

A global Hartman-Grobman theorem

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We showed that for any bounded neighborhood of a hyperbolic equilibrium point x_0 , there is a transformation which is locally homeomorphism, such that the system is changed into a linear system in this neighborhood.

If the eigenvalues of $Df(x_0)$ are all located in the left-half complex plane, then there is a homeomorphism on the whole region of attraction such that the nonlinear system on the region of attraction is changed into a linear system under such a coordinate change.

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1. Introduction

Reducing a nonlinear system

$$(1) \quad \dot{x} = f(x)$$

to a simpler form by choosing correct coordinates has always been a research direction. Since linear systems are simplest and well studied, it is always a desire to change it to a linear one. Poincaré in his dissertation showed that if f is analytic at the equilibrium point x_0 , and the eigenvalues of $Df(x_0)$ are nonresonant, then there is a formal power series of change of variable to change (1) to a linear system [7, 1]. Hartman and Grobman showed that if f is continuously differentiable, then there is a neighborhood of a hyperbolic equilibrium point and a homeomorphism on this neighborhood, such that the system in this neighborhood is changed to a linear system under such a homeomorphism [2, 3, 4, 6].

In this paper, we extend the Hartman-Grobman theorem to any bounded neighborhood of a hyperbolic equilibrium point, and show that for any bounded neighborhood of a hyperbolic equilibrium point x_0 , there is a transformation which is a local homeomorphism, such that the system is changed into a linear system in this neighborhood. Of course, such a transformation

can not be a homeomorphism on this neighborhood because one could enlarge the neighborhood to include another equilibrium point. However we are able to show that in the cases that either all the eigenvalues of $Df(x_0)$ are located in the left-half complex plane, or in the right-half complex plane, there is a homeomorphism such that the system on the region of attraction (or on the region of repulsion if the eigenvalues are located in the right-half complex plane) is changed to a linear system.

2. Hartman-Grobman theorem on any bounded region

By applying a translation, we can always assume 0 is the hyperbolic equilibrium point of (1).

Theorem 2.1. *Let E be an open set of \mathbb{R}^n containing the origin, $f : E \rightarrow \mathbb{R}^n$ be a C^1 function on E , 0 be a hyperbolic equilibrium point of the system (1) (i.e. none of the eigenvalues of $DF(0)$ is on the imaginary axis), and $N_M = \{x : \|x\| < M\}$ be the neighborhood of the origin of radius M . For any $M, \epsilon > 0$ such that $\overline{N}_{M+\epsilon} \subset E$, there exists a transformation $y = H(x)$, $H(0) = 0$ and H is a homeomorphism in a neighborhood of 0, such that the system (1) is changed into the linear system*

$$\dot{y} = Ay, \quad A = Df(0)$$

in N_M .

Proof. Without lose of generality by applying a linear change of coordinates, assume $A = Df(0)$ has the form of

$$A = \begin{bmatrix} P & 0 \\ 0 & N \end{bmatrix}$$

where the eigenvalues of P are located in the right complex plane, and the eigenvalues of N are located in the left complex plane. Write

$$f(x) = \begin{bmatrix} P & 0 \\ 0 & N \end{bmatrix} x + \begin{bmatrix} W_1(x) \\ W_2(x) \end{bmatrix}$$

where $\begin{bmatrix} W_1(x) \\ W_2(x) \end{bmatrix} = f(x) - Df(0)x$ has the properties that $W_i(0) = 0$ and $DW_i(0) = 0$, $i = 1, 2$.

For any $M, \epsilon > 0$ such that $\overline{N}_{M+\epsilon} \subset E$, let $\alpha(x)$ be a C^∞ function such that

$$(2) \quad \alpha(x) = \begin{cases} 1, & \|x\| \leq M \\ 0, & \|x\| \geq M + \epsilon \end{cases} \quad \text{and} \quad \hat{W}_i(x) = W_i(\alpha(x)x), \quad i = 1, 2.$$

Then $W_i(x) = \hat{W}_i(x)$, $i = 1, 2$, for all x with $\|x\| < M$. Let $\phi_t(x)$ be the flow of the vector field $f(x)$, i.e. $x(t) = \phi_t(x_0)$ is the solution of the initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0.$$

Define

$$(3) \quad H(x) = x + \begin{bmatrix} \int_0^\infty e^{-Ps} \hat{W}_1(\phi_s(x)) \, ds \\ - \int_{-\infty}^0 e^{-Ns} \hat{W}_2(\phi_s(x)) \, ds \end{bmatrix}.$$

Since \hat{W}_i are continuous on $\overline{N}_{M+\epsilon}$, $\hat{W}_i(\phi_s(x))$ is bounded for all s . So the integrals in the definition of H converge, and H is well defined.

