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A survey of estimation algebras in application of nonlinear filtering problems

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Ever since the technique of Kalman-Bucy filter was popularized, due to its limitations that it needs a Gaussian assumption on the initial data and it acts on linear systems, there has been an intense interest in finding new classes of finite dimensional recursive filters. In the late seventies of last century, the idea of using estimation algebra to construct finite-dimensional nonlinear filters was first proposed by Brockett, Clark, and Mitter independently. It has been proven to be an invaluable tool in the study of nonlinear filtering problems. Since then, Yau and his coworkers were devoted to the researches of the classification of finite dimensional estimation algebras (FDEAs) with maximal rank and clarified the complete classification of it. Moreover, they shed some light on the structure of the finite dimensional estimation algebras at most dimension six. In addition, they also got some progress on the classification of FDEAs with non-maximal rank. In this survey, we shall briefly go through the development of the researches on the nonlinear filtering problems, and put emphases on the results of complete classification of FDEAs with maximal rank. And it is also presented that how to use Lie algebra method to the nonlinear filtering problems by Wei-Norman approach. Further, the recent results are given out about the structure of FDEAs with non-maximal rank.

Keywords and phrases: Finite-dimensional filter, estimation algebras, non-maximal rank, nonlinear filtering problems.

1. Introduction

The field of nonlinear filtering (NLF) problem stems from tracking and signal processing problems. The underlying formulation is so general and ubiquitous that it has wide applications to various complex dynamical systems modeled by stochastic processes. The aim of filtering is to obtain good estimates of the states in the stochastic dynamical system recursively in time,

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based on the arrival of noisy observations of the states. The states are also called signals, which can represent all kinds of quantities in various applications. And the good estimates of the states are meant to be in some sense, like in the least mean square error rule. In terms of the formulation of NLF, there is no doubt that Bayesian theory is one of the main tools, which was originally discovered by [8] in 1963. Bayesian theory is the most commonly used method for the study of the dynamic systems. Besides the Bayesian framework, the conditional density function of the states can also be obtained by numerically solving the so-called Kusher's or Duncan-Mortensen-Zakai's (DMZ) equation. It is shown in [32] that based on the observation history $\mathcal{Y}_t$, the conditional density $p(\mathbf{x}_t|\mathcal{Y}_t)$ of the states $\mathbf{x}_t$ satisfies an Itô stochastic differential equation (SDE), which is called Kusher's equation. By means of the change of measure, the unnormalized conditional density $\pi(\mathbf{x}_t|\mathcal{Y}_t)$ satisfies a linear Itô SDE, which is the so-called DMZ equation [25], [40], [69].

Solving the DMZ equation which is satisfied by the unnormalized conditional density of the system state has long been the research focus of general NLF problems. In some sense, the NLF problems are said to be completely solved, if one can solve the DMZ equation in real time and in a memoryless way because all the statistical information can be extracted from the conditional density function of the states. For the past several decades, as we know, there are two approaches to solve the DMZ equation explicitly. The first one is to take advantage of Lie algebraic method to solve DMZ equation, which method will be elaborately introduced in the sequel of this section. The second approach to solve DMZ equation is the direct method. In [59] and [66], the direct method was newly introduced to study the linear filtering and exact filtering systems with arbitrary initial condition. There are many works about direct method for filtering problems, like as [60] and [67], which systems are limited in time-invariant cases. Recently, the authors in [20] extend the direct method so that it is applicable to time-varying cases. This direct approach offers several advantages. It is easy and the derivation no longer needs controllability and observability. Thus, the algorithm is universal for any linear filtering system. Furthermore, it eliminates the necessity of integrating n first-order linear partial differential equations, as was the case in the Lie algebra method. Finally, the number of sufficient statistics required to compute the conditional probability density of the state in this direct method is n. In all these direct methods referred to as [59], [60], [66], and [67], they need to assume that all the observation terms $h_i(x)$, $1 \leq i \leq m$, are degree one polynomials.

However, it is well known that the exact solution to the DMZ equation, generally speaking, can not be written in a closed form. With the well-posed theory of the DMZ equation in mind, many mathematicians make efforts to seek an efficient algorithm to construct a "good" approximate solution to the DMZ equation. Yau and Yau in [68] developed a novel algorithm to solve the "pathwise-robust" DMZ equation, where the boundedness of the drift term and observation term is replaced by some mild growth conditions on f and g. Nevertheless, they still made the assumption that the drift term, the observation term and the diffusion term are "time-invariant". That is, f, h and g in (2.1) are not explicitly time-dependent. In [33], Luo and Yau generalized Yau-Yau's algorithm to the most general settings of the NLF problems, i.e., the "time-varying" case, where f, h and g could be explicitly time-dependent. Time-invariant system can only be seen as an ideal model of practical applications. Therefore, it is more meaningful to solve time-varying NLF problems. By extending the algorithm developed in [68] to the most general settings of NLF, Luo and Yau in [34], [36] investigated the Hermite spectral method to numerically solve the forward Kolmogorov equation, which is closely related to the implementation of the algorithm developed in [33]. In [19], Chen, Luo and Yau by transforming the Kolmogorov forward equation into a time-varying Schrodinger equation with respect to the time-varying nonlinear systems.

Ever since 1960, there are numerous research activities in NLF problems, after Kalman and Bucy first established the finite dimensional filters for linear-filtering systems with Gaussian initial distributions [30], [31]. In the later 1960s and early 1970s, the basic approach to NLF theory was via the "innovations method" originally purposed by Kailath and subsequently rigorously developed by Fujisaki, Kallianpur, and Kunita [26]. However, as pointed out by Mitter, the weakness of this approach is that in general it is not explicitly computable. In view of this weakness, in the late 1970s and early 1980s, Brockett and Clark [3], Brockett [4], and Mitter [38] independently proposed the idea of using estimation algebras to construct finite dimensional nonlinear filters, which become a basic approach to NLF problems.

