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On ergodic control of switching processes

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An ergodic control problem of switching Markov-Feller processes is considered with a control of impulse type. We discuss the extent to which a solution can be obtained from existing results, in particular when the control acts only on the discrete component, and we study the handling of weaker assumptions. As a particular case, a detailed study of a general class of switching reflected diffusion with jumps having oblique boundary conditions is carried out.

1. Introduction

In this paper, "switching processes" designate Markov processes where the state $x = (z, n)$ has two components, x belongs to $Z \times N$, with an infinitesimal generator of the form $A^n + Q^z$ where A^n gives the evolution of z_t as a general Markov-Feller process when n is fixed, and Q^z, with z is fixed, is the generator of a continuous time Markov chain. It may be useful to mention that this type of processes has been considered for a long time in various contexts and with different names: in automatic control "Markov jump linear systems" have been used since the early 1960s (cf. Costa et al. [11, 12], Yin and Zhang [48] and the references therein), in probability, Griego and Hersh [24] developed the theory of "random evolution" in the late 1960s, (see also Ethier and Kurtz [17] and their references), and in Gihman and Skorohod [23, Vol II, Ch III, Sec 4], this corresponds to "Markov processes with a discrete component". More recently, much attention has been paid to the properties of "switching diffusions", also named "regime-switching diffusions", i.e. A^n generates a diffusion in $\mathbb{R}^d$ (for n fixed), with a great variety of applications with hybrid models involves such processes particularly in the field of manufacturing and, in the recent years, in finance, e.g., see Yin and Zhu [49] and their references. Moreover, the properties of regime-switching jump diffusions have been studied in Chen et al. [9], Xi and Zhu [47].

For control problems with discounted cost (or with finite horizon), the impulse control of diffusions with jumps has been treated in Bensoussan and Lions [5], and [30]. It can be noted that the particular case $Q = 0$ of switching processes is studied in [5, Ch 4, Sec 6.4)]. The general case of switching diffusions with Dirichlet boundary condition, is considered in Bensoussan and Lions [6]. Specific examples are studied in Song et al. [44], Wei et al. [46], among others. For ergodic continuous control of switching diffusions, we refer to Arapostathis et al. [1] (and their references). The ergodic impulse control for the particular case $Q = 0$ is treated in a joint paper with Perthame [31], while Palczewski and Stettner [38], Gatarek and Stettner [22] have obtained general results for the ergodic impulse control of Markov-Feller processes which can be applied to some cases of switching processes.

When Z is a compact metric space and N a finite set, we consider the control of such processes by means of interventions, namely, the control is a sequence of instants θ_k at which an impulse or switching ξ_k is applied to the state. We use both impulse and switching to designate the intervention, although the second term is better understood when the control acts only on the second component. A cost $c(x, \xi)$ is incurred for each action and the objective is to minimize a total ergodic cost taking in account a running cost and the cost of the control.

One aim of the present paper is to discuss to what extent existing results can be applied to our situation, in particular when the control acts only on the discrete component, and to study what can be done when the assumptions are weakened, namely when the cost is not strictly positive everywhere and the possibility of multiple simultaneous impulses. Another aim is to show that a large class of switching reflected diffusions with jumps satisfies the assumptions for which the ergodic control problem can be solved.

The content of the paper is organized as follows: the switching processes are defined in the second section. The third section presents the control problem and discusses cases where a solution can be obtained. Section 4 studies situations with weaker assumptions on the switching cost and the possibility of a finite number of simultaneous impulses. A detailed study of a general class of switching reflected diffusions with jumps is given to obtain a strictly positive transition density and, consequently, results on the ergodic problem for these processes. Section 6 mentions several possible extensions which would need further works.

2. The uncontrolled process

Let Z be a locally compact Polish space (i.e., a locally compact complete metric space) with its Borel σ-algebra $\mathcal{B}(Z)$, and denotes by $B(Z)$ the space of real-valued bounded Borel functions on Z, $C_b(Z)$ the subspace of continuous functions within $B(Z)$, $C_0(Z)$ the subspace of functions in $C_b(Z)$ vanishing at infinity (i.e., for every $\varepsilon > 0$ there exists a compact set $K = K_\varepsilon \subset Z$

such that $|f(z)| \leq \varepsilon$ for every $z \in Z \smallsetminus K$). Let us set $N = \{1, \ldots, N\}$ (abusing notation for simplicity) endowed with the discrete topology, and $E = Z \times N$ the state space of the uncontrolled process. Also $\mathcal{D}(\mathbb{R}^+, E) = \mathcal{D}([0, \infty[, Z \times N)$ the canonical space of cad-lag functions with its canonical process $x_t(\omega) = \omega(t)$ for any $\omega \in \mathcal{D}(\mathbb{R}^+, E)$ with its canonical filtration $\mathbb{F}^0 = \{\mathcal{F}_t^0 : t \geq 0\}$, $\mathcal{F}_t^0 = \sigma\{x_s : 0 \leq s \leq t\}$, and use $x_t(\omega) = (z_t(\omega), n_t(\omega))$ to distinguish between the two components. It is clear any function $h \in B(E)$ can be written in a vector-form $\vec{h}(x) = (h_i(z) : i = 1, \ldots, N)$, $x = (z, i) \in E$, i.e., $\vec{h} \in B(Z,]0, \infty[^N)$, and that either $h \in C_b(E)$ or $h \in C_0(E)$ means either $h(\cdot, i) \in C_b(Z)$ or $h(\cdot, i) \in C_0(Z)$, for any $i = 1, \ldots, N$.

Assumption 1. Suppose $\{\Omega, \mathbb{F}, x_t, P_x\}$ is a given a homogeneous Markov process on $\Omega = \mathcal{D}(\mathbb{R}^+, E)$ with its canonical filtration universally completed $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ and $x_t = (z_t, n_t)$ being the canonical process with values in $E = Z \times N$. Moreover, if $\{\Phi(t) : t \geq 0\}$, $(\Phi(t)h)(x) = \mathbb{E}_x\{h(x_t)\}$, denotes its associated the semigroup initially defined on $B(Z)$ then $\Phi(t)C_0 \subset C_0$, for any $t > 0$, with $C_0 = C_0(E)$. Furthermore, its infinitesimal generator has the form

$$Lh(z,i) = A^i h(z,i) + \sum_{j=1}^{N} q_{ij}(z)[h(z,j) - h(z,i)],$$

where, for any fixed i, the operator A^i is the infinitesimal generator of a C_0-semigroup on $B(Z)$ denoted by $\{\Phi^i(t) : t \geq 0\}$, and for any fixed z, the matrix $Q(z) = (q_{ij}(z) : i, j)$ is either null or the infinitesimal generator of an *irreducible* Markov chain with states in N satisfying

$$\sum_{j=1}^{N} q_{ij}(z) = 0, \qquad q_{ij}(z) \geq 0, \quad \forall i \neq j.$$

Also, for any fixed i, j, the functions $z \mapsto q_{ij}(z)$ are assumed *bounded and continuous.* It is clear that the expression $Lh(z,i)$ is written assuming that $h \in B(E)$ and $h(\cdot, i) \in \mathcal{D}^i(A^i)$ the domain of A^i. □

Note that the special case $q_{ij}(z) = 0$, for every i, j and z, is useful for some applications, and this means that discrete variable n remains constant in the uncontrolled process. In this case, the probability transition

$$P\{z_{t+s} \in B,\, n_{t+s} = j \mid (z_s, n_s) = (z,i)\} = G_i(z,t,B)\delta_{ij},$$

for every $t, s > 0$, $B \in \mathcal{B}(Z)$, where G_i is the probability transition corresponding to A^i and $\delta_{ij} = \mathbb{1}_{i=j}$.

Before we continue, let us make some remarks on the construction of the 'Markov-Feller' process x_t. Given a family $\{A^i : i \in N\}$ of infinitesimal generators and a (non-zero) matrix-valued function $Q(z)$, $z \in Z$, several methods can be used to construct the so-called 'switching process'. For example:

(a) We refer to the classic construction of Gihman and Skorohod [23, Vol 2, pp. 239–243].
(b) Using perturbation results, assuming (for simplicity) that $\mathcal{D}^i(A^i) = \mathcal{D}$ for every i, one can proceed as follows. Define the generator A_0 by $A_0 h(z,i) = A^i h(z,i)$ corresponding to a process in $Z \times N$ with the *second* component being constant, which generates a $C_0(Z \times N)$-semigroup, assuming that each $\Phi^i(t)$ is a $C_0(Z)$-semigroup. Similarly, consider $B = Q(z)$ as a generator corresponding to a process in $Z \times N$ with the *first* component being constant. Since $Q(z)$ is continuous, it easy to see that B generates a $C_0(Z \times N)$-semigroup. Now, regarding B as a bounded perturbation of A_0 (e.g., see Ethier and Kurtz [17, Thm 7.1, pp. 37–40]) we deduce that $A_0 + B$ generates a $C_0(Z \times N)$-semigroup. Note that with this method, the Feller property is obtained directly by construction.
(c) When A^i corresponds to a diffusion process in $\mathbb{R}^d$, the process (z_t, n_t) can be constructed as strong solution of a pair of SDE (e.g., see Yin and Zhu [49], Arapostathis et al. [1]) or as solution of a martingale problem (e.g., see Bensoussan and Lions [6]). For diffusion processes with jumps in $\mathbb{R}^d$, see Xi and Zhu [47].

3. The control problem

Although the Assumption 1 of Section 2 is given for Z locally compact, from now on, and until Section 5 included, we assume that Z is a compact Polish space and therefore $C_b(Z) = C_0(Z)$ is denoted by $C(Z)$.

