threefold $X$. Its mathematical proof was achieved by Zinger, much later. First, Li-Zinger generalized Kontsevich's hyperplane theorem to the case of genus one, after introducing the reduced GW invariants [**LZ09**]. Using this theory, Zinger proved the BCOV formula for $F_1$ [**Zi09**] in 2009.

In 1993, Bershadsky-Cecotti-Ooguri-Vafa (BCOV) developed their theory of Kodaira-Spencer [**BCOV94**], a theory on a $C^\infty$ section $\mathcal{F}_g^Z(q,\bar q)$ on the moduli of Calabi-Yau threefolds of a certain line bundle. They derived their holomorphic anomaly equation for $\mathcal{F}_g^Z(q,\bar q)$, and related its (holomorphic) limit to the generating function $F_g^{\hat Z}$ of the genus $g$ GW invariants of a CY manifold $\hat Z$, a mirror of $Z$:

$$\lim_{\bar q\to\infty} \mathcal{F}_g^Z(q,\bar q) = F_g^{\hat Z}(q).$$

Applying their theory, they obtained their well-known Feynman rule for $F_g^{\hat Z}$.

In 2004, applying special geometry to BCOV's solution of the Holomorphic Anomaly Equations, Yamaguchi-Yau (YY) derived the polynomiality of $F_g(q)$ [**YY04**], and derived their functional equation for $F_g(q)$, explicitly for the quintic $X$.

In 2009, with additional arguments from physics, Huang-Klemm-Quackenbush demonstrated how to obtain $F_g(q)$ effectively, for $g$ up to 51 [**HKQ09**].

Recently, two groups have made progress in uncovering much of the mystery of the GW invariants of the quintics CY threefold. One group is Fan-Guo-Janda-Ruan, who constructed the Fan-Jarvis-Ruan-Witten (FJRW) invariants, proved the YY holomorphic anomaly equation, among other things [**FJR07, FJR13, CR10, GJR17, GJR18**]. During the decade of 2000-2010, Fan-Jarvis-Ruan (based on a proposal of Witten) established the FJRW theory. In the CY setting, FJRW theory is the analytic continuation, or the Landau-Ginzburg-dual, of the GW theory. Together with its generalization (Gauged-Linear-Sigma-Model), it provided a new perspective to the GW theory and is one of the key elements to the recent developments.

The other group is Chang-Guo-Kiem-Li-Li-Liu, who developed the theory of cosection localized virtual cycles [**KL13**], the GW invariants of stable maps with $p$-fields [**CL12**], the FJRW invariants using the cosection localized cycles [**CLL15**], and in the end, developed the Mixed-Spin-P field theory [**CLLL15, CLLL16**] and the N-Mixed-Spin-P field theory [**NMSP1**] that eventually led to a mathematical proof, by Chang-Guo-Li, of both the

Feynman rule of BCOV and the polynomiality and the functional equations of YY [NMSP2, NMSP3].

This note will be a brief survey of this development, following the work of the later group.

0.0.2. *GW-invariants of CY threefolds.* Let $X$ be a smooth projective variety. For a homology class $d \in H_2(X,\mathbb{Z})$ and nonnegative integers $g$, $n$, one forms the moduli of stable morphisms from genus $g$, $n$-pointed nodal curves to $X$ of fundamental class $d$:

$$\overline{\mathcal{M}}_{g,n}(X,d) = \{f : C \to X,\ p_1,\ldots,p_n \in C : f_*([C]) = d,\ f \text{ stable}\}/\sim .$$

Here a morphism $f$ is stable if the automorphism group of $(C, p_\bullet, f)$ is finite.

The moduli space $\overline{\mathcal{M}}_{g,n}(X,d)$ admits an obvious perfect obstruction theory, which induces a virtual fundamental class [LT98, BF97]

$$[\overline{\mathcal{M}}_{g,n}(X,d)]^{\text{vir}} \in H_{2\nu}(\overline{\mathcal{M}}_{g,n}(X,d),\mathbb{Q}),$$

of degree twice of the virtual dimension of $\overline{\mathcal{M}}_{g,n}(X,d)$:

$$\nu := \text{vir.dim}\, \overline{\mathcal{M}}_{g,n}(X,d) = (g-1)(3-\dim X) + d\cdot c_1(X) + n.$$

We now let $X$ be a quintic CY threefold, and let $n = 0$. The the above is a degree zero class. We define the degree $d$ genus $g$ GW-invariants of $X$ to be

$$N_{g,d} = \deg[\overline{\mathcal{M}}_{g,n}(X,d)]^{\text{vir}} \in \mathbb{Q}.$$

The challenge is to determine, for $g \geq 0$, the generating function

$$F_g(q) = \sum N_{g,d}\cdot q^d.$$

0.0.3. *Genus zero mirror symmetry.* Let $X$ be a quintic CY threefold, say defined by the vanishing locus of the quintic Fermat polynomial; let $H$ be the hyperplane class of $X$. We let the $I$-function for $X$ be

$$I(q,z) := z\sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{5d}(5H+mz)}{\prod_{m=1}^{d}(H+mz)^5} = \sum_{i=0}^{3} I_i(q)H^i z^{1-i};$$

let the $J$-function for $X$ be

$$J_i(q) := I_i(q)/I_0(q), \quad i = 0,\ldots,3.$$

We let $c_1 := \int_X H^3$.

