From dimension-free manifolds to dimension-varying control systems

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Starting from the vector multipliers, the inner product, norm, distance, as well as addition of two vectors of different dimensions are proposed, which makes the spaces into a topological vector space, called the Euclidean space of different dimension (ESDD). An equivalence is obtained via distance. As a quotient space of ESDDs w.r.t. equivalence, the dimension-free Euclidean spaces (DFESs) and dimension-free manifolds (DFMs) are obtained, which have bundled vector spaces as its tangent space at each point. Using the natural projection from a ESDD to a DFES, a fiber bundle structure is obtained, which has ESDD as its total space and DFES as its base space. Classical objects in differential geometry, such as smooth functions, (co-)vector fields, tensor fields, etc., have been extended to the case of DFMs with the help of projections among different dimensional Euclidean spaces. Then the dimension-varying dynamic systems (DVDSs) and dimension-varying control systems (DVCSs) are presented, which have DFM as their state space. The realization, which is a lifting of DVDSs or DVCSs from DFMs into ESDDs, and the projection of DVDSs or DVCSs from ESDDs onto DFMs are investigated.

1. Introduction

Dimension-varying dynamic systems (DVDSs) and dimension-varying control systems (DVCSs) exist widely in nature and man-made equipments or environments. For instance, on the internet users are joining and withdrawing frequently. In a biological system, cells are producing and dying

from time to time. Some man-made mechanical systems are also of varying dimensions. For instance, the docking and undocking of spacecrafts [26, 17]; the connecting and disconnecting of vehicle clutch systems while speed changes [9]. The DVDS models are also used for specious population dynamics [25, 14].

Another interesting phenomenon which stimulates our interest is: a geometrical object, or a complex system, may be described by models of different dimensions. For instance, in power systems a single generator can be modeled as a 2, 3, or 5, 6, or even 7-dimensional dynamic system [21]. In contemporary physics, the sting theory assumes the dynamics of strings to be the model for universe of time-space. But this model may have dimension 4 (special relativity), 5 (Kaluza-Klein theory), 10 (type 1 string), 11 (M-theory) or even 26 (Bosonic model) [19]. One observes from this phenomenon that two models with different dimensions might be very similar or even equivalent. In other words, dimension-varying model may be proper to describe such dynamics.

So far, a classical way to deal with DVDSs and DVCSs is switching [26]. This approach ignores the dynamics of the system during the dimension-varying process. In practice, the transient period may be long enough so that the dynamics during this process is not ignorable. For instance, automobile clutch takes about 1 second to complete a connection or separation action; docking and undocking of spacecrafts take even longer. Not to mention that some processes might be continuously dimension-varying. In the latter cases, switching is almost meaningless.

To our best knowledge, there are few proper theories in existing mathematics to model DVDSs and DVCSs. In ordinary or partial differential equations or difference equations, only dynamic models of fixed dimensions can be treated. To provide a proper model for formulating DVDSs and DVCSs, a new framework should be created.

The purpose of this paper is twofold. One is to build the dimension-free Euclidean spaces (DFESs) and dimension-free manifolds (DFMs), which provides a mathematical framework for DVDSs or DVCSs. The DFM is a completely new "manifold", where each point has its own dimension. It is the base space of DFES while the total space of DFES is considered as the tangent space of the DFM. It makes the "state space" of the DVDSs or DVCSs a mathematically well-posed geometrical object; The other issue is to use the geometric structure of DFMs to model, analyze and/or design controls for DVDSs and DVCSs, either linear or nonlinear, are unambiguously defined.

This paper is a follow-up research of our previous works. In [6, 7] the dimension-free matrix theory and mix-dimensional vector spaces have been proposed and investigated. As an application of dimension-free matrix theory, the dimension-varying linear (control) systems have been investigated [9]. The basic concept used there was the equivalence of vectors of different dimensions [8]. This idea is also one of the key techniques in this paper.

To build up a theory of dimension-free manifolds, the key issue is to construct a connected topological space allowing the dimensions of the points in it to vary. We first construct an equivalence relation on the ESDD $\mathcal{V} := \bigcup_{n=1}^{\infty} \mathcal{V}$, (equivalently, $\mathbb{R}^{\infty} := \bigcup_{n=1}^{\infty} \mathbb{R}^n$) and take the quotient space Ω as the model space, which is a topological vector space. it is called "dimension-free" Euclidean space, which means each vector $\bar{x} \in \Omega$ has its dimension $1 \leq \dim(\bar{x}) < \infty$, but this dimension varies from point to point. We refer to [6, 7] for details. Though these results have been discussed before, they are re-organized systematically in Section 2 here. Then the projection of a vector to a certain Euclidian space is recalled, which was firstly proposed in [9] and was discussed in detail by [11, 12, 27, 28]. Then the cross-dimensional linear control systems are investigated as S-systems (semi-group systems).

For each point $\bar{x} \in \Omega$ a neighborhood coordinate bundle is proposed. Using neighborhood coordinate bundles, differentiable structure is obtained. Unlike the classical differential manifold, this differential structure poses on each point a bundled Euclidean spaces of different dimensions as its "tangent space". A DFM is a fiber bundle locally homeomorphic to a coordinate neighborhood bundle of a DFES.

Over a DFES (or DFM), the smooth (C^r) functions, vector fields, co-vector fields, distributions, co-distributions, tensor fields, etc. are proposed. The integral curves of vector fields, or integral manifolds of distributions, are also properly defined. In a word, a dimension-free differential geometric structure is proposed for DFMs. Dimension-free tensor fields are also introduced, and using degree-2 covariant tensor fields, dimension-free Riemannian manifolds and symplectic manifolds are also proposed.

With these geometric constructions, this paper attempts to explore the dynamics and control of DVDSs. The basic idea is as follows: Projection and lift connect the DVDSs on DFMs (as the base space) with the DVDSs on DFESs (as the total space): lifting the trajectory of a DVDS (or DVCS) to ESDD, a set of trajectories over ESDD are obtained. Using them a dimension-varying trajectory of the DVDS (or DVCS) can be constructed. Conversely, if a classical dynamic (control) system is defined on a Euclidean space, it can also be projected to DFES via natural projection. Therefore, the fiber bundle $(\mathbb{R}^{\infty}, \mathrm{Pr}, \Omega)$ provides the bearing state space for both DVDSs

and DVCSs. In addition to general DVDSs, particular attention has been paid to the dynamics of the transient process of classical dimension-varying systems, which have invariant dimensions except during the transient period.

The STP of matrices was proposed by the author and his colleagues [4, 5]. It was the fundamental method in previous works [6, 7, 9] on DVDSs and DVCSs. It is also a basic tool in this paper, where the default matrix product is assumed to be STP. We refer to [4, 5] for notions and basic results.

Before ending this section, lists of notations and abbreviations are provided as follows.

(1) List of Notations:

- $\mathbb{R}$: set of real numbers.
- $\mathcal{M}_{m \times n}$: set of $m \times n$ dimensional real matrices.
- $a \vee b$: the least common multiple of two positive integers a and b.
- $a \wedge b$: the greatest common divisor of two positive integers a and b.
- $\vec{\boxplus}$: (left) vector addition.
- $\vec{\vdash}$: (left) vector subtraction.
- $\mathbf{1}_n$: $\underbrace{[1, \cdots, 1]}_{n}^T$; $\mathbf{1}_{m \times n}$: $m \times n$ matrices with all entries being 1.
- δ_k^i: The i-th column of identity matrix I_k. δ_k^0 is for a zero vector of dimension k.
- $\leftrightarrow$: vector equivalence.
- $\Omega := \mathbb{R}^\infty / \leftrightarrow$.

(2) List of Abbreviations:

- ESDD: Euclidean space of different dimensions.
- DFES: dimension-free Euclidean space.
- DFM: dimension-free manifold.
- DFEB: dimension-free Euclidean bundle.
- DFRM: dimension-free Riemannian manifold.
- DVDS: dimension-varying dynamic system.
- DVCS: dimension-varying control system.

The rest of this paper is organized as follows. The ESDDs are investigated in Section 2. The inner product, norm, distance, topologies and equivalence relations on them are introduced. In Section 3 a vector space structure is given to ESDDs to form the DFESs using equivalence among vectors of different dimensions. Using the natural projection from ESDDs to DFES, a fiber bundle structure is obtained. Section 4 considers the projections among

Euclidian Spaces of different dimensions. First, the projection of a vector onto another vector, which may have different dimensions, is proposed. Using the coordinates from ESDD, smooth functions over DFESs are constructed. Then the least square approximation of a linear (control) system is considered. The DFMs are considered in Section 5. After DFESs are endowed with a differential structure, the (co)-vector fields, (co)-distributions, and the integral curses of vector fields over them are proposed and investigated. In Section 6, the tensor fields over DFMs are constructed first. Using proper symmetric and skew-symmetric covariant tensor fields, the dimension-free Riemannian manifold and dimension-free symplectic manifold are constructed respectively. As an application, Section 7 considers DVDSs and DVCSs. First, the projection of a nonlinear (control) system on an Euclidean space onto another Euclidean space of different dimension is proposed. Then the nonlinear (control) systems over DFESs are considered, which is then used to model dimension-varying nonlinear (control) systems over ESDDs. Finally, the control problems of dimension-varying linear and nonlinear systems are considered in principle. Section 8 contains some concluding remarks. First, the construction of DFES (DFM) is summarized step by step. Then the modeling and control design of DVDSs are also summarized. Finally, a conjecture is presented, which claims that DFESs (DFMs) might be used as the framework for string theory.

2. Euclidean space of different dimensions (ESDDs)

In this section we introduce the notion of ESDDs, which is constructed by choosing $\{\mathbf{1}_n\}_{n=1,2,\cdots}$ as vector multipliers [8].

2.1. Mix-dimensional sets and mix-dimensional vector spaces

Consider an n dimensional real vector space, denoted by $\mathcal{V}_n$. For simplicity, one can take $\mathcal{V}_n = \mathbb{R}^n$. To construct mix-dimensional state space, the set of mix-dimensional vectors, called ESDD, is defined as

$$\mathcal{V} := \bigcup_{n=1}^{\infty} \mathcal{V}_n.$$

We may view $\mathcal{V}$ as $\mathbb{R}^\infty := \bigcup_{n=1}^{\infty} \mathbb{R}^n$ since they are isomorphic.

First, we define "addition" and "scalar product" over $\mathcal{V}$ to turn it into a pseudo vector space.

Definition 1.   *(i) Let $x \in \mathcal{V}_m \subset \mathcal{V}$, $r \in \mathbb{R}$. Then the scalar product is defined as follows:*

(1) $$r \times x := rx \in \mathcal{V}_m.$$

(ii) Let $x \in \mathcal{V}_m$, $y \in \mathcal{V}_n$, and $t = m \vee n$ be the least common multiple of m and n. Then the addition of x and y is defined as follows:

(2) $$x \vec{\vDash} y := (x \otimes \mathbf{1}_{t/m}) + (y \otimes \mathbf{1}_{t/n}) \in \mathcal{V}_t.$$

Correspondingly, the subtraction of y from x is defined as $x \vec{\vDash} y := x \vec{\vDash} (-y)$.

Proposition 2. *Set $\mathcal{V}$ with scalar multiplication defined as in (1), addition as in (2) is a pseudo-vector space [1], where the set of zero elements is*

$$\mathbf{0} := \{ [\underbrace{0, 0, \cdots, 0}_{n}]^T \mid n = 1, 2, \cdots \}.$$

Remark 3. *For notational ease, when $x \in \mathcal{V}_n$ we assume $-x \in \mathcal{V}_n$ where $-x$ is the vector satisfying $x + (-x) \in \mathbf{0}$. Since such elements are not unique, this $-x$ is considered as a representative of the set of them.*

2.2. Norm and distance on ESDDs

Definition 4. *Let $x \in \mathcal{V}_m \subset \mathcal{V}$, $y \in \mathcal{V}_n \subset \mathcal{V}$, and $t = m \vee n$. Then the inner product of x and y is defined by*

(3) $$\langle x , y \rangle_{\mathcal{V}} := \frac{1}{t} \left\langle x \otimes \mathbf{1}_{t/m} , y \otimes \mathbf{1}_{t/n} \right\rangle,$$

where $\langle \cdot , \cdot \rangle$ is the conventional inner product on $\mathbb{R}^t$. That is, if $x, y \in \mathbb{R}^t$, then $\langle x, y \rangle = \sum_{i=1}^{t} x_i y_i$. The inner product defined by (3) is called the weighted inner product, because there is a weight coefficient $1/t$.

Using inner product, the norm of $x \in \mathcal{V}$ can be defined.

Definition 5. *The norm of $x \in \mathcal{V}$ is defined by*

(4) $$\|x\|_{\mathcal{V}} := \sqrt{\langle x , x \rangle_{\mathcal{V}}}.$$

One sees easily that $\|\cdot\|_\mathcal{V}$ defined by (4) satisfies linearity and triangle inequality, however, $\|x\| = 0 \Rightarrow x \in \mathbf{0}$. It is also called a pseudo-norm.

Finally, we define the distance on $\mathcal{V}$.

Definition 6. *Let* $x,\ y \in \mathcal{V}$*. The distance between* x *and* y *is defined by*

$$(5) \qquad\qquad d_\mathcal{V}(x,y) := \|x \vec{\vdash} y\|_\mathcal{V}.$$

It is easily verified that $d_\mathcal{V}$ satisfies symmetry and triangle inequality, however, $d_\mathcal{V}(x,y) = 0 \Rightarrow x \vec{\vdash} y \in \mathbf{0}$. Hence, this distance is called a pseudo-distance.

Remark 7. *The distance defined on a vector space is, in general, required to be invariant under displacement. That is,*

$$(6) \qquad\qquad d(x+z, y+z) = d(x,y), \quad x,y,z \in X.$$

It is easy to verify that the $d_\mathcal{V}$ *defined by (5) satisfies (6).*

2.3. Topology on ESDDs

This subsection considers the topology on $\mathcal{V}$. We refer to any standard textbook of topology for the basic topological concepts involved in this subsection, for instance, [10, 20]. In the following some topologies are considered.

- Natural Topology:

Naturally, the topology on each $\mathbb{R}^n$ is considered as conventional topology. To be precise, the open balls in $\mathbb{R}^n$ with center at $c = (c_1, c_2, \cdots, c_n)$, and radius $r > 0$, are defined by

$$B_r^n(c) := \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n \ \Big| \ \sqrt{\sum_{i=1}^n (x_i - c_i)^2} < r \right\}.$$

Taking

$$B^n := \{ B_r^n(c) \mid c \in \mathbb{R}^n, r > 0 \}$$

as topological basis, the topology on $\mathbb{R}^n$ generated by B^n is the conventional topology on $\mathbb{R}^n$.

Then each $\mathbb{R}^n$, $n = 1, 2 \cdots$, are considered as a set of clopen subsets in $\mathcal{V}$. Such a topology is called the natural topology on $\mathcal{V}$, denoted by $\mathbf{N}$.

The following properties are obvious.

Proposition 8. *(i) Assume $\emptyset \neq O_n \in \mathbb{R}^n$ is an open set, then it is also open in $(\mathcal{V}, \mathbf{N})$.*

(ii) $(\mathcal{V}, \mathbf{N})$ is a second countable Hausdorff space.

- Distance Topology:

Define open balls in $\mathcal{V} = \mathbb{R}^\infty$ by

$$B_r(c) := \{x \in \mathbb{R}^\infty \mid d_\mathcal{V}(x, c) < r\}, \quad c \in \mathbb{R}^\infty.$$

Using $B := \{B_r(c) \mid c \in \mathbb{R}^\infty, r > 0\}$ as a topological basis, the topology generated by B is called the distance topology on $\mathbb{R}^\infty$ deduced from $d_\mathcal{V}$, denoted by $\mathbf{D}$.

Remark 9. *(i) Assume $\emptyset \neq O_n \in \mathbb{R}^n$ is an open set, it is not open under distance-deduced topology, i.e., not open in $(\mathcal{V}, \mathbf{D})$. This is because $\forall x \in O_n$ there exists a point $y = x \otimes \mathbf{1}_s \notin O_n$. But $d(x, y) = 0$, which means x is not an interior point of O_n. Hence, O_n is not open in $(\mathcal{V}, \mathbf{D})$.*

(ii) $(\mathcal{V}, \mathbf{D})$ is not a Hausdorff space. To see this, consider x and $x \otimes \mathbf{1}_s$, $s > 1$, which are two different points. But they are not separable in $(\mathcal{V}, \mathbf{D})$. It is clear that $(\mathcal{V}, \mathbf{D})$ is not even T_0.

(iii) It is easy to see that if O is open in $(\mathcal{V}, \mathbf{D})$, then O is also open in $(\mathcal{V}, \mathbf{N})$. Hence $\mathbf{D} \subset \mathbf{N}$, that is, the distance-deduced topology $\mathbf{D}$ is rougher than the natural topology $\mathbf{N}$.

- Product Topology:

One way to understand $\mathcal{V} = \mathbb{R}^\infty$ is to consider $\mathbb{R}^\infty = \prod_{n=1}^\infty \mathbb{R}^n$, then the product topology is generated by the topological basis

$$B = \left\{ \prod_{n=1}^\infty O_n \;\middle|\; O_n \subset \mathbb{R}^n \text{ is open, and } O_n = \mathbb{R}^n \text{ except for finite } n \right\}.$$

The product topology is denoted by $\mathbf{P}$. It is easy to see that $\mathbf{P} = \mathbf{N}$.

2.4. Equivalent vectors

Definition 10. *(i) Let x, $y \in \mathcal{V}$. x and y are said to be equivalent, denoted by $x \leftrightarrow y$, if there exist two one-vectors $\mathbf{1}_\alpha$ and $\mathbf{1}_\beta$, such that*

$$(7) \qquad\qquad x \otimes \mathbf{1}_\alpha = y \otimes \mathbf{1}_\beta.$$

(ii) The equivalence class of x is denoted by $\bar{x} := \{y \mid y \leftrightarrow x\}$.

Remark 11. *Obviously $\leftrightarrow$ is an equivalence relation. Assume $x,\ y \in \mathcal{V}$. Then $x \leftrightarrow y$, if and only if $x \vec{\vdash} y \in \mathbf{0}$.*

For the equivalence we have the following properties.

Theorem 12.   *(i) If $x \leftrightarrow y$, then there exists a $\gamma \in \mathcal{V}$ such that*

$$(8) \qquad\qquad x = \gamma \otimes \mathbf{1}_\beta, \quad y = \gamma \otimes \mathbf{1}_\alpha.$$

(ii) In each equivalence class $\bar{x}$ there exists unique smallest element $x_1 \in \bar{x}$, such that $\bar{x} = \{x_1 \otimes \mathbf{1}_k \mid k = 1, 2, \cdots\}$.

The proofs follow similarly as in [8].
A partial order can be defined on $\mathcal{V}$.

Definition 13. *A partial order, denoted by $\prec$, is defined as follows: Let $x, y \in \mathcal{V}$. If there exists a one-vector $\mathbf{1}_s$ such that $x \otimes \mathbf{1}_s = y$, then $x \prec y$. For any equivalence class $\bar{x}$, $x_1 \in \bar{x}$ is called the smallest element of $\bar{x}$, if $\forall y \in \bar{x}$, $y \prec x_1$ implies $y = x_1$.*

Remark 14.   *(i) If $x = y \otimes \mathbf{1}_s$, then y is called a divisor vector of x, and x is called a multiplier vector of y. This relation determines the order $y \prec x$.*

(ii) If (8) holds, and α, β are co-prime, then the γ in Eq. (8) is called the maximum common divisor vector of x and y, denoted by $\gamma = \gcd(x, y)$. It is easy to prove that if z is also a common divisor vector of x and y, then $z \prec \gamma$. Moreover, the maximum common divisor vector is unique.

(iii) If (7) holds and α, β are co-prime, then $\xi := x \otimes \mathbf{1}_\alpha = y \otimes \mathbf{1}_\beta$ is called the least common multiple vector of x and y, denoted by $\xi = \mathrm{lcm}(x, y)$. It is also easy to prove that if z is also a common multiple vector of x and y, then $\xi \prec z$. Moreover, the least common multiple vector is also unique.

Proposition 15.   *(i) Assume $x \in \mathcal{V}$, then $(\bar{x}, \prec)$ is a lattice [8].*

(ii) Assume $x, y \in \mathcal{V}$, then $(\bar{x}, \prec) \cong (\bar{y},\ \prec)$, where $\cong$ stands for lattice isomorphism. That is, any two equivalence classes as lattices are isomorphic.

Proof. It is straightforward verifiable that $\forall u,\ v \in \bar{x}$, $\sup(u, v) = \mathrm{lcm}(u, v)$; $\inf(u, v) = \gcd(u, v)$. Then the first part is obvious. Assume $\bar{x} = \{x_1, x_2, \cdots\}$ and $\bar{y} = \{y_1, y_2, \cdots\}$, where $x_i = x_1 \otimes \mathbf{1}_i$, $i = 1, 2, \cdots$, etc. Define $\pi : \bar{x} \to \bar{y}$, $\pi(x_i) = y_i$, $i = 1, 2, \cdots$, Then one sees easily that π is a lattice isomorphism.   □

The above arguments can be considered as special cases of that in Section 7 of [8] by choosing the vector multiplier as $\{\mathbf{1}_n\}$.