Assumption 2. **(a)** There are a running cost $f(z,n) = f(x)$ and a cost per impulse (or controlled switching) $c(x,\xi)$ satisfying $f \in C_b(E)$, $f \geq 0$, and $c \in C_b(E \times E)$, $c \geq c_0$ with a constant $c_0 > 0$.
(b) For any $x \in E$, all possible impulses ξ must be in

$$\Gamma(x) = \{\xi \in E : (x,\xi) \in E_2\},$$

where E_2 is a given analytic set in $E \times E$ (recall that Z is compact and $E = Z \times N$, with N being a finite discrete set), and such that the following properties hold true

$$
\begin{aligned}
&\emptyset \neq \Gamma(x) \text{ is closed}, \quad \Gamma(\xi) \subset \Gamma(x),\ \forall \xi \in \Gamma(x), \quad \text{and} \\
&c(x,\xi) + c(\xi,\xi') \geq c(x,\xi'), \quad \forall \xi \in \Gamma(x),\ \forall \xi' \in \Gamma(\xi).
\end{aligned} \tag{1}
$$

(c) If the operator operator M is defined by

$$
Mv(x) = \inf_{\xi\in\Gamma(x)} \{c(x,\xi) + v(\xi)\}, \tag{2}
$$

then the condition

$$
\begin{aligned}
&M \text{ maps } C_b(E) \text{ into } C_b(E), \text{ and there exists a measurable} \\
&\text{selector } \hat{\xi}(x) = \hat{\xi}(x,v) \text{ realizing the infimum in } Mv(x),\ \forall x, v.
\end{aligned} \tag{3}
$$

is assumed. □

This general framework allows many possibilities for the transfer of $x = (z,n)$ to $\xi = (z',n')$, with a cost $c(x,\xi) = c(z,n;z',n')$. For instance, when the problem is to control separately z and n, we could have

$$
\Gamma(z,n) = \{z' \in K(z)\} \times \{n \in N_1(n)\},
$$

with $K(z)$ a compact subset of Z and $N_1(n)$ a (finite) subset of N, and the cost could be

$$
c(z,n;z',n') = c_1(z,z') + c_2(n,n').
$$

A case of particular interest is the situation when the control acts only on n, considered as the operating mode, i.e., $\Gamma(z,n) = \{z\} \times \{n \in N_1(n)\}$.Actually, this last case is called sometimes ‘switching control’ in a strict sense, and one can see that it can be formulated as an impulse control problem as soon as a strict positive instantaneous cost is associated to the change of the discrete component. For the case $Q = 0$, e.g., see Bensoussan and Lions [5, p. 33 and Sec 6.4].

For the controlled process, we refer to Bensoussan and Lions [5, p. 668], Robin [39] for a detailed construction on $\Omega^\infty = [\mathcal{D}(\mathbb{R}^+, E)]^\infty$, which is only summarized here. Define the product σ-algebras $\mathcal{F}_t^n = \mathcal{F}_t^{n-1} \times \mathcal{F}_t$, $n \geq 1$, where $\mathcal{F}_t^0 = \mathcal{F}_t$ is the universal completion of the canonical filtration in Assumption 1. An admissible impulse control (or switching control) is a sequence of $\nu = \{(\theta_k, \xi_k) : k \geq 1\}$ where θ_k is a $\mathcal{F}_t^{k-1}$-stopping time and ξ_k is a $\mathcal{F}_{\theta_k}^{k-1}$-measurable random variable with values in Γ. Abusing notation $\{x_{\theta_{k-1}+s}^{k-1} : s \geq 0\} = \{\tilde{x}_t^{k-1} : t \geq \theta_{k-1}\}$ (with $\tilde{x}_t$ being the shifted process) represents the controlled process in the k-copy of Ω, after the first $k-1$ impulses, which is also denoted by $\{x_t : \theta_{k-1} \le t < \theta_k\}$, so that by convention, $x_{\theta_k -} = x^{k-1}_{\theta_k}$ on $\theta_k < \infty$, i.e., the value of the controlled process just before θ_k. Let $S \subset E$ be a closed 'stopping region', and $\xi(x) \in \Gamma(x)$ be a Borel 'impulse selection' for every $x \in E$. These S and $\xi(\cdot)$ define a 'feedback' impulse control as follows: The first stopping $\theta_1 = \inf\{t \ge 0 : x^0_t \in S\}$ with $\{x^0_t : t \ge 0\}$ being the uncontrolled process, the first impulse $\xi_1 = \xi(x^0_{\theta_1})$, and hence, the process $\{x^1_{\theta_1+s} : s \ge 0\}$ is defined using ξ_1 as the initial condition at $t = \theta_1$. By induction, if $\{x^k_{\theta_k+s} : s \ge 0\}$ and $k \ge 1$ is given then $\theta_{k,1} = \inf\{s \ge 0 : x^k_{\theta_k+s} \in S\}$, $\theta_{k+1} = \theta_k + \theta_{k,1} \circ \vartheta_{\theta_k}$, and $\xi_{k+1} = \xi(x^k_{\theta_{k+1}})$, where $\{\vartheta_t : t \ge 0\}$ is the shift operator.

For instance, the reader may check Davis [13, Ch 5, pp. 186–255] for a discussion on the conditions (1) and (3), and it should be clear that the following setting is thought as an impulse control model.

The control problem is to minimize the ergodic cost

$$J_x(\nu) = \liminf_{T\to\infty} \frac{1}{T} \mathbb{E}^\nu_x \Big\{ \int_0^T f(x_t)\mathrm{d}t + \sum_{k=1}^\infty \mathbb{1}_{\theta_k \le T}\, c(x_{\theta_k -}, \xi_k) \Big\} \tag{4}$$

over all admissible impulse controls ν.

Usual heuristic arguments on the discounted cost problem lead to the following *weak form* Hamilton-Jacobi-Bellman (HJB) equation

$$w(x) = \inf_\theta \liminf_{T\to\infty} \mathbb{E}^\nu_x \Big\{ \int_0^{T\wedge\theta} [f(x_t) - \lambda]\mathrm{d}s + Mw(x_{T\wedge\theta}) \Big\}, \tag{5}$$

with $w(x)$ in $B(E)$, where M is the operator by (2).

The ergodic cost is given by (4), and the associated α-discounted problem is defined by

$$u_\alpha(x) = \inf_\nu \mathbb{E}^\nu_x \Big\{ \int_0^\infty \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \sum_{k=1}^\infty \mathrm{e}^{-\alpha\theta_k} c(x_{\theta_k -}, \xi_k) \Big\}.$$

Standard results on impulse controls (e.g., see [39, Thm V.2.1]) show that u_α is the maximal solution of

$$u_\alpha = \inf_\theta \mathbb{E}_x \Big\{ \int_0^\infty \mathrm{e}^{-\alpha t} f(x_t)\mathrm{d}t + \mathrm{e}^{-\alpha\theta} M u_\alpha(x_\theta) \Big\}.$$

Define $m_\alpha = \inf_{x\in E} u_\alpha(x)$ and $w_\alpha(x) = u_\alpha(x) - m_\alpha$.

In Gatarek and Stettner [22] it is shown that

Theorem 3. *Under the* Assumptions 1 *and* 2 *if* $\{w_\alpha : \alpha > 0\}$ *is bounded and the resolvent* $h \mapsto R_1 h$*, with*

$$R_1 h(x) = \mathbb{E}_x\Big\{ \int_0^\infty \mathrm{e}^{-t} h(x_t)\mathrm{d}t \Big\},$$

is compact from C *into itself, then* **(i)** $\lim_{\alpha\to 0} \alpha u_\alpha(x) = \lambda$*, uniformly in* $x \in E$*, with* $\lambda = \inf_\nu J_x(\nu)$*;* **(ii)** $w_\alpha \to w$ *in* C*, for some suitable sequence;* **(iii)** *there exists an optimal control* $\hat{\nu}$ *defined as* $\hat{\theta}_1 = \inf\{s \geq 0 : w(x_s) = Mw(x_s)\}$*,* $\hat{\theta}_{k+1} = \hat{\theta}_k + \hat{\theta}_1 \circ \vartheta_{\hat{\theta}_k}$*,* $k \geq 1$*,* $\hat{\xi}_k = \hat{\xi}(x_{\hat{\theta}_k})$*,* ϑ_t *being the shift operator and* $\hat{\xi}(x)$ *a Borel function satisfying* $Mw(x) = c(x, \hat{\xi}(x)) + w(\hat{\xi}(x))$*.* □

Moreover, several cases where the set $\{w_\alpha : \alpha > 0\}$ is bounded are also given. It is easy to see that one can apply these results to the switching process defined above.

However, considering the situation when the control acts only on the discrete component, that is $\Gamma(z, n) = \{z\} \times \{n \in N_1\}$ with a fixed $N_1 \subset N$, it might be difficult to use Gatarek and Stettner [22, Prop 1]. Indeed, for this $\Gamma(z, n)$, the conditions of Proposition 1 involve hitting times T_s of sets of the form

$$\begin{aligned} S(z_0, n_0) = \{(z, n) : (z_0, n_0) \in \{z\} \times N_1\} = \\ = \{(z, n) : z = z_0, n \text{ such that } n_0 \in N_1\}, \end{aligned}$$

i.e., $S(z_0, n_0) = \emptyset$ if $n_0 \in N \smallsetminus N_1$ and $S(z_0, n_0) = \{z_0\} \times N$ if $n_0 \in N_1$. This means in particular that the first component z_t should reach the point z_0. This is clearly a restrictive condition on z_t.

Nevertheless, if we consider a slight modification of $\Gamma(z, n)$, namely $\Gamma_\varepsilon(z, n) = \{z : |z' - z| \leq \varepsilon\} \times N_1$, then we could use Theorem 3.

Another direction to investigate is the following: Assume $Mv(z, n) = c + \min_{j\in N} v(z, j)$, $c > 0$ constant. Define $m_\alpha = \inf_{z,j} u_\alpha(z, j)$, $w_\alpha = u_\alpha - m_\alpha$ (which is positive), and $E_0 = \{(z, n) : u_\alpha(z, n) \leq m_\alpha + c\}$. If we also assume that the measure of E_0 is bounded below by a constant $\beta > 0$ independent of α, and that the uncontrolled process satisfies

$$\sup_{z,n} \mathbb{E}_{z,n}\{\tau_0\} < \infty,$$

independent of α, where τ_0 is the hitting time of E_0, then one can show that $\{w_\alpha : \alpha\}$ is bounded and one can apply Theorem 3 above.