The (genus zero) Mirror Theorem is

Theorem 1 ([Giv96, LLY97]). *For the quintic CY threefold $X$, the genus zero GW potential*

$$F_0(Q) := \frac{c_1}{6}(\log Q)^3 + \sum_{d>0} N_{0,d}\cdot Q^d$$

*is given by*

$$F_0(Q) = \frac{c_1}{6}\big(\log Q^3 - J_1(q)^3\big) + \frac{c_1}{2}\big(J_1(q)J_2(q) - J_3(q)\big),$$

*where the variables $Q$ and $q$ are related via the mirror map*

$$Q = q\exp\tau_X(q), \quad \tau_X(q) := J_1(q).$$

Following Zagier-Zinger, for $n \geq m \geq 0$, we define $I_{m,n}$ inductively

$$I_{0,n} =: I_n,$$
$$I_{m,n} := \frac{I_{m-1,n-1}}{I_{m-1,m-1}} + q\frac{d}{dq}\frac{I_{m-1,n}}{I_{m-1,m-1}}, \quad m > 0.$$

Using the above notation, the 3-point function for the Calabi-Yau 3-fold $X$ becomes

$$\langle H, H, H\rangle_{0,3}^X(Q) = (Q\tfrac{d}{dQ})^3F_0(Q) = (Q\tfrac{d}{dQ})^2J_2 = I_{2,2}/I_{1,1}.$$

0.0.4. *BCOV Feynman rule.* The *BCOV's Feynman rule* provides an effectively algorithm to evaluate $F_g$, up to finite ambiguity [**BCOV94**]. We now state its exact form. To this end, we introduce

$$A_p := \frac{D^pI_{11}(q)}{I_{11}(q)}, \quad B_p := \frac{D^pI_0(q)}{I_0(q)}, \quad X := 1 - Y = \frac{-5^5q}{1-5^5q},$$

where $D = q\frac{\mathrm{d}}{\mathrm{d}q}$. We usually write $A = A_1$ and $B = B_1$. It is proved in [**YY04**] that they all lie in the ring $\mathcal{R}$ generated by five generators

$$\mathcal{R} = \mathbb{Q}[A_1, B_1, B_2, B_3, X].$$
$$\mathcal{R} = \mathbb{Q}[A, B, B_2, B_3, X].$$

Further, $\mathcal{R}$ is closed under the differential $D$. Following [**BCOV94, YY04**], we let $E_\psi := B_1$, and

$$E_{\varphi\varphi} := A + 2\,B_1, \quad E_{\varphi\psi} := -B_2, \quad E_{\psi\psi} := -B_3 + (B - X)\cdot B_2 - \tfrac{2}{5}\,B_1X.$$
$$E_{\varphi\varphi} := A + 2\,B, \quad E_{\varphi\psi} := -B_2, \quad E_{\psi\psi} := -B_3 + (B - X)\cdot B_2 - \tfrac{2}{5}\,BX.$$

These $E_{**}$ are called *propagators* in [**BCOV94**].

We further introduce

$$P_{g,m,n} := \begin{cases} (2g+m+n-3)_n\cdot\frac{(c_1Y)^{g-1}(I_{11})^m}{(I_0)^{2g-2}}\big(Q\tfrac{d}{dQ}\big)^mF_g(Q) & \text{if } 2g-2+m>0 \\ (n-1)!\big(\frac{X}{24}-1\big) & \text{if } (g,m)=(1,0)\end{cases}.$$

Here $\chi = -200$; $(a)_k := a(a-1)\cdots(a-k+1)$. As $\mathcal{R}$ is closed under $D$, all $P_{g,m,n} \in \mathcal{R}$.

For positive $g$, we set

$$G_g^{BCOV} = \{\text{genus } g \text{ stable dual graphs with decorated line edges}\}$$

Here decorated line edges take three forms: solid, dotted-solid, and dotted line edges. For each

$$\Gamma \in G_g^{BCOV},$$

we will do the following placements:

**Edge:** at each solid *edge* (resp. dotted-solid; resp. dotted), *place* $E_{\varphi\varphi}$ (resp. $E_{\varphi\psi}$; resp. $E_{\psi\psi}$);

**Vertex:** at a genus $g$ *vertex*, with $m$ solid and $n$ dotted edges attached, *place* $P_{g,m,n}$.

We define

$$\mathbf{Cont}_\Gamma = \text{the product of the edge and the vertex placements.}$$

We define

$$f_g^{BCOV} := \sum_{\Gamma \in G_g^{BCOV}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \mathrm{Cont}_\Gamma \in \mathbb{Q}[\![q]\!].$$

CONJECTURE 2 (BCOV Feynman rule [**BCOV94**]). *For* $g > 1$, $f_g^{BCOV}$ *is a degree* $3g-3$ *polynomial in* $X$, *where* $X = \frac{-5^5 q}{1-5^5 q}$.

The constant term of $f_g^{BCOV}$ is known; so the polynomial $f_g^{BCOV}$ has $3g-3$ unknown coefficients. Based on this conjecture, once all genus $< g$ GW generating functions $F_{<g}$ are known, to determine $F_g$ we only need to determine $f_g^{BCOV}$. The coefficients of these polynomials are thus called the ambiguity.

0.0.5. *Yamaguchi-Yau (YY) conjectures.* By definition the $E_{**}$'s together with $X$ also generate $\mathcal{R}$:

$$\mathcal{R} = \mathbb{Q}[E_\psi, E_{\varphi\varphi}, E_{\varphi\psi}, E_{\psi\psi}, X].$$

We form a subring

$$\tilde{\mathcal{R}} := \mathbb{Q}[E_{\varphi\varphi}, E_{\varphi\psi}, E_{\psi\psi}, X] \subset \mathcal{R}.$$

We state YY *finite generation conjecture*:

CONJECTURE 3 (Polynomial structure [**YY04**]). *For positive* $g$, *the "normalized" GW potential*

$$P_g(q) := \frac{(c_1 Y)^{g-1}}{(I_0)^{2g-2}} F_g(Q) \in \tilde{\mathcal{R}}. \tag{1}$$

Based on BCOV's results, YY derived their functional equation.

