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Applications of the Girsanov theorem for multivariate fractional Brownian motions

Monika Camfrlová and Petr Čoupek*

The article is dedicated to Professor Tyrone E. Duncan on the occasion of his 80th birthday

In this article, multivariate fractional Brownian motions with possibly different Hurst indices in different coordinates are considered and a Girsanov-type theorem for these processes is given. Two applications of this theorem to stochastic differential equations driven by multivariate fractional Brownian motions are presented. The first is an existence result for weak solutions to stochastic differential equations with a drift coefficient that can be written as a sum of a regular and singular part and an autonomous diffusion coefficient. The second application concerns a maximum likelihood estimate of a drift parameter in stochastic differential equations with additive multivariate fractional noise.

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1. Introduction

In the article, multivariate fractional Brownian motions (mfBms) and stochastic differential equations (SDEs) driven by them are studied.

Let $\mathbb{H} = (H_1, H_2, \ldots, H_n)^\top \in (0,1)^n$. Generally, a multivariate fractional Brownian motion is an $\mathbb{R}^n$-valued stochastic process $B^{\mathbb{H}}$ whose k-th component is a standard H_k-fractional Brownian motion. The family of such processes provides a natural generalization of the family of standard $\mathbb{R}^n$-valued fractional Brownian motions (fBms) but it allows for higher flexibility as far as the regularity and fractal properties of the processes are concerned. This is of course relevant in applications. For example, the mfBm is used

*Corresponding author.

to describe the relationship between returns and volatility of the DAX index in [4, Section 3.1] or the structural properties of deformability, stacking energy, propeller twist, and position preference sequences of the Escherichia coli chromosome in [4, Section 3.2]. It has also been argued that it is an interesting model for functional magnetic resonance imaging; see [2]. We refer to the works [3] and [23] for various properties of mfBms, and to [11] and the references therein for a more general context.

In the present paper, the particular case when the components of $B^{\mathbb{H}}$ are mutually independent is considered and the stochastic differential equations

$$X_t = x_0 + \int_0^t b(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B^{\mathbb{H}}, \quad t \in [0,T], \tag{1}$$

where $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \to \mathscr{L}(\mathbb{R}^n)$ are Borel measurable deterministic functions are studied.[1]

A main tool in our analysis is a Girsanov-type theorem for mfBms. This result is a generalization of the known results in the univariate case (see, e.g., [9, Theorem 4.9], [32, Theorem 2], [10, Theorem 2.2], or [42, Theorem 1]) and of the multivariate case with common Hurst index (see [40, Theorem 4.1]) to the multivariate case with possibly different Hurt indices.

Two applications of the Girsanov theorem to SDEs are subsequently given. In the first, an existence result for a weak solution to the equation

$$X_t = x_0 + \int_0^t [b_1(r, X_r) + b_2(r, X_r)]\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B^{\mathbb{H}}_r, \quad t \in [0,T], \tag{2}$$

is presented (see Proposition 12). Here, $\sigma : [0,T] \to \mathscr{L}(\mathbb{R}^n)$ is a Borel measurable function with invertible values such that its integral with respect to $B^{\mathbb{H}}$ has a continuous version and b_1 and b_2 are two Borel measurable functions $[0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ that correspond to the "regular" and "singular" part of the drift. More precisely, while the function b_1 is assumed to be locally Lipschiz and of at most linear growth in the space variable, it is only assumed for the function b_2 that the map $[(t,x) \mapsto \sigma(t)^{-1} b_2(t,x)]$ is of at most linear growth in those coordinates that corresponds to the singular coordinates of $B^{\mathbb{H}}$, i.e. those with $H_k \leq 1/2$; and is Hölder continuous in both time and space in those components that correspond to the regular components of $B^{\mathbb{H}}$, i.e. those with $H_k > 1/2$.

Existence of weak solutions to SDEs driven by fBms has been already studied by many authors and we refer, for example, to [43] where the Wiener

[1] Throughout the article, the symbol $\mathscr{L}(\mathbb{R}^n; \mathbb{R}^m)$ denotes the space of $m \times n$ real matrices identified with linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$. If $m = n$, we write $\mathscr{L}(\mathbb{R}^n)$.

case $H = 1/2$ is studied; to [10], [24], [28], [32], and [33] where equation (1) is studied in dimension one with $\sigma \equiv 1$; or to [7], [25], and [40] where the particular case of (2) is treated. In fact, the article [40] is a main inspiration for the first part of the present paper and it is shown that the arguments from [40] can be used even in the case of a mfBm with different Hurst indices in different coordinates provided that the singular Hurst indices differ from the regular ones by at most one half.

The second application of the Girsanov theorem treated in this article is estimation of a drift parameter for equation (1) with additive noise, i.e. with $\sigma \equiv 1$. More precisely, the equation

$$X_t = \theta \int_0^t b(r, X_r)\,\mathrm{d}r + B_t^{\mathbb{H}}, \quad t \geq 0, \tag{3}$$

is considered and a maximum likelihood estimate (MLE) of θ that is based on a continuous observation of one trajectory of the solution is proposed and sufficient conditions for its strong consistency and asymptotic normality in the spirit of [36, section 2.4] are found.

Statistical inference for SDEs driven by the Wiener process is now a classical subject and as such, it has been treated extensively; see, e.g., the monographs [21], [26], and [27] and the references therein. On the other hand, the literature concerning inference for fractional diffusions is much more scarce. Even though a somewhat general treatment can be found in [36], only specific problems are usually considered and among these, estimation of the drift parameter θ in (3) seems to have received the most attention. From the results directly related to our problem of drift parameter estimation from a continuous observation of the trajectory, we refer, for example, to [9] and [29] where a MLE of the drift parameter of an fBm is studied; to [5], [8], and [19] where a MLE of the drift parameter of a fractional Ornstein-Uhlenbeck process is treated; to [42] where a MLE of θ in the general equation (3) with $b(t, x) \equiv b(x)$ is considered (see also [22]); to [15], [16], and [41] where a least-square estimate (LSE) of the drift parameter in a fractional Ornstein-Uhlenbeck process is analysed; and to [17] where a LSE of θ in the general equation (3) with $b(t, x) \equiv b(x)$ is considered.

The paper is organized as follows. Section 2 contains some preliminaries on (m)fBms and section 3 contains the Girsanov-type formula. SDEs are treated in section 4 – the existence result for the weak solution to equation (2) is given in part 4.2 and the MLE of the parameter θ in equation (3) is considered in part 4.3.

2. Preliminaries

The definition of one-dimensional fractional Brownian motions is recalled initially. Let $H \in (0,1)$ and $T > 0$. A stochastic process $(B_t^H)_{t\in[0,T]}$ defined on some probability space $(\Omega, \mathscr{F}, \mathsf{P})$ is called an *H-fractional Brownian motion* if it is centered, Gaussian, and if it satisfies

$$\mathsf{E}B_s^H B_t^H = R_H(s,t) := \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$$

for every $(s,t) \in [0,T]^2$. The covariance function R_H can be described via a certain Volterra-type kernel. If $H = 1/2$, define the kernel $K_H : [0,T]^2 \to \mathbb{R}$ by

$$K_H(t,r) := \mathbf{1}_{(0,t)}(r).$$

If $H \neq 1/2$, define the kernel $K_H : [0,T]^2 \to \mathbb{R}$ by

$$K_H(t,r) := \frac{c_H}{\Gamma\left(H+\frac{1}{2}\right)}(t-r)^{H-\frac{1}{2}} \times {}_2F_1\left(H-\frac{1}{2}, \frac{1}{2}-H, H+\frac{1}{2}, 1-\frac{t}{r}\right)$$

for $(t,r) \in [0,T]^2$ such that $0 < r < t \leq T$ and by $K_H(t,r) := 0$ otherwise. Here, the constant c_H is given by

$$c_H := \sqrt{\frac{\pi H(1-2H)}{\Gamma(2-2H)\cos(\pi H)}},$$

Γ is the Gamma function, and ${}_2F_1$ is the Gauss hypergeometric function; see, e.g., [39, Chapter 1]. It is well-known that the equality

$$R_H(s,t) = \int_0^{s\wedge t} K_H(s,r)K_H(t,r)\,\mathrm{d}r$$

is satisfied for every $(s,t) \in [0,T]^2$; see [9, Lemma 3.1]. This fact is a key result in the theory of fractional Brownian motions and it is also crucial in the present article.