3. Constructing DFESs from ESDDs

3.1. Quotient spaces as vector spaces

Definition 16. *The quotient space of $\mathcal{V} = \mathbb{R}^\infty$ under equivalence relation $\leftrightarrow$ defined on it by (7), denoted by Ω, is called the DFES. That is,*

$$(9) \qquad\qquad \Omega := \mathcal{V}/ \leftrightarrow .$$

  (i) Let $\bar{x} \in \Omega$. The scalar product on Ω is defined by

$$(10) \qquad\qquad a\bar{x} := \overline{ax}, \quad a \in \mathbb{R}.$$

  (ii) Let $\bar{x}, \bar{y} \in \Omega$. Then the addition of $\bar{x}$ and $\bar{y}$ is defined by

$$(11) \qquad\qquad \bar{x} \vec{\Vdash} \bar{y} := \overline{x \vec{\Vdash} y}.$$

   Correspondingly, the subtraction is defined by $\bar{x} \vec{\dashv} \bar{y} := \bar{x} \vec{\Vdash} (-\bar{y})$, where $-\bar{y} := \overline{-y}$.

It is easy to verify that the scalar product (10) and the addition (11) are well defined, that is, if $x \leftrightarrow x'$ and $y \leftrightarrow y'$, then $\overline{ax} \leftrightarrow \overline{ax'}$, $\forall a \in \mathbb{R}$, and $x \vec{\Vdash} y \leftrightarrow x' \vec{\Vdash} y'$. One may refer to [8] for proofs.

Theorem 17. *Using the addition defined as in (11) and the scalar product as in (10), Ω is a vector space.*

Consider the subspaces of ESDD and the corresponding subspaces of DFES.

Definition 18. *(i) Let $p \in \mathbb{Z}_+$ be a positive integer. Define the p-upper truncated ESDD as*

$$\mathbb{R}^{[p,\cdot]} := \bigcup_{\{s \,\mid\, p \mid s\}} \mathbb{R}^s.$$

  (ii) Define p-upper truncated DFES as

$$\Omega^p := \mathbb{R}^{[p,\cdot]}/ \leftrightarrow \ = \left\{ \bar{x} \mid x_1 \in \mathbb{R}^{pr}, \ r \geq 1 \right\}.$$

(iii) *Define p-lower truncated ESDD as*

$$(12) \qquad\qquad \mathbb{R}^{[\cdot, p]} := \bigcup_{\{s \,\mid\, s|p\}} \mathbb{R}^s.$$

(iv) *Define p-lower truncated DFES as*

$$\Omega_p := \mathbb{R}^{[\cdot, p]}/ \leftrightarrow \ = \left\{ \bar{x} \,\middle|\, x_1 \in \mathbb{R}^s, \ s|p \right\}.$$

The next proposition is an immediate consequence of the definition.

Proposition 19. *(i) Ω^p, and Ω_p, $p = 1, 2, \cdots$ are subspaces of Ω;*
(ii) If $i|j$, Then, Ω^j is a subspace of Ω^i, Ω_i is a subspace of Ω_j.

Example 20. *The lattice structure on $\mathbb{R}^\infty$ can be transferred to Ω:*

(i) *Define $\Omega_{(n)} := \mathbb{R}^n/ \leftrightarrow, \quad n = 1, 2, \cdots$. Then $\Omega = \bigcup_{n=1}^\infty \Omega_{(n)}$. Define $\Omega_{(m)} \prec \Omega_{(n)} \Leftrightarrow \mathbb{R}^m \prec \mathbb{R}^n$, then it is obvious that $(\Omega, \prec)$ is a lattice with $\sup(\Omega_{(p)}, \Omega_{(q)}) = \Omega_{(p \vee q)}$, $\inf(\Omega_{(p)}, \Omega_{(q)}) = \Omega_{(p \wedge q)}$.*
(ii) *Ω_p is an ideal of Ω.*
(iii) *Ω^p is a filter of Ω.*

In fact, Ω has the same lattice structure as its filters.

Proposition 21. *Let $p > 1$. The filter Ω^p is lattice isomorphic to Ω.*

Proof. Define a mapping $\varphi : \Omega^p \to \Omega$, $\varphi(\Omega_{(np)}) := \Omega_{(p)}$. Then it is easy to verify that φ is a lattice isomorphism. $\qquad\square$

3.2. Topology on DFESs

First, we extend the inner product over ESDD $\mathbb{R}^\infty$ to DFES Ω.

Definition 22. *Let $\bar{x}$, $\bar{y} \in \Omega$. Define their inner product as*

$$(13) \qquad\qquad \langle \bar{x} \,,\, \bar{y} \rangle_\mathcal{V} := \langle x \,,\, y \rangle_\mathcal{V}, \quad x \in \bar{x}, \ \ y \in \bar{y},$$

where $\langle \cdot, \cdot \rangle_\mathcal{V}$ is defined as in (3).

The following proposition shows Definition 22 is well defined.

Proposition 23. *(13) is properly defined. That is, it is independent of the choice of representatives x and y.*

Proof. Assume $x_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow y_2$. According to Theorem 12, there exist $x_0 \in \mathbb{R}^s$ and $y_0 \in \mathbb{R}^t$, such that

$$
\begin{aligned}
x_1 &= x_0 \otimes \mathbf{1}_\alpha; & x_2 &= x_0 \otimes \mathbf{1}_\beta, \\
y_1 &= y_0 \otimes \mathbf{1}_p; & y_2 &= y_0 \otimes \mathbf{1}_q.
\end{aligned}
$$

First, we prove two facts:

- Fact 1: Let $s \wedge t = \xi$, and $s = a\xi$, $t = b\xi$, where $\alpha \wedge b = 1$. If f, g satisfy $sf = tg$, then, $af\xi = bg\xi$, i.e., $af = bg$. Since $a \wedge b = 1$, there exists a c such that $f = cb, \quad g = ca$.
- Fact 2: $\langle x, y \rangle_\mathcal{V} = \langle x \otimes \mathbf{1}_s, y \otimes \mathbf{1}_s \rangle_\mathcal{V}$.

These facts can be verified by definition directly.

Next, we consider

$$
\begin{aligned}
\langle x_1, y_1 \rangle_\mathcal{V} &= \langle x_0 \otimes \mathbf{1}_\alpha, y_0 \otimes \mathbf{1}_p \rangle_\mathcal{V} \\
&= \left\langle x_0 \otimes \mathbf{1}_\alpha \otimes \mathbf{1}_{\frac{s\alpha \vee tp}{s\alpha}}, y_0 \otimes \mathbf{1}_p \otimes \mathbf{1}_{\frac{s\alpha \vee tp}{tp}} \right\rangle_\mathcal{V} \\
&= \left\langle x_0 \otimes \mathbf{1}_{\frac{s\alpha \vee tp}{s}}, y_0 \otimes \mathbf{1}_{\frac{s\alpha \vee tp}{t}} \right\rangle_\mathcal{V}.
\end{aligned}
$$

Hence we have $s\frac{s\alpha \vee tp}{s} = t\frac{s\alpha \vee tp}{t}$. Using Fact 1, one sees that

$$
\frac{s\alpha \vee tp}{s} = cb; \quad \frac{s\alpha \vee tp}{t} = ca.
$$

Using Fact 2 yields

$$
\langle x_1, y_1 \rangle_\mathcal{V} = \langle x_0 \otimes \mathbf{1}_{cb}, y_0 \otimes \mathbf{1}_{ca} \rangle_\mathcal{V} = \langle x_0 \otimes \mathbf{1}_b, y_0 \otimes \mathbf{1}_a \rangle_\mathcal{V}.
$$

Similarly, we have $\langle x_2, y_2 \rangle_\mathcal{V} = \langle x_0 \otimes \mathbf{1}_b, y_0 \otimes \mathbf{1}_a \rangle_\mathcal{V}$. The conclusion follows. $\qquad\square$

Since Ω is a vector space, (13) defines an inner product on Ω. This inner product has the following properties.

Proposition 24. Ω *with the inner product defined by* (13) *is an inner product space. But it is not a Hilbert space.*

Proof. Obviously Ω is an inner product space. To see that it is not a Hilbert space, we construct a sequence as follows:

$$
x_1 = a \in \mathbb{R}; \ x_{i+1} = x_i \otimes \mathbf{1}_2 + \frac{1}{2^{i+1}} \left(\delta_{2^{i+1}}^1 - \delta_{2^{i+1}}^2 \right), \quad i = 1, 2, \cdots.
$$

It is obvious that this sequence is a Cauchy sequence. But it does not converge to any point $x \in \mathcal{V}$. Let $\bar{x}_i := \overline{x_i}$. According to Proposition 23, it is easy to see that $\{\bar{x}_i\}$ is also a Cauchy sequence in Ω, but it can not converge to any point in Ω. □

$\forall x \in \mathbb{R}^\infty$, $\varphi_x : y \mapsto \langle x , y \rangle_\mathcal{V}$ gives a mapping $\varphi_x : \mathbb{R}^\infty \to \mathbb{R}$. Similarly, a point $\bar{x} \in \Omega$ can be used to construct a mapping $\varphi_{\bar{x}} : \Omega \to \mathbb{R}$, $\bar{y} \mapsto \langle \bar{x} , \bar{y} \rangle_\mathcal{V}$. Conversely, not every linear mapping $\varphi : \Sigma \to \mathbb{R}$ can be expressed as a mapping deduced by an element as $\varphi_{\bar{x}}$. This is because Ω is an infinite dimensional vector space, while each element $\bar{x} \in \Omega$ is a finite dimensional element.

Using the inner product defined by (13), the norm and distance on Ω are also well defined.

Definition 25. *(i) Let $\bar{x} \in \Omega$. The norm of $\bar{x}$ is defined by*

$$\|\bar{x}\|_\mathcal{V} := \|x\|_\mathcal{V}. \tag{14}$$

(ii) Let $\bar{x}$, $\bar{y} \in \Omega$. The distance between $\bar{x}$ and $\bar{y}$ is defined as

$$d_\mathcal{V}(\bar{x}, \bar{y}) := d_\mathcal{V}(x, y). \tag{15}$$

According to Proposition 23, (14)(15) are both well defined.

Finally, as a topological space, the topology on Ω is deduced by the distance. This topology is equivalent to the quotient topology of $(\mathbb{R}^\infty, \mathbf{N})$ over equivalence. That is, the glued topology inherited from $(\mathbb{R}^\infty, \mathbf{N})$.

As a topological space, Ω has the following properties.

Proposition 26. Ω *is a second countable Hausdorff space.*

Proof. Since $\mathbb{R}^n$ is second countable, denote by $\{O_i^n \mid i = 1, 2, \cdots\}$ its countable topological bases. Then $\bigcup_{n=1}^\infty \bigcup_{i=1}^\infty O_i^n$ is a topological basis of $\mathbb{R}^\infty$, which is also countable. Hence, as its quotient space, $\Omega = \mathcal{V}/\leftrightarrow$ is also second countable.

Since Ω is a metric space, then $\bar{x} \neq \bar{y}$, if and only if, $d_\mathcal{V}(\bar{x}, \bar{y}) > 0$. It is obvious that this space is a Hausdorff space. (In fact, it is easy to see that this space is T_4.) □

Definition 27. *Let $\bar{x} \in \Omega$. The dimension of $\bar{x}$, denoted by $\dim(\bar{x})$, is the dimension of the smallest element in $\bar{x}$. That is, $\dim(\bar{x}) = \dim(x_1) = \min_{x \in \bar{x}} \dim(x)$.*

Remark 28.   *(i) Note that $\bar{x} = \{x_1 \otimes \mathbf{1}_n \mid n = 1, 2, \cdots\}$, it is clear that $\bar{x}$ can be considered as x_1 and the images of merging $x_1 \in \mathbb{R}^s$ into $\mathbb{R}^{ns}$, $n = 2, 3, \cdots$. Hence x_1 is the essential element in $\bar{x}$, which determines $\bar{x}$ completely. This fact shows that the Definition 27 is reasonable.*

*(ii) It is surprising that Ω is a topological vector space with each point $\bar{x} \in \Omega$ having its own dimension $1 \leq \dim(\bar{x}) < \infty$. Hence, the **DFES (dimension-free Euclidean space) is a totally new mathematical object**.*

3.3. Fiber bundle structure on ESDDs and DFESs

First, we recall the definition of a fibre bundle.

Definition 29. *[15] Let T and B be two topological spaces, $\mathrm{Pr} : T \to B$ is a continuous surjective mapping. Then (T, Pr, B) is called a fiber bundle, where T is the total space, B is the base space. $\forall b \in B$, $\mathrm{Pr}^{-1}(b)$ is called the fiber at b.*

The following result comes from the definition immediately.

Proposition 30. *Let $T = (\mathcal{V}, \mathbf{N})$ be the total space, $B = (\Omega, \mathbf{D})$ be the base space, and $\mathrm{Pr} : T \to B$ be the natural projection, i.e., $x \mapsto \bar{x}$. Then $(\mathcal{V}, \mathbf{N}) \xrightarrow{\mathrm{Pr}} (\Omega, \mathbf{D})$ is a fiber bundle, which is called the dimension-free Euclidean bundle (DFEB).*

The DFEB is said to be a discrete bundle, because the bundle at each point $\bar{x}$ is a discrete countable (topological) subspace of ESDD $\mathbb{R}^\infty$.

Definition 31.   *(i) Two fiber bundles $(T_i, \mathrm{Pr}_i, B_i)$, $i = 1, 2$ are called homomorphic, if there exist two continuous mappings $\pi : T_1 \to T_2$ and $\varphi : B_1 \to B_2$, such that the diagram (16) is commutative. In addition, if both π and φ are bijective, and $\pi^{-1} : T_2 \to T_1$ and $\varphi^{-1} : B_2 \to B_1$ are also making (16) commutative, (T_i, Pr_i, B_i), $i = 1, 2$ are said to be isomorphic.*

$$(16) \qquad\qquad \begin{array}{ccc} T_1 & \xrightarrow{\ \pi\ } & T_2 \\ {\scriptstyle \mathrm{Pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{Pr}_2} \\ B_1 & \xrightarrow{\ \varphi\ } & B_2 \end{array}$$

(ii) Two fiber bundles on B, denoted by (T_i, Pr_i, B), $i = 1, 2$, are called homomorphic, if there exists a continuous mapping $\pi : T_1 \to T_2$, such

that the diagram (17) is commutative. In addition, if π is bijective, and $\pi^{-1} : T_2 \to T_1$ making (17) commutative, (T_i, Pr_i, B), $i = 1, 2$ are said to be isomorphic.

(17)

$$
\begin{array}{ccc}
T_1 & \xrightarrow{\ \pi\ } & T_2 \\
\ \ \downarrow{\scriptstyle \mathrm{Pr}_1} & \swarrow{\scriptstyle \mathrm{Pr}_2} & \\
B_1 & &
\end{array}
$$

Example 32. *Consider $\left(\mathbb{R}^{[p,\cdot]}, \mathrm{Pr}, \Omega\right)$ and $(\mathbb{R}^\infty, \mathrm{Pr}, \Omega)$. Define $\pi : \mathbb{R}^{[p,\cdot]} \hookrightarrow \mathbb{R}^\infty$ as the including mapping. Then it is obvious that π is a fiber bundle homomorphism, since the following diagram commutes.*

$$
\begin{array}{ccc}
\mathbb{R}^{[p,\cdot]} & \xrightarrow{\ \pi\ } & \mathbb{R}^\infty \\
\ \ \downarrow{\scriptstyle \mathrm{Pr}} & \swarrow{\scriptstyle \mathrm{Pr}} & \\
\Omega & &
\end{array}
$$

3.4. Coordinate neighbourhoods

To establish a differential structure on DFES, we need a "local coordinate neighborhood" for each point $\bar{x} \in \Omega$. Since Ω is a dimension-free space, the coordinate neighborhoods are not classical ones in standard differential manifold. In fact, they are sub-bundles of DFEB.

Definition 33. *Let $\bar{x} \in \Omega$, and $\dim(\bar{x}) = p$. Assume $O_{\bar{x}}$ is an open neighborhood of $\bar{x} \in \Omega$. That is, $\bar{x} \in O_{\bar{x}}$, and $O_{\bar{x}} \subset \Omega$ is open. Then*

$$
\mathcal{V}_{O_{\bar{x}}} := \mathrm{Pr}^{-1}\left(O_{\bar{x}}\right) \bigcap \mathbb{R}^{[p,\cdot]}
$$

with $\mathbb{R}^{[p,\cdot]}$ defined as in (12), is called the set of coordinate charts of $\bar{x}$, $(\mathcal{V}_{O_{\bar{x}}}, \mathrm{Pr}, O_{\bar{x}})$ is called the bundle of coordinate neighborhood of $\bar{x}$.

$$
\mathcal{V}^r_{O_{\bar{x}}} := \mathrm{Pr}^{-1}\left(O_{\bar{x}}\right) \bigcap \mathbb{R}^{rp}, \quad r = 1, 2, \cdots
$$

is called a leaf of the bundle of coordinate neighborhood bundle of $\bar{x}$.

An example is given in the following to depict the bundle of coordinate neighborhood.

Example 34. *Assume $x = (\alpha, \alpha, \beta, \beta)^T \in \mathbb{R}^4$, then $\bar{x} = \{x_1, x_2, \cdots\}$, where, $x_1 = (\alpha, \beta) \in \mathbb{R}^2$. Hence $\dim(\bar{x}) = 2$. Consider $O_{\bar{x}} = B_r(\bar{x}) \subset \Omega$, which is an open ball neighborhood of $\bar{x}$. Then the set of coordinate charts, deduced by*
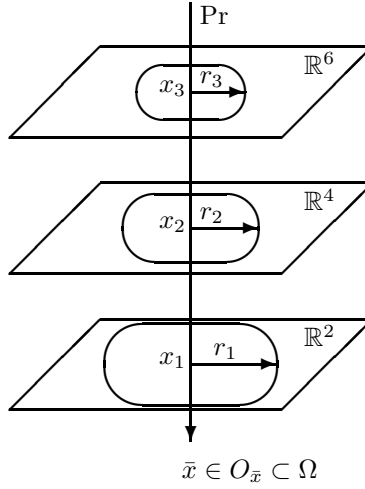
Figure 1: Bundle of coordinate neighborhood.

$O_{\bar{x}}$, is $\mathcal{V}_O = \{B_{r_1}(x_1), B_{r_2}(x_2), \cdots\}$, where $r_k = 1/\sqrt{2k}$, $x_k = (\alpha, \beta)^T \otimes \mathbf{1}_k$, $k = 1, 2, \cdots$. The bundle of coordinate neighborhood of $\bar{x}$ is $(\mathcal{V}_{O_{\bar{x}}}, \mathrm{Pr}, O_{\bar{x}})$.
    Fig. 1 demonstrates the bundle of coordinate neighborhood of $\bar{x}$.

    Note that the set of coordinate charts $\mathcal{V}_O$ does not include all the inverse image of O, i.e., $\mathcal{V}_O \subsetneq \mathrm{Pr}^{-1}(O)$. But it can provide coordinates for all points within O. The following proposition shows this.

Proposition 35. *Assume $\bar{y} \in O$, then $\mathrm{Pr}^{-1}(\bar{y}) \bigcap \mathcal{V}_O \neq \emptyset$.*

Proof. Assume $\bar{y} \in O$, $\dim(\bar{x}) = p$, $\dim(\bar{y}) = q$, $r = p \vee q$, then $y_{r/q} \in \mathrm{Pr}^{-1}(O) \bigcap \mathbb{R}^r \subset \mathcal{V}_O$. $\qquad\square$

Remark 36. *Assume $\bar{x} \in \Omega$ with $\dim(\bar{x}) = p$. Then $\mathcal{V}_{\Omega_{\bar{x}}} := \mathrm{Pr}^{-1}(\Omega) \bigcap \mathbb{R}^{[p,\cdot]}$ $= \mathbb{R}^{[p,\cdot]}$ is a coordinate neighborhood of $\bar{x}$, which is the largest coordinate neighborhood of $\bar{x}$. When the DFES is considered, we can simply use this coordinate neighborhood, then the corresponding coordinates are called the global coordinates. The general definition is mainly for DFM.*

4. Projections on ESDDs and continuous functions on DFESs

In this section we introduce the cross-dimensional projections on ESDD, which is crucial for prolonging functions on a finite-dimensional Euclidean space to construct continuous functions on DFES.

4.1. Cross-dimensional projection of a vector

Definition 37. *Assume $\xi \in \mathcal{V}_n$. A cross-dimensional projection of ξ to $\mathcal{V}_m$, denoted by $\pi_m^n(\xi)$, is defined as follows:*

$$(18) \qquad \pi_m^n(\xi) := \underset{x \in \mathcal{V}_m}{\mathrm{argmin}} \|\xi - x\|_{\mathcal{V}}.$$

Assume $t = \mathrm{lcm}(n, m) = t$ and denote $\alpha := t/n$, $\beta := t/m$. Then the distance between ξ and $x \in \mathcal{V}$ is $\Delta := \|\xi - x\|_{\mathcal{V}}^2 = \frac{1}{t}\|\xi \otimes \mathbf{1}_\alpha - x \otimes \mathbf{1}_\beta\|^2$.