The basic motivation originated from the Wei-Norman approach [50] which uses the Lie algebraic method to solve time varying linear differential equations. Brockett, Clark and Mitter's idea of using Lie algebras for solving NLF problems is to imitate the Wei-Norman approach to solve the Duncan-Mortensen-Zakai (DMZ) equation, which the unnormalized conditional probability of the state must satisfy. For more details about the Wei-Norman approach and its connection with the NLF problem, we refer the readers to paper [22], [48] and the survey article by Marcus [37]. The most important advantage of the Lie algebraic approach is that as long as the estimation algebra is finite dimensional, not only can the finite dimensional recursive filters can be constructed, but also the filter so constructed is universal in the sense of [9]. Therefore, it is very meaningful to study the estimation algebras method.

In 1981, Benés established exact finite-dimensional filters for certain diffusions with nonlinear drift which is the first important breakthrough in Lie algebra approach [6]. Later, Wong in [51] constructed some new FDEAs and used the Wei-Norman approach to construct finite dimensional filters. Another class of finite dimensional filters was found by Charalambous and Elliott [10] in advantage of the gauge transformation method, where Benés exact filtering systems were extended by inserting linear combinations of $\mathrm{d}x(t)$ in the observations. There are also some results about new finite dimensional filters with respect to various background scattered in [2], [21], [27], and [44]. However, many researchers have found that not all NLF problems allow finite dimensional filters, for instance, there exists no finite dimensional filters for the cubic sensor problem [28]. Actually, it is only a few NLF problems that can allow FDF. Mitter in [39] discussed the existence and the nonexistence of the FDF, and the sufficient condition for FDF in discrete time partially observable systems was studied in [45].

Due to the practical importance of the estimation algebra method, Brockett [5] proposed the problem of classifying all FDEAs at the 1983 International Congress of Mathematics in order to find new classes of finite dimensional filters besides the Benés exact filtering. Since then, a lot of efforts have been devoted to classifying FDEAs. Under quite severe conditions, Wong [52] proved that all FDEAs of (2.1) are solvable and the observation $h(x)$ is a polynomial of degree one. Besides, he was able to describe the structure of FDEAs under these conditions. In Wong [53], Wong introduced a fundamental notion of the Wong's Ω-matrix which plays an significant role in subsequent researches. Since the 1990s, Yau and his coworkers begun to study the algebraic structure of several general classes of estimation algebras. In [59], [58], and [62] Yau proved that the number of sufficient statistics in the Lie algebra method, which is required in the computation of conditional probability density, is linear in n, where n is the dimension of the state space. On the one hand, Yau [58] and his co-workers in a series of research works [62], [48], [64], [14], [65], and [13] have completely classified all FDEAs of maximal rank with arbitrary state space dimension [62] which included both Kalman-Bucy and Benés filtering systems as special cases. In particular, they have proved that all the observations terms $h_i(x)$, $1 \leq i \leq m$ must be degree one polynomials. On the other hand, they were able to classify all FDEAs with dimension at most six [11, 29, 57]. Due to the difficulty of the problem, Brockett suggested to understand the low-dimensional estimation algebras at first. Rasoulian and Yau in [57] have got the classification of the estimation algebras with dimension at most four. And Chiou, Chiueh, and Yau in [11] have classified the estimation algebras with dimension five. Further, Yang, Yau, and Chiou were able to give a structure theorem for FDEAs with dimension six [29].

When the rank of FDEA is not maximal, the problem is still open. Wu and Yau [55] have classified FDEAs with state dimension 2. One of the key steps that Yau and his coworkers were able to classify all finite dimensional maximal rank estimation algebras is that they were able to show that Wong's Ω-matrix is a constant matrix. And Yau and Rasoulian in [43] gave some construction of non-maximal rank FDEAs. In [46], Shi and Yau found that there exists a linear structure of Wong matrix in state dimension 3 and rank 2. And Shi, Chen, Dong and Yau further considered a new classes of finite dimensional filters with non-maximal rank of state dimension 3 and linear rank of 1 in [47]. For higher state dimensions $n \geq 4$, the problem remains to be unsolved.

Due to the derivation of the DMZ equation, the discovery of the innovation process, and the introduction of the Lie-algebraic and geometric techniques, much of the Lie-algebraic approach to the filtering problem is encapsulated into the concept of an estimation algebra. The surveys in the field of NLF are numerous, there are also a verity of survey papers about the estimation algebra theory in the NLF problems. There exists an excellent survey paper by Marcus of the earlier results [37]. Later on, Wong and Yau in chapter 2 of the book [7] presented some of the advances in estimation algebra and its application to NLF problems, which is not meant to a comprehensive account of all existing work concerning the estimation algebra idea or other related work, like as the invariance group method or Mallian calculus. Luo and Yau in [35] also have a survey paper which goes through the existing three major global approaches for nonlinear filtering: finite-dimensional nonlinear filtering, sequential Monte Carlo methods and the Yau-Yau's on-line and off-line solver of DMZ equation [68]. In this survey paper, we comprehensively described a more recent advances in the classification of FDEAs with maximal rank and non-maximal rank.

The paper is organized as follows: some basic concepts about estimation algebras are described in section 2. The classification of FDEAs with maximal rank and non-maximal rank are given in section 3 and section 5, respectively. We elaborate the structural results of FDEAs with dimension at most six in section 4. Furthermore, we use the structure results to derive finite-dimensional filters for the robust-DMZ equation by the Wei-Norman approach in section 6. And we finally arrive at conclusion in the last section.