CONJECTURE 4 (Yamaguchi-Yau equations [**YY04**]). *For positive* $g$, *the following two identities hold*

$$-\partial_A P_g = \frac{1}{2} P_{g-1,2} + \frac{1}{2} \sum_{g_1+g_2=g} P_{g_1,1} P_{g_2,1},$$

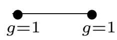
$$\Big(-2\partial_A + \partial_B + (A+2B)\partial_{B_2} + \big((B-X)(A+2B) - B_2 - \frac{2}{5}X\big)\partial_{B_3}\Big) P_g = 0.$$

0.0.6. *Calculating* $F_2$ *using BCOV Feynman rule.* In this case the conjecture becomes

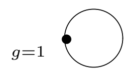
$$\begin{aligned} P_2 &+ \frac{1}{2} T^{\varphi\varphi} P_{1,1}^2 + \frac{1}{2} T^{\varphi\varphi} P_{1,2} + \frac{1}{2} (T^{\varphi\varphi})^2 P_{1,1} + \frac{1}{8} (T^{\varphi\varphi})^2 P_{0,4} + \frac{1}{8} (T^{\varphi\varphi})^3 + \\ &+ \frac{1}{12} (T^{\varphi\varphi})^3 + \frac{\chi}{24} T^{\varphi} P_{1,1} + \frac{1}{2} \frac{\chi}{24} T^{\varphi} T^{\varphi\varphi} + \frac{1}{2} \frac{\chi}{24} (\frac{\chi}{24} - 1) T \\ &= f_2(X), \end{aligned}$$

using the complete list of $g = 2$ decorated stable graphs listed below

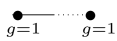

$F_2\,,$




$\frac{1}{2}\,F_{1,1}^2 \cdot T^{\varphi\varphi}\,,$

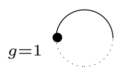



$\frac{1}{2}\,F_{1,2} \cdot T^{\varphi\varphi}\,,$

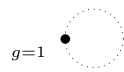



$F_{1,1} \cdot T^{\varphi} \cdot \left(\frac{\chi}{24} - 1\right),$

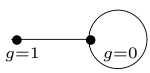

$\frac{1}{2}\left(\frac{\chi}{24} - 1\right)^2 \cdot T\,,$



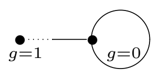

$F_{1,1} \cdot T^{\varphi}\,,$



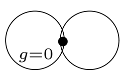

$\frac{1}{2}\left(\frac{\chi}{24} - 1\right) \cdot T\,,$



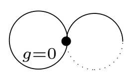

$\frac{1}{2}\,F_{1,1} \cdot (T^{\varphi\varphi})^2 \cdot F_{0,3}\,,$



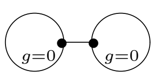

$\frac{1}{2}\left(\frac{\chi}{24} - 1\right) \cdot T^{\varphi} \cdot F_{0,3} \cdot T^{\varphi\varphi}\,,$



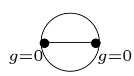

$\frac{1}{8}\,F_{0,4} \cdot (T^{\varphi\varphi})^2\,,$




$\frac{1}{2}\,F_{0,3} \cdot T^{\varphi\varphi} \cdot T^{\varphi}\,,$




$\frac{1}{8}\,F_{0,3}^2 \cdot (T^{\varphi\varphi})^3$




$\frac{1}{12}\,F_{0,3}^2 \cdot (T^{\varphi\varphi})^3$

Using the genus one formula, and the divisor equation, after finding out the initial conditions $N_{2,0}$, $N_{2,1}$, $N_{2,2}$, and $N_{2,3}$ ([**NMSP3**]), we obtain

$$f_2^{BCOV}(X) = -\frac{1}{240}X^3 + \frac{113}{7200}X^2 + \frac{487}{300}X - \frac{11771}{7200}.$$

Hence one solves

$$\begin{aligned} -P_2 = &-\frac{350\,B_3}{9} - \Big(\frac{25\,A}{6} + \frac{425\,B}{9} + \frac{625}{36}\Big)B_2 \\ &+ \frac{5\,A^3}{24} + \frac{65\,A^2B}{12} + \frac{1045\,AB^2}{18} + \frac{865\,B^3}{9} + \frac{25}{144} \\ &+ \Big(\frac{A^2}{6} + \frac{49\,AB}{36} + \frac{167\,A}{720} + \frac{37\,B^2}{18} - \frac{1811\,B}{120} - \frac{475\,B_2}{12} - \frac{5759}{3600}\Big)X \\ &+ \frac{25\,A^2}{24} + \frac{775\,AB}{36} + \frac{350\,B^2}{9} + \frac{625}{288}\,(A + 2B) \\ &+ \left(\frac{13\,A}{288} + \frac{13\,B}{144} + \frac{41}{3600}\right)X^2 + \frac{X^3}{240}. \end{aligned}$$

**0.1. NMSP theorey.** We briefly recall the road to the Mixed-Spin-P fields, which is the theory for proving the BCOV Feynman rule conjecture.

0.1.1. *GW invariants of stable maps with fields.* The moduli of stable maps with fields is

$$\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p = \{[C, \mathcal{L}, \varphi, \rho] \mid \varphi \in H^0(\mathcal{L})^{\oplus 5}, \rho \in H^0(\mathcal{L}^{\otimes -5} \otimes \omega_C)\}$$

such that $\deg \mathcal{L} = d$, and $\varphi$ are nowhere vanishing ([**CL12**]). Because of these requirements, elements

$$[C, \mathcal{L}, \varphi, \rho] \in \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$$

are in 1-1 correspondence with

$$([\varphi] : C \to \mathbb{P}^4; \rho),$$

consisting of a morphism $f = [\varphi]$ together with a field $\rho \in H^0(f^*\mathcal{O}(-5) \otimes \omega_C)$. Because of this, we call it the moduli of stable maps with fields.