2.1. Multivariate fractional Brownian motion

In what follows, a multivariate fractional Brownian motion is defined. To this end, let $n \in \mathbb{N}$ and $\mathbb{H} = (H_i)_{i=1}^n \in (0,1)^n$. *A (multivariate) $\mathbb{H}$-fractional*

Brownian motion is an $\mathbb{R}^n$-valued process $B^{\mathbb{H}} = (B^{H_1}, B^{H_2}, \dots, B^{H_n})^\top$ if for every $i \in \{1, 2, \dots, n\}$, B^{H_i} is an H_i-fractional Brownian motion and if B^{H_i} is independent of B^{H_j} whenever $i, j \in \{1, 2, \dots, n\}$ are such that $i \neq j$. Clearly, it holds that

$$\mathsf{E} B_s^{\mathbb{H}} (B_t^{\mathbb{H}})^\top = R_{\mathbb{H}}(s,t)$$

for every $(s,t) \in [0,T]^2$ where $R^{\mathbb{H}} : [0,T]^2 \to \mathscr{L}(\mathbb{R}^n)$ is defined by

$$R^{\mathbb{H}}(s,t) := \operatorname{diag}\{R_{H_i}(s,t)\}_{i=1}^n.$$

Now, if we define the matrix-valued kernel $K_{\mathbb{H}} : [0,T]^2 \to \mathscr{L}(\mathbb{R}^n)$ by

$$K_{\mathbb{H}}(t,r) := \operatorname{diag}\{K_{H_i}(t,r)\}_{i=1}^n,$$

then we obtain that the equality

$$R_{\mathbb{H}}(s,t) = \int_0^{t \wedge s} K_{\mathbb{H}}(s,r) K_{\mathbb{H}}(t,r) \,\mathrm{d}r$$

is satisfied for every $(s,t) \in [0,T]^2$.

2.2. Wiener integration

As well-known, unless $H = 1/2$, the H-fractional Brownian motion is not a semimartingale; see, e.g., [9, p. 178]. Therefore, the standard (Itô's) integration theory cannot be applied and an integration theory has to be developed. For our purposes, however, it suffices to consider only deterministic integrands.

Let $\mathbb{H} \in (0,1)^n$ and let $B^{\mathbb{H}}$ be a multivariate $\mathbb{H}$-fractional Brownian motion. Let $m \in \mathbb{N}$ and denote by $\mathscr{E}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))$ the space of step functions on the interval $[0,T]$ with values in the space $\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m)$, i.e. every $f \in \mathscr{E}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))$ is of the form

$$f = \sum_{i=0}^{N-1} A_i \mathbf{1}_{[t_i, t_{i+1})} \tag{4}$$

for some $N \in \mathbb{N}$, some partition $\{t_i\}_{i=0}^N$ of the interval $[0,T]$ such that $0 = t_0 < t_1 < \ldots < t_N = T$, and some set $\{A_i\}_{i=0}^m \subset \mathscr{L}(\mathbb{R}^n;\mathbb{R}^m)$. (If $m = n = 1$, we simply write $\mathscr{E}(0,T)$.) For a step function f that is represented by (4), the *Wiener integral* with respect to the multivariate $\mathbb{H}$-fractional Brownian motion $B^{\mathbb{H}}$ is defined by

$$I_T(f) := \sum_{i=0}^{N-1} A_i(B^{\mathbb{H}}_{t_{i+1}} - B^{\mathbb{H}}_{t_i}). \tag{5}$$

In what follows, this definition of the integral I_T is extended from the space of step functions to a larger space of admissible integrands. Define the integral operator $\partial K^*_{\mathbb{H}} : \mathscr{E}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m)) \to L^2(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))$ by

$$(\partial K^*_{\mathbb{H}} f)(s) := f(s)K_{\mathbb{H}}(T,s) + \int_s^T [f(r) - f(s)](\partial_1 K_{\mathbb{H}})(r,s)\,\mathrm{d}r$$

for $f \in \mathscr{E}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))$ and $s \in [0,T]$. Here, $\partial_1 K_{\mathbb{H}} = \operatorname{diag}\{\partial_1 K_{H_i}\}_{i=1}^n$ where $\partial_1 K_{H_i}$ denotes the partial derivative of K_{H_i} in the first variable. It follows by using the fact that

$$K_{\mathbb{H}}(t,r) = (\partial K^*_{\mathbb{H}})(\mathbf{1}_{[0,t]}\mathrm{Id}_n)(r) \tag{6}$$

holds for $(t,r) \in [0,T]^2$ that there is the isometry

$$\langle I_T(f), I_T(g)\rangle_{L^2(\Omega;\mathbb{R}^m)} = \langle \partial K^*_{\mathbb{H}} f, \partial K^*_{\mathbb{H}} g\rangle_{L^2(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))} \tag{7}$$

for $f, g \in \mathscr{E}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))$.

Remark 1. For $H \in (0,1)$, the operator $\partial K^*_H : \mathscr{E}(0,T) \to L^2(0,T)$ is injective and it can be described by the fractional operators defined by (33) and (34). In particular, it follows that for $f \in \mathscr{E}(0,T)$, $\partial K^*_H f$ is given by

$$\partial K^*_H f = c_H r^{\frac{1}{2}-H} I^{H-\frac{1}{2}}_{T-} r^{H-\frac{1}{2}} f,$$

and for $f \in (\partial K^*_H)(\mathscr{E}(0,T))$, the inverse $(\partial K^*_H)^{-1} f$ is given by

$$(\partial K^*_H)^{-1} f = c_H^{-1} r^{\frac{1}{2}-H} I^{\frac{1}{2}-H}_{T-} r^{H-\frac{1}{2}} f,$$

cf. [6, p. 30 and p.36] and [1, section 8].

By Remark 1, the operator $\partial K^*_{\mathbb{H}}$ is injective so that

$$\langle f, g\rangle_{\mathscr{D}^{\mathbb{H}}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))} := \langle \partial K^*_{\mathbb{H}} f, \partial K^*_{\mathbb{H}} g\rangle_{L^2(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))},$$

$f, g \in \mathscr{E}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))$, defines an inner product on $\mathscr{E}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))$ (with $\|\cdot\|_{\mathscr{D}^{\mathbb{H}}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))}$ being the induced norm). We obtain from equality (7) that

$$\|I_T(f)\|_{L^2(\Omega;\mathbb{R}^m)} = \|f\|_{\mathscr{D}^{\mathbb{H}}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))} \tag{8}$$

holds for every $f \in \mathscr{E}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))$ and therefore, the operator

$$I_T : \mathscr{E}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m)) \to L^2(\Omega;\mathbb{R}^m)$$

defined by formula (5) is a linear isometry. As such, it admits a unique extension to a linear isometry, denoted again by I_T, from the the completion of $\mathscr{E}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))$ with respect to the norm $\|\cdot\|_{\mathscr{D}^{\mathbb{H}}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))}$ (we denote this completion by $\mathscr{D}^{\mathbb{H}}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m)))$ to a closed linear subspace of the space $L^2(\Omega;\mathbb{R}^m)$. For $f \in \mathscr{D}^{\mathbb{H}}(0,T;\mathscr{L}(\mathbb{R}^n;\mathbb{R}^m))$, the $\mathbb{R}^m$-valued random variable $I_T(f)$ is called the *Wiener integral* of f with respect to the process $B^{\mathbb{H}}$. Whenever convenient, the following notation will also be used:

$$\int_0^T f(r)\,\mathrm{d}B_r^{\mathbb{H}} := I_T(f).$$

Remark 2. The above described procedure is a generalization of the case when the integrator is a scalar H-fractional Brownian motion and we refer, for example, to the works [1], [9], [34], or to the monograph [6] and the many references therein. It is well-known that the abstract completion procedure of the space of step functions with respect to the norm induced by the operator ∂K_H^* can and will in general produce admissible integrands that are not functions. More specifically, if $H \in (0, 1/2]$, then $\mathscr{D}^H(0,T)$ contains only functions, and there is, for example, the continuous embedding

$$\mathscr{C}^\delta([0,T]) \hookrightarrow \mathscr{D}^H(0,T)$$

for any $\delta \in (1/2 - H, 1)$. If $H \in (1/2, 1)$, then the space $\mathscr{D}^H(0,T)$ contains distributions; however, there is the continuous embedding

$$L^{\frac{1}{H}}(0,T) \hookrightarrow \mathscr{D}^H(0,T)$$

which provides a convenient space of functions to which one can restrict the domain of I_T; see, e.g., [31, section 2.1].