2. Basic concepts and preliminary results

The filtering problem we consider is based on the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))dt + dw(t) & y(0) = 0, \end{cases} \tag{2.1}$$

where x, v, y, w are respectively $\mathbb{R}^n, \mathbb{R}^p, \mathbb{R}^m, \mathbb{R}^m$ valued process, and v and w are independent, standard Brownian motion. Moreover, assume f and h are C^∞ smooth, and g is an orthogonal matrix with assumption of $p = n$. $x(t)$ is referred to as the state of the system at time t and $y(t)$ as the observation at time t. Also, x_0 is the initial state, and independent of v and w.

Let $\rho(t, x)$ denote the conditional probability density of the state $x(t)$ given the observation $\{y(s) : 0 \leq s \leq t\}$. $\rho(t, x)$ is the normalized version of $\sigma(t, x)$ which satisfies the well-known Duncan-Mortensen-Zakai (DMZ) equation. Under the Stratonovich calculus, the DMZ equation can be written as

$$\begin{cases} d\sigma(t, x) = L_0\sigma(t, x)dt + \sum_{i=1}^{m} L_i\sigma(t, x)dy_i(t), \\ \sigma(0, x) = \sigma_0, \end{cases} \tag{2.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2,$$

for $i = 1, \ldots, m$, L_i is the zero-order differential operator of multiplication by h_i. And σ_0 is the probability density of the initial point x_0.

If we define

$$D_i = \frac{\partial}{\partial x_i} - f_i, \quad \eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2 \tag{2.3}$$

then we have

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right), \tag{2.4}$$

which is in a more compact form.

The normalized conditional density $\rho(t,x)$ is then given by

$$\rho(t,x) = \frac{\sigma(t,x)}{\int \sigma(t,x)}.$$

Therefore in the subsequent sections we aim to solve the DMZ equation (2.2). We need the following preliminary definitions and properties.

Definition 2.1. If X and Y are differential operators, the Lie bracket of X and Y, $[X,Y]$ is defined by $[X,Y]\phi = X(Y\phi) - Y(X\phi)$ for any C^∞ function ϕ.

Recall that a vector space $\mathcal{F}$ with the Lie bracket operation $\mathcal{F}\times\mathcal{F}\to\mathcal{F}$ denoted by $(x,y)\mapsto [x,y]$ is called a Lie algebra if the following axioms are satisfied:

(1) The Lie bracket operation is bilinear;
(2) $[x,x]=0$ for all $x\in\mathcal{F}$;
(3) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \quad (x,y,z\in\mathcal{F})$.

Definition 2.2. Let U be the set of differential operators in the form

$$A = \sum_{|(i_1,i_2,\dots,i_n)|=l+1} a_{i_1,i_2,\dots,i_n} D_1^{i_1} D_2^{i_2} \dots D_n^{i_n} \tag{2.5}$$

where nonzero functions $a_{i_1,i_2,\dots,i_n}\in\mathbb{C}^\infty(\mathbb{R}^n)$ and I_A is the finite index set of A. Each element of the index set is an n-tuple $(i_1,i_2,\dots,i_n)$ of nonnegative integers. The norm of an index $i=(i_1,i_2,\dots,i_n)$ is defined by $|i|=\sum_{l=1}^n i_l$. The order of A is denoted by $ord\,A = \max_{i=(i_1,i_2,\dots,i_n)\in I_A}|i|$. If $A=0$, $ord\,A$ is defined to be $-\infty$. It is clear that for A, $B\in U$

$$ord\,(AB) = ord\,(BA) = ord\,A + ord\,B, \tag{2.6}$$

$$ord\,(A\pm B) \le max(ord\,A, ord\,B). \tag{2.7}$$

U is a Lie algebra under the Lie bracket $[\cdot,\cdot]$ defined earlier. Two differential operators A and B in U are equal if they have the identical index sets $I_A = I_B$ and $a_{i_1,i_2,\dots,i_n} = b_{i_1,i_2,\dots,i_n}$, $\forall a_{i_1,i_2,\dots,i_n}\in I_A$. Let U_k denote the subspace of U consisting of the elements with order less than or equal to k.

In particular, $U_0 = \mathbb{C}^\infty(\mathbb{R}^n)$. As usual, *mod* is used to denote the equivalence class, i.e., if V is a subspace of U,

$$A = B,\ mod\, V \Leftrightarrow A - B \in V.$$

If $A, B \in U$, define

$$Ad_A B = [A, B],\ Ad_A^l B = [A, Ad_A^{l-1} B],\ l \geq 1, \tag{2.8}$$

where Ad_A^0 is the identity operator by standard convention.

Definition 2.3. (Chiou-Yau [13], Chen-Yau [17]). The estimation algebra E of a filtering problem (2.1) is defined to be the Lie algebra generated by $\{L_0, L_1, \cdots, L_m\}$. E is said to be the estimation algebra with maximal rank if, for any $1 \leq i \leq n$, there exists a constant c_i such that $x_i + c_i \in E$. If in addition, E is of maximal rank, then E is a real vector space of dimension $2n+2$ with basis given by $1, x_1, \cdots, x_n, D_1, \cdots, D_n$ and L_0.

In real applications, we are interested in considering robust state estimator from observed sample paths with some properties of robustness. Since the problem of designing filters with some nice continuity or robustness properties in important, Davis in [24] pointed out that by using the transformation

$$\xi(t, x) = exp\big(-\sum_{i=1}^{m} h_i(x) y_i(t)\big)\sigma(t, x), \tag{2.9}$$

one can obtain a robust form of DMZ equation,

$$\begin{cases} \dfrac{d}{dt}\xi(t, x) = L_0 \xi(t, x) + \displaystyle\sum_{i=1}^{m} y_i(t)[L_0, L_i]\xi(t, x) \\ \qquad\qquad + \dfrac{1}{2}\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t) y_j(t)[[L_0, L_i], L_j]\xi(t, x), \\ \xi(0, x) = \sigma_0. \end{cases} \tag{2.10}$$

As we can know that the (2.10), called as robust DMZ equation, is also a time-varying partial differential equation. One can define an estimation algebra for the robust version to be the Lie algebra generated by $\{L_0, [L_0, L_i], [[L_0, L_i], L_j], i, j = 1, 2, \cdots, m\}$.

