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PSL(2; $\mathbb{C}$) connections on 3-manifolds with L^2 bounds on curvature

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Karen Uhlenbeck's compactness theorem for sequences of connections with L^2 bounds on curvature applies only to connections on principal bundles with compact structure group. This article states and proves an extension of Uhlenbeck's theorem that describes sequences of connections on principal PSL(2; $\mathbb{C}$) bundles over compact three dimensional manifolds.

Suppose that M is a compact Riemannian manifold and P is a principal bundle over M with fiber a Lie group to be denoted by G. Fix an integer no less than $\frac{1}{2}$ of the dimension of M, this denoted by p. In the case when G is compact, Karen Uhlenbeck's foundational paper *Connections with* L^p *bounds on curvature* [U] explained the sense in which the space of connections on the given principal bundle with an a priori L^p bound on the norm of the curvature is compact. This paper is the first of a planned series of papers that provide a generalization of Uhlenbeck's theorem in the case when G is not compact. This paper considers only the case when M has dimension 2 or 3, the group G is PSL(2; $\mathbb{C}$) and p = 2. The generalization of Uhlenbeck's theorem for the case of dimension 3 is the upcoming Theorem 1.1. The dimension 2 case is subsumed by Theorem 1.1 by taking the manifold in Theorem 1.1 to be the product of the given surface with the circle. The dimension 2 case is also stated separately as Theorem 1.2.

The space of automorphism equivalence classes of irreducible, flat PSL(2; $\mathbb{C}$) connections need not be compact. Morgan and Shalen [MS1]–[MS3] construct a compactification of the latter space using certain equivariant maps from the universal cover of M to certain sorts of $\mathbb{R}$-trees, an $\mathbb{R}$-tree being a metric space where by any two points are connected by a unique path. Daskoloupoulus, Dostoglu and Wentworth [DDW1]–[DDW3] subsequently proved that the Morgan-Shalen maps can be taken to be harmonic. Some brief remarks are made near the end of Section 1 about what is said in Theorem 1.1 and what is said in [MS1]–[MS4] and [DDW1]–[DDW3].

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A generalization of Uhlenbeck's theorem to the case when G is PSL(2; ℂ) and M has dimension 3 is of specific, topical interest for reasons that are described at the end of Section 1 of this paper. Moreover, close kin to the techniques that are introduced to prove Theorem 1.1 will likely play crucial roles in proofs of Theorem 1.1's analog when G has rank greater than 2, when M has dimension greater than 3, and/or when p is not equal to 2. It is also likely that close kin of these same techniques can help characterize the moduli spaces of solutions to various generalizations of the 3 and 4 dimensional Seiberg-Witten equations that involve more than one spinor and more than one U(1) connection. The reason is that these Seiberg-Witten equations have the same formal structure as those that assert the flatness of a connection on a PSL(2; ℂ) principal bundle. Compactness theorems for solutions to these other equations are subjects for possible sequels to this article.

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1. The PSL(2; ℂ) extension of Uhlenbeck's theorem

This section has six subsections. The first, Section 1.a, presents Uhlenbeck's theorem and then states its PSL(2; ℂ) generalization, this being Theorem 1.1. Section 1.b states and proves the dimension 2 analog of Theorem 1.1. Section 1.c first explains the relationship between what is said in Theorem 1.1 and what is said in [MS1]–[MS4] and [DDW1] about the flat connections. It then gives a very brief account how the data supplied by Theorem 1.1 leads to some central notions in 3-manifold topology. Section 1.d provides a short outline of the proof of Theorem 1.1. Section 1.e describes some questions in 3 and 4 dimensional differential topology and geometry that may well require some sort of PSL(2; ℂ) extension of Uhlenbeck's theorem. This section also has a paragraph that says something about extensions of Uhlenbeck's theorem to PSL(n; ℂ) for $n > 2$. Section 1.f supplies a table of contents for this article and it states certain notational conventions that are subsequently invoked with no further comments.

1.a. Theorem 1.1

Theorem 1.1 is stated below after some necessary stage setting to define the notation and supply some needed background. The stage setting has nine parts.

Part 1: The group SL(2; ℂ) is viewed here as the group of 2×2 complex matrices with determinant 1. Viewed in this light, its Lie algebra is the vector space of trace zero, 2×2 complex matrices. The latter space can be written as the direct sum $\mathfrak{su}(2) \oplus i\mathfrak{su}(2)$ with $\mathfrak{su}(2)$ denoting the vector space of skew hermitian complex matrices and with i denoting the square root of -1. Keep in mind that the Lie algebra of SU(2) is the vector space $\mathfrak{su}(2)$. The linear form on the vector space of 2×2 complex matrices given by -1 times the trace is denoted in what follows by $\langle\, , \rangle$. The square of the Hermitian norm on $\mathfrak{su}(2)$ can be written using this notation as the function $\mathfrak{u} \to \langle \mathfrak{u}\, \mathfrak{u} \rangle$.

Part 2: Assume henceforth that M is a compact 3-dimensional manifold with a given principal PSL(2; ℂ) bundle. As PSL(2; ℂ) bundles have reductions to principal SO(3) bundles, choose once and for all such a reduction so as to write the given principal PSL(2; ℂ) bundle as $P \times_{SO(3)} PSL(2;\mathbb{C})$ with $P \to M$ a principal SO(3) bundle.

Any given connection on $P \times_{SO(3)} PSL(2;\mathbb{C})$ can be written as $A + i\mathfrak{a}$ with A being a connection on P and with $\mathfrak{a}$ denoting a 1-form with values in the associated vector bundle $P \times_{SO(3)} \mathfrak{su}(2)$. The curvature of the connection $\mathbb{A} = A + i\mathfrak{a}$ is denoted by $F_{\mathbb{A}}$ and that of A by F_A. The former is a section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes (\wedge^2 T^*M)$ and it can be written in terms of A's curvature as $F_{\mathbb{A}} = F_A - \mathfrak{a} \wedge \mathfrak{a} + i d_A \mathfrak{a}$ with d_A denoting here the exterior covariant derivative that is defined by A.

Part 3: Fix once and for all a Riemannian metric on M. Unless instructed to the contrary, assume that all inner products on TM, T^*M and their tensor products are defined by this metric. The metric Hodge star is denoted by $*$. Likewise, assume that all covariant derivatives on these tensor bundles are those induced by the associated Levi-Civita connection. These covariant derivatives are denoted by ∇. The metric's Ricci curvature tensor defines a symmetric, bilinear form on T^*M that is denoted by Ric. Integration on M is defined using the metric's volume element.

Norms of tensor bundle valued sections of $P \times_{SO(3)} \mathfrak{su}(2)$ are defined using the Riemannian metric and the norm on $\mathfrak{su}(2)$. Let A denote a given connection on P. The covariant derivative defined by A and the Levi-Civita connection on $P \times_{SO(3)} \mathfrak{su}(2)$ valued sections of tensor bundles is denoted by ∇_A. The Hermitian adjoint of this operator is denoted by $\nabla_A^\dagger$.

Part 4: Sobolev spaces of connections are front and center in Uhlenbeck's theorem, and so they are front and center in Theorem 1.1. This part of the subsection defines these spaces.

Fix a 'fiducial' connection on P to be denoted by A_0. The connection A_0 is used to define a given $k \in \{0, 1, 2, \ldots\}$ version of the Sobolev L^2_k norm on tensors with values in $P \times_{SO(3)} \mathfrak{su}(2)$, this being the norm whose square assigns to a given tensor $\mathfrak{t}$ the integral over M of $\sum_{0\le m\le k} |(\nabla_{A_0})^{\otimes m}\mathfrak{t}|^2$. An L^2_k *tensor* with values in $P \times_{SO(3)} \mathfrak{su}(2)$ is an almost everywhere defined section of the relevant vector bundle with finite Sobolev L^2_k norm.

Let Conn(P) denote the space of smooth connections on P. Let k denote for the moment a given non-negative integer. The L^2_k topology on Conn(P) is defined as follows: Write a given connection on P as $A_0 + \hat{a}$ with $\hat{a}$ being a section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. Doing so identifies Conn(P) with the vector space of sections of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. The L^2_k topology on Conn(P) is the topology that is induced by this identification from the L^2_k metric metric topology on the space of sections of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. This topology does not depend on the chosen fiducial connection.

A connection on P is said to be an L^2_k *connection* if it has the form $A_0 + \hat{a}$ with $\hat{a}$ denoting an L^2_k section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. This notion is likewise independent of the choice of A_0. A sequence $\{A_n\}_{n=1,2,\ldots}$ of connections on P is said to converge *weakly* in the L^2_k topology on P when it can be written as $\{A_n = A_0 + \hat{a}_n\}_{n=1,2,\ldots}$ with the sequence $\{\hat{a}_n\}_{n=1,2,\ldots}$ having bounded L^2_k norm and converging weakly with respect to the L^2_k norm to an L^2_k section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$.

Part 5: Karen Uhlenbeck's [U] theorem applies to connections on P and in particular makes the following assertion:

Uhlenbeck's Theorem. *Suppose that* $\{A_n\}_{n=1,2,\ldots}$ *is a sequence of connections on* P *with the corresponding sequence*

$$\left\{\int_M |F_{A_n}|^2\right\}_{n=1,2,\ldots}$$

being bounded. There is a subsequence of $\{A_n\}_{n=1,2,\ldots}$*, hence renumbered consecutively from* 1*, and a corresponding sequence of automorphisms of* P*, this denoted by* $\{g_n\}_{n=1,2,\ldots}$*, such that* $\{{g_n}^* A_n\}_{n=1,2,\ldots}$ *converges weakly in the* L^2_1 *topology to an* L^2_1 *connection on* P*.*

By way of a reminder, the space of automorphisms of P acts on Conn(P) by pull-back. This space of automorphism can be identified in a canonical fashion with the space of sections of $P \times_{SO(3)} SO(3)$. Let g denote such an automorphism and let A denote a given connection. The pull-back g^*A can be written as $A + \hat{g}^{-1} d_A \hat{g}$ where $\hat{g}$ is a (locally defined) section of the associated bundle to P with fiber the group of 2×2 unitary matrices with determinant 1, this being SU(2). The group SO(3) acts on SU(2) via conjugation. The automorphism g need not lift over the whole of M to a section of the fiber bundle $P \times_{Ad(SO(3))} SU(2)$, but a lift does exists over any contractible subset of M. Even so, the section $\hat{g}^{-1} d_A \hat{g}$ of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ is defined everywhere on M because any two lifts of g differ by the action of multiplication by 1 or -1.

Part 6: Uhlenbeck's theorem is a godsend because its assumptions are invariant under the action on Conn(P) of P's automorphism group; the reason being that the norm given by $\mathfrak{u} \to -\langle \mathfrak{u}\, \mathfrak{u} \rangle$ on $\mathfrak{su}(2)$ is ad-invariant. This implies that the norm of the curvature of a connection on P is pointwise identical to the norm of its pull-back via any automorphism of P. This being the case, the L^2 norm of the curvature of any given connection is is the same as that of its pull-back via an automorphism.

The Lie algebra of SL(2; ℂ) does not have a norm that is invariant under the adjoint action of SL(2; ℂ). Even so, there is a useful generalization of the L^2 norm of the curvature of an SL(2; ℂ) connection that is invariant under the action of the group of PSL(2; ℂ) automorphisms of $P \times_{SO(3)} PSL(2; \mathbb{C})$. The definition is given momentarily in (1.1). This generalization plays the role in Theorem 1.1 that is played by the curvature L^2 norm in Uhlenbeck's theorem.

The upcoming definition uses $\mathrm{Conn}(P \times_{SO(3)} PSL(2; \mathbb{C}))$ to denote the space of connections on $P \times_{SO(3)} PSL(2; \mathbb{C})$ The group of automorphisms of $P \times_{SO(3)} PSL(2; \mathbb{C})$ is denoted by $\mathcal{G}_{\mathbb{C}}$; and the $\mathcal{G}_{\mathbb{C}}$-orbit in $\mathrm{Conn}(P \times_{SO(3)} PSL(2; \mathbb{C}))$ of a given connection $\mathbb{A}$ is denoted by $\mathcal{G}_{\mathbb{C}}(\mathbb{A})$.

The generalization to $\mathrm{Conn}(P \times_{SO(3)} PSL(2; \mathbb{C}))$ of the square of the L^2 norm of the curvature is the function on $\mathrm{Conn}(P \times_{SO(3)} PSL(2; \mathbb{C}))$ assigns to any given connection $\mathbb{A}$ the infimum over connections $A + i\mathfrak{a} \in \mathcal{G}_{\mathbb{C}}(A)$ of a function that is denoted by $\mathfrak{F}$ and defined by the rule

$$\mathbb{A} \to \mathfrak{F}(\mathbb{A}) = \int_M \left(|F_A - \mathfrak{a} \wedge \mathfrak{a}|^2 + |d_A \mathfrak{a}|^2 + |d_A {*} \mathfrak{a}|^2\right). \tag{1.1}$$

Part 7: Fix $k \in \{0, 1, 2, \dots\}$. The L^2_k topology on $\mathrm{Conn}(P \times_{SO(3)} PSL(2;\mathbb{C}))$ is defined by first identifying the latter space with $\mathrm{Conn}(P) \times C^\infty(M; (P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M)$ with it understood that a given pair $(A, \mathfrak{a})$ in the latter space corresponds to the connection $A + i\mathfrak{a}$. Having done so, the topology is defined to be the product of the L^2_k topologies on Conn(P) and $C^\infty(M; (P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M)$. An L^2_k connection on the principal $PSL(2;\mathbb{C})$ bundle $P \times_{SO(3)} PSL(2;\mathbb{C})$ is defined by a pair $(A, \mathfrak{a})$ of L^2_k connection on P and L^2_k section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$.

A part of Theorem 1.1 describes a connection on an open set in M as being an $L^2_{k;loc}$ connection. Let U denote the given open set. A connection on $P|_U \times_{SO(3)} PSL(2;\mathbb{C})$ is said to be an $L^2_{k;loc}$ connection if it can be written as $A_0 + \hat{a} + i\mathfrak{a}$ with $\hat{a}$ and a being almost everywhere defined sections over U of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ with the following property: If $V \subset U$ is any given open set with compact closure, then the L^2_k norm on V of $(\hat{a}, \mathfrak{a})$ is finite, the latter being the square root of the integral on V of $\sum_{0 \le m \le k} |(\nabla_{A_0})^{\otimes m} \hat{a}|^2$ and $\sum_{0 \le m \le k} |(\nabla_{A_0})^{\otimes m} \mathfrak{a}|^2$. This notion of an $L^2_{k;loc}$ connection does not depend on the fiducial connection A_0.

A sequence of connections on $P \times_{SO(3)} PSL(2;\mathbb{C})$ is said to converge weakly in the L^2_k topology to an L^2_k connection when the Conn(P) part of the corresponding sequence in $\mathrm{Conn}(P) \times C^\infty(M; (P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M)$ converges weakly in the L^2_k topology to an L^2_k connection on the principal SO(3) bundle P and the $C^\infty(M; (P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M)$ part has bounded L^2_k norm and converges weakly to an L^2_k section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. If $U \subset M$ is a given open set, a sequence of connections on $P \times_{SO(3)} PSL(2;\mathbb{C})$ is said to converge weakly to an $L^2_{k;loc}$ connection on U if both the Conn(P) part and the $C^\infty(M; (P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M)$ part of the connection converge weakly with respect to the L^2_k topology on all open subsets in U with compact closure. This is to say that the Conn(P) part differ from A_0 on U by a sequence of sections over U of the vector bundle $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ with bounded L^2_k norm on open sets in U with compact closure; and that it converges weakly on such open sets to an $L^2_{k;loc}$ section of this vector bundle. Meanwhile, the $C^\infty(M; (P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M)$ part of the sequence has bounded L^2_k norm on open sets in U with compact closure and it also converges weakly on such open sets to an $L^2_{k;loc}$ section of $(P|_U \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$.

Part 8: The term *real line bundle* is used here to describe the associated line bundle to a principal $\mathbb{Z}/2\mathbb{Z}$ bundle. With this term understood, let $U \subset M$ denote a given open set and suppose that $\mathcal{I} \to U$ is a real line bundle. An $\mathcal{I}$ valued tensor field on U is a section of the tensor product bundle with $\mathcal{I}$. The Riemannian metric defines the pointwise norm of an $\mathcal{I}$ valued p-form or any $\mathcal{I}$ valued tensor field. Meanwhile, the Levi-Civita covariant derivative defines a covariant derivative of such a tensor field.

Fix $p \in \{0, 1, 2, 3\}$ and let q denote for the moment an $\mathcal{I}$ valued p-form. The definition of the exterior derivative of q is canonical and is denoted by dq. This $\mathcal{I}$ valued p-form is said to be closed if $dq = 0$ and it is said to be coclosed if $d{*}q = 0$. An $\mathcal{I}$ valued p-form that is closed and coclosed is said to be harmonic.

If A is a connection on P, then A defines a covariant derivative on $\mathrm{Hom}(\mathcal{I}; P \times_{SO(3)} \mathfrak{su}(2))$. A given section is said to be A-covariantly constant if it is annihilated by A's covariant derivative.

Keep in mind that if q is an $\mathcal{I}$-valued p-form and σ is a section of the vector bundle $(\mathcal{I} \otimes P) \times_{\mathbb{Z}/2\mathbb{Z} \times SO(3)} \mathfrak{su}(2)$, then $q\sigma$ is a p-form on U with values in $P \times_{SO(3)} \mathfrak{su}(2)$.

Part 9: Theorem 1.1 describes a certain subset of an open set in M as an *embedded Lipshitz curve*. For the present purposes, an *embedded Lipshitz curve* in a given open set U is closed in U and characterized as follows: Let γ denote such a curve and let p denote a given point in γ. Let $x = (x_1, x_2, x_3)$ denote Euclidean coordinates for $\mathbb{R}^3$. There is a coordinate chart for M centered at p that depicts γ near p as the small $|x|$ part of the graph $t \to (x_1 = t, x_2 = \varphi_2(t), x_3 = \varphi_3(t))$ with $\varphi = (\varphi_1, \varphi_2)$ being a Lipshitz map from $\mathbb{R}$ to $\mathbb{R}^2$. By way of a reminder, a continuous map from an interval $I \subset \mathbb{R}$ into a Riemannian manifold is said to be Lipshitz under the following circumstances: Let γ denote the map in question. Then γ is Lipshitz when $\sup_{t,t' \in I} \mathrm{dist}(\gamma(t), \gamma(t')) \leq c_\gamma |t - t'|$ with c_γ being a constant.

A subset of an open set of M is said to be a Lipshitz curve if it is the image of a Lipshitz map from either the circle or an open set in $\mathbb{R}$. A Lipshitz curve is almost everywhere differentiable, and it has finite length if its domain has compact closure. An embedded Lipshitz curve is a priori a Lipshitz curve.

Granted all of this notation, here is the promised PSL(2; ℂ) generalization of Uhlenbeck's theorem:

Theorem 1.1. *Suppose that* $\{\mathbb{A}_n = A_n + i\mathfrak{a}_n\}_{n=1,2\ldots}$ *is a sequence of connections on* $P \times_{SO(3)} PSL(2; \mathbb{C})$ *with the corresponding sequence* $\{\mathfrak{F}(\mathbb{A}_n)\}_{n=1,2,\ldots}$ *being bounded. For each* $n \in \{1, 2, \ldots\}$, *use* r_n *to denote the* L^2 *norm of* $\mathfrak{a}_n$.

- *If the sequence* $\{r_n\}_{i=1,2,\ldots}$ *has a bounded subsequence, then there exists a subsequence of* $\{\mathbb{A}_n\}_{n=1,2,\ldots}$*, hence renumbered consecutively from 1, and a corresponding sequence of automorphisms of* P*, this denoted by* $\{g_n\}_{n=1,2,\ldots}$*, such that* $\{g_n{}^*\mathbb{A}_n\}_{n=1,2,\ldots}$ *converges weakly in the* L^2_1 *topology to an* L^2_1 *connection on* $P \times_{SO(3)} PSL(2;\mathbb{C})$.
- *If the sequence* $\{r_n\}_{n=1,2,\ldots}$ *has no bounded subsequence, then there exists a subsequence of* $\{\mathbb{A}_n\}_{n=1,2,\ldots}$*, hence renumbered consecutively from 1, a corresponding sequence of automorphisms of* P*, this denoted by* $\{g_n\}_{n=1,2,\ldots}$*, plus the following additional data: A closed set* $Z_S \subset M$*, a real line bundle* $\mathcal{I} \to M{-}Z_S$*, and a harmonic* $\mathcal{I}$ *valued 1-form on* $M{-}Z_S$*. The latter is denoted by* ν*. These are such that*

  1) *The norm* $|\nu|$ *of* ν *extends to the whole of* M *as a continuous,* L^2_1 *function. The set* Z_S *is contained in the zero locus of* $|\nu|$*, the latter being a closed set in* M *that is contained in a countable union of 1-dimensional Lipshitz curves. Moreover, given* $\varepsilon > 0$*, there exists a finite set of balls whose total volume is less than* ε *and with pairwise disjoint closures such that* $|\nu|$*'s zero locus in the complement of their union is a properly embedded finite length Lipshitz curve with a finite set of components.*
  2) *The sequence* $\{g_n{}^*A_n\}_{n=1,2,\ldots}$ *converges weakly in the* $L^2_{1;loc}$ *topology on* $M{-}Z_S$ *to an* $L^2_{1;loc}$ *connection on* $P|_{M-Z_S}$*, this denoted by* A.
  3) *The sequence* $\{r_n^{-1}g_n{}^*\mathfrak{a}_n\}_{n=1,2,\ldots}$ *converges weakly in the* $L^2_{1;loc}$ *topology on* $M{-}Z_S$ *to* $\nu\sigma$ *with* σ *being a unit length,* A*-covariantly constant homorphism over* $M{-}Z_S$ *from* $\mathcal{I}$ *into* $P \times_{SO(3)} \mathfrak{su}(2)$*. Meanwhile,* $\{r_n^{-1}|\mathfrak{a}_n|\}_{n=1,2,\ldots}$ *converges to* $|\nu|$ *in the weak* L^2_1 *topology and the* C^0 *topology on the whole of* M.

Let Z denote the zero locus of $|\nu|$. It is not out of the question that there is a 'generic metric' theorem to the effect that Z and perhaps Z_S are necessarily the union of a finite set of points with a compact, embedded Lipshitz or even smooth curve if the metric on M is chosen from a suitable dense set. It is also possible that Z and perhaps Z_S are as just described if the sequence $\{\mathbb{A}_n\}_{n=1,2,\ldots}$ is chosen to have some additional properties; a case to consider is that where the sequence sits on an integral curve of $\mathfrak{F}$'s gradient vector field and the corresponding sequence of $\mathfrak{F}$ values converges to the infimum of $\mathfrak{F}$.

There is a certain topological significance in Z, $\mathcal{I}$ and ν that are illustrated by two comparisons. The first comparison is that between what

Theorem 1.1 says and what is said by the analog of Theorem 1.1 for manifolds of dimension 2. The dimension 2 analog of Theorem 1.1 is in the next subsection. This second comparison is that between what is said by Theorem 1.1 with no added assumptions and what can be said when the sequence $\{\mathbb{A}_n\}_{n=1,2,\ldots}$ is a sequence of flat $PSL(2;\mathbb{C})$ connections. Section 1.c contains some very brief remarks about the latter case of Theorem 1.1 and then about the topological significance in Z, I and ν in the general case.

1.b. The case of dimension 2

Let Σ denote a compact, Riemann surface and let $P \to \Sigma$ denote a given principal SO(3) bundle. The function $\mathfrak{F}$ in (1.1) has its analog for connections on the principal $PSL(2;\mathbb{C})$ bundle $P \times_{SO(3)} PSL(2;\mathbb{C})$, the formula being identical but for the fact that the integration domain is Σ and the Hodge dual is defined using the metric on Σ.

The upcoming Theorem 1.2 is the analog of Theorem 1.1 for oriented surfaces. The dimension 2 case of Theorem 1.1 for an non-orientable surface can be deduced from the theorem below by considering pull-backs to the double cover. The statement of the non-orientable version is left to the reader.

By way of background for Theorem 1.2, keep in mind first of all that the chosen orientation and metric for Σ define a complex structure for $T\Sigma$ and thus a corresponding splitting of $T^*\Sigma \otimes_{\mathbb{R}} \mathbb{C}$ as the direct sum of two complex line bundles, $T^{1,0} \oplus T^{0,1}$. If q is a given $P \times_{SO(3)} \mathfrak{su}(2)$ valued 1-form, then q has a corresponding decomposition as $q^{1,0} + q^{0,1}$ with $q^{1,0}$ being the part of q in $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes_{\mathbb{R}} T^{1,0}$ and with $q^{0,1}$ denoting the part of q in $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes_{\mathbb{R}} T^{0,1}$.

Theorem 1.2 refers to what is known as a quadratic differential. The latter is a section of the complex line bundle $T^{2,0} = T^{1,0} \otimes_{\mathbb{C}} T^{1,0}$. This bundle has a canonical holomorphic structure and this is used to define the notion of a holomorphic quadratic differential. The Riemann-Roch theorem asserts that the space of holomorphic quadratic differentials is a complex vector space of dimension zero if Σ is a sphere, dimension 1 over $\mathbb{C}$ if Σ is a torus, and dimension $3\textsc{g} - 3$ over $\mathbb{C}$ with $\textsc{g}$ denoting the genus of Σ when this genus is greater than 1. Let μ denote a non-trivial, holomorphic quadratic differential. The zero locus of μ consists of $4\textsc{g} - 4$ points counted with multiplicity.

It may or may not be the case that μ is the square of a holomorphic section of $T^{1,0}$. This is the case if each of μ's zeros has even multiplicity. In any event, the square root of μ can be defined on the complement of its zero locus as a section of a real line bundle on this complement. The zero locus of μ is denoted in Theorem 1.2 by Z_μ, and the square root of μ on $\Sigma - Z_\mu$ is denoted by $\mu^{1/2}$. The corresponding real line bundle is denoted by $\mathcal{I}_\mu$.

Theorem 1.2. *Let Σ denote a compact, oriented Riemann surface and let $P \to \Sigma$ denote a principal $SO(3)$ bundle. Suppose that $\{\mathbb{A}_n = A_n + i\mathfrak{a}_n\}_{n=1,2...}$ is a sequence of connections on $P \times_{SO(3)} PSL(2;\mathbb{C})$ with the corresponding sequence $\{\mathfrak{F}(\mathbb{A}_n)\}_{n=1,2,...}$ being bounded. For each $n \in \{1, 2, \dots\}$, use r_n to denote the L^2 norm of $\mathfrak{a}_n$.*

- *If the sequence $\{r_n\}_{n=1,2,...}$ has a bounded subsequence, then there exists a subsequence of $\{\mathbb{A}_n\}_{n=1,2,...}$, hence renumbered consecutively from 1, and a corresponding sequence of automorphisms of P, this denoted by $\{g_n\}_{n=1,2,...}$, such that $\{g_n{}^*\mathbb{A}_n\}_{n=1,2,...}$ converges weakly in the L^2_1 topology to an L^2_1 connection on $P \times_{SO(3)} PSL(2;\mathbb{C})$.*
- *If the sequence $\{r_n\}_{n=1,2,...}$ has no bounded subsequence, then there exists a subsequence of $\{\mathbb{A}_n\}_{n=1,2,...}$, hence renumbered consecutively from 1, a corresponding sequence of automorphisms of P, this denoted by $\{g_n\}_{n=1,2,...}$, and a non-trivial, holomorphic quadratic differential, this denoted by μ. These are such that*
  1) *The sequence $\{g_n{}^*A_n\}_{n=1,2,...}$ converges weakly in the $L^2_{1;loc}$ topology on $\Sigma - Z_\mu$ to an $L^2_{1;loc}$ connection on $P|_{M-Z_\mu}$ this denoted by A.*
  2) *The sequence $\{r_n^{-1/2} g_n{}^*\mathfrak{a}_n\}_{n=1,2,...}$ converges weakly in the $L^2_{1;loc}$ topology on $M - Z_\mu$ to a section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*\Sigma$ whose $T^{1,0}$ part is $\mu^{1/2}\sigma$ with σ being an isometric, A-covariantly constant homomorphism over $M - Z_\mu$ from $\mathcal{I}$ to $P \times_{SO(3)} \mathfrak{su}(2)$.*

By way of a parenthetical remark, a connection on a principal bundle over a Riemann surface defines a holomorphic structure on any associated bundle with complex fibers. This understood, the conclusions of Theorem 1.2 are foreshadowed by Simon Donaldson's paper [D1] about stable holomorphic bundles on surfaces.

Proof of Theorem 1.2. As explained momentarily, this theorem constitutes a special case to Theorem 1.1. Even so, it can be proved independently of Theorem 1.1 and with much less effort. To obtain Theorem 1.2 from Theorem 1.1, take M in Theorem 1.1 to be the product $S^1 \times \Sigma$ with the metric being the product metric. The pull-back of the principal SO(3) bundle P on Σ to M via the projection map to Σ defines a principal SO(3) bundle over M, the latter is denoted also by P. The bundle P is the SO(3) bundle to use for Theorem 1.1. The corresponding pull-back of $\{\mathbb{A}_i\}_{i=1,2,...}$ defines a sequence of connections on the incarnation of $P \times_{SO(3)} PSL(2;\mathbb{C})$ as a bundle on M. This pull-back sequence is also denoted by $\{\mathbb{A}_n\}_{n=1,2,\ldots}$. Keep in mind that the value of Σ's version of $\mathfrak{F}$ on a connection is the same up to a multipicative factor as that of (1.1) on the pulled back connection. By the same token, the Σ version of the sequence of L^2 norms $\{r_n\}_{n=1,2,\ldots}$ differs by an index independent multiplicative factor from the corresponding M version of $\{r_n\}_{n=1,2,\ldots}$.

Granted what was said in the preceding paragraph, the first bullet of Theorem 1.2 follows directly from the first bullet of Theorem 1.1 provided an argument can be made to the effect that the sequence of automorphisms $\{g_n\}_{n=1,2,\ldots}$ that Theorem 1.1 provides can be assumed to be pull-backs of a corresponding sequence that is defined on Σ. The fact that this is so follows from the fact that an L^2_1 section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*(S^1 \times \Sigma)$ restricts to almost every constant $t \in S^1$ slice as an L^2_1 section along the slice, and it restricts to half of these slices with L^2_1 norm no greater than twice that of its L^2_1 norm on the whole of $S^1 \times \Sigma$.

An analogous argument can be used to prove that Theorem 1.1's set Z is the product of S^1 and a closed set $Z_\Sigma \subset \Sigma$ and that Theorem 1.1's real line bundle $\mathcal{I}$ is isomorphic to the pull-back via the projection map of a real line bundle defined on the complement in Σ of Z_Σ, this denoted for now by $\mathcal{I}_\Sigma$. Moreover, such an isomorphism identifies Theorem 1.1's version of ν with the pull-back of a harmonic, $\mathcal{I}_\Sigma$ valued 1-form on $\Sigma - Z_\Sigma$ with Z_Σ being the locus where its norm is zero. Let ν_Σ denote the latter. Use the splitting of $T^*\Sigma \otimes_{\mathbb{R}} \mathbb{C}$ as $T^{1,0} \oplus T^{0,1}$ to write ν_Σ as $e + \bar{e}$ with e denoting the $T^{1,0} \otimes_{\mathbb{R}} \mathcal{I}_\mu$ part of ν_Σ. The section e is holomorphic, this being a consequence of the fact that ν_Σ is harmonic. Since e is holomorphic, its square e^2 is a holomorphic section of $T^{2,0}$ over $\Sigma - Z_\Sigma$ that vanishes on Σ. This last observation implies that e^2 extends over Z_Σ so as to define a holomorphic quadratic differential on Σ. The latter is Theorem 1.2's quadratic differential μ. Since the zero locus of e is Z_Σ, this is likewise the zero locus of μ. Thus Z_Σ is Theorem 1.2's set Z_μ and Theorem 1.2's principal $\mathbb{Z}/2\mathbb{Z}$ bundle $\mathcal{I}_\mu$ is $\mathcal{I}_\Sigma$.

1.c. The topological significance of Z, $\mathcal{I}$ and ν

There are two parts to what follows. The first part talks about Theorem 1.1 when the connections in the sequence $\{\mathbb{A}_n\}_{n=1,2,\ldots}$ are irreducible, flat connections. The second part gives a very brief account of how the data Z, $\mathcal{I}$ and ν in the general case lead to structures that are mainstays of 3-dimensional topology.

Part 1: Morgan and Shalen in [MS1]–[MS4] use $\mathbb{R}$-trees to define a compactification of the equivalence classes of irreducible representations of $\pi_1(\mathrm{M})$ in $\mathrm{PSL}(2;\mathbb{C})$ with the equivalence relation defined by conjugation by $\mathrm{PSL}(2;\mathbb{C})$. The latter space is a disjoint union of subspaces with each equivalent to the space of automorphism classes of irreducible, flat $\mathrm{PSL}(2;\mathbb{C})$ connections on some principal $\mathrm{PSL}(2;\mathbb{C})$ bundle over M. As noted in the introduction, the Morgan-Shalen compactification involves $\pi_1(\mathrm{M})$-equivariant maps from M's universal cover to $\mathbb{R}$-trees and Daskoloupoulus, Dostoglu and Wentworth [DDW1] proved that the Morgan-Shalen maps can be taken to be harmonic. The next paragraph gives a rough summary of [DDW1]'s construction of this map.

Suppose that $\{\mathbb{A}_n\}_{n=1,2,\ldots}$ is a sequence of flat connections on $\mathrm{P}\times_{\mathrm{SO}(3)}\mathrm{PSL}(2;\mathbb{C})$ that is described by the second bullet in Theorem 1.1. Donaldson [D2] and Corlette [Co] prove that there is a connection on the $\mathrm{Aut}(\mathrm{P}\times_{\mathrm{SO}(3)}\mathrm{PSL}(2;\mathbb{C}))$ orbit of each member of this sequence with $\mathfrak{F} = 0$. This understood, Daskoloupoulus, Dostoglu and Wentworth start with a sequence $\{\mathbb{A}_n\}_{n=1,2,\ldots}$ with just this property. Daskoloupoulus, Dostoglu and Wentworth use what is known as the 'developing map' to obtain a π_1-equivariant harmonic map from M's universal cover to the hyperbolic 3-ball from each connection in the sequence. They then define a sequence of index n dependent rescalings of the hyperbolic metric on the hyperbolic 3-ball and then view the hyperbolic ball with this sequence of rescaled metrics as a sequence of pointed metric spaces. Having taken this view, a theorem of Korevaar and Schoen [KS] is invoked to conclude that the sequence of metric spaces converges in a suitable sense to an $\mathbb{R}$ tree with an action of $\pi_1(\mathrm{M})$ and that the sequence of harmonic maps to the hyperbolic ball has a subsequence that converges to a $\pi_1(\mathrm{M})$ equivariant harmonic map to this $\mathbb{R}$-tree. Daskoloupoulus, Dostoglu and Wentworth subsequently show that this limit $\mathbb{R}$-tree is of the sort that appears in the work of Morgan and Shalen.