3. Girsanov theorem

In this subsection, a Girsanov-type theorem for the multivariate fractional Brownian motion is given. We begin with some preliminaries on an integral operator associated with the kernel $K_{\mathbb{H}}$ and a Volterra-type representation of the multivariate fractional Brownian motion.

The first preparatory result is that a certain integral operator can be associated with the kernel K_H. Define for $H \in (0,1)$ and $f \in L^2(0,T)$ the operator K_H by

$$(K_H f)(t) := \int_0^t K_H(t,r) f(r)\,\mathrm{d}r, \quad t \in [0,T].$$

Remark 3. The operator K_H can be described as follows. It holds that the operator K_H is an isomorphism from the space $L^2(0,T)$ to the space $I_{0+}^{H+\frac{1}{2}}(L^2(0,T))$ and for $f \in L^2(0,T)$ we have that

$$K_H f = \begin{cases} c_H I_{0+}^{2H} r^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} r^{H-\frac{1}{2}} f, & H \in (0, 1/2), \\ c_H I_{0+}^{1} f, & H = 1/2, \\ c_H I_{0+}^{1} r^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} r^{\frac{1}{2}-H} f, & H \in (1/2, 1); \end{cases}$$

by [38, Theorem 10.4]. (The formula in the singular case $H \in (0, 1/2)$ is obtained by noting that by [9, formula (4)], the equality

$${}_2F_1\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}; z\right) = {}_2F_1\left(H-\frac{1}{2}, \frac{1}{2}-H, H+\frac{1}{2}; z\right)$$

holds for any $H \in (0,1)$, $H \neq 1/2$, and $z \in \mathbb{C}$ such that $|\arg(1-z)| < \pi$.) See also [9, Theorem 2.1]. The inverse K_H^{-1} is given for $f \in I_{0+}^{H+\frac{1}{2}}(L^2(0,T))$ by

$$K_H^{-1} f = \begin{cases} c_H^{-1} r^{\frac{1}{2}-H} I_{0+}^{H-\frac{1}{2}} r^{H-\frac{1}{2}} I_{0+}^{-2H} f, & H \in (0, 1/2), \\ c_H^{-1} I_{0+}^{-1} f, & H = 1/2, \\ c_H^{-1} r^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} r^{\frac{1}{2}-H} I_{0+}^{-1} f, & H \in (1/2, 1); \end{cases}$$

see [10, formulas (5),(6), and (9),(10)]. Note also that if the function f is absolutely continuous, then is it proved in [32, p. 108] that for $H \in (0, 1/2)$, $K_{\mathbb{H}}^{-1} f$ can also be computed as

$$K_H^{-1} f = c_H^{-1} r^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} r^{\frac{1}{2}-H} I_{0+}^{-1} f.$$

The multivariate extension of the operator K_H is defined for $\mathbb{H} \in (0,1)^n$ and $f \in L^2(0,T;\mathbb{R}^n)$ by

$$K_{\mathbb{H}} f := \operatorname{diag}\{K_{H_i}\}_{i=1}^n f. \tag{9}$$

In order to describe the operator we set for $\mathbb{H} \in (0,1)^n$ and $f \in L^2(0,T;\mathbb{R}^n)$

$$I_{0+}^{\mathbb{H}+\frac{1}{2}} f := \operatorname{diag}\{I_{0+}^{H_i+\frac{1}{2}}\}_{i=1}^n f,$$

and it follows by Remark 3 that the operator $K_{\mathbb{H}}$ defined by (9) is an isomorphism from the space $L^2(0,T;\mathbb{R}^n)$ onto the space $I_{0+}^{\mathbb{H}+\frac{1}{2}}(L^2(0,T;\mathbb{R}^n))$. Moreover, its inverse is given for $f \in I_{0+}^{\mathbb{H}+\frac{1}{2}}(L^2(0,T;\mathbb{R}^n))$ by

$$K_{\mathbb{H}}^{-1} f = \operatorname{diag}\{K_{H_i}^{-1}\}_{i=1}^n f.$$

As a second preparatory result, let us mention that there is a one-to-one correspondence between an $\mathbb{R}^n$-valued Wiener process and a multivariate fractional Brownian motion. To be more precise, let $\mathbb{H} \in (0,1)^n$ and note that if $(W_t)_{t\in[0,T]}$ is an $\mathbb{R}^n$-valued Wiener process, then the process $(B_t^{\mathbb{H}})_{t\in[0,T]}$ defined by

$$B_t^{\mathbb{H}} := \int_0^t K_{\mathbb{H}}(t,r)\,\mathrm{d}W_r, \quad t \in [0,T],$$

is a multivariate $\mathbb{H}$-fractional Brownian motion (recall equality (6)). On the other hand, if $B^{\mathbb{H}}$ is a multivariate $\mathbb{H}$-fractional Brownian motion, then it follows that the process $(W_t)_{t\in[0,T]}$ defined by

$$W_t := \int_0^t (\partial K_{\mathbb{H}}^*)^{-1}(\mathbf{1}_{[0,t]}\mathrm{Id}_n)(r)\,\mathrm{d}B_r^{\mathbb{H}}, \quad t \in [0,T], \tag{10}$$

is an $\mathbb{R}^n$-valued Wiener process. Moreover, in both cases, their augmented generated filtrations coincide; cf., e.g., [9, Corollary 3.1, Remark 3.2, and Theorem 4.8], [12, Theorem 1], [32, formulas (5) and (6)], [10, formulas (3) and (4)], and [40, formula (2.4) and the remark following Definition 2.1].

The Girsanov-type theorem for multivariate fractional Brownian motion can now be formulated. It is an adaptation of the Girsanov-type theorem for scalar fractional Brownian motions (see, e.g., [9, Theorem 4.9], [32, Theorem 2], [10, Theorem 2.2], or [42, Theorem 1]) to the multivariate case with different Hurst indices (cf. [40, Theorem 4.1]).

Proposition 4. *Let $\mathbb{H} \in (0,1)^n$. Let $(B_t^{\mathbb{H}})_{t\in[0,T]}$ be a multivariate fractional Brownian motion defined on some probability space $(\Omega, \mathscr{F}, \mathsf{P})$ and let $(W_t)_{t\in[0,T]}$ be the Wiener process defined by formula (10). Let $(u_t)_{t\in[0,T]}$ be an $(\mathscr{F}_t^{B^{\mathbb{H}}})$-adapted $\mathbb{R}^n$-valued stochastic process such that*

$$u_\cdot \in L^1(0,T;\mathbb{R}^n) \quad \text{and} \quad \int_0^\cdot u_r \, \mathrm{d}r \in I_{0+}^{\mathbb{H}+\frac{1}{2}}\left(L^2(0,T;\mathbb{R}^n)\right)$$

are satisfied P-almost surely. Define the process $(v_t)_{t\in[0,T]}$ by

$$v_t := K_{\mathbb{H}}^{-1}\left(\int_0^\cdot u_r \, \mathrm{d}r\right)(t), \quad t \in [0,T],$$

and the random variable $\mathcal{E}_T$ by

$$\mathcal{E}_T := \exp\left\{\int_0^T v_r^\top \, \mathrm{d}W_r - \frac{1}{2}\int_0^T \|v_r\|_{\mathbb{R}^n}^2 \, \mathrm{d}r\right\}.$$