Since $\hat{W}_i(0) = W_i(0) = 0$ and $D\hat{W}_i(0) = DW_i(0) = 0$, we have $DH(0) = I$, and by the inverse function theorem, H is a local homeomorphism in a neighborhood of 0.

For any solution $x(t)$ of the differential equation $\dot{x} = f(x)$,

$$\begin{aligned} H(x(t)) &= x(t) + \begin{bmatrix} \int_0^\infty e^{-Ps} \hat{W}_1(\phi_s(x(t))) \, ds \\ - \int_{-\infty}^0 e^{-Ns} \hat{W}_2(\phi_s(x(t))) \, ds \end{bmatrix} \\ &= x(t) + \begin{bmatrix} \int_0^\infty e^{-Ps} \hat{W}_1(x(t+s)) \, ds \\ - \int_{-\infty}^0 e^{-Ns} \hat{W}_2(x(t+s)) \, ds \end{bmatrix} \\ &= x(t) + \begin{bmatrix} \int_t^\infty e^{-P(\tau-t)} \hat{W}_1(x(\tau)) \, d\tau \\ - \int_{-\infty}^t e^{-N(\tau-t)} \hat{W}_2(x(\tau)) \, d\tau \end{bmatrix}. \end{aligned}$$

So under the transformation $y = H(x)$

$$\dot{y} = \frac{d}{dt} H(x(t))$$

$$\begin{aligned}
&= \dot{x} - \begin{bmatrix} \hat{W}_1(x(t)) \\ \hat{W}_2(x(t)) \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \int_t^\infty e^{-P(\tau-t)} \hat{W}_1(x(\tau)) d\tau \\ -\int_{-\infty}^t e^{-N(\tau-t)} \hat{W}_2(x(\tau)) d\tau \end{bmatrix} \\
&= Ax + \begin{bmatrix} W_1(x) \\ W_2(x) \end{bmatrix} - \begin{bmatrix} \hat{W}_1(x) \\ \hat{W}_2(x) \end{bmatrix} + A \begin{bmatrix} \int_0^\infty e^{-Ps} \hat{W}_1(\phi_s(x)) ds \\ -\int_{-\infty}^0 e^{-Ns} \hat{W}_2(\phi_s(x)) ds \end{bmatrix} \\
&= Ay
\end{aligned}$$

for all $x \in N_M$. □

Theorem 2.1 improves the classical Hartman-Grobman Theorem. Unlike the classical Hartman-Grobman Theorem in which the transformation is defined on a small neighborhood of the equilibrium point, the transformation $y = H(x)$ given by Theorem 2.1 is defined on arbitrarily large bounded region. At first glance, the result seems too good to be true. However it should be pointed out that the transformation $y = H(x)$ is not necessary a homeomorphism on N_M . Clearly H maps all the equilibrium points of $\dot{x} = f(x)$ in N_M to 0, because for any equilibrium $x_0 \in N_M$

$$\begin{aligned}
H(x_0) &= x_0 + \begin{bmatrix} \int_0^\infty e^{-Ps} \hat{w}_1(\phi_s(x_0)) ds \\ -\int_{-\infty}^0 e^{-Ns} \hat{w}_2(\phi_s(x_0)) ds \end{bmatrix} \\
&= x_0 + \begin{bmatrix} \int_0^\infty e^{-Ps} ds w_1(x_0) \\ -\int_{-\infty}^0 e^{-Ns} ds w_2(x_0) \end{bmatrix} \\
&= x_0 + \begin{bmatrix} P^{-1}w_1(x_0) \\ N^{-1}w_2(x_0) \end{bmatrix} \\
&= A^{-1}f(x_0) = 0
\end{aligned}$$