To see the Daskoloupoulus, Dostoglu and Wentworth construction in context of Theorem 1.1, note first that the hyperbolic 3-space can be viewed as the space of Hermitian, 2×2 complex matrices with determinant equal to 1, this denoted in what follows by $\mathbb{H}$. The hyperbolic metric is that defined by the norm on $T\mathbb{H}$ given by the trace of the square of a matrix. Let $\{\mathbb{A}_n\}_{n=1,2\ldots}$ denote the sequence considered by Daskoloupoulus, Dostoglu and Wentworth. Fix an index $n\in\{1,2,\ldots\}$. Let $\mathfrak{u}_n$ denote the map from M's universal cover $\mathbb{H}$ that is constructed by Daskoloupoulus, Dostoglu and Wentworth using the developing map with input being $\mathbb{A}_n$. This map appears in the context of Theorem 1.1 as follows: Write $\mathbb{A}_n$ as $A_n + i\mathfrak{a}_n$. The push-forward to M of the differential of u_n is the 1-form $i\mathfrak{a}_n$.

The Daskoloupoulus, Dostoglu and Wentworth renormalization of the hyperbolic metric on $\mathbb{H}$ multiplies the latter by the inverse of Theorem 1.1's constant r_n, this being the L^2 norm of the differential of $\mathfrak{u}_n$. Multiplying the metric on $\mathbb{H}$ by the factor r_n^{-1} is accounted for in Theorem 1.1 by the appearance of $r_n^{-1}\mathfrak{a}_n$ in Item 3) of Theorem 1.1's second bullet.

Theorem 1.1's limit 1-form ν and its singular set Z_S have the following interpretation in the context of [DDW1]: Use $\mathbb{T}$ to denote the [DDW1] limit $\mathbb{R}$ tree and u their limit harmonic map from M's universal cover to $\mathbb{T}$. Gromov and Schoen [GS] (see also [S]) proved that u can be viewed as an honest harmonic function on small balls in the complement of a set with Hausdorff dimension at most 1. The latter set appears in the context of Theorem 1.1 as the inverse image in M's universal cover of the set Z_S. Meanwhile, the differential of u where it is an honest harmonic function is the pull-back of ν to M's universal cover.

Daskoloupoulus, Dostoglu and Wentworth [DDW2], [DDW3] tell a fascinating story about sequences of equivalence classes of flat PSL(2; ℂ) connections in the context of Theorem 1.2, these being sequences of connections on a principal PSL(2; ℂ) bundle over a Riemann surface.

Part 2: A tetrad of closely related notions in 3-manifold topology are singular measured foliations, measured laminations, weighted branched surfaces and maps to $\mathbb{R}$-trees. To paraphrase Hatcher and Oertel [HO], measured laminations are extremely useful generalizations of two central notions in 3-manifold topology, incompressible surfaces and foliations without Reeb components. Weighted branched surfaces and certain sorts of measured laminations are in some sense, two sides of the same coin. These notions were developed extensively by a number of people in concert and separately, among others Oertel [O], Hatcher [Hat], Morgan and Shalen [MS2], [MS3], and Gabai with Ortel [GO]. As explained by [HO], certain sorts of measured singular foliations give rise to measured laminations and weighted branched surfaces, and vice versa. Measured laminations are also closely related to maps from the universal cover to $\mathbb{R}$-trees, this being the central theme in [MS2]. Meanwhile, Bowditch [B] explains how to use measured, singular foliations to define maps to $\mathbb{R}$-trees. The reader should consult these references and the myriad of more recent articles to learn more about this tetrad of notions and their use in 3-manifold topology. As this author is a neophyte on this subject, no more will (or can) be said here except to point out that the notions from the tetrad serve as the dimension 3 generalization of notions that play central roles in research on the structure of Teichmuller space and the mapping class group for surfaces. An elegant account of the 2-dimensional story can found in the beautiful book by Calegari [Ca].

The singular foliation member of the tetrad appears in the context of Theorem 1.1. To say more about how this comes about, suppose for the moment that $\Gamma \to M$ is a compact, embedded Lipshitz curve, that $\mathcal{I}_\Gamma$ is a real line bundle on $M-\Gamma$ and that ν_Γ is a smooth, closed $\mathcal{I}_\Gamma$-valued 1-form on $M-\Gamma$ with zero locus being the union of Γ with a finite set of points in $M-\Gamma$. Data of this sort can be found if M has a suitable branched cover with branch locus being Γ. Let Z_Γ denote the zero locus of ν_Γ. The kernel of ν_Γ defines a 2-plane subbundle in M's tangent bundle over $M-Z_\Gamma$. This subbundle is integrable because ν_Γ is closed, and so it is everywhere tangent to the leaves of a foliation of $M-Z_\Gamma$. Moreover, the folation defined by ν_Γ is transversely measured with the measure given by integration along curves that are transversal to the leaves. This foliation can be viewed as a singular, transversely measured foliation on M. More to the point, if the structure of ν_Γ near Z_Γ is reasonable in a certain precise sense, then this singular foliation will have the local structure that is needed so as to invoke what is said in [HO].

Let Z, $\mathcal{I}$ and ν be as described in the second bullet of Theorem 1.1. Being closed, the $\mathcal{I}$-valued 1-form ν defines a transversely measured foliation on $M-Z$ and so a singular, transversely measured foliation on M. It is a consequence of what is said in the upcoming Propositions 8.1 and in Sections 8.h and 10.d that ν near most of its zero locus has the local form that is required by [HO]. This is the part of Z that is described as a finite length Lipshitz curve with finitely many components by Item 1) from the second bullet of Theorem 1.1. As the complementary part of Z is contained in a finite set of small, disjoint balls, it has little by way of topological significance. In particular, the $\mathcal{I}$-valued 1-form ν can likely be approximated by one that behaves in the desired manner near its zero set.

1.d. An overview of the proof of Theorem 1.1

The first bullet of Theorem 1.1 is proved in Section 2.a. A fundamental Bochner-Weitzenboch formula makes the first bullet little more than a corollary to Uhlenbeck's theorem for connections on principal SO(3) bundles.

The proof of the second bullet of Theorem 1.1 has three components. The first component obtains an L^2_1 limit of a subsequence of the rescaled sequence $\{r_n^{-1}|\mathfrak{a}_n|\}_{n=1,2,\ldots}$, this being the content of Lemma 2.1. The subsequence is hence renumbered from 1. As it turns out, the limit function is bounded, but the convergence is not L^∞ convergence. For this and for other reasons, it proves necessary to modify the sequence $\{r_n^{-1}\mathfrak{a}_n\}_{n=1,2,\ldots}$ so as to obtain a new sequence, this denoted by $\{\mathit{a}_n\}_{n=1,2,\ldots}$, with $\{|\mathit{a}_n|\}_{n=1,2,\ldots}$ being uniformly bounded, converging in L^2_1 to the same limit function, but with the following additional property: The limit L^2_1 function, now denoted by $|\hat{a}_\diamond|$, is defined at each point in M by the rule $|\hat{a}_\diamond| = \limsup_{n\to\infty} |\hat{\mathit{a}}_n|$. The properties of $\{\mathit{a}_n\}_{n=1,2,\ldots}$ are described in Proposition 2.2.

This notion of pointwise convergence brings up a subtle but central issue, which is that pointwise convergence to an L^∞ limit does not imply that the limit is C^0. The second component of the proof of Theorem 1.1 consists of a proof that $|\hat{a}_\diamond|$ is a continuous function. The assertion that $|\hat{a}_\diamond|$ is C^0 is made by Proposition 6.1. The intervening Sections 3, 4 and 5 develop the tools that are needed to prove Proposition 6.1. This proof that $|\hat{a}_\diamond|$ is continuous brings to bear, among other things, Uhlenbeck's theorem for SO(3) connections, the properties of a certain canonical, constant coefficient, first order elliptic operator on $\mathbb{R}^3$ that is defined by the non-linear structure of the curvature, and a gauge theoretic notion of the frequency function that was introduced by Almgren [Al] and used to great success by [HHL] and others to study the singularities of nodal sets of eigenfunctions of the Laplacian.

Uhlenbeck's theorem is brought to bear in Section 3 to study the behavior of the sequence $\{(A_n, \hat{\mathit{a}}_n)\}_{n=1,2,\ldots}$ on sequences of small radius balls about each point of M; the radii of the balls in any given such sequence depend on the chosen point and the index n. With a given point in M fixed, Section 4 uses the analysis in Section 3 to draw conclusions about the sequence whose n'th term is the curvature of the connection A_n on the corresponding index n dependent radius ball about the point. This analysis brings to bear the aforementioned, constant coefficient first order operator. Section 5 defines the gauge theory analog of Almgren's frequency function and proves that it obeys an approximate monotonicity formula, the latter being the gauge theory analog of the monotonicity formula that is exploited by Almgren and others to study the nodal sets of eigenfunctions of the Laplacian. Section 6 uses the conclusions of Sections 3, 4 and 5 to prove that $|\hat{a}_\diamond|$ is continuous. It follows from what is said in Lemma 7.2 that $\{|\hat{\mathit{a}}_n|\}_{n=1,2,\ldots}$ converges to $|\hat{a}_\diamond|$ in the exponent $\frac{1}{4}$ Holder topology near any point where $|\hat{a}_\diamond|$ is positive.

Let Z denote the zero locus of $|\hat{a}_\diamond|$. The third component of the proof of Theorem 1.1 begins in Section 7 with the construction of Theorem 1.1's real line bundle $\mathcal{I}$, this defined over M−Z, and Theorem 1.1's $\mathcal{I}$ valued 1-form ν. Note in this regard that the fact that $|\hat{a}_\diamond|$ is continuous implies that Z is a *closed* subset of M and thus that M−Z is an open set. The fact that Z is closed rules out all sorts of terrible pathologies, among them the possibility that M−Z has empty interior.

Proposition 7.1 asserts in part that $|\nu| = |\hat{a}_\diamond|$ and that ν is a harmonic $\mathcal{I}$ valued 1-form on the complement of its zero locus, Z. Proposition 7.1 also defines a version of Almgren's frequency function for ν, the latter playing the central role in the proof of Theorem 1.1's assertions about Lipshitz curves. Lemma 7.2 proves what is asserted by Items 2) and 3) of Theorem 1.1. Theorem 1.1's assertion about Lipshitz curves is restated as Proposition 10.1 and proved in Section 10. The intervening Sections 8 and 9 supply the tools that are needed for the proof of Proposition 10.1. More is said about this in next paragraph.

If $\mathcal{I}$ is a product bundle, then it extends across Z and the extension writes ν as an $\mathbb{R}$-valued 1-form on the whole of M. In the latter case, what Theorem 1.1 says about Lipschitz graphs is little more than a corollary to what is said in [HHL]. The story when $\mathcal{I}$ is not a product bundle is far more complicated for two reasons, the first being that the derivative of ν on Z is not a priori defined. This is because the derivative of an $\mathcal{I}$ valued section of a vector bundle makes no sense where $\mathcal{I}$ is not defined. This issue can be circumvented when Z is a reasonable set, for example a smooth curve, by looking at ν on a suitable two-fold branched cover over Z. Even so, a strategy of this sort will run afoul of the second complication, which is that Z is determined by ν and to know if Z is nice requires knowing that ν is nice. Since ν determines Z and Z determines ν, there is a chicken versus egg problem to wrestle with. What is done in Sections 8–10 solves this problem by augmenting and reworking strategies from [HHL] so as to avoid their use of comparison functions and linear vector space structures.

1.e. Extensions of Theorem 1.1

This subsection briefly describes various contexts where the techniques and strategies that are used prove Theorem 1.1 could prove useful.

Groups with rank greater than one: As noted in the introduction, there is likely some sort of analog of Theorem 1.1 for Lie groups such as $PSL(n;\mathbb{C})$ for $n > 2$ The statement of a hypothetical $n > 2$ version of Theorem 1.1 will almost surely be more involved by virtue of the fact that the Lie algebra of such a group has non-Abelian subalgebras. In particular, the role that is played in Item 3 of Theorem 1.1's second bullet by $\nu\sigma$ will likely involve a 1-form with values in a twisted vector bundle whose fiber is in a non-Abelian subalgebra of the group's Lie algebra. The singular locus

Z_S may well extend beyond the zero locus of this $\nu\sigma$ analog to account for possible ranks of its stabilizer in the group.

Manifolds of dimension greater than three: There is likely an L^p version of Theorem 1.1 for any p greater than half the dimension of the ambient manifold. A case of special concern is the $p = 2$ case for a manifold of dimension 4. The reason being that this case is relevant to any attempt to define $PSL(2;\mathbb{C})$ analogs of Floer homology and $PSL(2;\mathbb{C})$ analogs of Donaldson's invariants. A bit more is said below about these analogs. The analog of Theorem 1.1 in the case where p is half the dimension of the manifold will be more complicated because this is so for the analogous version of Uhlenbeck's theorem. There may well be additional complications. In any event, the singular set in the case where the dimension is greater than 3 will likely involve some sort of union of rectifiable, codimension 2, Lipshitz submanifolds.

$PSL(2;\mathbb{C})$ Floer homology: Of interest here are two sorts of equations for maps from $\mathbb{R}$ to the space of connections on $P \times_{SO(3)} PSL(2;\mathbb{C})$, these being

$$(1.2)\qquad \bullet\ \tfrac{d}{dt}A = -{*}(F_A - \mathfrak{a}\wedge\mathfrak{a}) \quad \textit{and} \quad \tfrac{d}{dt}\mathfrak{a} = {*}d_A\mathfrak{a}.$$
$$\bullet\ \tfrac{d}{dt}A = -{*}d_A\mathfrak{a} \quad \textit{and} \quad \tfrac{d}{dt}\mathfrak{a} = -{*}(F_A - \mathfrak{a}\wedge\mathfrak{a}).$$

The former is the gradient flow for the real part of the Chern-Simons functional

$$(1.3)\qquad \mathbb{A} \to \mathrm{CS}(\mathbb{A}) = \tfrac{1}{2}\int_M \mathrm{tr}\big(\mathbb{A}\wedge\big(F_{\mathbb{A}} - \tfrac{1}{3}\mathbb{A}\wedge\mathbb{A}\big)\big).$$

The second equation in (1.2) is the gradient flow for the imaginary part of CS, this being the Hamiltonian flow for the real part as defined using a certain canonical symplectic form on the space of $P \times_{SO(3)} PSL(2;\mathbb{C})$ connections. The function CS is decreasing with respect to the gradient flow and constant along the Hamiltonian flow. The imaginary part of CS is constant under the gradient flow and is monotonic under the Hamiltonian flow. Witten [W1], [W2] conjectured that a certain $\mathbb{CP}^1$ parameterized family of linear combinations of the four equations in the first and second bullets of (1.2) can be used to give a gauge theoretic construction of Khovanov homology. See also [GW]. A similar suite of equations were introduced in [Hay]. Such linear combinations also enter in the work of Kapustin and Witten [KW] on the geometric Langlands program. Witten proposed in [W3], [W4] that the equations in (1.2) could be used to compute certain formal path integrals of the Chern Simons functional.

Solutions to the $\mathfrak{a} = 0$ version of the equation in the top bullet of (1.2) are used to define the differential for the SO(3) Floer homology on M; and the L^2 version of Uhlenbeck's theorem for the manifold $\mathbb{R} \times M$ plays a central role in the proof that this differential has square zero. See [Fl] and also [D3]. This being the case, it is almost sure bet that some sort of $PSL(2;\mathbb{C})$ extension of Uhlenbeck's theorem will be needed to define analogous algebraic structures using solutions to the equations in (1.2) and to the sorts of linear combinations that are introduced by Witten.

What follows is a likely relevant observation with regards to any such extension: The functional $\mathbb{A} \to \int_M |d_A{*}\mathfrak{a}|^2$ is constant along all of these flows.

$PSL(2;\mathbb{C})$ Self duality: Let X denote a smooth, compact and oriented 4 dimensional Riemannian manifold. As noted by Witten [W1], the equations in (1.2) have analogs on X, these being equations for a pair consisting of a connection on a principal SO(3) bundle over X and a 1-form with values in the associated vector bundle with fiber $\mathfrak{su}(2)$ given by the adjoint representation. Let $(A, \mathfrak{a})$ denote such a pair. Use the metric to define the respective bundles of self-dual and anti-self dual 2-forms. The orthogonal projection to these respective bundles are denoted by Π^+ and Π^-. The analog of the top equation in (1.2) reads

$$\Pi^+(F_A - \mathfrak{a} \wedge \mathfrak{a}) = 0 \quad \textit{and} \quad \Pi^- d_A \mathfrak{a} = 0 \quad \textit{and} \quad d_A{*}\mathfrak{a} = 0. \tag{1.4}$$

There are corresponding analogs to the lower equation in (1.2) and to suitable linear combinations of the four equations in (1.2). See [DK] to read about the applications of the $\mathfrak{a} = 0$ version of (1.4)

Witten and Vafa [VW] proposed an alternate generalization of the equations in (1.2), this being a system of equations for a connection on a principal SO(3) bundle and a self-dual 2-form with values in the same associated bundle with fiber $\mathfrak{su}(2)$. Let $(A, \mathfrak{w})$ denote such a pair. The Witten-Vafa equations are written schematically as

$$\Pi^+(F_A - [\mathfrak{w}; \mathfrak{w}]) = 0 \quad \textit{and} \quad d_A \mathfrak{w} = 0, \tag{1.5}$$

where $[\cdot, \cdot]$ here denotes a certain canonical, bilinear, symmetric fiber preserving map that is defined by the metric's Hodge dual and the commutator on $\mathfrak{su}(2)$.

1.f. Table of contents and conventions

What follows is a table of contents for this article.

The paper employs two conventions throughout. The first convention has c_0 denoting a number that is greater than 100 whose value does not depend on any of the salient issues under consideration in a given assertion. The value of c_0 in any given appearance can depend on the particulars of M and its Riemannian metric; and also on the upper bound for Theorem 1.1's sequence $\{\mathfrak{F}(\mathbb{A}_n)\}_{n=1,2,\ldots}$. However, under no circumstances does it depend on the index n. The value of c_0 in successive appearances can be assumed to increase.

The second convention concerns what is denoted by χ. This is a fixed, smooth and nonincreasing function on $\mathbb{R}$ that equals 1 on $(-\infty, \frac{1}{4}]$ and equals 0 on $[\frac{3}{4}, \infty)$. A favorite version is chosen now and used throughout the paper.

2. L^2_1 and pointwise limits

The four subsections that follow comprise what Section 1.c described as the Part 1 of the proof of Theorem 1.1. The first Section 2.a begins with a Bochner-Weitzenboch formula that plays a central role in the remaining subsections. It then uses this formula to prove the first bullet of Theorem 1.1. The Section 2.b begins the proof of Theorem 1.1's second bullet with the construction an L^2_1 limit from the sequence $\{r_i^{-1}|\mathfrak{a}_i|\}_{i=1,2,\ldots}$. The third subsection modifies the sequence $\{\mathfrak{a}_i\}_{i=1,2,\ldots}$ to obtain a new sequence that allows greater control over the limit in Section 1.b. The salient features of this new sequence are summarized by Proposition 2.2. The final subsection proves a central lemma that is used to prove Proposition 2.2.

2.a. The Bochner-Weitzenboch formula

Let A denote a given connection on P and $\mathfrak{a}$ denote a section of $P\times_{SO(3)}\mathfrak{su}(2)$. It is important to keep in mind that there is a Bochner-Weitzenbock formula that writes

$$(2.1)\qquad *d_A{*}d_A\mathfrak{a} - d_A{*}d_A{*}\mathfrak{a} = \nabla_A^\dagger\nabla_A\mathfrak{a} + *({*}F_A\wedge\mathfrak{a} + \mathfrak{a}\wedge{*}F_A) + \mathrm{Ric}(\mathfrak{a}),$$

with Ric(·) denoting here the Ricci curvature in its guise as a homomorphism of T^*M.

It proves useful to write (2.1) as an equality between integrals over M. To this end, let f denote a chosen C^2 function on M and let $r\in[1,\infty)$ denote a chosen positive number. Take the inner product of both sides of (2.1) with the section f a of $P\times_{SO(3)}\mathfrak{su}(2)$ and integrate over M. An integration by parts to obtain an expression with only first derivatives of $\mathfrak{a}$ leads to the resulting integral identity to the following one:

$$(2.2)\qquad \int_M f(|d_A\mathfrak{a}|^2 + |d_A{*}\mathfrak{a}|^2 + |r^{-1}F_A - r\mathfrak{a}\wedge\mathfrak{a}|^2)$$
$$= \tfrac{1}{2}\int_M d{*}df|\mathfrak{a}|^2 + \int_M f\big(|\nabla_A\mathfrak{a}|^2 + r^2|\mathfrak{a}\wedge\mathfrak{a}|^2 + r^{-2}|F_A|^2 + \mathrm{Ric}(\langle\mathfrak{a}\otimes\mathfrak{a}\rangle)\big)$$
$$- \int_M (df\wedge *\langle\mathfrak{a}({*}d_A{*}\mathfrak{a})\rangle + df\wedge\langle\mathfrak{a}\wedge{*}d_A\mathfrak{a}\rangle).$$

The notation here has $\langle\mathfrak{a}\otimes\mathfrak{a}\rangle$ denoting the symmetric section of $T^*M\otimes T^*M$ that is defined as follows: Fix an orthonormal frame for T^*M at any given point and use $\{\mathfrak{a}_\alpha\}_{\alpha\in\{1,2,3\}}$ to denote the coefficients of a when written using the chosen frame. The corresponding coefficients of $\langle\mathfrak{a}\otimes\mathfrak{a}\rangle$ are $\{\langle\mathfrak{a}_\alpha\mathfrak{a}_\beta\rangle\}_{\alpha,\beta\in\{1,2,3\}}$. Meanwhile, the Ricci tensor in (2.2) is viewed using the metric as a linear functional on $T^*M\otimes T^*M$.

Proof of the first bullet of Theorem 1.1. Take $f = 1$ and $r = 1$ in (2.2) to see that

$$(2.3)\qquad \mathfrak{F}(A+i\mathfrak{a}) = \int_M (|\nabla_A\mathfrak{a}|^2 + |\mathfrak{a}\wedge\mathfrak{a}|^2 + |F_A|^2 + \mathrm{Ric}(\langle\mathfrak{a}\otimes\mathfrak{a}\rangle)).$$

It follows as a consequence that

$$(2.4)\qquad \int_M (|\nabla_A\mathfrak{a}|^2 + |\mathfrak{a}\wedge\mathfrak{a}|^2 + |F_A|^2) \le \mathfrak{F}(A+i\mathfrak{a}) + c_0\int_M |\mathfrak{a}|^2.$$

To exploit (2.4), suppose that $\{\mathbb{A}_n = A_n + i\mathfrak{a}_n\}_{n=1,2,\ldots}$ is a sequence with both $\{\mathfrak{F}(\mathbb{A}_n)\}_{n=1,2,\ldots}$ and $\{\int_M |\mathfrak{a}_n|^2\}_{n=1,2,\ldots}$ being bounded. Then the sequence $\{\int_M |F_{A_n}|^2\}_{n=1,2,\ldots}$ is bounded and so Uhlenbeck's theorem finds a subsequence of $\{\mathbb{A}_n\}_{n=1,2,\ldots}$ (hence renumbered consecutively from 1) and corresponding sequence of automorphisms of P, this denoted by $\{g_n\}_{n=1,2,\ldots}$ such that $\{g_n{}^*A_n\}_{n=1,2,\ldots}$ converges weakly in the L^2_1 topology to an L^2_1 connection on P. Denote the latter by A. Meanwhile, the sequence $\{\int_M |\nabla_{A_n}\mathfrak{a}_n|^2\}_{n=1,2,\ldots}$ is also bounded, and this implies that $\{g_n{}^*a_n\}_{n=1,2,\ldots}$ has a subsequence that converges weakly in the L^2_1 topology on the space of sections of $P\times_{SO(3)}\mathfrak{su}(2)$ to an L^2_1 section, this denoted by $\mathfrak{a}$. The pair $(A, \mathfrak{a})$ define the desired limit L^2_1 connection on the bundle $P\times_{SO(3)} PSL(2;\mathbb{C})$.

2.b. Renormalization and L^2_1 convergence

This subsection begins the proof of the second bullet of Theorem 1.1. The input is a sequence $\{\mathbb{A}_n = A_n + i\mathfrak{a}_n\}_{n=1,2,\ldots}$ with $\{\mathfrak{F}(\mathbb{A}_n)\}_{n=1,2,\ldots}$ being bounded, but not so the sequence whose n'th term is the L^2 norm of $\mathfrak{a}_n$. The latter sequence is assumed to be unbounded with no finite limits. Fix n and set $\hat{a}_n = r_n^{-1}\mathfrak{a}_n$ with r_n denoting the L^2 norm of the section $\mathfrak{a}_n$. Fix a C^2 function, f, on M. Multiply the $(A, \mathfrak{a}) = (A_n, \mathfrak{a}_n)$ and $r = 1$ version of (2.2) by r_n^{-2} to see that

$$(2.5)\quad \lim_{n\to\infty} \tfrac{1}{2}\int_M d^\dagger df|\hat{a}_n|^2 + \lim_{n\to\infty}\int_M f\big(|\nabla_{A_n}\hat{a}_n|^2 + r_n^2|\hat{a}_n\wedge\hat{a}_n|^2 + r_n^{-2}|F_{A_n}|^2 + \mathrm{Ric}(\langle\hat{a}_n\otimes\hat{a}_n\rangle)\big) = 0.$$

This is so because the terms with $r_n^{-1}d_{A_n}\mathfrak{a}_n$ and $r_n^{-1}d_{A_n}{*}\mathfrak{a}_n$ and $r_n^{-2}|F_{A_n} - \mathfrak{a}_n\wedge\mathfrak{a}_n|$ have limit zero as $n\to\infty$.

Lemma 2.1. *There exists $\kappa > 1$ with the following significance: Let $\{\mathbb{A}_n = A_n + i\mathfrak{a}_n\}_{n=1,2,\ldots}$ denote a sequence of connections on $P\times_{SO(3)} SL(2;\mathbb{C})$ with $\{\mathfrak{F}(\mathbb{A}_n)\}_{n=1,2,\ldots}$ being bounded, but with the sequence whose n'th term is the L^2 norm of $\mathfrak{a}_n$ being unbounded with no finite limits. For each $n\in\{1,2,\ldots\}$, set r_n to be the L^2 norm of $\mathfrak{a}_n$ and set $\hat{a}_n = r_n^{-1}\mathfrak{a}_n$. There is a subsequence in $\{1,2,\ldots\}$, hence renumbered consecutively from 1 with the properties that are listed below.*

- *The sequence $\{|\hat{a}_n|\}_{n=1,2,\ldots}$ converges weakly in the L^2_1 topology and strongly in each $p < 6$ version of the L^p topology to an L^2_1 function on M, this denoted by $|\hat{a}|$. The L^2 norm of $|\hat{a}|$ is 1 and its L^2_1 norm* *is bounded by κ. Moreover, $|\hat{a}|$ defines an L^∞ function with $|\hat{a}| < \kappa$ almost everywhere.*

- *The sequence $\{\langle \hat{a}_n \otimes \hat{a}_n \rangle\}_{n=1,2,\ldots}$ converges strongly in each $q \leq 3$ version of the L^q topology on the space of sections of $T^*M \otimes T^*M$ and weakly in the L^3 topology. The limit section of $T^*M \otimes T^*M$ is denoted by $\langle \hat{a} \otimes \hat{a} \rangle$; it is in L^∞ and its trace is the function $|\hat{a}|^2$.*
- *Let f denote any given C^2 function on M. The three sequences $\{\int_M f|\nabla_{A_n}\hat{a}_n|^2\}_{n=1,2,\ldots}$, $\{\int_M r_n^2 f|\hat{a}_n \wedge \hat{a}_n|^2\}_{n=1,2,\ldots}$ and $\{\int_M r_n^{-2} f|F_{A_n}|^2\}_{n=1,2,\ldots}$ each converge. These limits are denoted by $Q_{\nabla,f}$, $Q_{\wedge,f}$ and $Q_{F,f}$.*
  - *a) Each of $Q_{\nabla,f}$, $Q_{\wedge,f}$ and $Q_{F,f}$ is bounded by κ times the sum of the supremum norm and the L^2_1 norm of f. Moreover, $Q_{\wedge,f} = Q_{F,f}$; and if $f \geq 0$, then $Q_{\nabla,f} \geq \int_M f|d|\hat{a}||^2$.*
  - *b) $\frac{1}{2}\int_M d^\dagger df|\hat{a}|^2 + Q_{\nabla,f} + 2Q_{\wedge,f} = -\int_M f\mathrm{Ric}(\langle \hat{a} \otimes \hat{a} \rangle)$.*

Proof of Lemma 2.1. The assertion in the first bullet to the effect that $\{|\hat{a}_n|\}_{n=1,2,\ldots}$ has a subsequence which converges weakly in the L^2_1 topology follows if the sequence has uniformly bounded L^2_1 norm. That this is the case follows from the $f = 1$ version because

$$|d|\mathfrak{b}|| \leq |\nabla_A \mathfrak{b}| \tag{2.6}$$

with A being any connection on P and $\mathfrak{b}$ being any tensor valued section of $P \times_{SO(3)} \mathfrak{su}(2)$. The L^∞ assertions are proved momentarily. The proof of the remaining bullets do not require the L^∞ assertion in the first bullet.

With regards to the second bullet, the inequality in (2.6) with (2.4) implies that the sequence $\{|\hat{a}_n|^{-1}\langle \hat{a}_n \otimes \hat{a}_n \rangle\}_{n=1,2,\ldots}$ is bounded in the L^2_1 topology. It follows as a consequence that a subsequence converges weakly in the L^2_1 topology and strongly in the L^p topology for $p < 6$. Since this is also the case for $\{|\hat{a}_n|\}_{n=1,2,\ldots}$, the product sequence $\{\langle \hat{a}_n \otimes \hat{a}_n \rangle\}_{n\in 1,2,\ldots}$ converges strongly in the L^q topology for $q < 3$.

The existence of a subsequence that makes the third bullet true follows from (2.5) because the space of C^2 functions on M has a dense, countable subset. The equality $Q_{\wedge,f} = Q_{F,f}$ follows because $\lim_{n\to\infty} r_n^{-2}\int_M |F_{A_n} - r_n^2 \hat{a}_n \wedge \hat{a}_n|^2 = 0$; and the $f \geq 0$ upper bound on $Q_{\nabla,f}$ follows from (2.6).

The four steps that follow momentarily prove that $|\hat{a}|$ defines an L^∞ function with the asserted norm bound. By way of a reminder, a nonnegative measurable function defines an L^∞ function if it has an upper bound on the complement of a measure zero set.

Step 1. Fix $p \in M$ and let G_p denote the Green's function with pole at p for the operator $d^\dagger d + 1$, this being a smooth function on $M-p$ which extends to the whole of M as an L^q function for any $q < 3$. More to the point, $G_p(\cdot) \le c_0 \mathrm{dist}(p, \cdot)^{-1}$, a fact that is exploited more than once in this section. Keep in mind as well that $|dG_p| \le c_0 \mathrm{dist}(\mathfrak{p}, \cdot)^{-2}$.

What follows directly describes a $(0, 1]$ parametrized sequence of C^2 approximations to G_p that converge to G_p as the parameter limits to 0 in the L^q topology on $C^2(M)$ for $q < 3$ and in the C^2 topology on compact subsets of $M-p$. The $\varepsilon \in (0, 1]$ member of this sequence is denoted by $f_{p,\varepsilon}$. To define $f_{p,\varepsilon}$, let $v_{p,\varepsilon}$ denote the volume of the ball of radius ε centered at p and define $\delta_{p,\varepsilon}$ to be the function given by be $v_{p,\varepsilon}^{-1}$ on this radius ε ball and zero on its complement. The function $f_{p,\varepsilon}$ is the solution on M to the equation $d^\dagger d f + f = \delta_{p,\varepsilon}$.

A depiction of $f_{p,\varepsilon}$ near p is given momentarily. This depiction introduces a constant, z_p, with norm bounded by c_0. This constant is defined by the depiction of G_p in Gaussian coordinates near p, this having the form $x \to G_p(x) = \frac{1}{4\pi|x|} - z_p + \dots$ where the unwritten terms have norm bounded by $c_0|x|$. Note in this regard that the $\mathbb{R}^3$ analog of G_p with pole at the origin is the function $x \to \frac{1}{4\pi|x|}e^{-|x|}$ and so the analog on $\mathbb{R}^3$ of z_p is $-\frac{1}{4\pi}$.

With z_p understood, then the function $f_{p,\varepsilon}$ can be written using Gaussian coordinates near p as $f_{p,\varepsilon} = g_\varepsilon + \mathfrak{e}_p$ where the function $x \to g_\varepsilon(x)$ is defined by the rule

$$
\begin{aligned}
g_\varepsilon(x) &= \frac{1}{4\pi|x|} - z_p \quad \textit{for } |x| > \varepsilon \quad \textit{and} \\
g_\varepsilon(x) &= \frac{3}{8\pi\varepsilon}\left(1 - \frac{|x|^2}{3\varepsilon^2}\right) - z_p \quad \textit{for } |x| \le \varepsilon;
\end{aligned}
\tag{2.7}
$$

and where $\mathfrak{e}_p$ is a continuous function that is smooth on the complement of the origin and such that $|\mathfrak{e}_p| \le c_0|x|$ and $|d\mathfrak{e}_p| \le c_0$.

Step 2. The $f = f_{p,\varepsilon}$ version of the equality given by Item b) of the third bullet of Lemma 2.1 reads

$$
\tfrac{1}{2}\int_M \delta_{p,\varepsilon}|\hat{a}|^2 + Q_{\nabla, f_{p,\varepsilon}} + 2Q_{\wedge, f_{p,\varepsilon}} = \int_M f_{p,\varepsilon}\big(\tfrac{1}{2}|\hat{a}|^2 - \mathrm{Ric}(\langle \hat{a} \otimes \hat{a}\rangle)\big). \tag{2.8}
$$

The right hand side of (2.8) converges as $\varepsilon \to 0$ because $|\hat{a}|^2$ is an L^2 function. It follows from the lemma's first two bullets that the limit is the integral of $G_p(\frac{1}{2}|\hat{a}|^2 - \mathrm{Ric}(\langle \hat{a} \otimes \hat{a}\rangle))$.