If $\mathsf{E}\mathcal{E}_T = 1$, then the process $(\tilde{B}_t^{\mathbb{H}})_{t\in[0,T]}$ defined by

$$\tilde{B}_t^{\mathbb{H}} := B_t^{\mathbb{H}} - \int_0^t u_r \, \mathrm{d}r, \quad t \in [0,T],$$

is a multivariate $\mathbb{H}$-fractional Brownian motion under the probability measure $\tilde{\mathsf{P}}$ that is defined by

$$\frac{\mathrm{d}\tilde{\mathsf{P}}}{\mathrm{d}\mathsf{P}} := \mathcal{E}_T.$$

Proof. First note that by the standard Girsanov theorem (see, e.g., [18, Theorem 3.5.1]), it follows that the process $(\tilde{W}_t)_{t\in[0,T]}$ defined by

$$\tilde{W}_t := W_t - \int_0^t v_r \, \mathrm{d}r, \quad t \in [0,T],$$

is an $(\mathscr{F}_t^{B^{\mathbb{H}}})$-Wiener process under the probability measure $\tilde{\mathsf{P}}$. Moreover, we have that

$$\begin{aligned}\tilde{B}_t^{\mathbb{H}} &= B_t^{\mathbb{H}} - \int_0^t u_r \, \mathrm{d}r \\ &= \int_0^t K_{\mathbb{H}}(t,r) \, \mathrm{d}W_r - \int_0^t K_{\mathbb{H}}(t,r) v_r \, \mathrm{d}r\end{aligned}$$

$$= \int_0^t K_{\mathbb{H}}(t,r)\,\mathrm{d}\tilde{W}_r$$

for every $t \in [0,T]$, $\tilde{\mathsf{P}}$-almost surely which proves the claim. □

4. Stochastic differential equations

Let $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a Borel measurable function that satisfies the following two conditions:

(I) There exists a finite positive constant K_b such that for every $t \in [0,T]$ and every $x \in \mathbb{R}^n$ it holds that

$$\|b(t,x)\|_{\mathbb{R}^n} \leq K_b(1 + \|x\|_{\mathbb{R}^n}).$$

(II) For every $N \in \mathbb{N}$ there exists a finite positive constant K_N such that for every $t \in [0,T]$ and every $x, y \in \mathbb{R}^n$ that satisfy $\|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^n} \leq N$ it holds that

$$\|b(t,x) - b(t,y)\|_{\mathbb{R}^n} \leq K_N \|x - y\|_{\mathbb{R}^n}.$$

Assume further that $(Z_t)_{t\in[0,T]}$ is an $\mathbb{R}^n$-valued process with continuous sample paths sample paths defined on some probability space $(\Omega, \mathscr{F}, \mathsf{P})$ and that $x_0 \in \mathbb{R}^n$. It follows by standard Picard iteration scheme that there exists an $\mathbb{R}^n$-valued continuous stochastic process $(X_t)_{t\in[0,T]}$, unique in the sense of indistinguishability, that satisfies the random differential equation

$$X_t = x_0 + \int_0^t b(r, X_r)\,\mathrm{d}r + Z_t \tag{11}$$

for every $t \in [0,T]$ P-almost surely. Additionally, it can be shown exactly as in [40, Theorem 3.6] that if there exists a $\nu \in (0,1)$ such that the stochastic process Z has γ-Hölder continuous sample paths for every $\gamma \in (0,\nu)$, then the process $(X_t)_{t\in[0,T]}$ has a version with γ-Hölder continuous sample paths for every $\gamma \in (0,\nu)$; and, moreover, for every $\gamma \in (0,\nu)$, the inequality[2]

$$\|X\|_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)} \lesssim 1 + \|Z\|_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)} \tag{12}$$

holds P-almost surely with a constant that depends on T, x_0, K_b, and γ.

[2]If there exists a constant C such that $A \leq CB$ and the value of this constant is not important, we simply write $A \lesssim B$ throughout the article.

4.1. Strong solutions

Let us fix $\mathbb{H} \in (0,1)^n$ and a multivariate $\mathbb{H}$-fractional Brownian motion $B^{\mathbb{H}}$ defined on some probability space $(\Omega, \mathscr{F}, \mathsf{P})$ for this subsection. Let $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \to \mathscr{L}(\mathbb{R}^n)$ be Borel measurable functions and let $x_0 \in \mathbb{R}^n$. Assume that $\sigma \in \mathscr{D}^{\mathbb{H}}(0,T;\mathscr{L}(\mathbb{R}^n))$. In this section, we find sufficient conditions for the existence of an $(\mathscr{F}_t^{B^{\mathbb{H}}})$-adapted process $(X_t)_{t\in[0,T]}$ defined on $(\Omega, \mathscr{F}, \mathsf{P})$ with continuous sample paths that satisfies the equation

$$X_t = x_0 + \int_0^t b(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B_r^{\mathbb{H}} \tag{13}$$

for every $t \in [0,T]$ P-almost surely. If such a process exists, we say that problem (13) admits a *strong solution*. We aim to use the facts described at the beginning of section 4 with the process Z being defined by

$$Z_t := \int_0^t \sigma(u)\,\mathrm{d}B_u^{\mathbb{H}}, \quad t \in [0,T]. \tag{14}$$

Therefore, we first need to find conditions under which the integral process has a version with continuous sample paths. Let us fix the following notation:

Notation In what follows, we will write $f \in S^{H+}$ with $H \in (0,1)$ if there exists $\delta > 0$ such that $f \in S^{H+\delta}$ where

$$S^{H+\delta} := \begin{cases} \mathscr{C}^{\frac{1}{2}-H+\delta}([0,T];\mathbb{R}^n), & H \in (0, 1/2), \\ L^{\frac{1}{H}+\delta}(0,T;\mathbb{R}^n), & H \in [1/2, 1), \end{cases}$$

Similarly, we will write $f \in S^{\mathbb{H}+}$ for $\mathbb{H} = (H_1, H_2, \dots, H_n)^\top \in (0,1)^n$ if there exists $\delta = (\delta_1, \delta_2, \dots, \delta_n)^\top \in (0,\infty)^n$ such that $f \in \bigtimes_{i=1}^n S^{H_i+\delta_i}$.

Proposition 5. *If $\sigma \in S^{\mathbb{H}+}$, then the integral process $(Z_t)_{t\in[0,T]}$ defined by formula (14) has a version with continuous sample paths.*

Proof. Write $\sigma = (\sigma_{\cdot 1}, \sigma_{\cdot 2}, \dots, \sigma_{\cdot n})$ and let $0 \le s < t \le T$ be fixed. Since the coordinates of $B^{\mathbb{H}}$ are independent fractional Brownian motions and the columns of σ are deterministic functions, the integrals $\int_s^t \sigma_{\cdot k}(r)\,\mathrm{d}B_r^{H_k}$ and $\int_s^t \sigma_{\cdot l}(r)\,\mathrm{d}B_r^{H_l}$ are uncorrelated $\mathbb{R}^n$-valued random variables whenever $k \neq l$. Using this property and the isometry (8), the equality

$$\|Z_t - Z_s\|^2_{L^2(\Omega;\mathbb{R}^n)} = \sum_{k=1}^n \mathsf{E}\left\| \int_s^t \sigma_{\cdot k}(r)\,\mathrm{d}B_r^{H_k} \right\|^2_{\mathbb{R}^n} = \sum_{k=1}^n \|\sigma_{\cdot k}\|^2_{\mathscr{D}^{H_k}(s,t;\mathbb{R}^n)}$$

is easily obtained. Now, for $k \in \{1, 2, \dots, n\}$, we have by the definition of the norm $\|\cdot\|_{\mathscr{D}^{H_k}(s,t;\mathbb{R}^n)}$ that

$$\|\sigma_{\cdot,k}\|^2_{\mathscr{D}^{H_k}(s,t;\mathbb{R}^n)} = \|\partial K^*_{H_k}\sigma_{\cdot\, k}\|^2_{L^2(s,t;\mathbb{R}^n)}.$$

If $H_k \in (0, 1/2)$, it follows by the exact same arguments as in the proof of [40, Proposition 3.1] that

$$\|\partial K^*_{H_k}\sigma_{\cdot\, k}\|^2_{L^2(s,t;\mathbb{R}^n)} \lesssim \|\sigma_{\cdot\, k}\|^2_{\mathscr{C}^{\frac{1}{2}-H_k+\delta_k}([0,T];\mathbb{R}^n)}(t-s)^{2H_k}.$$

On the other hand, if $H_k \in [1/2, 1)$, we have by the Hardy-Littlewood-Sobolev inequality (see, e.g., [38, Theorem 3.5]) and the Hölder inequality that

$$\begin{aligned}\|\partial K^*\sigma_{\cdot\, k}\|^2_{L^2(s,t;\mathbb{R}^n)} &\lesssim \left(\int_s^t \|\sigma_{\cdot\, k}(r)\|^{\frac{1}{H_k}}_{\mathbb{R}^n}\,\mathrm{d}r\right)^{2H_k} \\ &\leq \|\sigma_{\cdot\, k}\|^2_{L^{\frac{1}{H}+\delta_k}(0,T;\mathbb{R}^n)}(t-s)^{\frac{2\delta_k H_k^2}{1+\delta_k H_k}}.\end{aligned}$$

The proof is concluded by a standard argument based on the the fact that higher-order moments of Gaussian random variables can be estimated by the second moment [30, Corollary 2.8.14] and the Kolmogorov-Chentsov continuity criterion [18, Theorem 2.2.8]. □

Remark 6. It follows from the proof of Proposition 5 that there exists a $\nu > 0$ such that the integral process $(Z_t)_{t\in[0,T]}$ admits a version with γ-Hölder continuous sample paths for every $\gamma \in (0, \nu)$. In particular, if we denote

$$G_k := \begin{cases} H_k, & H_k \in (0, 1/2), \\ \frac{\delta_k H_k^2}{1+\delta_k H_k}, & H_k \in [1/2, 1), \end{cases}$$

for $k \in \{1, 2, \dots, n\}$, then the claim holds with

$$\nu := \min_{k\in\{1,2,\dots,n\}} G_k$$

that we call, in the sequel, the *Hölder bound for* Z for simplicity.

