Communications in Information and Systems
Volume 21, Number 2, 209–218, 2021

A note on generalized CIR equations

Michał Barski and Jerzy Zabczyk

Dedicated to Professor Tyron Duncan

The note is a complement to the paper [2] by the authors on the generalized CIR equation. We provide here a stochastic analysis proof of a crucial step of the proof in [2] which required there some advanced results on infinitesimal generators of a class of Markov processes.

AMS 2000 subject classifications: Primary 60H10, 60G51; secondary 91G30.
Keywords and phrases: Cox-Ingersoll-Ross model, bond market, short rate, positivity of stochastic equations.

Solutions to the classical Cox, Ingersol, Ross (CIR) equation

$$dR(t) = (aR(t) + b)dt + c\sqrt{R(t)}dW(t), \tag{1}$$

where $a, c \geq 0$ and W is a Wiener process, are nonnegative, paths continuous processes which generate affine term structure of the mathematical finance, see Cox, Ingersoll, Ross [7], Björk [5, p. 375, 380, 383], Filpović [8, p. 87], Carmona, Tehranchi [6, p. 53]. This means that there exist smooth nonnegative deterministic functions $A(s), B(s), s \geq 0$ such that the process R has the following martingale property (MP): for each $T > 0$, the discounted bond price process

$$\hat{P}(t,T) = P(t,T)e^{-\int_0^t R(s)ds}, \quad t \in [0,T], \tag{2}$$

is a local martingale. The bond prices $P(t,T)$ are of the form:

$$P(t,T) = e^{-A(T-t)-B(T-t)R(t)}, \quad t \in [0,T], \tag{3}$$

and R is interpreted as the short rate process. If R is a solution of (1) then the bond market with the bond prices (3) is arbitrage free.

The paper [2] by the authors drops the requirement that the short rate process has continuous paths leaving all the other properties valid. In [2] the equations for short rate processes are assumed to be of the form

$$(4) \qquad dR(t) = F(R(t))dt + G(R(t-))dZ(t), \quad R(0) = x, \quad t > 0,$$

where Z stands for a Lévy process which is also a martingale.

Let us recall that the Laplace exponent $J(z)$ of Z is defined, for z wherever it is finite, by the identities:

$$\mathbb{E}[e^{-zZ(t)}] = e^{tJ(z)}, \qquad t \geq 0$$

and it admits the representation:

$$J(z) = \frac{1}{2}qz^2 + \int_{-\infty}^{+\infty} (e^{-zy} - 1 + zy)\nu(dy),$$

see Proposition 5.3.4 in [3].

The measure $\nu(dy)$ is called a Lévy measure or a jump measure of Z and $q \geq 0$. In fact, $J(z)$ is well defined for z satisfying

$$\int_{\{|y|>1\}} e^{-zy}\nu(dy) < +\infty,$$

see formula 5.2.8 in [3].

In particular, if $Z = Z^\alpha$ is an α-stable martingale, $\alpha \in (1, 2]$, with positive jumps, then

$$J(z) = c_\alpha z^\alpha, \quad z \geq 0.$$

In the case $\alpha \in (1, 2)$ the Lévy measure of Z^α has the form

$$\nu(dy) = \frac{1}{y^{1+\alpha}}\mathbf{1}_{(0,+\infty)}(y)dy,$$

and $c_\alpha := \Gamma(2-\alpha)/\alpha(\alpha-1)$, where Γ stands for the Gamma function. For $\alpha = 2$ the jump measure disappears, Z^2 is a standard Wiener process W and $c_2 = \frac{1}{2}q$, $q = \mathbb{E}[W(1)]^2$.

The aim of [2] was to determine all equations (4) having nonnegative solutions for each $x \geq 0$ and satisfying (MP). The following two theorems were established in [2]. Theorem 1 determines necessary conditions which the coefficients and the noise process should satisfy if (MP) holds and Theorem 2 shows that they are also sufficient.