Theorem 2.1 is an existence type result, the proof is not constructive because it is generally impossible to find an analytic solution of a nonlinear system. Nevertheless, we give some examples of nonlinear systems with analytic solutions to demonstrate the transformations. Note that in the proof, the \hat{W}_1, \hat{W}_2 are introduced in order to guarantee the convergence of the integrals. If the integrals converge for either $s \rightarrow \infty$ or $s \rightarrow -\infty$, then we do not need replace W_1, W_2 with \hat{W}_1, \hat{W}_2

Example 2.2. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2^2 \\ -x_2 + x_3^2 \\ -x_3 \end{bmatrix}.$$

The flow of the vector field is given by

$$\phi_t(x) = \begin{bmatrix} e^t \alpha(x) - \frac{e^{-2t}}{3} \beta(x) + \frac{e^{-3t}}{2} (x_3^2 x_2 + x_3^4) - \frac{e^{-4t}}{5} x_3^4 \\ e^{-t} (x_2 + x_3^2) - e^{-2t} x_3^2 \\ e^{-t} x_3 \end{bmatrix},$$

where

$$\alpha(x) = x_1 + \frac{x_2^3}{3} + \frac{x_3^2 x_2}{6} + \frac{x_3^4}{30}, \quad \beta(x) = x_2^2 + 2x_3^2 x_2 + x_3^4.$$

We have

$$W_1(x) = x_2^2, \quad W_2(x) = \begin{bmatrix} x_3^2 \\ 0 \end{bmatrix},$$

and

$$W_1(\phi_s(x)) = (e^{-s} (x_2 + x_3^2) - e^{-2s} x_3^2)^2, \quad W_2(\phi_s(x)) = \begin{bmatrix} (x_3 e^{-s})^2 \\ 0 \end{bmatrix}.$$

Since both

$$\begin{aligned} \int_0^\infty e^{-s} W_1(\phi_s(x)) ds &= \int_0^\infty e^{-s} (e^{-s} (x_2 + x_3^2) - e^{-2s} x_3^2)^2 ds \\ &= \frac{1}{3} x_2^2 + \frac{1}{6} x_2 x_3^2 + \frac{1}{30} x_3^4 \end{aligned}$$

and

$$\int_0^\infty \begin{bmatrix} e^s & 0 \\ 0 & e^s \end{bmatrix} W_2(\phi_s(x)) ds = \begin{bmatrix} \int_0^\infty e^s (x_3 e^{-s})^2 ds \\ 0 \end{bmatrix} = \begin{bmatrix} x_3^2 \\ 0 \end{bmatrix}$$

converge,

$$H(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \int_0^\infty e^{-s} W_1(\phi_s(x)) ds \\ \int_0^\infty \begin{bmatrix} e^s & 0 \\ 0 & e^s \end{bmatrix} W_2(\phi_s(x)) ds \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + \frac{1}{3}x_2^2 + \frac{1}{6}x_2x_3^2 + \frac{1}{30}x_3^4 \\ x_2 + x_3^2 \\ x_3 \end{bmatrix},$$

and the transformation is given by

$$\begin{aligned} y_1 &= x_1 + \frac{1}{3}x_2^2 + \frac{1}{6}x_2x_3^2 + \frac{1}{30}x_3^4, \\ y_2 &= x_2 + x_3^2, \\ y_3 &= x_3. \end{aligned}$$

We can verify

$$\begin{aligned} \dot{y}_1 &= \dot{x}_1 + \frac{2}{3}x_2\dot{x}_2 + \frac{1}{6}(2x_2x_3\dot{x}_3 + \dot{x}_2x_3^2) + \frac{2}{15}x_3^3\dot{x}_3 \\ &= x_1 + x_2^2 + \frac{2}{3}x_2(-x_2 + x_3^2) + \frac{1}{6}(2x_2x_3(-x_3) \\ &\quad + (-x_2 + x_3^2)x_3^2) + \frac{2}{15}x_3^3(-x_3) \\ &= x_1 + \frac{1}{3}x_2^2 + \frac{1}{6}x_2x_3^2 + \frac{1}{30}x_3^4 = y_3 \\ \dot{y}_2 &= \dot{x}_2 + 2x_3\dot{x}_3 = -x_2 + x_3^2 + 2x_3(-x_3) \\ &= -x_2 - x_3^2 = -y_2 \\ \dot{y}_3 &= \dot{x}_3 = -x_3 = -y_3 \end{aligned}$$

and the system becomes linear.