Definition 2.4. Let $L(E) \subset E$ be the vector space consisting of all the homogeneous degree one polynomials in E. Then the linear rank of estimation algebra E is defined by $r := \dim L(E)$. So estimation algebra of maximal rank is in fact linear rank n estimation algebra.

Definition 2.5. The Wong matrix first introduced in [53] is a certain skew-symmetric matrix $\Omega = (\omega_{ij})$, where

$$\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}, \quad \forall 1 \le i, j \le n.$$

Obviously $\omega_{ij} = -\omega_{ji}$.

As we will notice in the following sections, Ω matrix plays a crucial role in the structure of an estimation algebra.

Let Q be the space of quadratic forms in n variables, i.e., real vector space spanned by $x_i x_j$, with $1 \le i, j \le n$. Let $X = (x_1, \cdots, x_n)^\top$, for any quadratic form $p \in Q$, there exists a symmetric matrix B such that $p(x) = X^\top B X$. The rank of the quadratic form of p is denoted by $rk(p)$ and is defined to be the rank of matrix B.

Definition 2.6. (Chen-Yau [17]). A fundamental quadratic form of the estimation algebra E is an element $p_0 \in E \bigcap Q$ with the greatest positive rank, i.e., $rk(p_0) \ge rk(p)$ for any $p \in E \bigcap Q$. The quadratic rank of the estimation algebra E is defined to be $rk(p_0)$.

Theorem 2.7. *(Wu-Yau [55], Yau-Hu [62]). Let E be a finite dimensional estimation algebra, and D_i is defined as in (2.3). If $l \ge 0$ and*

$$A = \sum_{|(i_1,i_2,\dots,i_n)|=l+1} a_{i_1,i_2,\dots,i_n} D_1^{i_1} D_2^{i_2} \dots D_n^{i_n}, mod \quad U_l$$

is in E, then $a_{i_1,i_2,\dots,i_n}$ are polynomials.

Lemma 2.8. *(Wu-Yau [55]). Let g, $h \in \mathbb{C}^\infty(\mathbb{R}^n)$ and let $i_1, \dots, i_n, j_1, \dots, j_n$ be nonnegative integers with $\sum_{l=1}^n i_l = r$, $\sum_{l=1}^n j_l = s$, and $r + s \ge 2$. Let δ_{ij} be the Kronecker symbol, then*

$$\begin{aligned}&[gD_1^{i_1} \dots D_n^{i_n}, hD_1^{j_1} \dots D_n^{j_n}] \\ &= \sum_{l=1}^n (i_l g \frac{\partial h}{\partial x_l} - j_l h \frac{\partial g}{\partial x_l}) D_1^{i_1+j_1-\delta_{1l}} \dots D_n^{i_n+j_n-\delta_{nl}}, \quad mod \ U_{r+s-2}.\end{aligned}$$

Theorem 2.9. *(Yau [64]). Let $F(x_1, \cdots, x_n)$ be a $\mathbb{C}^\infty$-function in $\mathbb{R}^n$. Suppose that there exists a path $c : \mathbb{R} \longrightarrow \mathbb{R}^n$ and $\delta > 0$ such that $\lim_{t\longrightarrow\infty} ||c(t)|| = \infty$ and $\lim_{t\longrightarrow\infty} sup_{B_\delta(c(t))} F = -\infty$, where $B_\delta(c(t)) = x \in \mathbb{R}^n : ||x - c(t)|| < \delta\}$. Then there are no $\mathbb{C}^\infty$-functions $f_1, f_2, \cdots, f_n$ on $\mathbb{R}^n$ satisfying*

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F. \tag{2.11}$$

Theorem 2.10. *(Yau [64]). Let $F(x_1, \cdots, x_n)$ be a polynomial on $\mathbb{R}^n$. Suppose that there exists a polynomial path $c : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $\lim_{t\longrightarrow\infty} ||c(t)|| = \infty$ and $\lim_{t\longrightarrow\infty} F(c(t)) = -\infty$. Then there are no $\mathbb{C}^\infty$-functions $f_1, \cdots, f_n$ on $\mathbb{R}^n$ satisfying (2.9).*

Theorem 2.11. *(Ocone [42]). Let E be a finite dimensional estimation algebra. If a function ϕ is in E, then ϕ is a polynomial of degree less than or equal to 2.*

3. Classification of finite dimensional estimation algebras with maximal rank

The concept of estimation algebras has proven to be an invaluable tool in the study of NLF problems. In 1983, Brockett proposed classifying all FDEAs. Before many researchers begin to tackle this problem, Mitter conjectured that the observation terms $h_i(x)$ are affine polynomials.

Yau [64] has begun to study a filtering system such that all entries of Ω are constants. He was able to classify all FDEAs with maximal rank and proved Mitter conjecture for such a filtering system. The program of classifying FDEAs of maximal rank was begun in 1990 by S. S.-T. Yau. There are four crucial steps [61].