Theorem 1. *Assume that solutions to the equation (4) with functions F, G which are continuous on $[0, +\infty)$ have the martingale property (MP) with some functions A, B satisfying*

$$A(0) = 0, \quad B(0) = 0, \quad A'(0) = 0, \quad B'(0) = 1.$$

I) If G is differentiable on $(0, +\infty)$ and $G(\bar{x}) > 0$, $G'(\bar{x}) \neq 0$ for some $\bar{x} > 0$, then

a) $Z = Z^{\alpha}$ is an α-stable martingale, for some $\alpha \in (1, 2]$, with positive jumps,

b) $F(x) = ax + b$ with $a \in \mathbb{R}$, $b \geq 0$, $x \geq 0$,

c) $G(x) = c^{\frac{1}{\alpha}} x^{\frac{1}{\alpha}}$, $c > 0$, $x \geq 0$.

II) If G is a positive constant σ, then

d) Z has no Wiener part,

e) the martingale Z has positive jumps and $\int_0^{+\infty} y\nu(dy) < +\infty$,

f) $F(x) = ax + b$, $x \geq 0$, with $a \in \mathbb{R}$, $b \geq \sigma \int_0^{+\infty} y\nu(dy)$.

Theorem 2. *I) For arbitrary $\alpha \in (1, 2]$ and $a \in \mathbb{R}, b \geq 0$, $c > 0$, the equation*

$$dR(t) = (aR(t) + b)dt + c^{\frac{1}{\alpha}} R(t-)^{\frac{1}{\alpha}} dZ^{\alpha}(t), \quad R(0) = x \geq 0, \tag{5}$$

has a unique non-negative strong solution and satisfies, with some functions A and B, the martingale property (MP). Moreover, the functions A, B in (3) are such that B solves the equation

$$B'(v) = -cc_{\alpha}[B(v)]^{\alpha} + aB(v) + 1, \quad v \geq 0, \quad B(0) = 0, \tag{6}$$

and A is given by $A'(v) = bB(v), v \geq 0$, $A(0) = 0$.

II) If G is a positive constant σ and Z is such that (d), (e), (f) in Theorem 1 hold, then the equation

$$dR(t) = (aR(t) + b) + \sigma dZ(t), \;\; R(0) = x \geq 0, \;\; t > 0,$$

has the martingale property (MP) and its solutions are non-negative processes. Moreover, A, B are given by

(7)

$$B'(v) = B(v)a + 1, \;\; B(0) = 0,$$

(8)
$$A'(v) = B(v)(b - \sigma \int_0^{+\infty} y\nu(dy)) + \int_0^{+\infty} (1 - e^{-\sigma B(v) y})\nu(dy), \ A(0) = 0.$$

Open problem A generalization of the above theorems to equations

$$dR(t) = F(R(t))dt + \sum_{j=1}^{d} G_j(R(t-))dZ_j(t), \quad R(0) = x, \quad t > 0, \tag{9}$$

with a d-dimensional Lévy martingale (Z_j), is an open problem. For some partial results see [3].

The proofs were given in [2]. However, the proof of the final crucial step 4 of the Theorem 1 was based on a result due to Filipović [9] on Markovian infinitesimal generators, whereas the steps 1, 2 and 3, were based on stochastic analysis arguments only.

It is our aim here to give a direct, stochastic analysis proof of the step 4 without referring to the general theory of Markov processes. Our proof is a consequence of the following Theorem 3 on nonnegative solutions to (4). We think that the theorem and its proof are of independent interest and might be useful for a multidimensional extension. They were announced and established in our earlier arxiv-preprint [1].

Theorem 3. *Consider a stochastic equation*

$$dR(t) = F(R(t))dt + G(R(t-))dZ^{\alpha}(t), \tag{10}$$

where G is a Lipschitz function, F continuous on $[0, +\infty)$ and Z^{α} is an α-stable martingale with index $\alpha \in (1, 2)$. If solutions of (10) are nonnegative then necessarily $G(0) = 0$.