4. Weaker assumptions

4.1. Switching cost

Assumption 2 on the switching cost $c(x,\xi) \geq c_0 > 0$ is designed mainly for impulse control models, and it is not always suitable for switching models, since the possibility of allowing some switching to have a zero-cost could be important in practical applications e.g., Blankenship and Menaldi [7], [32], among others.

The condition $c(x,\xi) \geq c_0 > 0$ plays an essential role to show the convergence of the iterations to solve the equations for u_α and its continuity, but also to show that the control constructed from the continuation region $\{x \in E : u(x) < Mu(x)\}$ satisfies $\hat{\theta}_k \to \infty$ as $k \to \infty$. Thus, if the assumption is only $c(x,\xi) \geq 0$ then some additional conditions are needed to solve the control problem. This difficulty can be (partially) overcome assuming

$$
(6) \qquad \begin{aligned} &\emptyset \neq \Gamma(x) \text{ is closed } \forall x \in E, \quad \Gamma(\xi) \subset \Gamma(x) \quad \forall \xi \in \Gamma(x), \\ &\text{and } \ c(x,\xi) + c(\xi,\xi') > c(x,\xi'), \quad \forall \xi \in \Gamma(x),\ \forall \xi' \in \Gamma(\xi), \end{aligned}
$$

which is a stronger (or strict) version of condition (1). As shown in [35],

Lemma 4. *Let* Assumption 2, (a) *with* $c_0 = 0$, *and* (b) *with* (6) *in lieu of* (1), *and* (c) *hold true. If* v *is a function in* $B^+(E)$ *and* $v(x) = Mv(x)$ *for some* x *in* E, *then its continuation region* $C_v = \{x \in E : v(x) < Mv(x)\}$ *is nonempty and any selector* $\hat{\xi}(x) = \hat{\xi}(x,v)$ *as in condition* (3) *satisfies* $v(\hat{\xi}(x,v)) < Mv(\hat{\xi}(x,v))$. □

This gives $\theta_k \to \infty$, however, this is not sufficient to ensure the uniform convergence of the iterates (of optimal stopping time problems) approximating u_α, but if the class $\mathcal{V} = \mathcal{V}_x$ of switching controls (available at the state x) is limited to $\nu = \{(\theta_k, \xi_k) : k \geq 1\}$ satisfying: for every fixed $x = (z,n) \in E$ and $T > 0$,

$$
(7) \qquad \sup_{z \in Z} \sup_{\nu \in \mathcal{V}_x} P_x\{\theta_k < T\} \to 0 \ \text{ as } \ k \to \infty
$$

Note that with these conditions, we can revise and adapt the arguments in Gatarek and Stettner [22, Proposition 2].

Let us mention that several classes of control problems satisfy the above conditions on θ_k: Control problems with a deterministic (or stochastic) time delay like [39], or with a 'state delay' like [30], and even more recent problems as in [33, 35, 36].

Let us give an example: Assume that the control acts on the discrete component of $x = (z, n)$ and that the control ξ_k decided at θ_k is implemented after a delay $h > 0$ (deterministic). Moreover, between θ_k and $\theta_k + h$ (i) no decision can be taken (i.e., necessarily $\theta_{k+1} \geq \theta_k + h$), and (ii) the running cost is still present. Thus, if a decision n' is taken when the state is (z, n), the future cost is (for discounted cost)

$$c(n, n') + \varphi(z, n) + \mathbb{E}_{z,n}\{\mathrm{e}^{-\alpha h} u(z_h, n')\}$$

with

$$\varphi(z, n) = \mathbb{E}_{z,n}\{\int_0^h \mathrm{e}^{-\alpha t} f(z_t, n_t)\mathrm{d}t\}.$$

Therefore, the operator M becomes

$$Mn(z, n) = \min_{n'}\{c(n, n') + \mathbb{E}_{z,n}\{\mathrm{e}^{-\alpha h} u(z_h, n')\} + \varphi(z, n)\}.$$

It is clear that the condition $\theta_{k+1} \geq \theta_k + h$ yields the property (7).

Remark 5. Regarding the ‘strict’ conditions (6), note that $c(x, \xi) \geq 0$ does not forbid $c(x, \xi) = 0$ for every x, ξ, but having $c(x, \xi)$ identically zero does not satisfies (6). So, this situation is implicitly excluded. □

4.2. *k*-simultaneous impulses

Clearly, under the condition (1) on $c(x, \xi)$, in Assumption 2 (b), there is no need to consider multiple simultaneous impulses. Therefore, throughout this section, it is only assumed that

$$\emptyset \neq \Gamma(x) \text{ is closed}, \quad \Gamma(\xi) \subset \Gamma(x),\ \forall \xi \in \Gamma(x),$$

instead of (1).

A formal setting allowing multiple simultaneous impulses may be formulated as follows:

Definition 6. A *k-simultaneous impulse* (or in short, *k-impulse*) from x to ξ is a k-uple $\xi = (\xi_1, \ldots, \xi_k)$ such that $x = x_1$, $\xi_1 \in \Gamma(x_1)$, $x_2 = \xi_1$, $\xi_2 \in \Gamma(x_2), \ldots, x_k = \xi_{k-1}$, $\xi_k \in \Gamma(x_k)$, ξ_k, e.g., a (single) impulse as used in previous sections is an 1-impulse from x to ξ. A ‘multiple simultaneous impulses’ is used as alternative name when the integer k of simultaneous impulses is not necessarily mentioned, and implicitly understood that only a finite number of simultaneous impulses is used. Also, define the iterates of $\Gamma(x)$ as

$$\Gamma_k(x) = \{(\xi_1, \ldots, \xi_k) : \xi_1 \in \Gamma(x), \xi_2 \in \Gamma(\xi_1), \ldots, \xi_k \in \Gamma(\xi_{k-1})\} \subset E^k,$$

for any $k \geq 1$. Clearly, $\Gamma(x)$ is identified with $\Gamma_1(x)$ and for any integer $\kappa \geq 1$ define

$$\Gamma^\kappa(x) = \Gamma_1(x) \cup \Gamma_1(x) \cup \cdots \cup \Gamma_{\kappa-1}(x) \cup \Gamma_\kappa(x),$$

the set of possible k-impulses with $1 \leq k \leq \kappa$. Similarly, if the set $\{x \in E : x \in \Gamma(x)\}$ is non-empty then it may be useful to define the set $\Gamma'_k(x) \subset \Gamma_k(x)$ of all k-impulses with $\Gamma'(x) = \{\xi \in \Gamma(x) : \xi \neq x\}$ in lieu of $\Gamma(x)$, which are referred to as *strict* k-simultaneous impulse. The function $c(x, \xi)$ is initially defined for any $x \in E$ and $\xi \in \Gamma(x)$ and therefore, extended to any $(\xi_1, \ldots, \xi_k) \in \Gamma_k(x)$ by linearity, i.e.,

$$c(x, \xi_1, \ldots, \xi_k) = c(x, \xi_1) + c(\xi_1, \xi_2) + \cdots + c(\xi_{k-1}, \xi_k),$$

and eventually conveniently extended to $E \times E^k$, fro any $k \geq 1$. Thus, the same notation $c(x, \xi)$ for any $x \in E$ and $\xi \in \Gamma_k(x)$ can still be used, with the previous meaning, i.e., for ξ belongs to $\Gamma_k(k) \subset E^k$ the expression of $c(x, \xi)$ changes accordingly. □

Now, besides the operator M as given by (2), in order to consider simultaneous impulses (or switchings) the iterations of M are useful, and it is convenient to define the operator

$$(8) \quad \begin{aligned} M_k v(x) = \inf \big\{ & c(x_1, \xi_1) + \cdots + c(x_k, \xi_k) + v(\xi_k) : x = x_1 \in E, \\ & x_1 \neq x_2 = \xi_1 \in \Gamma(x_1), \ldots, x_{k-1} \neq x_k = \xi_{k-1} \in \Gamma(x_{k-1}), \\ & x \neq \xi = \xi_k \in \Gamma(x_k) \big\}, \end{aligned}$$

which agrees with the (power) expression $M^k v = M(M^{k-1})v$, and the cost of k-impulses is $c(x, \xi_1, \ldots, \xi_k)$ defined above.

Setting up an impulse (or switching) control model imposes (implicitly) the restriction $x \notin \Gamma(x)$, i.e., an impulse (or switching) that does not actually move the state is not allowed (and unnecessary).

Also, it may be expected that a positive cost should be associated with any intervention (impulse or switching), i.e., $c(x, \xi) > 0$ for any $x \neq \xi \in \Gamma(x)$, when a switching control model is in mind, e.g., the cost-per-switching may be associated with beginning some operation (i.e., starting a machine), and therefore, stopping the operation, may have no cost (i.e., a zero cost), which yields $c(x,\xi) = 0$ for some $\xi \neq x$. For instance, if there is no cost for two interventions then switching forward and backward between them, produces an undesired situation: the system may be trapped at a finite time.

This justify the assertion that the extension of the function $c(x,\xi)$ to the whole product space $E \times E$ could be not necessarily continuous on the diagonal $x = \xi$, unless the diagonal is an isolated region, like in an usual switching control model.

Now, if $u_\alpha(x) = \inf\{J_x(\nu) : \nu \in \mathcal{V}\}$, where $\mathcal{V}$ is the set of controls with at most a finite number of simultaneous impulses (switchings), then u_α is expected to be the maximum solution of

$$u_\alpha(x) = \inf_\theta \mathbb{E}_x\Big\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x_s)\mathrm{d}s + \mathrm{e}^{-\alpha\theta} \inf_{k\geq 1}\{M_k u_\alpha(x_\theta)\}\Big\}$$

within a suitable class of functions u_α. This analysis could be worked out under the condition: there exists a positive integer κ and a constant $c_\kappa > 0$ such that

$$c(x,\xi_1,\ldots,\xi_k) \geq c_\kappa, \quad \forall(\xi_1,\ldots,\xi_k) \in \Gamma_k(x),\ \forall k > \kappa, \tag{9}$$

since this would implies that any impulse control $\{\theta_n, \xi_n\}$ in $\mathcal{V}$ cannot have a finite cost and also an infinite number of simultaneous k-interventions with $k > \kappa$, namely, an impulse control model with possible κ-simultaneous impulses.