- **Step1**. In 1990, Yau first observed that Wong's Ω-matrix plays an important role. As the first crucial step, he classifies all FDEAs of maximal rank if Wong's matrix has entries in constant coefficients. His results announced in CDC 1990 [56] and published in 1994 [64]. In 1991 Chiou and Yau [63] formally introduced the concept of FDEA of maximal rank and gave classification when the state space dimension n is at most 2. Their results were published in 1994 [13].
- **Step2**. The second crucial step was due to Chen and Yau in 1996 [14]. They developed quadratic structure theory in FDEAs of maximal rank

and laid down all the ingredients which are needed to give classification of FDEAs of maximal rank. In particular, they proved that all the entries of Wong's matrix are degree one polynomials. They also introduced the notion of quadratic rank k. In this way the Wong's matrix is divided into 3 parts: (i) (ω_{ij}), $1 \le i, j \le k$, (ii) (ω_{ij}), $k+1 \le i, j \le n$ and (iii) (ω_{ij}), $1 \le i \le k, k+1 \le j \le n$, or $k+1 \le i \le n, 1 \le j \le k$. In [14], Chen and Yau proved among many other things that part (i) (ω_{ij}), $1 \le i, j \le k$ is a matrix with constant coefficients.

- **Step3**. In their 1997 paper [16], Chen, Yau and Leung proved the weak Hessian matrix non-decomposition theorem for $n \le 4$. As a result, the part (ii) of the Wong's matrix, (ω_{ij}), $k+1 \le i, j \le n$ is a matrix with constant coefficients. In their paper [54], Wu, Yau and Hu proved the weak Hessian matrix non-decomposition theorem for general n. Thus part (ii) of the Wong's matrix is a matrix with constant coefficients for arbitrary n. Later on, Yau, Wu and Wong in [65] established the strong Hessian matrix non-decomposition theorem which implies the weak Hessian matrix non-decomposition theorem as a special case.
- **Step4**. In 2005, Yau and Hu in [62] using the full power of the quadratic structure theory developed by Chen and Yau [14] to prove that part (iii) of the Wong's matrix (ω_{ij}), $1 \le i \le k, k+1 \le j \le n$ and the matrix (ω_{ij}), $k+1 \le i \le n, 1 \le j \le k$ are with constant coefficients.

Some parts of results are listed below.

Theorem 3.1. *(Yau [64]). Let E be a finite dimensional estimation algebra of (2.1) satisfying $\dfrac{\partial f_j}{\partial x_i} - \dfrac{\partial f_i}{\partial x_j} = c_{ij}$, where c_{ij} are constants for all $1 \le i, j \le n$.*

(i) If η is a polynomial of degree at most two, then E is finite dimensional and has a basis consisting of $E_0 = L_0$, differential operators $E_1, \cdots, E_p$ (for some p) of the form

$$\sum_{j=1}^{n} \alpha_{ij} D_j + \beta_i, \quad 1 \le i \le p,$$

where α_{ij}'s are constants and β_i's are affine in x, and zero degree differential operators $E_{p+1}, \cdots, E_q$ (for some $q > p$) where E_i's are affine in x for $p+1 \le i \le q$. Moreover the quadratic part of $\eta - \sum_{i=1}^{m} h_i^2$ is positive semi-definite.

(ii) Conversely, if E is finite dimensional, then $h_1, \cdots, h_m$ are affine in x, i.e., the observation matrix is a constant matrix. Furthermore if the

observation matrix has rank n (in particular $m \geq n$), then η is a polynomial of degree at most two.

If the dimension of state space is two or three, then Chiou and Yau [13] and Chen, Leung, and Yau [15] have shown, respectively, that all entries of Ω are constants as long as the estimation algebra of maximal rank and finite dimension.

In [14], Chen and Yau have shown that Ω matrix is an affine matrix that every entry in Ω is an affine polynomial if the estimation algebra is of maximal rank and finite dimensional, which is a fundamental step in classifying FDEAs of maximal rank. Chen and Yau further have proven that the Mitter conjecture for finite-dimensional estimation algebra of maximal rank with arbitrary state space dimension in [17]. In the process of proving this, they also showed that the $\Omega = (\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j})$ matrix, where f denotes the drift term, has special linear structure which generalizes their previous results in [14].

Theorem 3.2. *(Chen-Yau [17]). Let E be a finite-dimensional estimation algebra of maximal rank. Let k be the quadratic rank of E. Then*

(i) the observation terms $h_i(x)$, $1 \leq i \leq m$, are affine polynomials.

(ii) for $1 \leq i \leq k$ or $1 \leq j \leq k$, ω_{ij} are constant; for $k+1 \leq i, j \leq n$, ω_{ij} are degree-one polynomials in $x_{k+1}, \cdots, x_n$.

(iii) $\eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2$ is a homogeneous polynomial of degree four. Moreover, η_4 (homogeneous polynomial of degree-four part of η) depends only on $x_{k+1}, \cdots, x_n$ variables.

If E is a FDEA, Ocone's theorem says that h_i, $1 \leq i \leq m$, are polynomials of degree at most two. Mitter conjecture asserts that h_i has to be affine (i.e., degree-one polynomial). In [17], Chen and Yau also proved that the Mitter conjecture for FDEAs of maximal rank with arbitrary state space dimension.

Theorem 3.3. *(Chen-Yau [17]). If E is a finite-dimensional estimation algebra of maximal rank, then h_i, $1 \leq i \leq m$, are degree-one polynomials.*

The classification of the FDEAs with maximal rank has been completed in [58], [62]. The following theorem can describe the complete classification of FDEAs with maximal rank.