Remark 7. Proposition 5 holds with obvious modifications for $\delta_k = \infty$. For example, if σ is such that $\sigma_{\cdot\, k} \in L^\infty(0, T; \mathbb{R}^n)$ for every k such that $H_k \in [1/2, 1)$, then $\nu = \min_k H_k$. Thus, [40, Proposition 3.2] is recovered as a particular case of Proposition 5.

Remark 8. For $\lambda \in (0,1]$, consider the space

$$\tilde{\mathscr{C}^\lambda}([0,T];\mathbb{R}^n) := \left\{ f \in \mathscr{C}^\lambda([0,T];\mathbb{R}^d) \,\middle|\, \forall \varepsilon > 0\, \exists \delta > 0 \right.$$
$$\left. \forall s,t \in (0,T), 0 < |t-s| < \delta \implies \frac{|f(t)-f(s)|}{|t-s|^\lambda} < \varepsilon \right\}$$

equipped with the norm $\|\cdot\|_{\mathscr{C}^\lambda([0,T];\mathbb{R}^n)}$. It follows that $\tilde{\mathscr{C}^\lambda}([0,T];\mathbb{R}^n)$ is separable (see [20, Theorem 1.4.11]) and there are the inclusions

$$\mathscr{C}^{\kappa_2}([0,T];\mathbb{R}^n) \subset \tilde{\mathscr{C}^{\kappa_1}}([0,T];\mathbb{R}^n) \subset \mathscr{C}^{\kappa_1}([0,T];\mathbb{R}^n)$$

whenever $0 < \kappa_1 < \kappa_2 \leq 1$ (see [20, Exercise 1.2.10 (ii)]). Therefore, if the assumptions of Proposition 5 are satisfied, the integral process $(Z_t)_{t\in[0,T]}$ can be viewed as a $\tilde{\mathscr{C}^\gamma}([0,T];\mathbb{R}^n)$-valued Gaussian random variable for any $\gamma \in (0,\nu)$ where ν is the Hölder bound for Z. As a consequence, it follows by Fernique's theorem (see [13, Théorème d' integrabilité]) that for every $\gamma \in (0,\nu)$,

$$\mathsf{E}\exp\{K\|Z\|^2_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)}\} < \infty$$

is satisfied with some positive constant K. This fact will be used in the proof of Proposition 10.

Combining Proposition 5 with the existence and uniqueness result for equation (11) discussed at the beginning of section 4 allows to obtain a strong solution to the stochastic differential equation (13).

Proposition 9. *Assume that b satisfies conditions (I) and (II) and that σ belongs to the space $S^{\mathbb{H}+}$. Then there exists a unique strong solution to* (13). *Moreover, the solution has γ-Hölder continuous sample paths for every $\gamma \in (0,\nu)$ where ν is the Hölder bound for the integral process Z defined by* (14).

4.2. Weak solutions

Let $\mathbb{H} \in (0,1)^n$ be fixed in this subsection. Let also $b_1, b_2 : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \to \mathscr{L}(\mathbb{R}^n)$ be Borel measurable functions and $x_0 \in \mathbb{R}^n$. Assume that $\sigma \in \mathscr{D}^{\mathbb{H}}(0,T;\mathscr{L}(\mathbb{R}^n))$. In what follows, we find sufficient conditions for the existence of a probability space $(\Omega, \mathscr{F}, \mathsf{P})$, a multivariate $\mathbb{H}$-fractional Brownian motion $B^{\mathbb{H}}$, and an $(\mathscr{F}_t^{B^{\mathbb{H}}})$-adapted process $(X_t)_{t\in[0,T]}$ with continuous sample paths that satisfy the equation

$$X_t = x_0 + \int_0^t [b_1(r,X_r) + b_2(r,X_r)]\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B_r^{\mathbb{H}} \tag{15}$$

for every $t \in [0,T]$ P-almost surely. If this is the case, we say that problem (15) admits a *weak solution*. The main tool will be the Girsanov-type theorem from Proposition 4 and initially, its application is given.

Proposition 10. *Let $(B_t^{\mathbb{H}})_{t\in[0,T]}$ be a multivariate $\mathbb{H}$-fractional Brownian motion defined on some probability space $(\Omega, \mathscr{F}, \mathsf{P})$. Assume that the function b_1 satisfies conditions (I) and (II). Assume also that the function σ belongs to the space $S^{\mathbb{H}+}$ and that for every $t \in [0,T]$, the matrix $\sigma(t)$ is invertible. Let $\nu > 0$ be the Hölder bound for the integral process Z that is defined by (14) and assume additionally that the functions b_2 and σ satisfy the following condition:*

(III) If, for $k \in \{1,2,\dots,n\}$, the parameter H_k belongs to $(0, 1/2]$, then there exists a constant $K_k > 0$ such that for every $t \in [0,T]$ and every $x \in \mathbb{R}^n$ it holds that

$$|[\sigma(t)^{-1}b_2(t,x)]_k| \leq K_k(1 + \|x\|_{\mathbb{R}^n});$$

and if H_k belongs to $(1/2, 1)$, then $H_k < \nu + 1/2$ and there exist constants $\alpha_k \in (H_k - 1/2, 1]$, $\beta_k \in (\frac{2H_k-1}{2\nu}, 1]$, and $K_k > 0$ such that for every $s,t \in [0,T]$ and every $x, y \in \mathbb{R}^n$ it holds that

$$|[\sigma(t)^{-1}b_2(t,x)]_k - [\sigma(s)^{-1}b_2(s,y)]_k| \leq \\ \leq K_k(|t-s|^{\alpha_k} + \|x-y\|_{\mathbb{R}^n}^{\beta_k}).$$

Here, $[z]_k$ denotes the k-th component of $z \in \mathbb{R}^n$.

Denote by $(X_t)_{t\in[0,T]}$ the strong solution[3] to the equation

$$X_t = x_0 + \int_0^t b_1(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B_r^{\mathbb{H}}, \quad t \in [0,T].$$

Then the process $(\tilde{B}_t^{\mathbb{H}})_{t\in[0,T]}$ given by

$$\tilde{B}_t^{\mathbb{H}} := B_t^{\mathbb{H}} - \int_0^t \sigma(r)^{-1} b_2(r, X_r)\,\mathrm{d}r, \quad t \in [0,T],$$

is a multivariate $\mathbb{H}$-fractional Brownian motion under the probability measure $\tilde{\mathsf{P}}$ that is defined by

$$\frac{\mathrm{d}\tilde{\mathsf{P}}}{\mathrm{d}\mathsf{P}} := \mathcal{E}_T := \exp\left\{\int_0^T v_r^\top\,\mathrm{d}W_r - \frac{1}{2}\int_0^T \|v_r\|_{\mathbb{R}^n}^2\,\mathrm{d}r\right\} \tag{16}$$

[3] The existence and uniqueness of such solution is ensured by Proposition 9.

where $(v_t)_{t\in[0,T]}$ *is the stochastic process given by*

$$v_t := K_{\mathbb{H}}^{-1}\left(\int_0^{\cdot} \sigma(r)^{-1} b_2(r, X_r)\,\mathrm{d}r\right)(t), \quad t \in [0,T].$$