5. Switching reflected diffusions with jumps

The aim of this section is to establish properties of a class of reflected switching diffusions with jumps. These properties allow to fully apply Gatarek and Stettner [22] results, in particular the regularity of the transition density and the exponential ergodicity. To the best of our knowledge the results below are new.

5.1. Preliminary setting

First recall a result in Garroni and Menaldi [20, Ch 5, pp. 159–161] relative to the transition density of a diffusion with jumps, i.e., for each A^i with i fixed. The notation and assumptions are as follows:

Assumption 7. **(a)** First, let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ with a boundary $\partial\mathcal{O}$ of class $C^{2+\alpha}$ and $Z = \overline{\mathcal{O}}$.

(b) For each $i \in N$, $A^i = A_0^i - I^i$, where A_0^i is a second order elliptic operator in $\mathcal{O}$, i.e.,

$$A_0^i \varphi(z) = \sum_{k,\ell=1}^{d} a_{k\ell}^i(z) \partial_{k\ell} \varphi(z) + \sum_{k=1}^{d} a_k^i(z) \partial_k \varphi(z),$$

with $a_{k\ell}^i \in C^\alpha(\overline{\mathcal{O}})$, $a_k^i \in L^\infty(\mathcal{O})$ for every k, ℓ, i, and satisfying a uniform ellipticity condition

$$\exists\, \mu > 0 \ / \ \sum_{k,\ell=1}^{d} a_{k\ell}^i(z) \xi_k \xi_\ell \geq \mu |\xi|^2, \quad \forall\, \xi \in \mathbb{R}^d,\ z \in \overline{\mathcal{O}},\ i \in N,$$

and I^i is an integro-differential operator having the structure given below. The boundary operator is given by

$$B^i \varphi(z) = \sum_{k=1}^{d} b_k^i(z) \partial_k \varphi(z) + b_0^i(z) \varphi(z), \quad b_k^i \in C^\alpha(\partial \mathcal{O}),\ \forall k, i,$$

with $b_0^i \geq 0$, and satisfying the oblique derivative condition

$$\exists\, \mu > 0 \ / \ \sum_{k=1}^{d} b_k^i(z) n_k(z) \geq \mu, \quad \forall\, z \in \partial \mathcal{O},\ i \in N,$$

where $n = (n_1, \ldots, n_d)$ is the unit outward normal at the point $z \in \partial \mathcal{O}$.

(c) The integro-differential operators I^i (also called Waldenfelts or pseudo-differential operators, e.g. see Bony et al. [8], Eidelman et al. [15], Jacob [28], Galakhov and Skubachevskiĭ [18], Taira [45]) have a general form

$$I^i \varphi(z) = \int_{\mathcal{O}} [\varphi(\zeta) - \varphi(z) - (\zeta - z) \cdot \nabla \varphi(z)] M^i(z, \mathrm{d}\zeta),$$

where the kernel $M^i(z, \mathrm{d}\zeta)$ integrates $\zeta \mapsto |\zeta - z|^\gamma$, $0 \leq \gamma \leq 2$, uniformly in $z \in \mathcal{O}$. The structure of the kernel is as used in Garroni and Menaldi [20], i.e., there are a jump-intensity $\mathsf{j}^i(z, \zeta)$ and a jump-density $\mathsf{m}^i(z, \zeta)$ coefficients, which are Borel measurable functions defined on $\mathbb{R}^d \times \mathbb{R}_*^d$ with values in $\mathbb{R}_*^d$ and $[0, \infty[$. Here $\mathbb{R}_*^d = \mathbb{R}^d \smallsetminus \{0\}$ is endowed with[1] a σ-finite measure π having

[1]But any other σ-finite measure space could be used instead of $\mathbb{R}_*^d$.

a strictly positive measurable function $\bar{\mathsf{j}}(\zeta)$ such that

$$\begin{gathered}|\mathsf{j}^i(z,\zeta)| \le \bar{\mathsf{j}}(\zeta), \quad 0 \le \mathsf{m}^i(z,\zeta) \le 1, \quad \forall \zeta \in \mathbb{R}^d_*,\ i \in N, \\ \int_{\bar{\mathsf{j}}<1} [\bar{\mathsf{j}}(\zeta)]^\gamma \pi(\mathrm{d}\zeta) + \int_{\bar{\mathsf{j}}\ge 1} \pi(\mathrm{d}\zeta) = C_0 < \infty,\end{gathered}$$

with some $\gamma < 2$. Moreover, $z \mapsto \mathsf{j}(z,\zeta)$ is continuously differentiable and there exists a constant $c_0 > 0$ such that for every z, z', ζ, and $0 \le \theta \le 1$,

$$c_0|z - z'| \le |(z - z') + \theta[\mathsf{j}(z,\zeta) - \mathsf{j}(z',\zeta)]| \le c_0^{-1}|z - z'|,$$

and only interior jumps are allowed, i.e., if $z \in \mathcal{O}$, $\zeta \in \mathbb{R}^d_*$, $\mathsf{m}(z,\zeta) \neq 0$ then $z + \theta\mathsf{j}(z,\zeta) \in \mathcal{O}$, for any $\theta \in [0,1]$.

(d) The previous conditions allows us to work on Sobolev/Lebesgue spaces $W^{2,p}(\mathcal{O})$, $1 < p < \infty$, but to obtain 'solutions' in the Hölder space $C^{2+\alpha}(\overline{\mathcal{O}})$, $0 < \alpha < 1$, three more conditions are added: $a^i_k \in C^\alpha(\overline{\mathcal{O}})$, $b^i_k \in C^{1+\alpha}(\partial\mathcal{O})$ and

$$\begin{gathered}|\mathsf{j}(z,\zeta) - \mathsf{j}(z',\zeta)| \le \bar{\mathsf{j}}(\zeta)|z - z'|^\alpha, \quad \forall z, z', \\ |\mathsf{m}(z,\zeta) - \mathsf{m}(z',\zeta)| \le M_0|z - z'|^\alpha, \quad \forall z, z',\end{gathered}$$

for some constant $M_0 > 0$ and the same function $\bar{\mathsf{j}}(\zeta)$ used earlier, as well as the convenient condition $0 \le \gamma < 2 - \alpha$ and $0 \le \alpha < 1$. □

Consider the Cauchy problem corresponding to the differential operators A^i_0 and B^i with a fixed i in N, and its associated fundamental (or Green) function $G_{0,i}(z,t,\mathsf{z})$ or equivalently the transition function, i.e.,

$$\begin{aligned}&\partial_t G_{0,i}(z,t,\mathsf{z}) = A^i_0 G_{0,i}(z,t,\mathsf{z}), \quad \forall (z,t) \in \mathcal{O}\times]0,\infty[, \\ &G_{0,i}(z,0,\mathsf{z}) = \delta_0(z - \mathsf{z}), \quad \forall z \in \mathcal{O}, \\ &B^i G_{0,i}(z,t,\mathsf{z}) = 0, \quad \forall (z,t) \in \partial\mathcal{O}\times]0,\infty[,\end{aligned} \tag{10}$$

for each fixed $\mathsf{z} \in \mathcal{O}$, with δ_0 being the Dirac measure. For instance, in Il'in et al. [25], Garroni and Solonnikov [21], Solonnikov [41, 42] it is proved that under the Assumption 7 (only relative to A^i_0 and B^i) that problem (10) has a unique solution satisfying suitable heat-kernel type estimates, i.e., for $F_0 = G_{0,i}$, $F_1 = \partial_z G_{0,i}$, $F_2 = \partial^2_z G_{0,i}$ and $F_2 = \partial_t G_{0,i}$, there exists constants $C_0 \ge c_0 > 0$ such that

$$|F_k(z,t,\mathsf{z})| \le C_0 t^{-(k+d)/2} \mathrm{e}^{-c_0|z-\mathsf{z}|^2/t},$$

and

$$|F_k(z,t,\mathsf{z}) - F_k(z',t,\mathsf{z})| \le C_0|z-z'|^{\alpha} t^{-(k+\alpha+d)/2} \times \\ \times \left[\mathrm{e}^{-c_0|z-\mathsf{z}|^2/t} + \mathrm{e}^{-c_0|z'-\mathsf{z}|^2/t}\right],$$

for any $z, z', \mathsf{z} \in \mathcal{O}$, $t \ge 0$, and $k = 0,1,2$. This last α-Hölder type estimate is relevant only for highest derivatives $\partial_z^2 G_{0,i}$ and $\partial_t G_{0,i}$, and the last part (d) of Assumption 7 is used only to obtain this estimate, i.e., it is not necessary for solutions in Sobolev spaces like $W^{2,p}(\mathcal{O})$.

Moreover, besides these upper bound heat-kernel type estimates, also a lower bound heat-kernel type estimate holds true, namely, there exist constants $c_0' > 0$ such that

$$G_{0,i}(z,t,\mathsf{z}) \ge c_0' t^{-d/2} \mathrm{e}^{-c_0'|z-\mathsf{z}|^2/t}, \quad \forall\, z, \mathsf{z} \in \mathcal{O},\ t \ge 0. \tag{11}$$

Certainly, this implies that $G_{0,i}(z,t,\mathsf{z})$ is strictly positive.