THEOREM 5 (Chang-Li [**CL12**]).

$$\deg[\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p]^{\mathrm{vir}} = (-1)^{g+d+1} N_{g,d}.$$

A few words on the degree of

$$[\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p]^{\mathrm{vir}}$$

are in order. First, it is direct to see that $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ has a standard perfect obstruction theory, thus its virtual cycle is well-defined. However, $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ is non-proper for positive $g$. Thus taking the degree of its virtual cycle is problematic. We resolve this issue by using the cosection localized virtual cycles constructed by Y.-H. Kiem and the author [**KL13**]. Using the cosection coming from the LG function of $K_{\mathbb{P}^4}$, we obtain the cosection localized virtual cycle

$$[\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p]^{\mathrm{vir}} \in A_0\big(\overline{\mathcal{M}}_g(X, d)\big),$$

where the inclusion

$$\overline{\mathcal{M}}_g(X, d) \subset \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$$

is via the embedding $X \subset \mathbb{P}^4$ and setting the fields $\rho = 0$. This way, we can define the GW invariants of stable maps with fields to be

$$\deg[\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p]^{\mathrm{vir}} \in \mathbb{Q}.$$

The theorem proves that, up to a sign, it is equal to the GW invariants of $X$.

The upshot of this construction is that it transforms the GW invariants $N_{g,d}$, a highly "non-linear" theory, to a field theory of $[C, \mathcal{L}, \varphi, \rho]$, which in principle is a "quantization" of

$$K_{\mathbb{P}^4} = [(\mathbb{C}^5 - 0) \times \mathbb{C}/\mathbb{C}^*],$$

where the action is with weights $(1^5, -5)$

Inspired by Witten's vision that the GW invariants of quintics and the FJRW invariants of the Fermat quintic are the two ends of a single theory ([**Wit93**]), and realizing that the GW invariants end is the quantization of the aforementioned GIT quotient of the Artin stack

$$[\mathbb{C}^5 \times \mathbb{C}/\mathbb{C}^*], \tag{2}$$

we began to realize that what we need is a grand "single theory" realizing Witten's vision in algebraic geometry.

What came out is the following: we construct the FJRW invariants end as a "quantization" of another GIT quotient of the Artin stack (2), we construct the grand "single theory" as a quantization of the the master space that detects the wall-crossing between the two GIT quotients of this Artin stack. This is the moduli of stable Mixed-Spin-P fields introduced by Chang-Li-Li-Liu in [**CLLL15, CLLL16**].

0.1.2. *The FJRW invariants and the MSP theory.* We construct the Fan-Javis-Ruan-Witten invariants in the narrow case using the moduli space

$$\overline{\mathcal{M}}_{g,\gamma}^{1/5,5p} = \{[\Sigma, \mathcal{C}, \mathcal{L}, \varphi, \rho] \mid \varphi \in H^0(\mathcal{L})^{\oplus 5}, \rho : \mathcal{L}^{\otimes 5} \cong \omega_{\mathcal{C}}(\Sigma)\},$$

where $\mathcal{C}$ is a twisted curve with orbifold markings $\Sigma$.

THEOREM 6 (H.-L. Chang, L. and W.-P. Li [**CLLL15**]). *The cosection localized virtual cycle provides an algebraic construction of the (narrow) FJRW invariants.*

Without going into the details, let us point out how the GW and FJRW theories are the quantizations of the two GIT quotients of the Artin stack (2). Consider the moduli space defined below, which is an Artin stack:

$$\mathcal{M} = \{[\Sigma, \mathcal{C}, \mathcal{L}, \varphi, \rho] \mid \varphi \in H^0(\mathcal{L})^{\oplus 5}, \rho \in H^0(\mathcal{L}^{\otimes -5} \otimes \omega_{\mathcal{C}}(\Sigma))\}.$$

- the moduli $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, d)^p$ is a DM open substack of $\mathcal{M}$ after imposing the condition that $\varphi$ is never zero, thus $f = [\varphi] : \mathcal{C} \to \mathbb{P}^4$ is a morphism;
- the moduli $\overline{\mathcal{M}}_{g,\gamma}^{1/5,5p}$ is a DM open substack of $\mathcal{M}$ after imposing the condition that $\rho$ is nowhere zero, thus $\rho : \mathcal{L}^{\otimes 5} \cong \omega_{\mathcal{C}}(\Sigma)$.

These are the quantizations of the two GIT quotients, respectively,

$$[(\mathbb{C}^5 - 0) \times \mathbb{C}/\mathbb{C}^*] \quad \text{and} \quad [\mathbb{C}^5 \times (\mathbb{C} - 0)/\mathbb{C}^*],$$

and they are the two GIT quotients of the quotient Artin stack (2).

The theory of Mixed-Spin-P Fields is the quantization of the master space relating the two GIT quotients of (2). Precisely, a Mixed-Spin-P (MSP) field is a tuple

$$\xi = [\Sigma, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \mu, \nu] \tag{3}$$

such that

(1) $(\Sigma, \mathcal{C})$ is a pointed twisted curves, $(\mathcal{L}, \mathcal{N})$ is a pair of invertible sheaves on $\mathcal{C}$;
(2) $\varphi \in H^0(\mathcal{L})^{\oplus 5}$; $\rho \in H^0(\mathcal{L}^{\otimes -5} \otimes \omega_{\mathcal{C}}(\Sigma))$;
(3) $\mu \in \mathcal{L} \otimes \mathcal{N}$, $\nu \in \mathcal{N}$;
(4) plus non-vanishing conditions from the GIT master space.

We say an MSP field $\xi$ is stable if its automorphism group is finite ([**CLLL15**]).