Before proving Theorem 3 we sketch the proof of the step 4 using Theorem 3. Let us recall that $\bar{I} = (\bar{a}, \bar{b})$ denoted in the paper [2] the maximal interval containing $\bar{x}$ such that $G(x) > 0$ for $x \in \bar{I}$. The aim of the step 4 was to show that $\bar{I} = (0, +\infty)$ and that

$$G(x) = c^{\frac{1}{\alpha}} x^{\frac{1}{\alpha}}, \quad c > 0, \quad x \in [0, +\infty). \tag{11}$$

Proof of the step 4. From the steps 1, 2 and 3 of the proof of Theorem 1 we know that Z is an α-stable martingale, $\alpha \in (1, 2]$, with Laplace exponent $J(z) = c_{\alpha} z^{\alpha}, z \in [0, +\infty)$ and that

$$\alpha c_{\alpha} G^{\alpha-1}(x) G'(x) B^{\alpha}(v) = B(v)a + 1 - B'(v), \quad x \in \bar{I}, \quad v \geq 0,$$

where a is a constant. We can find $\tilde{v} > 0$ such that $B(\tilde{v}) \neq 0$. Then

$$\alpha c_\alpha G^{\alpha-1}(x)G'(x) = M, \quad x \in \bar{I}, \tag{12}$$

with $M := (B(\tilde{v})a + 1 - B'(\tilde{v}))/B^\alpha(\tilde{v})$. Now we show that $\bar{I} = (0, +\infty)$. Assume that $\bar{a} > 0$. Since, by definition, $\lim_{x\downarrow\bar{a}} G(x) = 0$, we see from (12) that $\lim_{x\downarrow\bar{a}} G'(x) = \pm\infty$, which contradicts the differentiability of G on $(0, +\infty)$. Similarly one can exclude the case $\bar{b} < +\infty$. Solving (12) we obtain

$$G(x) = \Big(G(\bar{x}) - \frac{M}{c_\alpha}\bar{x} + \frac{M}{c_\alpha}x\Big)^{\frac{1}{\alpha}} := (m_1 + m_2 x)^{\frac{1}{\alpha}}, \quad x \in (0, +\infty),$$

with $m_1 \geq 0, m_2 > 0$. If $m_1 > 0$ then G is Lipschitz at zero and by Theorem 3, $G(0) = 0$ which is a contradiction. Hence (11) follows with $c := m_2$. □

Proof of Theorem 3. In the proof we use the classical maximal inequality

$$\mathbb{P}(\sup_{s\in[0,t]} \mid X(s) \mid \geq r) \leq \frac{3}{r}\mathbb{E} \mid X(t) \mid, \quad t > 0, \tag{13}$$

where X is a càdlàg submartingale, see Proposition 7.12 in [11] and the following auxiliary lemma:

Lemma 1. *Let us assume that $g(s), s \geq 0$ is a predictable process satisfying*

$$\mathbb{E}\int_0^t \mid g(s) \mid^p ds < +\infty, \quad t \geq 0,$$

with $2 \geq p > \alpha > 1$. Then

$$\mathbb{E} \mid \int_0^t g(s)dZ_0^\alpha(s) \mid^p \leq \frac{c_p}{p-\alpha}\mathbb{E}\int_0^t \mid g(s) \mid^p ds, \quad t \geq 0, \tag{14}$$

with some $c_p > 0$.

Here Z_0^α is a modified α-stable martingale Z_0^α with the Lévy measure

$$\nu(dy) = \mathbf{1}_{(0,1)}(y)\frac{1}{y^{1+\alpha}}dy.$$

Its jumps are thus bounded by 1 and it is identical with the process Z^α on the interval $[0, \tau_1)$, where τ_1 is the first jump of Z^α exceeding 1.