Denote $\xi \otimes \mathbf{1}_\alpha := (\eta_1, \eta_2, \cdots, \eta_t)^T$, where $\eta_j = \xi_i$, $(i-1)\alpha + 1 \le j \le i\alpha$; $i = 1, \cdots, n$. Then, $\Delta = \frac{1}{t} \sum_{i=1}^{m} \sum_{j=1}^{\beta} \left(\eta_{(i-1)\beta+j} - x_i \right)^2$. Setting $\frac{\partial \Delta}{\partial x_i} = 0$, $i = 1, \cdots, m$ yields

$$(19) \qquad x_i = \frac{1}{m} \left(\sum_{j=1}^{\beta} \eta_{(i-1)\beta+j} \right), \quad i = 1, \cdots, m.$$

That is, $\pi_m^n(\xi) = x$. Moreover, it is easy to verify the following orthogonality, i.e. $\left\langle \xi \vdash\!\!\rightarrow x, x \right\rangle_{\mathcal{V}} = 0$. The above argument leads to the following conclusion:

Proposition 38. *Let $\xi \in \mathcal{V}_n$. Then the projection of ξ on $\mathcal{V}_m$, say, x, can be calculated by (19). Moreover, $\xi \vdash\!\!\rightarrow x$ and x are orthogonal.*

Example 39. *Assume $\xi = [1, 0, -1, 0, 1, 2, -2]^T \in \mathbb{R}^7$. Consider its projection on $\mathbb{R}^3$, denoted by $\pi_3^7(\xi) := x$. Then $\eta = \xi \otimes \mathbf{1}_3$. Denote by $x = [x_1, x_2, x_3]^T$, then*

$$x_1 = \frac{1}{7} \sum_{j=1}^{7} \eta_j = 0.2857, \quad x_2 = \frac{1}{7} \sum_{j=8}^{14} \eta_j = 0, \quad x_3 = \frac{1}{7} \sum_{j=15}^{21} \eta_j = 0.1429.$$

Moreover,

$$\begin{aligned}
\xi \vdash\!\!\rightarrow x \quad = \quad & [0.7143, 0.7143, 0.7143, -0.2857, -0.2857, \\
& -0.2857, -1.2857, -1, -1, 0, 0, 0, 1, 1, 0.8571, 1.8571, \\
& 1.8571, 1.8571, -2.1429, -2.1429, -2.1429].
\end{aligned}$$

Since the projection of a vector to a space of different dimension π_m^n is a linear mapping, it can be expressed by a matrix. Assume there exists a matrix Π_m^n, such that the projection of $\xi \in \mathbb{R}^n$ to $\mathbb{R}^m$ can be expressed as

$$(20) \qquad \pi_m^n(\xi) = \Pi_m^n \xi, \quad \xi \in \mathcal{V}_n.$$

We try to figure out this matrix.

Let $\mathrm{lcm}(n, m) = t$, $\alpha := t/n$, and $\beta := t/m$. Then

$$\eta = \xi \otimes \mathbf{1}_\alpha = (I_n \otimes \mathbf{1}_\alpha)\,\xi, \quad x = \frac{1}{\beta}\left(I_m \otimes \mathbf{1}_\beta^T\right)\eta = \frac{1}{\beta}\left(I_m \otimes \mathbf{1}_\beta^T\right)(I_n \otimes \mathbf{1}_\alpha)\,\xi.$$

Hence we have

$$(21) \qquad \Pi_m^n = \frac{1}{\beta}\left(I_m \otimes \mathbf{1}_\beta^T\right)(I_n \otimes \mathbf{1}_\alpha).$$

Using this structure, we have the following result.

Lemma 40. *(i) Let $n \geq m$. Then Π_m^n is of full row rank. Therefore, $\Pi_m^n(\Pi_m^n)^T$ is invertible.*
(ii) Let $n \leq m$. Then Π_m^n is of full column rank. Hence, $(\Pi_m^n)^T \Pi_m^n$ is invertible.

Proof. (i) Assume $n \geq m$. When $n = m$, $\Pi_m^n(\Pi_m^n)^T$ is an identity matrix, the conclusion is trivial. We, therefore, need only to consider the case when $n > m$. According to the structure of Π_m^n determined by (21), it is easy to see that each row of Π_m^n contains at least two non-zero elements. Moreover, when $j > i$ the column of non-zero element in row i is prior to the column of non-zero element in row j, and only when $j = i + 1$ there is an overlapped column. This structure ensures the full row rank of Π_m^n. Hence, $\Pi_m^n(\Pi_m^n)^T$ is invertible.

(ii) According to (21), one sees easily that $\Pi_n^m = \frac{\beta}{\alpha}(\Pi_m^n)^T$, hence, the full column rank of Π_m^n comes from the full row rank of Π_n^m. $\qquad\square$

The following proposition shows that the projection from factor dimension space to multiple dimension space does not lose information.

Proposition 41. *Let $X \in \mathbb{R}^m$. Project it to $\mathbb{R}^{km}$ and then project the image back to $\mathbb{R}^m$, the vector X remains unchanged. That is,*

$$(22) \qquad \Pi_m^{km}\Pi_{km}^m = I_m.$$

Proof. $\Pi_m^{km}\Pi_{km}^m = \frac{1}{k}\left(I_m \otimes \mathbf{1}_k^T\right)(I_m \otimes \mathbf{1}_k) = \frac{1}{k}\left(I_m \otimes \mathbf{1}_k^T \mathbf{1}_k\right) = I_m.$ $\qquad\square$

4.2. Continuous functions on DFESs

Now we are ready to define continuous functions on Ω.

Definition 42. *Let $f : \Omega \to \mathbb{R}$ be a real function on Ω.*

(i) *Define $f : \mathbb{R}^\infty \to \mathbb{R}$, $x \mapsto f(\bar{x})$.*

(ii) *If for each point $\bar{x} \in \Omega$ there exists a neighborhood $O_{\bar{x}}$ of $\bar{x}$ such that on each leaf $\mathcal{V}^r_{O_{\bar{x}}} \subset \mathbb{R}^{rp}$ $f \in C(\mathcal{V}^r_{O_{\bar{x}}})$, then f is called a continuous function on Ω.*

(iii) *If on each leaf of the bundle of coordinate neighborhood $f \in C^r(\mathcal{V}^r_{O_{\bar{x}}})$, then f is called a C^r function on Ω, where $r = 1, 2, \cdots, \infty, \omega$, $r = \omega$ means f is an analytic function.*

Remark 43. *In definition 33, the set of coordinate neighborhood is used. In fact, up to now only global coordinates are used. So the definition can also use global coordinates. That is, consider $\mathbb{R}^{rp}$ as each leaf of the bundle of coordinate neighborhood.*

Constructing a differentiable function on Ω directly is very difficult. Our technique to construct such a function is to transfer a smooth function on $\mathbb{R}^\infty$ to Ω. Note that $\mathbb{R}^n$ is a clopen subset of $\mathbb{R}^\infty$. $f : \mathbb{R}^\infty \to \mathbb{R}$ is continuous, if and only if, $f_n := f|_{\mathbb{R}^n}$, $n = 1, 2, \cdots$, are continuous. Hence, it is reasonable to transfer an $f \in C^r(\mathbb{R}^n)$ to Ω.

Definition 44. *Let $f \in C^r(\mathbb{R}^n)$. Define $\bar{f} : \Omega \to \mathbb{R}$ as follows: Let $\bar{x} \in \Omega$ and $\dim(\bar{x}) = m$. Then*

$$\text{(23)} \qquad \bar{f}(\bar{x}) := f(\Pi^m_n(x_1)), \quad \bar{x} \in \Omega,$$

where $x_1 \in \bar{x}$ is the smallest element in $\bar{x}$.

Proposition 45. *Assume $f \in C^r(\mathbb{R}^n)$, then the function $\bar{f}$ defined by (23) is C^r, that is, $\bar{f} \in C^r(\Omega)$.*

Proof. Given $\bar{x} \in \Omega$, where $\dim(\bar{x}) = m$. Consider a leaf of the bundle of coordinate neighborhood $\mathcal{V}^r_{O_{\bar{x}}}$ of $\bar{x}$. Assume $y \in \mathcal{V}^r_{O_{\bar{x}}}$, consider the following two cases:

- Case 1: $y \in \mathbb{R}^{rm}$ is the smallest element of $\bar{y}$. By definition,

$$\text{(24)} \qquad \bar{f}(y) = f(\Pi^{rm}_m y).$$

- Case 2: $y_1 \in \bar{y}$ is the smallest element of $\bar{y}$ and $\dim(y_1) = \xi$. Then there exists s such that $y = y_1 \otimes \mathbf{1}_s$. Since $y \in \mathbb{R}^{rm}$, then $\xi s = mr$. By definition,

$$\text{(25)} \qquad \bar{f}(y) = f(\bar{y}) = f(\Pi^\xi_m y_1).$$

Denote $z_0 := \Pi_m^\xi y_1 \in \mathbb{R}^m$. Then z_0 is the point on $\mathbb{R}^m$, which is closest to y_1. Since $y \leftrightarrow y_1$, According to Proposition 23, we know $d_\mathcal{V}(z, y) = d_\mathcal{V}(z, y_1)$, $z \in \mathbb{R}^m$. Hence, z_0 is also the point on $\mathbb{R}^m$ which is closest to y. That is, $\Pi_m^{mr} y = z_0 = \Pi_m^\xi y_1$. Hence, (25) becomes (24). It is obvious that $\bar{f}$ is a C^r function on $\mathcal{V}_{O_{\bar{x}}}^r$. $\qquad\square$

The following is a simple example.

Example 46. *Given*

$$(26) \qquad\qquad f(x_1, x_2, x_3) = x_1 + x_2^2 - x_3 \in C^\omega(\mathbb{R}^3).$$

(i) *Assume* $\bar{y} \in \Omega$, *where* $y_1 = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)^T \in \mathbb{R}^5$. *It is easy to calculate that*

$$\Pi_3^5 = \frac{1}{5} \left(I_3 \otimes \mathbf{1}_5^T \right) (I_5 \otimes \mathbf{1}_3) = \frac{1}{5} \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix}.$$

Hence we have

$$\bar{f}(\bar{y}) = f(\Pi_3^5 y_1) = \frac{1}{5}(3\xi_1 + 2\xi_2) + \frac{1}{25}(\xi_2 + 3\xi_3 + \xi_4)^2 - \frac{1}{5}(2\xi_4 + 3\xi_5).$$

(ii) *Assume* $\bar{y} \in \Omega$, *where* $y_1 = (\xi_1, \xi_2) \in \mathbb{R}^2$.

Consider $\mathcal{V}_O^1$: *since* $\Pi_3^2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \\ 0 & 1 \end{bmatrix}$, *we have* $\bar{f}|_{\mathcal{V}_O^1} = \xi_1 + \frac{1}{4}(\xi_1 + \xi_2)^2 - \xi_2$.

Consider $\mathcal{V}_O^2$: *since* $\Pi_3^4 = \frac{1}{4}\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, *we have* $\bar{f}|_{\mathcal{V}_O^2} = \frac{1}{4}(3\xi_1 + \xi_2) + \frac{1}{16}(\xi_2 + \xi_3)^2 - \frac{1}{4}(\xi_3 + 3\xi_4)$.

4.3. Least square approximation of linear systems

Consider a linear system

$$(27) \qquad\qquad \xi(t+1) = A\xi(t), \quad \xi(t) \in \mathbb{R}^n.$$

Our goal is to find a matrix $A_\pi \in \mathcal{M}_{m \times m}$, and construct a linear system on $\mathbb{R}^m$ as

$$(28) \qquad\qquad x(t+1) = A_\pi x(t), \quad x(t) \in \mathbb{R}^m.$$

Then take (28) as the projected system of (27) on $\mathbb{R}^m$.

We are mainly concerning about the trajectories. The trajectory of the idea projected system should satisfy the same projection relation. That is,

$$(29) \qquad x(t, \pi(\xi_0)) = \pi_m^n(\xi(t, \xi_0)).$$

Unfortunately, it is, in general, impossible to realize this. So we can only search such a system that makes the error of (29) smallest. Then a practical way is that we can search for the least square approximation.

Proposition 47. *Let system (27) be approximated by system (28). Then the least square approximation satisfies*

$$(30) \qquad A_\pi = \begin{cases} \Pi_m^n A(\Pi_m^n)^T \left(\Pi_m^n(\Pi_m^n)^T\right)^{-1} & n \geq m \\ \Pi_m^n A \left((\Pi_m^n)^T \Pi_m^n\right)^{-1} (\Pi_m^n)^T & n < m. \end{cases}$$

Proof. From (29) we have $x(t) = \Pi_m^n \xi(t)$, with $x_0 = \Pi_m^n \xi_0$. Plugging it into (28) yields

$$(31) \qquad \Pi_m^n \xi(t+1) = A_\pi \Pi_m^n \xi(t).$$

Using (27) and noting that $\xi(t)$ is arbitrary, we have

$$(32) \qquad \Pi_m^n A = A_\pi \Pi_m^n.$$

Assume $n \geq m$, right multiplying by $(\Pi_m^n)^T \left(\Pi_m^n(\Pi_m^n)^T\right)^{-1}$ both sides of (32) yields the first equality of (30).

Assume $n < m$. Then we search for a solution of the following form: $A_\pi = \tilde{A}(\Pi_m^n)^T$. Then the least square solution $\tilde{A}$ can be obtained as $\tilde{A} = \Pi_m^n A \left((\Pi_m^n)^T \Pi_m^n\right)^{-1}$. Hence, $A_\pi = \Pi_m^n A \left((\Pi_m^n)^T \Pi_m^n\right)^{-1} (\Pi_m^n)^T$, which is the second equality of (30) $\qquad \square$

Using a similar argument to continuous time linear system, we have the following result:

Corollary 48. *Consider a continuous time linear system*

$$(33) \qquad \dot{\xi}(t) = A\xi(t), \quad \xi(t) \in \mathbb{R}^n.$$

Its least square projected system on $\mathbb{R}^m$ is

$$(34) \qquad \dot{x}(t) = A_\pi x(t), \quad x(t) \in \mathbb{R}^m,$$

where, A_π is the same as in (30).

As an application, assume n is very large, that is, system (27) is a large scale one. Then we may project it onto a lower dimensional space $\mathcal{V}_m$, where, $m << n$. That is, we have a lower dimensional trajectory to approximate the original one, which might reduce the computational complexity. In the sequel one may see that the projection of lower dimensional system into a higher dimensional vector space is sometimes also necessary.

Similarly, the projection of linear control systems can also be obtained.

Corollary 49. *(i) Consider a discrete-time linear control system*

$$
(35) \qquad \begin{cases} \xi(t+1) = A\xi(t) + Bu, & \xi(t) \in \mathbb{R}^n \\ y(t) = C\xi(t), & y(t) \in \mathbb{R}^p. \end{cases}
$$

Its least square projected system on $\mathbb{R}^m$ is

$$
(36) \qquad \begin{cases} x(t+1) = A_\pi x(t) + \Pi_m^n Bu, & x(t) \in \mathbb{R}^m \\ y(t) = C_\pi x(t), \end{cases}
$$

where, A_π is defined as in (30). Moreover,

$$
(37) \qquad C_\pi = \begin{cases} C(\Pi_m^n)^T \left(\Pi_m^n (\Pi_m^n)^T \right)^{-1}, & n \geq p \\ C \left((\Pi_m^n)^T \Pi_m^n \right)^{-1} (\Pi_m^n)^T, & n < p. \end{cases}
$$

(ii) Consider a continuous time linear control system

$$
\begin{cases} \dot{\xi}(t) = A\xi(t) + Bu, & \xi(t) \in \mathbb{R}^n \\ y(t) = C\xi(t), & y(t) \in \mathbb{R}^p. \end{cases}
$$

Its least square projected system on $\mathbb{R}^m$ is

$$
\begin{cases} \dot{x}(t) = A_\pi x(t) + \Pi_m^n Bu, & x(t) \in \mathbb{R}^m \\ y(t) = C_\pi x(t), & y(t) \in \mathbb{R}^p, \end{cases}
$$

where A_π is defined as in (30), C_π as in (37).

4.4. Approximation of linear dimension-varying systems

Consider a discrete-time linear dimension-varying system

$$
(38) \qquad \xi(t+1) = A(t)\xi(t),
$$

where $\xi(t) \in \mathbb{R}^{n(t)}$, $\xi(t+1) \in \mathbb{R}^{n(t+1)}$, $A(t) \in \mathcal{M}_{n(t+1) \times n(t)}$.

We search its least square projection on $\mathbb{R}^m$ as

$$(39) \qquad x(t+1) = A_\pi x(t).$$

Similarly to the constant dimensional case, the following result can be obtained.

Proposition 50. *Let (39) be the least square projected system of (38) on $\mathbb{R}^m$. Then*

$$(40) \qquad A_\pi = \begin{cases} \Pi_m^{n(t+1)} A (\Pi_m^{n(t)})^T \left(\Pi_m^{n(t)} (\Pi_m^{n(t)})^T \right)^{-1} & n(t) \geq m \\ \Pi_m^{n(t+1)} A \left((\Pi_m^{n(t)})^T \Pi_m^{n(t)} \right)^{-1} (\Pi_m^{n(t)})^T & n(t) < m. \end{cases}$$

An obvious advantage of this projection is the projected system is of constant dimension.

Consider a continuous-time linear dimension-varying system

$$(41) \qquad \dot{\xi}(t) = A(t)\xi(t),$$

where $\xi(t) \in \mathbb{R}^{n(t)}$, $\xi(t+1) \in \mathbb{R}^{n(t+1)}$, $A(t) \in \mathcal{M}_{n(k) \times n(k)}$, $k \leq t < k+1$.

We search its least square projection on $\mathbb{R}^m$ as

$$(42) \qquad \dot{x}(t) = A_\pi x(t).$$

To use the previous technique, we assume the dimension of $x(t)$ is piecewise constant. Precisely speaking, we assume:

$$(43) \qquad \dim(\xi(t)) = \dim(\xi(n)), \quad n \leq t < n+1.$$

Then the following result can be obtained.

Proposition 51. *Let (42) be the lease square projected system of (41) on $\mathbb{R}^m$. Under the assumption that (43) holds, the A_π is as in (40).*

Next, we consider dimension-varying linear control systems. Using similar technique, it is easy to obtain the following projected control systems:

Proposition 52. *(i) Consider a discrete-time linear dimension-varying control system*

$$(44) \qquad \begin{cases} \xi(t+1) = A(t)\xi(t) + B(t)u \\ y(t) = C(t)\xi(t), \end{cases}$$

where $\xi(t) \in \mathbb{R}^{n(t)}$, $A(t)$, $B(t) \in \mathcal{M}_{n(t+1) \times n(t)}$, $C(t) \in \mathcal{M}_{p \times n(t)}$. Then its least square projected control system is

$$(45) \qquad \begin{cases} x(t+1) = A_\pi(t)x(t) + \Pi_m^{n(t+1)}Bu, & x(t) \in \mathbb{R}^m \\ y(t) = C_\pi(t)x(t), & y(t) \in \mathbb{R}^p, \end{cases}$$

where A_π is defined as in (40). Moreover,

$$(46) \qquad C_\pi = \begin{cases} C(t)(\Pi_m^{n(t)})^T \left(\Pi_m^{n(t)}(\Pi_m^{n(t)})^T \right)^{-1}, & n(t) \geq p \\ C(t) \left((\Pi_m^{n(t)})^T \Pi_m^{n(t)} \right)^{-1} (\Pi_m^{n(t)})^T, & n(t) < p. \end{cases}$$

(ii) Consider a continuous-time linear dimension-varying control system

$$(47) \qquad \begin{cases} \dot{\xi}(t) = A(t)\xi(t) + B(t)u \\ y(t) = C(t)\xi(t), \end{cases}$$

where $\xi(t) \in \mathbb{R}^{n(t)}$, $A(t), B(t) \in \mathcal{M}_{n(k) \times n(k)}$, $k \leq t < k+1$, $C(t) \in \mathcal{M}_{p \times n(t)}$. Assume (43) holds, then its least square projected system is

$$(48) \qquad \begin{cases} \dot{x}(t) = A_\pi(t)x(t) + \Pi_m^{n(t+1)}Bu, & x(t) \in \mathbb{R}^m \\ y(t) = C_\pi(t)x(t), & y(t) \in \mathbb{R}^p, \end{cases}$$

where A_π is defined as in (40), C_π as in (46).

Later on, it will be seen that the fixed-dimensional projected system is a very useful realization of dimension-varying systems.

In the following an example is presented to depict projected system.