Step 3. The functions in the family $\{g_\varepsilon\}_{\varepsilon\in(0,1]}$ are such that $g_\varepsilon \geq g_{\varepsilon'}$ for $\varepsilon < \varepsilon'$. This has the following consequence for the left hand side of (2.8): Let $\bullet$ denote either ∇ or $\wedge$. Then $Q_{\bullet,f_{p,\varepsilon}}$ can be written as $Q_{\bullet,g_\varepsilon} + \mathrm{P}_{\bullet(p,\varepsilon)}$ where the sequence $\{\mathrm{P}_{\bullet,(p,\varepsilon)}\}_{\varepsilon\in(0,1]}$ converges as ε limits to zero and where the sequence $\{Q_{\bullet,g_\varepsilon}\}_{\varepsilon\in(0,1]}$ is bounded and monotonically increasing as ε decreases to zero. It follows that this sequence also has a unique limit as ε limits to zero. Thus, the sequence $\{Q_{\bullet,f_{p,\varepsilon}}\}_{\varepsilon\in(0,1]}$ has a unique limit as ε limits to zero. The limit is denoted by $Q_{\bullet,G_p}$.

Step 4. Consider now the integral of $\delta_{p,\varepsilon}|\hat{a}|^2$ that appears on the left hand side of (2.8). This integral is positive, and it follows from (2.8) that the various $\varepsilon \in (0,1]$ versions are uniformly bounded as $\varepsilon \to 0$ with a $p \in M$ independent upper bound. This implies in particular that $|\hat{a}|$ is bounded by c_0 on the complement of a measure zero set. Meanwhile, the $\varepsilon \to 0$ limit of the integral of $\delta_{p,\varepsilon}|\hat{a}|^2$ converges to $|\hat{a}|^2$ on the complement of such a set ([Fo], Theorem 3.18). This being the case, it follows from what was said in Steps 2 and 3 that $|\hat{a}|$ can be modified on a measure zero set so that

$$\frac{1}{2}|\hat{a}|^2(p) + Q_{\nabla,G_p} + 2Q_{\wedge,G_p} = -\int_M G_p\big(\tfrac{1}{2}|\hat{a}|^2 - \langle \mathrm{Ric}(\hat{a},\hat{a})\rangle\big) \tag{2.9}$$

for each $p \in M$. This formula implies the asserted norm bound.

2.c. Second derivative bounds

Let A denote for the moment a connection on P and let $\mathfrak{a}$ denote a section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. Define $\mathcal{q}_A(\mathfrak{a})$ to be the section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ given by

$$\mathcal{q}_A(\mathfrak{a}) = \nabla_A^\dagger \nabla_A \mathfrak{a} + *(*F_A \wedge \mathfrak{a} + \mathfrak{a} \wedge *F_A) + \mathrm{Ric}((\cdot)\otimes\mathfrak{a}). \tag{2.10}$$

Note in particular that $\mathcal{q}_A(\mathfrak{a})$ is the expression on the right hand side of (2.1).

Let $\{(A_n, \hat{a}_n)\}_{n\in 1,2,\ldots}$ denote the sequence from Theorem 1.1. By way of a look ahead, the subsequent analysis of Lemma 2.1's limit function $|\hat{a}|$ requires a uniform bound for the sequence whose n'th term is the L^2 norm of the $A = A_n$ and $\mathfrak{a} = \hat{a}_n$ version of $\mathcal{q}_A(\mathfrak{a})$. The sad fact is that Theorem 1.1's assumptions are not strong enough to guarantee such a bound. The next proposition circumvents this problem. This proposition uses $\|\cdot\|_2$ to denote the L^2 norm of an indicated tensor field valued section of $P \times_{SO(3)} \mathfrak{su}(2)$.

Proposition 2.2. *Suppose that* $\{\mathbb{A}_n = A_n + i\mathfrak{a}_n\}_{n=1,2,\ldots}$ *is a sequence of connections on* $P \times_{SO(3)} SL(2;\mathbb{C})$ *with* $\{\mathfrak{F}(\mathbb{A}_n)\}_{n=1,2,\ldots}$ *being bounded. For each* $n \in \{1, 2, \ldots\}$*, let* r_n *denote the* L^2 *norm of* $\mathfrak{a}_n$ *and assume that* $\{r_n\}_{n=1,2,\ldots}$ *is divergent with no finite limits. There exists a number* $\kappa > 1$ *that depends only on the upper bound for the sequence* $\{\mathfrak{F}(\mathbb{A}_n)\}_{n=1,2,\ldots}$*, and there exists a sequence* $\{\hat{a}_n\}_{n=1,2\ldots}$ *of sections of* $P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ *such that each* $n \in \{1, 2, \ldots\}$ *version of Items a)–e) below holds.*

a) $\|\nabla_{A_n}(\hat{a}_n - r_n^{-1}\mathfrak{a}_n)\|_2 + \kappa^2 r_n\|\hat{a}_n - r_n^{-1}\mathfrak{a}_n\|_2 < \kappa r_n^{-1}$.
b) $\int_M (|\nabla_{A_n}\hat{a}_n|^2 + r_n^2|\hat{a}_n \wedge \hat{a}_n|^2 + r_n^{-2}|F_{A_n}|^2 + \mathrm{Ric}(\langle \hat{a}_n \otimes \hat{a}_n\rangle)) < \kappa r_n^{-2}$.
c) $\|d_{A_n}\hat{a}_n\|_2^2 + \|d_{A_n}{*}\hat{a}_n\|_2^2 + r_n^{-2}\|F_{A_n} - r_n^2\hat{a}_n \wedge \hat{a}_n\|_2^2 < \kappa r_n^{-2}$.
d) $\|\mathfrak{q}_{A_n}(\hat{a}_n)\|_2 < \kappa$.
e) $\sup_M |\hat{a}_n| < \kappa$.

Moreover, there is a subsequence $\Lambda \subset \{1, 2, \ldots\}$ *with the properties listed below.*

- *The sequence* $\{|\hat{a}_n|\}_{n\in\Lambda}$ *is bounded in* L^2_1*, it converges weakly in the* L^2_1 *topology and strongly in* L^q *topology for* $q < 6$*. No member vanishes on an open set in* M.
- *The* L^2 *limit of* $\{|\hat{a}_n|\}_{n\in\Lambda}$ *is in* L^∞*. The limit is denoted in what follows by* $|\hat{a}_\diamond|$*. This function is defined pointwise by the rule* $|\hat{a}_\diamond|(p) = \limsup_{n\in\Lambda} |\hat{a}_n|(p)$.
- *The sequence* $\{\langle \hat{a}_n \otimes \hat{a}_n\rangle\}_{n\in\Lambda}$ *converges strongly in any* $q < 6$ *version of the* L^6 *topology on the space of sections of* $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$*, and it converges weakly in the* L^6 *topology. The limit section is* $\langle \hat{a}_\diamond \otimes \hat{a}_\diamond\rangle$.
- *Let* f *denote a given* C^2 *function. The three sequences* $\{\int_M f|\nabla_{A_n}\hat{a}_n|^2\}_{n\in\Lambda}$*,* $\{\int_M r_n^2 f|\hat{a}_n \wedge \hat{a}_n|^2\}_{n\in\Lambda}$ *and* $\{\int_M r_n^{-2} f|F_{A_n}|^2\}_{n\in\Lambda}$ *converge with respective limits that are denoted in what follows by* $Q_{\nabla,f}$*,* $Q_{\wedge,f}$ *and* $Q_{F,f}$*. These are such that* $Q_{\wedge,f} = Q_{F,f}$ *and any* $f \geq 0$ *version of* $Q_{\nabla,f}$ *is no less than* $\int_M f|d|\hat{a}_\diamond||^2$*. Moreover,*

$$\tfrac{1}{2}\int_M d{*}df|\hat{a}_\diamond|^2 + Q_{\nabla,f} + 2Q_{\wedge,f} + \int_M f\mathrm{Ric}(\langle \hat{a}_\diamond \otimes \hat{a}_\diamond\rangle) = 0.$$

- *The sequence* $\{\mathfrak{q}_{A_n}(\hat{a}_n)\}_{n\in\Lambda}$ *has a weak limit in the* L^2 *topology. Moreover, if* f *is any* L^2 *function, then* $\lim_{n\in\Lambda}\int_M f\langle \hat{a}_n \wedge {*}\mathfrak{q}_{A_n}(\hat{a}_n)\rangle = 0$.
- *Fix* $p \in M$*. The two sequences indexed by* Λ *with respective n'th terms given by the integral of* $G_p|\nabla_{A_n}\hat{a}_n|^2$ *and the integral of* $G_p r_n^2|\hat{a}_n \wedge \hat{a}_n|^2$ *are less than* κ*. Introduce by way of notation* $Q_{\diamond,p}$ *to denote the lim-inf of*

the sequence with n'th term the integral of $G_p(|\nabla_{A_n}\hat{a}_n|^2 + 2r_n^2|\hat{a}_n \wedge \hat{a}_n|^2)$. *The function* $|\hat{a}_\Diamond|^2$ *obeys the equation*

$$\tfrac{1}{2}|\hat{a}_\Diamond|^2(p) + Q_{\Diamond,p} = -\int_M G_p\big(\tfrac{1}{2}|\hat{a}_\Diamond|^2 - \mathrm{Ric}(\langle \hat{a}_\Diamond \otimes \hat{a}_\Diamond\rangle)\big).$$

The proof of Proposition 2.2 requires a preliminary lemma that directly asserts a part of the proposition.

Lemma 2.3. *Let* $\{(A_n, \hat{a}_n)\}_{\{n\in 1,2,\dots\}}$ *denote the sequence in Lemma 2.1. There exists* $\kappa > 1$ *that depends only on the upper bound for the sequence* $\{\mathfrak{F}(\mathbb{A}_n)\}_{n=1,2,\dots}$ *and, given* $z > \kappa$, *a sequence* $\{\hat{a}_n\}_{n=1,2\dots}$ *of sections of* $P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ *such that for each* $n \in \{1, 2, \dots\}$,

- $\|\hat{a}_n - \hat{a}_n\|_2 \le z^{-1/2} r_n^{-2}$.
- $\|d_{A_n}\hat{a}_n\|_2^2 + \|d_{A_n}{*}\hat{a}_n\|_2^2 \le \kappa r_n^{-2}$.
- $\|q_{A_n}(\hat{a}_n)\|_2 \le \kappa z$.

This lemma is proved in the next subsection. Accept as gospel truth for the moment.

Proof of Proposition 2.2. The proof has three parts. Fix $z \ge c_0$ so as to invokes Lemma 2.3. As is evident in the proof, a large choice for z, but in any event less than c_0 suffices to prove the assertions of the proposition. This said, view z for now as a chosen parameter. By way of notation, the proof introduces $\|\cdot\|_q$ to denote a given $q \in [1, \infty]$ version of the L^q norm on tensor valued sections of $P \times_{SU(2)} \mathfrak{su}(2)$. The proof also denotes any given $n \in \{1, 2, \dots\}$ version of $r_n^{-1}\mathfrak{a}_n$ by $\hat{a}_n$.

Part 1: This first part proves Item e) of Proposition 2.2 and the assertion in the proposition's sixth bullet that the integrals of each $n \in \Lambda$ version of $G_p|\nabla_{A_n}\hat{a}_n|^2$ and $G_p r_n^2|\hat{a}_n \wedge \hat{a}_n|^2$ has a $p \in M$ and index n independent upper bound. There are four steps.

Step 1. Integrate by parts to rewrite the second bullet of Lemma 2.3 as

$$\int_M (|\nabla_{A_n}\hat{a}_n|^2 + 2\langle *F_{A_n} \wedge \hat{a}_n \wedge \hat{a}_n\rangle + \mathrm{Ric}(\langle \hat{a}_n \otimes \hat{a}_n\rangle)) \le \kappa r_n^{-2}. \tag{2.11}$$

To exploit (2.11), use the first bullets of Lemmas 2.1 and 2.3 to see that $\|\hat{a}_n\|_2 = 1 + \mathfrak{e}$ with $|\mathfrak{e}| \le c_0 z^{-1/2} r_n^{-2}$. Next, use the $f = 1$ version of Lemma 2.1's third bullet to bound the integral of $|\langle *F_{A_n} \wedge \hat{a}_n \wedge \hat{a}_n\rangle|$ by c_0. Then write $\hat{a}_n$ as $\hat{a}_n + (\hat{a}_n - \hat{a}_n)$ and use the first bullet of Lemma 2.3 to go from the preceding bound to a bound on the integral of $|\langle *F_{A_n} \wedge \hat{a}_n \wedge \hat{a}_n\rangle|$

by $c_0 + c_0\|F_{A_n}\|_2\|\hat{a}_n - \hat{a}_n\|_6^{1/2}(\|\hat{a}_n\|_6 + \|\hat{a}_n\|_6)^{3/2}$. Here and below, $\|\cdot\|_6$ denotes the L^6 norm. Note that $\|F_{A_n}\|_2\|\hat{a}_n - \hat{a}_n\|_6^{1/2} \le c_0$ by Lemmas 2.1 and 2.3; and $\|\hat{a}_n\|_6 \le c_0$ due to Lemma 2.1. The L^6 norm of $\hat{a}_n$ is bounded by $c_0(\|\nabla_{A_n}\hat{a}_n\|_2 + \|\hat{a}_n\|_2)$ using (2.6) and the Sobolev inequality that bounds a function's L^6 norm by c_0 times its L^2_1 norm. These bounds give a $c_0(1 + \|\nabla_{A_n}\hat{a}_n\|_2^{3/2})$ bound for the integral of $|\langle *F_{A_n} \wedge \hat{a}_n \wedge \hat{a}_n\rangle|$ and thus (2.11) gives a c_0 bound for $\|\nabla_{A_n}\hat{a}_n\|_2$. Note for use below that this leads back to a c_0 bound $\|\hat{a}_n\|_6$.

Use the c_0 bounds for the L^6 norm of $\hat{a}_n$ and the L^2 norm of $F_{A_n} - r_n^2\hat{a}_n \wedge \hat{a}_n$ in (2.11) to see that

$$\int_M (|\nabla_{A_n}\hat{a}_n|^2 + 2r_n^2\langle *(\hat{a}_n \wedge \hat{a}_n) \wedge \hat{a}_n \wedge \hat{a}_n\rangle) \le c_0. \tag{2.12}$$

Write $\hat{a}_n = \hat{a}_n - \hat{a}_n + \hat{a}_n$ and use this decomposition in (2.12) to see that

$$\int_M (|\nabla_{A_n}\hat{a}_n|^2 + 2r_n^2|\hat{a}_n \wedge \hat{a}_n|^2) \le c_0(1 + r_n^2\|\hat{a}_n - \hat{a}_n\|_2(\|\hat{a}_n\|_6 + \|\hat{a}_n\|_6)\|\hat{a}_n\|_6^2). \tag{2.13}$$

Lemma 2.3's first bullet, the c_0 bounds for the L^6 norms of $\hat{a}_n$ and $\hat{a}_n$ and (2.13) imply that

$$\int_M (|\nabla_{A_n}\hat{a}_n|^2 + 2r_n^2|\hat{a}_n \wedge \hat{a}_n|^2) \le c_0. \tag{2.14}$$

This last inequality implies that $\|\nabla_{A_n}\hat{a}_n\|_2^2$ and $r_n^2\|\hat{a}_n \wedge \hat{a}_n\|_2^2$ are both bounded by c_0.

Step 2. Let $(A, \mathfrak{a})$ denote for the moment a given pair of connection on P and section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. Let f denote a given C^2 function. Integrate by parts in the right most integral on its right hand side of (2.2) and then use (2.1) with the definition of $q_A(\mathfrak{a})$ to obtain the equality

$$\frac{1}{2}\int_M d^\dagger df|\mathfrak{a}|^2 + \int_M f(|\nabla_A\mathfrak{a}|^2 + 2\langle *F_A \wedge \mathfrak{a} \wedge \mathfrak{a}\rangle + \mathrm{Ric}(\langle \mathfrak{a} \otimes \mathfrak{a}\rangle)) - \int_M f\langle \mathfrak{a} \wedge *q_A(\mathfrak{a})\rangle = 0. \tag{2.15}$$

Since A and $\mathfrak{a}$ are smooth, this equality also holds when f is such that $d^\dagger df$ is a distribution. In particular, it holds with f being the Green's function of $d^\dagger d + 1$ with pole at any given point. This understood, fix $p \in M$ and let G_p again denote the Green's function for $d^\dagger d + 1$ with pole at p. The corresponding version of (2.11) reads

$$\tag{2.16} \tfrac{1}{2}|\mathfrak{a}|^2(p) + \int_M G_p(|\nabla_A \mathfrak{a}|^2 + 2\langle *F_A \wedge \mathfrak{a} \wedge \mathfrak{a}\rangle) = \int_M G_p(\tfrac{1}{2}|\mathfrak{a}|^2 - \mathrm{Ric}(\langle \mathfrak{a} \otimes \mathfrak{a}\rangle) + \langle \mathfrak{a} \wedge * \mathcal{q}_A(\mathfrak{a})\rangle).$$

A bound for the right hand side of this inequality can be had by using the fact that $G_p(\cdot)$ is bounded by $c_0 \mathrm{dist}(p, \cdot)^{-1}$. This being the case, the left most two terms on the right hand side of (2.16) contribute at most $c_0\|\mathfrak{a}\|_4^2$ to the absolute value of the right hand side. What with (2.6), a dimension 3 Sobolev inequality bounds this by $c_0(\|\nabla_A \mathfrak{a}\|_2^2 + \|\mathfrak{a}\|_2^2)$.

Meanwhile, the term with $\mathcal{q}_A(\mathfrak{a})$ in (2.16) contributes at most

$$\tag{2.17} c_0\Big(\sup_{p\in M} \|\mathrm{dist}(p, \cdot)^{-1}\mathfrak{a}\|_2\Big)\|\mathcal{q}_A(\mathfrak{a})\|_2$$

to the absolute value of the right hand side of (2.16). As explained in the next paragraph, $\sup_{p\in M} \|\mathrm{dist}(p, \cdot)^{-1}\mathfrak{a}\|_2$ is no greater than $c_0(\|\nabla_A \mathfrak{a}\|_2 + \|\mathfrak{a}\|_2)$. Granted such a bound, then the absolute value of the right hand side of (2.16) is no greater than

$$\tag{2.18} c_0(\|\nabla_A \mathfrak{a}\|_2^2 + \|\mathfrak{a}\|_2^2 + \|\mathcal{q}_A(\mathfrak{a})\|_2^2).$$

The assertion in the preceding paragraph about the supremum in (2.17) invokes Hardy's inequality: Let f denote any given L^2_1 function on M. Then

$$\tag{2.19} \sup_{p\in M} \int_M \frac{1}{\mathrm{dist}(p, \cdot)^2} f^2 \le c_0(\|df\|_2^2 + \|f\|_2^2).$$

The latter and (2.6) imply the asserted bound for $\sup_{p\in M} \|\mathrm{dist}(p, \cdot)^{-1}\mathfrak{a}\|_2$.

Step 3. Fix $n \in \{1, 2, \ldots\}$. Use the first and third bullets of Lemma 2.3 and the bound for $\|\nabla_{A_n} \hat{a}_n\|_2$ from Step 1 to bound the ($A = A_n$, $\mathfrak{a} = a_n$) version of (2.18) by $c_0 z^2$ when $z > c_0$. Given this bound, it then follows that the absolute value of the right hand side of the ($A = A_n$, $\mathfrak{a} = \hat{a}_n$) version of (2.16) is also bounded by $c_0 z$.

The integral of $G_p|\nabla_{A_n} \hat{a}_n|^2$ that appears on the left hand side of the ($A = A_n$, $\mathfrak{a} = \hat{a}_n$) version of (2.16) is nonnegative, and this is all that need be said about this integral for the time being. The other integral on the left hand side of this same version of (2.16) is that of $2G_p\langle *F_{A_n} \wedge \hat{a}_n \wedge \hat{a}_n\rangle$. This integral is not manifestly nonnegative and so more needs to be said about it. To start the story on this integral, write F_{A_n} as the sum of two terms, these being $F_{A_n} - r_n^2 \hat{\mathrm{a}}_n \wedge \hat{\mathrm{a}}_n$ and $r_n^2 \hat{\mathrm{a}}_n \wedge \hat{\mathrm{a}}_n$. Use this decomposition to write

$$\int_M G_p \langle *F_{A_n} \wedge \hat{a}_n \wedge \hat{a}_n \rangle = r_n^2 \int_M G_p \langle *(\hat{\mathrm{a}}_n \wedge \hat{\mathrm{a}}_n) \wedge (\hat{\mathrm{a}}_n \wedge \hat{\mathrm{a}}_n)\rangle + \mathfrak{e}, \tag{2.20}$$

with $\mathfrak{e}$ being the contribution from $F_{A_n} - r_n^2 \hat{\mathrm{a}}_n \wedge \hat{\mathrm{a}}_n$. The bound by c_0 on the latter's L^2 norm leads to the bound $|\mathfrak{e}| \le c_0 \|\mathrm{dist}(p,\cdot)^{-1} \hat{a}_n\|_2 \|\hat{a}_n\|_\infty$. This last bound with (2.19), (2.16) and Step 1's bound for $\|\nabla_{A_n} \hat{a}_n\|_2$ implies that $|\mathfrak{e}| \le c_0 \|\hat{a}_n\|_\infty$.

To see about the integral on the right hand side of (2.20), write $\hat{\mathrm{a}}_n \wedge \hat{\mathrm{a}}_n$ as a sum of two terms, these being $\hat{a}_n \wedge \hat{a}_n$ and $\hat{\mathrm{a}}_n \wedge \hat{\mathrm{a}}_n - \hat{a}_n \wedge \hat{a}_n$. Use this splitting to write (2.20) as

$$\int_M G_p \langle *F_{A_n} \wedge \hat{a}_n \wedge \hat{a}_n \rangle = r_n^2 \int_M G_p |\hat{a}_n \wedge \hat{a}_n|^2 + \mathfrak{e}' + \mathfrak{e}, \tag{2.21}$$

with $\mathfrak{e}'$ being the contribution from $\hat{\mathrm{a}}_n \wedge \hat{\mathrm{a}}_n - \hat{a}_n \wedge \hat{a}_n$. Of particular note is that

$$|\mathfrak{e}'| \le c_0 r_n^2 \|\hat{a}_n - \hat{\mathrm{a}}_n\|_2 (\|\mathrm{dist}(p,\cdot)^{-1} \hat{\mathrm{a}}_n\|_2 + \|\mathrm{dist}(p,\cdot)^{-1} a_n\|_2) \|\hat{a}_n\|_\infty^2, \tag{2.22}$$

and thus $|\mathfrak{e}'|$ is no greater that $c_0 z^{-1/2} \|\hat{a}_n\|_\infty^2$. This bound follows from the first bullet of Lemma 2.3 using what was said already about the integrals that involve $\mathrm{dist}(p,\cdot)^{-1}$.

Step 4. Use the bounds on $|\mathfrak{e}|$ and $|\mathfrak{e}'|$ in (2.21) with the bound in Step 2 for the right hand side of (2.16) to see that the latter equation implies the bound

$$(1 - c_0 z^{-1/2}) \|\hat{a}_n\|_\infty^2 + \sup_{p \in M} \int_M G_p (|\nabla_{A_n} \hat{a}_n|^2 + r^2 |\hat{a}_n \wedge \hat{a}_n|^2) \le c_0 z. \tag{2.23}$$

Any $z > c_0$ version of (2.23) supplies an index n independent bound for the $\|\hat{a}_n\|_\infty$ and an index n and $p \in M$ independent bound for the integral of $G_p |\nabla_{A_n} \hat{a}_n|^2$ and $G_p r_n^2 |\hat{a}_n \wedge \hat{a}_n|^2$.

Part 2: This part of the subsection proves Items a), b) and c) of Proposition 2.2. This is done in three steps.

Step 1. This step explains why Item b) follows from Item c) and the first bullet of Lemma 2.3. To start, fix $n \in \{1, 2, \ldots\}$ for the moment and define $\mathcal{A}_n$ to be the connection $\mathcal{A}_n = A_n + i r_n \hat{a}_n$, this being a connection on

$(P \times_{SO(3)} PSL(2;\mathbb{C})$. The assertion in Item c) is equivalent to the assertion that $\mathfrak{F}(\mathcal{A}_n) \leq c_0$. Meanwhile, the first bullet of Lemma 2.3 implies that $\|\hat{a}_n\|_2 = 1 + \mathfrak{e}$ with $|\mathfrak{e}| \leq c_0 r_n^{-2}$.

The just stated bounds imply that the sequence $\{\mathcal{A}_n\}_{n=1,2,\ldots}$ can be used in lieu of $\{\mathbb{A}_n\}_{n=1,2,\ldots}$ as input for Lemma 2.1. Item b) of Proposition 2.2 follow directly from the second bullet of the $\{\mathcal{A}_n\}_{n=1,2,\ldots}$ version of Lemma 2.1.

Step 2. This step and Step 3 prove Items a) and c) in tandem. Note in this regard that it suffices to prove that both $\|\nabla_{A_n}(\hat{a}_n - \hat{\mathrm{a}}_n)\|_2^2$ and $r_n^{-2}\|F_{A_n} - r_n^2 \hat{a}_n \wedge \hat{a}_n\|_2^2$ are at most $c_0 r_n^{-2}$. This is a consequence of the first and second bullets of Lemma 2.3.

To start the proof, use the second bullet of Lemma 2.3 to see that

$$\|d_{A_n}(\hat{a}_n - \hat{\mathrm{a}}_n)\|_2^2 + \|d_{A_n}{*}(\hat{a}_n - \hat{\mathrm{a}}_n)\|_2^2 \leq c_0 r_n^{-2}. \tag{2.24}$$

Integration by parts leads from (2.24) to the inequality

$$\begin{aligned} \|\nabla_{A_n}(\hat{a}_n - \hat{\mathrm{a}}_n)\|_2^2 + 2\int_M \langle *F_{A_n} \wedge (\hat{a}_n - \hat{\mathrm{a}}_n) \wedge (\hat{a}_n - \hat{\mathrm{a}}_n)\rangle \\ + \int_M \mathrm{Ric}(\langle(\hat{a}_n - \hat{\mathrm{a}}_n) \otimes (\hat{a}_n - \hat{\mathrm{a}}_n)\rangle) \leq c_0 r_n^{-2}. \end{aligned} \tag{2.25}$$

The absolute value of the integral with Ricci curvature tensor is bounded by $c_0 r_n^{-4}$, this being a consequence of the first bullet of Lemma 2.3. To see about the middle integral on the left hand side of (2.24), write F_{A_n} as $(F_{A_n} - r_n^2 \hat{a}_n \wedge \hat{a}_n) + r_n^2 \hat{a}_n \wedge \hat{a}_n$. The contribution to the middle integral on the left hand side of (2.24) from $(F_{A_n} - r_n^2 \hat{a}_n \wedge \hat{a}_n)$ is no greater than $2\|F_{A_n} - r_n^2 \hat{a}_n \wedge \hat{a}_n\|_2 \|\hat{a}_n - \hat{\mathrm{a}}_n\|_4^2$. This, in turn, is no greater than

$$c_0 \|F_{A_n} - r_n^2 \hat{a}_n \wedge \hat{a}_n\|_2 \|\hat{a}_n - \hat{\mathrm{a}}_n\|_2^{1/2} \|\hat{a}_n - \hat{\mathrm{a}}_n\|_6^{3/2}. \tag{2.26}$$

To say more about the expression in (2.26), use the first bullet of Lemma 2.3 to bound it by c_0 times the product $z^{-1/2}(r_n^{-1}\|F_{A_n} - r_n^2 \hat{a}_n \wedge \hat{a}_n\|_2)\|\hat{a}_n - \hat{\mathrm{a}}_n\|_6^{3/2}$. Meanwhile,

$$\|\hat{a}_n - \hat{\mathrm{a}}_n\|_6^{3/2} \leq c_0(\|\nabla_{A_n}(\hat{a}_n - \hat{\mathrm{a}}_n)\|_2 + r_n^{-1}). \tag{2.27}$$

To derive this, note that $\|\hat{a}_n - \hat{\mathrm{a}}_n\|_6^{3/2} \leq \|\hat{a}_n - \hat{\mathrm{a}}_n\|_6(\|\hat{a}_n\|_6 + \|\hat{\mathrm{a}}_n\|_6)^{1/2}$. This understood, then (2.27) follows from (2.6) and a standard Sobolev inequality given that the L^2 norms of $\hat{a}_n$ and $\hat{\mathrm{a}}_n$ and their respective A_n-covariant derivatives are bounded by c_0. Such a bound for the A_n-covariant derivative of $\hat{a}_n$ is supplied by Step 1 of Part 1; and that of $\hat{\mathrm{a}}_n$ is supplied by Lemma 2.1.

Use what is said in the preceding two paragaphs with (2.25) to see that

$$\|\nabla_{A_n}(\hat{a}_n-\hat{\mathrm{a}}_n)\|_2^2 \le c_0 z^{-1/2}\|F_{A_n}-r_n^2\hat{a}_n\wedge\hat{a}_n\|_2(\|\nabla_{A_n}(\hat{a}_n-\hat{\mathrm{a}}_n)\|_2+r_n^{-1})+c_0 r_n^{-2}. \tag{2.28}$$

This last inequality is invoked in at the very end of Step 3.

Step 3. The L^2 norm of $r_n^{-1}(F_{A_n} - r_n^2\hat{a}_n \wedge \hat{a}_n)$ is no greater than the sum of the L^2 norms of $r_n^{-1}(F_{A_n} - r_n^2\hat{\mathrm{a}}_n \wedge \hat{\mathrm{a}}_n)$ and $r_n(\hat{\mathrm{a}}_n \wedge \hat{\mathrm{a}}_n - \hat{a}_n \wedge \hat{a}_n)$. The L^2 norm of the former is bounded by $c_0 r_n^{-1}$. Meanwhile, that of the latter is no greater than the sum of the L^2 norms of $r_n(\hat{a}_n - \hat{\mathrm{a}}_n) \wedge (\hat{a}_n - \hat{\mathrm{a}}_n)$ and twice that of $r_n(\hat{\mathrm{a}}_n - \hat{a}_n) \wedge \hat{a}_n$.

The preceding observations imply directly that

$$r_n^{-2}\|F_{A_n} - r_n^2\hat{a}_n\wedge\hat{a}_n\|_2^2 \le c_0(r_n^2\|\hat{a}_n-\hat{\mathrm{a}}_n\|_4^4 + r_n^2\|\hat{a}_n\|_\infty^2\|\hat{a}_n-\hat{\mathrm{a}}_n\|_2^2 + r_n^{-2}). \tag{2.29}$$

Part 1 bounds $\|\hat{a}_n\|_\infty$ by c_0 and Lemma 2.3 bounds $\|\hat{a}_n - \hat{\mathrm{a}}_n\|_2$ by $c_0 r_n^{-2}$. This leads to a bound on the right hand side of (2.28) by $c_0 r_n^2\|\hat{a}_n - \hat{\mathrm{a}}_n\|_4^4 + c_0 r_n^{-2}$. The latter is no greater than $c_0(r_n^2\|\hat{a}_n - \hat{\mathrm{a}}_n\|_2\|\hat{a}_n - \hat{\mathrm{a}}_n\|_6^3 + r_n^{-2})$. This understood, use the first bullet of Lemma 2.3 and the bound on $\|\hat{a}_n - \hat{\mathrm{a}}_n\|_6$ given by (2.27) to see that

$$r_n^{-2}\|F_{A_n} - r_n^2\hat{a}_n \wedge \hat{a}_n\|^2 \le c_0 z^{-1}\|\nabla_{A_n}(\hat{a}_n - \hat{\mathrm{a}}_n)\|_2^2 + c_0 r_n^{-2}. \tag{2.30}$$

Taken together, the inequalities in (2.28) and (2.30) imply that

$$(1 - c_0 z^{-1})(\|\nabla_{A_n}(\hat{a}_n - \hat{\mathrm{a}}_n)\|_2^2 + r_n^{-2}\|F_{A_n} - r_n^2\hat{a}_n \wedge \hat{a}_n\|_2^2 \le c_0 r_n^{-2}. \tag{2.31}$$

This inequality supplies the desired bounds if $z > c_0$.

Part 3: The nine steps that follow in this part of the subsection prove the bulleted assertions of Proposition 2.2.

Step 1. Reintroduce the sequence $\{\mathcal{A}_n = A_n + i r_n \hat{a}_n\}_{n=1,2,\dots}$ from Step 1 of Part 2, this being a sequence of connections on $P \times_{SO(3)} PSL(2;\mathbb{C})$. As noted therein, the corresponding sequence $\{\mathfrak{F}(\mathcal{A}_n)\}_{n=1,2,\dots}$ is bounded and so it can be used in lieu of $\{\mathbb{A}_n\}_{n=1,2,\dots}$ as input for Lemma 2.1.

Except for the assertion about vanishing on an open set, what is said by the first bullet of Proposition 2.2 constitutes a part of the first bullet the $\{\mathcal{A}_n\}_{n=1,2,\dots}$ version of Lemma 2.1. Note that the limit L_1^2 function has L^2 norm equal to 1. The reason is that the top bullet of Lemma 2.3 implies that L^2 norm of each $n \in \{1, 2, \dots\}$ version of $\hat{a}_n$ differs from 1 by at most $z^{-1/2} r_n^{-2}$. The nonvanishing condition can readily be satisfied by making a very small perturbation of a sequence that has all of the other properties that are required by the proposition. This understood, no more will be said about the nonvanising on an open set requirement.

The first bullet of the $\{\mathcal{A}_n\}_{n=1,2,\dots}$ version of Lemma 2.1 asserts what is said in the second bullet of Proposition 2.2 to the effect that the limit L^2_1 function of the sequence $\{|\hat{a}_n|\}_{n=1,2,\dots}$ is an L^∞ function. In fact, it follows from Item a) of Proposition 2.2 that the limit functions of $\{|\hat{a}_n|\}_{n=1,2,\dots}$ and $\{|\hat{\mathrm{a}}_n|\}_{n=1,2,\dots}$ have the same weak L^2_1 limits with it understood that the subsequences that are chosen for the respective $\{\mathcal{A}_n\}_{n=1,2,\dots}$ and $\{\mathbb{A}_n\}_{n=1,2,\dots}$ versions of Lemma 2.1 are identical.

Step 2. The definition of the function $|\hat{\mathrm{a}}_\diamondsuit|$ by the second bullet of Proposition 2.2 raises a subtle point, this being the distinction between elements in $L^2_1 \cap L^\infty$ and functions that are defined pointwise. An element in the Banach space of $L^2_1 \cap L^\infty$ is an *equivalence class* of functions that are defined almost everywhere with two functions being equivalent if they agree on the complement of a set of measure zero. The distinction between a pointwise defined function and an equivalence class of functions that differ on sets with measure zero is at issue with regards to the definition in the second bullet of Proposition 2.2 of $|\hat{\mathrm{a}}_\diamondsuit|$. In particular, this bullet of Proposition 2.2 *defines* an honest function, $|\hat{\mathrm{a}}_\diamondsuit|$; and in so doing, this bullet makes the following implicit assertion:

(2.32) *The function defined by the rule* $p \to \limsup_{n\to\infty} |\hat{a}_n|(p)$ *is in the equivalence class of the* L^2_1 *limit of the sequence* $\{|\hat{a}_n|\}_{n=1,2\dots}$.

The proof of the second bullet of Proposition 2.2 requires a proof of (2.32).