0.1.3. *The road to BCOV.*

THEOREM 7 (H.-L. Chang, L., W.-P. Li and C.-C. Liu [**NMSP1**]). *The moduli of stable MSP fields is a $\mathbb{C}^*$-DM stack; it has a cosection localized (compactly supported) $\mathbb{C}^*$-equivariant virtual cycle.*

Because the virtual cycle is $\mathbb{C}^*$-equivariant, we can apply the virtual localization formula to get a collection of polynomial relations among

a. the GW invariants of the quintic CY threefolds, and
b. the FJRW invariants of the quintic Fermat polynomial.

On first sight, these relations seem impossible to penetrate. With tour de force, H.-L. Chang, S. Guo, W.-P. Li and J. Zhou recovered BCOV-Zinger's formula $F_1$ for the genus one GW invariants for the quintics ([**CGLZ18**]).

We saw "the light at the end of the tunnel" when, after some manipulation and careful packaging, we saw that it recovers BCOV's Feynman rule. To actually prove it, we need to modify the MSP fields to allow a better localization packaging. This led to the theory of NMSP fields.

THEOREM 8 (H.-L. Chang, S. Guo, L. and W.-P. Li [**NMSP1**]). *The theory of MSP fields can be extended to NMSP fields, where an NMSP field is a collection of data*

$$\xi = [\Sigma, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \mu, \nu]$$

*such that all but $\mu$ are as in the definition of MSP fields, while*

$$\mu = (\mu_1, \dots, \mu_N) \in H^0(\mathcal{L} \otimes \mathcal{N})^{\oplus N}.$$

*The moduli of stable NMSP fields is a $(\mathbb{C}^*)^N$-DM stack, and satisfies all the other properties as in the case of MSP fields.*

This way, one can use the $\mathbb{C}^*$ action

$$t \cdot \xi = [t \cdot (\Sigma, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho), t \cdot \mu, t \cdot \nu], \quad t \in \mathbb{C}^*,$$

where $t \cdot \mu := (t, \mu_1 \zeta_N t \mu_2, \zeta_N^2 t \mu_3, \dots)$, $t \cdot \nu = t\nu$, and $\zeta_N$ is a primitive $N$-th root of unity.

With this $\mathbb{C}^*$ action, we proved that the BCOV Feynman sum $f_g^{BCOV}$ is a polynomial in $X$.

THEOREM 9 (H.-L. Chang, S. Guo, and J. Li [**NMSP2, NMSP3**]). *For positive $g$, $f_g^{BCOV}$ is a polynomial in $X$ of degree at most $3g-3$.*

**0.2. A brief outline.** We outline the main idea of our proof of the BCOV Feynman rule conjecture. The materials of this part are drawn from [**CLLL15, CLLL16, CL17, NMSP1, NMSP2, NMSP3**].

0.2.1. *Localization graphs of $\mathbb{C}^*$ fixed NMSP fields.* As was mentioned, the proof of the BCOV Feynman rule conjecture is based on the moduli of stable NMSP fields, via the $\mathbb{C}^*$ localization formula. Thus it is essential to understand the geometry of the $\mathbb{C}^*$-fixed NMSP fields.

We denote by

$$\mathcal{W}_{g,\gamma,(d_0,d_\infty)}$$

the moduli of stable NMSP fields; we denote a field

$$\xi = \big(\Sigma, \mathcal{C}, \mathcal{L}, \mathcal{N}, \rho, \varphi_1, \dots, \varphi_5, \mu_1, \dots, \mu_{\mathrm{N}}, \nu\big) \tag{4}$$

parallel to (3), where $g$ is the genus of $\mathcal{C}$, $\gamma$ is the monodromy assignment of the markings $\Sigma$ of $\xi$, $d_0 = \deg(\mathcal{L} \otimes \mathcal{N})$ and $d_\infty = \deg \mathcal{N}$, and the fields are as stated in Theorem 8. (We drop $\gamma$ when $\Sigma = \emptyset$.) The $G = (\mathbb{C}^*)^N$ action on $\xi$ is via scaling the field $\mu$, component-wise, and we denote by

$$[\mathcal{W}_{g,n,(d_0,d_\infty)}]^{\mathrm{vir}}_{\mathrm{loc}} \in A^G_*(\mathcal{W}^-_{g,\gamma,(d_0,d_\infty)})$$

the cosection localized virtual cycle, supported on the degeneracy locus $\mathcal{W}^-_{g,\gamma,(d_0,d_\infty)}$ of the cosection, which is proper.

We characterize the geometry of a $G$-fixed field

$$\xi = [\Sigma, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \mu, \nu] \in \mathcal{W}^G_{g,\gamma,(d_0,d_\infty)} \subset \mathcal{W}_{g,\gamma,(d_0,d_\infty)}.$$

As $\xi$ is $G$-fixed, it is $G$-equivariant. Let $\mathcal{C}_0 \subset \mathcal{C}$ be any irreducible component of the underlying curve $\mathcal{C}$ (of $\xi$) on which the induced $G$ action is trivial, then one of the following three situations occur:

i. $\mu|_{\mathcal{C}_0} = 0$; we then call the curve $\mathcal{C}_0$ at level 0;
ii. $\varphi|_{\mathcal{C}_0} = \rho|_{\mathcal{C}_0} = 0$; we then call the curve $\mathcal{C}_0$ at level 1;
iii. $\nu|_{\mathcal{C}_0} = 0$; we then call the curve $\mathcal{C}_0$ at level $\infty$.