Example 53. *Consider a dimension-varying system*

$$(49) \qquad \begin{cases} \xi(t+1) = A(t)\xi(t) + B(t)u \\ y(t) = C(t)\xi(t), \end{cases}$$

where

$$\xi(t) \in \begin{cases} \mathbb{R}^5, & t \text{ is even,} \\ \mathbb{R}^4, & t \text{ is odd.} \end{cases}$$

$$A(t) = \begin{cases} A_1 = \begin{bmatrix} 1 & 0 & -1 & 2 & 1 \\ 2 & -2 & 1 & 1 & -1 \\ 1 & 2 & -1 & -2 & 0 \\ 0 & 1 & 0 & -1 & 2 \end{bmatrix}, & t \text{ is even} \\[4em] A_2 = \begin{bmatrix} 0 & -1 & 2 & 1 \\ 2 & 1 & 1 & -1 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 \end{bmatrix}, & t \text{ is odd} \end{cases},$$

$$B(t) = \begin{cases} B_1 = \begin{bmatrix} 2 & 1 \\ 2 & -1 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}, & t \text{ is even} \\[4em] B_2 = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 2 & -1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}, & t \text{ is odd} \end{cases},$$

$$C(t) = \begin{cases} C_1 = \begin{bmatrix} -1 & 2 & 1 & 1 & -1 \\ 2 & -1 & -2 & -1 & 2 \end{bmatrix}, & t \text{ is even} \\[2em] C_2 = \begin{bmatrix} 2 & 1 & 2 & -1 \\ 0 & 1 & 0 & -2 \end{bmatrix}, & t \text{ is odd} \end{cases}.$$

A straightforward computation shows that

$$\Pi_3^4 = (I_3 \otimes \mathbf{1}_4^T)(I_4 \otimes \mathbf{1}_3)/3 = \begin{bmatrix} 1 & 1/3 & 0 & 0 \\ 0 & 2/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 1 \end{bmatrix}.$$

$$\Pi_3^5 = (I_3 \otimes \mathbf{1}_5^T)(I_5 \otimes \mathbf{1}_3)/3 = \begin{bmatrix} 1 & 2/3 & 0 & 0 & 0 \\ 0 & 1/3 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 2/3 & 1 \end{bmatrix}.$$

Then the projected system becomes

(50)
$$\begin{cases} x(t+1) = A_\pi(t)x(t) + B_\pi(t)u \\ y(t) = C_\pi(t)x(t), \end{cases}$$

where,

$$A(t) = \tilde{A}_1; \quad B(t) = \tilde{B}_1; \quad C(t) = \tilde{C}_1 \qquad t \text{ is even,}$$
$$A(t) = \tilde{A}_2; \quad B(t) = \tilde{B}_2; \quad C(t) = \tilde{C}_2 \qquad t \text{ is odd,}$$

where

$$\tilde{A}_1 = \Pi_3^4 A_1 (\Pi_3^5)^T \left(\Pi_3^5 (\Pi_3^5)^T\right)^{-1} = \begin{bmatrix} 0.9316 & -0.5556 & 1.6239 \\ 1.4325 & -0.3111 & -0.7214 \\ 1.0923 & -0.6000 & 0.7077 \end{bmatrix};$$

$$\tilde{A}_2 = \Pi_3^5 A_2 (\Pi_3^4)^T \left(\Pi_3^4 (\Pi_3^4)^T\right)^{-1} = \begin{bmatrix} 0.8333 & 1.3333 & 0.8333 \\ 2.0500 & 1.2500 & -1.0500 \\ 0.9167 & -0.5833 & 0.4167 \end{bmatrix};$$

$$\tilde{B}_1 = \Pi_3^4 B_1 = \begin{bmatrix} 2.6667 & 1.3333 \\ 2.0000 & 2.0000 \\ 0.3333 & -0.3333 \end{bmatrix}; \quad \tilde{B}_2 = \Pi_3^5 B_2 = \begin{bmatrix} 2.6667 & 0.3333 \\ 2.3333 & -1.6667 \\ 1.0000 & -0.6667 \end{bmatrix};$$

$$\tilde{C}_1 = C_1 (\Pi_3^5)^T \left(\Pi_3^5 (\Pi_3^5)^T\right)^{-1} = \begin{bmatrix} -0.0359 & 1.7333 & -0.4974 \\ 1.3333 & -2.6667 & 1.3333 \end{bmatrix};$$

$$\tilde{C}_2 = C_2 (\Pi_3^4)^T \left(\Pi_3^4 (\Pi_3^4)^T\right)^{-1} = \begin{bmatrix} 1.7000 & 2.0000 & -0.7000 \\ 0.0500 & 1.2500 & -2.0500 \end{bmatrix}.$$

5. Differential structures on DFMs

With the notion of continuous functions over DFESs, we proceed to endow differential structures to a DFES, generalizing it to a dimension-free manifold.

5.1. From DFESs to DFMs

Definition 54. *Given a fiber bundle* (T, π, B).

(i) *Let* $\emptyset \neq O \subset B$ *be an open set of* B. *Then* $(\pi^{-1}(O), \pi, O)$ *is called the open sub-bundle (over* O).

(ii) *Let* O_λ, $\lambda \in \Lambda$ *be an open cover of* B, *that is,* $\bigcup_{\lambda \in \Lambda} O_\lambda = B$, *then* $\left\{ \left(\pi^{-1}(O_\lambda), \pi, O_\lambda \right) \right\}_{\lambda \in \Lambda}$ *is called an open cover of the bundle* (T, π, B).

Definition 55. *Assume* (T, π, B) *is a fiber bundle, where both* T *and* B *are second countable Hausdorff spaces.* (T, π, B) *is called a* C^r *DFEB with* B *as a* C^r *DFM, if the following conditions are satisfied.*

(i) *There is an open cover* $\{(W_\lambda, \pi, B_\lambda)\}_{\lambda \in \Lambda}$ *of* (T, π, B).

(ii) *For each $(W_\lambda, \pi, B_\lambda)$ there is an open sub-bundle $\left(\mathrm{Pr}^{-1}(O_\lambda), \mathrm{Pr}, O_\lambda\right)$ of $(\mathbb{R}^\infty, \mathrm{Pr}, \Omega)$ with two bijective mappings*

$$\Psi_\lambda : W_\lambda \to \mathrm{Pr}^{-1}(O_\lambda), \quad \varphi_\lambda : B_\lambda \to O_\lambda,$$

such that $(W_\lambda, \pi, B_\lambda)$ and $(\mathrm{Pr}^{-1}(O_\lambda), \mathrm{Pr}, O_\lambda)$ are fiber bundle isomorphic, that is, diagram (51) commutes.

(51)
$$
\begin{array}{ccc}
W_\lambda & \xrightarrow{\ \Psi_\lambda\ } & \mathrm{Pr}^{-1}(O_\lambda) \\
\downarrow{\scriptstyle \mathrm{Pr}} & & \downarrow{\scriptstyle \pi} \\
B_\lambda & \xrightarrow{\ \varphi_\lambda\ } & O_\lambda
\end{array}
$$

(iii) *Assume $B_{\lambda_1} \bigcap B_{\lambda_2} \neq \emptyset$. Then the map $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(B_{\lambda_1} \bigcap B_{\lambda_2}) \to \varphi_2(B_{\lambda_1} \bigcap B_{\lambda_2})$ is C^r.*

The following proposition comes from definition immediately.

Proposition 56. *Let $T \xrightarrow{\pi} B$ be a C^r DFEB. Set $M_n := \bigcup_{\lambda \in \Lambda} \Psi_\lambda^{-1}(O_\lambda^n)$, then M_n is an n-dimensional C^r manifold. Moreover, $M = \bigcup_{n=1}^\infty M_n$.*

The following example provides a DFM.

Example 57. *Consider $S_\infty := \bigcup_{n=1}^\infty S_n$, where S_n is the n-dimensional unit sphere in $\mathbb{R}^{n+1}$, $n = 1, 2, \cdots$. Denote by $P_n = (\underbrace{0, \cdots, 0}_{n}, -1)$ and $Q_n = (\underbrace{0, \cdots, 0}_{n}, 1)$ the north and south poles of the n-dimensional sphere respectively.*

(i) *Set $M_n := S_n \backslash P_n$, and define a mapping $\Psi_n : M_n \to \mathbb{R}^n$ by $\xi_i = \frac{x_i}{1+x_{n+1}}$, $i = 1, 2, \cdots, n$.*
Define $M = \bigcup_{n=1}^\infty M_n$, and using the inherent topology from $\mathbb{R}^{n+1}$ for M_n, and assume M_n are clopen in M. Then the mapping $\Psi : M \to \mathbb{R}^\infty$ is a topological isomorphism.
To see this, we have only to show that Ψ is bijective and Ψ^{-1} is also continuous. It is clear from definition that

$$(\xi_1^2 + \cdots + \xi_n^2)(1 + x_{n+1})^2 = \sum_{i=1}^{n} x_i^2.$$

Then we have

(52)
$$\|\xi\|^2 (1 + x_{n+1})^2 + x_{n+1}^2 = 1.$$

Solving Equation ([52](#)) and noting that $x_{n+1} \neq -1$ yield

$$(53) \qquad x_{n+1} = \frac{1 - \|\xi\|^2}{1 + \|\xi\|^2},$$

and

$$(54) \qquad x_i = (1 + x_{n+1})\xi_i, \quad i \in [1, n].$$

([53](#))-([54](#)) show that Ψ^{-1} is also continuous.
Next, for $a, b \in M$ we define

$$(55) \qquad a \sim_M b \Leftrightarrow \Psi(a) \leftrightarrow \Psi(b).$$

Then we can define a mapping $\psi : M/\sim_M \to \Omega$ by

$$(56) \qquad \Psi(a) = x \Rightarrow \psi(\bar{a}) := \bar{x}.$$

Because of ([55](#)), ([56](#)) is properly defined.
Finally, we define $\pi_1 : M \to M/\sim_M$ as $\pi_1 = \psi^{-1} \circ \mathrm{Pr} \circ \Psi$. Then it is ready to verify that $(M, \pi_1, M/\sim_M)$ is a DFEB and $M/\sim_S$ is a DFM.

(ii) *Set $N_n := S_n \backslash Q_n$, and define a mapping $\Phi_n : N_n \to \mathbb{R}^n$ by*

$$(57) \qquad \eta_i = \frac{x_i}{1 - x_{n+1}}, \quad i = 1, 2, \cdots, n.$$

Similarly to case (i), from definition ([57](#)) we have

$$(\eta_1^2 + \cdots + \eta_n^2)(1 - x_{n+1})^2 = \sum_{i=1}^{n} x_i^2.$$

Then we have

$$(58) \qquad \|\eta\|^2 (1 - x_{n+1})^2 + x_{n+1}^2 = 1.$$

Solving Equation ([58](#)) and noting that $x_{n+1} \neq 1$ yield

$$(59) \qquad x_{n+1} = \frac{\|\eta\|^2 - 1}{1 + \|\eta\|^2},$$

and

(60)
$$x_i = (1 - x_{n+1})\eta_i, \quad i \in [1, n].$$

(59)(60) show that Φ^{-1} is also continuous.
Next, for $x, y \in N$ we define

$$x \sim_N y \Leftrightarrow \Phi(x) \leftrightarrow \Phi(y),$$

and $\pi_2 : N \to N/\sim_N$ as $\pi_2 = \phi^{-1} \circ \mathrm{Pr} \circ \Phi$, where ϕ can be constructed similarly as for ψ. Then $(N, \pi_2, N/\sim_N)$ is a DFEB and $N/\sim_N$ is a DFM.

(iii) Consider

$$S_\infty = \bigcup_{n=1}^{\infty} S_n = M \bigcup N.$$

It is clear that $\{M, N\}$ is an open cover of S_∞. Consider $(x_1, \cdots, x_{n+1})$ $\in M \bigcap S_n$. From (53)(54) we can solve $x_1, \cdots, x_{n+1}$ out as $x_i(\xi)$, $i \in [1, n+1]$. Similarly, for $(x_1, \cdots, x_{n+1}) \in N \bigcap S_n$, using (59)(60) we can also express $x_1, \cdots, x_{n+1}$ as $x_i(\eta)$, $i = 1, \cdots, n+1$.
Now, assume $\bar{\xi} \in M/\sim_M$ and $\bar{\eta} \in N/\sim_N$, $\dim(\bar{\xi}) = \dim(\bar{\eta})$, and the smallest elements in $\bar{\xi}$ and $\bar{\eta}$ are ξ_1 and η_1 respectively. Let $\dim(\xi_1) = \dim(\eta_1) = n$. Then $\bar{\xi}$ is said to be equivalent to $\bar{\eta}$, denoted by $\bar{\xi} \leftrightharpoons \bar{\eta}$, if $x_i(\xi_1) = x_i(\eta_1), \quad \forall i = 1, \cdots, n+1$.
It is easy to see that $\leftrightharpoons$ is an equivalence relation. Then we define

(61)
$$B := \left(M/\sim_M \bigcup N/\sim_N \right) / \leftrightharpoons .$$

Define

$$B_1 := (M/\sim_M) / \leftrightharpoons,$$
$$B_2 := (N/\sim_N) / \leftrightharpoons .$$

Then it is easy to verify that $\{B_1, B_2\}$ is an open cover of B. Moreover, it is ready to see that $\pi_1 : M \to B_1$, $\pi_2 : N \to B_2$ are consistent. Hence $\pi : S_\infty \to B$ can be defined as

(62)
$$\pi(x) := \begin{cases} \pi_1(x), & x \in M, \\ \pi_2(x), & x \in N. \end{cases}$$

We conclude that (S_∞, π, B) is a DFEB with B a DFM.

Remark 58. *Hereafter, to avoid notational mess, we consider only DFES. In fact, all the following arguments can easily be extended to DFM. Hence in the following the DFES can be considered as a bundle of coordinate chart, which has a set of fixed coordinate frames on each leaves.*

5.2. Vector fields on DFESs

First, we define the tangent space of Ω.

Definition 59. *Let $\bar{x} \in \Omega$ and $\dim(\bar{x}) = m$. Then the tangent space of $\bar{x}$, called the tangent bundle at $\bar{x}$ and denoted by $T_{\bar{x}}(\Omega)$, is defined by*

$$(63) \qquad\qquad T_{\bar{x}}(\Omega) := \mathbb{R}^{[m,\cdot]}.$$

Remark 60. *(i) When Ω is replaced by a DFM, Definition 59 can only be considered as for a given fixed set of coordinate charts.*

(ii) Consider Ω. Then we assume on each leaf of $T_{\bar{x}}(\Omega)$, say $\mathbb{R}^s$, where $s = km$, the coordinate frame is fixed as $(x_1, x_2, \cdots, x_s)$. Then the basis of the leaf is $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_s}\}$. Hence each vector at $T_{\bar{x}}(\Omega)$, denoted be $(a_1, a_2, \cdots, a_s)$, is an operator $\sum_{i=1}^{s} a_i \frac{\partial}{\partial x_i}$. Of course, this operator can be extended to coordinate-free form for DFM. But restricting on Ω can avoid the complexity of expression.

Recall the definition of bundle of coordinate neighborhood of DFES. (refer to Figure 1), it is easily seen that for each $\bar{x} \in \Omega$ the bundle of coordinate neighborhood coincides with its tangent bundle.

When a DFM M is considered, Let $\bar{x} \in M$ and $\dim(\bar{x}) = m$, then the tangent bundle $T_{\bar{x}}(M)$ is depicted at Figure 2, where $T_{\bar{x}}^i = \mathbb{R}^{im}$, $i = 1, 2, \cdots$. That is, $T_{\bar{x}}(M) = \mathbb{R}^{[m,\cdot]}$.

If we consider the tangent space over whole Ω, that is,

$$T(\Omega) := \bigcup_{\bar{x} \in \Omega} T_{\bar{x}},$$

Then it is obvious that $T(\Omega) = \mathbb{R}^\infty$.

Next, we define vector fields on Ω. The following definition is also available for DFMs.

Definition 61. *$\bar{X}$ is called a C^r vector field on Ω, denoted by $\bar{X} \in V^r(\Omega)$, if it satisfies the following condition:*
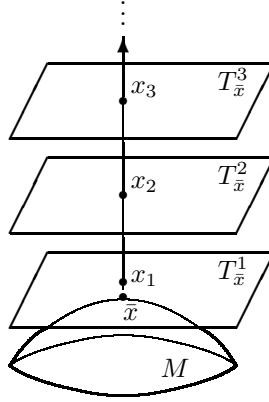
Figure 2: Tangent Bundle on Dimension-Free Manifold.

(i) *At each point $\bar{x} \in \Omega$, there exists $p = p_{\bar{x}} = \mu_{\bar{x}} \dim(\bar{x})$, called the dimension of the vector field $\bar{X}$ at $\bar{x}$ and denoted by $\dim(\bar{X}_{\bar{x}})$, such that $\bar{X}$ assigns to the bundle of coordinate neighborhood at $\bar{x}$ a p sub-lattice, $\mathcal{V}_O^{[p,\cdot]} = \{O^p, O^{2p}, \cdots\}$, then at each leaf of this sub-lattice the vector field assigns a vector $X^j \in T_{x_{j\mu}}(O^{jp})$, $j = 1, 2, \cdots$.*

(ii) *$\{X^j \mid j = 1, 2, \cdots\}$ satisfy consistence condition, that is, $X^j = X^1 \otimes \mathbf{1}_j$, $j = 1, 2, \cdots$.*

(iii) *At each leaf $O^{jp} \subset \mathbb{R}^{j\mu \dim(\bar{x})}$,*

$$(64) \qquad\qquad \bar{X}|_{O^{jp}} \in V^r(O^{jp}).$$

Definition 62. *A vector field $\bar{X} \in V^r(\Omega)$ is said to be dimension bounded, if $\max\limits_{\bar{x} \in \Omega} \dim(\bar{X}_{\bar{x}}) < \infty$.*

In the following a method is presented to construct a C^r vector field on Ω. The method is similar to the construction of continuous functions. It is first built on $\mathcal{V}^m = \mathbb{R}^m$, and then extended to $T(\Omega) = \mathbb{R}^\infty$.

Algorithm 63. • *Step 1: Assume there exists a smallest dimension $m > 0$, such that $\bar{X}$ is defined over whole $\mathbb{R}^m$. That is,*

$$(65) \qquad\qquad \bar{X}|_{\mathbb{R}^m} := X \in V^r(\mathbb{R}^m).$$

From the constructing point of view: A vector field $X \in V^r(\mathbb{R}^m)$ is firstly given, such that the value of $\bar{X}$ at leaf $\mathbb{R}^m$ is uniquely determined by (65).

- Step 2: Extend X to $T_{\bar{y}}$. Assume $\dim(\bar{y}) = s$, denote $m \vee s = t$, $t/m = \alpha$, $t/s = \beta$. Then $\dim(T_{\bar{y}}) = t$. Let $y \in \bar{y} \bigcap R^{[t,\cdot]}$, and $\dim(y) = kt$, $k = 1, 2, \cdots$. Define

$$(66) \qquad \bar{X}(y) := \Pi_{kt}^m X(\Pi_m^{kt} y), \quad k = 1, 2, \cdots.$$

Theorem 64. *(i) The $\bar{X}$ generated by Algorithm 63 is a C^r vector field, that is, $\bar{X} \in V^r(\Omega)$.*

(ii) If $\bar{X} \in V^r(\Omega)$ is dimension bounded, then $\bar{X}$ can be generated by Algorithm 63.

Proof. (i) By definition, for any $\bar{y} \in \Omega$ and assume $\dim(\bar{y}) = s$, then on a sub-lattice $\mathbb{R}^{[t,\cdot]}$ of the bundle of coordinate neighborhood of $\bar{y}$ (Since only the DFES is considered now, each leaf of the bundle of coordinate neighborhood can be whole Euclidean space.) a vector $\bar{X}_y$ is assigned. In the following we prove that the set of such vectors are consistent. Assume $\dim(y) = kt = k\beta m$, when $k = 1$, $y = y_\beta$, then

$$\begin{aligned} \bar{X}(y_\beta) &= \Pi_{\beta m}^m X(\Pi_m^{\beta m} y_\beta) \\ &= (I_{\beta m} \otimes \mathbf{1}_1^T)(I_m \otimes \mathbf{1}_\beta) X(\Pi_m^{\beta m} y_\beta) \\ &= (I_m \otimes \mathbf{1}_\beta) X(\Pi_m^{\beta m} y_\beta) \\ &= X(\Pi_m^{\beta m} y_\beta) \otimes I_\beta. \end{aligned}$$

Similar calculation shows that

$$\bar{X}(y_{k\beta}) = X(\Pi_m^{k\beta m} y_{k\beta}) \otimes I_{k\beta}.$$

Since $y_\beta \leftrightarrow y_{k\beta}$, then $\Pi_m^{k\beta m} y_{k\beta} = \Pi_m^{\beta m} y_\beta$. Hence,

$$\bar{X}(y_{k\beta}) = \bar{X}(y_\beta) \otimes \mathbf{1}_k.$$

The consistence is proved.

Finally, we show (64) holds. That is, to show that on leaf $\mathbb{R}^{jp}$, $\bar{X}$ is a C^r vector field. Since on a leaf all the points are of the same dimension, then the construction (66) ensures $\bar{X}|_{\mathbb{R}^{jp}}$ is a C^r vector field.

(ii) Assume $\bar{X}$ is dimension bounded, set

$$m := \mathrm{lcm}\left\{ \dim(\bar{X}_{\bar{x}}) \mid \bar{x} \in \Omega \right\}.$$

Then it is clear that $X := \bar{X}|_{\mathbb{R}^m} \in C^r(\mathbb{R}^m)$. Moreover, since $\bar{X}$ satisfies Definition 61, then starting from this X, the vector field constructed by (66) coincides with $\bar{X}$. $\qquad\square$

Hereafter, we consider only dimension bounded vector fields. This is because note only they are easily constructible, but also they are practically useful in modeling dynamic systems.

We construct an example.

Example 65. *Let* $X = (x_1 + x_2, x_2^2)^T \in C^\omega(\mathbb{R}^2)$. *Assume* $\bar{X} \in C^\omega(\Omega)$ *is generated by* X.

(i) *Consider* $\bar{y} \in \Omega$, $\dim(\bar{y}) = 3$, *Denote* $y_1 = (\xi_1, \xi_2, \xi_3)^T \in \mathbb{R}^3$. *Since* $2 \wedge 3 = 6$, $\bar{X}$ *at*

$$\bar{y} \bigcap \mathbb{R}^{6k} = \{y_2, y_4, y_6, \cdots\}$$

is well defined.
Now consider y_2.

$$\bar{X}(y_2) = \Pi_6^2 X(\Pi_2^6(y_2)) = (I_2 \otimes \mathbf{1}_3) X \left(\tfrac{1}{3}(I_2 \otimes \mathbf{1}_3^T)(y_1 \otimes \mathbf{1}_2)\right)$$

$$= \begin{bmatrix} \frac{2}{3}(\xi_1 + \xi_2 + \xi_3) \\ \frac{2}{3}(\xi_1 + \xi_2 + \xi_3) \\ \frac{2}{3}(\xi_1 + \xi_2 + \xi_3) \\ \frac{1}{9}(\xi_2 + 2\xi_3)^2 \\ \frac{1}{9}(\xi_2 + 2\xi_3)^2 \\ \frac{1}{9}(\xi_2 + 2\xi_3)^2 \end{bmatrix}.$$

Consider y_4, *similar calculation shows that*

$$\bar{X}(y_4) = \Pi_{12}^2 X(\Pi_2^{12}(y_4)) = \bar{X}(y_2) \otimes \mathbf{1}_2.$$

In fact, we have

$$\bar{X}(y_{2k}) = \bar{X}(y_2) \otimes \mathbf{1}_k, \quad k = 1, 2, \cdots.$$

(ii) *Consider* $\bar{X}|_{\mathbb{R}^6}$:
Assume $z = (z_1, z_2, z_3, z_4, z_5, z_6)^T \in \mathbb{R}^6$. *Then*

$$(67) \quad X^6 := \bar{X}_z = \Pi_6^2 X(\Pi_2^6 z) = \begin{bmatrix} \frac{1}{3}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) \\ \frac{1}{3}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) \\ \frac{1}{3}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) \\ \frac{1}{9}(z_4 + z_5 + z_6)^2 \\ \frac{1}{9}(z_4 + z_5 + z_6)^2 \\ \frac{1}{9}(z_4 + z_5 + z_6)^2 \end{bmatrix}.$$

$X^6 \in V^\omega(\mathbb{R}^6)$ *is a standard vector field.*

Next, we consider the integral curve of a vector field on Ω.