The distinction between a pointwise defined function and an equivalence class of functions that agree on the complement of a measure zero set is also at issue with regards to the proof of the sixth bullet of Proposition 2.2. The assertion in (2.32) and the fifth bullet of Proposition 2.2 are proved simultaneously in Steps 4–8. In the meantime, let $|\hat{\mathrm{a}}_\diamondsuit|$ denote a chosen function from the L^2_1 equivalence class of the weak limit of $\{|\hat{a}_n|\}_{n=1,2,\dots}$.

Step 3. The assertions made by the third bullet of Proposition 2.2 follow because the sequence $\{\langle \hat{a}_n \otimes \hat{a}_n \rangle\}_{n\in\{1,2\dots\}}$ is bounded in the L^2_1 topology; a priori bounds on the L^2_1 norms of its elements come via Items b) and e) of Proposition 2.2.

The proof of the assertions made by the fourth bullet of Proposition 2.2 are almost verbatim identical to those made by the third bullet of Lemma 2.1.

To prove the assertion made by the fifth bullet, let f denote for the moment a smooth function. Fix $n \in \{1, 2, \dots\}$. The right hand side (2.1) is $q_A(\mathfrak{a})$. This being the case, an integration by parts writes

$$(2.33)\quad \int_M f\langle \hat{a}_n \wedge *q_{A_n}(\hat{a}_n)\rangle \\ = \int_M f((|d_{A_n}\hat{a}_n|^2 + |d_{A_n}*\hat{a}_n|^2) + df \wedge (*\langle \hat{a}_n(*d_{A_n}*\hat{a}_n) + \hat{a}_n \wedge *d_{A_n}\hat{a}_n\rangle)).$$

Granted (2.33), then the second bullet of Lemma 2.3 and Item e) of Proposition 2.2 finds

$$(2.34)\qquad \left|\int_M f\langle \hat{a}_n \wedge *q_{A_n}(\hat{a}_n)\rangle\right| \le c_0(\|f\|_\infty r_n^{-2} + \|df\|_2 r_n^{-1}).$$

Let g denote a given L^2 function. Fix $\varepsilon > 0$ and let f denote a smooth function such that $\|f - g\|_2 \le \varepsilon$. Then

$$(2.35)\quad \left|\int_M g\langle \hat{a}_n \wedge *q_{A_n}(\hat{a}_n)\rangle\right| \le c_0\varepsilon\|\hat{a}_n\|_\infty\|q_{A_n}(\hat{a}_n)\|_2 + \left|\int_M f\langle \hat{a}_n \wedge *q_{A_n}(\hat{a}_n)\rangle\right|.$$

Now use (2.34) with (2.35) and Item e) of Proposition 2.2 to conclude that

$$(2.36)\qquad \int_M g\langle \hat{a}_n \wedge *q_{A_n}(\hat{a}_n)\rangle \le c_0(\varepsilon + r_n^{-2}\|f\|_\infty + r_n^{-1}\|df\|_2).$$

Taking n sufficiently large bounds the right hand side of (2.36) by ε and so (2.36) implies that

$$(2.37)\qquad \lim_{n\in\Lambda}\int_M g\langle \hat{a}_\diamond \wedge *q_{A_n}(\hat{a}_n)\rangle = 0.$$

Step 4. This step with Steps 5–9 prove the assertion in (2.32) and the assertion made by the sixth bullet of Proposition 2.2. To start, fix $n \in \{1, 2, \dots\}$ and a point $p \in M$ so as to consider the ($A = A_n$, $\mathfrak{a} = \hat{a}_n$) version of (2.16). Of particular concern in this step is the integral of $G_p\langle \hat{a}_n \wedge *q_{A_n}(\hat{a}_n)\rangle$ that appears on the right hand side of the $(A_n, \hat{a}_n)$ version. As explained directly, the various $p \in M$ versions are such that

$$(2.38)\qquad \sup_{p\in M}\left|\int_M G_p\langle \hat{a}_n \wedge *q_{A_n}(\hat{a}_n)\rangle\right| < c_0 r_n^{-1/5}.$$

To see that (2.38) holds, fix for the moment $\mathrm{p} \in \mathrm{M}$ and $\rho \in (0,1)$. Having done so, let $\chi_{\mathrm{p},\rho}$ denote the function on M given by $\chi(2-\rho^{-1}\mathrm{dist}(\mathrm{p},\cdot))$. This function equals 1 where the distance to p is greater than 2ρ and it equals 0 where the distance is less than ρ. Write $\mathrm{G_p}$ as $(1-\chi_{\mathrm{p},\rho})\mathrm{G_p} + \chi_{\mathrm{p},\rho}\mathrm{G_p}$ so as to split a given $\mathrm{n} \in \{1,2,\dots\}$ version of the integral in (2.38) into two integrals.

The absolute value of the integral of $(1-\chi_{\mathrm{p},\rho})\mathrm{G_p}\langle \hat{a}_\mathrm{n} \wedge *\mathfrak{q}_{\mathrm{A_n}}(\hat{a}_\mathrm{n})\rangle$ is no greater than $\mathrm{c}_0\rho^{1/2}\|\hat{a}_\mathrm{n}\|_\infty\|\mathfrak{q}_{\mathrm{A_n}}(\hat{a}_\mathrm{n})\|_2$, this because $\mathrm{G_p} \leq \mathrm{c}_0\mathrm{dist}(\mathrm{p},\cdot)^{-1}$. Meanwhile, the absolute value of the integral of $\chi_{\mathrm{p},\rho}\mathrm{G_p}\langle \hat{a}_\mathrm{n} \wedge *\mathfrak{q}_{\mathrm{A_n}}(\hat{a}_\mathrm{n})\rangle$ is no greater than $\mathrm{c}_0\rho^{-2}r_\mathrm{n}^{-1}$, this being a consequence of (2.34). Granted these bounds, take $\rho = r_\mathrm{n}^{-2/5}$ and invoke Items d) and e) of Proposition 2.2 to conclude that the absolute value in (2.38) is no greater than $\mathrm{c}_0 r_\mathrm{n}^{-1/5}$ as claimed.

Step 5. Fix $\mathrm{n} \in \{1,2,\dots\}$ and $\mathrm{p} \in \mathrm{M}$ again so as to return to the $(\mathrm{A} = \mathrm{A_n}, \mathfrak{a} = \hat{a}_\mathrm{n})$ version of (2.16). Of particular concern here is the integral of $\mathrm{G_p}\langle *\mathrm{F}_{\mathrm{A_n}} \wedge \hat{a}_\mathrm{n} \wedge \hat{a}_\mathrm{n}\rangle$ that appears on the left hand side of this version. As explained directly,

$$\int_\mathrm{M} \mathrm{G_p}\langle *\mathrm{F}_{\mathrm{A_n}} \wedge \hat{a}_\mathrm{n} \wedge \hat{a}_\mathrm{n}\rangle = \int_\mathrm{M} \mathrm{G_p} r_\mathrm{n}^2|\hat{a}_\mathrm{n} \wedge \hat{a}_\mathrm{n}|^2 + \mathfrak{e}, \tag{2.39}$$

n with $\mathfrak{e}$ having absolute value no greater than $\mathrm{c}_0 r_\mathrm{n}^{-2/3}$. To see why this is, decompose $\mathrm{F}_{\mathrm{A_n}}$ as the sum $r_\mathrm{n}^2\hat{a}_\mathrm{n}\wedge\hat{a}_\mathrm{n} + (\mathrm{F}_{\mathrm{A_n}} - r_\mathrm{n}^2\hat{a}_\mathrm{n}\wedge\hat{a}_\mathrm{n})$ so as to decompose the integral on the left hand side of (2.39) as a sum of two integrals. The term designated as $\mathfrak{e}$ in (2.39) is the second of these two integrals. To bound $|\mathfrak{e}|$, fix for the moment $\rho > 0$ and again write $\mathrm{G_p}$ as the sum $(1-\chi_{\mathrm{p},\rho})\mathrm{G_p} + \chi_{\mathrm{p},\rho}\mathrm{G_p}$. The absolute value of the contribution of $(1-\chi_{\mathrm{p},\rho})\mathrm{G_p}$ to $\mathfrak{e}$ is no greater than $\mathrm{c}_0\rho^{-1}\|\mathrm{F}_{\mathrm{A_n}} - r_\mathrm{n}^2\hat{a}_\mathrm{n} \wedge \hat{a}_\mathrm{n}\|_2\|\hat{a}_\mathrm{n} \wedge \hat{a}_\mathrm{n}\|_2$ because $\mathrm{G_p} \leq \mathrm{c}_0\mathrm{dist}(\mathrm{p},\cdot)^{-1}$. This understood, Items b) and c) of Proposition 2.2 bound this contribution by $\mathrm{c}_0\rho^{-1}r_\mathrm{n}^{-2}$. Meanwhile, the absolute value of the contribution of $\chi_{\mathrm{p},\rho}\mathrm{G_p}$ to $\mathfrak{e}$ is no greater than $\mathrm{c}_0\rho^{1/2}\|\mathrm{F}_{\mathrm{A_n}} - r_\mathrm{n}^2\hat{a}_\mathrm{n} \wedge \hat{a}_\mathrm{n}\|_2\|\hat{a}_\mathrm{n}\|_\infty^2$. Items c) and e) of Proposition 2.2 imply that this is no greater than $\mathrm{c}_0\rho^{1/2}$. Granted these bounds, take $\rho = r_\mathrm{n}^{-4/3}$ to see that $|\mathfrak{e}| \leq \mathrm{c}_0 r_\mathrm{n}^{-2/3}$.

Step 6. Fix once again $\mathrm{n} \in \{1,2,\dots\}$ and $\mathrm{p} \in \mathrm{M}$. Use the $(\mathrm{A} = \mathrm{A_n}, \mathfrak{a} = \hat{a}_\mathrm{n})$ version of (2.16) with what is said in Steps 4 and 5 to see that

$$\tfrac{1}{2}|\hat{a}_\mathrm{n}|^2(\mathrm{p}) + \int_\mathrm{M} \mathrm{G_p}(|\nabla_{\mathrm{A_n}}\hat{a}_\mathrm{n}|^2 + 2r_\mathrm{n}^2|\hat{a}_\mathrm{n} \wedge \hat{a}_\mathrm{n}|^2) = -\int_\mathrm{M} \mathrm{G_p}\big(\tfrac{1}{2}|\hat{a}_\mathrm{n}|^2 - \mathrm{Ric}(\langle \hat{a}_\mathrm{n} \otimes \hat{a}_\mathrm{n}\rangle)\big) + \mathfrak{e}_\mathrm{n}, \tag{2.40}$$

with $\mathfrak{e}_n$ having absolute value no greater than $c_0 r_n^{-1/5}$. Let $I_n(p)$ denote the integral that appears on the right hand side of (2.40). Introduce by way of notation

$$(2.41)\qquad I_\Diamond(p) = -\int_M G_p\big(\tfrac{1}{2}|\hat{a}_\Diamond|^2 - \mathrm{Ric}(\langle \hat{a}_\Diamond \otimes \hat{a}_\Diamond\rangle)\big).$$

As explained directly, $\lim_{n\to\infty}\sup_{p\in M}|I_\Diamond(p) - I_n(p)| = 0$. That this is so follows from the first and third bullets of Proposition 2.2 with the fact that G_p is square integrable.

Step 7. Introduce for the time being v to denote the function on M that is defined by the rule $p \to v(p) = \limsup_{n\to\infty}|\hat{a}_n|(p)$. Let $Q_\Diamond$ denote the function on M that is defined by the sixth bullet of Proposition 2.2. As explained momentarily, the identity in (2.40) and the fact that $\lim_{n\to\infty}\sup_{p\in M}|I_\Diamond(p) - I_n(p)| = 0$ imply that

$$(2.42)\qquad \tfrac{1}{2}v(p)^2 + Q_\Diamond(p) = I_\Diamond(p).$$

The final assertion of sixth bullet of Proposition 2.2 follows immediately from (2.42) if it is the case that $v = |\hat{a}_\Diamond|$ on the complement of a measure zero set.

To see about (2.42), fix $\Lambda \in \{1, 2, \ldots\}$ such that $\lim_{n\in\Lambda}|\hat{a}_n|(p) = v(p)$. Suppose that there exists $\delta > 0$ and a subsequence $\Lambda' \subset \Lambda$ such that

$$(2.43)\qquad \int_M G_p(|\nabla_{A_n}\hat{a}_n|^2 + 2r_n^2|\hat{a}_n \wedge \hat{a}_n|^2) > Q_\Diamond(p) + 2\delta$$

when $n \in \Lambda'$. If this is the case, then the left hand side of (2.40) for $n \in \Lambda'$ will be greater than $\frac{1}{2}v(p)^2 + Q_\Diamond(p) + 2\delta$ when n is large, and this implies that $I_\Diamond(p)$ is no less than $\frac{1}{2}v(p)^2 + Q_\Diamond(p) + 2\delta$. Now fix a second subsequence, $\Theta \subset \{1, 2, \ldots\}$ such that $n \in \Theta$ versions of the integral on the left hand side of (2.44) converges to $Q_\Diamond(p)$. The $n \in \Theta$ versions of $|a_n|(p)$ must in any event be less than $v(p) + \delta$ when n is large, and this implies that $I_\Diamond(p)$ is no greater than $\frac{1}{2}v(p)^2 + Q_\Diamond(p) + \delta$. This last conclusion is incompatible with the lower bound just stated for $I_\Diamond$.

Step 8. This step proves that $|\hat{a}_\Diamond| \geq v$ on a set of full measure. To start the proof, fix $\varepsilon > 0$ and reintroduce the functions $\delta_{(\cdot),\varepsilon}$ and $f_{(\cdot),\varepsilon}$ that are defined in Step 1 of the proof of Lemma 2.1. The function on M given by the rule

$$(2.44)\qquad p \to \int_M \delta_{p,\varepsilon}|\hat{a}_\Diamond|^2$$

is smooth. As noted in Step 3 of Lemma 2.1's proof, the function depicted in (2.44) converges as $\varepsilon \to 0$ to an L^∞ function that equals $|\hat{a}_\diamond|^2$ on the complement of a measure zero set. Of particular note is that the function defined by (2.44) obeys (2.8).

Write $f_{p,\varepsilon}$ as $g_\varepsilon + \mathfrak{e}_p$ with g_ε defined using Gaussian coordinates centered at p by the formulas in (2.7). The functions g_ε and G_p are such that

$$g_\varepsilon \le G_p + \mathfrak{r}_p, \tag{2.45}$$

with $\mathfrak{r}_p$ such that $|\mathfrak{r}_p| \le c_0|x|$. The inequality in (2.45) and the fact that $\{f_{p,\varepsilon}\}_{\varepsilon \in (0,1]}$ converges to G_p as $\varepsilon \to 0$ in the C^2 topology on compact subsets of $M - p$ has the following consequence: Fix $\rho \in (0, c_0^{-1}]$ and there exists $c_\rho > 1$ that is independent of the point p and such that there is a continuous function on M, to be denoted by $\mathfrak{r}_{p,\varepsilon,\rho}$, with norm less than ρ and such that

$$f_{p,\varepsilon} \le G_p + \mathfrak{r}_{p,\varepsilon,\rho} \quad \textit{when } \varepsilon < c_\rho^{-1}. \tag{2.46}$$

Fix $\rho \in (0, c_0^{-1}]$ and $\varepsilon < c_\rho^{-1}$. Given $n \in \{1, 2, \dots\}$ and $p \in M$, multiply both sides of (2.40) by $\delta_{p,\varepsilon}$, and integrate the result over M and use (2.46) to see that

$$\tfrac{1}{2}\int_M \delta_{p,\varepsilon}|\hat{a}_n|^2 + \int_M G_p(|\nabla_{A_n}\hat{a}_n|^2 + 2r_n^2|\hat{a}_n \wedge \hat{a}_n|^2) \ge I_\diamond(p) - \mathfrak{e}_{n,\varepsilon,\rho}, \tag{2.47}$$

where $\mathfrak{e}_{n,\varepsilon,\rho}$ is such that $\lim_{n\to\infty} \sup_{p\in M} |\mathfrak{e}_{n,\varepsilon,\rho}| \le c_0\rho$. Meanwhile, what is said in the third bullet of Proposition 2.2 has the following implication: Given $\varepsilon < c_\rho^{-1}$ and $\varepsilon' \in (0, 1]$, there exists $n_{\varepsilon,\varepsilon'} \ge 1$ such that

$$\sup_{p\in M}\left|\int_M \delta_{p,\varepsilon}|\hat{a}_\diamond|^2 - \int_M \delta_{p,\varepsilon}|\hat{a}_n|^2\right| < \varepsilon' \quad \textit{when } n > n_{\varepsilon,\varepsilon'}. \tag{2.48}$$

This follows from the fact that $\{|\hat{a}_n|\}_{n\in\{1,2,\dots\}}$ converges strongly in the L^2 topology to $|\hat{a}_\diamond|$.

Fix $p \in X$, take $\varepsilon' = \rho$ and then $n > n_{\varepsilon,\varepsilon'=\rho}$ with two additional constraints, both giving lower bounds: The first is that n's version of the G_p integral on the right hand side of (2.47) differs from $Q_\diamond(p)$ by at most ρ. The second is that $\sup_{p\in M} |\mathfrak{e}_{n,\varepsilon,\rho}| \le c_0\rho$. With n so chosen, invoke the $\varepsilon' = \rho$ version of (2.48) to see that

$$\tfrac{1}{2}\int_M \delta_{p,\varepsilon}|\hat{a}_\diamond|^2 + Q_\diamond(p) \ge I_\diamond(p) - c_0\rho. \tag{2.49}$$

Since ρ and then ε can be chosen as small as desired, and since this inequality holds for each $p \in X$ a comparison between (2.49) and (2.42) leads to the conclusion that $|\hat{a}_\Diamond| \geq v$ on the complement of a measure zero set.

Step 9. To see that $|\hat{a}_\Diamond| \leq v$ on a set of full measure, fix $\varepsilon > 0$, $p \in M$ and a positive integer n. Having done so, multiply n's version of (2.40) by $\delta_{p,\varepsilon}$ and integrate the result over M. Do this for choices of $n > n_{\varepsilon,\varepsilon'=\varepsilon}$ that obey two additional lower bound constraints. The first constraint asks that n's version of the G_p integral on the right hand side of (2.47) is greater than $Q_\Diamond(p) - \varepsilon$; and the second asks that $\sup_{p\in M} |I_\Diamond(p) - I_n(p)| < \varepsilon$. With n so chosen, invoke the $\varepsilon' = \varepsilon$ version of (2.48) to see that

$$\tag{2.50} \frac{1}{2}\int_M \delta_{p,\varepsilon}|\hat{a}_\Diamond|^2 + Q_\Diamond(p) \leq I_\Diamond(p) + c_0\varepsilon.$$

Since ε can be as small as desired, a comparison between (2.50) and (2.42) leads to the conclusion that $|\hat{a}_\Diamond| \leq v$ on the complement of a measure zero set.

2.d. Proof of Lemma 2.3

The proof has four parts. By way of a look ahead, any given $\hat{a}_n$ for $n \in \{1, 2, \dots\}$ is a particular $t \in [0, \infty)$ value of the solution to a certain heat equation whose initial value is $\hat{a}_n$.

Part 1: Fix a pair $(A, \mathfrak{a})$ with A being a connection on P and $\mathfrak{a}$ being a section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. Standard existence and uniqueness theorems for parabolic differential equations prove that there is a unique section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ over $[0, \infty) \times M$, this denoted by a, that obeys the linear heat equation

$$\tag{2.51} \frac{\partial}{\partial t} a = -(\nabla_A{}^\dagger \nabla_A a + *(*F_A \wedge a + a \wedge *F_A) + \mathrm{Ric}(a) \quad \textit{with } a|_{t=0} = \mathfrak{a}.$$

This equation can be written equivalently in two ways,

$$\tag{2.52} \frac{\partial}{\partial t} a = -\mathcal{q}_A(a) \quad \textit{and} \quad \frac{\partial}{\partial t} a = -(*d_A*d_A a - d_A*d_A*a).$$

Use $\|\cdot\|_2$ as before to denote the L^2 norm on M. It follows from the right most identity in (2.52) that the function $[0, \infty)$ given by $\|a\|_2$ is non-increasing, and in particular, that

$$\tag{2.53} \frac{d}{dt}\|a\|_2^2 = -2(\|d_A a\|_2^2 + \|d_A*a\|_2^2).$$

Let $\mathfrak{E}$ denote the function on $[0,\infty)$ given by $\frac{1}{2}(\|\mathrm{d}_A a\|_2^2 + \|\mathrm{d}_A{*}a\|_2^2)$. It follows from the left most identity in (2.52) that

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathfrak{E} = -\|\mathfrak{q}_A(a)\|_2^2. \tag{2.54}$$

The equations in (2.53) and (2.54) play central roles in Parts 2 and 3 of the proof.

Part 2: Define the function $\mathfrak{n}$ on $[0,\infty)$ by the rule

$$\mathrm{t} \to \mathfrak{n}(\mathrm{t}) = \mathrm{t}^{-1}\int_0^{\mathrm{t}} \|\mathfrak{q}_A(a)\|_2^2. \tag{2.55}$$

Having done so, integrate (2.55) to see that

$$\mathfrak{E}(\mathrm{t}) = \mathfrak{E}(0) - \mathrm{t}\mathfrak{n}(\mathrm{t}). \tag{2.56}$$

As $\mathfrak{E}(\mathrm{t}) \geq 0$ in any event, the preceding identity requires that

$$\mathfrak{n}(\mathrm{t}) \leq \mathfrak{E}(0)\mathrm{t}^{-1}. \tag{2.57}$$

Given the definition of $\mathfrak{n}$, this implies in particular that there exists $\mathrm{s} \in (0,\mathrm{t}]$ such that

$$\|\mathfrak{q}_A(a|_{\mathrm{s}})\|_2^2 \leq \mathfrak{E}(0)\mathrm{t}^{-1}. \tag{2.58}$$

Meanwhile, for any $\mathrm{s} \in [0,\infty)$, use of the left most identity in (2.52) gives the bound

$$\|a|_{\mathrm{s}} - \mathfrak{a}\|_2^2 \leq \mathrm{s}^2\mathfrak{n}(\mathrm{s}). \tag{2.59}$$

Indeed, (2.59) follows by noting that $\|a|_{\mathrm{s}} - \mathfrak{a}\|_2 \leq \int_0^{\mathrm{s}} \|\frac{\partial}{\partial \mathrm{s}} a\|_2 = \int_0^{\mathrm{s}} \|\mathfrak{q}_A(a)\|_2$.

Part 3: Let E denote an upper bound for the sequence $\{\mathfrak{F}(A_n + i\mathfrak{a}_n)\}_{n=1,2,\ldots}$. Now fix $n \in \{1,2,\ldots\}$ and take $(A,\mathfrak{a})$ in Parts 1 and 2 to be $(A_n, \hat{\mathfrak{a}}_n)$. If $\mathrm{E} = 0$, take $\hat{a}_n = \hat{\mathfrak{a}}_n$.

Assume now that E is positive. The relevant version in this case of the function $\mathfrak{E}$ has $\mathfrak{E}(0) \leq \mathrm{E}r_n^{-2}$. Granted that such is the case, take t in (2.58) to be $z^{-1}\mathrm{E}r_n^{-2}$. The resulting version of (2.58) asserts that there exists $\mathrm{s} \in (0,\mathrm{t}]$ such that $\|\mathfrak{q}_A(a|_{\mathrm{s}})\|_2 \leq z$; and for such s, the resulting version of (2.59) asserts that $\|a|_{\mathrm{s}} - \mathfrak{a}\|_2^2 \leq z^{-1}r_n^{-4}$. Meanwhile, (2.56) implies that $\|\mathrm{d}_A a|_{\mathrm{s}}\|_2^2 + \|\mathrm{d}_A{*}a|_{\mathrm{s}}\|_2^2 \leq \mathrm{E}r_n^{-2}$ for this same choice of s.

What is said in the preceding paragraph implies directly that all requirements of Lemma 2.3 are met by taking a_n to equal $a|_s$.

3. Scaling limits

Theorem 1.3 and Lemma 2.5 from [U] play a central role in what is done in this section. The required parts are stated momentarily in (3.1). The notation uses $B \subset \mathbb{R}^3$ to denote the radius 1 ball centered on the origin and θ_0 to denote the product connection on the product principal bundle $B \times SO(3)$. The assertion that follows restates the relevant parts Theorem 1.3 and Lemma 2.5 in [U].

(3.1) *There exists $\kappa_U > 1$ with the following significance: Suppose that A is a connection on the product SO(3) bundle over B whose curvature has L^2 norm at most κ_U^{-1}. There is an automorphism of this bundle that pulls A back as $\theta_0 + \hat{a}_A$ with $\hat{a}_A$ an $\mathfrak{su}(2)$ valued 1-form on B with the properties listed below.*

- *The 1-form $\hat{a}_A$ is coclosed, thus $d{*}\hat{a}_A = 0$.*
- *The 2-form $*\hat{a}_A$ pulls back as zero to the boundary of the closure of B.*
- *The Sobolev L^2_1 norm of $\hat{a}_A$ is bounded by κ_U times the L^2 norm of F_A.*

Conversely, if $A = \theta_0 + \hat{a}_A$ is a connection on $B \times SO(3)$ such that $\hat{a}_A$ obeys the first two bullets and has L^2_1 norm bounded by $\frac{1}{2}\kappa_U^{-1}$, then the third bullet is also obeyed.

By way of a guide for those unfamiliar with [U], the proof of (3.1) uses an open/closed argument that exploits four facts, the first being that the L^2_1 norm dominates the L^4 norm. The second fact is that the curvature of $\theta_0 + \hat{a}$ differs from $d\hat{a}$ by a term that is quadratic in $\hat{a}$. The third fact is as follows: If $\hat{a}$ is an $\mathfrak{su}(2)$ valued 1-form on B with $*\hat{a}$ pulling back as zero to the boundary of B, then the sum of the L^2 norms $d\hat{a}$ and $d{*}\hat{a}$ is greater than c_0^{-1} times the L^2_1 norm of $\hat{a}$. The fourth fact asserts that the differential of an $\mathfrak{su}(2)$ valued function can be added to any given $\mathfrak{su}(2)$ valued 1-form so that the result obeys the first and second bullets in (3.1).

This section uses Uhlenbeck's theorem to analyze the behavior of suitably constrained connections on P and sections of the bundle $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ on balls with radius chosen to guarantee an apriori L^2 bound for the curvature of the connection.

3.a. The constraints

Fix $r \geq 1$ and $\mathrm{E} \geq 1$, and let $(\mathrm{A}, \hat{a}) \in \mathrm{Conn}(\mathrm{P}) \times \mathrm{C}^{\infty}(\mathrm{M}; (\mathrm{P} \times_{\mathrm{SO}(3)} \mathfrak{su}(2)) \otimes \mathrm{T}^*\mathrm{M})$ denote a pair that obeys the following constraints:

(3.2)
a) $\int_{\mathrm{M}} |\hat{a}|^2 = 1.$
b) $\int_{\mathrm{M}} (|d_{\mathrm{A}} \hat{a}|^2 + |d_{\mathrm{A}} {*} \hat{a}|^2) < \mathrm{E} r^{-2}$
c) $\int_{\mathrm{M}} |\mathrm{F}_{\mathrm{A}} - r^2 \hat{a} \wedge \hat{a}|^2 < \mathrm{E}.$
d) $\int_{\mathrm{M}} |\mathcal{q}_{\mathrm{A}}(\hat{a})|^2 < \mathrm{E}.$

The subsections that follow and Sections 4 and 5 use pairs that obey Items a)–d) in (3.2). The rest of this subsection derives some direct implications, these summarized by

(3.3)
- $\int_{\mathrm{M}} (|\nabla_{\mathrm{A}} \hat{a}|^2 + r^2 |\hat{a} \wedge \hat{a}|^2 + r^{-2} |\mathrm{F}_{\mathrm{A}}|^2) \leq c_0 \mathrm{E},$
- $\sup_{\mathrm{p} \in \mathrm{M}} \int_{\mathrm{dist}(\mathrm{p}, \cdot) \leq \mathrm{r}} \mathrm{G}_{\mathrm{p}} (|\nabla_{\mathrm{A}} \hat{a}|^2 + 2r^2 |\hat{a} \wedge \hat{a}|^2) \leq c_0 \mathrm{E}^2,$
- $\sup_{\mathrm{p} \in \mathrm{M}} |\hat{a}|(\mathrm{p}) \leq c_0 \mathrm{E}.$

Items a)–c) with the $f = 1$ and $(\mathrm{A}, \hat{a})$ version of (2.2) lead to the bounds in the first bullet. The bounds in the second bullet follow with the addition of Item d) using arguments that are very much like those in Steps 2–4 of the proof of Proposition 2.2. The next paragraph gives the details.

Since $(\mathrm{A}, \hat{a})$ are smooth, the function f in the $(\mathrm{A}, \hat{a})$ version of (2.15) can be replaced by the Green's function G_{p} so as to obtain the corresponding version of (2.16). Given the top bullet of (3.3), then what is said in Step 2 of the proof of Proposition 2.2 bounds the right hand side of the $(\mathrm{A}, \hat{a})$ version of (2.6) by $c_0 \mathrm{E}$. The term on the integral on the left hand side with the integrand $\mathrm{G}_{\mathrm{p}} \langle {*}\mathrm{F}_{\mathrm{A}} \wedge \hat{a} \wedge \hat{a} \rangle$ is written as a sum of two integrals, the first with integrand $\mathrm{G}_{\mathrm{p}} \langle {*}(\mathrm{F}_{\mathrm{A}} - r^2 \hat{a} \wedge \hat{a}) \wedge \hat{a} \wedge \hat{a} \rangle$, and the second with integrand $\mathrm{G}_{\mathrm{p}} r^2 |\hat{a} \wedge \hat{a}|^2$. The latter is non-negative and it contributes in any event to the left hand side of the second bullet in (3.3). The absolute value of the former is at most

$$c_0 \|\mathrm{F}_{\mathrm{A}} - r^2 \hat{a} \wedge \hat{a}\|_2 \|\hat{a}\|_{\infty} \|\mathrm{dist}(\cdot, \mathrm{p})^{-1} \hat{a}\|_2. \tag{3.4}$$

Use the first bullet of (3.3) with (2.19) to bound $\|\mathrm{dist}(\cdot, \mathrm{p})^{-1} \hat{a}\|_2$ by $c_0 \mathrm{E}^{1/2}$. This bound and Item c) in (3.2) bound (3.4) by $c_0 \mathrm{E} \|\hat{a}\|_{\infty}$. Given what was said about the right hand side of the $(\mathrm{A}, \hat{a})$ version of (2.16), the latter bound leads directly to a bound by $c_0 \mathrm{E}^2$ for $\|\hat{a}\|_{\infty}^2$ and for the integral of any $\mathrm{p} \in \mathrm{M}$ version of $\mathrm{G}_{\mathrm{p}} r^2 |\hat{a} \wedge \hat{a}|^2$.

3.b. The parameter $r_\diamond$

Fix a point $p \in M$. With $r \in (0, c_0^{-1})$ specified, introduce by way of notation B_r to denote the ball of radius r centered at p. Denote by $r_\diamond$ the largest value of r such that

$$\int_{B_r} |F_A|^2 \leq \tfrac{1}{100}\kappa_U^{-2} r^{-1} \tag{3.5}$$

Note that $r_\diamond \geq c_0^{-1} r^{-1}$, this being a consequence of Item c) in (3.2) and the third bullet of (3.3). The upcoming Propositions 3.1 and 3.2 give an indication of the significance of $r_\diamond$.

To set the stage for Proposition 3.1 and for the discussion in the subsequent subsections, fix $p \in M$ and Gaussian coordinates centered on p. Use these coordinates to identify a radius c_0^{-1} ball centered at p with the radius c_0^{-1} ball centered at the origin in $\mathbb{R}^3$. Let ϕ denote the map from the radius $c_0^{-1} r_\diamond^{-1}$ ball about the origin in $\mathbb{R}^3$ to the radius c_0^{-1} ball about p that is obtained by composing first the map $x \to r_\diamond x$ from $\mathbb{R}^3$ to itself, and then the map that is defined by the Gaussian coordinates.

Pull the pair $(A, r_\diamond^{-1}\hat{a})$ back to the radius $c_0^{-1} r_\diamond^{-1}$ ball about the origin in $\mathbb{R}^3$ using ϕ to define a pair consising of a connection on ϕ^*P and a $\phi^*(P \times_{SO(3)} \mathfrak{su}(2))$ valued 1-form. Denote this pair by $(A_\diamond, \hat{a}_\diamond)$. The definition of $\hat{a}_\diamond$ as $r_\diamond^{-1}\phi^*\hat{a}$ implies that $|\hat{a}_\diamond| \leq \|\hat{a}\|_\infty$ and that $|\hat{a}_\diamond|(0) = |\hat{a}|(p)$ with it understood that the norm of $\hat{a}_\diamond$ is defined by using the metric that is given by $r_\diamond^{-2}$ times ϕ's pull-back of M's metric. The latter metric is denoted by m_ϕ. Note in any event that the metric m_ϕ and the Euclidean metric differ by a term whose norm and first two derivatives are bounded by $c_0 r_\diamond^2$, and whose derivatives to each order $k > 2$ are bounded by a k-dependent constant time $r_\diamond^k$.

The number $r_\diamond$ is defined so that the curvature of $A_\diamond$ obeys

$$\int_{|x|\leq 1} |F_{A_\diamond}|^2 \leq \tfrac{1}{100}\kappa_U^{-2} \quad \textit{with equality if } r_\diamond < c_0^{-1}, \tag{3.6}$$

with it again understood that m_ϕ is used to define the norm and volume form.

The proposition also introduces θ_0 to denote the product connection on principal SO(3) bundle $\mathbb{R}^3 \times SO(3)$. Proposition 3.1's Hodge star operator is the Euclidean metric's Hodge star, not m_ϕ's Hodge star. Meanwhile, the norm and volume form used in Proposition 3.1 can be either those defined by m_ϕ of those defined by the Euclidean metric.