Theorem 3.4. *(Yau-Hu [62]). Suppose that the state space of the filtering model (2.1) is of dimension n. If E is the finite-dimensional estimation algebra with maximal rank, then $f = \nabla\phi + (\alpha_1, \cdots, \alpha_n)$, where ϕ is a smooth* *function and α_i, $1 \leq i \leq n$ are affine functions and E is a real vector space of dimension $2n+2$ with basis given by $1, x_1, \cdots, x_n, D_1, \cdots, D_n$ and L_0.*

There are some stories about the complete classification of FDEAs. Following the program of Stephen Yau, Tang gave a proof of the classification of FDEA of maximal rank with arbitrary state-space dimension, i.e., Theorem 1.1 in [49]. Unfortunately, the proof of the Theorem 1.1 there was not clear. First, in Lemma 3.5 of Tang's paper, only the lower right corner of the Ω-matrix, i.e., $\omega_{ij}, k+1 \leq i, j \leq n$ were proved to be constants. This result had already been proven in Theorem 2.4 in Yau's paper [65] which appeared one year earlier. In fact, Yau's Hessian non-decomposition theorem is a much stronger result and has independent interest other than nonlinear filtering. Second, the proof of Theorem 1.1 of Tang's paper was wrong. In his paper, Lemma 3.4 combined with the Main Theorem in Chen and Yau's paper [17] gave the proof of Theorem 1.1 of Tang's paper. Nevertheless, the proof of the Main Theorem in Chen and Yau's paper [17] which states that ω_{ij}, $1 \leq i \leq k$ or $1 \leq j \leq n$ is constant was known to be wrong. Yau and Hu later gave a correct proof of ω_{ij}, $1 \leq i \leq k$ or $1 \leq j \leq n$ is constant in [62]. Finally the classification of FDEAs of maximal rank with arbitrary state-space dimension was completed after the publication of Yau and Hu's paper [62].

On the one hand, Nie's paper [41] is a weaker form of the main result in Yau's paper [65]. On the other hand, Nie's paper curiously coincided with Yau's paper [54] that circulated in China during 1997. In fact, Yau did send this paper to the academician Professor Chao-Hao Gu at Fudan University at that time. The academician Professor Jiaxing Hong knew about this fact. The second author of the paper was invited by the academicians Professor Lo Yang and Professor Lei Guo to give eight hour lectures on nonlinear filtering theory in October 1997 at Morningside Institute of Academia Sinica (now the name is Morningside Center of Mathematics, Chinese Academy of Sciences) at Beijing. The result of [54] was presented during these lecture series. Although the paper [54] was written early, where the weak form Hessian matrix non-decomposition theorem was proven, actually, it was published later, on account that the strong Hessian matrix non-decomposition theorem in [65] was discovered within two months after the success of the proof of weak Hessian matrix non-decomposition theorem in [54].

4. Structure of finite estimation algebra with dimension at most six

The problem of classification of FDEAs was formally proposed by Brockett in his lecture as International Congress of Mathematicians in 1983. However, due to the difficulty of the problem, in the early 1990s, Brockett suggested that one should understand the low-dimensional estimation algebras first. A decade later, Yau and Rosoulian in [57] have classified all four-dimensional estimation algebras for arbitrary state space dimension. Wu and Yau in [55] have discussed that FDEAs can arise when the underlying stochastic system (2.1) have state dimension 2 are completely classified. It has been shown that an estimation algebra with state dimension 2 have only $1, 2, 4, 5$ or 6 dimensions.

Theorem 4.1. *(Yau-Rosoulian [57]). Suppose the state space of the filtering system (2.1) is of dimension one. Then, the observation function, $h(x)$, is linear and the linear span of $\nabla h_1, \cdots, \nabla h_m$ is 1-D. Assume $h_1(x) = x_1$. Then, the 4-D estimation algebra has a basis given by $1, x, D = (\partial/\partial x) - f(x)$ and $L_0 = \frac{1}{2}(D^2 - \eta)$. Moreover, $[L_0, x] = D, [D, x] = 1, [L_0, D] = \frac{1}{2}(\partial\eta/\partial x)$, where $\eta = \alpha x^2 + 2\beta x + \gamma$. Here, α, β, γ are constants.*

In particular, f has to satisfy the equation $f' + f^2 = (\alpha - 1)x^2 + 2\beta x + \gamma$, where $\alpha - 1 \geq 0$ and $\sqrt{\alpha - 1} \geq (\beta^2/\alpha - 1) - \gamma$.

Theorem 4.2. *(Yau-Rosoulian [57]). Suppose the state space of the filtering system (2.1) is a dimension greater than one. Then, the observation function, $h(x)$, is linear and the linear span of $\nabla h_1, \cdots, \nabla h_m$ is 1-D. Assume $h_1(x) = x_1$. Then, the 4-D estimation algebra has a basis given by $1, x_1, D_1 = (\partial/\partial x_1) - f_1(x_1, \cdots, x_n)$ and $L_0 = \frac{1}{2}(\sum_{i=1}^n D_i^2 - \eta)$. Moreover, $\omega_{12} = \omega_{13} = \cdots = \omega_{1n} = 0$, $[L_0, x_1] = D_1, [D_1, x_1] = 1, [L_0, D_1] = \frac{1}{2}(\partial\eta/\partial x_1) = \alpha x_1 + \beta$, where α, β are constants. Also, $\eta = \alpha x_1^2 + 2\beta x_1 + q(x_2, \cdots, x_n)$, where $q(x_2, \cdots, x_n)$ is in $\mathbb{C}^\infty(\mathbb{R}^{n-1})$.*

In particular, $f_1, \cdots, f_n$ have to satisfy the equation $\sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 = (\alpha - 1)x_1^2 + 2\beta x_1 + q(x_2, \cdots, x_n)$, where $\alpha \geq 1$.

Hopefully, the results given in [11, 29] shed some light on the non-maximal rank FDEAs. The authors in [11] gave a structure theorem for estimation algebras of dimension five, and by using this structure theorem, they have found a new class of FDEAs. Corresponding to this, in [29], Yang and Yau also gave a structure theorem for estimation algebras with dimension six. We just go through some parts of results about the Mitter conjecture for the low-dimensional estimation algebras in nonlinear filtering below.