Example 2.3. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= -x_2 + x_1x_3^2 \\ \dot{x}_3 &= -x_3. \end{aligned}$$

The flow of the system is given by

$$\phi_t(x) = \begin{bmatrix} e^t x_1 \\ e^{-t}(x_2 + x_1x_3^2t) \\ e^{-t}x_3 \end{bmatrix}.$$

We have

$$W_1(x) = 0, \quad W_2(x) = \begin{bmatrix} x_1x_3^2 \\ 0 \end{bmatrix},$$

and

$$W_2(\phi_s(x)) = \begin{bmatrix} (e^s x_1)(e^{-s} x_3)^2 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-s} x_1 x_3^2 \\ 0 \end{bmatrix}$$

Both integrals

$$\begin{aligned} \int_0^\infty \begin{bmatrix} e^s & 0 \\ 0 & e^s \end{bmatrix} W_2(\phi_s(x)) ds &= \begin{bmatrix} \int_0^\infty x_1 x_3^2 ds \\ 0 \end{bmatrix} \\ - \int_{-\infty}^0 \begin{bmatrix} e^s & 0 \\ 0 & e^s \end{bmatrix} W_2(\phi_s(x)) ds &= \begin{bmatrix} - \int_{-\infty}^0 x_1 x_3^2 ds \\ 0 \end{bmatrix} \end{aligned}$$

diverge, so we need to modify the W_2 .

Instead of using a C^∞ $\alpha(x)$ to construct \hat{W} , here we use a simpler but discontinuous modification of $W(x)$. For any $M > 0$, define

$$\hat{W}_2(x) = \begin{cases} W_2(x), & \text{if } \|W_2(x)\| \leq M \\ 0, & \text{otherwise.} \end{cases}$$

Then $\hat{W}_2(\phi_s(x)) = \begin{bmatrix} v_1(s, x) \\ 0 \end{bmatrix}$ where

$$\begin{aligned} v_1(s, x) &= \begin{cases} x_1 x_3^2 e^{-s}, & \text{if } |x_1 x_3^2| e^{-s} \leq M \\ 0, & \text{if } |x_1 x_3^2| e^{-s} > M, \end{cases} \\ &= \begin{cases} x_1 x_3^2 e^{-s}, & \text{if } s \geq -\ln \frac{M}{|x_1 x_3^2|} \\ 0, & \text{if } s < -\ln \frac{M}{|x_1 x_3^2|} \end{cases} \end{aligned}$$

and

$$\begin{aligned} H(x) &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ - \int_{-\infty}^0 \begin{bmatrix} e^s & 0 \\ 0 & e^s \end{bmatrix} \hat{W}_2(\phi_s(x)) ds \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} x_1 \\ x_2 - x_1 x_3^2 \int_{-\ln \frac{M}{|x_1 x_3^2|}}^0 ds \\ x_3 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} x_1 \\ x_2 - x_1 x_3^2 \ln \frac{M}{|x_1 x_3^2|} \\ x^3 \end{bmatrix}.$$

Note that $H(x)$ is continuous everywhere. $H(x)$ is certainly continuous at the place where $|x_1 x_3^2| \neq 0$, and since

$$\lim_{u \rightarrow 0} u \ln \frac{M}{|u|} = 0$$

$H(x)$ is also continuous at the place where $|x_1 x_3^2| = 0$, and $H(x)$ approaches to the identity map when $|x_1 x_3^2| \rightarrow 0$. On the other hand, the subspaces defined by $|x_1 x_3^2| = 0$, i.e. either $x_1 = 0$ or $x_3 = 0$, are all invariant subspaces of the system, and on these subspaces, the system is linear.