The heat-kernel type estimates are lost when the integro-differential operator I intervene. However, keeping tract of the $L^1(\mathcal{O})$ norm (for z and z) and the $L^\infty(\mathcal{O}\times\mathcal{O})$ norm of $G_{0,i}(z,t,\mathsf{z})$ and its derivatives in z, as functions of t, it is proved in Garroni and Menaldi [19, 20] that the Cauchy problem

$$\begin{aligned} &\partial_t G_i(z,t,\mathsf{z}) = A^i G_i(z,t,\mathsf{z}), \quad \forall (z,t) \in \mathcal{O}\times]0,\infty[, \\ &G_i(z,0,\mathsf{z}) = \delta_0(z-\mathsf{z}), \quad \forall z \in \mathcal{O}, \\ &B^i G_i(z,t,\mathsf{z}) = 0, \quad \forall (z,t) \in \partial\mathcal{O}\times]0,\infty[, \end{aligned} \tag{12}$$

for each fixed $\mathsf{z} \in \mathcal{O}$, has a one and only one solution in a Banach space (so-called Green space), which contains a number of properties pertinent to the heat-kernel, e.g., it includes the semi-norms $C(\varphi,k)$ and $K(\varphi,k) = K_1(\varphi,k) + K_2(\varphi,k)$, $k \ge 0$, for functions $\varphi(z,t,\zeta)$ from $Z\times]0,\infty[\times Z$ into $\mathbb{R}$), i.e.,

$$C(\varphi,k) = \inf\left\{C \ge 0 : |\varphi(z,t,\zeta)| \le C t^{-1+(k-d)/2},\ \forall z,t,\zeta\right\} \tag{13}$$

and

$$\begin{aligned} &K_1(\varphi,k) = \inf\left\{K_1 \ge 0 : \int_Z |\varphi(z,t,\zeta)|\mathrm{d}\zeta \le K_1 t^{-1+k/2},\ \forall z,t\right\}, \\ &K_2(\varphi,k) = \inf\left\{K_2 \ge 0 : \int_Z |\varphi(z,t,\zeta)|\mathrm{d}z \le K_2 t^{-1+k/2},\ \forall t,\zeta\right\}, \end{aligned} \tag{14}$$

where G^i uses $k = 2$, $\partial_z G^i$ uses $k = 1$ and $\partial_z^2 G^i$, $\partial_t G^i$ uses $k = 2$. As discussed later, it may be convenient to write the Cauchy problem (10) in the form of the Chapman-Kolmogorov equation, i.e.,

$$(15) \qquad \begin{aligned} &(\partial_t - A^i)G_i(z,t,\mathsf{z}) = \delta_{zt}, \quad \forall (z,t) \in \mathcal{O}\times]0,\infty[, \\ &B^i G_i(z,t,\mathsf{z}) = 0, \quad \forall (z,t) \in \partial\mathcal{O}\times]0,\infty[, \end{aligned}$$

for any fixed $\mathsf{z} \in \mathcal{O}$ and with δ_{zt} being the Dirac measure in z and t, to express the solution as a series converging in the Green space. In this way, the transition function corresponding to the integro-differential operator $A^i = A_0^i - I^i$ with B^i oblique boundary conditions if obtained.

In the above references, the operators A_0^i and B^i contains a zero-order coefficient $a_0^i(z) \leq 0$ and $b_0^i(z) \geq 0$, both satisfying the same regularity as a_k^i and b_k^i, $k \geq 1$. This corresponds to sub-Markov processes (with a finite life), i.e., its semigroup satisfies $\Phi(1)1 \leq 1$ instead of $\Phi(1)1 = 1$.

For instance, if a d-dimensional stochastic differential equation (SDE) with drift g, diffusion σ and jumps j on $\mathbb{R}^d$ is given, i.e.,

$$z_t = z_0 + \int_0^t g(z_r)\mathrm{d}r + \int_0^t \sigma(z_r)\mathrm{d}w_r + \iint_{]0,t[\times\mathbb{R}_*^d} \mathsf{j}(z_r,\zeta)\tilde{\mathsf{p}}(\mathrm{d}r,\mathrm{d}\zeta),$$

with a (standard) Wiener process $\{w_t : t \geq 0\}$ in $\mathbb{R}^d$ and a Poisson measure p having Lévy measure π in $\mathbb{R}_*^d$, then its corresponding transition density, still denoted by $G_i(z,t,\mathsf{z})$ is found as the solution of a Cauchy problem in $\mathbb{R}^d$, instead of a bounded and smooth domain $\mathcal{O}$, as an element in a suitable Banach space with property similar to the typical heat-kernel estimates. In this case, $g = (a_k^i : k)$, $\frac{1}{2}\mathrm{Tr}(\sigma\sigma*) = (a_{k\ell}^i : k,\ell)$ and j is the same, with $\mathsf{m} = 1$. Boundary conditions corresponding to a normal reflection (i.e., with $b_k^i = n_k$) are introduced by transforming the SDE into a stochastic variational inequality or a so-called Skorohod problem, and even more delicate is the case of oblique boundary conditions.

Actually, our current interest is on the lower bound of the transition density $G_i(z,t,\mathsf{z})$. Based on the lower bound heat-kernel estimate (11), Garroni and Menaldi [20, Thm 4.2.4, Thm 5.1.1, pp. 134, 162],

Theorem 8. *Under the Assumption 7, without the last part (d), the transition function, denoted by $G_i(z,t,\mathrm{d}\mathsf{z})$, corresponding to the integro-differential operator A^i with oblique boundary conditions given by the differential operator B^i, has a density $G_i(z,t,\mathsf{z})$ which in particular is continuous in (z,t) belonging to $\overline{\mathcal{O}}\times]0,\infty[$, for any z fixed in $\mathcal{O}$. Moreover, for each $\varepsilon > 0$ there exists a constant $c(\varepsilon) > 0$ such that $G_i(z,t,\mathsf{z}) \geq c(\varepsilon)$, for every (z,t,z) in $\overline{\mathcal{O}} \times [\varepsilon, 1/\varepsilon] \times \mathcal{O}$.* □

5.2. Switching diffusion processes

Let us focus when there is not an integro-differential operator I, namely, within Assumption 7, $A^i\varphi = A^i_0\varphi - a^i_0(z)\varphi(z)$ (i.e., sub-Markov), in a smooth domain $\mathcal{O}$ non necessarily bounded. The system of linear second order differential equations

$$\begin{aligned}
&(\partial_t - \mathbf{L})\vec{u}(z,t) = \vec{h}(z,t), \quad \forall (z,t) \in \mathcal{O}\times]0,T[,\\
&\vec{u}(z,0) = 0, \quad \forall z \in \mathcal{O},\\
&\mathbf{B}\vec{u}(z,t) = 0, \quad \forall (z,t) \in \partial\mathcal{O}\times]0,T[,
\end{aligned}$$

with $\mathbf{L} = \mathbf{A} + \mathbf{Q}$ is a parabolic system in the sense of Petrovskii (i.e., all roots are strictly on the complex semi-plain of negative real part), which has been studied in classic papers and books, e.g., Eidelman [14], Eidelman and Zhitarashu [16], Ivasišen [26, 27], Solonnikov [42, 43] and several other, either in $\mathbb{R}^d$ or in $\mathcal{O}$ with B. Usually, the Green function (or sub-Markov transition density) of this system is first constructed for the whole space or half-space with constant coefficients ($a^i_k = 0$ and $b^i_0 = 0$) with explicit expressions (using the heat kernel and/or Laplace transform). Then, via the so-called 'parametrix method' and successive approximations, the Green function is obtained with coefficients $a^i_{k\ell}$ in C^α and b^i_k in $C^{1+\alpha}$. The other coefficients are introduced as lower order perturbations of the previous constructions. Certainly, the key part is to have suitable estimates to accomplish this task, since it is the same as searching for the solution $\vec{u}$ corresponding to a given $\vec{h}$, even in diagonal form.

In any way, the matrix Green function $\mathbf{G}(z,t,\zeta) = (G_{ij}(z,i,t,\zeta) : i,j)$ corresponding to the second order differential operator $\mathbf{L} = \mathbf{A} + \mathbf{Q}$ is found and shown to satisfies the upper bound heat-kernel type estimates in matrix form. However, we were not able to find a good reference to the lower bound heat-kernel type estimates (or even its strictly positivity) for the matrix Green function $\mathbf{G}(z,t,\zeta)$. An alternative possibility is to extend the arguments in Bensoussan [4, Sec II.4, Prop 4.1, p. 152] to a boundary operator B^i and eventually to a system of second order differential equations. For this reason, we decided to develop the following section based on the book of Garroni and Menaldi [20], to include diffusion with jumps with oblique boundary conditions.

From the modeling point of view, the complicate boundary conditions given by a first order differential operator B^i as in Assumption 7 may get simplified to none at all $\mathcal{O} = \mathbb{R}^d$, stopping at the boundary $b^i_k = 0$ with $b^i_0 = 1$, or normal reflected $b^i_k = n_k$ (the outward normal) with $b^i_0 = 0$.

Therefore, a direct construction via SDEs (or stochastic variational inequalities) is possible, driven by a Wiener process and a Poisson measure (e.g., the reader may take a look at [29] and the references therein). When there is not diffusion component, the switching process becomes part (some components) of the stochastic integral with respect to the Poisson (martingale) measure, somehow similar to looking at a piecewise deterministic process as a degenerate diffusion with jumps. In this case, the Poisson measure corresponding to the jumps is obtained from a compound Poisson process and its stochastic integral becomes a pathwise integral. From this viewpoint, most of the existing results concerning diffusion processes with jumps (with possible no diffusion component) could be translated (or adapted) to switching diffusion processes. By no means this is a trivial task, but perhaps, it is a point to be consider. For instance, the reader finds in Bakhtin and Hurth [2] a clear treatment of the invariant density for processes that could be called *piecewise deterministic switching process.* The above arguments concerning degenerate diffusion processes with jumps refer to what could be called *piecewise diffusion switching process*, as a generalization of piecewise deterministic 'switching' process, where the ordinary differential equation is replaced with a SDE driven by Wiener and Poisson processes; and perhaps results like those of Benaïm et al. [3] could be extended.

5.3. The density of the switching process

Based on Theorem 8, for each fixed i, there is a suitable transition function $G_i(z,t,\mathrm{d}\zeta)$ with density $G_i(z,t,\zeta)$ corresponding to the integro-differential operator A^i with oblique boundary conditions B^i, which corresponds to the continuous-type component. While, the switching-type component is governed by the operator Q^z.