Definition 66. *Assume $\bar{X} \in C^r(\Omega)$, $X \in C^r(R^n)$ is its generator, if $X = \bar{X}|_{\mathbb{R}^n}$. The generator of smallest dimension is called the minimum generator.*

The following result is an immediate consequence of the definition and Theorem 64.

Proposition 67. *Assume $\bar{X} \in V^r(\Omega)$.*

(i) *If $X \in V^r(\mathbb{R}^n)$ is its generator, then $X \otimes \mathbf{1}_s \in V^r(\mathbb{R}^{sn})$ is also its generator.*
(ii) *If $X \in V^r(\mathbb{R}^n)$ is its generator, $Y \in V^r(\mathbb{R}^m)$, $m < n$ is also its generator, then $m|n$, and $X = Y \otimes \mathbf{1}_{n/m}$.*
(iii) *Assume $\bar{X} \in V^r(\Omega)$ is dimension bounded, then it has at least one generator, and hence has a minimum generator.*

Definition 68. *Let $\bar{X} \in C^r(\Omega)$. $\bar{x}(t, \bar{x}_0)$ is called the integral curve of $\bar{X}$ with initial value $\bar{x}_0$, denoted by $\bar{x}(t, \bar{x}_0) = \Phi_t^{\bar{X}}(\bar{x}_0)$, if for each initial value $x_0 \in \bar{x}_0 \bigcap \mathbb{R}^n$, and each generator of $\bar{X}$, denoted by $X = \bar{X}|_{\mathbb{R}^n}$, the following condition holds:*

$$(68) \qquad \Phi_t^{\bar{X}}(\bar{x}_0)|_{\mathbb{R}^n} = \Phi_t^X(x_0), \quad t \geq 0.$$

Next, we consider the existence and the properties of integral curve. First, assume $X = \bar{X}|_{\mathbb{R}^n}$ is the smallest generator of $\bar{X}$. Then, all the generators of $\bar{X}$ are $X_k = \bar{X}|_{\mathbb{R}^{kn}}$, $k = 1, 2, \cdots$. Now assume $\bar{x}_0 \in \Omega$, $\dim(\bar{x}_0) = j$, and $j \vee n = s$, then,

$$\bar{x} \bigcap \mathbb{R}^\ell \neq \emptyset,$$

if and only if, $\ell = ks$, $k = 1, 2, \cdots$. Denote $x_0^s = \bar{x}_0 \bigcap \mathbb{R}^s$, then

$$\Phi_t^{X_s}(x_0^s) = \Phi_t^{\bar{X}}(\bar{x}_0)|_{\mathbb{R}^s}.$$

Moreover,

$$\Phi_t^{\bar{X}}(\bar{x}_0)|_{\mathbb{R}^{ks}} = \Phi_t^{X_{ks}}(x_0^{ks}) = \Phi_t^{X_s}(x_0^s) \otimes \mathbf{1}_k.$$

Hence, the integral curve of $\bar{X}$ with initial value $\bar{x}_0$ is a set of integral curves defined on the sublattice bundle $\mathbb{R}^{[s,\cdot]} = \{\mathbb{R}^{ks} \mid k = 1, 2, \cdots\}$, and they are all equivalent. That is, for any $0 \leq k$, $k' < \infty$

$$\Phi_t^{X_{ks}}(x_0^{ks}) \leftrightarrow \Phi_t^{X_{k's}}(x_0^{k's}), \quad \forall t \geq 0.$$

Example 69. *Recall Example 65 Let $\bar{X} \in \Omega$ be generated by $X = (x_1 + x_2, x_2^2)^T \in C^\omega(\mathbb{R}^2)$, and assume the initial value is $\bar{x}_0 \in \Omega$, $\dim(\bar{x}_0) = 3$, i.e., $x_1 = (\xi_1, \xi_2, \xi_3)^T$. Find the integral curve of $\bar{X}$ initiated at $\bar{x}_0$.*

Since $2 \vee 3 = 6$, the integral curve is a set of equivalent curves defined on $\mathbb{R}^{6k}$, $k = 1, 2, \cdots$. We can first calculate the one defined on $\mathbb{R}^6$, $\mathcal{X}|_{\mathbb{R}^6} := X^6$, it is calculated by (67). Note that $x_2^0 := \bar{x}_0 \bigcap \mathbb{R}^6$, then $x_2^0 = (\xi_1, \xi_1, \xi_2, \xi_2, \xi_3, \xi_3)^T$. Hence the integral curve is $\Phi_t^{X^6}(x_2^0)$. It follows that

$$(69) \qquad \Phi_t^{\bar{X}}(\bar{x}_0) = \left\{ \Phi_t^{X^6}(x_2^0) \otimes \mathbf{1}_k \mid k = 1, 2, \cdots \right\}.$$

5.3. Distributions on DFESs

Definition 70. *A distribution $\bar{D}$ on Ω is a rule, which assigns at each point $\bar{x} \in \Omega$ a sub-lattice of its bundle of coordinate neighborhood $O_{\bar{x}}$, denoted by $O_j = O_{\bar{x}} \cap \mathbb{R}^{jrs}$, $r \in \mathbb{Z}_+$, $s = \dim(\bar{x})$, $j = 1, 2, \cdots$, and on the tangent space of $x_{jr} \in O_{jr}$, $T_{x_{jr}}(\mathbb{R}^{jrs})$, a subspace $D_j(x_{rj}) \subset T_{x_{rj}}(\mathbb{R}^{rjs})$. Moreover, this set of subspaces satisfies the consistence condition, i.e.,*

$$(70) \qquad D_j(x_{rj}) = D_1(x_r) \otimes \mathbf{1}_j, \quad j = 1, 2, \cdots.$$

Similarly to vector fields, a distribution can be construct as follows: First, a distribution can be defined on a leaf $\mathcal{V}^m = \mathbb{R}^m$, then it is extended to $T(\Omega) = \mathbb{R}^\infty$.

Algorithm 71. • *Step 1: Assume m is the smallest one, such that $\bar{D}$ is defined on leaf $\mathbb{R}^m$. That is,*

$$(71) \qquad \bar{D}|_{\mathbb{R}^m} := D(x) \subset T^r(\mathbb{R}^m).$$

• *Step 2: Extend $D(x)$ to $T_{\bar{y}}(\Omega)$. Let $\dim(\bar{y}) = s$, and $m \vee s = t$, $t/m = \alpha$, $t/s = \beta$. Then $\dim(T_{\bar{y}}) = t$. Assume $y \in \bar{y} \bigcap R^{[t,\cdot]}$, and $\dim(y) = kt$, $k = 1, 2, \cdots$. Define*

$$(72) \qquad D(y) := \Pi_{kt}^m D(\Pi_m^{kt} y), \quad k = 1, 2, \cdots.$$

Similarly to vector fields, the following result can be obtained.

Theorem 72. *The $\bar{D}$ constructed by Algorithm 71 is a distribution on Ω. That is, $\bar{D}(\bar{x}) \subset T_{\bar{x}}(\Omega)$, $\forall \bar{x} \in \Omega$.*

The most commonly used distributions are expanded by a set of vector fields.

Definition 73. *Assume $\bar{X}_i \in V^r(\Omega)$ and $\dim(\bar{X}_i) = m_i$, $i \in [1, n]$, and $m = \mathrm{lcm}\{m_i \mid i \in [1, n]\}$. Moreover, let $\bar{X}_i|_{\mathbb{R}^m} = X_i$, and $D_m(x) \subset T(\mathbb{R}^m)$ be the distributions generated by the expansions of X_i, $i \in [1, n]$. Then the distribution $\bar{D} \subset T(\Omega)$ constructed by $D_m(x)s$, $s = 1, 2, \cdots$ is called the distribution spanned by $\bar{X}_i$, $i \in [1, m]$.*

Definition 74. *Assume $\bar{X}_i \in V^\infty(\Omega)$, $\dim(\bar{X}_i) = m_i$, $i = 1, 2$, and $m = m_1 \vee m_2$. Then the Lie bracket of $\bar{X}_1$ and $\bar{X}_2$ is defined by*

(73)
$$[\bar{X}_1, \bar{X}_2] := \bar{X} \in V^\infty(\Omega)$$

where $\bar{X}$ is the vector field determined by generator X, and

$$X = [\bar{X}_1\big|_{\mathbb{R}^m}, \bar{X}_2\big|_{\mathbb{R}^m}].$$

Example 75. *Assume $\bar{X}, \bar{Y} \in V^\infty(\Omega)$, $\bar{X}$ and $\bar{Y}$ are generated by $X_0 \in V^\infty(\mathbb{R}^2)$ and $Y_0 \in V^\infty(\mathbb{R}^3)$, where*

(74)
$$\begin{aligned} X_0(x) &= [x_1 + x_2, x_2^2]^T, \\ Y_0(y) &= [y_1^2, 0, y_2 + y_3]^T. \end{aligned}$$

Then $m = 2 \vee 3 = 6$. On leaf $\mathbb{R}^6$, we have

$$X(z) := \Pi_6^2 X_0(\Pi_2^6 z) = [\alpha, \alpha, \alpha, \beta, \beta, \beta]^T,$$

where,

$$\alpha = \frac{1}{3}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6), \quad \beta = \frac{1}{9}(z_4 + z_5 + z_6).$$

$$Y(z) := \Pi_6^3 Y_0(\Pi_3^6 z) = [\gamma, \gamma, 0, 0, \mu, \mu]^T,$$

where,

$$\gamma = \frac{1}{4}(z_1 + z_2)^2, \quad \mu = \frac{1}{2}(z_3 + z_4 + z_5 + z_6).$$

Then $\bar{Z} := [\bar{X}, \bar{Y}]$, which is generated by $Z_0 \in V^\infty(\mathbb{R}^6)$, and

$$\begin{aligned} Z_0 &= [X, Y] = \frac{\partial Y}{\partial z} X - \frac{\partial X}{\partial z} Y \\ &= [a, a, b, c, d, d]^T. \end{aligned}$$

where

$$\begin{aligned}
a &= \tfrac{1}{3}(z_1 + z_2)(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) \\
&\quad - \tfrac{1}{6}(z_1 + z_2)^2 - \tfrac{1}{3}(z_3 + z_4 + z_5 + z_6) \\
b &= -\tfrac{1}{6}(z_1 + z_2)^2 - \tfrac{1}{3}(z_3 + z_4 + z_5 + z_6) \\
c &= -\tfrac{2}{9}(z_4 + z_5 + z_6)(z_3 + z_4 + z_5 + z_6) \\
d &= \tfrac{1}{6}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) + \tfrac{1}{6}(z_4 + z_5 + z_6)^2 \\
&\quad - \tfrac{2}{9}(z_4 + z_5 + z_6)(z_3 + z_4 + z_5 + z_6)
\end{aligned}$$

Definition 76. *(i) Distribution $\bar{D}(\bar{x}) \subset T_{\bar{x}}(\Omega)$, $\bar{x} \in \Omega$ is called an involutive distribution, if any two vector fields $\bar{X}, \bar{Y} \in \bar{D}$ satisfy $[\bar{X}, \bar{Y}] \in \bar{D}$.*

(ii) Let $\bar{X}_i$, $i \in [1, n]$ be given. The involutive distribution generated by $\{\bar{X}_i \mid i \in [1, n]\}$, or equivalently, the smallest involutive distribution containing $\{\bar{X}_i \mid i \in [1, n]\}$, is called the Lie algebra generated by $\{\bar{X}_i \mid i \in [1, n]\}$, denoted by $\left\langle \bar{X}_i \mid i \in [1, n] \right\rangle_{LA}$.

6. Dimension-free Riemannian manifolds

Parallel to the previous discussions, this section introduce the tensor fields and co-distributions on a DFES, and use them to define the Riemannian and symplectic structure over DFESs.

6.1. Co-vector fields and co-distributions on DFES

First, we define the cotangent space on DFES Ω.

Definition 77. *Let $\bar{x} \in \Omega$ and $\dim(\bar{x}) = m$. Then the cotangent space at $\bar{x}$, called the cotangent bundle at $\bar{x}$ and denoted by $T_{\bar{x}}^*(\Omega)$, is defined by*

$$\tag{75} T_{\bar{x}}^*(\Omega) := \mathcal{V}^{*[m,\cdot]}.$$

Remark 78. *When Ω is replaced by DFM, Definition 77 remains available.*

Similarly to tangent bundle, for a given $\bar{x} \in \Omega$, each leaf of its cotangent bundle is an Euclidean space. Moreover, the cotangent bundle at each point is a sub-lattice of $\mathbb{R}^\infty$. If Ω is replaced by a DFM, then the cotangent bundle is similar to tangent bundle. Hence, the Figure 2 may also be considered as a description of cotangent bundle of DFM.

Next, we define co-vector field on Ω. The following definition is also applicable to DFM.

Definition 79. *$\bar{\omega}$ is called a C^r co-vector field on Ω, denoted by $\bar{\omega} \in V^{*r}(\Omega)$, if it satisfies the following conditions:*

(i) *At each point $\bar{x} \in \Omega$, there exists a $p = p_{\bar{x}} = \mu_{\bar{x}} \dim(\bar{x})$, called the dimension of co-vector field $\bar{\omega}$ at $\bar{x}$, denoted by $\dim(\bar{\omega}_{\bar{x}})$, such that $\bar{\omega}$ assigns a p-upper sub-lattice of the bundle of coordinate neighborhood at $\bar{x}$ as $\mathcal{V}_O^{[p,\cdot]} = \{O^p, O^{2p}, \cdots\}$, and then a set of co-vectors $\omega^j \in T^*_{x_{j\mu}}(O^{jp})$, $j = 1, 2, \cdots$.*

(ii) *$\{\omega^j \mid j = 1, 2, \cdots\}$ satisfy consistent condition, that is,*

$$(76) \qquad \omega^j = \omega^1 \otimes \frac{1}{j} \mathbf{1}_j^T, \quad j = 1, 2, \cdots.$$

(iii) *On each leaf $O^{jp} \subset \mathbb{R}^{j\mu \dim(\bar{x})}$,*

$$(77) \qquad \bar{\omega}|_{O^{jp}} \in V^{*r}(O^{jp}).$$

Definition 80. *$\bar{\omega} \in V^{*r}(\Omega)$ is said to be dimension bounded, if*

$$(78) \qquad \max_{\bar{x} \in \Omega} \dim(\bar{\omega}_{\bar{x}}) < \infty.$$

Similarly to the vector field, the co-vector field can be constructed as follows: First, define it on a leaf $\mathcal{V}^m = \mathbb{R}^m$, then extend it to $T^*(\Omega) = \mathbb{R}^\infty$.

Algorithm 81.   • *Step 1: Assume there exists a smallest m, such that $\bar{\omega}$ is defined on $\mathbb{R}^m$. That is*

$$(79) \qquad \bar{\omega}|_{\mathbb{R}^m} := \omega \in V^{*r}(\mathbb{R}^m).$$

From constructing point of view, assume $\omega \in V^{*r}(\mathbb{R}^m)$, then $\bar{\omega}$ is defined on $\mathbb{R}^m$ as in (79).

• *Step 2: Extend ω to $T^*_{\bar{y}}$. Assume $\dim(\bar{y}) = s$, $m \vee s = t$, $t/m = \alpha$, $t/s = \beta$, and then $\dim(T_{\bar{y}}) = t$. Let $y \in \bar{y} \bigcap R^{[t,\cdot]}$, and $\dim(y) = kt$, $k = 1, 2, \cdots$. Define*

$$(80) \qquad \bar{\omega}(y) := \omega(\Pi_m^{kt} y)\Pi_m^{kt}, \quad k = 1, 2, \cdots.$$

Similarly to the case of vector fields, we can prove the following theorem:

Theorem 82.   (i) *The $\bar{\omega}$ constructed by Algorithm 81 is a C^r co-vector field, that is, $\bar{\omega} \in V^{*r}(\Omega)$.*

(ii) *If $\bar{\omega} \in V^{*r}(\Omega)$ is dimension bounded, then any $\bar{\omega}$ can be constructed through Algorithm 81.*

Similarly to vector fields, hereafter we consider only dimension bounded co-vector fields.

Similarly to vector fields, co-vector fields are defined on a sub-lattice of the bundle of coordinate neighborhood of a point $\bar{x} \in \Omega$. Assume $\omega = \bar{\omega}|_{\mathbb{R}^n}$ is the smallest generator of $\bar{\omega}$. Then, all the generators of $\bar{\omega}$ are $\omega_k = \bar{\omega}|_{\mathbb{R}^{kn}}$, $k = 1, 2, \cdots$. Now assume $\bar{x}_0 \in \Omega$ and $\dim(\bar{x}_0) = j$. Denote $j \vee n = s$, then,

$$\bar{x} \bigcap \mathbb{R}^\ell \neq \emptyset,$$

if and only if, $\ell = ks$, $k = 1, 2, \cdots$.

In fact, a co-vector field can also be considered as a function of vector field. Hence, the consistence of co-vector fields and vector fields is important. The following proposition shows this consistence.

Proposition 83. *Let $\bar{X} \in V^r(\Omega)$, $\bar{\omega} \in V^{*r}(\Omega)$, and $\dim(\bar{X}) = \dim(\bar{\omega})$. Then at any point $\bar{x} \in \Omega$ and the sub-lattice of the bundle of coordinate neighborhood of $\bar{x}$ where both $\bar{X}$ and $\bar{\omega}$ are well defined, the action of $\bar{\omega}$ on $\bar{X}$, denoted by $\bar{\omega}(\bar{X})$, is uniquely defined. That is, on $x_k = \bar{x} \bigcap \mathbb{R}^{kp}$, $k = 1, 2, \cdots$,*

$$(81) \qquad \bar{\omega}(\bar{X})|_{x_k} = const., \quad k = 1, 2, \cdots .$$

Proof. Denote $\dim x = s$, $\dim(\bar{X}) = \dim(\bar{\omega}) = m$. According to the previous argument, it is clear that the sub-lattice, where both $\bar{X}$ and $\bar{\omega}$ are defined, is $\{x_p, x_{2p}, \cdots\}$, where, $p = s \vee m$. To prove (81), it is enough to show that

$$(82) \qquad \bar{\omega}(\bar{X})|_{x_k} = \bar{\omega}(\bar{X})|_{x_1}, \quad k = 1, 2, \cdots .$$

Assume $p = rm$, then

$$\omega_1 = \omega\left(\Pi_m^{rm}(x_p)\right)\Pi_m^{rm}, \ \omega_k = \omega\left(\Pi_m^{rkm}(x_{kp})\right)\Pi_m^{rkm},$$
$$X_1 = \Pi_{rm}^m X\left(\Pi_m^{rm}(x_p)\right), \ X_k = \Pi_{rkm}^m X\left(\Pi_m^{rkm}(x_{kp})\right),$$

Using which we have

$$(83) \qquad \begin{aligned} \omega_1(X_1) &= \omega\left(\Pi_m^{rm}(x_p)\right)\Pi_m^{rm}\Pi_{rm}^m X\left(\Pi_m^{rm}(x_p)\right) \\ \omega_k(X_k) &= \omega\left(\Pi_m^{rkm}(x_{kp})\right)\Pi_m^{rkm}\Pi_{rkm}^m X\left(\Pi_m^{rkm}(x_{kp})\right) \end{aligned}$$

Since $x_p \leftrightarrow x_{kp}$, then $\Pi_m^{rm}(x_p) = \Pi_m^{rkm}(x_{kp})$. Hence, to prove (83) it is enough to show

$$\Pi_m^{rm}\Pi_{rm}^m = \Pi_m^{rkm}\Pi_{rkm}^m.$$

A straightforward computation shows

$$\Pi_m^{rm}\Pi_{rm}^m = \Pi_m^{rkm}\Pi_{rkm}^m = 1. \qquad \qquad \square$$

Co-vector field is also called one-form. Assume $\bar{f} \in C^r(\Omega)$, then on each leaf R^m $f^m := \bar{f}|_{\mathbb{R}^m}$ has its differential

$$(84) \qquad df^m = (\frac{\partial f^m}{\partial x_1}, \frac{\partial f^m}{\partial x_2}, \cdots, \frac{\partial f^m}{\partial x_m}).$$

Then one sees easily that

Proposition 84. *(84) generates a co-vector field.*

Proof. Taking df^m as the smallest generator of a co-vector field. Consider the differential of $\bar{f}$ on $\mathbb{R}^{km}$. Assume $y \in \mathbb{R}^{km}$, consider $f(\Pi_m^{km}y)$. A simple computation shows that

$$(85) \quad df(\Pi_m^{km}y) = \left(\frac{\partial f^m}{\partial x_1}|_{\Pi_m^{km}y}, \cdots, \frac{\partial f^m}{\partial x_m}|_{\Pi_m^{km}y} \right) \Pi_m^{km} = df(\Pi_m^{km}y)\Pi_m^{km}y.$$

This fact shows that the differential of $\bar{f}$ on leaf $\mathbb{R}^{km}$ is exactly the co-vector field deduced by df^m. $\qquad \square$

Definition 85. *A co-distribution $\bar{D}^*$ on Ω is a rule, which assigns at each point $\bar{x} \in \Omega$ a sub-lattice $O_j = O_{\bar{x}} \cap \mathbb{R}^{jrs}$, $r \in \mathbb{Z}_+$, $s = \dim(\bar{x})$, $j = 1, 2, \cdots,$ of its bundle of coordinate neighborhood $O_{\bar{x}}$, and a sub-space $D_j^*(x_{rj}) \subset T_{x_{rj}}^*(\mathbb{R}^{rjs})$ at $x_{jr} \in O_{jr}$. Moreover, this set of sub-spaces of $T_{x_{jr}}^*(\mathbb{R}^{jrs})$ satisfy the consistence condition, i.e.,*

$$(86) \qquad D_j^*(x_{rj}) = D_1^*(x_r) \otimes \mathbf{1}_j^T, \quad j = 1, 2, \cdots.$$

Similarly to distribution, a C^r co-distribution on Ω can be constructed as follows:

Algorithm 86. • *Step 1: Assume m is the smallest one, such that $\bar{D}^*$ is defined on $\mathbb{R}^m$. That is,*

$$\bar{D}^*|_{\mathbb{R}^m} := D^*(x) \subset T^{*r}(\mathbb{R}^m).$$

• *Step 2: Extend $D^*(x)$ to $T_{\bar{y}}^*(\Omega)$. Assume $\dim(\bar{y}) = s$, $m \vee s = t$, $t/m = \alpha$, $t/s = \beta$, and $\dim(T_{\bar{y}}^*) = t$. Let $y \in \bar{y} \bigcap R^{[t,\cdot]}$, and $\dim(y) = kt$, $k = 1, 2, \cdots$. Define*

$$D^*(y) := D^*(\Pi_m^{kt}y)\Pi_m^{kt}, \quad k = 1, 2, \cdots.$$

Similarly to distributions, the most important co-distributions are generated by a set of co-vector fields.