Lemma 4.3. *(Chiou-Chiueh-Yau [12]). For any $1 \leq l \leq n$, if $\phi_i, i = 1, \cdots, l$, are polynomials in $x_1, \cdots, x_l$ with coefficients in $\mathbb{C}^\infty$ functions of*

$x_{l+1}, \cdots, x_n$ *satisfying*

$$\frac{\partial \phi_j}{\partial x_i} + \frac{\partial \phi_i}{\partial x_j} = 0, \quad for \quad all \quad 1 \le i, j \le l, \tag{4.1}$$

then each ϕ_i *is necessary of the form*

$$\phi_i = \sum_{1 \le j \le l} c_i^j(x_{l+1}, \cdots, x_n) x_j + d_i(x_{l+1}, \cdots, x_n), \tag{4.2}$$

where $c_i^j(x_{l+1}, \cdots, x_n)$ *and* $d_i(x_{l+1}, \cdots, x_n)$ *are* $\mathbb{C}^\infty$ *functions and* $c_i^j = -c_j^i$.

Lemma 4.4. *(Chiou-Chiueh-Yau [12]). If* $\dim E = 5$*, then* E *cannot contain two linear independent degree one polynomials. Furthermore,* E *cannot contain any degree two polynomial.*

Theorem 4.5. *(Chiou-Chiueh-Yau [12]). Let* E *be a finite dimensional estimation algebra associate to the filtering model (2.1) with arbitrary state space dimension. Then an function in* E *is a polynomial of degree at most one if* $\dim E \le 5$.

5. Classification of estimation algebras with non-maximal rank

Although the classification of FDEAs of maximal rank was completed by Yau and his coworkers Chen, Chiou, Hu, Wong and Wu, the finite dimensional filter can also be constructed from the FDEAs with non-maximal rank, see [43]. However, due to the difficulty of the problem, the classification of the non-maximal rank is still wide open, except some partial results, including those for low-dimensional estimation algebra with arbitrary states' dimension [57], [11], [29]; the classification with state dimension 2 and arbitrary dimensional estimation algebra [55]. In [46], Shi and Yau considered FDEAs with state dimension three and rank equal to 2.

Theorem 5.1. *(Chiou-Chiueh-Yau [11]). Suppose that the state space of the filtering model (2.1) is of dimension at least two. Then the five-dimensional estimation algebra is isomorphic to a Lie algebra generated by* L_0 *and an observation function* $h = x_1$ *with a basis given by* $1, x_1, D_1 = (\partial/\partial x_1) - f_1(x_1, \cdots, x_n), Y_1 = [L_0, D_1] = \sum_{i=1}^n \omega_{i1} D_i + \frac{1}{2}(\partial \eta / \partial x_1), L_0 = \frac{1}{2}(\sum_{i=1}^n D_i^2 - \eta)$*. Moreover, the following holds:*

(1) $\omega_{1i} \neq 0$ for some $i = 2, \cdots, n$ and each ω_{1i} is of the form

$$\omega_{1i} = \sum_{i=2}^{n} e_{ik} x_k + e_i, \; for \, 2 \leq i \leq n, \tag{5.1}$$

where $e_{ij} = -e_{ji}$, for $2 \leq i, j \leq n$, e_{ij} and e_i are constants.

(2) η is of the form

$$\eta = \left(\sum_{j=2}^{n} \omega_{1j}^2 + C_1 \right) x_1^2 + \beta(x_2, \cdots, x_n) x_1 + \phi(x_2, \cdots, x_n), \tag{5.2}$$

where $C_1 \geq 1$ is a constant and $\beta(x_2, \cdots, x_n)$ and $\phi(x_2, \cdots, x_n)$ are $\mathbb{C}^\infty$ functions.

(3) There exists a constant C_2 such that

$$\sum_{j=1}^{n} \omega_{1j} \omega_{ji} + \frac{1}{2} \frac{\partial^2 \eta}{\partial x_i \partial x_1} = C_2 \omega_{1i}, for 2 \leq i \leq n, \tag{5.3}$$

(4) There exists constants C_0 and C_3 such that

$$-\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial \omega_{1j}}{\partial x_i} \omega ji + \frac{1}{2} \sum_{j=1}^{n} \omega_{1j} \frac{\partial \eta}{\partial x_j} = C_0 x_1 + \frac{C_2}{2} \frac{\partial \eta}{\partial x_1} + C_3. \tag{5.4}$$

In particular, $f_1, \cdots, f_n$ have to satisfy the following equation:

$$\frac{\partial f_i}{\partial x_i} + \sum_{j=1}^{n} f_j^2 = \left(\sum_{j=2}^{n} \omega_{1j}^2 + C_1 - 1 \right) x_1^2 + \beta(x_2, \cdots, x_n) x_1 + \phi(x_2, \cdots, x_n). \tag{5.5}$$

The following result in [46] considered by Shi and Yau, is the most recent result about FDEAs with non-maximal rank.

Theorem 5.2. *(Shi-Yau [46]). Let E be the finite dimensional estimation algebra of (2.1) with state dimension 3 and rank 2. Then the Ω-matrix has linear structure, i.e., all the entries in the Ω-matrix are degree one polynomials.*

6. Explicit construction of nonlinear filters for NLF problem

The objective of constructing a robust finite-dimensional filter to system (2.1) is equivalent to finding a smooth manifold, $\mathcal{M}$, and complete $\mathbb{C}^\infty$ vector fields, μ_i, on $\mathcal{M}$ and $\mathbb{C}^\infty$ functions, ν, on $\mathcal{M}\times\mathbb{R}\times\mathbb{R}^n$ and w_i's on $\mathbb{R}^m$, such that $\xi(t,x)$ can be represented in the form:

$$\begin{cases} \dfrac{dz(t)}{dt} = \sum_{i=1}^{k} \mu_i(z(t))w_i(y(t)), \quad z(0)\in\mathcal{M} \\ \xi(t,x) = \nu(z(t),t,x). \end{cases} \tag{6.1}$$

Yau in [64] has constructed a class of finite-dimensional filter for NLF problem by utilization of estimation algebra techniques. It is referred as Yau filter in [18], which includes the Kalman-Bucy filter and Beneš filter as special cases. Yau also gave a necessary and sufficient condition to guarantee the estimation algebra to be finite-dimensional. In this section, we will use the structural results of previous sections to derive finite-dimensional filters by Wei-Norman approach.