Proof. Let $(u_t)_{t\in[0,T]}$ be defined by $u_t := \sigma(t)^{-1} b_2(t, X_t)$ for $t \in [0,T]$. In order to show that

$$u_{\cdot} \in L^1(0,T;\mathbb{R}^n) \quad \text{and} \quad \int_0^{\cdot} u_r\,\mathrm{d}r \in I_{0+}^{H+\frac{1}{2}}\left(L^2(0,T;\mathbb{R}^n)\right)$$

hold P-almost surely, it will be proved in the first step that $v \in L^2(0,T;\mathbb{R}^n)$ P-almost surely. In the second step it will be shown that there exists $\Delta > 0$ and a partition $0 = t_0 < t_1 < \ldots < t_{N(\Delta)} = T$ of the interval $[0,T]$ whose mesh size is smaller than Δ and it holds that

$$\mathsf{E}\exp\left\{\int_{t_i}^{t_{i+1}} \|v_r\|_{\mathbb{R}^n}^2\,\mathrm{d}r\right\} < \infty$$

for every $i \in \{0,1,\ldots,N(\Delta)-1\}$. This is because this last condition implies by [14, Lemma 7.1.3] that

$$\mathsf{E}\left[\exp\left\{\int_{t_i}^{t_{i+1}} v_r^\top\,\mathrm{d}W_r - \frac{1}{2}\int_{t_i}^{t_{i+1}} \|v_r\|_{\mathbb{R}^n}^2\,\mathrm{d}r\right\}\middle|\,\mathscr{F}_{t_i}^{B^{\mathbb{H}}}\right] = 1$$

holds P-almost surely for every $i \in \{0,1,\ldots,N(\Delta)-1\}$ and using the above equality iteratively yields $\mathsf{E}\mathcal{E}_T = 1$. Thus, the assumptions of Proposition 4 will be verified and the claim of the proposition will follow.

Step 1. Let $0 \le s < t \le T$. We have by Remark 3 that

$$\begin{aligned}\int_s^t \|v_u\|_{\mathbb{R}^n}^2\,\mathrm{d}u &= \sum_{k=1}^n \int_s^t \left|K_{H_k}^{-1}\left(\int_0^{\cdot} [\sigma(r)^{-1} b_2(r,X_r)]_k\,\mathrm{d}r\right)(u)\right|^2\,\mathrm{d}u \\ &= \sum_{k=1}^n \int_s^t \left|u^{H_k-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H_k}\left(r^{\frac{1}{2}-H_k}[\sigma(r)^{-1} b_2(r,X_r)]_k\right)(u)\right|^2\,\mathrm{d}u\end{aligned}$$

where I_{0+}^{α} is the fractional operator defined by formulas (31) and (32). Now, set

$$I_k(s,t) := \int_s^t \left|u^{H_k-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H_k}\left(r^{\frac{1}{2}-H_k}[\sigma(r)^{-1} b_2(r,X_r)]_k\right)(u)\right|^2\,\mathrm{d}u \tag{17}$$

for $k \in \{1, 2, \dots, n\}$. If $H_k \in (0, 1/2]$, we obtain that for every $\gamma \in (0, \nu)$, the inequality

$$I_k(s,t) \leq C_k^{(1)}(1 + \|Z\|^2_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)})(t-s) \tag{18}$$

holds with some finite positive constant $C_k^{(1)}$ that depends on H, T, α_k, β_k, γ, K_b, K_k, and x_0 by using assumption (III) and estimate (12) as in the proof of [40, Theorem 4.2, formula (4.8)]. On the other hand, if $H_k \in (1/2, 1)$, then for every $\gamma \in (\frac{2H_k-1}{2\beta_k}, \nu)$, the estimate

$$I_k(s,t) \leq C_k^{(2)}(1 + \|Z\|^{2\beta_k}_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)})(t-s)^{2(1-H_k)}$$

holds with some finite positive constant $C_k^{(2)}$ that depends on H, T, α_k, β_k, γ, K_b, K_k, x_0, and $\sigma^{-1}b_2$ by using assumption (III) and estimate (12) as in the proof of [40, Theorem 4.3]. Set $\beta_k := 1$ for any k such that $H_k \in (0, 1/2]$ and denote

$$H_0 := \max_{k: H_k \in (1/2, 1)} H_k \quad \text{and} \quad \nu_0 := \max_{k: H_k \in (1/2, 1)} \frac{2H_k - 1}{2\beta_k}.$$

We see from inequalities (17) and (18) that for every $\gamma \in (\nu_0, \nu)$ the estimate

$$\int_s^t \|v_u\|^2_{\mathbb{R}^n}\,\mathrm{d}u \leq C_0 \left(1 + \sum_{k=1}^n \|Z\|^{2\beta_k}_{\mathscr{C}^\gamma([0,T];\mathbb{R}^d)}\right)(t-s)^{2(1-H_0)}.$$

holds with some finite positive constant C_0. It follows from this estimate that $v \in L^2(0,T;\mathbb{R}^n)$ P-almost surely by choosing $s = 0$ and $t = T$.

Step 2. Let $\gamma \in (\nu_0, \nu)$ and let K_0 be the constant for which

$$\mathsf{E}\exp\{K_0\|Z\|^2_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)}\} < \infty \tag{19}$$

(cf. Remark 6). Let also $\Delta > 0$ be such that $(1+n)C_0\Delta^{2(1-H_0)} < K_0$ and let $0 = t_0 < t_1 < \ldots < t_{N(\Delta)} = T$ be a partition of the interval $[0,T]$ whose mesh size is smaller than Δ. For $i \in \{0, 1, \dots, N(\Delta) - 1\}$, we have that

$$\begin{aligned}\mathsf{E}\exp&\left\{\int_{t_i}^{t_{i+1}} \|v_u\|^2_{\mathbb{R}^n}\,\mathrm{d}u\right\} \leq \\ &\leq \mathsf{E}\exp\left\{C_0\left(1 + \sum_{k=1}^n \|Z\|^{2\beta_k}_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)}\right)(t_{i+1}-t_i)^{2(1-H_0)}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \mathsf{E}\exp\left\{\left(1+\sum_{k=1}^{n}\|Z\|^{2\beta_k}_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)}\right)\frac{K_0}{1+n}\right\}\mathbf{1}_{[\|Z\|_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)}\le 1]}\\
&\qquad+\mathsf{E}\exp\left\{\left(1+\sum_{k=1}^{n}\|Z\|^{2\beta_k}_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)}\right)\frac{K_0}{1+n}\right\}\mathbf{1}_{[\|Z\|_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)}> 1]}\\
&\le \mathrm{e}^{K_0}\mathsf{P}(\|Z\|_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)}\le 1)\\
&\qquad+\mathsf{E}\exp\left\{K_0\|Z\|^{2}_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)}\right\}\mathbf{1}_{[\|Z\|_{\mathscr{C}^\gamma([0,T];\mathbb{R}^n)}> 1]}
\end{aligned}
$$

which is finite by (19). Thus, the claim is proved. □

Remark 11. Assume that $\mathbb{H}$ contains at least one element larger than $1/2$ and one element smaller than $1/2$ and let $\sigma \in S^{\mathbb{H}+}$ be such that $\sigma_{\cdot k} \in L^\infty(0,T;\mathbb{R}^n)$ whenever k is such that $H_k \in (1/2, 1)$ as in Remark 7. Then it follows that $\nu = \min_{k:H_k\in(0,1/2]} H_k$. On the other hand, condition (III) in Proposition 10 says that ν has to be greater than $\max_{k:H_k\in(1/2,1)} H_k - 1/2$. Therefore, Proposition 10 can be applied if (besides the remaining conditions) the condition

$$\min_{k:H_k\in(0,1/2]} H_k > \max_{k:H_k\in(1/2,1)} H_k - \frac{1}{2}$$

is satisfied. Roughly speaking, this means that Proposition 10 can be applied if the singular values of Hurst indexes contained in $\mathbb{H}$ do not differ from the regular values by more than ½.