6.2. Tensor fields on quotient spaces

Let

$$\phi : \underbrace{V(\mathbb{R}^m) \times \cdots \times V(\mathbb{R}^m)}_{r} \times \underbrace{V^*(\mathbb{R}^m) \times \cdots \times V^*(\mathbb{R}^m)}_{s} \to \mathbb{R}$$

be an (r,s) th order tensor field on $\mathbb{R}^m$, where r is the covariant order and s is the contro-variant order. The set of (r,s) th order tensor fields is denoted by $T_s^r(\mathbb{R}^m)$. Let $\{e_1, e_2, \cdots, e_m\}$ be a basis of $V(\mathbb{R}^m)$, and $\{d_1, d_2, \cdots, d_m\}$ be a basis of $V^*(\mathbb{R}^m)$. Then

$$(87) \qquad \begin{aligned} \gamma_{j_1,j_2,\cdots,j_s}^{i_1,i_2,\cdots,i_r} \quad &:= \quad \phi\left(e_{i_1}, e_{i_2}, \cdots, e_{i_r}, d_{j_1}, d_{j_2}, \cdots, d_{j_s}\right), \\ &\quad 1 \le i_1, \cdots, i_r \le m, \ 1 \le j_1, \cdots, j_s \le m, \end{aligned}$$

are called the structure parameters of ϕ. Using structure parameters, the structure matrix is constructed as follows:

$$(88) \qquad \Gamma_\phi := \begin{bmatrix} \gamma_{11\cdots1}^{11\cdots1} & \cdots & \gamma_{11\cdots1}^{11\cdots m} & \cdots & \gamma_{11\cdots1}^{mm\cdots1} & \cdots & \gamma_{11\cdots1}^{mm\cdots m} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \gamma_{11\cdots m}^{11\cdots1} & \cdots & \gamma_{11\cdots m}^{11\cdots m} & \cdots & \gamma_{11\cdots m}^{mm\cdots1} & \cdots & \gamma_{11\cdots m}^{mm\cdots m} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \gamma_{mm\cdots1}^{11\cdots1} & \cdots & \gamma_{mm\cdots1}^{11\cdots m} & \cdots & \gamma_{mm\cdots1}^{mm\cdots1} & \cdots & \gamma_{mm\cdots1}^{mm\cdots m} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \gamma_{mm\cdots m}^{11\cdots1} & \cdots & \gamma_{mm\cdots m}^{11\cdots m} & \cdots & \gamma_{mm\cdots m}^{mm\cdots1} & \cdots & \gamma_{mm\cdots m}^{mm\cdots m} \end{bmatrix}$$

Using this structure matrix, we have the evaluation formula for ϕ as

$$(89) \qquad \phi(X_1, \cdots, X_r, \omega_1, \cdots, \omega_s) = \omega_s \cdots \omega_1 \Gamma_\phi X_1 \cdots X_r.$$

In the following we construct a tensor field on Ω, denoted by $\bar{\Xi} \in T_s^r(\Omega)$. Assume $\Xi \in T_s^r(\mathbb{R}^m)$ is the smallest generator of $\bar{\Xi} \in T_s^r(\Omega)$, and denote

$$\bar{\Xi}|_{\mathbb{R}^{km}} := \Xi_k.$$

Then it is enough to construct the structure matrix Ξ_k, $k = 1, 2, \cdots$.

It is clear that Ξ_k should satisfy the following requirement: for any $X_1, \cdots, X_r \in V^r(\mathbb{R}^m)$ and $\omega_1, \cdots, \omega_s \in V^{*r}(\mathbb{R}^m)$, their tensor value of Ξ

should be the same as the value of Ξ_k with its arguments as the projected vectors and co-vectors to $\mathbb{R}^{km}$. That is,

$$
\begin{aligned}
&\Xi(x)(X_1(x), \cdots, X_r(x), \omega_1(x), \cdots, \omega_s(x)) \\
(90)\quad =\ &\Xi_k(y)(\pi_{km}^k(X_1(x(y))), \cdots, \pi_{km}^k(X_r(x(y))), \\
&\pi_{km}^k(\omega_1(x(y))), \cdots, \pi_{km}^k(\omega_s(x(y)))),
\end{aligned}
$$

where $y = \Pi_{km}^m(x)$, $x(y) = \Pi_m^{km}(y)$.

Let $\Gamma(x)$ be the structure matrix of Ξ and $\Gamma_k(y)$ be the structure matrix of Ξ_k. Then (90) can be expressed as

$$
\begin{aligned}
&\omega_s(x) \cdots \omega_1(s) \Gamma(x) X_1(x) \cdots X_r(x) \\
=\ &\omega_s(\Pi_m^{km} y) \Pi_m^{km} \cdots \omega_1(\Pi_m^{km} y) \Pi_m^{km} \Gamma_k(y) \\
&\Pi_{km}^m X_1(\Pi_m^{km}(y)) \cdots \Pi_{km}^m X_r(\Pi_m^{km}(y)) \\
=\ &\omega_s(x) \cdots \omega_1(x) \left(I_{(s-1)m} \otimes \Pi_m^{km}\right) \cdots \\
&\left(I_m \otimes \Pi_m^{km}\right) \Pi_m^{km} \Gamma_k(y) \Pi_{km}^m \left(I_m \otimes \Pi_{km}^m\right) \cdots \\
&\left(I_{(r-1)m} \otimes \Pi_{km}^m\right) X_1(x) \cdots X_r(x).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
(91)\quad \Gamma(x)\ =\ &\left(I_{(s-1)m} \otimes \Pi_m^{km}\right) \cdots \\
&\left(I_m \otimes \Pi_m^{km}\right) \Pi_m^{km} \Gamma_k(y) P_{km}^m \left(I_m \otimes \Pi_{km}^m\right) \cdots \\
&\left(I_{(r-1)m} \otimes \Pi_{km}^m\right).
\end{aligned}
$$

It follows immediately that

$$
\begin{aligned}
(92)\quad \Gamma_k(y)\ :=\ &\Pi_{km}^m \left(I_m \otimes \Pi_{km}^m\right) \cdots \\
&\left(I_{(s-1)m} \otimes \Pi_{km}^m\right) \Gamma(\Pi_m^{km}(y)) \left(I_{(r-1)m} \otimes \Pi_m^{km}\right) \\
&\cdots \left(I_m \otimes \Pi_m^{km}\right) \Pi_m^{km}.
\end{aligned}
$$

A straightforward verification shows that the Γ_k defined by (92) satisfies (91).

Next, set $\bar{x} \in \Omega$, $\dim(\bar{x}) = s$, $s \vee m = p$, and let $p = \mu s = \lambda m$. Then, $\bar{\Xi}$ is defined at $\bar{x} \bigcap \mathbb{R}^{kp}$, $k = 1, 2, \cdots$. Denote $x_k = \bar{x} \bigcap \mathbb{R}^{kp}$, then

$$
\begin{aligned}
(93)\quad \bar{\Gamma}(x_k)\ :=\ &\Pi_{k\lambda m}^m \left(I_m \otimes \Pi_{k\lambda m}^m\right) \cdots \\
&\left(I_{(s-1)m} \otimes \Pi_{k\lambda m}^m\right) \Gamma(\Pi_m^{k\lambda m}(x_k)) \\
&\left(I_{(r-1)m} \otimes \Pi_m^{k\lambda m}\right) \cdots \left(I_m \otimes \Pi_m^{k\lambda m}\right) \Pi_m^{k\lambda m}.
\end{aligned}
$$

Example 87. *Assume $\bar{\bar{\Xi}} \in T_1^2(\Omega)$, and its smallest generator is $\Xi \in T_1^2(\mathbb{R}^2)$, which has its structure matrix as*

$$(94) \quad \Gamma(x) = \begin{bmatrix} 0 & \sin(x_1 + x_2) & 0 & \cos(x_1 + x_2) \\ -\cos(x_1 + x_2) & 0 & \sin(x_1 + x_2) & 0 \end{bmatrix}$$

(i) *Find the structure matrix of $\bar{\bar{\Xi}}|_{\mathbb{R}^4}$.*
   Using formula (92), we have

$$\begin{aligned} \bar{\bar{\Xi}}|_{\mathbb{R}^4} &= \Pi_4^2 \Gamma(\Pi_2^4(y)) \left(I_2 \otimes \Pi_2^4\right) \Pi_2^4 \\ &= \Pi_4^2 \Gamma([\tfrac{y_1+y_2}{2}, \tfrac{y_3+y_4}{2}]) \left(I_2 \otimes \Pi_2^4\right) \Pi_2^4 \\ &= \tfrac{1}{4} \begin{bmatrix} \mathbf{0} & S & \mathbf{0} & S & \mathbf{0} & C & \mathbf{0} & C \\ -C & \mathbf{0} & -C & \mathbf{0} & S & \mathbf{0} & S & \mathbf{0} \end{bmatrix}, \end{aligned}$$

where

$$\mathbf{0} = \left\{ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right\}$$

$$S = \left\{ \begin{array}{cc} \sin\left(\tfrac{y_1+y_2+y_3+y_4}{2}\right) & \sin\left(\tfrac{y_1+y_2+y_3+y_4}{2}\right) \\ \sin\left(\tfrac{y_1+y_2+y_3+y_4}{2}\right) & \sin\left(\tfrac{y_1+y_2+y_3+y_4}{2}\right) \end{array} \right\}$$

$$C = \left\{ \begin{array}{cc} \cos\left(\tfrac{y_1+y_2+y_3+y_4}{2}\right) & \cos\left(\tfrac{y_1+y_2+y_3+y_4}{2}\right) \\ \cos\left(\tfrac{y_1+y_2+y_3+y_4}{2}\right) & \cos\left(\tfrac{y_1+y_2+y_3+y_4}{2}\right) \end{array} \right\}$$

(ii) *Assume $\bar{x} \in \Omega$ and $\dim(\bar{z}) = 3$. Then $\bar{\bar{\Xi}}$ is defined only on z_{kp}, $p = 2 \vee 3 = 6$, $k = 1, 2, \cdots$. Using formula (93), we have*

$$\begin{aligned} \bar{\bar{\Xi}}|_{z_2} &= \Pi_6^2 \Gamma(\Pi_2^6(z_2)) \left(I_2 \otimes \Pi_2^6\right) \Pi_2^6 \\ &= \Pi_6^2 \Gamma([\tfrac{2z_1+z_2}{3}, \tfrac{z_+2z_3}{3}]) \left(I_2 \otimes \Pi_2^6\right) \Pi_2^6 \\ &= \tfrac{1}{9}[A, A, A, B, B, B] \otimes \mathbf{1}_{3\times 3}, \end{aligned}$$

where

$$A = \left\{ \begin{array}{cc} 0 & \sin\left(\tfrac{2(z_1+z_2+z_3)}{3}\right) \\ \cos\left(\tfrac{2(z_1+z_2+z_3)}{3}\right) & 0 \end{array} \right\}$$

$$B = \left\{ \begin{array}{cc} 0 & \cos\left(\tfrac{2(z_1+z_2+z_3)}{3}\right) \\ \sin\left(\tfrac{2(z_1+z_2+z_3)}{3}\right) & 0 \end{array} \right\}$$

Remark 88. (i) *The above constructing technique is applicable to tensor fields on DFMs. Hence, we assume tensor fields on DFMs are also properly defined.*

(ii) *A tensor field T_s^0 is called a covariant tensor field. A tensor field T_0^r is also called an r-form.*

6.3. Dimension-free Riemannian manifolds and dimension-free symplectic manifolds

Definition 89. *Let $\Xi \in T_0^r(\Omega)$ be an r th order covariant tensor field.*

(i) *Ξ is said to be symmetric, if*

$$(95) \qquad \Xi(X_1, \cdots, X_r) = \Xi(X_{\sigma(1)}, \cdots, X_{\sigma(r)}), \quad \sigma \in \mathbf{S}_r.$$

(ii) *Ξ is said to be skew-symmetric, if*

$$(96) \qquad \Xi(X_1, \cdots, X_r) = sign(\sigma)\Xi(X_{\sigma(1)}, \cdots, X_{\sigma(r)}), \quad \sigma \in \mathbf{S}_r.$$

Second order covariant tensor fields (or 2-forms) are of special importance. Their structure matrices can also be expressed into a quadratic form as

$$(97) \qquad M_\Xi = \begin{bmatrix} \gamma^{11} & \gamma^{12} & \cdots & \gamma^{1n} \\ \gamma^{21} & \gamma^{22} & \cdots & \gamma^{2n} \\ \vdots & & & \\ \gamma^{n1} & \gamma^{n2} & \cdots & \gamma^{nn} \end{bmatrix}$$

Using this structure matrix, the tensor field can be the expressed into a classical quadratic form as

$$(98) \qquad \Xi(X_1, X_2) = X_1^T M_\Xi X_2.$$

Then we have

Proposition 90. *$\bar{\Xi} \in T^2(\Omega)$ is symmetric (skew-symmetric), if and only if, its smallest generator Ξ is symmetric (skew-symmetric). That is, it has a symmetric (skew-symmetric) structure matrix.*

Definition 91. *Consider Ω. Assume there is an order 2 covariant tensor field $\bar{\Xi} \in T^2(\Omega)$.*

(i) *$(\Omega, \bar{\Xi})$ is called a dimension-free Riemannian manifold, if $\bar{\Xi}$ is generated by $\Xi_n = \bar{\Xi}|_{\mathbb{R}^n}$ and $(\mathbb{R}^n, \Xi_n)$ is a Riemannian manifold. That is, $\Xi_n := \bar{\Xi}|_{\mathbb{R}^n} \in T^2(\mathbb{R}^n)$ has a symmetric positive definite structure matrix M_{Ξ_n}.*

(ii) $(\Omega, \bar{\bar{\Xi}})$ *is called a dimension-free symplectic manifold, if $\bar{\bar{\Xi}}$ is generated by $\Xi_n = \bar{\bar{\Xi}}|_{\mathbb{R}^n}$, where $n = 2m$ is even, and $(\mathbb{R}^n, \Xi_n)$ is a symplectic manifold. That is, $\Xi_{2m} := \bar{\bar{\Xi}}|_{\mathbb{R}^{2m}} \in T^2(\mathbb{R}^{2m})$ has a skew-symmetric, non-singular structure matrix $M_{\Xi_{2m}}$, and Ξ is closed.*

Remark 92. *An r form Ξ is closed if $d\Xi = 0$ [2]. Let $\Xi \in T^2(\mathbb{R}^n)$. Ξ is closed, if and only if, its structure coefficients satisfy [13]*

$$(99) \qquad \frac{\partial}{\partial x_i}(\gamma^{jk}) + \frac{\partial}{\partial x_j}(\gamma^{ki}) + \frac{\partial}{\partial x_k}(\gamma^{ij}) = 0, \quad 1 \le i, j, k \le n.$$

Definition 93. *(i) A DFM M with a two-form $\bar{\Theta}$ is called a dimension-free Riemannian manifold, if there exists an open sub-bundle cover of M such that each open sub-bundle is bundle isomorphic to an open sub-bundle of a Riemannian manifold of Ω with Riemannian two-form $\bar{\bar{\Xi}}$. Moreover, $\bar{\Theta}$ is isomorphic consistently to $\bar{\bar{\Xi}}$.*

(ii) A DFM M with a two-form $\bar{\Theta}$ is called a dimension-free symplectic manifold, if there exists an open sub-bundle cover of M such that each open sub-bundle is bundle isomorphic to an open sub-bundle of a symplectic manifold of Ω with symplectic two-form $\bar{\bar{\Xi}}$. Moreover, $\bar{\Theta}$ is isomorphic consistently to $\bar{\bar{\Xi}}$.

Remark 94. *Let M_1 and M_2 be two DFMs, and $\Psi : M_1 \to M_2$ be an isomorphism. $\bar{\bar{\Xi}}_i \in T^r_s(M_i)$, $i = 1, 2$ are said to be isomorphic consistently, if*

$$(100) \qquad \begin{aligned} &\bar{\bar{\Xi}}_1(X_1, \cdots, X_r; \omega_1, \cdots, \omega_s) \\ &= \bar{\bar{\Xi}}_2(\Psi_*(X_1), \cdots, \Psi_*(X_r), \Psi^{-1^*}(\omega_1), \cdots, \Psi^{-1^*}(\omega_s)), \\ &\qquad X_1, \cdots, X_r \in V(M_1), \ \omega_1, \cdots, \omega_s \in V^*(M_1). \end{aligned}$$

Example 95. *Consider DEES Ω.*

(i) Assume Ω has a two form $\bar{\sigma} \in T^2(\Omega)$ with its generator $\sigma \in T^2(\mathbb{R}^2)$, and the structure matrix of σ is $M_\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Since M_σ is a symplectic matrix, it is clear that $(\Omega, \bar{\sigma})$ is a dimension-free symplectic manifold.

Assume $\bar{x} \in \Omega$ and $\dim(\bar{x}) = 2$. Then $\bar{\sigma}$ is defined on $T_{\bar{x}} = \{\mathbb{R}^{2k} \mid k = 1, 2, \cdots\}$. Moreover, the structure matrix of $\bar{\sigma}(x_k)$ is

$$M_k := M|_{x_k} = \Pi^2_{2k} M_\sigma \Pi^{2k}_2 = M_\sigma \otimes I_k.$$

Assume $\bar{y} \in \Omega$ and $\dim(\bar{y}) = 3$. Then $\bar{\sigma}$ is defined on $T_{\bar{y}} = \{\mathbb{R}^{6k} \mid k = 1, 2, \cdots\}$. Moreover, the structure matrix of $\bar{\sigma}(y_{2k})$ is

$$M_k := M|_{y_{2k}} = \Pi_{6k}^2 M_\sigma \Pi_2^{6k} = M_\sigma \otimes I_{3k}.$$

(ii) *Assume Ω has a two form $\bar{\omega} \in T^2(\Omega)$, where ω is deduced from part of sphere $S_2 \backslash P_2$. We refer to Example 57 for notations. Assume S_2 has the standard distance inherited from $\mathbb{R}^3$. Then the structure matrix of ω is*

$$M_\omega = (\frac{\partial x}{\partial \xi})^T I_3 (\frac{\partial x}{\partial \xi}) = \begin{bmatrix} \left\|\frac{\partial x}{\partial \xi_1}\right\|^2 & \left\langle \frac{\partial x}{\partial \xi_1}, \frac{\partial x}{\partial \xi_2} \right\rangle \\ \left\langle \frac{\partial x}{\partial \xi_1}, \frac{\partial x}{\partial \xi_2} \right\rangle & \left\|\frac{\partial x}{\partial \xi_2}\right\|^2 \end{bmatrix},$$

where

$$\left\|\frac{\partial x}{\partial \xi_1}\right\|^2 = \left(\frac{\partial x_1}{\partial \xi_1}\right)^2 + \left(\frac{\partial x_2}{\partial \xi_1}\right)^2 + \left(\frac{\partial x_3}{\partial \xi_1}\right)^2$$
$$\left\langle \frac{\partial x_1}{\partial \xi}, \frac{\partial x_2}{\partial \xi} \right\rangle = \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_1}{\partial \xi_2} + \frac{\partial x_2}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} + \frac{\partial x_3}{\partial \xi_1}\frac{\partial x_3}{\partial \xi_2}$$
$$\left\|\frac{\partial x}{\partial \xi_2}\right\|^2 = \left(\frac{\partial x_1}{\partial \xi_2}\right)^2 + \left(\frac{\partial x_2}{\partial \xi_2}\right)^2 + \left(\frac{\partial x_3}{\partial \xi_2}\right)^2,$$

and

$$\frac{\partial x_1}{\partial \xi_1} = \frac{2 - 2\xi_1^2 + 2\xi_2^2}{(1 + \xi_1^2 + \xi_2^2)^2}, \quad \frac{\partial x_1}{\partial \xi_2} = \frac{-4\xi_1\xi_2}{(1 + \xi_1^2 + \xi_2^2)^2}$$
$$\frac{\partial x_2}{\partial \xi_1} = \frac{-4\xi_1\xi_2}{(1 + \xi_1^2 + \xi_2^2)^2}, \quad \frac{\partial x_2}{\partial \xi_2} = \frac{2 + 2\xi_1^2 - 2\xi_2^2}{(1 + \xi_1^2 + \xi_2^2)^2}$$
$$\frac{\partial x_3}{\partial \xi_1} = \frac{-4\xi_1}{(1 + \xi_1^2 + \xi_2^2)^2}, \quad \frac{\partial x_3}{\partial \xi_2} = \frac{-4\xi_2}{(1 + \xi_1^2 + \xi_2^2)^2}$$