Proof of Lemma 1. Since the quadratic variation of the integral $\int g(s)dZ_0^\alpha(s)$ equals

$$\Big[\int g(s)dZ_0^\alpha(s)\Big](t) = \int_0^t\int_0^1 g^2(s)y^2\pi_0(ds,dy)$$

where π_0 stands for the jump measure of Z_0^α, by the Burkholder-Davis-Gundy inequality we obtain, for some $c_p > 0$,

$$\begin{aligned}\mathbb{E}\mid\int_0^t g(s)dZ_0^\alpha(s)\mid^p &\le c_p\mathbb{E}\Big[\int g(s)dZ_0^\alpha(s)\Big]^{\frac{p}{2}}(t)\\ &= c_p\mathbb{E}\Big(\int_0^t\int_0^1 g^2(s)y^2\pi_0(ds,dy)\Big)^{\frac{p}{2}},\end{aligned}$$

and further, since $p/2 \le 1$,

$$\begin{aligned}\mathbb{E}\mid\int_0^t g(s)dZ_0^\alpha(s)\mid^p &\le c_p\mathbb{E}\int_0^t\int_0^1 \mid g(s)\mid^p y^p ds\frac{1}{y^{1+\alpha}}dz\\ &\le c_p\mathbb{E}\int_0^t \mid g(s)\mid^p ds\cdot\int_0^1\frac{y^p}{y^{1+\alpha}}dy\\ &\le c_p\mathbb{E}\int_0^t \mid g(s)\mid^p ds\cdot\int_0^1\frac{y^p}{y^{1+\alpha-p}}dy\\ &\le \frac{c_p}{p-\alpha}\mathbb{E}\int_0^t\mid g(s)\mid^p ds.\end{aligned}$$

□

Now we continue the proof of Theorem 3. We adopt the proof of Milian [12] for the Wiener noise, which goes back to Gihman, Skorohod [10]. Let us consider (10) with $x = 0$. Then we can write R in the form

$$R(t) = \int_0^t F(R(s))ds + \int_0^t (G(R(s-)) - G(0))dZ^\alpha(s) + G(0)Z^\alpha(t), \quad t > 0.$$

Dividing by $t^{\frac{1}{\alpha}}$ yields

$$\begin{aligned}\frac{1}{t^{\frac{1}{\alpha}}}R(t) &= \frac{1}{t^{\frac{1}{\alpha}}}\int_0^t F(R(s))ds + \frac{1}{t^{\frac{1}{\alpha}}}\int_0^t (G(R(s-)) - G(0))dZ^\alpha(s)\\ &\quad + \frac{1}{t^{\frac{1}{\alpha}}}G(0)Z^\alpha(t),\end{aligned} \tag{15}$$

for $t>0$. Since

$$\liminf_{t\to 0}\frac{1}{t^{\frac{1}{\alpha}}}Z^{\alpha}(t)=-\infty,\quad \limsup_{t\to 0}\frac{1}{t^{\frac{1}{\alpha}}}Z^{\alpha}(t)=+\infty,$$

see [4], Theorem 5 in Section VIII, the last term in (15) becomes negative for some sequence $t_n\downarrow 0$ providing that $G(0)\neq 0$. Since

$$\frac{1}{t^{\frac{1}{\alpha}}}\int_0^t F(R(s))ds \underset{t\to 0}{\longrightarrow} 0,$$

the assertion is true if we show that

$$\frac{1}{t^{\frac{1}{\alpha}}}\int_0^t (G(R(s-))-G(0))dZ^{\alpha}(s) \underset{t\to 0}{\longrightarrow} 0.$$