Proposition 12. *Assume that the function b_1 satisfies conditions (I) and (II). Assume also that the function σ belongs to the space $S^{\mathbb{H}+}$ and that for every $t \in [0,T]$, the matrix $\sigma(t)$ is invertible. Assume finally, that the functions b_2 and σ satisfy condition (III). Then problem (15) admits a weak solution.*

Proof. Let $(B^{\mathbb{H}}_t)_{t\in[0,T]}$ be a multivariate $\mathbb{H}$-fractional Brownian motion that is defined on some probability space $(\Omega, \mathscr{F}, \mathsf{P})$. By Proposition 9, there exists an $(\mathscr{F}^{B^{\mathbb{H}}}_t)$-adapted process $(X_t)_{t\in[0,T]}$ with continuous sample paths that satisfies the equation

$$X_t = x_0 + \int_0^t b_1(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}B^{\mathbb{H}}_r \tag{20}$$

for every $t \in [0,T]$ P-almost surely. On the other hand, by Proposition 10, the process $(\tilde{B}_t^{\mathbb{H}})_{t\in[0,T]}$ defined by

$$\tilde{B}_t^{\mathbb{H}} := B_t^{\mathbb{H}} - \int_0^t \sigma(r)^{-1} b_2(r, X_r)\,\mathrm{d}r$$

is a multivariate $\mathbb{H}$-fractional Brownian motion under the probability measure $\tilde{\mathsf{P}}$ that is given by formula (16). Moreover, by a standard approximation argument, it can be shown that for every $f \in S^{\mathbb{H}+}$, the equation

$$\int_0^t f(r)\,\mathrm{d}B_r^{\mathbb{H}} = \int_0^t f(r)\,\mathrm{d}\tilde{B}_r^{\mathbb{H}} + \int_0^t f(r)\sigma(r)^{-1} b_2(r, X_r)\,\mathrm{d}r \tag{21}$$

is satisfied for every $t \in [0,T]$ $\tilde{\mathsf{P}}$-almost surely (cf. [40, Proposition 5.1]). It follows from equations (20) and (21) that

$$X_t = x_0 + \int_0^t b_1(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r)\,\mathrm{d}\tilde{B}_r^{\mathbb{H}} + \int_0^t b_2(r, X_r)\,\mathrm{d}r$$

holds for every $t \in [0,T]$ $\tilde{\mathsf{P}}$-almost surely. Consequently, it is seen that the triplet $((\Omega, \mathscr{F}, \tilde{\mathsf{P}}), \tilde{B}^{\mathbb{H}}, X)$ is a weak solution to problem (15). □

4.3. Estimation of the drift

In this section, a maximum likelihood estimate (MLE) of a drift parameter in a stochastic differential equation with additive multivariate fractional Brownian motion is found by means of the Girsanov-type theorem.

Let $\mathbb{H} \in (0,1)^n$ and let $(B_t^{\mathbb{H}})_{t\geq 0}$ be a multivariate $\mathbb{H}$-fractional Brownian motion on some probability space $(\Omega, \mathscr{F}, \mathsf{P})$. Let $b : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be a Borel measurable function that satisfies the condition

(IV) For every $T > 0$ there exists a finite positive constant C_T such that for every $s, t \in [0,T]$ and $x, y \in \mathbb{R}^n$ it holds that

$$\|b(t,x) - b(s,y)\|_{\mathbb{R}^n} \leq C_T \|x - y\|_{\mathbb{R}^n}.$$

Consider the equation

$$X_t = \theta \int_0^t b(r, X_r)\,\mathrm{d}r + B_t^{\mathbb{H}}, \quad t \geq 0, \tag{22}$$

where $\theta \in \mathbb{R}$ is an unknown parameter. Clearly, conditions (I) and (II) are satisfied and therefore for every $T > 0$, there exists a unique strong solution $(X_t^{\theta,T})_{t\in[0,T]}$ to equation (22) on the interval $[0,T]$ by Proposition 9. By uniqueness of the solution, $X_t^{\theta,T} = X_t^{\theta,S}$ for every $t \in [0,S]$ P-almost surely whenever $S \in (0,T)$. Therefore, if we define the process $(X_t^\theta)_{t\geq 0}$ by $X_t^\theta := X_t^{\theta,N+1}$ for $t \in [N,N+1)$, $N \in \mathbb{N}$, we obtain that X^θ is the unique solution to (22) on the interval $[0,\infty)$. The aim is to find a MLE of the parameter θ based on a continuous observation of a trajectory of the solution.

Let $T > 0$ and set $\mathsf{P}_0 := \mathsf{P}$. Notice first that the process $(X_t^0)_{t\in[0,T]}$ is a $\mathbb{H}$-fractional Brownian motion $B^{\mathbb{H}}$ on $(\Omega, \mathscr{F}, \mathsf{P}_0)$. On the other hand, if $\theta \neq 0$, define the stochastic process $(v_t)_{t\in[0,T]}$ by

$$v_t := K_{\mathbb{H}}^{-1}\left(\int_0^\cdot [-\theta b(r, X_r^\theta)]\,\mathrm{d}r\right)(t), \quad t \in [0,T].$$

Since the assumptions of Proposition 10 are satisfied, the stochastic process $(\tilde{B}_t^{\mathbb{H}})_{t\in[0,T]}$ defined by

$$\tilde{B}_t^{\mathbb{H}} := B_t^{\mathbb{H}} + \theta \int_0^t b(t, X_r^\theta)\,\mathrm{d}r, \quad t \in [0,T],$$

is a $\mathbb{H}$-fractional Brownian motion under the probability measure P_θ that is given by formula

$$\frac{\mathrm{d}\mathsf{P}_\theta}{\mathrm{d}\mathsf{P}_0} := \mathcal{E}_T := \exp\left\{\int_0^T v_r^\top\,\mathrm{d}W_r - \frac{1}{2}\int_0^T \|v_r\|_{\mathbb{R}^n}^2\,\mathrm{d}r\right\}$$

where $(W_t)_{t\in[0,T]}$ is the Wiener process constructed from $B^{\mathbb{H}}$ by formula (10). Moreover, it follows from equation (22) that $X_t^\theta = \tilde{B}_t^H$ holds for every $t \in [0,T]$ P_0-almost surely and therefore, the solution $(X_t^\theta)_{t\in[0,T]}$ is a $\mathbb{H}$-fractional Brownian motion on the probability space $(\Omega, \mathscr{F}, \mathsf{P}_\theta)$.

Proposition 13. *The MLE of parameter θ in equation (22) based on a continuous observation of a trajectory of its solution X on $[0,T]$ is given by*

$$\hat{\theta}_T = -\frac{\int_0^T Q_r^\top\,\mathrm{d}W_r}{\int_0^T \|Q_r\|_{\mathbb{R}^n}^2\,\mathrm{d}r} \tag{23}$$

where $(Q_t)_{t\in[0,T]}$ is given by

$$Q_t = K_{\mathbb{H}}^{-1}\left(\int_0^\cdot b(r, X_r)dr\right)(t), \quad t \in [0,T],$$

and where $(W_t)_{t\in[0,T]}$ *is the Wiener process constructed from the* $\mathbb{H}$*-fractional Brownian motion* $(B_t^{\mathbb{H}})_{t\in[0,T]}$ *by formula* (10)*.*

Proof. By Proposition 10, we have that

$$\mathsf{P}_\theta(X^\theta \in A) = \int_{\{\omega\in\Omega: X^\theta(\omega)\in A\}} \mathcal{E}_T(\omega)\,\mathrm{d}\mathsf{P}_0(\omega), \quad A \in \mathcal{B}(\mathscr{C}([0,T];\mathbb{R}^n)),$$

so that the MLE can be found by maximizing the function

$$F(\theta) := \log \frac{\mathrm{d}\mathsf{P}_\theta}{\mathrm{d}\mathsf{P}_0} = -\theta \int_0^T Q_r^\top \,\mathrm{d}W_r - \frac{\theta^2}{2}\int_0^T \|Q_r\|_{\mathbb{R}^n}^2\,\mathrm{d}r. \tag{24}$$