Hence, $(\Omega, \bar{\omega})$ is a dimension-free Riemannian Manifold (DFRM).
Assume Θ has a two form $\bar{\theta} \in T^2(\Theta)$, where θ is deduced from part of sphere $S_2 \backslash Q_2$. Assume S_2 has the standard distance inherited from $\mathbb{R}^3$. Then the structure matrix of Θ is

$$M_\theta = (\frac{\partial x}{\partial \eta})^T I_3 (\frac{\partial x}{\partial \eta}) = \begin{bmatrix} \left\|\frac{\partial x}{\partial \eta_1}\right\|^2 & \left\langle \frac{\partial x}{\partial \eta_1}, \frac{\partial x}{\partial \eta_2} \right\rangle \\ \left\langle \frac{\partial x}{\partial \eta_1}, \frac{\partial x}{\partial \eta_2} \right\rangle & \left\|\frac{\partial x}{\partial \eta_2}\right\|^2 \end{bmatrix},$$

where

$$\left\|\frac{\partial x}{\partial \eta_1}\right\|^2 = \left(\frac{\partial x_1}{\partial \eta_1}\right)^2 + \left(\frac{\partial x_2}{\partial \eta_1}\right)^2 + \left(\frac{\partial x_3}{\partial \eta_1}\right)^2$$
$$\left\langle \frac{\partial x_1}{\partial \eta}, \frac{\partial x_2}{\partial \eta} \right\rangle = \frac{\partial x_1}{\partial \eta_1}\frac{\partial x_1}{\partial \eta_2} + \frac{\partial x_2}{\partial \eta_1}\frac{\partial x_2}{\partial \eta_2} + \frac{\partial x_3}{\partial \eta_1}\frac{\partial x_3}{\partial \eta_2}$$
$$\left\|\frac{\partial x}{\partial \eta_2}\right\|^2 = \left(\frac{\partial x_1}{\partial \eta_2}\right)^2 + \left(\frac{\partial x_2}{\partial \eta_2}\right)^2 + \left(\frac{\partial x_3}{\partial \eta_2}\right)^2,$$

and

$$\begin{array}{ll}
\frac{\partial x_1}{\partial \eta_1} = \frac{2-2\eta_1^2+2\eta_2^2}{(1+\eta_1^2+\eta_2^2)^2}, & \frac{\partial x_1}{\partial \eta_2} = \frac{-4\eta_1\eta_2}{(1+\eta_1^2+\eta_2^2)^2} \\[2mm]
\frac{\partial x_2}{\partial \eta_1} = \frac{-4\eta_1\eta_2}{(1+\eta_1^2+\eta_2^2)^2}, & \frac{\partial x_2}{\partial \eta_2} = \frac{2+2\eta_1^2-2\eta_2^2}{(1+\eta_1^2+\eta_2^2)^2} \\[2mm]
\frac{\partial x_3}{\partial \eta_1} = \frac{-4\eta_1}{(1+\eta_1^2+\eta_2^2)^2}, & \frac{\partial x_3}{\partial \eta_2} = \frac{-4\eta_2}{(1+\eta_1^2+\eta_2^2)^2}
\end{array}$$

Hence, $(\Theta, \bar{\theta})$ is also a DFRM.
Combining Ω with Θ, one sees that S_∞ is a DFRM too.

7. Dimension-varying dynamic (control) systems

7.1. Dynamic (control) systems over DFMs

7.1.1. Projection of dynamic (control) systems   Consider a dynamic system over $\mathbb{R}^p$, described as

(101) $$\Sigma : \ \dot{x} = F(x), \quad x \in \mathbb{R}^p.$$

Definition 96. *Consider dynamic system (101). Its projection onto $\mathbb{R}^q$ is a dynamic system over $\mathbb{R}^q$, described as*

(102) $$\pi_q^p(\Sigma) : \ \dot{z} = \tilde{F}(z), \quad z \in \mathbb{R}^q,$$

where

(103) $$\tilde{F}(z) = \Pi_q^p F(\Pi_p^q(z)).$$

Consider a control system

(104) $$\Sigma^C : \ \dot{x} = F(x, u), \quad x \in \mathbb{R}^p, \ u \in \mathbb{R}^r.$$

Definition 97. *Consider control system (104). The $u = u_1, \cdots, u_r$ can be considered as parameters. Then its projection to R^q can still be considered as a projection of vector field as*

(105) $$\pi_q^p(\Sigma^C) : \ \dot{z} = \tilde{F}(z, u), \quad z \in \mathbb{R}^q, \ u \in \mathbb{R}^r,$$

where

(106) $$\tilde{F}(z, u) = \Pi_q^p F(\Pi_p^q(z), u).$$

Remark 98. *The projection from $\mathbb{R}^p$ to $\mathbb{R}^q$ can be extended to a projection from p dimensional manifold to q dimensional manifold. Then the above descriptions can be considered as the expression over local coordinate charts.*

The following is an example.

Example 99. *Consider the following control system Σ:*

$$(107) \qquad \begin{cases} \dot{x}_1 = u_1 \sin(x_1 + x_2), \\ \dot{x}_2 = u_2 \cos(x_1 + x_2). \end{cases}$$

(i) *Project ([107](#)) onto $\mathbb{R}^3$. It is ready to calculate that*

$$\Pi_2^3 = \frac{1}{3} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad \Pi_3^2 = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then the projected system $\pi_3^2(\Sigma)$ is calculated as

$$(108)$$
$$\begin{cases} \dot{z}_1 = u_1 \sin(\frac{2}{3}(z_1 + z_2 + z_3)), \\ \dot{z}_2 = \frac{1}{2} \left(u_1 \sin(\frac{2}{3}(z_1 + z_2 + z_3)) + u_2 \cos(\frac{2}{3}(z_1 + z_2 + z_3)) \right), \\ \dot{z}_3 = u_2 \cos(\frac{2}{3}(z_1 + z_2 + z_3)). \end{cases}$$

(ii) *Project ([107](#)) onto $\mathbb{R}^4$. We have*

$$\Pi_2^4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \Pi_4^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then the projected system $\pi_4^2(\Sigma)$ is easily obtained as

$$(109) \qquad \begin{cases} \dot{z}_1 = u_1 \sin(\frac{1}{2}(z_1 + z_2 + z_3 + z_4)), \\ \dot{z}_2 = u_1 \sin(\frac{1}{2}(z_1 + z_2 + z_3 + z_4)), \\ \dot{z}_3 = u_2 \cos(\frac{1}{2}(z_1 + z_2 + z_3 + z_4)), \\ \dot{z}_4 = u_2 \cos(\frac{1}{2}(z_1 + z_2 + z_3 + z_4)). \end{cases}$$

(iii) Project ([108](#)) (i.e., $\pi_3^2(\Sigma)$) back to $\mathbb{R}^2$, we have

$$(110) \qquad \begin{cases} \dot{x}_1 = \frac{1}{6}\left(5u_1\sin(x_1+x_2)+\cos(x_1+x_2)\right), \\ \dot{x}_2 = \frac{1}{6}\left(u_1\sin(x_1+x_2)+5\cos(x_1+x_2)\right). \end{cases}$$

System ([110](#)) differs from the original system, which means the transfer loses information.

(iii) Project ([109](#)) (i.e., $\pi_4^2(\Sigma)$) back to $\mathbb{R}^2$, we have Σ, which means the transfer is lossless.

Motivated by the above example, we can prove the following result.

Proposition 100. *Let $f(x) \in V^\infty(\mathbb{R}^p)$ and $q = kp$. Then*

$$(111) \qquad \pi_p^q \circ \pi_q^p(f(x)) = f(x).$$

Proof. First, a straightforward computation can prove the following equality:

$$(112) \qquad \Pi_p^{kp}\Pi_{kp}^p = I_p.$$

Using it, we have that

$$\begin{aligned} f(x) \ &\xrightarrow{\ \pi_{kp}^p\ } \ \Pi_{kp}^p f\left(\Pi_p^{kp}z\right) \\ &\xrightarrow{\ \pi_p^{kp}\ } \ \Pi_p^{kp}\Pi_{kp}^p f\left(\Pi_p^{kp}\Pi_{kp}^p x\right) = f(x). \end{aligned} \qquad \square$$

Remark 101. *(i) Proposition [100](#) shows that when a vector field is projected onto its multiple-dimension Euclidean space there is no information lost. This is essential for constructing a control system on dimension-free manifolds.*

(ii) In previous sections, according to the definition of a vector field on $\mathbb{R}^\infty$, for a vector field on $\mathbb{R}^p$ only its integral curves over $\mathbb{R}^{kp}$ are considered. That means only the projection of the vector field to $\mathbb{R}^{kp}$ are considered. In current definition, the projection to any $\mathbb{R}^s$ is allowed. In fact, only when $s = kp$, the extension is lossless. When $s \neq kp$, the projected system can only be considered as an approximated system of the original one. Its integral curve can not be considered as the integral curve of the original system, but only an approximation too.

7.1.2. Nonlinear control systems over Ω   To avoid counting the degrees of differentiability, the functions, vector fields, etc. are assumed to be of C^∞.

Definition 102.   *(i) A nonlinear control system over Ω, denoted by $\bar{\Sigma}$, is described by*

$$(113) \qquad \begin{cases} \dot{\bar{x}} = \overline{F}(\bar{x}, u) \\ \bar{y}_s = \bar{h}_s(\bar{x}), \quad s \in [1, p], \end{cases}$$

where $\overline{F}(\bar{x}, u) \in V^\infty(\Omega)$, $\bar{h}_s \in C^\infty(\Omega)$, $s \in [1, p]$, $u = (u_1, u_2, \cdots, u_m)$ are controls, which can be considered as parameters in F. $\bar{y}_s$, $s \in [1, m]$ are outputs.
(ii) Let $\bar{f}$, $\bar{g}_j$, $j \in [1, m] \in V^\infty(\Omega)$,

$$(114) \qquad \begin{cases} \overline{F}(\bar{x}, u) = \bar{f}(\bar{x}) + \sum_{j=1}^{m} \bar{g}_j(\bar{x}) u_j, \\ \bar{y}_s = \bar{h}_s(\bar{x}), \quad s \in [1, p]. \end{cases}$$

Then (114) is called an affine nonlinear control system over Ω.
(iii) Assume

$$q := \mathrm{lcm}\left(\dim(\overline{F}(u)), \ \dim(\bar{h}_s), \ s \in [1, p]\right).$$

Then $\bar{\Sigma}|_{\mathbb{R}^q} := \Sigma$ is called the minimum generator of $\bar{\Sigma}$, denoted by

$$(115) \qquad \begin{cases} \dot{x} = F(x, u), \quad x \in \mathbb{R}^q \\ y_s = h_s(x), \quad s \in [1, p]. \end{cases}$$

(iv) $\bar{\Sigma}$ is said to be completely controllable (observable), if Σ is completely controllable (observable).

Remark 103.   *(i) If the state space of minimum generator is on $\mathbb{R}^q$, then, $\bar{\Sigma}$ is well posed on $\mathbb{R}^{kq}$, $k = 1, 2, \cdots$. They will be called the realizations of $\overline{F}(u)$. Unfortunately, the control properties, such as controllability, observability, etc., of the realizations with different dimensions are not the same. Hence the controllability and observability of $\bar{\Sigma}$ are defined by corresponding properties of its minimum generator.*
(ii) Hereafter all the control properties of $\bar{\Sigma}$ are referred to the corresponding properties of its minimum generator.

7.2. Linear systems over DFMs

- Linear vector fields

Let $\bar{X} \in V^{\infty}(\Omega)$ be a linear vector field and $\dim(\bar{X}) = m$. Then there exists $A \in \mathcal{M}_{m \times m}$ such that $X := \bar{X}|_{\mathbb{R}^m} = Ax$. Consider $\bar{X}|_{\mathbb{R}^{km}}$: Let $y \in \mathbb{R}^{km}$. Then

$$(116) \qquad X_k := \bar{X}(y) = \Pi_{km}^m \left(X \left(\Pi_m^{km}(y) \right) \right) = \Pi_{km}^m A \Pi_m^{km} y := A_k y,$$

where,

$$(117) \qquad A_k = \Pi_{km}^m A \Pi_m^{km} = \frac{1}{k} \left(I_m \otimes \mathbf{1}_k \right) A \left(I_m \otimes \mathbf{1}_k^{\mathrm{T}} \right).$$

Then we consider the integral curve of $\bar{X}$.

Assume $\bar{X} \in V^{\infty}(\Omega)$ is a linear vector field and $\dim(\bar{X}) = m$. $X := \bar{X}|_{\mathbb{R}^m} = Ax$. Consider $\bar{x}^0 \in \Omega$.

Case 1: Assume $\dim(\bar{x}^0) = m$. Then the integral curve of $\bar{X}$ with initial value $\bar{x}^0$ is defined only on a filter of the tangent bundle

$$T_{\bar{x}^0} \bigcap \mathbb{R}^{km}, \quad k = 1, 2, \cdots.$$

On $T_{\bar{x}^0} \bigcap \mathbb{R}^{km}$, at $x_k^0 = x_1^0 \otimes \mathbf{1}_k$, the vector field is determined by (117). Then the integral curve becomes

$$\begin{aligned}
x_k(t, x_k^0) &= e^{X_k t} x_k^0 \\
&= \left(I_{km} + t(I_m \otimes \mathbf{1}_k) A (I_m \otimes \mathbf{1}_k^{\mathrm{T}}) + \right. \\
&\quad \left. \frac{t^2}{2!} (I_m \otimes \mathbf{1}_k) A^2 (I_m \otimes \mathbf{1}_k^{\mathrm{T}}) + \cdots \right) (x_1^0 \otimes \mathbf{1}_k) \\
&= \frac{1}{k} (I_m \otimes \mathbf{1}_k) e^{At} (I_m \otimes \mathbf{1}_k^{\mathrm{T}}) (x_1^0 \otimes \mathbf{1}_k) \\
&= (I_m \otimes \mathbf{1}_k) e^{At} x_0 \\
&= e^{At} x_0 \otimes \mathbf{1}_k.
\end{aligned}$$

Case 2: Assume $\dim(\bar{x}^0) = s$, $m \vee s = p = km = rs$. Then the integral curve of $\bar{X}$ with initial value $\bar{x}^0$ is defined on a filter of its tangent bundle

$$T_{\bar{x}^0)} \bigcap \mathbb{R}^{jp}, \quad j = 1, 2, \cdots.$$

On leaf $T_{\bar{z}^0} \bigcap \mathbb{R}^p$, the initial value $z_r^0 = z_1^0 \otimes \mathbf{1}_r$, where $z_1^0 \in \bar{z}^0$ is its smallest element. The vector field is $A_k z$, where A_k is determined by (117). Hence, the integral curve is

$$z_r(t, z_r^0) = e^{X_k t} z_r^0 = \frac{1}{k}(I_m \otimes \mathbf{1}_k) e^{At} (I_m \otimes J_k^{\mathrm{T}})(z_1^0 \otimes \mathbf{1}_r).$$

On leaf $T_{\bar{z}^0} \bigcap \mathbb{R}^{jp}$, the integral curve with initial value $z_{jr}^0 = z_1^0 \otimes \mathbf{1}_{jr}$ is

$$z_{jr}(t, z_{jr}^0) = e^{X_{jk} t} z_{jr}^0 = \frac{1}{k}(I_m \otimes \mathbf{1}_k) e^{At} (I_m \otimes J_k^{\mathrm{T}})(z_1^0 \otimes \mathbf{1}_{jr}).$$

Summarizing the above argument, we have the following result.

Proposition 104. *Let $\bar{X} \in V^\infty(\Omega)$ be a linear vector field, and $\dim(\bar{X}) = m$. $X := \bar{X}|_{\mathbb{R}^m} = Ax$. Assume $\bar{x}^0 \in \Omega$, $\dim(\bar{x}^0) = s$.*

(i) If $s = m$, then the integral curve of $\bar{X}|_{\mathbb{R}^m}$ is

$$\Phi_t^X(x_1^0) = e^{Xt} x_1^0.$$

Hence, the integral curve of $\bar{X}|_{\mathbb{R}^{rm}}$ becomes

$$\Phi_t^{X_r}(x_r^0) = \left[e^{Xt} x_1^0 \right] \otimes \mathbf{1}_r.$$

Finally the integral curve of $\bar{X}$ with initial value $\bar{x}^0$ is $\overline{\Phi_t^X(x_1^0)} \subset \Omega$.

(ii) If $s = km$, then the integral curve of $\bar{X}|_{\mathbb{R}^{km}}$ is

$$\Phi_t^{X_k}(x_1^0) = e^{X_k t} x_1^0,$$

where, X_k is determined by (116). Hence the integral curve of $\bar{X}$ with initial value $\bar{x}^0$ is $\overline{\Phi_t^{X_k}(x_1^0)} \subset \Omega$.

(iii) If $m \vee s = p = km = rs$, then the integral curve of $\bar{X}|_{\mathbb{R}^p}$ is

$$\Phi_t^{X_k}(x_r^0) = e^{X_k t}(x_1^0 \otimes I_s).$$

Hence, the integral curve of $\bar{X}$ with initial value $\bar{x}^0$ is $\overline{\Phi_t^{X_k}(x_1^0 \otimes I_s)} \subset \Omega$.

- Linear control systems

First, we consider the relationship among equivalent matrices, equivalent vectors, and linear vector fields.

Definition 105. *Let A, $B \in \mathcal{M}$.*

(i) *The matrices A and B are said to be type 1 left equivalent, denoted by $A \sim B$, if there exist I_α and I_β, such that $A \otimes I_\alpha = B \otimes I_\beta$. The equivalence class of A is denoted by $\langle A \rangle_\ell$.*

(iii) *The matrices A and B are said to be type 2 left equivalent, denoted by $A \approx B$, if there exist J_α and J_β, such that $A \otimes J_\alpha = B \otimes J_\beta$, where $J_i := \frac{1}{i}\mathbf{1}_{i \times i}$, $i = 1, \cdots$. The equivalence class of A is denoted by $\langle\langle A \rangle\rangle_\ell$.*

More general notions on matrix equivalence can be found in [8]. With these concepts we can define the linear vector fields over Ω.

Proposition 106. *Let $\bar{X} \in V^\infty(\Omega)$ be a linear vector field and $\dim(\bar{X}) = m$. $X := \bar{X}|_{\mathbb{R}^m} = Ax$. Assume $\bar{x}^0 \in \Omega$, $\dim(\bar{x}^0) = s$. $m \vee s = p = km = rs$. Then $\bar{X}$ is defined only on the filter of its tangent bundle*

$$x_{jr}^0 = T_{\bar{x}^0} \bigcap \mathbb{R}^{jp}, \quad j = 1, 2, \cdots.$$

Moreover, on the leaf containing x_r^0 it is $\bar{X}(x_r^0) = A_k x_r^0$, where A_k is determined by (116). On the leaf containing x_{jr}^0 it is $\bar{X}(x_r^0) = A_{jk} x_{jr}^0$, $j = 1, 2, \cdots$, where the two sets of consistent matrices are

$$(118) \qquad A_{jk} = A_k \otimes I_j \sim A_k,$$

$$(119) \qquad A_{jk} = A_k \otimes J_j \approx A_k,$$

respectively. The available variables are

$$(120) \qquad x_{jr}^0 = x_r^0 \otimes \mathbf{1}_j \leftrightarrow x_r^0.$$

Proof. In fact, what do we need to show is that the tangent vectors on the bundle leaves are consistent. That is,

$$(121) \qquad \bar{X}(x_{jr}^0) = \bar{X}(x_r^0) \otimes \mathbf{1}_j, \quad j = 1, 2, \cdots.$$

It is obvious that (118) together with (120), or (119) together with (120) can ensure (121) to be true. $\square$

Next, we consider the linear control system on Ω. Recall a classical linear system [18]

$$(122) \qquad \begin{cases} \dot{x} = Ax + \sum\limits_{i=1}^{m} b_i u_i, \\ y = Cx, \quad x \in \mathbb{R}^n, \ y \in \mathbb{R}^p. \end{cases}$$

One sees that a classical linear control system consists of three ingredients: linear vector field Ax, a set of constant vector fields $B = \{b_1, \cdots, b_m\}$, and linear function Cx. To extend a classical linear control system to Ω, it is enough to create these three kind of objects to Ω. The key of this extension is to make them consistent at each $\bar{x} \in \Omega$.

(i) Linear Vector Field: Assume the smallest generator of linear vector field $\bar{X}$ is $X = Ax \in V^\infty(\mathbb{R}^m)$. $\dim(\bar{x}^0) = s$, $m \vee s = p = \mu m = rs$. Then according to the argument in previous subsection, we know

$$(123) \qquad \bar{X} \bigcap T_{\bar{x}^0} = \{\bar{X}(x_{jr}) \mid j = 1, 2, \cdots\}.$$

Moreover,

$$(124) \qquad \bar{X}(x_{jr}) = A_{j\mu} x_{jr}, \quad j = 1, 2, \cdots,$$

where, $A_{j\mu}$ is defined by (117) with $k = j\mu$.