We let $\mathcal{C}^{\mathrm{fix}} \subset \mathcal{C}$ (resp. $\mathcal{C}^{\mathrm{mv}} \subset \mathcal{C}$) be closed substack on which the induced $G$ action is trivial (resp. non-trivial). Note that $\mathcal{C}^{\mathrm{fix}}$ may include 0-dimensional substacks, while $\mathcal{C}^{\mathrm{mv}}$ has pure dimension one. We then represent each connected component of $\mathcal{C}^{\mathrm{fix}}$ by a vertex, and represent each connected component of $\mathcal{C}^{\mathrm{mv}}$ by an edge; we build a graph by attaching an edge $e$ to a vertex $v$ if their associated substacks $\mathcal{C}_e$ and $\mathcal{C}_v$ has the property that $\mathcal{C}_e \cup \mathcal{C}_v \subset \mathcal{C}$ is connected; we add a leg $\ell_k$ to a vertex $v$ if the marking $\Sigma_k$ lies on the curve $\mathcal{C}_v$. We label the vertices by their levels, as specified before, and we decorate the graph with the geometry of the field $\xi$. This way we get a decorated graph $\Theta_\xi$. Finally, a decorated graph labels a connected component of

$$\mathcal{W}^G_{g,\gamma,(d_0,d_\infty)}.$$

We call such graphs localization graphs.

Following this convention, we can write the virtual localization as

$$[\mathcal{W}_{g,n,(d_0,d_\infty)}]^{\mathrm{vir}}_{\mathrm{loc}} = \sum_\Theta \frac{[F_\Theta]^{\mathrm{vir}}}{e(N^{\mathrm{vir}}_\Theta)}, \tag{5}$$

where $F_\Theta$ is the connected component of $\mathcal{W}^{\mathbb{C}^*}_{g,n,(d_0,d_\infty)}$, indexed by the localization graph $\Theta$, and $N^{\mathrm{vir}}_\Theta$ is the virtual normal bundle of $F_\Theta$.

One remark: If a graph has an edge connecting two vertices of level 0 and $\infty$, then the contribution to the localization formula from its associated component will be zero. This is the "irregular vanishing" refereed to in [**CLLL15, CL17**]. Thus in (5), we only need to sum over all regular graphs.

0.2.2. *Evaluation maps and state spaces.* Given an NMSP field $\xi$ as in (4) with $\gamma$ non-orbifold markings, then $\xi$ is an NMSP field implies that $\rho|_{\Sigma_k} = 0$ at every marking $\Sigma_k$, which implies that $\nu|_{\Sigma_k} \neq 0$, $(\rho, \nu)$ has no base point. and that $(\mu_j/\nu)|_{\Sigma_k} \in \mathcal{L}|_{\Sigma_k}$ is well defined. Thus

$$(\varphi_1|_{\Sigma_k}, \dots, \varphi_5|_{\Sigma_k}, \frac{\mu_1}{\nu}|_{\Sigma_k}, \dots, \frac{\mu_{\mathrm{N}}}{\nu}|_{\Sigma_k}) \neq 0 \in \mathcal{L}|_{\Sigma_k}^{\oplus(5+\mathrm{N})},$$

defining a point in $\mathbb{P}^{4+\mathrm{N}}$. This process associates to each marking $\Sigma_k$ a $G$-equivariant evaluation map

$$ev_k : \mathcal{W}_{g,n,(d_0,d_\infty)} \to \mathbb{P}^{4+\mathrm{N}},$$

where $G$ acts on $\mathbb{P}^{4+\mathrm{N}}$ via scaling the last N homogeneous coordinates.

For $\tau_i(z) \in H^*_G(\mathbb{P}^{4+\mathrm{N}})[z]$, $1 \leq i \leq n$, we define the NMSP generating function to be

$$\langle \bigotimes_{i=1}^{n} \tau_i(\psi_i) \rangle^{NMSP}_{g,n,d_\infty} = \sum_{d \geq 0} (-1)^{d+1-g} q^d \int_{[\mathcal{W}_{g,n,(d,d_\infty)}]^{\mathrm{vir}}_{\mathrm{loc}}} \prod_{i=1}^{n} \mathrm{ev}_i^* \tau_i(\psi_i) \in \mathbb{F}[\![q]\!].$$

For virtual dimension reason, the integrand in the above formula vanishes whenever

$$\mathrm{N}(d+1-g) + d_\infty + n = \mathrm{vir.}\dim \mathcal{W}_{g,n,(d,d_\infty)} > \sum_{i=1}^{n} \deg \tau_i(\psi_i).$$

This implies that

$$\langle \bigotimes_{i=1}^{n} \tau_i(\psi_i) \rangle^{NMSP}_{g,n,d_\infty} \tag{6}$$

is a polynomial in $q$.

On the other hand, for $\xi \in (\mathcal{W}_{g,n,(d_0,d_\infty)})^G \cap \mathcal{W}^-_{g,n,(d_0,d_\infty)}$,

$$ev_k(\xi) \in Q \cup \{\mathrm{pt}_1, \dots, \mathrm{pt}_{\mathrm{N}}\} := \aleph,$$

where $Q \subset \mathbb{P}^4$ is the Fermat quintic, and $\mathrm{pt}_i$ are the N $G$-fixed points of $\mathbb{P}^{\mathrm{N}-1} \subset \mathbb{P}^{4+\mathrm{N}}$.

By abuse of notation, we use the same symbol to denote the evaluation map

$$ev_k : F_\Theta \cap \mathcal{W}^-_{g,n,(d_0,d_\infty)} \longrightarrow \aleph.$$

Let $\mathbb{F} := \mathbb{Q}(\zeta_{\mathrm{N}})(t)$, where $\zeta_{\mathrm{N}}$ is a primitive N-th root of unity. We define the state space of our NMSP theory to be

$$\mathcal{H} := H^*(\aleph, \mathbb{F}),$$

after setting $t_\alpha = -\zeta_{\mathrm{N}}^\alpha t$, where $t_\alpha$ are the equivariant parameters of $G$.