On the region defined by $0 < |x_1 x_3^2|$, we have

$$\frac{d}{dt} x_1 x_3^2 = \dot{x}_1 x_3^2 + 2x_1 x_3 \dot{x}_3 = -x_1 x_3^2$$

and

$$x_1 x_3^2 \frac{d}{dt} \ln \frac{M}{|x_1 x_3^2|} = \begin{cases} -\frac{x_1 x_3^2}{|x_1 x_3^2|} \frac{d}{dt} x_1 x_3^2 & x_1 x_3^2 > 0 \\ \frac{x_1 x_3^2}{|x_1 x_3^2|} \frac{d}{dt} x_1 x_3^2 & x_1 x_3^2 < 0 \end{cases} = -\frac{d}{dt} x_1 x_3^2 = x_1 x_3^2.$$

Therefore for $y(t) = H(x(t))$,

$$\begin{aligned} \dot{y} &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 - \frac{d}{dt}(x_1 x_3^2) \ln \frac{M}{|x_1 x_3^2|} - x_1 x_3^2 \frac{d}{dt} \ln \frac{M}{|x_1 x_3^2|} \\ \dot{x}^3 \end{bmatrix} \\ &= \begin{bmatrix} x_1 \\ -x_2 + x_1 x_3^2 + x_1 x_3^2 \ln \frac{M}{|x_1 x_3^2|} - x_1 x_3^2 \\ -x_3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} x_1 \\ -\left(x_2 - x_1 x_3^2 \ln \frac{M}{|x_1 x_3^2|}\right) \\ -x^3 \end{bmatrix} \\
&= \begin{bmatrix} y_1 \\ -y_2 \\ -y_3 \end{bmatrix}
\end{aligned}$$

3. Global Hartman-Grobman theorem

In this section, we consider the special case that all the eigenvalues of $Df(0)$ are located in the left-half complex plane. In such cases, we can give a homeomorphism on the region of attraction of the origin which changes the nonlinear system to a linear system inside the whole region of attraction.

By reverse the time, the same is also true if the eigenvalues of $Df(0)$ are located in the right-half complex plane and region of attraction is replaced by the region of repulsion.

Theorem 3.1 was basically appeared in our conference paper [8] in 1999 (Theorem 2.1 and Theorem 2.5) phrased differently. The result was also proved independently by Y. Lan and I. Mezič in [5]. Here we fill in all the details in the proofs of [8].

Theorem 3.1. *Under the condition of Theorem 2.1, either if all the eigenvalues of $Df(0)$ are located in the left-half complex plane or right-half complex plane, then there is transformation $y = H(x)$, $H(0) = 0$, H is homeomorphism from the region of attraction/repulsion of the origin of (1) to \mathbb{R}^n , and the system (1) is changed into the linear system*

$$\dot{y} = Ay, \quad A = Df(0)$$

on the region of attraction/repulsion of 0 under $y = H(x)$.

Proof. By reversing the time, we only need to consider the case when all the eigenvalues of $Df(0)$ are located in the left-half complex plane.

Let

$$W(x) = f(x) - Df(0)x$$

then $f(x) = Ax + W(x)$.

Consider the partial differential equations

$$(4) \quad Ah(x) - W(x) = Dh(x)(Ax + W(x))$$

$$h(0) = 0, \quad Dh(0) = 0.$$

For a solution $h(x)$ of (4), if we define $H(x) = x + h(x)$, then $H(x)$ is a local homeomorphism in a neighborhood of 0 by the inverse function theorem. Under the transformation $y(t) = H(x(t))$,

$$\begin{aligned} \dot{y} &= \dot{x} + Dh(x(t))\dot{x} = Ax + W(x) + Dh(x)(Ax + W(x)) \\ &= Ax + W(x) + Ah(x) - W(x) = Ay \end{aligned}$$

and the system becomes a linear system.

The characteristic equations of (4) is

$$(5) \quad \dot{x} = Ax + W(x)$$

$$(6) \quad \dot{z} = Az - W(x)$$

The invariant manifold $z = h(x)$ of the system whose tangent space at 0 is the x -space gives us the solution of (4). Note that (5) is the original system and (6) is a linear system driven by the solution of (5).