(ii) Constant Vector Field: Assume the smallest generator of the constant vector field $\bar{X}$ is $X = b \in V^\infty(\mathbb{R}^m)$, where $\dim(\bar{x}^0) = s$, $m \vee s = p = \mu m = rs$, that is, (123) holds, and

$$\bar{X}(x_{jr}) = \Pi^m_{j\mu m} X(\Pi^{j\mu m}_m x_{jr}) = \Pi^m_{j\mu m} b = b \otimes \mathbf{1}_{j\mu}.$$

(iii) Linear Function: Let $\bar{h} \in C^\infty(\Omega)$. $\dim(\bar{x}^0) = m$, $\bar{h}$ is expressed at x_1^0 as $hx = c_m x$, where $c_m^{\mathrm{T}} \in R^m$. Let $\bar{z} \in \Omega$, $\dim(\bar{z}) = s$, $m \vee s = p = rs = \mu m$. Then $\bar{h}$ is expressed at z_1 as

$$\bar{h}(z_1) = \bar{h}(\Pi^s_m z_1) = \frac{1}{\mu} c_m \left(I_m \otimes \mathbf{1}_\mu^{\mathrm{T}} \right) \left(I_s \otimes \mathbf{1}_r \right) z_1$$

Hence $\bar{h}$ can be expressed on leaf $\mathbb{R}^s$ as

$$\bar{h}|_{\mathbb{R}^s} = c_s z,$$

where $c_s = \frac{1}{\mu} c_m \left(I_m \otimes \mathbf{1}_\mu^{\mathrm{T}} \right) \left(I_s \otimes \mathbf{1}_r \right)$.

Particularly, when $s = km$, we have $c_{km} = \frac{1}{k} c_m \left(I_m \otimes \mathbf{1}_k^{\mathrm{T}} \right)$.

Definition 107. *Assume $\bar{f}(x)$ is a linear vector field, $\bar{B} = [\bar{b}_1, \cdots, \bar{b}_m]$ is a set of constant vector fields, $\bar{C} = [\bar{c}_1, \cdots, \bar{c}_p]^{\mathrm{T}}$ is a set of linear functions, then*

$$\begin{cases} \dot{\bar{x}} = \bar{f}(x) + \bar{B}u, \\ \bar{y} = \bar{C}\bar{x}, \end{cases}$$

is a linear control system over Ω.

Example 108. *Consider a linear control system $\bar{\Sigma}$ over Ω, which has its dynamic equation as (114), where the smallest generator of $\bar{f}$ is $f(x) = 2[x_1 + x_2, x_2]^{\mathrm{T}} \in V^\infty(\mathbb{R}^2)$, $m = 2$, the smallest generators of $\bar{g}_1$ and $\bar{g}_2$ are $g_1 = [1, 0, 0, 1]^{\mathrm{T}} \in V^\infty(\mathbb{R}^4)$, $g_2 = [0, 1, 0, 0]^{\mathrm{T}} \in V^\infty(\mathbb{R}^4)$ respectively. $p = 1$, $\bar{h}|_{\mathbb{R}^2} = x_2 - x_1$.*

   Then, $q = 4$.

$$\bar{f}|_{\mathbb{R}^4} = \Pi_4^2 f\left(\Pi_2^4 [z_1, z_2, z_3, z_4]^{\mathrm{T}}\right) = \begin{bmatrix} z_1 + z_2 + z_3 + z_4 \\ z_1 + z_2 + z_3 + z_4 \\ z_3 + z_4 \\ z_3 + z_4 \end{bmatrix} := Az,$$

where,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

$$\bar{h}|_{\mathbb{R}^4} = h(\Pi_2^4 z) = h(z_1 + z_2, z_3 + z_4) = z_1 + z_2 - z_3 - z_4 := Cz,$$

where $C = [1, 1, -1, -1]$.

   Then the smallest generator of system $\bar{\Sigma}$, denoted by $\Sigma := \bar{\Sigma}|_{\mathbb{R}^4}$, is

$$\begin{cases} \dot{z} = Az + Bu, \\ y = Cz. \end{cases}$$

It is easy to calculate that the controllability matrix of Σ is

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 2 & 1 & 6 & 2 & 16 & 4 \\ 0 & 1 & 2 & 1 & 6 & 2 & 16 & 4 \\ 0 & 0 & 1 & 0 & 2 & 0 & 4 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 & 4 & 0 \end{bmatrix}.$$

Since $\mathrm{rank}(\mathcal{C}) = 4$, Σ is completely controllable. By definition, $\bar{\Sigma}$ is completely controllable.

   The observability matrix of Σ is

$$\mathcal{O} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 2 & 0 & 0 \\ 4 & 4 & 4 & 4 \\ 8 & 8 & 16 & 16 \end{bmatrix}$$

Since $\text{rank}(\mathcal{O}) = 2 < 4$, Σ *is not completely observable, and so is* $\bar{\Sigma}$.

7.3. Dimension-varying dynamic (control) systems

Consider a continuous time dynamic system

$$\dot{x} = F(x), \quad x \in \mathcal{X}, \tag{125}$$

where F is considered as a vector field on a manifold $\mathcal{X}$. Then the solution (integral curve) is expressed as $x(t, x_0) = \Phi_t^F(x_0)$. It is well known that if (125) is a dynamic system, then $x(t, x_0)$ must be continuous with respect to t [22]. Hence a continuous time dimension-varying dynamic system can not be defined on ESDD $\mathbb{R}^\infty$. It can only be defined on DFES Ω (or in general, DFM).

Definition 109. *Consider a dynamic system*

$$\dot{\bar{x}} = \bar{F}(\bar{x}), \quad \bar{x} \in \Omega. \tag{126}$$

$$\dot{x} = F(x), \quad x \in \mathbb{R}^n \subset \mathbb{R}^\infty, \tag{127}$$

is called a realization (or a lifting) of (126), if for each $\bar{x}$ *there exists* $x \in \bar{x}$, *such that the corresponding vector field* $F(x) \in \bar{F}(\bar{x})$. *Meanwhile, system (126) is called the project system of (127).*

The following result is an immediate consequence of the definition.

Proposition 110. $\bar{x}(t) = \bar{x}(t, \bar{x}_0)$ *is the solution of (126), if and only if,* $x(t) = x(t, x_0)$ *is the solution of (127), where* $x(t) \in \bar{x}(t)$, $t \in [0, \infty)$.

It is obvious that the lifting system of (126) is not unique. Assume at $t \in [0, T)$ the system (126) is lifted to R^n and at $t \in [T, \infty)$ the system is lifted to $\mathbb{R}^m$ ($m \neq n$), then the overall lifting system becomes a dimension-varying system.

Definition 111. *System (127) is called a dimension-varying system, if there are at least two points* x_1, x_2 *such that* $F(x_1) \in V(\mathbb{R}^{d_1})$, $F(x_2) \in V(\mathbb{R}^{d_2})$ *and* $d_1 \neq d_2$.

Remark 112. *The Definitions 109 and 111 can easily be extended to corresponding control systems in a natural way. The Proposition 110 has also its corresponding version for control systems.*

In the following we consider how to construct dimension-varying control systems. We consider two cases.

- Case 1: switching dimension-varying control systems

Assume the original control system is

(128) $\dot{x} = F(x, u), \quad x \in \mathbb{R}^m, \ u \in \mathbb{R}^p.$

The target system is

(129) $\dot{z} = G(z, v), \quad z \in \mathbb{R}^n, \ v \in \mathbb{R}^q.$

Our purpose is to switch system (128) to system (129) at time $t = T$. To get a continuous trajectory over Ω, the following condition is necessary:

(130) $\bar{x}(T) = \bar{z}(T) \in \Omega.$

Proposition 113. *Assume (130) is satisfied, and assume system (128) is controllable. Then the dynamic switching from system (128) to system (129) at time $t = T$ is realizable.*

Proof. We construct the following system over Ω:

(131)
$$\dot{\bar{\xi}} = \begin{cases} \bar{F}(\bar{\xi}, u), & t < T, \\ \bar{G}(\bar{\xi}, v), & t > T. \end{cases}$$

Since system (128) is controllable, there exists $u(t)$, $t < T$, such that (131) is controllable to $\bar{\xi}(T) = \bar{x}(T) = \bar{z}(T)$, where $\dim(\bar{\xi}) = p \wedge q$.

Then the minimum realization of (131) becomes the required dimension-varying system. $\square$

Example 114. *Consider two systems*

$$\begin{aligned} \Sigma_1: \quad & \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x \in \mathbb{R}^2, \ x(0) = (0,0)^T, \\ \Sigma_2: \quad & \dot{z} = Az + Bv, \quad z \in \mathbb{R}^3. \end{aligned}$$

Design a control such that Σ_1 is switched to Σ_2 at $T = 1$.

Since Σ_1 is completely controllable, and $2 \wedge 3 = 1$, so we have to design a control which can drive the system from $x(0)$ to $x(T)$ with $\dim(\bar{x}(T)) =$

1. *We may choose $x(T) = (1,1)^T$. Then it is easy to calculate that the controllability Gramian matrix is*

$$W_C(t) = \int_0^t e^{-A\tau} BB^T e^{-A^T\tau} d\tau = \frac{1}{6} \begin{bmatrix} 2t^3 & -3t^2 \\ -3t^2 & t \end{bmatrix}.$$

Then the control is

$$u(t) = -B^T e^{-A^T t} W_C^{-1}(T) \left(x(0) - e^{-AT} x(T) \right) = -6t.$$

Using this control, the system can be switched from Σ_1 to Σ_2 at $T = 1$.

- Case 2: continuous dimension-varying control systems

In this case we require the designed dimansion-varying system has continuous $\bar{F}(\bar{x}, u)$. For instance, in a docking/undocking process, we want the dimension-transient process to be as smooth as possible.

First, let us see what a dimension-varying system with "continuous" vector field means.

Let $\bar{v}_0$ and $\bar{v}_2$ be two vector fields on Ω. Our purpose is to design a new vector field which continuously transfer from $\bar{v}_0$ to $\bar{v}_2$. Define

$$\bar{v} := \begin{cases} \bar{v}_0, & t \in [t_0, 0, t_1), \\ \bar{v}_1 = (1-\lambda)\bar{v}_0 + \lambda\bar{v}_2, & t \in (t_1, t_2), \\ \bar{v}_2, & t \in (t_2, \infty), \end{cases}$$

where $\lambda = \frac{t-t_1}{t_2-t_1}$.

Assume the minimum realization of $\bar{v}_0$ is $v_0 \in V^r(\mathbb{R}^p)$, the minimum realization of $\bar{v}_2$ is $v_2 \in V^r(\mathbb{R}^q)$. Then the minimum realization of $\bar{v}_1$ is $v_1 \in V^r(\mathbb{R}^{p\vee q})$. Then the integral curve of $\bar{v}$ can be lifted as shown in Fig 3.

- Docking:

Assume there are two original control systems as

(132) $$\Sigma_1: \quad \dot{x} = F(x, u), \quad x \in \mathbb{R}^m, \ u \in \mathbb{R}^p;$$

(133) $$\Sigma_2: \quad \dot{z} = G(z, v), \quad z \in \mathbb{R}^n, \ u \in \mathbb{R}^q.$$

They will be docked into the target system Ω as

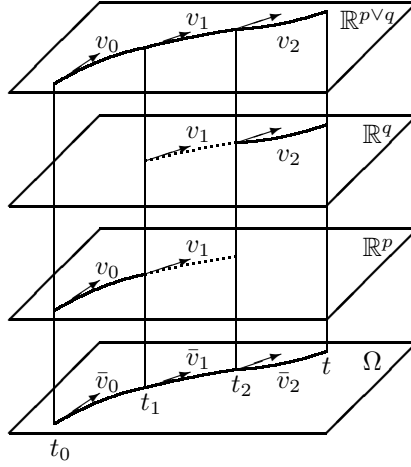(134) $$\dot{\xi} = H(\xi, w), \quad \xi \in \mathbb{R}^s, \ w \in \mathbb{R}^\ell.$$

Figure 3: Lift and projection of integral curves.

It is required that the docking happens during a transient period $[T_0, T_1]$, and the process is smooth.

Definition 115. *System ([132](#)) and system ([133](#)) are said to be docked into ([134](#)) smoothly during the transient period $[T_0, T_1]$. If the following requirements are satisfied.*

    (i) There exists a smooth monotonically non-decreasing function $\lambda(t)$, $t \in [T_0, T_1]$, such that

$$\lambda(t) = \begin{cases} 0, & t = T_0, \\ 1, & t = T_1. \end{cases}$$

    (ii) There exists a control deformation function $w = \varphi(u, v)$, and using it a control system over Ω, called a transient system, is constructed as

(135)
$$\dot{\bar{\xi}} = (1 - \lambda(t)) \begin{bmatrix} \bar{F}(\bar{x}, u) \\ \bar{G}(\bar{z}, v) \end{bmatrix} + \lambda(t)\bar{H}(\bar{\xi}, w) + \Psi(\bar{x}, \bar{z}, \bar{\xi}, u, v), \quad t \in [T_0, T_1].$$

    (iii) The transient system is controllable to

$$\begin{bmatrix} \bar{x}(T_1) \\ \bar{z}(T_1) \end{bmatrix} = \bar{\xi}(T_1),$$

with $\dim(\xi(\bar{T}_1)) = p \wedge q \wedge s$.

Remark 116. *In (135) $\Psi(\bar{x}, \bar{z}, \bar{\xi}, u, v)$ is a (fictitious) virtual force, caused by the interaction of docking objects, satisfying*

$$\Psi(\bar{x}, \bar{z}, \bar{\xi}, u, v) = 0, \quad if \begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix} \vec{\vdash} \bar{\xi} = 0.$$

In control of Clutch System the virtual force is chosen as [23, 24]

$$\tau_c = FcR_a\psi(\omega_i, \omega_0).$$

Particularly, it was chosen as [9]: $\tau_c = sign(\omega_i - \omega_0)F$.

- Undocking:

Assume there is an original system Σ_1 as:

$$(136) \qquad \dot{\xi} = H(\xi, w), \quad \xi \in \mathbb{R}^s, \; w \in \mathbb{R}^\ell.$$

It will be undocked into two systems as

$$(137) \qquad \Sigma_2: \quad \dot{x} = F(x, u), \quad x \in \mathbb{R}^m, \; u \in \mathbb{R}^p;$$

$$(138) \qquad \Sigma_3: \quad \dot{z} = G(z, v), \quad z \in \mathbb{R}^n, \; u \in \mathbb{R}^q.$$

It is required that the docking happens during a transient period $[T_0, T_1]$, and the process is smooth.

Definition 117. *System (136) is said to be un-docked into (137) and (138) smoothly during the transient period $[T_0, T_1]$. If the follow requirements are satisfied.*

(i) There exists a smooth monotonically non-decreasing function $\lambda(t)$, $t \in [T_0, T_1]$, such that

$$\lambda(t) = \begin{cases} 0, & t = T_0, \\ 1, & t = T_1. \end{cases}$$

(ii) There exist two control deformation functions $u = \varphi(w)$, $v = \varphi(w)$, and using them a control system over Ω, called a transient system, is constructed as

$$(139) \qquad \begin{aligned} \dot{\bar{\xi}} &= (1 - \lambda(t))\bar{H}(\bar{\xi}, w) \\ &\quad + \lambda(t) \begin{bmatrix} \bar{F}(\bar{x}, u) \\ \bar{G}(\bar{z}, v) \end{bmatrix} + \Psi(\bar{x}, \bar{z}, \bar{\xi}, w), \; t \in [T_0, T_1]. \end{aligned}$$

(iii) The transient system is controllable to state $\bar{\xi}(T_1) = \begin{bmatrix} \bar{x}(T_1) \\ \bar{z}(T_1) \end{bmatrix}$, *where*

$\dim(\xi(\bar{T}_1)) = p \wedge q \wedge s.$

Remark 118. *In (139) $F(\bar{x}, \bar{z}, \bar{\xi}, w)$ is a virtual force, caused by the interaction of docking objects, satisfying*

$$\Psi(\bar{x}, \bar{z}, \bar{\xi}, w) = 0, \quad if \begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix} \vec{\vdash} \bar{\xi} = 0.$$

8. Concluding remarks

The main purpose of this paper is to construct a new geometric object called the DFES (or DFM), which provides a framework (i.e., the state space) for DVDS. We briefly summarize the results.

The DFES is constructed as follows:

- Step 1: Define an inner product for two vectors of different dimensions. It turns the ESDD $\mathcal{V} = \mathbb{R}^\infty = \bigcup_{n=1}^\infty \mathbb{R}^n$ into a distance space.
- Step 2: Two vectors $x, y \in \mathbb{R}^\infty$ are said to be equivalent, denoted by $x \leftrightarrow y$, if their distance is zero. The quotient space $\Omega = \mathbb{R}^\infty / \leftrightarrow$ is called the DFES. In fact, $d_\mathcal{V}(x, y) = 0$, if and only if, there exist $\mathbf{1}_\alpha$ and $\mathbf{1}_\beta$ such that $x \otimes \mathbf{1}_\alpha = y \otimes \mathbf{1}_\beta$. It is clear that two vectors are equivalent if they contain the same information. In other words, vector form is a way for a set of data to express itself. It may be expressed as vectors of different dimensions, but from information point of view, they are equivalent.
- Step 3: By posing scalar multiplication and addition Ω becomes a topological real vector space. Let $\mathrm{Pr} : \mathbb{R}^\infty \to \Omega$ be the natural projection. Then $(\mathbb{R}^\infty, \mathrm{Pr}, \Omega)$ becomes a fiber bundle.
- Step 4. Using the fiber bundle structure of $(\mathbb{R}^\infty, \mathrm{Pr}, \Omega)$, each $\bar{x} \in \Omega$ has a coordinate neighborhood, which is a set of coordinate charts of various dimensions. It is called a bundle of coordinate charts. Hence Ω is called a DFES.
- Step 5. Using the bundles of coordinate charts, a differentiable structure can be posed on Ω, making it a DFM. Then the continuous functions, (co)-vector fields, (co)-distributions, and tensor fields can be built for Ω. Eventually, the dimension-free Riemannian manifolds and dimension-free symplectic manifolds can be properly constructed.

Note that the gluing topology on the DFES $\Omega = \mathbb{R}^\infty / \leftrightarrow$ makes it a pathwise connected topological space. Therefore, intuitively, the trajectories of dynamic systems over Ω can continuously move "across" Euclidian spaces of

different dimensions. This is the main idea for using DFM to design DVDS and DVCS. Lifting the trajectory of a dynamic system over Ω to a leaf (a Euclidian space of fixed dimension) is called a realization. As a dynamic system over Ω is lifted onto leafs of different dimensions, a dimension-varying realization is obtained. Conversely, we can also project the trajectory of a dynamic system on a Euclidian space of fixed dimension onto Ω.

The design of dimension-varying dynamic (control) systems can be described as follows:

- Step 1: Project a dimension-varying dynamic system, which has broken vector fields over Euclidian spaces of different dimensions, onto Ω to form a dynamic system over Ω, which consists of several (finite number) of vector fields.
- Step 2: Lifting the dynamic system on Ω to a Euclidian space of proper dimension, where all the vector fields involved by the dynamic system on Ω can be properly lifted into this Euclidian space.
- Step 3: All the analysis and control design can be done in conventional way for this lifting system on its Euclidian space.
- Step 4: Project the resulting manipulated system back to Ω and then lifting it into several original Euclidean spaces, where the original dimension-varying system lies on.

It is noteworthy that our construction may have a physical interpretation from the viewpoint of multi-agent system synchronization: if there are m independent agents moving on an n-dimensional manifold and obtaining complete synchronization, then the dimension of the system can be reduced from mn to n, which gives the equivalence defined by (7) in this paper when the state space is Euclidean.

Finally, we would like to present a conjecture: The DFM might provide a framework (i.e., the state space) for string theory in physics. The idea is sketched as follows:

Consider a subspace of DFM as $\Omega_3 := \{\bar{x} \in \Omega \mid \dim(\bar{x}) \leq 3\}$. We choose 3 because it is the dimension of real physical world.

Now $(\mathbb{R}^\infty, Pr, \Omega_3)$ is a sub-bundle of the fiber-bundle $(\mathbb{R}^\infty, \mathrm{Pr}, \Omega)$. If we consider all possible realization of dynamic systems over Ω_3, then the minimum total subspace which allow all possible realizations is $\mathbb{R}^{[\cdot,6]}$. Hence, if we desire a space which is of minimum dimension and contains all moves (or dynamic systems), then it is $\mathcal{B} := (\mathbb{R}^{[\cdot,6]} \to \Omega_3)$. Since this manifold is of dimension 9, plus a dimension for t, a manifold of dimension 10 is reasonable for describing state-motion-time. This might be the string space.

Some further arguments are the following:

(i) It is well known in classical differential geometry that an n-dimension manifold M has an n-dimensional tangent space at each point. Hence, if taking both M and $T(M)$ into consideration, an n-dimensional manifold with its tangent bundle is a $2n$ dimensional manifold, which is a well known fact. So consider the bundle $\mathcal{B}$ as a 9-dimensional manifold is reasonable.

(ii) It seems that there is no static particle in the world. That is, particles are always joined with their moves. Moves can be described by vector fields. So to describe a particle, a position plus a vector field on its tangent space may be reasonable to describe it, as the particle is small and its movement is very fast. Using string to describe a particle might essentially be a description for both the position and the trajectory of a particle.

(iii) Now taking the position and moving trajectory into consideration. We may consider the extra 6 dimensions being used to describe open string (or open movement of a particle). In addition, we need $SU(3)$ to describe the gauge group and $SU(1)$ for rotation. Then we have $\mathcal{B} + SU(3) + SU(1) +$ time, which is of dimension 26. This manifold might be proper for Bosonic super-sting model.

In brief, DFM could provide a framework for systems with arbitrary dimensions. DFM with a Reimannian structure becomes a DFRM. The investigation of DFRM in this paper is very elementary. A continuous study is necessary. It is promising that DFRM might overcome the crisis of classical Riemannian geometry [29].

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