Let us denote $g(s):=G(R(s-))-G(0)$. In the neighborhood of zero we can replace Z^{α} by Z_0^{α}. Then, by (13), for the submartingale $\mid\int_0^s g(u)dZ_0^{\alpha}(u)\mid^p$, with $2>p>\alpha>1$, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le s\le t}\frac{1}{t^{\frac{1}{\alpha}}}\left|\int_0^s g(u)dZ_0^{\alpha}(u)\right|>\varepsilon\right)&=\mathbb{P}\left(\sup_{0\le s\le t}\left|\int_0^s g(u)dZ_0^{\alpha}(u)\right|^p>(\varepsilon t^{\frac{1}{\alpha}})^p\right)\\ &\le\frac{3}{(\varepsilon t^{\frac{1}{\alpha}})^p}\mathbb{E}\left|\int_0^t g(u)dZ_0^{\alpha}(u)\right|^p.\end{aligned}\tag{16}$$

It follows from (14) that

$$\frac{3}{(\varepsilon t^{\frac{1}{\alpha}})^p}\mathbb{E}\mid\int_0^t g(u)dZ_0^{\alpha}(u)\mid^p\le\frac{3c_p}{(\varepsilon t^{\frac{1}{\alpha}})^p(p-\alpha)}\int_0^t\mathbb{E}\mid g(u)\mid^p du.\tag{17}$$

Since G is Lipschitz, so

$$\mathbb{E}\mid g(u)\mid^p=\mathbb{E}\mid G(R(u-))-G(R(0))\mid^p\le K\cdot\mathbb{E}\mid R(u-)\mid^p,\tag{18}$$

with some constant $K>0$. By (16), (17) and (18) we obtain thus

$$\mathbb{P}\left(\sup_{0\le s\le t}\frac{1}{t^{\frac{1}{\alpha}}}\mid\int_0^s g(u)dZ_0^{\alpha}(u)\mid>\varepsilon\right)\le\frac{3Kc_p}{(\varepsilon t^{\frac{1}{\alpha}})^p(p-\alpha)}\int_0^t\mathbb{E}\mid R(u-)\mid^p du.\tag{19}$$

Therefore, for a sequence $\{a_k\}$ we obtain

$$H(k) := \mathbb{P}\left(\sup_{2^{-k}\le s\le 2^{-k+1}} \frac{1}{s^{\frac{1}{\alpha}}} \mid \int_0^s g(u)dZ_0^\alpha(u) \mid > a_k \right)$$
$$\le \mathbb{P}\left(\sup_{2^{-k}\le s\le 2^{-k+1}} \frac{1}{(2^{-k+1})^{\frac{1}{\alpha}}} \Big(\frac{2^{-k+1}}{s}\Big)^{\frac{1}{\alpha}} \mid \int_0^s g(u)dZ_0^\alpha(u) \mid > a_k \right)$$
$$\le \mathbb{P}\left(\sup_{0\le s\le 2^{-k+1}} 2^{\frac{1}{\alpha}} \frac{1}{(2^{-k+1})^{\frac{1}{\alpha}}} \mid \int_0^s g(u)dZ_0^\alpha(u) \mid > a_k \right)$$
$$\le \mathbb{P}\left(\sup_{0\le s\le 2^{-k+1}} \frac{1}{(2^{-k+1})^{\frac{1}{\alpha}}} \mid \int_0^s g(u)dZ_0^\alpha(u) \mid > \frac{a_k}{2^{\frac{1}{\alpha}}} \right), \quad k = 0, 1, \ldots,$$

and, consequently, by (19),

(20)
$$H(k) \le \frac{3Kc_p}{\left(\frac{a_k}{2^{\frac{1}{\alpha}}}(2^{-k+1})^{\frac{1}{\alpha}}\right)^p (p-\alpha)} \int_0^{2^{-k+1}} \mathbb{E} \mid R(u-) \mid^p du, \quad k = 0, 1, \ldots.$$