Let us regard A^i, Q^z and G_i in a vector/matrix notation as follows:

- $\mathbf{A}\vec{v}(z) = (A^i v_i(z) : i)$, which abusing notation is written also as $Av(z,i)$,
- $\mathbf{B}\vec{v}(z) = (B^i v_i(z) : i)$, which abusing notation is written also as $Bv(z,i)$,
- $\mathbf{Q}^z\vec{v}(z) = \big(\sum_j q_{ij}(z)v(z,j) : i\big)$, and abusing notation, the matrix $Q = (q_{ij}(z) : i,j)$ for each given z.
- $\mathbf{G}_0(z,t,\mathrm{d}\zeta) = (\mathrm{diag}(G_i(z,t,\mathrm{d}\zeta) : i)$ and $\mathbf{G}_0(z,t,\zeta) = (\mathrm{diag}(G_i(z,t,\zeta) : i)$, where 'diag' means to form a diagonal matrix with the given entries.

Thus the transition density $\mathbf{G}_0(z,t,\zeta)$ is the solution of the Chapman-Kolmogorov equation (15) regarded as a diagonal system of equations corresponding to $\mathbf{A}$ with oblique boundary conditions $\mathbf{B}$ and $Z = \overline{\mathcal{O}}$, i.e.,

$$(\partial_t - \mathbf{A})\mathbf{G}_0(z,t,\mathsf{z}) = \delta_{zt}\mathbf{I}, \quad \forall (z,t) \in \mathcal{O}\times]0,\infty[,$$

$$\mathbf{B}\mathbf{G}_0(z,t,\mathsf{z}) = 0, \quad \forall (z,t) \in \partial\mathcal{O}\times]0,\infty[.$$

To solve the Chapman-Kolmogorov equation associated with the operator $\mathbf{L} = \mathbf{A} + \mathbf{Q}$ instead of $\mathbf{A}$, i.e., for the transition density $\mathbf{G}(z,t,\mathsf{z})$

$$\begin{aligned}&(\partial_t - \mathbf{L})\mathbf{G}(z,t,\mathsf{z}) = \delta_{zt}\mathbf{I}, \quad \forall (z,t) \in \mathcal{O}\times]0,\infty[,\\ &\mathbf{B}\mathbf{G}(z,t,\mathsf{z}) = 0, \quad \forall (z,t) \in \partial\mathcal{O}\times]0,\infty[,\end{aligned}$$

we can use the classic 'iteration' methods, i.e., expressing its transition function $\mathbf{G}(z,t,\mathrm{d}\zeta)$, corresponding to the switching process, as

$$\mathbf{G}(z,t,\mathrm{d}\zeta) = \mathbf{G}_0(z,t,\mathrm{d}\zeta) + \mathbf{G}_0(z,\cdot,\cdot) \bullet \mathbf{H}(\cdot,t-\cdot,\mathrm{d}\zeta), \tag{16}$$

where $\mathbf{G}_0(z,t,\mathrm{d}\zeta)$ is the transition function corresponding to A, i.e., solving the diagonal (uncoupled) system $(\partial_t - \mathbf{A})\mathbf{G}_0(z,t,\mathrm{d}\zeta) = \delta_{zt}\mathbf{I}$, and the $\bullet$ means a suitable convolution-type operation, i.e.,

$$\mathbf{G}_0(z,\cdot,\cdot) \bullet \mathbf{H}(\cdot,t-\cdot,\mathrm{d}\zeta) = \int_0^t \mathrm{d}t' \int_Z \mathbf{G}_0(z,t',\mathrm{d}z')\mathbf{H}(z',t-t',\mathrm{d}\zeta),$$

also written as $\mathbf{G}_0(z,t-\cdot,\cdot) \bullet \mathbf{H}(\cdot,\cdot,\mathrm{d}\zeta)$, and $\mathbf{H}(z,t,\mathrm{d}\zeta)$ is to be determined by solving the equation

$$\begin{aligned}\mathbf{H}(z,t,\mathrm{d}\zeta) = \mathbf{H}_0(z,t,\mathrm{d}\zeta) + \mathbf{H}_0(z,\cdot,\cdot) \bullet \mathbf{H}(\cdot,t-\cdot,\mathrm{d}\zeta),\\ \text{with } \mathbf{H}_0(z,t,\mathrm{d}\zeta) = \mathbf{Q}\mathbf{G}_0(z,t,\mathrm{d}\zeta),\end{aligned} \tag{17}$$

as a series

$$\begin{aligned}&\mathbf{H}(z,t,\mathrm{d}\zeta) = \sum_{k=0}^{\infty} \mathbf{H}_k(z,t,\mathrm{d}\zeta), \quad \text{with terms}\\ &\mathbf{H}_k(z,t,\mathrm{d}\zeta) = \mathbf{H}_0(z,\cdot,\cdot) \bullet \mathbf{H}_{k-1}(\cdot,t-\cdot,\mathrm{d}\zeta), \quad k \geq 1.\end{aligned} \tag{18}$$

Indeed, from the Chapman-Kolmogorov equation satisfied by $\mathbf{G}(z,t,\mathrm{d}\zeta)$ and $\mathbf{G}_0(z,t,\mathrm{d}\zeta)$ follows

$$\begin{aligned}\delta_{zt}\mathbf{I} + \mathbf{Q}\mathbf{G}(z,t,\mathrm{d}\zeta) &= (\partial_t - A)\mathbf{G}(z,t,\mathrm{d}\zeta) =\\ &= (\partial_t - A)[\mathbf{G}_0(z,t,\mathrm{d}\zeta) + \mathbf{G}_0(z,\cdot,\cdot) \bullet \mathbf{H}(\cdot,t-\cdot,\mathrm{d}\zeta)] =\\ &= \delta_{zt}\mathbf{I} + \mathbf{H}(z,t,\mathrm{d}\zeta),\end{aligned}$$

i.e., $\mathbf{H}(z,t,\mathrm{d}\zeta) = \mathbf{Q}\mathbf{G}(z,t,\mathrm{d}\zeta)$, and so,

$$\mathbf{H}(z,t,\mathrm{d}\zeta) = \mathbf{Q}\mathbf{G}(z,t,\mathrm{d}\zeta) = \\ = \mathbf{Q}\mathbf{G}_0(z,t,\mathrm{d}\zeta) + \mathbf{Q}\mathbf{G}_0(z,\cdot,\cdot) \bullet \mathbf{H}(\cdot,t-\cdot,\mathrm{d}\zeta),$$

i.e., equation (17), follows after applying $\mathbf{Q}$ to equation (16).

Without using the vector/matrix notation, Chapman-Kolmogorov equation is written as $(\partial_t + LG(z,i,t,\mathrm{d}\zeta,j) = \delta_{zt}\mathbb{1}_{i=j}$, and equation (16) becomes

$$G(z,i,t,\mathrm{d}\zeta,j) = \mathbb{1}_{i=j}G_i(z,t,\mathrm{d}\zeta) + G_i(z,\cdot,\cdot) \bullet H(\cdot,i,t-\cdot,\mathrm{d}\zeta,j), \tag{19}$$

and the $\bullet$ means another suitable convolution-type operation, i.e.,

$$\mathbb{1}_{i=\cdot}G_i(z,\cdot,\cdot) \bullet H(\cdot,\cdot,t-\cdot,\mathrm{d}\zeta,j) = \\ = \sum_{i'} \int_0^t \mathrm{d}t' \int_Z \mathbb{1}_{i=i'}G_i(z,t',\mathrm{d}z')H(z',i',t-t',\mathrm{d}\zeta,j) = \\ = \int_0^t \mathrm{d}t' \int_Z G_i(z,t',\mathrm{d}z')H(z',i,t-t',\mathrm{d}\zeta,j).$$

Since $\sum_{i'} q_{ii'}(z)\mathbb{1}_{i'=j}G_{i'} = G_j q_{ij}$, equation (17) can be written as

$$H(z,i,t,\mathrm{d}\zeta,j) = H_0(z,i,t,\mathrm{d}\zeta,j) + H_0(z,i,\cdot,\cdot,\cdot) \bullet H(\cdot,\cdot,t-\cdot,\mathrm{d}\zeta,j), \\ \text{with } H_0(z,i,t,\mathrm{d}\zeta,j) = q_{ij}(z)G_j(z,t,\mathrm{d}\zeta), \tag{20}$$

and the series (18) takes essentially the same form, namely,

$$H(z,i,t,\mathrm{d}\zeta,j) = \sum_{k=0}^{\infty} H_k(z,i,t,\mathrm{d}\zeta,j), \quad \text{with terms} \\ H_k(z,i,t,\mathrm{d}\zeta,j) = H_0(z,\cdot,\cdot,\cdot) \bullet H_{k-1}(\cdot,\cdot,t-\cdot,\mathrm{d}\zeta,j), \quad k \geq 1, \tag{21}$$

i.e., for any $k \geq 1$, $i,j \in N$, $z \in Z$ and $t > 0$ we have

$$H_k(z,i,t,\mathrm{d}\zeta,j) = \\ = \sum_{i'} \int_0^t \mathrm{d}t' \int_Z q_{ii'}(z)G_{i'}(z,t',\mathrm{d}\zeta')H_{k-1}(\zeta',i',t-t',\mathrm{d}\zeta,j).$$

Now, to state a precise result for the switching diffusion with jumps, we need some more notations. if $\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)$ denotes the space of real-valued $N \times N$ (square) matrices endowed with the operator-norm $|\cdot|$, and Z is a regular domain of $\mathbb{R}^d$ (the possibility $Z = \mathbb{R}^d$ is included), then consider a vector form of the semi-norms $C(\varphi,k)$ and $K(\varphi,k)=K_1(\varphi,k)+K_2(\varphi,k)$, $k\geq 0$, for functions (or *kernels*) $\varphi(z,t,\zeta)$ from $Z\times]0,\infty[\times Z$ into $\mathcal{L}(\mathbb{R}^{\bar{n}},\mathbb{R}^{\bar{n}})$, i.e.,

$$C(\varphi,k)=\inf\big\{C\geq 0: |\varphi(z,t,\zeta)|\leq Ct^{-1+(k-d)/2},\ \forall z,t,\zeta\big\} \tag{22}$$

and

$$\begin{aligned} K_1(\varphi,k)&=\inf\big\{K_1\geq 0: \int_Z |\varphi(z,t,\zeta)|\mathrm{d}\zeta\leq K_1t^{-1+k/2},\ \forall z,t\big\},\\ K_2(\varphi,k)&=\inf\big\{K_2\geq 0: \int_Z |\varphi(z,t,\zeta)|\mathrm{d}z\leq K_2t^{-1+k/2},\ \forall t,\zeta\big\}. \end{aligned} \tag{23}$$