Definition 6.1. (Wei-Norman [50], Yau [64]). Suppose X is a differential operator, ρ_0 is the domain of X, r is the continuous function, and $R(t) = \int_0^t r(s)ds$. We denote by $e^{R(t)X}\rho_0$ the solution at time T of the following equation

$$\frac{d\rho(t,x)}{dt} = r(t)X\rho(t,x), \quad \rho(0,x)=\rho_0(x) \tag{6.2}$$

if it is well defined.

For $1\le i\le n$, $e^{tD_i}\rho_0(x)$ can be expressed in the form:

$$e^{tD_i}\rho_0(x) = \rho_0(x_1,\cdots,x_i+t,\cdots,x_n)e^{-\int_0^t f_i(x_1,\cdots,x_i+t-s,\cdots,x_n)ds}. \tag{6.3}$$

In particular, the following theorem from [64] shows how to construct finite-dimensional filters from FDEAs with maximal rank.

Theorem 6.2. *(Yau [64]). Let E be an estimation algebra of (2.1) satisfying $\dfrac{\partial f_j}{\partial x_i} - \dfrac{\partial f_i}{\partial x_j} = c_{ij}$, where the c_{ij}'s are constants for all $1\le i,j\le n$. Suppose that E is a finite dimensional estimation algebra of maximal rank. Then E has a basis of the form $1, x_1,\cdots,x_n, D_1,\cdots,D_n$ and L_0, and $\sum_{i=1}^n \dfrac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{k=1}^m h_k^2$ is a degree two polynomial $\sum_{i,j=1}^n a_{ij}x_ix_j + \sum_{i=1}^n b_ix_i + d$, where D_i and L_0 are defines in (2.3) and (2.4). The robust DMZ equation (2.10) has a solution for all $t\ge 0$ of the form*

$$u(t,x) = e^{T(t)}e^{r_n(t)x_n}\cdots e^{r_1(t)x_1}e^{s_n(t)D_n}\cdots e^{s_1(t)D_1}e^{tL_0}\sigma_0,$$

where $T(t), r_1(t), \cdots, r_n(t), s_1(t), \cdots, s_n(t)$ satisfy the following ODEs:

$$\frac{ds_i}{dt}(t) = r_i(t) + \sum_{j=1}^{n} s_j(t)c_{ji} + \sum_{k=1}^{m} h_{ki}y_k(t), 1 \le i \le n;$$

$$\frac{dr_j}{dt}(t) = \frac{1}{2}\sum_{i=1}^{n} s_i(t)(a_{ij} + a_{ji}), 1 \le j \le n;$$

$$\begin{aligned}\frac{dT}{dt} &= \frac{1}{2}\sum_{i=1}^{n} r_i^2(t) - \frac{1}{2}\sum_{i=1}^{n} s_i^2(t)\left(\sum_{j=1}^{n} c_{ij}^2 - a_{ij}\right) + \sum_{i=1}^{n} r_i(t) - \sum_{j=2}^{n}\sum_{i=1}^{j} s_j(t)c_{ij} \\ &+ \sum_{1\le i<k\le n} s_i(t)s_k(t)\left[\sum_{j=1}^{n} c_{ij}c_{jk} + \frac{1}{2}(a_{ik} + a_{ki})\right] + \frac{1}{2}\sum_{i=1}^{n} s_i(t)b_i \\ &+ \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)\sum_{k=1}^{n} h_{ik}h_{jk} - \sum_{i,j=1}^{n} s_i(t)r_j(t)c_{ij},\end{aligned}$$

where $h_k(x) = \sum_{j=1}^{n} h_{kj}x_j + e_k$, $1 \le k \le m$, h_{kj} and e_k are constants. In particular, a universal finite-dimensional filter exists.

The characterization of the condition $\dfrac{\partial f_j}{\partial x_i} - \dfrac{\partial f_i}{\partial x_j} = c_{ij}$, where c_{ij} are constants for all $1 \le i, j \le n$, is also given in [64].

Theorem 6.3. *(Yau [64]). $\dfrac{\partial f_j}{\partial x_i} - \dfrac{\partial f_i}{\partial x_j} = c_{ij}$, where c_{ij} are constants for all $1 \le i, j \le n$, if and only if*

$$(f_1, \cdots, f_n) = (l_1, \cdots, l_n) + \left(\frac{\partial \psi}{\partial x_1}, \cdots, \frac{\partial \psi}{\partial x_n}\right),$$

where $l_1, \cdots, l_n$ are polynomials of degree one and ψ is a $\mathbb{C}^\infty$ function.

7. Conclusion

In this survey, we can know that the Lie algebraic method provides an important research direction for NLF theory. By interpreting the DMZ equation or its robust form as a partial differential equation with time varying parameters, one derives an approach to filtering based on Lie algebra as well as the theory of linear differential operators. The research and construction of the finite-dimensional filter are turned into the study of the structure of the estimation algebra. In return, the theory of estimation algebra provides a systematic tool to deal with questions concerning the finite-dimensional filters. It has led to a number of new results corresponding to finite-dimensional filters and to a deeper understanding of the structure of NLF in general. More importantly, the finite-dimensionality of the estimation algebra guarantees the explicit construction of the finite-dimensional recursive filter. In terms of the significant application of the Lie algebra method to a variety of NLF problems, it is urge to figure out the structure of the FDEAs. Therefore, in this survey, we go through the results of complete classification of FDEAs and how to use Lie algebra method to the NLF problems by Wei-Norman approach, and further give out the recent results on the structure of the FDEAs with non-maximal rank.

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