Now we estimate the integral $\int_0^t \mathbb{E} \mid R(u-) \mid^p du$ for $t > 0$. We can assume that F and G are bounded because we investigate the behaviour of R before it leaves a neighborhood of zero. Then

$$\mid R(t) \mid^p \le 2^{p-1} \left(\mid \int_0^t F(R(s))ds \mid^p + \mid \int_0^t G(R(s-))dZ_0^\alpha(s) \mid^p \right),$$

and, consequently,

$$\mathbb{E} \mid R(t) \mid^p \le 2^{p-1}(ct^p + \mathbb{E}\int_0^t \mid G(R(s-)) \mid^p ds) \le \tilde{c}t,$$

with some constants $c, \tilde{c}$. Hence

(21)
$$\int_0^t \mathbb{E} \mid R(u-) \mid^p du = \int_0^t \mathbb{E} \mid R(u) \mid^p du \le \tilde{c}\int_0^t ds = \frac{\tilde{c}}{2}t^2, \quad t > 0.$$

By (20) and (21) we obtain finally

$$H(k) \le \frac{3Kc_p}{\left(\frac{a_k}{2^{\frac{1}{\alpha}}}(2^{-k+1})^{\frac{1}{\alpha}}\right)^p (p-\alpha)} \frac{\tilde{c}}{2}(2^{-k+1})^2$$

$$= \frac{3Kc_p\tilde{c}2^{\frac{p}{\alpha}-1}}{p-\alpha} \cdot \frac{1}{a_k^p}(2^{-k+1})^{2-\frac{p}{\alpha}}, \quad k = 0, 1, \ldots.$$

Taking $a_k = \frac{1}{k}$ and $\delta := 2 - p/\alpha > 0$ we obtain that

$$\sum_{k=0}^{+\infty} H_k < +\infty,$$

and, by the Borel-Cantelli lemma,

$$\frac{1}{t^{\frac{1}{\alpha}}} \int_0^t G(R(s-) - G(0))dZ_0^\alpha(s) \underset{t\to 0}{\longrightarrow} 0,$$

as required. □

References

[1] M. Barski and J. Zabczyk, On generalized CIR equations, (2019), preprint, arXiv:1902.08976. MR4065191

[2] M. Barski and J. Zabczyk, On CIR equations with general factors, (2020), *SIAM Journal on Financial Mathematics*, 11, 1, 131–147. MR4065191

[3] M. Barski and J. Zabczyk, Mathematics of the Bond Market, (2020), Cambridge University Press.

[4] J. Bertoin, Lévy Processes, (1996), Cambridge University Press. MR1406564

[5] T. Björk, Arbitrage Theory in Continuous Time, (2009), Oxford University Press.

[6] R. A. Carmona and M. R. Tehranchi, Interest Rate Models: an Infinite Dimensional Stochastic Analysis Perspective, (2006), Springer. MR2235463

[7] J. Cox, J. Ingersoll and S. Ross, A theory of the term structure of interest rates, (1985), *Econometrica*, 53, 385–408. MR0785475

[8] D. Filpović, Term-Structure Models: A Graduate Course, (2009), Springer-Verlag. MR2553163

[9] D. Filipović, A general characterization of one factor affine term structure models, (2001), *Finance and Stochastics*, 5, 3, 389–412. MR1850789

[10] I. I. Gihman and A. V. Skorohod, Stochastic Differential Equations, (1972), Springer. MR0346904

[11] O. Kallenberg, Foundations of Modern Probability, (2001), 2-nd ed. Springer-Verlag. MR1876169

[12] A. Millian, Nagumo's type theorems for stochastic equations, (1994), PhD, Institute of Mathematics, Polish Academy of Sciences, (in Polish).

Michał Barski
Faculty of Mathematics Informatics and Mechanics
University of Warsaw
Poland
E-mail address: m.barski@mimuw.edu.pl

Jerzy Zabczyk
Institute of Mathematics
Polish Academy of Sciences
Warsaw
Poland
E-mail address: zabczyk@impan.pl

Received May 15, 2020