Proposition 3.1. *There exists $\kappa > 1$ with the following significance: Suppose that $\mathrm{E} \geq 1$, $r > 1$ and $(\mathrm{A}, \hat{a})$ is a pair of connection on P and section of $(\mathrm{P} \times_{\mathrm{SO}(3)} \mathfrak{su}(2)) \otimes \mathrm{T}^*\mathrm{M}$ that obey (3.2). Fix $\mathrm{p} \in \mathrm{M}$. There is an isomorphism, to be denoted by g, from the product $\mathrm{SO}(3)$ bundle over the $|\mathrm{x}| < \kappa^{-1}\mathrm{r}_\diamondsuit^{-1}$ ball in $\mathbb{R}^3$ to $\phi^*\mathrm{P}$ such that*

- *The connection $\mathrm{g}^*\mathrm{A}_\diamondsuit$ can be written as $\theta_0 + \hat{\mathrm{a}}_{\mathrm{A}_\diamondsuit}$ where $\hat{\mathrm{a}}_{\mathrm{A}_\diamondsuit}$ is an $\mathfrak{su}(2)$-valued 1-form on the $|\mathrm{x}| \leq 1$ ball in $\mathbb{R}^3$ with L^2_1 norm bounded by $\kappa \int_{|\mathrm{x}|\leq 1} |\mathrm{F}_{\mathrm{A}_\diamondsuit}|^2$. Moreover, $\hat{\mathrm{a}}_{\mathrm{A}_\diamondsuit}$ obeys $\mathrm{d}_{\mathrm{A}_\diamondsuit} {*}\hat{\mathrm{a}}_{\mathrm{A}_\diamondsuit} = 0$ and ${*}\hat{\mathrm{a}}_{\mathrm{A}_\diamondsuit}$ pulls back as zero to the $|\mathrm{x}| = 1$ sphere.*
- *The L^2_1 norm of $\mathrm{g}^*\hat{a}_\diamondsuit$ on the $|\mathrm{x}| < 1$ ball is bounded by $\kappa \mathrm{E}^2$. Moreover, given $\mathrm{r} \in [\frac{1}{2}, 1)$, there exists $\kappa_{\mathrm{E},r} > \kappa$ which is independent of $(\mathrm{A}, \hat{a})$ and is such that the L^2_2 norm of $\mathrm{g}^*\hat{a}_\diamondsuit$ on the $|\mathrm{x}| \leq \mathrm{r}$ ball in $\mathbb{R}^3$ is bounded by $\kappa_{\mathrm{E},\mathrm{r}}$.*

Granted (3.6) and some standard Sobolev inequalities, Proposition 3.1 amounts to little more than a corollary to Uhlenbecks theorem in (3.1). In any event, its proof is in the next subsection.

Proposition 3.1 can be said to see only the $\mathrm{SO}(3)$ subgroup in $\mathrm{PSL}(2;\mathbb{C})$, this being the maximal compact subgroup. The upcoming Proposition 3.2 can be said to see the whole of $\mathrm{PSL}(2;\mathbb{C})$. Proposition 3.2 is the key to Theorem 1.1's extension of Uhlenbeck's theorem to $\mathrm{PSL}(2;\mathbb{C})$.

Proposition 3.2. *Given $\mathrm{E} \geq 1$, $\mu \in (0, \frac{1}{2}]$ and $\varepsilon \in (0, 1]$, there exists $\kappa_{\mathrm{E},\mu,\varepsilon} > 1$ with the following significance: Suppose that $r > 1$ and $(\mathrm{A}, \hat{a})$ is a pair of connection on P and section of $(\mathrm{P} \times_{\mathrm{SO}(3)} \mathfrak{su}(2)) \otimes \mathrm{T}^*\mathrm{M}$ that obey (3.2). Fix $\mathrm{p} \in \mathrm{M}$. If both $\mathrm{r}_\diamondsuit < \kappa^{-1}_{\mathrm{E},\mu,\varepsilon}$ and $\int_{\mathrm{B}_{\mathrm{r}_\diamondsuit}} |\nabla_\mathrm{A} \hat{a}|^2 \leq \kappa^{-1}_{\mathrm{E},\mu,\varepsilon} \mathrm{r}_\diamondsuit^{-2} \int_{\mathrm{B}_{\mathrm{r}_\diamondsuit}} |\hat{a}|^2$, then the $\mathrm{r} = (1-\mu)\mathrm{r}_\diamondsuit$ version of $\int_{\mathrm{B}_\mathrm{r}} |\mathrm{F}_\mathrm{A}|^2$ is less than $\varepsilon \mathrm{r}^{-1}$.*

The proof of Proposition 3.2 will invoke Proposition 3.1. This being the case, it proves convenient to restate the proposition in terms of $\mathrm{A}_\diamondsuit$ and $\hat{a}_\diamondsuit$. To this end, introduce z to denote the L^2 norm of $\hat{a}_\diamondsuit$ on the $|\mathrm{x}| < 1$ ball; then define $\hat{a}_*$ to equal $z^{-1}\hat{a}_\diamondsuit$. Proposition 3.2's assertion is equivalent to the following:

(3.7) *Given $\mathrm{E} \geq 1$, $\varepsilon \in (0, 1]$ and $\mu \in (0, \frac{1}{2}]$ and there exists $\mathrm{K}_{\mathrm{E},\mu,\varepsilon} > 1$ with the following significance: Suppose that $(\mathrm{A}, \hat{a}) \in \mathrm{Conn}(\mathrm{P}) \times \mathrm{C}^\infty(\mathrm{M}; \mathrm{P} \times_{\mathrm{SO}(3)} \mathfrak{su}(2)) \otimes \mathrm{T}^*\mathrm{M}$ obeys (3.2). Fix $\mathrm{p} \in \mathrm{M}$. If $\mathrm{r}_\diamondsuit < \mathrm{K}^{-1}_{\mathrm{E},\mu,\varepsilon}$ and $\int_{|\mathrm{x}|\leq 1} |\nabla_{\mathrm{A}_\diamondsuit} \hat{a}_*|^2 \leq \mathrm{K}^{-1}_{\mathrm{E},\mu,\varepsilon}$, then $\int_{|\mathrm{x}|\leq 1-\mu} |\mathrm{F}_{\mathrm{A}_\diamondsuit}|^2 < \varepsilon$.*

The equivalence between the assertion in (3.7) and Proposition 3.2 follows directly from the scaling identities given in the next subsection. Note that the norms and volume form used in (3.7) can be either those defined by $\mathfrak{m}_\phi$ or those defined by the Euclidean metric. The proof of (3.7) will use those defined by the Euclidean metric.

The proof of Proposition 3.2 is in Section 4. Section 3.c states some rescaling identities and, as noted previously, it proves Proposition 3.1. Sections 3.d and 3.e state and prove a pair of lemmas that are used in Section 4 for the proof of Proposition 3.2.

3.c. Scaling identities

Proposition 3.1 is proved at the end of this subsection. The observations that follow directly in (3.8) are used in the proof and in the subsequent subsections. The upcoming (3.8) lists some rescaling identities. What is denoted by $\mathfrak{e}_G$ in the fourth bullet has norm bounded by $c_0 r_\diamondsuit$. All bullets refer to a chosen $R \in (0, c_0^{-1} r_\diamondsuit^{-1})$. The norms, inner products, volume form and Hodge dual in (3.8) are all defined using $\mathfrak{m}_\phi$.

(3.8)
- $\int_{|x|\le R}(|d_{A_\diamondsuit}\hat{a}_\diamondsuit|^2 + |d_{A_\diamondsuit}*\hat{a}_\diamondsuit|^2) \le E r_\diamondsuit^{-1} r^{-2}$.
- $\int_{|x|\le R}|F_{A_\diamondsuit} - r_\diamondsuit^2 r^2 \hat{a}_\diamondsuit \wedge \hat{a}_\diamondsuit|^2 \le E r_\diamondsuit$.
- $\int_{|x|\le R}|\nabla_{A_\diamondsuit}\hat{a}_\diamondsuit|^2 = r_\diamondsuit^{-1}\int_{B_{Rr_\diamondsuit}}|\nabla_A \hat{a}|^2$.
- $\int_{|x|\le R}|F_{A_\diamondsuit}|^2 = r_\diamondsuit \int_{B_{Rr_\diamondsuit}}|F_A|^2$.
- $\int_{|x|\le R}|\mathcal{q}_{A_\diamondsuit}(\hat{a}_\diamondsuit)|^2 = r_\diamondsuit \int_{B_{Rr_\diamondsuit}}|\mathcal{q}_A(\hat{a})|^2$.

Note that $|\hat{a}_\diamondsuit| \le c_0 E$, this being a consequence of the third bullet in (3.3). Moreover,

(3.9)
- $\int_{|x|\le R}|\nabla_{A_\diamondsuit}\hat{a}_\diamondsuit|^2 \le c_0 E^2 R$,
- $r_\diamondsuit^2 r^2 \int_{|x|\le R}|\hat{a}_\diamondsuit \wedge \hat{a}_\diamondsuit|^2 \le c_0 E^2 R$,
- $\int_{|x|\le R}|\mathcal{q}_{A_\diamondsuit}(\hat{a}_\diamondsuit)|^2 \le c_0 E r_\diamondsuit$.

The first and second bullets are consequences of the second bullet in (3.3); and the third bullet is a consequence of Item d) of (3.2). These inequalities hold with the norms, volume form and covariant derivative defined by either $\mathfrak{m}_\phi$ or the Euclidean metric.

A standard, dimension 3 Sobolev inequality is used in the proofs of both Proposition 3.1 and Proposition 3.2; and a two other inequalities are used only in the proof of Proposition 3.2. The first inequality is a version of the assertion that the L^2_1 norm dominates the L^4 norm, the second inequality is a local version of Hardy's inequality in (2.19) and the third is a version of the assertion that L^4_1 norm dominates the Holder exponent $\frac{1}{4}$ norm. These inequalities are given directly for future reference. To set the stage, fix $p \in M$ and $r \in (0, c_0^{-1})$. Let f denote a Lipshitz function on B_r. Then

(3.10)
- $\int_{B_r} |f|^4 \leq c_0(\int_{B_r} |f|^2)^{1/2}(\int_{B_r}(|df^2 + r^{-2}|f|^2))^{3/2}$.
- $\int_{B_r} \frac{1}{\mathrm{dist}(p,\cdot)^2} f^2 \leq c_0 \int_{B_r}(|df|^2 + r^{-2}|f|^2)$.
- *If* q_1 *and* q_2 *are any two points in* B_r, *then* $|f(q_2) - f(q_1)| \leq c_0 \mathrm{dist}(q_2, q_1)^{1/4}(\int_{B_r} |df|^4)^{1/4}$.

These same inequalities hold when f is a function on a ball about the origin in $\mathbb{R}^3$ with the only change being the use of the Euclidean metric to define the norms, volume form, and distance function.

It proves convenient at this point to use henceforth the following notational conventions. Unless stated explicitly to the contrary, the Euclidean metric is used to define the norms, Hodge star, covariant derivatives and volume form on the $|x| < c_0^{-1} r_\diamondsuit^{-1}$ part of $\mathbb{R}^3$. Covariant derivatives on $\mathfrak{su}(2)$ valued tensors on $\mathbb{R}^3$ that are defined using the connection θ_0 are denoted by ∇. What is denoted subsequently as c_E is a number that is greater than 16 and depends only on E. It can be assumed to increase between successive appearances. If $\mu \in (0, 1)$ has also been specified, $c_{E,\mu}$ is used to denote a number that is greater than 16 and depends only on E and μ. It can also be assumed to increase between successive appearances.

Proof of Proposition 3.1. Granted (3.3), then Uhlenbeck's theorem in the guise of (3.1) can be invoked. This theorem supplies an isomorphism from the product SO(3) bundle over the $|x| < 1$ ball to $\phi^* P$ that pulls $A_\diamondsuit$ back as $\theta_0 + \hat{a}_{A_\diamondsuit}$ with $\hat{a}_{A_\diamondsuit}$ as described by Proposition 3.1. Use g to denote this isomorphism.

The L^2 norm of $\nabla_{A_\diamondsuit} \hat{a}_\diamondsuit$ on any radius $R < c_0 r_\diamondsuit^{-1}$ ball in $\mathbb{R}^3$ centered at the origin is a priori bounded by $c_E R$ this being a consequence of the first bullet in (3.9). This L^2 norm bound on $\nabla_{A_\diamondsuit} \hat{a}_\diamondsuit$ with what was said about $g^* A_\diamondsuit$ implies that the L^2_1 norm of $g^* \hat{a}_\diamondsuit$ on the $|x| < 1$ ball is bounded by c_E.

Fix $\mu \in (0, \frac{1}{2}]$ and introduce by way of notation χ_μ to denote the function on $\mathbb{R}^3$ given by the rule $x \to \chi(\frac{1}{\mu}(|x| - 1 + \mu))$. This function equals 1 where $|x| \leq 1 - \frac{3}{4}\mu$ and it equals 0 where $|x| \geq 1 - \frac{1}{4}\mu$. Integrate by parts and use the L^∞ bound on $|\hat{a}_\diamondsuit|$ to see that

$$(3.11) \quad \int_{|x|\leq 1} |\nabla_{A_\diamondsuit}(\chi_\mu \nabla_{A_\diamondsuit} \hat{a}_\diamondsuit)|^2 \leq \int_{|x|\leq 1} \chi_\mu |\mathfrak{q}_\diamondsuit|^2 + c_0\Big(\mu^{-2}\|\nabla_{A_\diamondsuit}\hat{a}_\diamondsuit\|_2^2 + \int_{|x|\leq 1} |F_{A_\diamondsuit}|\chi_\mu^2 |\nabla_{A_\diamondsuit}\hat{a}_\diamondsuit|^2 + r_\diamondsuit^2\Big) + c_0 E^2 \int_{|x|\leq 1} |F_{A_\diamondsuit}|^2 \chi_\mu^2.$$

To exploit this inequality, let χ denote for the moment $\chi_\mu \nabla_{A_\diamond} \hat{a}_\diamond$. Then (3.6), the first and third bullets of (3.9) and (3.11) lead to the bound

$$\|\nabla_{A_\diamond} \chi\|_2^2 \leq c_E(1 + \mu^{-2} + \|\chi\|_4^2), \tag{3.12}$$

Given the Sobolev inequality asserted by the top bullet of (3.10), the inequality in (3.12) implies that $\chi_\mu|\nabla_{A_\diamond} \hat{a}_\diamond|$ is an L^2_1 function on the $|x| \leq 1 - \mu$ ball and a second appeal to (3.10) and (3.11) bounds $\|\nabla_{A_\diamond} \chi\|_2$ by $c_{E,\mu}$.

Granted the latter L^2 norm bound, write $g^* A_\diamond = \theta_0 + \hat{a}_{A_\diamond}$ to see that

$$\|\nabla(g^* \chi)\|_2 \leq c_0 \|\hat{a}_{A_\diamond}\|_4 \|\chi\|_4. \tag{3.13}$$

Use this last bound, the a priori L^2_1 bound for $\hat{a}_{A_\diamond}$ and the $f = |\hat{a}_{A_\diamond}|$ version of the top bullet of (3.10) to see that the L^2_1 norm of $g^*\chi$ is also bounded by $c_{E,\mu}$. Write $g^*(\nabla_{A_\diamond} \hat{a}_\diamond)$ as $\nabla_{g^* A_\diamond}(g^* \hat{a}_\diamond)$ to see that

$$|\nabla(\nabla(g^* \hat{a}_\diamond))| \leq c_0(|\nabla(g^*(\nabla_{A_\diamond} \hat{a}_\diamond))| + (|\nabla \hat{a}_{A_\diamond}| + |\hat{a}_{A_\diamond}|^2)|\hat{a}_\diamond| + |\hat{a}_{A_\diamond}||\nabla_{A_\diamond} \hat{a}_\diamond|). \tag{3.14}$$

Given this last inequality, then the apriori L^2_1 bounds for $g^*\chi$ and $\hat{a}_{A_\diamond}$ with the apriori L^∞ bound for $|\hat{a}_\diamond|$ lead directly to the desired $c_{E,\mu}$ bound for the L^2_2 norm of $g^* \hat{a}_\diamond$.

3.d. When $r_z = zr_\diamond r$ is small

Let z again denote the L^2 norm of $\hat{a}_\diamond$ over the $|x| \leq 1$ ball in $\mathbb{R}^3$. Introduce by way of notation r_z to denote the combination $zr_\diamond r$. This combination appears when writing the integral in the second bullet of (3.8) in terms of $\hat{a}_*$. Do so and it asserts the following:

$$\int_{|x| \leq R} |F_{A_\diamond} - r_z^2 \hat{a}_* \wedge \hat{a}_*|^2 \leq E r_\diamond. \tag{3.15}$$

The lemma that follows makes a formal statement to the effect that (3.7) is true if there is an a priori bound on r_z.

Lemma 3.3. *Given $E \geq 1$, $K \geq 1$, $\mu \in (0, \frac{1}{2}]$ and $\varepsilon \in (0, 1]$, there exists $\kappa > 1$ with the following significance: Fix $r > 1$ and a pair $(A, \hat{a})$ of connection on P and section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ obeying (3.2). Suppose that $p \in M$ is a point where $r_z \leq K$. If both $r_\diamond < \kappa^{-1}$ and $\int_{|x| \leq 1} |\nabla_A \hat{a}_*|^2 < \kappa^{-1}$, then $\int_{|x| \leq 1 - \mu} |F_{A_\diamond}|^2 < \varepsilon$.*

Proof of Lemma 3.3. Define χ_μ to be the function on $\mathbb{R}^3$ given by $x \to \chi(\frac{1}{\mu}(|x| - 1 + \mu))$. This function equals 1 where $|x| \leq 1 - \frac{3}{4}\mu$ and it equals 0 where $|x| \geq 1 - \frac{1}{4}\mu$. Fix constant orthonormal vectors e_1 and e_2 on $\mathbb{R}^3$ and integrate $\chi_\mu^2 \langle \hat{a}_*(e_1)[F_{A_\Delta}(e_2, e_1), \hat{a}_*(e_2)]\rangle$ over the radius 1 ball about the origin. The resulting integral is the same as the integral of $\chi_\mu^2\langle F_{A_\Delta}(e_2, e_1)[\hat{a}_*(e_2), \hat{a}_*(e_1)]\rangle$. It follows from (3.15) that c_0 times the latter integral plus $c_E r_z^{-2} r_\Diamond$ bounds $r_z^{-2}\|\chi_\mu F_{A_\Delta}(e_1, e_2)\|_2^2$.

Meanwhile, $[F_{A_\Delta}(e_2, e_1), \hat{a}_*(e_2)]$ can be written as the commutator of $A_\Diamond$ covariant derivatives of $\hat{a}_*(e_2)$. Do so and use this depiction with an integration by parts to see that

$$r_z^{-2}\|\chi_\mu F_{A_\Delta}(e_1, e_2)\|_2^2 \leq c_0(\|\nabla_{A_\Diamond}\hat{a}_*\|_2^2 + \mu^{-1}\|\nabla_{A_\Diamond}\hat{a}_*\|_2 + r_z^{-2} r_\Diamond c_E), \tag{3.16}$$

Suppose that $r_z < \kappa$. Then (3.15) asserts the desired bound $\|\chi_\mu F_{A_\Delta}\|_2^2 < \varepsilon$ when $r_\Diamond < c_E^{-1}\varepsilon$ and when $\|\nabla_{A_\Diamond}\hat{a}_*\| \leq c_0^{-1}\kappa^{-2}\mu\varepsilon$.

The remaining subsections assume unless stated to the contrary that $r_z \geq 1$.

3.e. First order equations

The 4 parts of this subsection explain how a certain almost tautological system of inhomogeneous, semi-linear first order differential equations can be used to prove Proposition 3.2. The linear terms in the homogeneous version of these equations define the first order, constant coefficient system of elliptic equations on $\mathbb{R}^3$ that is described in Part 1 of the subsection. The fully non-linear, inhomogeneous equation is described in Part 2. Part 3 explains how a suitable a priori estimate for a solution to these equations can be used to prove Proposition 3.2. The final part of the subsection states and then proves a lemma that is subsequently used to invoke the arguments in Part 3.

Part 1: To set the stage for what is to come, let τ denote a given, unit length element in $\mathfrak{su}(2)$. With τ chosen, introduce by way of notation $\mathbb{V} \subset (T^*\mathbb{R}^3 \oplus \mathbb{R}) \otimes \mathfrak{su}(2)$ to denote the subbundle that is annihilated by the homomorphism to $(T^*\mathbb{R}^3 \oplus \mathbb{R})$ that is defined by the rule $\mathfrak{f} \to \langle \tau\mathfrak{f}\rangle$. Let e denote a chosen, constant 1-form on $\mathbb{R}^3$ with norm 1.

Fix $m > 1$. Define $\mathcal{L}_m : C^\infty(\mathbb{R}^3; \mathbb{V} \oplus \mathbb{V}) \to C^\infty(\mathbb{R}^3; \mathbb{V} \oplus \mathbb{V})$ as follows: Write an element in $\mathbb{V}$ as $(\mathfrak{a}, \mathfrak{a}_0)$ with $\mathfrak{a}$ being an $\mathfrak{su}(2)$ valued 1-form and $\mathfrak{a}_0$ an element in $\mathfrak{su}(2)$. The operator $\mathcal{L}_m$ sends a given element $((\mathfrak{a}, \mathfrak{a}_0), (\mathfrak{b}, \mathfrak{b}_0))$ to one whose components in the left and right most factor of $\mathbb{V} \oplus \mathbb{V}$ are the respective pairs of $\mathfrak{su}(2)$ valued 1-form and valued function given by

(3.17) • $*(d\mathfrak{a} - me \wedge [\tau, \mathfrak{b}]) - d\mathfrak{a}_0 + me[\tau, \mathfrak{b}_0]$ *and* $*(d{*}\mathfrak{a} - me \wedge [\tau, *\mathfrak{b}])$,

• $*(d\mathfrak{b} - me \wedge [\tau, \mathfrak{a}]) - d\mathfrak{b}_0 + me[\tau, \mathfrak{a}_0]$ *and* $*(d{*}\mathfrak{b} - me \wedge [\tau, *\mathfrak{a}])$,

with $*$ denoting the Euclidean metric's Hodge star operator. The operator $\mathcal{L}_m$ is symmetric with respect to the L^2 inner product defined by the Euclidean metric and such that $\mathcal{L}_m^2 = (\nabla^\dagger\nabla + m^2)\mathbb{I}$ with $\mathbb{I}$ denoting the identity endomorphism of $\mathbb{V}$, with ∇ denoting the covariant derivative on $C^\infty(\mathbb{R}^3; \mathbb{V} \oplus \mathbb{V})$ as defined by the Euclidean metric with $\nabla^\dagger$ being its formal, L^2 adjoint.

This depiction of $\mathcal{L}_m^2$ has the following useful corollary: Let $\|\cdot\|_2$ denote the L^2 inner product on the space of L^2_1 map from $\mathbb{R}^3$ to $\mathbb{V} \oplus \mathbb{V}$. If $\mathfrak{k}$ is any such map, then

$$\|\mathcal{L}_m\mathfrak{k}\|_2^2 = \|\nabla\mathfrak{k}\|_2^2 + m^2\|\mathfrak{k}\|_2^2. \tag{3.18}$$

The positivity of the square of $\mathcal{L}_m$ implies that the equation $\mathcal{L}_m\mathfrak{f} = \mathfrak{h}$ has a unique, L^2_1 solution in $C^\infty(\mathbb{R}^3; \mathbb{V} \oplus \mathbb{V})$ when $\mathfrak{h} \in C^\infty(\mathbb{R}^3; \mathbb{V} \oplus \mathbb{V})$ is square integrable. This solution $\mathfrak{h}$ can be written explicitly using the Green's function for $-d^\dagger d + m^2$. The version of the latter with pole at a given point $y \in \mathbb{R}^3$ is denoted in what follows by G_y; it is the function on $\mathbb{R}^3 - \{y\}$ given by

$$G_y(\cdot) = \frac{1}{4\pi}\frac{1}{|(\cdot) - y|}e^{-m|(\cdot)-y|}. \tag{3.19}$$

As can be seen from (3.19), the function G_y obeys

$$|G_y| + |dG_y| + |\nabla(dG_y)| \leq c_0 e^{-m\delta/2} \tag{3.20}$$

at points with distance greater than δ from y. The Green's function for $\mathcal{L}_m$ with pole at y is the $\mathrm{End}(\mathbb{V} \oplus \mathbb{V})$-valued function on $\mathbb{R}^3 - y$ that is defined by $x \to (\mathcal{L}_m G_{(\cdot)})|_y(x)$. Note in particular that the bounds given above for G_y and its derivatives lead directly to the following observation: The Green's function for $\mathcal{L}_m$ with pole at y and those of its first derivatives are also bounded by $c_0\delta^{-2}e^{-m\delta/2}$ at points with distance greater than δ from y.

Part 2: Let (A_Δ, a_Δ) denote a pair consisting of a connection on the product SO(3) bundle over the $|x| \leq 1$ ball in $\mathbb{R}^3$ and an $\mathfrak{su}(2)$ valued 1-form on this ball. The connection A_Δ is written as $\theta_0 + \hat{a}_{A_\Delta}$. Fix $m \geq 1$ and, with $\tau \in \mathfrak{su}(2)$ as in (3.17), define

$$\mathfrak{a} = m^{-1}(\hat{a}_{A_\Delta} - \tau\langle\tau\hat{a}_{A_\Delta}\rangle \quad \textit{and} \quad \mathfrak{b} = \sqrt{\frac{4\pi}{3}}(a_\Delta - \tau\langle\tau a_\Delta\rangle). \tag{3.21}$$

With $\mathfrak{b}$ understood, define the 1-form e_Δ by writing a_Δ as $a_\Delta = \sqrt{\frac{3}{4\pi}}(\tau e_\Delta + \mathfrak{b})$.

Let $\mathfrak{su}_\perp \subset \mathfrak{su}(2)$ denote the kernel of the homorphism $\mathfrak{f} \to \langle\tau\mathfrak{f}\rangle$. Use $\mathfrak{s}$ to denote the $\mathfrak{su}_\perp$ part of the $\mathfrak{su}(2)$ valued 1-form $m^{-1}{*}(F_{A_\Delta} - \frac{4\pi}{3}m^2 a_\Delta \wedge a_\Delta)$. By the same token, use $\mathfrak{r}$ to denote the $\mathfrak{su}_\perp$ parts of $\sqrt{\frac{4\pi}{3}}d_{A_\Delta} a_\Delta$. Let $*_\phi$ denote the Hodge dual that is defined by the metric $\mathfrak{m}_\phi$ and introduce $\mathfrak{r}_0$ to denote the $\mathfrak{su}_\perp$ part of $\sqrt{\frac{4\pi}{3}}{*}(d_{A_\Delta} {*_\phi} a_\Delta)$. Granted this notation, define an $\mathfrak{su}_\perp$ valued 1-form S by writing $\mathfrak{s}$ as ${*}(d\mathfrak{a} - me \wedge [\tau, \mathfrak{b}]) - S$. Introduce a second $\mathfrak{su}_\perp$ valued 1-form R by writing $\mathfrak{r}$ as ${*}(d\mathfrak{b} - me \wedge [\tau, \mathfrak{a}]) - R$, and introduce an $\mathfrak{su}_\perp$ valued function R_0 by writing $\mathfrak{r}_0$ as ${*}(d{*}\mathfrak{b} - me \wedge [\tau, {*}\mathfrak{a}]) - R_0$. Use S_0 to denote the $\mathfrak{su}_\perp$ valued function ${*}(d{*}\mathfrak{a} - me \wedge [\tau, {*}\mathfrak{b}])$. By way of a summary the troika (S, R, R_0) are as follows:

(3.22)
- ${*}S = m(e_\Delta - e) \wedge [\tau, \mathfrak{b}] - \langle\tau\hat{a}_{A_\Delta}\rangle \wedge [\tau, \mathfrak{a}] + \mathfrak{s}$,
- ${*}R = m(e_\Delta - e) \wedge [\tau, \mathfrak{a}] + \langle\tau\hat{a}_{A_\Delta}\rangle \wedge [\tau, \mathfrak{b}] + \mathfrak{r}$,
- ${*}R_0 = {*}(d(\pi_\phi\mathfrak{b}) - me \wedge [\tau, \pi_\phi\mathfrak{a}]) + m(e_\Delta - e) \wedge [\tau, {*_\phi}\mathfrak{a}] + \langle\tau\hat{a}_{A_\Delta}\rangle \wedge [\tau, {*_\phi}\mathfrak{b}] + \mathfrak{r}_0$,

where the notation has π_ϕ denoting $*_\phi - *$.

Granted all of these definitions, view $\mathfrak{f} = ((\mathfrak{a}, 0), (\mathfrak{b}, 0))$ as mapping the $|x| < 1$ ball to $\mathbb{V} \oplus \mathbb{V}$. The map $\mathfrak{f}$ obeys the tautological equation $\mathcal{L}_m\mathfrak{f} = \mathfrak{h}$ with $\mathfrak{h} = ((S, S_0), (R, R_0))$. The respective left and write $\mathbb{V}$ summands of this equation are

(3.23)
- ${*}(d\mathfrak{a} - me \wedge [\tau, \mathfrak{b}]) = S$ *and* ${*}(d{*}\mathfrak{a} - me \wedge [\tau, {*}\mathfrak{b}]) = S_0$.
- ${*}(d\mathfrak{b} - me \wedge [\tau, \mathfrak{a}]) = R$ *and* ${*}(d{*}\mathfrak{b} - me \wedge [\tau, {*}\mathfrak{a}]) = R_0$.

The equation $\mathcal{L}_m\mathfrak{f} = \mathfrak{h}$ is introduced so that the properties of the Green's function for $\mathcal{L}_m$ can be used to obtain a priori bounds on $\mathfrak{f}$. This is done in Part 4 of the subsection.

Part 3: With the proof of Proposition 3.2 in mind, what follows constitutes a digression that explains how bounds on the L^2 norm of $\mathfrak{b}$ and L^∞ norm of a_Δ lead to a bound for the L^2 norm of F_{A_Δ}. This comes about by writing a_Δ as $\sqrt{\frac{3}{4\pi}}(\tau e_\Delta + \mathfrak{b})$ so as to write F_{A_Δ} as

$$F_{A_\Delta} = m^2 e_\Delta \wedge [\tau, \mathfrak{b}] + m^2\mathfrak{b} \wedge \mathfrak{b} + \left(F_{A_\Delta} - \frac{4\pi}{3}m^2 a_\Delta \wedge a_\Delta\right). \tag{3.24}$$

Fix $\mu \in (0, \frac{1}{2}]$. It follows directly from this depiction of F_{A_Δ} that its L^2 norm on the $|x| < 1 - \mu$ ball in $\mathbb{R}^3$ obeys

(3.25) $$\int_{|x|\le 1-\mu} |F_{A_\Delta}|^2 \le 2\Big(\sup_{|x|\le 1-\mu} |a_\Delta|^2\Big)\Big(m^4\int_{|x|\le 1-\mu} |\mathfrak{b}|^2\Big) + \int_{|x|\le 1-\mu}\Big|F_{A_\Delta} - \frac{4\pi}{3}m^2 a_\Delta \wedge a_\Delta\Big|^2.$$

This inequality has the following consequence: Fix $\varepsilon \in (0,1]$. The L^2 norm of F_{A_Δ} on the $|x| < 1-\mu$ ball will be less than ε if $\sup_{|x|\le 1-\mu} |a_\Delta| < D$, and if the L^2 norm of $|\mathfrak{b}|$ on the $|x| \le 1-\mu$ ball is less than $\frac{1}{4}m^{-2}D^{-1}\varepsilon$, and if that of $(F_{A_\Delta} - \frac{4\pi}{3}m^2 a_\Delta \wedge a_\Delta)$ is less than $\frac{1}{2}\varepsilon$.

Part 4: The tautological equation $\mathcal{L}_m\mathfrak{f} = \mathfrak{h}$ leads to a priori bound for $\mathfrak{f}$ when $\mathfrak{h}$ is small in a suitable sense. A precise statement as to what is meant by 'small' requires the introduction of parameters $M \ge 1$, $\mu \in (0, 2^{-20}]$ and $\rho \in (0,1]$. Having chosen their values, make the following assumptions:

(3.26)
- $\sup_{|x|\le 1-\mu}(|\mathfrak{b}| + |e_\Delta - e|) < \rho$.
- *The* L^2 *norm of* $\mathfrak{a}$ *on the* $|x| \le 1-\mu$ *ball is less than* ρ.
- *The* L^4 *norm of* $\langle \tau \hat{a}_{A_\Delta}\rangle$ *on the* $|x| \le 1-\mu$ *ball is less than* M.
- *The* L^2 *norms of* $\mathfrak{s}, \mathfrak{r}, \mathfrak{r}_0$ *and* s_0 *on the* $|x| \le 1-\mu$ *ball are less than* $m^{-1}\rho$.
- *The endomorphism* $\pi_\phi = *_\phi - *$ *and its covariant derivative obey* $|\nabla\pi_\phi| + |\pi_\phi| < \rho$.

The equation $\mathcal{L}_m\mathfrak{f} = \mathfrak{h}$ is used to prove the next lemma.

Lemma 3.4. *There exists* $\kappa > 1$ *and given* $\mu \in (0, 2^{-20}]$ *and* $\varepsilon \in (0,1)$, *there exists* $\kappa_* > 1$, *these with the following significance: Fix* M *and suppose that* $m > \kappa M + \kappa_*^{-1}$ *and* $\rho < \kappa_*^{-1}$. *Define* $(\mathfrak{a}, \mathfrak{b})$ *as in Part 2 and suppose that the bounds in (3.26) are satisfied. Then*

$$\int_{|x|\le 1-2048\mu} (|\nabla\mathfrak{a}|^2 + |\nabla\mathfrak{b}|^2 + m^2|\mathfrak{a}|^2 + m^2|\mathfrak{b}|^2) < \varepsilon m^{-2}.$$

Proof of Lemma 3.4. The proof has seven steps.

Step 1. Define $\mathfrak{f}_\mu : \mathbb{R}^3 \to \mathbb{V} \oplus \mathbb{V}$ to be the function $\chi_{4\mu}\mathfrak{f}$. The latter obeys an equation that has the schematic form $\mathcal{L}_m\mathfrak{f}_\mu = \mathfrak{h}_\mu$ with $\mathfrak{h}_\mu = \mathfrak{S}(d\chi_{4\mu})\mathfrak{f} + \chi_{4\mu}\mathfrak{h}$ where $\mathfrak{S}$ denotes the principal symbol of the operator $\mathcal{L}_m$. In this case, $\mathfrak{S}$ is constant and so an element in $\mathrm{Hom}(\mathbb{R}^3; \mathrm{End}(\mathbb{V} \oplus \mathbb{V}))$. The introduction of $\mathfrak{f}_\mu$ facilitates the use of the Green's function for $\mathcal{L}_m$ that is described in Part 1. The parameter 4μ is used with $\chi_{(\cdot)}$ because each point where $\chi_{4\mu} > 0$ has distance at least $\frac{1}{4}\mu$ from the $|x| > 1-\mu$ part of $\mathbb{R}^3$ and in particular from the support of $1-\chi_\mu$.

Step 2. Let $\mathfrak{g}$ denote the square integrable map from $\mathbb{R}^3$ to $\mathbb{V}\oplus\mathbb{V}$ that solves the equation $\mathcal{L}_m\mathfrak{g} = \mathfrak{S}(d\chi_\mu)\mathfrak{f}$. To bound the size of $|\mathfrak{g}|$, first use Part 1's Green's function for $\mathcal{L}_m$ to see that

(3.27)
- $|\mathfrak{g}|(y) \le c_0 \int_{\mathbb{R}^3}(\frac{1}{|(\cdot)-y|^2} + m\frac{1}{|(\cdot)-y|})e^{-m|(\cdot)-y|}|d\chi_{4\mu}||\mathfrak{f}|$ *at any given* $y\in\mathbb{R}^3$.
- $|\nabla\mathfrak{g}|(y) \le c_0 \int_{\mathbb{R}^3}(\frac{1}{|(\cdot)-y|^3} + m\frac{1}{|(\cdot)-y|^2} + m^2\frac{1}{|(\cdot)-y|})e^{-m|(\cdot)-y|}|d\chi_{4\mu}||\mathfrak{f}|$ *if* $d\chi_{4\mu} = 0$ *at* y.