Here is a quick comment on this substitution $t_\alpha = -\zeta_{\mathrm{N}}^\alpha t$. As $G = (\mathbb{C}^*)^{\mathrm{N}}$,

$$H^*_G(\mathrm{pt}, \mathbb{Q}) = \mathbb{Q}[\![t_1, \dots, t_{\mathrm{N}}]\!].$$

Thus (6) is a Laurent series in $t_1, \dots, t_{\mathrm{N}}$. After the aforementioned substitution, it becomes a Laurent series in $t$. We remark that it is this substitution that makes many explicit calculations possible.

0.2.3. *The invariants appearing in the localization formula.* We now look at the virtual localization formula (5) more closely. Let $\Theta$ be a localization graph and let $F_\Theta$ be its associated component in $\mathcal{W}^G_{g,n,(d_0,d_\infty)}$. Here, in place of $\gamma$ we use an integer $n$, meaning that all markings of $\xi \in F_\Theta$ are scheme points.

Let $v$ be a vertex of $\Theta$; let $\mathcal{C}_v$ be the associated substacks in the underlying curve $\mathcal{C}$ of a general $\xi \in F_\Theta$. Then according to the level of $v$, we have the following conclusions:

(1) level 0; then $\mu|_{\mathcal{C}_v} = 0$; the collection of all such $\xi|_{\mathcal{C}_v}$ becomes the moduli of stable maps with fields $\overline{\mathcal{M}}_{g',n'}(\mathbb{P}^4, d')^p$, whose virtual cycle gives the GW invariant of the quintic CY threefold;
(2) level 1: then $\varphi_i|_{\mathcal{C}_v} = \rho|_{\mathcal{C}_v} = 0$, one $\mu_\alpha = 1$, and all else $\mu_\beta = 0$; the collection of all such $\xi|_{\mathcal{C}_v}$ becomes the moduli of pointed stable curves $\overline{\mathcal{M}}_{g',n'}$;
(3) level $\infty$: then $\nu|_{\mathcal{C}_v} = 0$, $\mu_\alpha|_{\mathcal{C}_v} = 1$ for precisely one $\alpha$ in $1, \ldots, N$, and $\mu_\beta|_{\mathcal{C}_v} = 0$ for all $\beta \neq \alpha$ the collection of all such $\xi|_{\mathcal{C}_v}$ becomes the moduli $\mathcal{W}^{1/5,p}_{g',\gamma'}$ of 5-spin curve with five sections of the spin bundle.

Therefore, the localization (5), and thus the expression (6), are polynomial expressions involving the GW invariants of the quintic CY, the Hodge integrals and the FJRW invariants of quintic Fermat polynomial. In particular when $\Theta$ consists of a single vertex at lever 0 and no edges, $[F_\Theta]^{\mathrm{vir}}$ gives rise to the genus $g$ GW invariants of the quintic CY threefold.

0.2.4. *Genus zero* NMSP *invariants.* We first point out that when $g = 0$ and $d_\infty = 0$, the moduli of NMSP fields

$$\mathcal{W}_{0,n,(d,0)} = \overline{\mathcal{M}}_{0,n}(\mathbb{P}^{4+\mathrm{N}}, d),$$

where the identity is $G$-equivariant, and preserves the $G$-equivariant obstruction theories of both sides. This gives

Proposition 10. *The genus zero* NMSP *theory equals the genus zero $L_p^{\otimes 5}$-twisted $G$-equivariant GW theory of $\mathbb{P}^{4+\mathrm{N}}$:*

$$\langle \tau_i(\psi_i) \rangle^{\mathrm{msp}}_{0,n} = \sum_{d \geq 0} q^d \int_{\overline{\mathcal{M}}_{g=0,n}(\mathbb{P}^{4+\mathrm{N}},d)} e_G(\pi_{d*} f_d^* L_p^{\otimes 5}) \prod_{i=1}^n \tau_i(\psi_i). \tag{7}$$

Here $L_p := \mathcal{O}_{\mathbb{P}^{4+\mathrm{N}}}(1)$ with $c_1(L_p) = p$ is the Poincare dual of $(x_1 = 0)$; $(\pi_d, f_d) : \mathcal{C} \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^{4+\mathrm{N}}, d) \times \mathbb{P}^{4+\mathrm{N}}$ is the universal family, and

$$[\mathcal{W}_{0,n,(d,0)}]^{\mathrm{vir}} = (-1)^{d+1} e_G(\pi_* f^* L_p^{\otimes 5}) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{4+\mathrm{N}}, d)].$$

This simple observation implies that the String Equation (SE), Dilaton Equation (DE), Topological Recursion relation (TRR), and WDVV equation all hold for the NMSP $g = 0$ $[0, 1]$ theory (to be introduced below); so does Givental's theory of Lagrangian cones, $S$-matrices and $R$-matrices.

0.2.5. *The* $[0,1]$ *theory and its polynomiality.* Let $p$ be the hyperplane class in the NMSP theory and $\bar\psi_k$ be the ancestor classes of the $k$-th marking. For $2g-2+n>0$ and integers $m_i \in [0, \mathrm{N}+3]$, the NMSP-correlator

$$\big\langle p^{m_1}\bar\psi_1^{k_1},\dots,p^{m_n}\bar\psi_n^{k_n}\big\rangle^{\mathrm{msp}}_{g,n,0} \in \mathbb{Q}(t)[\![q]\!] \tag{8}$$

is a polynomial in $q$ of degree at most $g-1+r$, where $r=\frac{1}{\mathrm{N}}\big(3g-3+\sum_{i=1}^n m_i\big)$. We comment that this is a simple consequence of the virtual dimension

$$\mathrm{vir.}\dim \mathcal{W}_{g,n,(d_0,0)} = Nd_0 + N(1-g)+n.$$

We say a localization graph is supported on $[0,1]$ if all of its vertices are at levels 0 and 1. Let $G^{[0,1]}_{g,n,d}$ be the set of localization graphs of $\mathcal{W}^G_{g,n,(d,0)}$ supported on $[0,1]$. We introduce

$$[\mathcal{W}_{g,n,(d_0,0)}]^{[0,1]} = \sum_{\Theta\in G^{[0,1]}_{g,n,d}} \frac{[F_\Theta]^{\mathrm{vir}}}{e(N^{\mathrm{vir}}_\Theta)}. \tag{9}$$