As in (3), if we define

$$\hat{h}(x) = - \int_{-\infty}^0 e^{-As} \hat{W}(\phi_s(x)) ds$$

where $\phi_s(x)$ is flow of (5) and $\hat{W}(x) = W(\alpha(x)x)$ with the $\alpha(x)$ in (2), then for any solution $x(t)$ of (5),

$$\begin{aligned} \hat{h}(x(t)) &= - \int_{-\infty}^0 e^{-As} \hat{W}(\phi_s(x(t))) ds \\ &= - \int_{-\infty}^0 e^{-As} \hat{W}(x(t+s)) ds \\ &= - \int_{-\infty}^t e^{-A(\tau-t)} \hat{W}(x(\tau)) d\tau \end{aligned}$$

and for $z(t) = \hat{h}(x(t))$

$$\dot{z} = -\hat{W}(x) - A \int_{-\infty}^t e^{-A(\tau-t)} \hat{W}(x(\tau)) d\tau = Az - \hat{W}(x).$$

So

$$z(t) = \hat{h}(x(t))$$

is a solution of (6) if $x(t) \in N_M$.

For any solution $x(t)$ of (5) with the initial value $x(0)$ in the region of attraction, there exists a $t_0 > 0$ such that $x(t) \in N_M$ for all $t \geq t_0$, which means that $z(t) = \hat{h}(x(t))$ is a solution of (6) for all $t \geq t_0$.

For a solution of $x(t)$ of (5),

$$(7) \quad z(t) = e^{(t-\rho)A} \hat{h}(x(\rho)) - \int_{\rho}^t e^{(t-\sigma)A} W(x(\sigma)) d\sigma$$

is a solution of (6) for all $t \neq \rho$, and if $\rho > t_0$, $z(t) = \hat{h}(x(t))$ in a neighborhood of ρ because they all satisfy (6) with the same initial condition at $t = \rho$.

(7) can be rewritten as

$$\begin{aligned} z(t) &= -e^{(t-\rho)A} \int_{-\infty}^0 e^{-As} \hat{W}(\phi_s(x(\rho))) ds - \int_{\rho}^t e^{(t-\sigma)A} W(x(\sigma)) d\sigma \\ &= - \int_{-\infty}^0 e^{-(\rho+s-t)A} \hat{W}(\phi_s(x(\rho))) ds - \int_{\rho}^t e^{(t-\sigma)A} W(x(\sigma)) d\sigma \\ &= - \int_{-\infty}^{\rho-t} e^{-\tau A} \hat{W}(\phi_{\tau+t-\rho}(x(\rho))) d\tau + \int_0^{\rho-t} e^{-\tau A} W(\phi_{\tau}(x(t))) d\tau \\ &= - \int_{-\infty}^{\rho-t} e^{-\tau A} \hat{W}(\phi_{\tau}(x(t))) d\tau + \int_0^{\rho-t} e^{-\tau A} W(\phi_{\tau}(x(t))) d\tau \\ &= - \int_{-\infty}^0 e^{-\tau A} \hat{W}(\phi_{\tau}(x(t))) d\tau \\ &\quad + \int_0^{\rho-t} e^{-\tau A} \left(W(\phi_{\tau}(x(t))) - \hat{W}(\phi_{\tau}(x(t))) \right) d\tau \end{aligned}$$

Let $\rho \rightarrow \infty$

$$(8) \quad z(t) = - \int_{-\infty}^0 e^{-\tau A} \hat{W}(\phi_{\tau}(x(t))) d\tau \\ + \int_0^{\infty} e^{-\tau A} \left(W(\phi_{\tau}(x(t))) - \hat{W}(\phi_{\tau}(x(t))) \right) d\tau$$

Note that both integrals in (8) converge if $x(t)$ is a solution of (5) with initial condition $x(0)$ in the region of attraction. The first integral converges because that $\hat{W}(\phi_{\tau}(x(t)))$ is bounded for all $\tau \in (-\infty, 0]$. The second integral converges because when τ is large enough, $\phi_{\tau}(x(t)) \in N_M$ and $W(\phi_{\tau}(x(t))) - \hat{W}(\phi_{\tau}(x(t))) = 0$.