Let $D_\mu : \mathbb{R}^3 \to [0,\infty)$ denote the distance to the support of $d\chi_{4\mu}$. Since $|\mathfrak{f}| \le |\mathfrak{a}| + |\mathfrak{b}|$, the bounds in (3.26) and (3.27) imply that

(3.28)
- $|\mathfrak{g}|(y) \le c_0 D_\mu(y)^{-2} e^{-mD_\mu(y)/2}\mu^{-1}\rho$ *if* $d\chi_{4\mu} = 0$ *at* y.
- $|\nabla\mathfrak{g}|(y) \le c_0 D_\mu(y)^{-3} e^{-mD_\mu(y)/2}\mu^{-1}\rho$ *if* $d\chi_{4\mu} = 0$ *at* y.

Save these bounds for the moment.

Step 3. Introduce from (3.22) and (3.23) the terms that are denoted by $\mathfrak{s}, \mathfrak{r}, \mathfrak{r}_0$ and s_0. Let $\mathfrak{t}$ denote the solution in $L^2_1(\mathbb{R}^3; \mathbb{V}\oplus\mathbb{V})$ to the equation

$$\mathcal{L}_m\mathfrak{t} = \chi_{4\mu}((\mathfrak{s}, s_0), (\mathfrak{r}, \mathfrak{r}_0)). \tag{3.29}$$

Take the L^2 norm of both sides of (3.29) and (3.28) with (3.26)'s fourth bullet find

$$\|\nabla\mathfrak{t}\|_2^2 + m^2\|\mathfrak{t}\|_2^2 \le c_0 m^{-2}\rho^2. \tag{3.30}$$

This bound should also be saved.

Step 4. Let $\mathfrak{q} = \mathfrak{f}_\mu - \mathfrak{g} - \mathfrak{t}$. The latter obeys an inhomogeneous differential equation that can be written schematically as

$$\mathcal{L}_m\mathfrak{q} - \chi_{4\mu}P(\mathfrak{q}) = \mathfrak{d}, \tag{3.31}$$

where P is defined momentarily and $\mathfrak{d} = \chi_{4\mu}P(\mathfrak{g}+\mathfrak{t})$. The homomorphism P of $\mathbb{V}\oplus\mathbb{V}$ sends a given element $\mathfrak{s} = ((\mathfrak{s}_1, \mathfrak{s}_{01}), (\mathfrak{s}_2, \mathfrak{s}_{02}))$ to $P(\mathfrak{s}) = ((P_1(\mathfrak{s}), 0), (P_2(\mathfrak{s}), P_{02}(\mathfrak{s}))$ where

(3.32)
- $*P_1(\mathfrak{s}) = m(e_\Delta - e)\wedge[\tau, \mathfrak{s}_2] - \langle\tau\hat{a}_{A_\Delta}\rangle\wedge[\tau, \mathfrak{s}_1]$.
- $*P_2(\mathfrak{s}) = m(e_\Delta - e)\wedge[\tau, \mathfrak{s}_1] + \langle\tau\hat{a}_{A_\Delta}\rangle\wedge[\tau, \mathfrak{s}_2]$.
- $*P_{02}(\mathfrak{s}) = *(d(\pi_\phi\mathfrak{s}_2) - me\wedge[\tau, \pi_\phi\mathfrak{s}_1]) + m(e_\Delta - e)\wedge[\tau, *_\phi\mathfrak{s}_1] + \langle\tau\hat{a}_{A_\Delta}\rangle\wedge[\tau, *_\phi\mathfrak{s}_2]$.

As explained momentarily, the operator $\mathcal{L}_m - \chi_{4\mu}\mathrm{P}$ is invertible if $\rho < c_0^{-1}$ and $m > c_0 \mathrm{M}^2$. This implies in particular that (3.31) has a unique solution.

The asserted invertibility of $\mathcal{L}_m - \chi_{4\mu}\mathrm{P}$ is proved using the bounds stated below in (3.33). The notation in (3.33) has $\mathfrak{v}$ denoting a given L^2_1 section of $T^*\mathbb{R}^3 \otimes \mathfrak{su}(2)$.

(3.33)
- $m\|\chi_{4\mu}|e_\Delta - e|\mathfrak{v}\|_2 \le c_0 \rho m \|\mathfrak{v}\|_2$.
- $\|\chi_{4\mu}|\langle \tau \hat{a}_{A_\Delta}\rangle|\mathfrak{v}\|_2 \le c_0 \mathrm{M}(m^{-1}\|\nabla \mathfrak{v}\|_2 + m\|\mathfrak{v}\|_2)$.
- $\|\chi_{4\mu} d(\pi_\phi \mathfrak{v})\|_2 \le c_0\rho(\|\nabla\mathfrak{v}\|_2 + \|\mathfrak{v}\|_2)$ *and* $m\|\chi_{4\mu}\pi_\phi\mathfrak{v}\|_2 \le m\rho\|\mathfrak{v}\|_2$.

These bounds are derived in the next paragraph. The assertion that $\mathcal{L}_m - \chi_{4\mu}\mathrm{P}$ is invertible follows directly from the following assertion: If $\rho < c_0^{-1}$ and $m > c_0\mathrm{M}^2$, then

$$\|(\mathcal{L}_m - \chi_{4\mu}\mathrm{P})\mathfrak{k}\|_2^2 \ge \tfrac{3}{4}(\|\nabla\mathfrak{k}\|_2^2 + m^2\|\mathfrak{k}\|_2^2) \tag{3.34}$$

when $\mathfrak{k}$ is any given L^2_1 map from $\mathbb{R}^3$ to $\mathbb{V} \oplus \mathbb{V}$. This last assertion follows from (3.33) because the (3.33) implies the following: If $\mathfrak{k}$ is an L^2_1 map from $\mathbb{R}^3$ to $\mathbb{V} \oplus \mathbb{V}$, then

$$\|\chi_{4\mu}\mathrm{P}(\mathfrak{k})\|_2^2 \le c_0(m^{-1}\mathrm{M} + \rho^2)(\|\nabla\mathfrak{k}\|_2^2 + m^2\|\mathfrak{k}\|_2^2). \tag{3.35}$$

If $\rho < c_0^{-1}$ and $m > c_0\mathrm{M}$, then the right hand side is no greater than $\frac{1}{100}\|\mathcal{L}_m k\|_2^2$. The latter bound with (3.28) lead directly to the bound in (3.34).

The assertions of the first and third bullets in (3.33) follow directly from the bounds in the respective first and fifth bullets of (3.26). To see about the middle bullet, keep in mind that the L^2 norm of $\chi_{4\mu}|\langle\tau\hat{a}_{A_\Delta}\rangle||\mathfrak{v}|$ is bounded by the product of their L^4 norms. Use the $\mathfrak{f} = |\mathfrak{v}|$ version of the top bullet in (3.10) to bound the L^4 norm of $\mathfrak{v}$. Having done so, then the inequality in the middle bullet of (3.33) follows directly from the bound in the third bullet of (3.26).

Step 5. Assume henceforth that $\rho < c_0^{-1}$ and that $m > c_0\mathrm{M}$ so as to use (3.28) to bound the right hand side of (3.35) by $\frac{1}{100}\|\mathcal{L}_m\mathfrak{k}\|_2^2$. This being the case, then (3.34) holds. Use the $\mathfrak{k} = \mathfrak{q}$ version of (3.34) with (3.31) to conclude that

$$\|\nabla\mathfrak{q}\|_2^2 + m^2\|\mathfrak{q}\|_2^2 = \tfrac{4}{3}\|\chi_{4\mu}\mathrm{P}(\mathfrak{g} + \mathfrak{t})\|_2^2. \tag{3.36}$$

To exploit this last bound, first apply the bound for $\|\mathrm{P}(\cdot)\|_2 \le \frac{1}{10}\|\mathcal{L}_m(\cdot)\|_2$ to conclude that

$$\|\nabla\mathfrak{q}\|_2 + m\|\mathfrak{q}\|_2 \le \tfrac{1}{10}(\|\mathcal{L}_m\mathfrak{g}\|_2 + \|\mathcal{L}_m\mathfrak{t}\|_2). \tag{3.37}$$

Use (3.43) to bound $\|\mathcal{L}_m\mathfrak{t}\|_2$ by $c_0 m^{-1}\rho$. Use the identity $\mathcal{L}_m\mathfrak{g} = \mathfrak{S}(d\chi_{4\mu})\mathfrak{f}$ to bound $\|\mathcal{L}_m\mathfrak{g}\|_2$ by $c_0\mu^{-1}\|\chi_\mu\mathfrak{f}\|_2$. Substituting these bounds on the right hand side of (3.37) finds

$$\|\nabla\mathfrak{q}\|_2 + m\|\mathfrak{q}\|_2 \le c_0(m^{-1}\rho + \mu^{-1}\|\chi_\mu\mathfrak{f}\|_2) \tag{3.38}$$

when $\rho < c_0^{-1}$ and $m > c_0 M^2$.

Step 6. Write $\mathfrak{f}$ on the $|x| \le 1 - 4\mu$ part of $\mathbb{R}^3$ as $\mathfrak{f} = \mathfrak{g} + \mathfrak{t} + \mathfrak{q}$ to conclude that

$$\begin{aligned}\|\chi_{16\mu}\nabla\mathfrak{f}\|_2 &+ m\|\chi_{16\mu}\mathfrak{f}\|_2 \\ &\le \|\chi_{16\mu}\nabla\mathfrak{g}\|_2 + m\|\chi_{16\mu}\mathfrak{g}\|_2 + \|\nabla\mathfrak{t}\|_2 + m\|\mathfrak{t}\|_2 + \|\nabla\mathfrak{q}\|_2 + m\|\mathfrak{q}\|_2.\end{aligned} \tag{3.39}$$

Use (3.30) to bound the terms in (3.39) with $\mathfrak{t}$ and thus by $m^{-1}\rho$. Use (3.38) to bound those with $\mathfrak{q}$ by $c_0(m^{-1}\rho+\mu^{-1}\|\chi_\mu\mathfrak{f}\|_2)$. The pointwise norms in (3.28) lead to a bound on the terms with $\mathfrak{g}$ by $c_0\mu^{-4}e^{-m\mu/4}\rho$. These substitutions imply the bound

$$\|\chi_{16\mu}\nabla\mathfrak{f}\|_2+m\|\chi_{16\mu}\mathfrak{f}\|_2 \le c_0\mu^{-1}(\mu^{-4}e^{-m\mu/4}m+m^{-1})\rho+c_0\mu^{-1}\|\chi_\mu\mathfrak{f}\|_2. \tag{3.40}$$

Note in particular that $\mu^{-4}e^{-m\mu/4}m \le m^{-1}$ when $m > c_0\mu^{-2}$. Granted that such is the case, then (3.40) implies that

$$\|\chi_{16\mu}\nabla\mathfrak{f}\|_2 + m\|\chi_{16\mu}\mathfrak{f}\|_2 \le c_0\mu^{-1}(m^{-1}\rho + \|\chi_\mu\mathfrak{f}\|_2). \tag{3.41}$$

The first and second bullets of (3.26) bound $\|\chi_\mu\mathfrak{f}\|_2$ by $c_0\rho$ and so the right hand side of (3.41) is no larger than $c_0\mu^{-1}\rho$.

Step 7. Assume henceforth that $m > c_0\mu^{-2}$ so as to conclude from (3.41) that

$$\|\chi_{16\mu}\mathfrak{f}\|_2 \le c_0\mu^{-1}m^{-1}\rho. \tag{3.42}$$

With (3.42) in hand, repeat Steps 1–6 but with μ replaced by $\mu' = 128\mu$. The salient difference is the replacement of the factor $\|\chi_\mu\mathfrak{f}\|_2$ in (3.38), (3.40) and (3.41) by $\|\chi_{16\mu}\mathfrak{f}\|_2$. Where as the former can be bounded only by $c_0\rho$, the latter is bounded courtesy of (3.42) by $c_0 m^{-1}\rho$, this being an improvement by a factor of m^{-1}. Granted this replacement, then the $\mu' = 128\mu$ version of (3.41) reads

$$\|\chi_{2048\mu}\nabla\mathfrak{f}\|_2 + m\|\chi_{2048\mu}\mathfrak{f}\|_2 \le c_0\mu^{-1}m^{-1}\rho. \tag{3.43}$$

The assertion made by Lemma 3.4 follows directly from (3.43).

4. Unexpectedly small curvature

Lemma 3.3 and a version of Lemma 3.4 are brought to bear in Section 4.d to prove Proposition 3.2. The intervening subsections supply data that can be used by Lemma 3.4. The latter requires as input a parameter m, a connection on the product bundle over the $|x| < 1$ ball, and an $\mathfrak{su}(2)$ valued 1-form over this ball, these being A_Δ and a_Δ. The parameter m is taken to be $m = \sqrt{\frac{3}{4\pi}}\, r_z$ and the connection A_Δ is chosen to have the form $h^*A_\diamond$ with h being a suitable isomorphism from the product bundle on the $|x| < 1$ ball to the bundle ϕ^*P over this ball. The $\mathfrak{su}(2)$ valued 1-form a_Δ will be the pull back via h of a suitable perturbation of $\hat{a}_*$. Section 4.a constructs this perturbation and Section 4.c constructs h. The intervening subsection says more about the perturbation.

4.a. The heat equation on the $|x| < 1$ ball

The three parts of this subsection modify $\hat{a}_*$ over the $|x| < 1$ ball in $\mathbb{R}^3$ so as to obtain a $\phi^*(P \times_{SO(3)} \mathfrak{su}(2))$ valued 1-form whose pointwise norm and L^2_2 norm on concentric balls with radius less than 1 obey a priori bounds that can not be assumed to hold for $\hat{a}_*$. The modified version is denoted by $\tilde{a}_*$. The following proposition summarizes the salient features of $\tilde{a}_*$. The proposition uses the metric m_ϕ to define the norms, volume form, Hodge star and covariant derivatives on tensors.

Proposition 4.1. *Given $E \geq 1$, there exists $\kappa_E > \kappa$, and given also $\mu \in (0, \frac{1}{2}]$, there exists $\kappa_{E,\mu} > 1$; these having the following significance: Suppose that $r > 1$ and $(A, \hat{a})$ is a pair of connection on P and section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ that obey (3.2). Fix $p \in M$ such that $r_z = z r_\diamond r \geq 1$. There exists a $\phi^*(P \times_{SO(3)} \mathfrak{su}(2))$ valued 1-form on the $|x| \leq 1$ ball in $\mathbb{R}^3$ to be denoted by $\tilde{a}_*$ with the properties in the list below.*

- $\int_{|x|\leq 1} (|d_{A_\diamond} \tilde{a}_*|^2 + |d_{A_\diamond} {*\tilde{a}_*}|^2) < \kappa_E r_\diamond r_z^{-2}$.
- $\int_{|x|\leq 1} |\hat{a}_* - \tilde{a}_*|^2 < \kappa_E r_\diamond r_z^{-4}$.
- $\int_{|x|\leq 1} |\nabla_{A_\diamond}(\hat{a}_* - \tilde{a}_*)|^2 < \kappa_E r_\diamond r_z^{-2}$.
- *If $\mu \in (0, \frac{1}{2}]$, then* $\int_{|x|\leq 1-\mu} |\nabla_{A_\diamond}(\nabla_{A_\diamond} \tilde{a}_*)|^2 < \kappa_{E,\mu}$.
- *If $\mu \in (0, \frac{1}{2}]$, then* $\sup_{|x|\leq 1-\mu} |\tilde{a}_*| < \kappa_{E,\mu}$.
- *If $\mu \in (0, \frac{1}{2}]$, then* $\int_{|x|\leq 1-\mu} |F_{A_\diamond} - r_z^2 \tilde{a}_* \wedge \tilde{a}_*|^2 \leq \kappa_{E,\mu} r_\diamond$.

Proof of Proposition 4.1. The desired $\tilde{a}_*$ is constructed from $\hat{a}_*$ by mimicking what is done in Sections 2.c and 2.d. The construction has six steps. As in the statement of the proposition, these steps implicitly use the metric $\mathfrak{m}_\phi$ to define the norms, the volume form, the Hodge star and the covariant derivatives on tensors. This metric is also used to define the formal, L^2 adjoint of the covariant derivative.

Step 1. The section $\tilde{a}_*$ is obtained from the solution to an analog of the heat equation in (2.51). The heat equation in this case specifies a $\phi^*(P \times_{SO(3)} \mathfrak{su}(2))$ valued 1-form over the product of $[0, \infty)$ with the $|x| \leq 1$ ball in $\mathbb{R}^3$. The 1-form in question is denoted by a and obeys the following:

(4.1)
- $\frac{\partial}{\partial t} a = -(\nabla_{A_\diamond}{}^\dagger \nabla_{A_\diamond} a + *(*F_{A_\diamond} \wedge a + a \wedge *F_{A_\diamond}) + \mathrm{Ric}_{\mathfrak{m}_\phi}((\cdot) \otimes a)$ where $|x| < 1$.
- $a|_{t=0} = \hat{a}_*$ for all $|x| \leq 1$.
- $a|_{|x|=1} = a_*|_{|x|=1}$ for all $t \geq 0$.

What is denoted by $\mathrm{Ric}_{\mathfrak{m}_\varphi}$ in the top bullet of (4.1) is the Ricci curvature tensor of $\mathfrak{m}_\phi$. Note in particular that its pointwise norm is bounded by $c_0 r_\diamond^2$. The third bullet in (4.1) specifies both tangential and normal components of a on the boundary $|x| = 1$ sphere. Standard results about parabolic equations prove that there is a unique solution to (4.1).

The desired $\tilde{a}_*$ is given by $a|_s$ for an appropriate stopping time $s \in [0, \infty)$.

Step 2. The analog of the function that is denoted by $\mathfrak{E}$ in Part 1 of Section 2.d is the function on $[0, \infty)$ given by

$$\mathfrak{E} = \int_{|x| \leq 1} (|d_{A_\diamond} a|^2 + |d_{A_\diamond} * a|^2). \tag{4.2}$$

Since a is constant on the $|x| = 1$ sphere, integration by parts writes

$$\frac{d}{dt} \mathfrak{E} = -2 \| q_{A_\diamond}(a) \|_2^2, \tag{4.3}$$

where $\| \cdot \|_2$ denotes the L^2 norm on the $|x| < 1$ sphere and where $q_{A_\diamond}(a)$ is defined by writing the top bullet of (4.1) schematically as $\frac{\partial}{\partial t} a = -q_{A_\diamond}(a)$. Note that the integration by parts has no boundary contribution because $a = a_*$ for all t where $|x| = 1$.

Step 3. The first bullet in (3.8) implies that $\mathfrak{E}(\hat{a}_*) \leq c_E r_\diamond r_z^{-2}$, and so (4.3) finds

$$\mathfrak{E}(t) \leq c_0 r_\diamond r_z^{-2} - t\mathfrak{n}(t), \tag{4.4}$$

where $\mathfrak{n}(\mathrm{t})$ is defined by analogy with (2.53) as

$$(4.5)\qquad \mathfrak{n}(\mathrm{t}) = \mathrm{t}^{-1}\int_0^t \|\mathfrak{q}_{\mathrm{A}_\diamond}(\mathfrak{a})\|_2^2.$$

The inequality in (4.5) implies the assertion in the first bullet of Proposition 4.1 if $\tilde{\mathfrak{a}}_*$ is defined to be any $\mathrm{s} \geq 0$ version of $\mathfrak{a}|_\mathrm{s}$. In any event, (4.4) requires that $\mathfrak{n}(\mathrm{t}) \leq \mathrm{c_E r_\diamond} r_z^{-2}\mathrm{t}^{-1}$ if t is positive. Meanwhile, the formula in the first bullet of (4.1) for $\frac{\partial}{\partial \mathrm{t}}\mathfrak{a}$ implies that

$$(4.6)\qquad \|\hat{\mathfrak{a}}_* - \mathfrak{a}|_\mathrm{t}\|_2^2 \leq \mathrm{t}^2\mathfrak{n}(\mathrm{t}) \quad \textit{for any } \mathrm{t} \in [0,\infty).$$

Step 4. The definition in (4.5) with the bound $\mathfrak{n}(\mathrm{t}) \leq \mathrm{c_E r_\diamond} r_z^{-2}\mathrm{t}^{-1}$ have the following consequence: There exists $\mathrm{s} \in [\frac{1}{2}\mathrm{t}, \mathrm{t}]$ such that $\mathfrak{a}|_\mathrm{s}$ obeys

$$(4.7)\qquad \|\mathfrak{q}_{\mathrm{A}_\diamond}(\mathfrak{a}|_\mathrm{s})\|_2^2 \leq \mathrm{c_E r_\diamond} r_z^{-2}\mathrm{t}^{-1}.$$

This understood, take $\mathrm{t} = r_z^{-2}$ and fix $\mathrm{s} \in [\frac{1}{2}\mathrm{t}, \mathrm{t}]$ so that (4.7) holds. Set $\tilde{\mathfrak{a}}_*$ to equal $\mathfrak{a}|_\mathrm{s}$. It follows from (4.7) that $\|\mathfrak{q}_{\mathrm{A}_\diamond}(\tilde{\mathfrak{a}}_*)\|_2 \leq \mathrm{c_E r}_\diamond^{1/2}$ and (4.6) finds $\|\hat{\mathfrak{a}}_* - \tilde{\mathfrak{a}}_{*\mathrm{n}}\|_2 \leq \mathrm{c_E r}_\diamond^{1/2} r_z^{-2}$. The latter bound is the assertion in the second bullet of the proposition.

To see about the third bullet of the proposition, use the fact that both $\mathfrak{E}(\hat{\mathfrak{a}}_*)$ both $\mathfrak{E}(\tilde{\mathfrak{a}}_*)$ are bounded by $\mathrm{c_E r} r_z^{-2}$ to see that

$$(4.8)\qquad \int_{|\mathrm{x}|\leq 1} (|\mathrm{d}_{\mathrm{A}_\diamond}(\hat{\mathfrak{a}}_* - \tilde{\mathfrak{a}}_*)|^2 + |\mathrm{d}_{\mathrm{A}_\diamond}{*}(\hat{\mathfrak{a}}_* - \tilde{\mathfrak{a}}_*)|^2) \leq \mathrm{c_E r_\diamond} r_z^{-2}$$

also. This understood, an integration by parts with the fact that $\hat{\mathfrak{a}}_* = \tilde{\mathfrak{a}}_*$ where $|\mathrm{x}| = 1$ leads from (4.8) to the bound

$$(4.9)\qquad \int_{|\mathrm{x}|\leq 1} \big(|\nabla_{\mathrm{A}_\diamond}(\hat{\mathfrak{a}}_* - \tilde{\mathfrak{a}}_*)|^2 + 2\langle *\mathrm{F}_{\mathrm{A}_\diamond} \wedge (\hat{\mathfrak{a}}_* - \tilde{\mathfrak{a}}_*) \wedge (\hat{\mathfrak{a}}_{*\mathrm{n}} - \tilde{\mathfrak{a}}_{*\mathrm{n}})\rangle\big) \\ \leq \mathrm{c_E r}_\diamond^2\|\hat{\mathfrak{a}}_* - \tilde{\mathfrak{a}}_*\|_2^2 + \mathrm{c_E r_\diamond} r_z^{-2}.$$

In turn, this last inequality, the bound on the L^2 norm of $\mathrm{F}_{\mathrm{A}_\diamond}$ and the Sobolev inequality given in the top bullet in (3.10) lead to the bound

$$(4.10)\qquad \int_{|\mathrm{x}|\leq 1} |\nabla_{\mathrm{A}_\diamond}(\hat{\mathfrak{a}}_* - \tilde{\mathfrak{a}}_*)|^2 \leq \mathrm{c_E r_\diamond} r_z^{-2}.$$

This is the bound that is asserted by the third bullet of the proposition.

Step 5. To see about the fourth and fifth bullets, fix $\mu \in (0, \frac{1}{2}]$. Set χ_μ to denote again denote the function on $\mathbb{R}^3$ that is given by $\mathrm{x} \to \chi(\frac{1}{\mu}(|\mathrm{x}|-1+\mu))$. The bound on the L^2 norm of $q_{\mathrm{A}_\diamondsuit}(\tilde{a}_*)$, the bound on $|\mathrm{Ric}_{\mathfrak{m}_\phi}|$ by $\mathrm{c}_0\mathrm{r}_\diamondsuit^2$ and the fact that (4.10) implies that $\|\nabla_{\mathrm{A}_\diamondsuit}\tilde{a}_*\|_2 \le \mathrm{c}_{\mathrm{E}}$ implies via an integration by parts that

$$\|\nabla_{\mathrm{A}_\diamondsuit}(\nabla_{\mathrm{A}_\diamondsuit}(\chi_\mu\tilde{a}_*))\|_2^2 \le \mathrm{c}_{\mathrm{E},\mu} + \mathrm{c}_0\|\mathrm{F}_{\mathrm{A}_\diamondsuit}\|_2^2\|\chi_\mu\tilde{a}_*\|_\infty^2, \tag{4.11}$$

Note that (4.11) also uses the fact that $\mathrm{A}_\diamondsuit = \theta + \hat{\mathrm{a}}_{\mathrm{A}_\diamondsuit}$ with the bound given by Uhlenbeck's theorem for the L_1^2 norm of $\hat{\mathrm{a}}_{\mathrm{A}_\diamondsuit}$. Since $\|\mathrm{F}_{\mathrm{A}_\diamondsuit}\|_2 \le 1$, the bound in (4.11) implies that

$$\|\mathrm{d}|\nabla_{\mathrm{A}_\diamondsuit}(\chi_\mu\tilde{a}_*)|\|_2 \le \mathrm{c}_{\mathrm{E},\mu} + \mathrm{c}_0\|\chi_\mu\tilde{a}_*\|_\infty, \tag{4.12}$$

and so $|\nabla_{\mathrm{A}_\diamondsuit}(\chi_\mu\tilde{a}_*)|$ is an L_1^2 function with L_1^2 norm bounded by $(\mathrm{c}_{\mathrm{E},\mu} + \mathrm{c}_0\|\chi_\mu\tilde{a}_*\|_\infty)$.

Invoke the top bullet in (3.10) yet again to see that the L^4 norm of $|\nabla_{\mathrm{A}_\diamondsuit}(\chi_\mu\tilde{a}_*)|$ is also bounded by $(\mathrm{c}_{\mathrm{E},\mu} + \mathrm{c}_0\|\chi_\mu\tilde{a}_*\|_\infty)$, and so this is also the case for the L^4 norm of $\mathrm{d}|\chi_\mu\tilde{a}_*|$. With the preceding understood, invoke the the third bullet in (3.10) using $|\chi_\mu\tilde{a}_*|$ for f and then invoke the first bullet of (3.10) using $|\nabla_{\mathrm{A}_\diamondsuit}(\chi_\mu\tilde{a}_*)|$ conclude the following: Fix $\delta \in (0,1)$. Then

$$\|\chi_\mu\tilde{a}_*\|_\infty \le \delta^{-1}(\mathrm{c}_{\mathrm{E},\mu} + \|\nabla_{\mathrm{A}_\diamondsuit}\tilde{a}_*\|_2) + \delta\|\nabla_{\mathrm{A}_\diamondsuit}(\nabla_{\mathrm{A}_\diamondsuit}(\chi_\mu\tilde{a}_*))\|_2. \tag{4.13}$$

A $\delta = \mathrm{c}_0^{-1}$ version of (4.13), the fact that $\|\mathrm{F}_{\mathrm{A}_\diamondsuit}\|_2 \le 1$ with (3.26) and (3.25) lead directly to the assertion in the fourth bullet of Proposition 4.1. The assertion in the fifth bullet follows from the $\delta = 1$ version of (4.13) and the assertion in the fourth bullet.

Step 6. This step proves the assertion in the sixth bullet of the proposition. Given what is said by (3.15), it is only necessary to prove that

$$r_z^4\|\chi_\mu((\hat{a}_* \wedge \hat{a}_*) - (\tilde{a}_* \wedge \tilde{a}_*))\|_2^2 \le \mathrm{c}_{\mathrm{r}}\mathrm{r}_\diamondsuit. \tag{4.14}$$

The left hand side of (4.14) is no greater than

$$\mathrm{c}_0 r_z^4\||\hat{a}_* - \tilde{a}_*|^2\|_2^4 + \mathrm{c}_0 r_z^4\|\hat{a}_* - \tilde{a}_*\|_2^2\|\chi_\mu\tilde{a}_*\|_\infty^2. \tag{4.15}$$

Use the first bullet in (3.10) with the first two bullets of the proposition to see that the left most term in (4.15) is no greater than $\mathrm{c}_{\mathrm{E}}\mathrm{r}_\diamondsuit^{5/2} r_\diamondsuit^{-1}$. Invoke the first bullet of the proposition to bound the right most term by $\mathrm{c}_{\mathrm{E}}\mathrm{r}_\diamondsuit\|\chi_\mu\tilde{a}_*\|_\infty^2$; then use the proposition's fifth bullet to bound $\mathrm{c}_0\mathrm{r}_\diamondsuit\|\chi_\mu\tilde{a}_*\|_\infty^2$ by $\mathrm{c}_{\mathrm{E},\mu}\mathrm{r}_\diamondsuit$.

4.b. The pair $(\hat{a}_{A_\Delta}, \tilde{a}_*)$ and the bounds in (3.26)

Lemma 3.4 refers to parameters m, ρ and M. As noted at the outset, m will be taken to be $\sqrt{\frac{3}{4\pi}}r_z$. The parameters ρ and M are such that (3.26) holds. An appeal to a given $\varepsilon > 0$ version of Lemma 3.4 requires an upper bound for ρ. The upcoming Lemma 4.2 is the first of two lemmas that are used to obtain the required upper bound. This lemma reintroduces the isomorphism g given by Proposition 3.1. The lemma also writes $g^*A_\diamondsuit$ as $\theta_0 + \hat{a}_{A_\Delta}$ as done in Proposition 3.1.

Lemma 4.2. *There exists $M_0 > 1$, and given $E \geq 1$, $\mu \in (0, \frac{1}{2}]$ and $\rho \in (0, 1]$, there exists $\kappa_\rho > 1$ with the following significance: Suppose that $r > 1$ and $(A, \hat{a})$ is a pair of connection on P and section of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ that obey (3.2). Fix $p \in M$ where $r_z \geq \kappa_\rho$. Suppose in addition that $r_\diamondsuit < \kappa_\rho^{-1}$ and that $\|\nabla_{A_\diamondsuit} \tilde{a}_*\|_2 < \kappa_\rho^{-1}$. There exists a constant, unit length 1-form e on $\mathbb{R}^3$ and a constant, unit length element $\tau \in \mathfrak{su}(2)$ with the properties that are listed below.*

- *Write $g^*\tilde{a}_*$ as $\sqrt{\frac{3}{4\pi}}(\tau e_* + \mathfrak{b}_*)$ with $\langle \tau \mathfrak{b}_* \rangle = 0$. Then $\sup_{|x| \leq 1-\mu}(|\mathfrak{b}_*| + |e_* - e|) < \rho$.*
- *Write $\hat{a}_{A_\Delta}$ as $\tau\langle\tau\hat{a}_{A_\Delta}\rangle + m\mathfrak{a}_*$ with $\langle\tau\mathfrak{a}_*\rangle = 0$. The L^2 norm of $\mathfrak{a}_*$ on the $|x| \leq 1-\mu$ ball is less than ρ and the L^4 norm of $\langle\tau\hat{a}_{A_\Delta}\rangle$ on the $|x| \leq 1-\mu$ ball is less than M_0.*

Proof of Lemma 4.2. The assertions in the lower bullet follow directly from (3.26) and Proposition 3.1. The proof of the top bullet has four steps. These steps use δ to denote $\|\nabla_{A_\diamondsuit} \tilde{a}_*\|_2$.

Step 1. Proposition 3.1 asserts in part that the L^2_2 norm of $g^*\tilde{a}_*$ on the $|x| \leq 1-\mu$ ball is bounded by $c_{E,\mu}$. Use this bound with those given in the top bullet of (3.10) to conclude the following: Let x and y denote two points $\mathbb{R}^3$ with norm at most $1-\mu$. Then

$$|(g^*\tilde{a}_*)|_x - (g^*\tilde{a}_*)|_y| \leq c_{E,\mu}\delta^{1/4}. \tag{4.16}$$

As explained directly, this inequality implies that $||\tilde{a}_*|(0) - \sqrt{\frac{3}{4\pi}}| \leq c_{E,\mu}(\delta^{1/4} + r_\diamondsuit r_z^{-2})$.

To see about the latter claim, suppose for the moment that $c > 1$ is such that the $|\tilde{a}_*|(0) < \sqrt{\frac{3}{4\pi}} - c^{-1}$. If so, then (4.16) implies that $|\tilde{a}_*|^2 < (\sqrt{\frac{3}{4\pi}} - c^{-1} + c_{E,\mu}\delta^{1/4})^2$ on the whole of the $|x| < 1-\mu$ ball. This being the case, then its integral over this ball is no greater than $(1-\sqrt{\frac{4\pi}{3}}c^{-1}+\mathrm{c}_{\mathrm{E},\mu}\delta^{1/4})^2(1-\mu)^3$. Moreover, the integral of $|\tilde{a}_*|^2$ on the $|\mathrm{x}|=1-\mu$ sphere is no greater than $3(1-\sqrt{\frac{4\pi}{3}}c^{-1}+\mathrm{c}_{\mathrm{E},\mu}\delta^{1/4})^2(1-\mu)^2$. Granted this last bound, it then follows using the fundamental theorem of calculus that the integral of $|\tilde{a}_*|^2$ over the spherical annulus where $1-\mu\leq|\mathrm{x}|\leq 1$ is no greater than $3(1-\sqrt{\frac{4\pi}{3}}c^{-1}+\mathrm{c}_{\mathrm{E},\mu}\delta^{1/4})^2\mu+\mathrm{c}_0\delta\mu^{1/2}$. Add these various bounds to conclude that $\|\tilde{a}_*\|_2^2\leq(1-\mathrm{c}_0c^{-1}+\mathrm{c}_{\mathrm{E},\mu}\delta^{1/4})^2+\mathrm{c}_0\delta\mu^{1/2}$. This is nonsense if c^{-1} is less than $\mathrm{c}_{\mathrm{E},\mu}(\delta^{1/4}+\mathrm{r}_\Diamond r_z^{-2})$ because Proposition 4.1's second bullet implies that the L^2 norm of $\tilde{a}_*$ is no less than $1-\mathrm{c}_0\mathrm{r}_\Diamond r_z^{-2}$. Very much the same argument derives nonsense if $|\tilde{a}_*|(0)>\sqrt{\frac{3}{4\pi}}+\mathrm{c}_{\mathrm{E},\mu}(\delta^{1/4}+\mathrm{r}_\Diamond r_z^{-2})$.