A (non-commutative) kernel-convolution is defined between two kernels as

$$(\varphi\bullet\psi)(z,t,\zeta)=\int_0^t \mathrm{d}t'\int_Z \varphi(z,t',\zeta')\psi(\zeta',t-t',\zeta)\mathrm{d}\zeta',\quad \forall z,t,\zeta, \tag{24}$$

which could be also denoted by $\varphi(z,\cdot,\cdot)\bullet\psi(\cdot,t-\cdot,\zeta)$. Denote by $\mathcal{G}_r$ or $\mathcal{G}_r(Z\times]0,T],\mathbb{R}^{\bar{n}}\times\mathbb{R}^{\bar{n}})$ the vector space of all kernels φ satisfying $C(\varphi,r)+K(\varphi,r)<\infty$. It is simple to check that if φ is a *heat kernel type* of order k, i.e., there exit positive constants Λ, λ such that

$$|\varphi(z,t,\zeta)|\leq \Lambda t^{-1+(k-d)/2}\exp\big(-\frac{\lambda|z-\zeta|}{t}\big),\quad \forall z,t,\zeta'$$

then $C(\varphi,k)\leq\Lambda$ and $K(\varphi,k)\leq 2\pi^{d/2}\Lambda\lambda^{-d/2}$.

Proposition 9. *Assume that the infinitesimal generator* $\mathbf{Q}$ *satisfies* Assumption 1. *If the* $\mathbf{H}_0(z,t,\mathrm{d}\zeta)$ *has a density denoted by* $\mathbf{H}_0(z,t,\zeta)$ *belonging to* $\mathcal{G}_{r_0}$ *with* $0<r_0\leq 2$, *then the equation* (17) *can be written as*

$$\begin{aligned} \mathbf{H}(z,t,\zeta)=\mathbf{H}_0(z,t,\zeta)+\mathbf{H}_0(z,\cdot,\cdot)\bullet\mathbf{H}(\cdot,t-\cdot,\zeta),&\\ \textit{with}\ \ \mathbf{H}_0(z,t,\zeta)=\mathbf{Q}\mathbf{G}_0(z,t,\zeta),& \end{aligned} \tag{25}$$

where $\bullet$ *means the kernel-convolution* (24), *and the series*

$$\begin{aligned} &\mathbf{H}(z,t,\zeta)=\sum_{k=0}^{\infty}\mathbf{H}_k(z,t,\zeta),\quad \textit{with terms}\\ &\mathbf{H}_k(z,t,\zeta)=\mathbf{H}_0(z,\cdot,\cdot)\bullet\mathbf{H}_{k-1}(\cdot,t-\cdot,\zeta),\quad k\geq 1, \end{aligned} \tag{26}$$

converges uniformly on $\mathcal{G}_{k_0}$*, moreover,*

$$[C(\mathbf{H},r_0)\vee K(\mathbf{H},r_0)]\leq\sum_{k=1}^{\infty}C_k[C(\mathbf{H}_0,r_0)\vee K(\mathbf{H}_0,r_0)]^k,$$

with $C_{k+1}/C_k\to 0$ *as* $k\to\infty$*.*

Proof. Actually, the arguments are similar to those found in Garroni and Menaldi [19, Ch VIII, Prop 1.1, pp. 325–328]. Indeed, because the operator $\mathbf{Q}$ satisfies

$$[C(\mathbf{Q}\varphi,r)\vee K(\mathbf{Q}\varphi,r)]\leq|\mathbf{Q}|[C(\varphi,r)\vee K(\varphi,r)],\quad\forall\varphi\in\mathcal{G}_r,\ r>0,$$

where $|\mathbf{Q}|=\sup\{|\mathbf{Q}^z\vec{v}|:|\vec{v}|=1,\ z\in Z\}$ is the matrix/operator norm, and the following property holds for the kernel-convolution

$$\begin{aligned}&C(\varphi\bullet\psi,k+r)\leq c_d(k,r)[C(\varphi,k)K(\psi,r)+K(\varphi,k)C(\psi,r)],\\&K_i(\varphi\bullet\psi,k+r)\leq K_i(\varphi,k)K_i(\psi,r),\quad\forall k,r>0,\ i=1,2,\end{aligned}\tag{27}$$

with

$$c_r(k,r)=\begin{cases}2^{2+d}(\dfrac{1}{k}+\dfrac{1}{r}) & \text{if } 0<k\leq d,\\ \beta(\dfrac{r}{2},\dfrac{k-d}{2})=\displaystyle\int_0^1\theta^{\frac{r}{2}-1}(1-\theta)^{\frac{k-d}{2}-1}\mathrm{d}\theta & \text{if } k>d,\end{cases}$$

$$K_i(\varphi,k+r)\leq K_i(\varphi,k)K_i(\psi,r),\quad\forall k,r>0,\ i=1,2,\tag{28}$$

we can complete the argument. □

Theorem 10. *Under the* Assumption 7 *for* A^i*,* B^i*, without the last part* (d)[2]*, and the Assumption 1 for* Q^z*, the switching diffusion process with jumps has a transition density belonging to* $\mathcal{G}_1$*, which means in particular that* $\mathbf{G}(z,t,\zeta)=(G(z,i,t,\zeta,j):i.j)$ *is continuous in* z*,* ζ *and* $t>0$*.*

Proof. Indeed, as discussed in Proposition 9 and earlier, under Assumption 7 the transition density function $\mathbf{G}_0(z,t,\zeta)$ belongs to the Green space $\mathcal{G}_1$. Since $\mathbf{Q}$ is linear from $\mathbb{R}^N$ into itself, uniformly bounded in $z\in Z$, the kernel $\mathbf{H}_0(z,t,\zeta)=\mathbf{QG}_0(z,t,\zeta)$ belongs to $\mathcal{G}_1$. Hence, $\mathbf{H}(z,t,\zeta)$ has the same smoothness and the conclusion follows. □

[2]But if part (d) is assumed then the density $\mathbf{G}(z,t,\zeta)$ belongs to $\mathcal{G}_1^{2+\alpha}$.

5.4. Strict positivity of the transition density

The maximum principle can be used to check the strict positivity of the transition density function $\mathbf{G}(z,t,\zeta)$ and specific lower bound estimates can be obtained, e.g., instead of beginning with the transition function $\mathbf{G}_0(z,t,\zeta)$ we may begin with the sub-transition function $\mathbf{G}_=(z,t,\zeta)$ corresponding to the uncoupled system associated with the operator

$$\mathbf{A}+\mathbf{Q}_=, \quad \text{for} \quad \mathbf{Q}^z_= = \operatorname{diag}(q_{ii}(z):i), \quad \text{and} \quad \mathbf{Q}^z = \mathbf{Q}^z_= + \mathbf{Q}^z_{\neq}.$$

This sub-transition function $\mathbf{G}_=(z,t,\zeta) = \operatorname{diag}(G^=_i(z,t,\zeta):i)$ is a diagonal matrix and it is obtained from results of equations. This means that directly from the results quoted at the end of Subsection 5.1 if the coefficients $z \mapsto q_{ii}(z)$ are bounded and measurable then the transition density function $G^=_i(z,t,\zeta)$ has the same regularities as $G_i(z,t,\zeta)$. In particular, the lower bound estimates are satisfied, i.e., for every $\varepsilon > 0$ there exists a positive constant $c = c(\varepsilon) > 0$ such that

$$G^=_i(z,t,\zeta) \geq c, \quad \forall (z,t,\zeta) \in Z \times [\varepsilon, 1/\varepsilon] \times Z, \tag{29}$$

see Theorem 8.

On the other hand, the expressions (19), (20), (21) corresponding to the transition function $G^=_i(z,t,\mathrm{d}\zeta)$ and the operator $Q^z_{\neq}$ are valid, and they can be written in term of the transition density function $G^=_i(z,t,\zeta)$. Since the coefficients $q^{\neq}_{ij}(z) = \mathbb{1}_{i\neq j} q_{ij}(z)$ corresponding to $Q^z_{\neq}$ are non-negative, the first term

$$H_0(z,i,t,\mathrm{d}\zeta,j) = q^{\neq}_{ij}(z) G^=_j(z,t,\mathrm{d}\zeta)$$

is non-negative, i.e., now

$$H_k(z,i,t,\mathrm{d}\zeta,j) = \\ = \sum_{i'} \int_0^t \mathrm{d}t' \int_Z \mathbb{1}_{i\neq i'} q_{ii'}(z) G^=_{i'}(z,t',\mathrm{d}\zeta') H_{k-1}(\zeta',i',t-t',\mathrm{d}\zeta,j).$$

and by induction, every term $H_k(z,i,t,\mathrm{d}\zeta,j)$ in the series expression (21) is non-negative, as well as the series itself, i.e, $H(z,i,t,\mathrm{d}\zeta,j) \geq 0$. This implies that $G(z,i,t,\mathrm{d}\zeta,j) \geq 0$.