If we define

$$(9) \quad h(x) = - \int_{-\infty}^0 e^{-\tau A} \hat{W}(\phi_\tau(x)) \, d\tau + \int_0^\infty e^{-\tau A} \left(W(\phi_\tau(x)) - \hat{W}(\phi_\tau(x)) \right) \, d\tau,$$

then $z(t)$ in (8) has the form that $z(t) = h(x(t))$, i.e. $h(x)$ defined in (9) is an invariant manifold of (5, 6). Also since $\phi_\tau(0) \equiv 0$ for any τ , $W(0) = \hat{W}(0) = 0$ and $DW(0) = D\hat{W}(0) = 0$,

$$h(0) = 0, \quad Dh(0) = 0.$$

So $y = x + h(x) := H(x)$ satisfy $\dot{y} = Ay$ for any x in the region of attraction of the origin of (1).

Let U be a neighborhood of the origin such that H is a homeomorphism on U , then for any $\hat{y} \in \mathbb{R}^n$, since the linear system $\dot{y} = Ay$ is asymptotically stable, there is $t_0 > 0$ such that $y(t_0) \in H(U)$ for the solution $y(t)$ with initial condition $y(0) = \hat{y}$. Let $x(t)$ be the solution of (1) such that $x(t_0) = H^{-1}(y(t_0))$. Then $\hat{x} = x(0)$ is the unique point such that $H(\hat{x}) = \hat{y}$ by the uniqueness of the initial value problem. So H is one-to-one and onto. The continuities of H and H^{-1} follow the continuity of the flows with respect to the initial points. Let $\hat{\phi}_t(y)$ be the flow of $(y) = Ay$. For any x in the region of attraction, we can choose t large enough such that $\phi_t(x) \in U$. Then

$$H(x) = \hat{\phi}_{-t}(H(\phi_t(x))).$$

So $H(x)$ is continuous on the region of attraction. Conversely for any y , we can choose t large enough such that $\hat{\phi}_t(y) \in H(U)$. Then

$$H^{-1}(y) = \phi_{-t}(H^{-1}(\hat{\phi}_t(y))).$$

So $H^{-1}(y)$ is continuous at any y . □

Remark 3.2. The expression (9) can be simplified under two cases. If

$$\int_{-\infty}^0 e^{-\tau A} W(\phi_\tau(x)) \, d\tau$$

converges, then we can take $M = \infty$, so $\hat{W} \equiv W$ and

$$(10) \quad h(x) = - \int_{-\infty}^0 e^{-\tau A} W(\phi_\tau(x)) \, d\tau.$$

On the other hand, if

$$\int_0^\infty e^{-\tau A} W(\phi_\tau(x)) \, d\tau$$

converges, then we can take $M = \epsilon = 0$, so $\hat{W} \equiv 0$ and

$$(11) \quad h(x) = \int_0^\infty e^{-\tau A} W(\phi_\tau(x)) \, d\tau.$$

Example 3.3. Consider the system

$$\dot{x} = -\sin x.$$

Clearly the region of attraction of 0 is $(-\pi, \pi)$. We have

$$(12) \quad dt = \frac{-1}{\sin x} dx.$$

The flow $\phi = \phi_t(x)$ of the system satisfies

$$(13) \quad \frac{\sin \phi_t(x)}{1 - \cos \phi_t(x)} = \frac{\sin(x)}{1 - \cos(x)} e^t = \cot\left(\frac{x}{2}\right) e^t.$$

For this system, $A = -1$ and $W(x) = x - \sin(x)$. Applying the change of variable

$$t = \ln\left(\tan\left(\frac{x}{2}\right) \frac{\sin \phi}{1 - \cos \phi}\right)$$

on (11), we have (see (12) and (13))

$$dt = \frac{-1}{\sin \phi} d\phi, \quad e^t = \tan\left(\frac{x}{2}\right) \frac{\sin \phi}{1 - \cos \phi}, \quad \phi_0(x) = x, \quad \phi_\infty(x) = 0$$

and (11) becomes

$$h(x) = \int_0^\infty e^\tau (\phi_\tau(x) - \sin \phi_\tau(x)) \, d\tau = \tan\left(\frac{x}{2}\right) \int_0^x \frac{\phi - \sin(\phi)}{1 - \cos(\phi)} \, d\phi.$$

We can see that

$$y = H(x) = x + \tan\left(\frac{x}{2}\right) \int_0^x \frac{\phi - \sin(\phi)}{1 - \cos(\phi)} \, d\phi$$

maps the region of attraction $(-\pi, \pi)$ of 0 onto \mathbb{R} .

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