Step 2. Let τ denote a unit length element in $\mathfrak{su}(2)$ that maximizes the function on the unit sphere in $\mathfrak{su}(2)$ that assigns to any given element σ the norm of the corresponding covector $\langle\sigma(\mathrm{g}^*\tilde{a}_*)|_0\rangle$. Define the 1-form e_* to be $\langle\tau\mathrm{g}^*\tilde{a}_*\rangle$ and define the $\mathfrak{su}(2)$ valued 1-form $\mathfrak{b}_*$ by writing $\mathrm{g}^*\tilde{a}_*$ as $\tau e_*+\mathfrak{b}_*$.

Write $e_*|_0$ as a positive multiple of a unit length 1-form and denote the latter by e. As explained directly the inner product between e and $\mathfrak{b}_*|_0$ must be zero. To see why this is, note first that $(\mathrm{g}^*\tilde{a}_*)|_0$ can be written as

$$(\mathrm{g}^*\tilde{a}_*)|_0=\tau\gamma e+\tau_1(\alpha_1e_1+\nu e)+\tau_2(\alpha_2e_2+\beta e_1), \tag{4.17}$$

with $\gamma>\mathrm{c}_0^{-1}$, with $\{\tau_1,\tau_2\}$ being an orthonormal basis orthogonal complement of τ and with $\{e_1,e_2\}$ being an orthonormal basis in $\mathrm{T}^*\mathbb{R}^3$ for the orthogonal complement to e. Let τ' denote $(\gamma\tau+\nu\tau_1)/(\gamma^2+\nu^2)^{1/2}$. Look at (4.17) to see that $\langle\tau'(\mathrm{g}^*\tilde{a}_*)|_0\rangle$ is the 1-form $(\gamma^2+\nu^2)^{1/2}e+\alpha_1\nu(\gamma^2+\nu^2)^{-1/2}e_1$. The norm of the latter would be larger than γ were $\nu\neq 0$.

Write $\mathfrak{b}_*$ for the moment as $\mathfrak{b}_*=\sigma e+\mathfrak{b}_\perp$ with σ being a map from the $|\mathrm{x}|\leq 1-\mu$ ball in $\mathbb{R}^3$ to τ's version of $\mathfrak{su}^\perp$ and with $\mathfrak{b}_\perp$ such that $e\wedge*\mathfrak{b}_\perp=0$. It follows from (4.16) and from what was said in the preceding paragraph that $|\sigma|\leq\mathrm{c}_{\mathrm{E},\mu}\delta^{1/4}$.

Step 3. Use the $\mathrm{A}_\Delta=\mathrm{g}^*\mathrm{A}_\Diamond$ and $a_\Delta=\mathrm{g}^*\tilde{a}_*$ version of (3.24) with (3.15) and what is said in Part 1 to conclude that

$$\int_{|\mathrm{x}|\leq 1-\mu}|\mathrm{F}_{\mathrm{A}_\Diamond}|^2\geq\tfrac{1}{2}r_z^4\Big(\sup_{|\mathrm{x}|\leq 1-\mu}|\mathfrak{b}_\perp|^2-\mathrm{c}_{\mathrm{E},\mu}\delta^{1/2}\Big)-\mathrm{c}_{\mathrm{E},\mu}\mathrm{r}_\Diamond. \tag{4.18}$$

Note that this follows because $\mathfrak{b}_*\wedge\mathfrak{b}_*$ is proportional to τ and thus pointwise orthogonal to $[\tau,\mathfrak{b}_*]$. This lower bound with (3.6) have the following consequence: Fix for the moment $c\geq 1$. If $\mathrm{r}_\Diamond<\mathrm{c}_{\mathrm{E},\mu}^{-1}c^{-1}$ and $\delta<\mathrm{c}_{\mathrm{E},\mu}c^{-4}$ and $r_z^2>\mathrm{c}_{\mathrm{E},\mu}c^2$, then $|\mathfrak{b}_\perp|<c^{-1}$ on the $|\mathrm{x}|\leq 1-\mu$ ball.

Step 4. Fix $c > 1$ and suppose that $r_\diamond < c_{E,\mu}^{-1} c^{-1}$, $\delta < c_{E,\mu} c^{-4}$ and $r_z^2 \geq c_{E,\mu} c^2$ so as to conclude from Steps 2 and 3 that $|\mathfrak{b}_*| < c^{-1}$ on the $|x| \leq 1 - \mu$ ball. This understood, it then follows from what is in Step 1 that $|e_* - e| \leq c_{E,\mu}(\delta^{1/4} + c^{-1} + r_\diamond)$ on the $|x| \leq 1 - \mu$ ball.

Granted the bound in the preceding paragraph, the assertion made by the top bullet follows if $r_z \geq c_{E,\mu}\rho^{-2}$ and $\delta < c_{E,\mu}^{-1}\rho^4$ and $r_\diamond < c_{E,\mu}^{-1}\rho$.

4.c. The construction of h

Fix $\rho > 0$. Lemma 4.2 asserts that the conditions in the first, second and third bullets of (3.26) are obeyed for a suitable M if $r_\diamond < c_{E,\mu}^{-1}\rho^{-1}$, $\delta < c_{E,\mu}^{-1}\rho^4$ and $r_z \geq c_{E,\mu}\rho^{-2}$. The conditions in the fifth bullet are also met if $r_\diamond < c_0^{-1}\rho$ since the metric m_ϕ differs from the Euclidean metric on the $|x| \leq 1$ ball by at most $c_0 r_\diamond^2$ and the norms of its first derivatives are also bounded by $c_0 r_\diamond^2$. The L^2 norms of what are denoted by $\mathfrak{s}$, $\mathfrak{r}$ and $\mathfrak{r}_0$ are bounded respectively by c_0 times $r_z^{-1}\|F_{A_\diamond} - r_z^2\tilde{a}_* \wedge \tilde{a}_*\|_2$, $\|d_{A_\diamond}\tilde{a}_*\|_2$ and $\|d_{A_\diamond} {*}\tilde{a}_*\|_2$. A look at Proposition 4.1 finds the latter to be bounded by $c_E r_z^{-1} r_\diamond^{1/2}$. This being the case, then the L^2 norms of $\mathfrak{s}$, $\mathfrak{r}$ and $\mathfrak{r}_0$ will be bounded by $m^{-1}\rho$ if $r_\diamond \leq c_E^{-1}\rho^2$.

The problematic requirement in (3.26) is the one that concerns s_0, the reason being that the $(\hat{a}_{A_\diamond}, g^*\tilde{a}_*)$ version of s_0 is $-m{*}(e \wedge [\tau, {*}\mathfrak{b}_*])$ because the Coulomb gauge condition was imposed on $\hat{a}_{A_\diamond}$. Given what is said so far, the L^2 norm s_0 can be as large as $c_0 m\rho$. The lemma that follows will be used to circumvent this problem. This lemma finds an automorphism of the product bundle that pulls back the given pair $(A_\diamond, \tilde{a}_*)$ to one that satisfies all of the conditions in (3.26).

Lemma 4.3. *Given* $E \geq 1$, $\mu \in (0, \frac{1}{2}]$ *and* $\rho \in (0, 1]$, *there exists* $c > 1$ *with the following significance: Suppose that* $r > 1$ *and* $(A, \hat{a})$ *is a pair of connection on* P *and section of* $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$ *that obey (3.2). Fix* $p \in M$ *where* $r_z \geq c$. *Suppose in addition that* $r_\diamond < c^{-1}$ *and that* $\|\nabla_{A_\diamond}\tilde{a}_*\|_2 < c^{-1}$. *There exists an isomorphism,* h, *from the product principal* SO(3) *bundle over the* $|x| \leq 1 - \mu$ *ball to* ϕ^*P *over this ball such that the pair* $(A_\Delta = h^*A_\diamond, a_\Delta = h^*\tilde{a}_*)$ *satisfies the conditions in the* $(\rho, M = \rho r_z)$ *version (3.26).*

Proof of Lemma 4.3. The proof has five parts. Parts 1–4 construct h and Part 5 verifies that it has the required properties.

Part 1: Let κ_* denote the E and $\frac{1}{16}\mu$ version of Lemma 4.2's constant $\kappa_{\rho/16}$. Use δ again to denote $\|\nabla_{A_\diamond}\tilde{a}_*\|_2$. Assume henceforth that $r_z > \kappa_*$, that $r_\diamond < \kappa_*^{-1}$ and that $\delta < \kappa_*^{-1}$ so as to invoke the apriori bounds that Lemma 4.2 supplies on the $|x| < 1-\frac{1}{16}\mu$ ball with ρ replaced every where by $\frac{1}{16}\rho$. The desired isomorphism h is written as the composition of first g and then the restriction to the $|x| \leq 1-\mu$ ball of a certain isomorphism of the product bundle $\mathbb{R}^3 \times SO(3)$. An isomorphism of the product principal SO(3) bundle over $\mathbb{R}^3$ can be viewed as a map from $\mathbb{R}^3$ to SU(2) and vice-versa. The notation in what follows does not distinguish between such a map and the corresponding isomorphism of the product principal bundle $\mathbb{R}^3 \times SO(3)$. The desired map/isomorphism is denoted by u. The incarnation of u as a map from $\mathbb{R}^3$ to SU(2) is written as $u = e^{m\chi}$ with χ denoting in what follows a map from $\mathbb{R}^3$ to τ's version of $\mathfrak{su}^\perp$ that is chosen so that the corresponding $(A_\Delta = h^*A_\diamond, a_\Delta = h^*\tilde{a}_*)$ version of s_0 is 0 on the $|x| \leq 1-\mu$ ball.

Part 2: Let $u : \mathbb{R}^3 \to SO(3)$ denote a given map. The action of u on $(g^*A_\diamond, g^*\tilde{a}_*)$ sends this pair to a pair that is written as $(\theta_0 + \hat{a}_u, u(g^*\tilde{a}_*)u^{-1})$ with $\hat{a}_u = u\hat{a}_{A_\diamond}u^{-1} + udu^{-1}$. Define a pair, $(\mathfrak{a}_u, \mathfrak{b}_u)$ of maps from the $|x| < 1-\frac{1}{16}\mu$ ball in $\mathbb{R}^3$ to $\mathbb{V}\oplus\mathbb{V}$ by

$$\mathfrak{a}_u = m^{-1}(\hat{a}_u - \tau\langle\tau\hat{a}_u\rangle) \quad \textit{and} \tag{4.19}$$
$$\mathfrak{b}_u = \sqrt{\frac{4\pi}{3}}(u(g^*\tilde{a}_*)u^{-1} - \tau\langle\tau u(g^*\tilde{a}_*)u^{-1}\rangle),$$

these being the $A_\Delta = \theta_0 + \hat{a}_u$ and $a_\Delta = u(g^*\tilde{a}_*)u^{-1}$ versions of what (3.21) denote by $\mathfrak{a}$ and $\mathfrak{b}$. The plan for what follows is to construct a map $\chi : \mathbb{R}^3 \to \mathfrak{su}^\perp$ so that the $u = e^{m\chi}$ version of $(\mathfrak{a}_u, \mathfrak{b}_u)$ obey $d{*}\mathfrak{a}_u - me[\tau, {*}\mathfrak{b}_u] = 0$ where $|x| < 1-\mu$.

To see what the construction of the desired map involves, let $\chi : \mathbb{R}^3 \to \mathfrak{su}(2)$ denote a smooth map with $\langle\tau\chi\rangle = 0$ and with $|\chi| < m^{-1}c_0^{-1}$ at all points. With χ chosen, set $u = e^{m\chi}$. The resulting version of $\mathfrak{a}_u$ and $\mathfrak{b}_u$ can be written schematically as

$$\mathfrak{a}_u = \mathfrak{a} - d\chi + A(\chi, \mathfrak{a}) \quad \textit{and} \quad \mathfrak{b}_u = \mathfrak{b} - m[\tau, \chi]e + B(\chi, \mathfrak{b}). \tag{4.20}$$

The desired version of χ is constrained to obey the differential equation

$$d^\dagger d\chi + m^2\chi + \chi_\mu{*}(me \wedge [\tau, {*}\mathfrak{b}] - d{*}A + me \wedge [\tau, {*}B]) = 0. \tag{4.21}$$

If χ obeys (4.21), then $d{*}\mathfrak{a}_u - me \wedge [\tau, {*}\mathfrak{b}_u]) = 0$ is obeyed where $\chi_\mu = 1$ and thus on the ball where $|x| < 1-\frac{3}{4}\mu$. Note that (4.21) is the linear equation $d^\dagger d\chi + m^2\chi = 0$ where χ_μ is zero, and thus where $|x| \geq 1-\frac{1}{4}\mu$.

Part 3: It proves useful to write (4.21) as an equation for the fixed point of a self-map of a certain Banach space completion of the space of compactly supported maps from $\mathbb{R}^3$ to $\mathfrak{su}^\perp$. The norm for the Banach space is defined by the rule

$$\chi \to \sup_{\mathbb{R}^3}(m|\chi| + |d\chi|). \tag{4.22}$$

This norm is denoted by $\|\cdot\|_{\mathbb{B}}$ and the resulting Banach space is denoted by $\mathbb{B}$.

The specification of the relevant map from $\mathbb{B}$ to $\mathbb{B}$ is denoted by $\mathcal{T}$. This map is depicted in the upcoming (4.24). What follows directly explains the notation that is used in (4.24). To start, let u again denote $e^{m\chi}$. Then $m^{-1}udu^{-1}$ can be written schematically as $\mathcal{G}(d\chi)$ with $\mathcal{G}$ denoting the map from $\mathbb{R}^3$ to $\mathrm{End}(\mathfrak{su}(2))$ given by the rule

$$\alpha \to \mathcal{G}(\alpha) = -\int_0^1 e^{sm\chi}\alpha e^{-sm\chi}ds. \tag{4.23}$$

This formula has the following implication: If $|\chi| \le c_0^{-1}m^{-1}$, then $\mathcal{G}$ can be written as $1 + g$ with $|g(\alpha)| < \frac{1}{100}|\alpha|$. In particular, $\mathcal{G}$ is invertible if $|u| \le c_0^{-1}m^{-1}$ and its inverse can be written as $\mathcal{G}^{-1} = 1 + g_*$ with $|g_*(\alpha)| < c_0|\chi||\alpha|$. In addition to introducing g_*, the upcoming (4.24) introduces A_1 to denote the contribution to A in (4.20) from the term $m^{-1}u\hat{a}_{A_\diamond}u^{-1}$. This is to say that $A_1 = m^{-1}([duu^{-1}, u\hat{a}_{A_\diamond}u^{-1}] - \tau\langle\tau[duu^{-1}, u\hat{a}_{A_\diamond}u^{-1}]\rangle)$. By way of a final remark about notation, (4.24) reintroduces the Green's function G from (3.19).

Having set the notation, fix $c > c_0$ so that g_* is defined on the $\|\cdot\|_{\mathbb{B}} < c^{-1}$ ball about the origin in $\mathbb{B}$. Denote this ball by $\mathbb{B}_c$ and introduce the map $\mathcal{T} : \mathbb{B}_c \to C^0(\mathbb{R}^3; \mathfrak{su}(2))$ by defining $\mathcal{T}(\chi)$ at any given point $y \in \mathbb{R}^3$ to be

$$-\int_{\mathbb{R}^3} G_y\chi_\mu\big(m^2 g_*\chi - {*d*A_1} + {*m}(1+g_*)(e\wedge[\tau, *\mathfrak{b}] - \mathcal{G}(d\chi)\wedge *\mathcal{G}(d\chi) + e\wedge[\tau, *B])\big). \tag{4.24}$$

A C^2 fixed point of $\mathcal{T}$ is, by construction, a solution to (4.21).

Part 4: This part of the proof explains why the following assertion is true:

(4.25) *There exists* $\kappa > 1$ *with the following significance: Suppose that* $c > \kappa$ *and that* $m > \kappa$ *and that* $\sup_{|x|<1-\mu/16}|\mathfrak{b}_*| < \kappa^{-1}$. *Then* $\mathcal{T}$ *maps* $\mathbb{B}_c$ *to itself as a contraction mapping.*

Steps 1–3 of what follows prove the assertion in (4.25). A fourth step proves that the corresponding fixed point is C^∞. These steps use λ to denote $\sup_{|x|<1-\mu/16} |\mathfrak{b}_*|$.

Step 1. Fix $c > c_0$ so that $\mathcal{T}$ is defined on $\mathbb{B}_c$. This step derives an a priori bound for the supremum norm of any given $\chi \in \mathbb{B}_c$ version of $\mathcal{T}(x)$. To this end, use the fact that $d{*}\hat{a}_{A_\diamond} = 0$ to see that the integrand in (4.24) has absolute value less than

$$c_0(m\|\chi\|_{\mathbb{B}}^2 + m|\mathfrak{b}_*| + \|\chi\|_{\mathbb{B}}|\hat{a}_{A_\diamond}|). \tag{4.26}$$

As can be seen from (3.20), the integral $G_y(\cdot)$ is bounded by $c_0 m^{-2}$, and so the integral of G_y times the left most two terms in (4.26) is bounded by $c_0 m^{-1}(\|\chi\|_{\mathbb{B}}^2 + \lambda)$. Meanwhile, the integral of $G_y(\cdot)|\hat{a}_{A_{\diamond n}}|$ is bounded by c_0 times the product of the L^2 norm of the function $e^{-m|\cdot|/4}$ with the L^2 norm of $\frac{1}{|y-(\cdot)|}\chi_\mu|\hat{a}_{A_\diamond}|$. The former is bounded by $c_0 m^{-3/2}$ and the latter is bounded courtesy of the second bullet in (3.10) and Proposition 3.1 by c_0. Granted all of this, it then follows that

$$|\mathcal{T}(\chi)| \le c_0 m^{-1}(c^{-2} + \lambda + m^{-1/2}c^{-1}) \quad \textit{when } \chi \in \mathbb{B}_c. \tag{4.27}$$

This is the promised supremum norm for $\mathcal{T}(\chi)$ because it holds at all $x \in \mathbb{R}^3$.

Step 2. This step derives an a priori bound for the supremum norm of the 1-form $d\mathcal{T}(\chi)$. To start this task, fix $x \in \mathbb{R}^3$ and view the assignment $y \to G_y(x)$ as defining a function on $\mathbb{R}^3 - \{x\}$. Use $(dG)_y(x)$ to denote the 1-form on $\mathbb{R}^3$ that is obtained by differentiating this function. It follows from (3.20) that

$$|(dG)_y(\cdot)| \le c_0\left(\frac{1}{|(\cdot) - y|^2} + m\frac{1}{|(\cdot) - y|}\right)e^{-m|(\cdot)-y|}. \tag{4.28}$$

The integral of the right hand side of (4.28) is bounded by $c_0 m^{-1}$. This understood, then the integral of $|(dG)_y(\cdot)|$ times the left most two terms in (4.26) is at most $c_0(\|\chi\|_{\mathbb{B}}^2 + \lambda)$. Meanwhile, the L^2 norm of $(\frac{1}{|(\cdot)-y|} + m)e^{-m|(\cdot)-y|}$ is no greater than $c_0\ m^{-1/2}$. It follows from the latter bound that the integral of $|(dG_y(\cdot)|$ times the term $\|\chi\|_{\mathbb{B}}|\hat{a}_{A_{\diamond n}}|$ in (4.26) is no greater than $c_0 m^{-1/2}\|\chi\|_{\mathbb{B}}$ times the L^2 norm of $\frac{1}{|y-(\cdot)|}\chi_\mu|\hat{a}_{A_\diamond}|$. As noted previously, the latter norm is bounded by c_0. Therefore, the bounds just stated imply that

$$|d\mathcal{T}(\chi)| \le c_0(c^2 + \lambda + m^{-1/2}c^{-1}) \quad \textit{when } \chi \in \mathbb{B}_c. \tag{4.29}$$

This inequality holds for all $x \in \mathbb{R}^3$ and thus supplies a supremum norm for $|d\mathcal{T}(x)|$.

Step 3. It follows from (4.27) and (4.29) that $\mathcal{T}$ maps $\mathbb{B}_c$ to the space of Lipshitz maps from $\mathbb{R}^3$ to $\mathfrak{su}^\perp$. To see that the image is in $\mathbb{B}$, use the fact that the integrand in (4.24) is supported where $|x| < 1$ with (4.26) to conclude that $\mathcal{T}(\chi)$ is smooth where $|x| > 1$ and that the norms of $\mathcal{T}(\chi)$ and its derivatives to any given order are bounded a priori where $|x| > 1$ by an order dependent multiple of $\|\mathcal{T}(\chi)\|_{\mathbb{B}} e^{-m_n|x|/2}$. This implies that $\mathcal{T}$ maps $\mathbb{B}_c$ to $\mathbb{B}$.

Granted that $\mathcal{T}$ maps $\mathbb{B}_c$ to $\mathbb{B}$, it follows from (4.27) and (4.29) that

$$\|\mathcal{T}(\chi)\|_{\mathbb{B}} \leq c_0(c^{-2} + \lambda + m^{-1/2}c^{-1}). \tag{4.30}$$

This last bound has the following consequence: If $c > c_0$ and $\lambda < c_0^{-1}c^{-1}$ and $m > c_0$, then $\mathcal{T}$ maps $\mathbb{B}_c$ to the $\|\cdot\|_{\mathbb{B}} < \frac{1}{2}c^{-1}$ ball in $\mathbb{B}$.

Fix c and ρ and m so that the preceding conclusions apply. Let χ and χ' denote two elements in $\mathbb{B}_c$. Arguments that differ only cosmetically from those in Steps 1 and 2 prove that $\|\mathcal{T}(\chi) - \mathcal{T}(\chi')\|_{\mathbb{B}} \leq c_0(c^{-1} + \lambda + m^{-1/2})\|\chi - \chi'\|_{\mathbb{B}}$. This last bound implies that $\mathcal{T}$ defines a contraction self-map of $\mathcal{B}_c$ when $c > c_0$ and $\lambda < c_0^{-1}$ and $m > c_0$.

Step 4. Fix $c > c_0$ and $\lambda < c_0^{-1}c^{-1}$ and $m > c_0$ so that $\mathcal{T}$ maps $\mathbb{B}_c$ to itself as a contraction mapping. The contraction mapping theorem asserts that $\mathcal{T}$ has a unique fixed point in $\mathbb{B}_c$. Let χ now denote this fixed point. The fact that χ is smooth can be proved using a boot-strapping argument with (4.24). In particular an argument of this sort using a difference quotient construction gives Lipshitz bounds on successive orders of derivatives. This argument is straightforward; and since it is somewhat tedious, it is left to the reader except for the following brief remarks about the argument for the second derivatives. To start, fix $\varepsilon > 0 |\nabla d\chi|_\varepsilon$ denote

$$|\nabla d\chi|_\varepsilon = \varepsilon^{-1} \sup_{\{(y,y') \in \mathbb{R}^3 : |y-y'| > \varepsilon\}} |(d\chi)|_y - (d\chi)|_{y'}| \tag{4.31}$$

Since $\chi = \mathcal{T}(\chi)$, the bound in (4.31) for the $\|\cdot\|_{\mathbb{B}}$ norm of (4.24) and the definition in (4.23) for $\mathcal{G}$ can be used with the arguments that differ little from those in Step 2 to prove that

$$|\nabla d\chi|_\varepsilon \leq c_0(c^{-1} + m^{-1/2})|\nabla d\chi|_\varepsilon + c_0\Big(\sup_{\mathbb{R}^3}(|\nabla \mathfrak{b}| + |\nabla \hat{a}_{A_\diamond}|) + mc^{-1}\Big). \tag{4.32}$$

This last bound leads to the following conclusion: If $c > c_0^{-1}$ and $m > c_0$, then $|\nabla d\chi|_\varepsilon$ has a ε independent upper bound. Granted that such is the case, then $d\chi$ is Lipschitz.

Part 5: Fix $c > c_0$ and $\lambda < c_0^{-1}c^{-1}$ and $m > c_0$ so that the map $\mathcal{T}$ has a unique fixed point in $\mathbb{B}_c$. Let χ denote this fixed point. Let u denote $e^{m\chi}$. This part of the proof considers the extent to which the pair $(A_\Delta = \theta_0 + \hat{a}_u, a_\Delta = u(g^*\tilde{a}_*g^{-1})u^{-1})$ obey the conditions set forth in a given M and ρ version of (3.26).

The fifth bullet in (3.26) is again obeyed when $r_\diamond < c_0^{-1}\rho^{1/2}$, this because π_ϕ has no $(A_\Delta, \tilde{a}_\Delta)$ dependence. As for the fourth bullet, the L^2 norms of $\mathfrak{s}$, $\mathfrak{r}$ and $\mathfrak{r}_0$ for the case when $(A_\Delta = \theta_0 + \hat{a}_u, a_\Delta = u(g^*\tilde{a}_*g^{-1})u^{-1})$ are the same as those of their analogs for the case when $(A_\Delta = \theta_0 + \hat{a}_{A_\Delta}, a_\Delta = g^*\tilde{a}_*g^{-1})$. This is because the $(A_\Delta = \theta_0 + \hat{a}_u, a_\Delta = u(g^*\tilde{a}_*g^{-1})u^{-1})$ versions of $\mathfrak{s}$, $\mathfrak{r}$ and $\mathfrak{r}_0$ are the pointwise conjugations of their $(A_\Delta = \theta_0 + \hat{a}_{A_\Delta}, a_\Delta = g^*\tilde{a}_*g^{-1})$ namesakes by $e^{m\chi}$. Meanwhile, the $(A_\Delta = \theta_0 + \hat{a}_u, a_\Delta = u(g^*\tilde{a}_*g^{-1})u^{-1})$ version of s_0 is zero on the $|x| \leq 1-\mu$ ball. This being the case, the L^2 norm bounds in the the fourth bullet of the $(A_\Delta = \theta_0 + \hat{a}_u, a_\Delta = u(g^*\tilde{a}_*g^{-1})u^{-1})$ version of (3.26) are met if $r_\diamond \leq c_0^{-1}\rho^2$.

The subsequent discussion of the first three bullets in (3.26) invokes an important consequence of (4.30), this being that

$$|d\chi| + m|\chi| \leq c_0\lambda. \tag{4.33}$$

It follows directly from (4.33) that

(4.34)
- $|\mathfrak{b}_u - \mathfrak{b}_*| + |e_u - e_*| \leq c_0\lambda(\lambda + |\mathfrak{b}_*|)$,
- $|\mathfrak{a}_u - \mathfrak{a}_*| \leq c_0\lambda(\lambda + |\mathfrak{a}_*|)$,
- $|\langle\tau\hat{a}_u\rangle - \langle\tau\hat{a}_{A_\Delta}\rangle| \leq c_0\lambda^2 m + c_0\lambda|\langle\tau\hat{a}_{A_\Delta}\rangle|$.

The inequality in the first bullet in (4.34) with the assumed bound $|\mathfrak{b}_*| + |e_* - e| < \frac{1}{16}\rho$ implies that $\sup_{|x|\leq 1-\mu}(|\mathfrak{b}_u| + |e_u - e|) < \rho$ when $\lambda < c_0^{-1}\rho$. This is the requirement of the first bullet in the $(A_\Delta = \theta_0 + \hat{a}_u, a_\Delta = u(g^*\tilde{a}_*g^{-1})u^{-1})$ version of (3.26). The second bullet of (4.34) with the assumed $\frac{1}{16}\rho$ bound for the L^2 norm of $\mathfrak{a}_*$ on the $|x| < 1 - \frac{1}{16}\mu$ ball implies the following: If $\lambda < c_0^{-1}\rho$, then the L^2 norm of $\mathfrak{a}_u$ on the $|x| < 1-\mu$ ball is less than ρ. This is the requirement of the second bullet of the $(A_\Delta = \theta_0 + \hat{a}_u, a_\Delta = u(g^*\tilde{a}_*g^{-1})u^{-1})$ version of (3.26). If $\lambda < c_0^{-1}\rho$, then the third bullet in (4.34) implies the third bullet of the $(A_\Delta = \theta_0 + \hat{a}_u, a_\Delta = u(g^*\tilde{a}_*g^{-1})u^{-1})$ version of (3.26) in the case when $M = \rho r_z$.

4.d. The proof of Proposition 3.2

Fix small $\mu > 0$ and $\varepsilon' > 0$. Lemma 4.3 with the first and second bullets of Proposition 4.1 and Lemma 3.4 lead directly to the following: There exists $c > 1$ such that if $r_z > c$, $r_\diamondsuit < c^{-1}$ and $\|\nabla_{A_\diamondsuit} \hat{a}_*\|_2 < c^{-1}$, then $|u(g^*\tilde{a}_* g^{-1})u^{-1}| \le c_0$ on the $|x| < 1 - \mu$ ball and the $(A_\Delta = u^* g^* A_\diamondsuit$, $a_\Delta = u(g^*\tilde{a}_* g^{-1})u^{-1})$ version of what is denoted by $\mathfrak{b}$ in (3.21) obeys

$$\int_{|x|\le 1-\mu} |\mathfrak{b}|^2 < \varepsilon' r_z^{-4}. \tag{4.35}$$

Given what is said in the second bullet of (3.8), these bounds with the $m = \sqrt{\frac{3}{4\pi}} r_z$ version of (3.25) lead directly to the bound

$$\int_{|x|\le 1-\mu} |F_{A_\diamondsuit}|^2 \le c_0(\varepsilon' + r_\diamondsuit). \tag{4.36}$$

If ε' is taken equal to $c_0^{-1}\varepsilon$ and if $r_\diamondsuit < c_0^{-1}\varepsilon$, then (4.36) is the assertion of Proposition 3.2. Therefore, if $r_z > c$ and $r_\diamondsuit < \min(c^{-1}, c_0^{-1}\varepsilon)$ and $\|\nabla_{A_\diamondsuit} \hat{a}_*\|_2 < c^{-1}$, then Proposition 3.2 is true.

Let κ_c denote the $(\kappa = c, \mu, \varepsilon)$ version of Lemma 3.3's constant κ. If $r_z \le c$, then Lemma 3.3's conclusion holds if both $r_\diamondsuit < \kappa_c^{-1}$ and $\|\nabla_{A_\diamondsuit} \hat{a}_*\|_2 < \kappa_c^{-1}$. As Lemma 3.3's conclusion is the assertion of Proposition 3.2, it follows that Proposition 3.2 is true no matter the value of r_z when both $r_\diamondsuit$ and $\|\nabla_{A_\diamondsuit} \hat{a}_*\|_2$ are less than $\min(c^{-1}, c_0^{-1}\varepsilon, \kappa_c^{-1})$.

5. Integral identities and monotonicity

This section has two purposes, the first is to state and then prove a proposition that asserts a monotonicity property of two functions on $(0, c_0^{-1})$ that are associated to any given point in M and a pair $(A, \hat{a})$ of connection on P and section of $P \times_{SO(3)} \mathfrak{su}(2)$ that obeys the constraints in (3.2) for a given E and r. This monotonicity assertion constitutes the upcoming Proposition 5.1. The second purpose of the section is to establish various integral inequalities, some of which are used in the proof of Proposition 5.1 and some are used in Section 6.

To set the stage for Proposition 5.1, fix $c_0 > 1$ so that the ball of radius c_0^{-1} centered about any given point in M has compact closure in a Gaussian coordinate chart about the chosen point. Fix for the moment a point $p \in M$. Given $r \in (0, c_0^{-1})$, let $B_r \subset M$ denote radius r ball centered at p and let ∂B_r denote the boundary of the closure of B_r.

Reintroduce Uhlenbeck's constant κ_U from (3.1). Having specified a connection, A, on P and $p \in M$, define $r_\diamondsuit \in (0, c_0^{-1}]$ as in (3.5) to be the largest value of r such that

$$\int_{B_r} |F_A|^2 \le \tfrac{1}{100}\kappa_U^{-2} r^{-1}, \tag{5.1}$$

To continue the stage setting, suppose that $p \in M$ and $r \in (0, c_0^{-1})$ are given and that r is a chosen, positive number. Fix a pair $(A, \hat{a})$ with A being a connection on P's restriction to the closure of B_r and with $\hat{a}$ being a section over this closure of the bundle $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. The parameter r with A and $\hat{a}$ define functions h and H on $(0, c_0^{-1})$ by the rules

$$r \to h(r) = \int_{\partial B_r} |\hat{a}|^2 \quad \textit{and} \quad r \to H(r) = \int_{B_r} (|\nabla_A \hat{a}|^2 + 2r^2 |\hat{a} \wedge \hat{a}|^2). \tag{5.2}$$

Use these two to define the function N on the $h \neq 0$ part of $(0, c_0^{-1})$ by the rule

$$r \to N(r) = \frac{rH(r)}{h(r)}. \tag{5.3}$$

The function N is called the *frequency* function because it plays the role here that is played by an eponymous function that was introduced by Almgren [Al]. The latter incarnation of N plays a central role in proving various theorems about the nodal sets of eigenfunctions of the Laplacian on Riemannian manifolds, and on solutions to certain nonlinear equations, second order differential equations with Laplace symbol. See for example [Han], [DF] and [HHL].

Proposition 5.1. *Fix $r_0 > 0$ such that each $p \in M$ version of the functions h, H and N are defined on the radius r_0 ball centered at p. Given r_0, there exists $\kappa > 100$ whose significance is explained by what follows. Suppose that $E \ge 1$, that $r > \kappa E^{\kappa}$ and that $(A, \hat{a})$ is a pair in $\mathrm{Conn}(P) \times C^\infty(M; (P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M)$ obeying Items a)–d) in (3.2). Fix a point $p \in M$ so as to define the function h, H and N using r and $(A, \hat{a})$ on $(0, r_0]$.*

- *If $r \in [\frac{1}{2} r_\diamond, \kappa^{-1} r_0]$, then $\frac{d}{dr} h = 2r^{-1}(1 + N)h + \mathfrak{e}$ with $\mathfrak{e}$ being a function of r whose absolute value is less than $\kappa(E r^{-2} + E r^{-1/2} h^{1/2} + E^{1/2} r^{-1} H^{1/2})$.*
- *Let u denote the function on $[\frac{1}{2} r_\diamond, \kappa^{-1} r_0]$ that is defined by the rule $r \to u(r) = \kappa E^{\kappa} r^{1/\kappa}$. If $r_\dagger \in [\frac{1}{2} r_\diamond, \kappa^{-1} r_0]$ is such that $h(r) \ge r^{3-1/\kappa}$ when $r \in [r_\dagger, \kappa^{-1} r_0]$. Then*

  $$N(r_2) \ge e^{-u(r_2)+u(r_1)} N(r_1) - u(r_2) + u(r_1) - r^{-1/8}.$$

  when r_2 and r_1 are such that $r_\dagger \le r_1 \le r_2 \le \kappa^{-1} r_0$.