□

Remark 14. As noted before, the solution X^θ to equation (22) is a $\mathbb{H}$-fractional Brownian motion on $(\Omega,\mathscr{F},\mathsf{P}_\theta)$. Moreover, it follows by the proof of Proposition 4 that the process $(\tilde{W}_t)_{t\in[0,T]}$ defined by

$$\tilde{W}_t := W_t + \theta \int_0^t Q_s\,\mathrm{d}s, \quad t\in[0,T], \tag{25}$$

is a Wiener process on $(\Omega,\mathscr{F},\mathsf{P}_\theta)$ such that

$$X_t = \int_0^t K_{\mathbb{H}}(t,r)\,\mathrm{d}\tilde{W}_r$$

and

$$\tilde{W}_t = \int_0^t (\partial K_{\mathbb{H}}^*)^{-1}(\mathbf{1}_{[0,t]}\mathrm{Id}_n)(r)\,\mathrm{d}X_r^\theta$$

hold for every $t\in[0,T]$ P_θ-almost surely (and also P-almost surely since the measures P and P_θ are equivalent). Note that this last expression can be computed from the observed trajectory X. Now, it follows from formula (25) that

$$\int_0^T Q_r^\top\,\mathrm{d}W_r = \int_0^T Q_r^\top\,\mathrm{d}\tilde{W}_r - \theta\int_0^T \|Q_r\|_{\mathbb{R}^n}^2\,\mathrm{d}r \tag{26}$$

holds P_θ-almost surely (P-almost surely) and therefore, we have that the function F given by formula (24) satisfies

$$F(\theta) = -\theta\int_0^T Q_r^\top\,\mathrm{d}W_r + \frac{\theta^2}{2}\int_0^T \|Q_r\|_{\mathbb{R}^n}^2\,\mathrm{d}r$$

P_θ-almost surely (P-almost surely). Maximizing the last expression over θ yields the following alternative form of the MLE $\hat{\theta}_t$:

$$\hat{\theta}_T = \frac{\int_0^T Q_r^\top \, \mathrm{d}\tilde{W}_r}{\int_0^T \|Q_r\|_{\mathbb{R}^n}^2 \, \mathrm{d}r}. \tag{27}$$

This proves that the MLE can be computed from the observed trajectory.

In what follows, we give sufficient conditions for strong consistency and asymptotic normality of the MLE $\hat{\theta}_T$ in the spirit of [36, section 2.4].

Proposition 15. *If the convergence*

$$\int_0^T \|Q_r\|_{\mathbb{R}^n}^2 \, \mathrm{d}r \quad \xrightarrow[T\to\infty]{\mathsf{P}-\text{a.s.}} \quad \infty, \tag{28}$$

is satisfied, then the MLE $\hat{\theta}_T$ given by (23) is a strongly consistent estimate of the parameter θ in equation (22), i.e. there is the following convergence:

$$\hat{\theta}_T \quad \xrightarrow[T\to\infty]{\mathsf{P}-\text{a.s.}} \quad \theta.$$

Proof. By substituting equality (26) in equality (27), we obtain that

$$\hat{\theta}_T - \theta = \frac{\int_0^T Q_r^\top \, \mathrm{d}W_r}{\int_0^T \|Q_r\|_{\mathbb{R}^n}^2 \, \mathrm{d}r} \tag{29}$$

holds P-almost surely. The claim of the proposition follows by the strong law of large numbers for martingales; see, e.g., [37, Exercise V.1.6]. ☐

Remark 16. In the case $n = 1$ and $b(t, x) \equiv b(x)$, a sufficient condition for the validity of the convergence (28) is given in [42, Theorem 2] for $H \in (0, 1/2)$ and in [42, Theorem 3] for $H \in (1/2, 1)$. Moreover, it is shown in [42, Proposition 3] and in [19, Proposition 2.2] (see also [42, Section 5]) that the convergence (28) is satisfied for the particular case of the fractional Ornstein-Uhlenbeck process ($b(t, x) \equiv x$) for $H \in (0, 1/2)$ and $H \in [1/2, 1)$, respectively.

Proposition 17. *If there exists a finite positive constant C for which the convergence*

$$\frac{1}{T} \int_0^T \|Q_r\|_{\mathbb{R}^n}^2 \, \mathrm{d}r \quad \xrightarrow[T\to\infty]{\mathsf{P}-\text{a.s.}} \quad \frac{1}{C^2} \tag{30}$$

is satisfied, then the MLE $\hat{\theta}_T$ *given by* (23) *is an asymptotically normal estimate of the parameter* θ *in equation* (22)*, i.e. there is the following convergence in law:*

$$\sqrt{T}(\hat{\theta}_T - \theta) \quad \xrightarrow[T\to\infty]{\mathrm{D}} \quad Z$$

where $Z \sim \mathsf{N}(0, C^2)$*.*

Proof. It follows from formula (29) that

$$\sqrt{T}(\hat{\theta}_T - \theta) = \frac{\frac{1}{\sqrt{T}} \int_0^T Q_r^\top \,\mathrm{d}W_r}{\frac{1}{T} \int_0^T \|Q_r\|^2_{\mathbb{R}^n} \,\mathrm{d}r}$$

holds P-almost surely. The claim of the proposition follows by the central limit theorem for martingales; see, e.g., [35, Theorem 1.49]. □

Remark 18. In the case of one-dimensional fractional Ornstein-Uhlenbeck process ($n = 1$ and $b(t, x) = x$), asymptotic normality of the MLE $\hat{\theta}_T$ is proved in [8, Theorem 2] where a condition analogous to condition (30) is shown to be valid. See also [5] where a different method is used.

Appendix A. Fractional integrals and derivatives

The notions of fractional integrals and derivatives are recalled here. For $\alpha > 0$ and $f \in L^1(0, T)$, the *left-sided Riemann-Liouville fractional integral of order* α *on* $(0, T)$, I^α_{0+}, is defined by

$$(I^\alpha_{0+} f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t f(r)(t - r)^{\alpha-1} \,\mathrm{d}r \tag{31}$$

for almost every $t \in [0, T]$. This notion extends the usual iterated integrals of f. The operator I^α_{0+} is extended to allow for $\alpha = 0$ by setting I^0_{0+} to be the identity operator. Moreover, for $p > 1$, $\alpha \in (0, 1)$, and $f \in I^\alpha_{0+}(L^p(0, T))$ (which is the image of $L^p(0, T)$ by I^α_{0+}), the inverse operation $I^{-\alpha}_{0+}$ can be defined and satisfies

$$(I^{-\alpha}_{0+} f)(t) = \frac{1}{\Gamma(1 + \alpha)} \left(f(t) t^\alpha - \alpha \int_0^t [f(t) - f(r)](t - r)^{\alpha-1} \,\mathrm{d}r \right) \tag{32}$$

for almost every $t \in [0, T]$ (the convergence of the integrals at the singularity is understood in the L^p-sense). The operator $I^{-\alpha}_{0+}$ is usually called the *left-sided Riemann-Liouville fractional derivative*. The operator I^α_{0+} is extended to $\alpha = -1$ by setting I_{0+}^{-1} to be the first derivative in the L^p sense. We refer to I_{0+}^{α} with $\alpha \geq -1$ as the *(left-sided) fractional operator of order* α for simplicity.

Similarly, for $\alpha > 0$ and $f \in L^1(0,T)$, the *right-sided Riemann-Liouville fractional integral of order* α *on* $(0,T)$, I_{T-}^{α}, is defined by

$$(I_{T-}^{\alpha} f)(t) := \frac{1}{\Gamma(\alpha)} \int_t^T (r-t)^{\alpha-1} f(r)\,\mathrm{d}r \tag{33}$$

for almost every $t \in [0,T]$, the operator I_{T-}^{0} is defined as the identity operator. Moreover, for $p > 1$, $\alpha \in (0,1)$, and $f \in I_{T-}^{\alpha}(L^p(0,T))$, the inverse operation $I_{T-}^{-\alpha}$ can be defined and satisfies

$$(I_{T-}^{-\alpha} f)(t) = \frac{1}{\Gamma(1+\alpha)} \left(f(t)(T-t)^{\alpha} - \alpha \int_t^T [f(t) - f(r)](r-t)^{\alpha-1}\,\mathrm{d}r \right). \tag{34}$$

The operator $I_{T-}^{-\alpha}$ is usually called the *right-sided Riemann-Liouville fractional derivative*. As before, the operator I_{T-}^{α} is extended to $\alpha = -1$ by setting I_{T-}^{-1} to be the first derivative in the L^p sense. We refer to I_{T-}^{α} with $\alpha \geq -1$ as the *(right-sided) fractional operator of order* α for simplicity.

For a thorough discussion of fractional integrals, derivatives, and their properties, we refer to the monograph [38] and the references therein.

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Monika Camfrlová
Department of Probability and Mathematical Statistics
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
Prague 8, 186 75
Czech Republic
E-mail address: monika.camfrlova@seznam.cz

Petr Čoupek
Department of Probability and Mathematical Statistics
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
Prague 8, 186 75
Czech Republic
E-mail address: coupek@karlin.mff.cuni.cz

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