Proposition 11. *Suppose* Assumption 1 *for the infinitesimal generator* $\mathbf{Q} = \mathbf{Q}^z$ *and* Assumption 7 *for* A^i, B^i, *without the last part* (d). *Then* $G^=_i(z,t,\mathrm{d}\zeta)$

has a density denoted by $G_i^=(z,t,\zeta)$*, such that*

$$\mathbf{G}^= = diag(G_i^=(z,t,\zeta) : i)$$

belongs to $\mathcal{G}_1$*. Moreover, if the coefficients*[3] $q_{ij}(z)$ *are bounded below, i.e.,*

$$\text{there exists } q_0 \text{ such that } \quad q_{ij}(z) \geq q_0 > 0, \quad \forall i \neq j, \tag{30}$$

then $G(z,i,t,\mathrm{d}\zeta,j)$ *has a density denoted by* $G(z,i,t,\zeta,j)$ *and the inequality*

$$G(z,i,t,\zeta,j) \geq \mathbb{1}_{i=j} G_i^=(z,t,\zeta) + tq_0 \mathbb{1}_{i\neq j} G_i^=(z,t,\zeta), \quad \forall z,\ i,t,\ \zeta,\ j, \tag{31}$$

holds true. Furthermore, the transition density function $G(z,i,t,\zeta,j)$ *is strictly positive, actually, for every* $\varepsilon > 0$ *there exists a positive constant* $c = c(\varepsilon) > 0$ *such that*

$$G(z,i,t,\zeta,j) \geq c, \quad \forall (z,t,\zeta) \in Z \times [\varepsilon, 1/\varepsilon] \times Z,\ i,j \in N, \tag{32}$$

Proof. Since the equation (19) becomes

$$G(z,i,t,\zeta,j) = \mathbb{1}_{i=j} G_i^=(z,t,\zeta) + \\ + \int_0^t \mathrm{d}t' \int_Z G_i^=(z,t',\zeta') H(\zeta',i,t-t',\zeta,j) \mathrm{d}\zeta',$$

let us look at the second term G^1 in the series representing G as solution to (19), namely,

$$G^1(z,i,t,\zeta,j) = \int_0^t \mathrm{d}t' \int_Z G_i^=(z,t',\zeta') \mathbb{1}_{i\neq j} q_{ij}(z) G_i^=(\zeta',t-t',\mathrm{d}\zeta) \mathrm{d}\zeta',$$

and, in view of the Chapman-Kolmogorov equality, the transition function satisfies

$$\mathbb{1}_{i\neq j} q_{ij}(z) \int_Z G_i^=(z,t',\zeta') G_i^=(\zeta',t-t',\zeta) \mathrm{d}\zeta' = \mathbb{1}_{i\neq j} q_{ij}(z) G_i^=(z,t,\zeta).$$

Hence, assumption (30) implies that

$$G^1(z,i,t,\zeta,j) \geq tq_0 \mathbb{1}_{i\neq j} G_i^=(z,t,\zeta), \quad \forall z,\ i,t,\ \zeta,\ j,$$

and the inequality (31) follows. Therefore, it is clear that (29) yields (32). □

[3] Actually, the condition $q_{ii} = -\sum_{j\neq i} q_{ij}(z)$ is satisfied, so that condition (30) is 'almost' deduced from Assumption 1, since the Markov Chain (for any fixed parameter z) is irreducible

5.5. Invariant measure and exponential ergodicity

Everything is in place to give the desired result. Indeed, the strict positivity of the density allow to use Doob's classical argument as in Garroni and Menaldi [20, Sec 5.2, Thms 5.2.1 & 5.2.3, pp. 165–169].

Proposition 12. *Suppose Assumption 1 for the infinitesimal generator* $\mathbf{Q} = \mathbf{Q}^z$ *and Assumption 7 for* A^i, B^i, *without the last part* (d). *Then the switching process has a unique invariant probability* $\{\mu(\mathrm{d}z,i) : i \in N\}$ *and there exist constants* $C_0 > c_0 > 0$ *such that*

$$\Big|\sum_{j\in N}\int_Z G(z,i,t,\mathrm{d}\zeta,j)f(\zeta,j) - \sum_{j\in N}\int_Z f(\zeta,j)\mu(\mathrm{d}\zeta,j)\Big| \leq \\ \leq C_0\mathrm{e}^{-c_0 t}\sup\big\{|f(z,i)| : z\in Z,\, i\in N\big\},$$

for every (z,i) *in* $Z\times N$, $t>0$, *and any bounded Borel function* f. *Actually,* $\mu(\mathrm{d}z,i)$ *has a density denoted by* $\mu(z,i)$. □

Remark 13. An argument similar to Bensoussan [4, Thm 4.7, p. 146] shows that the support of the invariant probability $\mu(\mathrm{d}z,i)$ is the whole space E, i.e., the density $\mu(z,i)$ is strictly positive. □

The resolvent operator R_α associated with the switching diffusions with jumps having oblique boundary conditions is defined as the operator $h\mapsto u$ on $C_b(Z) = C_0(Z) = C(Z)$

$$u_\alpha(z,n) = R_\alpha h(z,n) := \mathbb{E}_{zn}\Big\{\int_0^\infty \mathrm{e}^{-t\alpha}h(z_t,n_t)\mathrm{d}t\Big\},$$

and therefore, $\vec{u}_\alpha = (u_\alpha(z,i) : i)$ solves the following integro-differential equation

$$\begin{aligned} &(\mathbf{A}+\mathbf{Q}-\alpha)\vec{u}_\alpha(z) = h(z), \quad \forall z\in\mathcal{O},\\ &\mathbf{B}\vec{u}_\alpha(z) = 0, \quad \forall z\in\partial\mathcal{O}, \end{aligned} \tag{33}$$

with $\vec{h} = (h(z,i : i)$, and it can be represented via the Green function $G(z,i,t,\zeta,j)$ as

$$u_\alpha(z,i) = \sum_{j\in N}\int_0^\infty \mathrm{e}^{-\alpha t}\mathrm{d}t\int_Z G(z,i,t,\zeta,j)h(\zeta,j)\mathrm{d}\zeta.$$

Following classical arguments,

Proposition 14. *Suppose* Assumption 1 *for the infinitesimal generator* $\mathbf{Q} = \mathbf{Q}^z$ *and* Assumption 7 *for* A^i, B^i, *without the last part* (d). *Then the resolvent* R_α, $\alpha > 0$ *is compact from* $C(Z)$ *into itself.*

Proof. Accepting the implicitly mentioned estimates on the matrix Green function $\mathbf{G}(z,t,\zeta)$ similar to the estimates on the Green function obtained in Garroni and Menaldi [19, 20][4], there exists a constant $C_\alpha > 0$ such that

$$\|\vec{u}_\alpha\|_{C^\alpha} \le C_\alpha \|\vec{h}\|_{C}, \quad \forall h \in C(Z, \mathbb{R}^N),$$

where $\|\cdot\|_C$ is the sup-norm in $C(Z,\mathbb{R}^N)$ and $\|\cdot\|_{C^\alpha}$ is the α-norm in Hölder space $C^\alpha(Z,\mathbb{R}^N)$. This estimate yields the compactness of the resolvent.

Alternatively, considering a problem similar to (33) with only $\mathbf{A}$, (i.e., without the 'coupling' operator $\mathbf{Q}$), the linear system becomes diagonal, and it can be solved in a standard way, e.g., Bensoussan and Lions [5] (assuming only normal reflected boundary conditions, i.e., $b_k^i = n_k$ the outward normal direction). A simple fixed point argument shows that the equation (33) has a unique solution $\vec{u} \in W^{2,p}(\mathcal{O},\mathbb{R}^N)$ for any given $\vec{h} \in L^p(\mathcal{O},\mathbb{R}^N)$, for every $1 < p < \infty$. Since $C(Z,\mathbb{R}^N) \subset L^p(\mathcal{O},\mathbb{R}^N)$ and the Sobolev embedding implies that the identity operator is a compact from $W^{2,p}(\mathcal{O},\mathbb{R}^N)$ into $C(Z,\mathbb{R}^N)$, for p sufficiently large (recall N is finite), the resolvent is indeed compact. □

6. Extensions

• The case Z locally compact Polish space with N denumerable (i.e., finite or countable). The most complete results in ergodic impulse control for Markov-Feller processes (in a locally compact Polish space) are in Palczewski and Stettner [37], assuming that $\Gamma(x) = \Gamma$, a fixed compact (as seen previously), but typical switching model requires a variable $\Gamma(x)$. Perhaps this extension is feasible, but we have not found such a result. Also they include an ergodic condition, a little more general than: If $P(x,t,B) = P\{x_t \in B : x_0 = x\}$ denotes the transition probability of the Markov-Feller process in Assumption 1 (in locally compact Polish spaces), then suppose that there exist a unique probability μ on $\mathcal{B}(E)$, and functions $K : Z \times N \to]0,\infty[$ *bounded on compact sets*, $h : \mathbb{R}^+ \to \mathbb{R}^+$ *integrable*, such that

$$\|P(x,t,\cdot) - \mu\|_{\text{TV}} \le K(x)h(t), \quad \forall x \in Z \times N, \tag{34}$$

[4] Only the C^α-estimates for $G(z,i,t,\zeta,j)$ in z are necessary.

and such that, the random variables $\{K(x_t) : t \geq 0\}$ are *uniformly integrable* for x on compact sets. There are several classes of switching processes satisfying the exponential convergence (34). The list below is by no means an exhaustive one:

– When N is finite: for switching diffusions, i.e., A^i corresponds to a diffusion, see Yin and Zhu [49]; for switching diffusion with jumps, see Chen et al. [9]. Moreover, Cloez and Hairer [10] show the exponential ergodicity condition for several types of Markov processes (including examples in infinite dimensional spaces).
– When N is infinite: for switching diffusions, see Shao [40], and for switching diffusion with jumps, see Xi and Zhu [47].

• The case of N infinite and Z compact does not seem to have been studied and could be a subject of future work.
• Allowing switching on the boundary for reflected diffusion with jumps. In Section 5, Assumption 7 (b) the boundary operator does not contain a coupling on the zero-order terms. To consider switching on the boundary, the boundary conditions becomes

$$\mathbf{B}\vec{v}(z) + \mathbf{Q}_\partial \vec{v}(z) = 0$$

include another operator $\mathbf{Q}_\partial = \{Q_\partial^z : z\}$, where

$$Q_\partial^z v_i(z) = \sum_j q_{ij}^\partial(z) v_j(z), \quad \vec{v}(z) = (v_i(z) : i),$$

for any $z \in \partial\mathcal{O}$.
• The extension of our previous works [33, 35, 36] on impulse controls with constraint could be done for switching processes, along the lines of [34].

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