For any $2g-2+n>0$, and for $\tau_i(z)\in\mathcal{H}[\![z]\!]$, we define the genus $g$ NMSP-$[0,1]$-correlators to be

$$\big\langle \otimes_i^n \tau_i(\psi_i)\big\rangle^{[0,1]}_{g,n,0} = \sum_{d\geq 0}(-1)^{d+1-g}q^d\int_{[\mathcal{W}_{g,n,(d,0)}]^{[0,1]}}\prod_{i=1}^n \mathrm{ev}_i^*\tau_i(\psi_i). \tag{10}$$

Furthermore, by introducing bipartite graphs, which are graphs whose vertices are colored black and white, we can show that the NMSP-correlator can be expressed as bipartite graph sums, which can also be viewed as the NMSP-$[0,1]$ theory with insertions from the infinity contribution:

$$\Big\langle p^{m_1}\bar\psi_1^{k_1},\dots,p^{m_n}\bar\psi_n^{k_n}\Big\rangle^{\mathrm{msp}}_{g,n,0} = \sum_{\Lambda\in\Xi_{g,n}}\frac{1}{|\operatorname{Aut}\Lambda|}\prod_{v\in V_b(\Lambda)}\mathrm{Cont}^\infty_{[v]}\Big(\prod_{i\in\mathbb{L}_v}\bar\psi^{k_i}_{c(i)}\Big)$$
$$\prod_{v\in V_w(\Lambda)}\Big\langle \bigotimes_{i\in L_v} p^{m_i}\prod_{i\in\mathbb{L}_v}\bar\psi^{k_i}_{c(i)}\bigotimes_{\substack{e\in E_v,\\ f=(e,v)}}\frac{1^{\alpha(e),\mathrm{tw}}}{\frac{\mathrm{wt}_{\alpha_e}}{a_e}-\psi_f}\Big\rangle^{[0,1]}_{g_v,n_v},$$

where $\Xi_{g,n}$ is the (finite) set of all decorated bipartite graphs of genus $g$ with $n$ legs, defined in [**NMSP2**].

By degree consideration, the contributions from infinity, i.e. $\mathrm{Cont}^\infty_{[v]}$, are polynomials in $q$ with an explicit degree bound. As mentioned before, by virtual dimension consideration, the NMSP-correlators are polynomial in $q$, thus as argued, the NMSP-$[0,1]$ correlator (10) is a polynomial in $q$.

0.2.6. *The* $[0]$ *and* $[1]$ *theories and their polynomiality.* Let $f_g^{[0,1]}(q)$ be the genus $g$ NMSP-$[0,1]$ correlator:

$$f_g^{[0,1]}(q) = \big\langle p^{m_1}\bar\psi_1^{k_1},\dots,p^{m_n}\bar\psi_n^{k_n}\big\rangle^{[0,1]}_{g,n} \in \mathbb{Q}(\mathfrak{t})[q]_{g-1+r},$$

where the subscript means the upper bound of the degree. For $X = 1 - Y = \frac{-5^5 q}{1-5^5 q}$, we get

$$Y^{g-1+r} \cdot f_g^{[0,1]}(q) \in \mathbb{Q}[Y]_{g-1+r}.$$

We define the NMSP [0]-correlator $f_g^{[0]}(q)$, similar to that of $f_g^{[0,1]}(q)$, via the localization formula summing over those $\Theta$ whose stabilizations $\Theta^{st}$ have only vertices at level 0.[1] We define $f_g^{[1]}(q)$ similarly.

What we proved is the following decomposition of NMSP-$[0,1]$ correlator: We have a stable graph sum formula which expresses the $[0,1]$-potential in terms of summations over bipartite graphs. The theories corresponding to the [0]- and [1]-vertices are defined via certain $R$-matrix action on the quintic CY GW theory and the N-points theory, with potential functions $f^{[0]}$ and $f^{[1]}$. The edge contributions are given by the $V$-matrix.

We then show that for $\mathrm{N} \gg 3g-3$, we have

$$f_g^{[1]}(q) \in \mathbb{Q}[X]_{3g-3}. \tag{11}$$

We also show that the edge contribution $V$-matrix has entries in $\mathbb{Q}[X]_{3g-3}$.

$$V \in \mathbb{Q}[X]_{3g-3}. \tag{12}$$

This way the polynomiality of NMSP $[0,1]$-correlator, together with (11) and (12), implies that [0]-correlator $f^{[0]}$ is a polynomial in $X$. With a more detailed analysis, we get that its degree is at most $3g-3$:

$$f_g^{[0]}(q) \in \mathbb{Q}[X]_{3g-3}.$$

0.2.7. *The NMSP Feynman rule conjecture.* Following the argument outlined before, $f_g^{[0]}$ is expressed as a sum indexed by stable genus $g$ graphs, by putting all (stable) vertices at level 0. For each edge, it is the result of contracting various double-rooted subtrees in $\Theta$. It is a miracle that using Givental's theory, combined with the NMSP theory for N sufficiently large, the sum of these double-rooted subtrees in $\Theta$ contracting to a particular edge gives exactly the same as those appear in the BCOV Feynman rule conjecture. This leads to a proof of the BCOV Feynman rule conjecture.

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