- *If* $r_* \in [\frac{1}{2}r_\diamondsuit, \kappa^{-1})$ *is such that* $h(r_*) = r_*^{2+1/16}$ *and* $h(r) > r^{2+1/16}$ *when* r *is greater than* r_* *but sufficiently close to* r_*, *then* $h(r) \leq \kappa r^{2+1/16}$ *when* $r \in [r_*, \kappa^{-1}r_0]$.
- *If* $r_* \in [\frac{1}{2}r_\diamondsuit, \kappa^{-1})$ *and* $h(r_*) \leq r_*^{2+1/16}$, *then* $h(r) \leq \kappa r^{2+1/16}$ *for all* $r \in [r_*, \kappa^{-1}r_0]$.

Note that the version of κ in this proposition depends on r_0. The case when r or $r_\dagger$ or r_* are less than $r_\diamondsuit$ is of no interest to the subsequent applications. Keep in mind in what follows that r is less than $r_\diamondsuit$ if $r > c_0E$ and $r \leq c_0^{-1}E^{-1}10^{-1/2}\kappa_U^{-1/2}r^{-1}$, this being a consequence Item c) in (3.2) and the third bullet of (3.3).

By way of an outline for the rest of this section, the proof of the first bullet of the proposition is in Section 5.b and that of the second is in Section 5.d. Section 5.e contains the proofs Proposition 5.1's third and fourth bullets. The intervening subsections establish some facts and observations that are used in the proof.

5.a. Integration by parts identities

The upcoming Lemma 5.2 states the two fundamental integration by parts identities. The second refers to the parameter r that is used to define the function H. By way of notation, the lemma introduces functions $\mathfrak{h}$ and $\mathfrak{f}$ on $(0, c_0^{-1})$, these defined by

$$\mathfrak{h}(r) = \int_{B_r} (|\nabla_A \hat{a}|^2 + 2\langle *F_A \wedge \hat{a} \wedge \hat{a}\rangle) \quad \textit{and} \tag{5.4}$$

$$\mathfrak{f}(r) = \int_{B_r} (r^{-2}|F_A - r^2\hat{a} \wedge \hat{a}|^2 + |d_A\hat{a}|^2 + |d_A{*}\hat{a}|^2).$$

The lemma also uses $\|\cdot\|_{2,r}$ to denote the L^2 norm on B_r, and it $\|\cdot\|_{\infty,r}$ to denote the supremum norm on B_r.

To set the rest of the notation, keep in mind that if $p \in M$ and if $r \in (0, c_0^{-1})$, then the ball B_r has compact closure in a Gaussian coordinate chart centered at p. This being the case, the outward pointing, unit length tangent vectors to the geodesic rays that start at p define a smooth vector field on B−p. This vector field is denoted by ∂_r. When $\hat{a}$ denotes a section over B_r of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$, then $\hat{a}_r$ is used to denote the section of $(P \times_{SO(3)} \mathfrak{su}(2))$ overy B_r-p that is given by pairing $\hat{a}$ with ∂_r. When A denotes a connection on P's restriction to B_r, then $\partial_{A,r}$ is used to denote the A's directional covariant derivative along the vector field ∂_r.

Lemma 5.2. *There exists $\kappa > 1$ with the following significance: Fix $\mathrm{p} \in \mathrm{M}$ and fix $\mathrm{r} \in (0, \kappa^{-1})$. Suppose that* A *is a connection on* P*'s restriction to the closure of* $\mathrm{B_r}$ *and that and* $\hat{a}$ *is a section over this closure of* $(\mathrm{P} \times_{\mathrm{SO(3)}} \mathfrak{su}(2)) \otimes \mathrm{T^*M}$*. Then*

- $\int_{\partial \mathrm{B_r}} \langle \partial_{\mathrm{A,r}} \hat{a} \wedge *\hat{a} \rangle = \hbar + \int_{\mathrm{B_r}} \mathrm{Ric}(\langle \hat{a} \otimes \hat{a} \rangle) - \int_{\mathrm{B_r}} \langle \hat{a} \wedge *q_{\mathrm{A}}(\hat{a}) \rangle$.
- $\int_{\partial \mathrm{B_r}} (|\nabla_{\mathrm{A}} \hat{a}|^2 + 2r^2 |\hat{a} \wedge \hat{a}|^2) = 2 \int_{\partial \mathrm{B_r}} (|\partial_{\mathrm{A,r}} \hat{a}|^2 + r^2 |[\hat{a}, \hat{a}_{\mathrm{r}}]|^2) + \frac{1}{\mathrm{r}} \mathrm{H} + \mathfrak{R}$,

with $\mathfrak{R}$ *obeying* $|\mathfrak{R}| \leq \mathrm{c_0}(\|q_{\mathrm{A}}(\hat{a})\|_{2,\mathrm{r}} + r\mathfrak{f}^{1/2} \|\hat{a}\|_{\infty,\mathrm{r}}) \mathrm{H}^{1/2} + \mathrm{c_0}(\mathrm{h} + \mathrm{rH}) + \mathrm{c_0}(\mathfrak{f}^{1/2} \|\hat{a}\|_{2,\mathrm{r}} + \|\hat{a}\|^2_{2,\mathrm{r}})$.

Proof of Lemma 5.2. Given the definitions of $\hbar$ and q_{A}, the identity in the first bullet of the lemma follows using Stoke's theorem. The proof of the assertion in the second bullet has two steps.

Step 1. Fix $\mathrm{p} \in \mathrm{M}$ and an orthonormal frame $\{\mathrm{e^i}\}_{\mathrm{i}=1,2,3}$ for $\mathrm{T^*M}$ on the radius $\mathrm{c_0^{-1}}$ ball centered at p. Write $\hat{a} = \hat{a}_{\mathrm{i}} \mathrm{e^i}$ and $\nabla_{\mathrm{A}} \hat{a} = (\nabla_{\mathrm{A,i}} \hat{a})_{\mathrm{k}} \mathrm{e^k} \otimes \mathrm{e^i}$ with it understood that repeated indices are to be summed. View the Riemann curvature tensor as a section of $\otimes^4 \mathrm{T^*M}$ and write it as $\mathrm{R_{ikmn}} \mathrm{e^i} \otimes \mathrm{e^k} \otimes \mathrm{e^m} \otimes \mathrm{e^n}$. Likewise, write the Ricci tensor as $\mathrm{Ric_{ik}} \mathrm{e^i} \otimes \mathrm{e^k}$. The metric tensor is $\delta_{\mathrm{ik}} \mathrm{e^i} \otimes \mathrm{e^k}$ in this notation with $\delta_{\mathrm{ik}} = 0$ if $\mathrm{i} \neq \mathrm{k}$ and $\delta_{11} = \delta_{22} = \delta_{33} = 1$. Granted this notation, a symmetric section of $\mathrm{T^*M} \otimes \mathrm{T^*M}$ to be denoted by $\mathrm{T} = \mathrm{T_{ij}} \mathrm{e^i} \otimes \mathrm{e^j}$ is defined by setting any given $\mathrm{i}, \mathrm{j} \in \{1, 2, 3\}$ version of $\mathrm{T_{ij}}$ to be

$$\begin{aligned}(5.5) \quad \mathrm{T_{ij}} = {} & \langle (\nabla_{\mathrm{A,i}} \hat{a})_{\mathrm{k}} (\nabla_{\mathrm{A,j}} \hat{a})_{\mathrm{k}} \rangle + r^2 \langle [\hat{a}_{\mathrm{i}}, \hat{a}_{\mathrm{k}}][\hat{a}_{\mathrm{j}}, \hat{a}_{\mathrm{k}}] \rangle + \mathrm{R_{ikjm}} \langle \hat{a}_{\mathrm{k}} \hat{a}_{\mathrm{m}} \rangle \\ & - \tfrac{1}{2} \delta_{\mathrm{ij}} (\langle (\nabla_{\mathrm{A,m}} \hat{a})_{\mathrm{k}} (\nabla_{\mathrm{A,m}} \hat{a})_{\mathrm{k}} \rangle + r^2 \langle [\hat{a}_{\mathrm{k}}, \hat{a}_{\mathrm{m}}][\hat{a}_{\mathrm{k}}, \hat{a}_{\mathrm{m}}] \rangle + \mathrm{Ric_{mk}} \langle \hat{a}_{\mathrm{k}} \hat{a}_{\mathrm{m}} \rangle),\end{aligned}$$

with it again understood that repeated indices are summed.

Let $\mathrm{d}^\dagger$ denote the map from $\mathrm{C^\infty(M; T^*M \otimes T^*M)}$ to $\mathrm{C^\infty(M; T^*M)}$ given by the formal $\mathrm{L^2}$ adjoint of the Levi-Civita covariant derivate. The 1-form $\mathrm{d^\dagger T}$ can be written schematically as

$$\begin{aligned}(5.6) \quad (\mathrm{d^\dagger T})_{\mathrm{i}} = {} & -\langle (\nabla_{\mathrm{A,i}} \hat{a})_{\mathrm{k}} q_{\mathrm{A}}(\hat{a})_{\mathrm{k}} \rangle + \mathcal{Q}_{\wedge \mathrm{i}}(r\mathfrak{q} \otimes \hat{a} \otimes r(\hat{a} \wedge \hat{a})) \\ & + \mathcal{Q}_{\nabla \mathrm{i}}(r\mathfrak{q} \otimes \hat{a} \otimes \nabla_{\mathrm{A}} \hat{a}) + \mathcal{R}_{\mathrm{i}}(\hat{a} \otimes \mathfrak{q}) + \mathcal{S}_{\mathrm{i}}(\hat{a} \otimes \hat{a}),\end{aligned}$$

where the notation has $\mathfrak{q}$ denoting the triple

$$(5.7) \qquad \mathfrak{q} = (r^{-1} *(\mathrm{F_A} - r^2 \hat{a} \wedge \hat{a}), *\mathrm{d_A} \hat{a}, \mathrm{d_A} * \hat{a}),$$

this being a section of $(\mathrm{P} \times_{\mathrm{SO(3)}} \mathfrak{su}(2)) \otimes (\mathrm{T^*M} \oplus \mathrm{T^*M} \oplus (\wedge^3 \mathrm{T^*M}))$. What are denoted by $\mathcal{Q}_\wedge$, $\mathcal{Q}_\nabla$, $\mathcal{R}$ and $\mathcal{S}$ in (5.6) are homomorphisms that involve only the metric, the Riemann curvature tensor and the latter's covariant derivative. In any event, each has norm bounded by $\mathrm{c_0}$.

Step 2. Fix $r \in (0, c_0^{-1})$. Let Δ denote for the moment the function $\mathrm{dist}(\cdot, p)^2$ on the ball B_r. Integrate $d^\dagger T \wedge *d\Delta$ on B_r and use Stoke's theorem to write

$$\int_{B_r} d^\dagger T \wedge *d\Delta = -2r \int_{\partial B_r} T(\partial_r \otimes \partial_r) + \int_{B_r} T_{ij} \nabla_i \nabla_j \Delta. \tag{5.8}$$

The identity in the second bullet of Lemma 5.2 follows directly from (5.8) given what is said in the subsequent paragraph about the integral on the left hand side and the two integrals on the right hand side.

The norm of $|d\Delta|$ on B_r is bounded by r and so it follows from (5.5) that the absolute value of the integral on the left hand side of (5.6) is no greater than r times

$$c_0(\|\mathcal{q}_A(\hat{a})\|_{2,r} + r\mathfrak{f}^{1/2}\|\hat{a}\|_{\infty,r})H^{1/2} + c_0(\mathfrak{f}^{1/2}\|\hat{a}\|_{2,r} + \|\hat{a}\|_{2,r}^2). \tag{5.9}$$

Meanwhile, the boundary integral on the right hand side of (5.8) differs from r times

$$\int_{\partial B_r} (|\nabla_A \hat{a}|^2 + 2r^2|\hat{a} \wedge \hat{a}|^2) - 2\int_{\partial B_r} (|\partial_{A,r}\hat{a}|^2 + r^2|[\hat{a}, \hat{a}_r]|^2) \tag{5.10}$$

by at most $c_0 r h$, this being a direct consequence of (5.5)'s formula for T. The right most integral in (5.8) differs from the integral of the trace of T by at most $c_0(r^2 H + \|a\|_{2,r}^2)$, this because $\nabla_i \nabla_k \rho$ differs from $2\delta_{ik}$ by at most $c_0 r^2$. A look at (5.5) finds that the trace of T differs from $-\frac{1}{2}(|\nabla_A \hat{a}|^2 + 2r^2|\hat{a} \wedge \hat{a}|^2)$ by at most $c_0|\hat{a}|^2$.

5.b. Proof of the first bullet of Proposition 5.1

The proof has five parts.

Part 1: Invoke the first bullet of Lemma 5.2 to write

$$\tfrac{d}{dr} h = 2r^{-1}(1 + \mathfrak{r})h + 2\hbar + \int_{B_r} \mathrm{Ric}(\langle \hat{a} \otimes \hat{a} \rangle) - \int_{B_r} \langle \hat{a} \wedge *\mathcal{q}_A(\hat{a}) \rangle, \tag{5.11}$$

with $\mathfrak{r}$ denoting a term whose absolute value is bounded by $c_0 r^2$. This term accounts for the fact that the second fundamental form of ∂B_r may differ from r^{-1} times the induced metric. The bound on $|\mathfrak{r}|$ is due to the fact that these two tensors can differ by at most $c_0 r$.

Part 2: Assume that A is a connection on P's restriction to the closure of B_r and that $\hat{a}$ is a section over this closure of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. Assume in addition that a positive number E has been specified and that $\|F_A - r^2\hat{a} \wedge \hat{a}\|_{2,r} \leq \mathrm{E}^{1/2}$. Write $\langle *F_A \wedge \hat{a} \wedge \hat{a}\rangle$ as the sum of $r^{-1}\langle *(F_A - r^2\hat{a} \wedge \hat{a}) \wedge (r\hat{a} \wedge a)\rangle$ and $r^2|a \wedge a|^2$ to derive the bound given below for the difference between $\hbar$ and H:

$$|\hbar - H| \leq c_0 \mathrm{E}^{1/2} r^{-1} H^{1/2}. \tag{5.12}$$

Use this bound with the definition of N to write (5.11) as

$$\frac{d}{dr}h = 2r^{-1}(1 + \mathrm{N} + \mathfrak{r})h + \mathfrak{r}_1 + \int_{B_r} \mathrm{Ric}(\langle \hat{a} \otimes \hat{a}\rangle) - \int_{B_r} \langle \hat{a} \wedge * q_A(\hat{a})\rangle \tag{5.13}$$

with $\mathfrak{r}_1$ being a term with norm at most $c_0\mathrm{E}^{1/2}r^{-1}H^{1/2}$.

Part 3: Assume as in Part 2 that $(A, \mathfrak{a})$ is a pair of connection on P's restriction to the closure of B_r and that $\hat{a}$ is a section over this closure of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. Then

$$\int_{B_r} |\hat{a}|^2 \leq 2rh(r) + 8r^2H(r). \tag{5.14}$$

This inequality is a direct consequence of the following integration by parts formula for the radius 1 ball in $\mathbb{R}^3$: Let f denote a smooth on the $|x| \leq 1$ ball. Fix $\varepsilon \in (0, 1)$. Then

$$(1 - \varepsilon)\int_{|x|\leq 1} \frac{1}{|x|^2} f^2 \leq \int_{|x|=1} f^2 + \varepsilon^{-1}\int_{|x|\leq 1} |df|^2. \tag{5.15}$$

The inequality in (5.15) leads to (5.14) by writing the integrals in (5.14) using a Gaussian coordinate chart centered at p. Having done so, appeal to (2.6) when invoking the $f = |\hat{a}|$ and $\varepsilon = \frac{1}{4}$ version of (5.15).

It follows directly from (5.14) that the contribution to (5.13) from the Ricci curvature integral on (5.13)'s right hand side has absolute value at most $c_0 r(1 + \mathrm{N})h$.

Part 4: Assume that A is a connection on P's restriction to the closure of B_r and that $\hat{a}$ is a section over this closure of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. Assume in addition that a number $\mathrm{E} > 1$ has been specified and that each of $\|q_A(a)\|_{2,r}$,

$r\|d_A\hat{a}\|_{2,r}$ and $r\|d_A{*}\hat{a}\|_{2,r}$ is less than $E^{1/2}$. As explained momentarily, $r > c_0E$ and if $r > c_0^{-1}E^{-1}10^{-1/2}\kappa_U^{-1/2}r^{-1}$, then

$$\left|\int_{B_r}\langle\hat{a}\wedge *q_A(\hat{a})\rangle\right| \le c_0(Er^{-2} + Er^{-1/2}h^{1/2} + E^{1/2}r^{-1}H^{1/2}). \tag{5.16}$$

By way of a parenthetical remark, (5.16) holds with $r > m^{-1}r$ for any $m > 0$ with c_0 replaced by an m-dependent constant. The given lower bound on r is used because r is necessarily less than $r_\diamond$ if $r > c_0E$ and $r \le c_0^{-1}10^{-1/2}\kappa_U^{-1/2}r^{-1}$. The proof of (5.16) has two steps.

Step 1. Fix $\varepsilon \in (0, \frac{1}{2})$ for the moment and let χ_ε denote the function on B_r given by $\chi(\varepsilon^{-1}(r^{-1}\text{dist}(\cdot, p)-1+\varepsilon))$. This function equals 1 where $\text{dist}(\cdot, p) \le (1-\varepsilon)r$ and it equals 0 where $\text{dist}(\cdot, p) \ge r$. Write the integral over B_r of $\langle\hat{a}\wedge *q_A(\hat{a})\rangle$ as the sum of two integrals, the first being the integral over B_r of $\chi_\varepsilon\langle\hat{a}\wedge *q_A(\hat{a})\rangle$ and the second being the integral of $(1-\chi_\varepsilon)\langle\hat{a}\wedge *q_A(\hat{a})\rangle$. The absolute values of these two integrals are bounded respectively by

$$c_0Er^{-2} + c_0E^{1/2}\varepsilon^{-1/2}r^{-1/2}r^{-1}\Big(\sup_{(1-\varepsilon)r\le s\le r} h(s)\Big)^{1/2} \quad \textit{and} \tag{5.17}$$
$$c_0E^{1/2}\varepsilon^{1/2}r^{1/2}\Big(\sup_{(1-\varepsilon)r\le s\le r} h(s)\Big)^{1/2}.$$

To prove the bound for the first integral, write $q_A(\hat{a})$ as $*d_A{*}d_A\hat{a} - d_A{*}d_A{*}\hat{a}$, integrate by parts and invoke the assumption that $\|d_A\hat{a}\|_{2,r} + \|d_A{*}\hat{a}\|_{2,r}$ is less than $r^{-1}E^{1/2}$. The bound for the second integral follows directly from the assumption that $\|q_A(\hat{a})\|_{2,r} \le E^{1/2}$.

Step 2. Use the fundamental theorem of calculus to see that

$$\sup_{(1-\varepsilon)r\le s\le r} h(s) \le c_0h(r) + c_0\varepsilon r\int_{B_r}|\nabla_A\hat{a}|^2. \tag{5.18}$$

This inequality with (5.17) leads directly to the following bound:

$$\left|\int_{B_r}\langle\hat{a}\wedge *q_A(\hat{a})\rangle\right| \tag{5.19}$$
$$\le c_0Er^{-2} + c_0E^{1/2}(\varepsilon^{-1/2}r^{-1/2}r^{-1} + \varepsilon^{1/2}r^{1/2})h^{1/2} + c_0(r^{-1}+\varepsilon r)(H^{1/2}+E^{1/2}r^{-1}).$$

Assume henceforth that $r > c_0^{-1}E^{-1}10^{-1}\kappa_U^{-1/2}r^{-1}$. Use $\varepsilon = c_0^{-1}\kappa_U^{-1/2}E^{-1}r^{-1}r^{-1}$ in the right hand side of (5.19) to go from the latter bound to the bound in (5.16).

Part 5: Use what is said in Parts 2–4 to conclude that the right hand side of (5.11) differs from $2\frac{1}{\mathrm{r}}(1+\mathrm{N})\mathrm{h}$ by at most $c_0\mathrm{r}(1+\mathrm{N})\mathrm{h} + c_0(\mathrm{E}r^{-2} + \mathrm{E}^{1/2}r^{-1}\mathrm{H}^{1/2}) + c_0\mathrm{E}r^{-1/2}\mathrm{h}^{1/2}$. This is what is claimed by the first bullet of Proposition 5.1.

5.c. A differential equation for N

The next lemma asserts a formula for $\frac{d\mathrm{N}}{d\mathrm{r}}$. This lemma also refers to Uhlenbeck's constant κ_{U} from (3.1).

Lemma 5.3. *There exists* $\kappa \geq 100$ *with the following significance: Suppose that* $\mathrm{E} \geq 1$*, that* $r > \kappa\mathrm{E}$ *and that* $(\mathrm{A}, \hat{a}) \in \mathrm{Conn}(\mathrm{P}) \times C^\infty(M; (\mathrm{P} \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M)$ *obeys the conditions in (3.2). Fix* $p \in M$ *so as to define the functions* h, H *and* N *using* r *and* $(\mathrm{A}, \hat{a})$*. If* $\mathrm{r} \geq \kappa^{-1}\kappa_{\mathrm{U}}^{-1/2} r^{-1}$*, then*

$$\frac{d\mathrm{N}}{d\mathrm{r}} \geq 2\frac{\mathrm{r}}{\mathrm{h}}\int_{\partial B_{\mathrm{r}}} r^2|[\hat{a}, \hat{a}_{\mathrm{r}}]|^2 - \mathfrak{r}\mathrm{N} - \kappa\mathrm{E}\left(\frac{\mathrm{r}}{\mathrm{h}}\right)^{1/2}\mathrm{N}^{1/2} - \kappa\left(\mathrm{r} + \kappa\mathrm{E}r^{-1/2}\mathrm{r}\left(\frac{\mathrm{r}}{\mathrm{h}}\right)^{1/2}\right),$$

with $\mathfrak{r}$ *obeying* $|\mathfrak{r}| \leq \kappa(\mathrm{E}r^{-1}(\frac{1}{\mathrm{rh}})^{1/2}\mathrm{N}^{1/2} + \mathrm{rN} + \mathrm{E}\frac{1}{\mathrm{h}}r^{-2} + c_0\mathrm{E}r^{-1/2}(\frac{1}{\mathrm{h}})^{1/2})$*.*

Proof of Lemma 5.3. The proof does its best to mimic what is done in [Al], [HHL] and [Han] to prove an analoguous monotonicity assertion for the version of N that arises when studying eigenfunctions of $d^\dagger d$. In any event, the starting point is the identity

$$\frac{d\mathrm{N}}{d\mathrm{r}} = \frac{\mathrm{H}}{\mathrm{h}} + \frac{\mathrm{r}}{\mathrm{h}}\frac{d\mathrm{H}}{d\mathrm{r}} - \frac{\mathrm{rH}}{\mathrm{h}^2}\frac{d\mathrm{h}}{d\mathrm{r}}. \tag{5.20}$$

The derivation of the lemma's lower bound from (5.20) has four parts.

Part 1: The derivative of H is

$$\frac{d\mathrm{H}}{d\mathrm{r}} = \int_{\partial B_{\mathrm{r}}} (|\nabla_{\mathrm{A}}\hat{a}|^2 + 2r^2|\hat{a} \wedge \hat{a}|^2), \tag{5.21}$$

and the second bullet of Lemma 5.2 is used to write this as

$$\frac{d\mathrm{H}}{d\mathrm{r}} = 2\int_{\partial B_{\mathrm{r}}} (|\partial_{\mathrm{A,r}}\hat{a}|^2 + r^2|[\hat{a}, \hat{a}_{\mathrm{r}}]|^2) + \frac{1}{\mathrm{r}}\mathrm{H} + \mathfrak{R}. \tag{5.22}$$

Use this last identity to write N's derivative as

$$\frac{d\mathrm{N}}{d\mathrm{r}} = 2\frac{\mathrm{r}}{\mathrm{h}}\int_{\partial B_{\mathrm{r}}} (|\partial_{\mathrm{A,r}}\hat{a}|^2 + r^2|[\hat{a}, \hat{a}_{\mathrm{r}}]|^2) + \frac{\mathrm{r}\mathfrak{R}}{\mathrm{h}} + 2\frac{\mathrm{H}}{\mathrm{h}} - \frac{\mathrm{rH}}{\mathrm{h}^2}\frac{d\mathrm{h}}{d\mathrm{r}}. \tag{5.23}$$

Save this identity for Part 3.

Part 2: Use the definition of h to write

$$\frac{dh}{dr} = 2(1+\mathfrak{r})\frac{h}{r} + 2\int_{\partial B_r} \langle \partial_{A,r}\hat{a} \wedge *\hat{a}\rangle\,, \tag{5.24}$$

with $\mathfrak{r}$ being the same as its namesake in (5.11). Thus, $|\mathfrak{r}| \le c_0 r^2$. This then leads to

$$2\frac{H}{h} - \frac{rH}{h^2}\frac{dh}{dr} = -2\frac{rH}{h^2}\int_{\partial B_r} \langle \partial_{A,r}\hat{a} \wedge *\hat{a}\rangle - 2\frac{1}{r}\mathfrak{r}N. \tag{5.25}$$

To continue, use (5.12) to write $H = \hbar - \mathfrak{z}_1$ with $\mathfrak{z}_1$ obeying $|\mathfrak{z}_1| \le c_0 E^{1/2} r^{-1} H^{1/2}$ and then use this to write the right hand side of (5.25) as twice

$$-\frac{r\hbar}{h^2}\int_{\partial B_r} \langle \partial_{A,r}\hat{a} \wedge *\hat{a}\rangle + \frac{r}{h^2}\mathfrak{z}_1\int_{\partial B_r} \langle \partial_{A,r}\hat{a} \wedge *\hat{a}\rangle - \frac{1}{r}\mathfrak{r}N, \tag{5.26}$$

Now use the first bullet of Lemma 5.2 to replace $\hbar$ in (5.26) and in doing so, replace (5.26) with the expression

$$\begin{aligned} &-\frac{r}{h^2}\left(\int_{\partial B_r} \langle \partial_{A,r}\hat{a} \wedge *\hat{a}\rangle\right)^2 \\ &+\frac{r}{h^2}\left(\mathfrak{z}_1 + \int_{B_r} \mathrm{Ric}(\langle \hat{a}\otimes\hat{a}\rangle) - \int_{B_r}\langle \hat{a}\wedge *\mathcal{q}_A(\hat{a})\rangle\right)\int_{\partial B_r} \langle \partial_{A,r}\hat{a} \wedge *\hat{a}\rangle - \frac{1}{r}\mathfrak{r}N. \end{aligned} \tag{5.27}$$

One last replacement is needed, this the replacement in (5.27) of the right most integral over ∂B_r of $\partial_{A,r}\hat{a} \wedge *\hat{a}$ using the first bullet of Lemma 5.2 to equate (5.27) with

$$-\frac{r}{h^2}\left(\int_{\partial B_r} \langle \partial_{A,r}\hat{a} \wedge *\hat{a}\rangle\right)^2 + \frac{r}{h^2}\mathfrak{z}(H+\mathfrak{z}) - \frac{1}{r}\mathfrak{r}N, \tag{5.28}$$

with $\mathfrak{z}$ denoting $\mathfrak{z} = \mathfrak{z}_1 + \int_{B_r} \mathrm{Ric}(\langle \hat{a}\otimes\hat{a}\rangle) - \int_{B_r}\langle \hat{a}\wedge *\mathcal{q}_A(\hat{a})\rangle$.

Part 3: Twice the expression in (5.28) is the same as the right hand side of (5.25) and thus the same as the left hand side of (5.25). The latter appears in (5.23). Replace its incarnation in (5.23) with (5.28) to write the derivative of N as

$$\frac{dN}{dr} = 2\frac{r}{h^2}\mathfrak{X} + \frac{r\mathfrak{R}}{h} + 2\frac{r}{h^2}\mathfrak{z}(H+\mathfrak{z}) - 2\frac{1}{r}\mathfrak{r}N, \tag{5.29}$$

with $\mathfrak{X} = (\int_{\partial B_r}|\hat{a}|^2)(\int_{\partial B_r}(|\partial_{A,r}\hat{a}|^2 + r^2|[\hat{a},\hat{a}_r]|^2)) - (\int_{\partial B_r}\langle \partial_{A,r}\hat{a}\wedge *\hat{a}\rangle)^2$.

The key observation with regards to (5.29) is that $\mathfrak{X}$ is non-negative. More to the point, writing the derivative of N as in (5.29) leads to the lower bound

$$\frac{d\mathrm{N}}{dr} \geq 2\frac{r}{h}\int_{\partial B_r} r^2|[\hat{a},\hat{a}_r]|^2 + \frac{r\mathfrak{R}}{h} + 2\frac{r}{h^2}\mathfrak{z}(H+\mathfrak{z}) - 2\frac{1}{r}\mathfrak{r}\mathrm{N}. \tag{5.30}$$

The left most term on the right hand side of (5.30) is nonnegative. Part 4 gives bounds for the absolute values of the remaining terms.

Part 4: With regards to (5.30), keep in mind that $|\mathfrak{r}| \leq c_0 r^2$. Keep in mind also that Lemma 5.2 with (5.14), the assumptions in Items a)–d) of (3.2) and the first plus third bullet of (3.3) give the bound

$$|\mathfrak{R}| \leq c_0 E H^{1/2} + c_0 h + c_0 r H + c_0 \mathrm{E} r^{-1} r^{1/2} h^{1/2} \tag{5.31}$$

Meanwhile, what is said above about $|\mathfrak{z}_1|$ and what is said in (5.14) and (5.16) imply that

$$|\mathfrak{z}| \leq c_0 E^{1/2} r^{-1} H^{1/2} + c_0(rh + r^2 H) + c_0(\mathrm{E} r^{-2} + \mathrm{E} r^{-1/2} h^{1/2}). \tag{5.32}$$

Use these bounds for $|\mathfrak{r}|$, $|\mathfrak{R}|$ and $|\mathfrak{z}|$ to bound the derivative of N from below by

$$\begin{aligned}\frac{d\mathrm{N}}{dr} \geq{}& 2\frac{r}{h}\int_{\partial B_r} r^2|[\hat{a},\hat{a}_r]|^2 - c_0 E r H^{1/2} h^{-1} - c_0 r - c_0 \mathrm{E} r^{-1/2} r^{3/2} h^{-1/2} \\ &- c_0(r + \mathrm{E} r^{-1} H^{1/2} h^{-1} + c_0 r^2 H h^{-1} + c_0 \mathrm{E} r^{-2} h^{-1} + c_0 \mathrm{E} r^{-1/2} h^{-1/2})\mathrm{N}.\end{aligned} \tag{5.33}$$

Write each occurrence of H in (5.33) as $r^{-1}\mathrm{N}h$ to obtain the bound asserted by Lemma 5.3.

5.d. Proof of the second bullet of Proposition 5.1

The proof of the second bullet has two parts.

Part 1: This part of the subsection states and then proves a lemma that asserts the conclusions of the second bullet on any interval in $[r_\ddagger, r_0]$ where N is not too large.

Lemma 5.4. *Given* $m > 1$, *there exists* $\kappa \geq 100$ *with the following significance: Suppose that* $\mathrm{E} \geq 1$, $r > \kappa \mathrm{E}^\kappa$ *and that* $(A, \hat{a}) \in$

$\mathrm{Conn}(P) \times C^\infty(M; (P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M)$ *obeys the conditions in Items a)–e) in the statement of Proposition 5.1. Suppose that* $r_2 > r_1$ *are from the interval* $[r_\diamond, r_0]$ *and such that both* $h(r) \geq r^{3-1/m}$ *and* $N(r) \leq r^{1/2m}$ *when* $r \in [r_1, r_2]$. *Let* u *denote the function on* $[r_1, r_2]$ *that is defined by the rule* $r \to u(r) = \kappa E^\kappa r^{1/\kappa}$. *Then* $N(r_2) \geq e^{-u(r_2)+u(r_1)} N(r_1) - u(r_2) + u(r_1)$.

Proof of Lemma 5.4. The proof has two steps.

Step 1. Keep in mind that the function H is uniformly bounded by c_0E, this being a consequence of the first bullet of (3.3). Write each explicit occurrence of $H^{1/2}$ on the right hand side of (5.33) as $(r^{-1}Nh)^{1/2}$ to obtain the bound

$$\begin{aligned}(5.34)\quad \frac{dN}{dr} &\geq -c_0Er^{1/2}h^{-1/2}N^{1/2} - c_0r - c_0Er^{-1/2}r^{3/2}h^{-1/2}\\ &- c_0(r + Er^{-1}(rh)^{-1/2}N^{1/2} + c_0r^2Hh^{-1} + c_0Er^{-2}h^{-1} + c_0Er^{-1/2}h^{-1/2})N.\end{aligned}$$

Replace the left most occurrence of $N^{1/2}$ on the right hand side of (5.34) by $(1 + N)$, replace the explicit occurrence of H on the right hand side by c_0E to obtain the lower bound

$$\begin{aligned}(5.35)\quad \frac{dN}{dr} &\geq -c_0Er^{1/2}h^{-1/2} - c_0r - c_0Er^{-1/2}r^{3/2}h^{-1/2}\\ &- c_0(r + Er^{1/2}h^{-1/2} + Er^{-1}(rh)^{-1/2}N^{1/2}\\ &+ c_0Er^2h^{-1} + c_0Er^{-2}h^{-1} + c_0Er^{-1/2}h^{-1/2})N.\end{aligned}$$

Now suppose $h(r) \geq r^{3-1/m}$ when $r \in [r_1, r_2]$. Use this assumption to replace all occurrences of h on the right hand side of (5.35) with $r^{3-1/m}$. Meanwhile, replace the occurrence of $N^{1/2}$ on the right hand side by $r^{1/4m}$. These replacements lead to the lower bound

$$\begin{aligned}(5.36)\quad \frac{dN}{dr} &\geq -c_0Er^{-1+1/2m} - c_0r - c_0Er^{-1/2}r^{1/2m}\\ &- c_0(r + Er^{-1+1/2m} + Er^{-1+1/4m}r^{-2+1/2m}\\ &+ c_0Er^{-2}r^{-3+1/m} + c_0Er^{-1/2}r^{-3/2+1/2m})N.\end{aligned}$$

This inequality plays the role of the monotonicity inequalities for N's namesakes in [Al], [HHL] and [Han].

Step 2. Integrate (5.36) to obtain the inequality

$$(5.37)\qquad N(r_2) \geq e^{-u(r_2)+u(r_1)}(N(r_1) - c_0(m + Er^{-1/2})(r_2^{1/2m} - r_1^{1/2m}),$$

where u is defined by the rule

(5.38)

$$r \to u(r) = c_0 E(m r^{1/2m} - \mathit{r}^{-1+1/4m} r^{-1+1/2m} - \mathit{r}^{-2} r^{-2+1/m} + \mathit{r}^{-1/2} r^{-1/2+1/2m}).$$

The right hand side of (5.38) is an increasing function of r, but it is no greater than $c_0 E m r^{1/2m}$ unless $r \le c_m E^{c_m} \mathit{r}^{-1-1/8m}$ with c_m depending only on m. This with the third bullet in (3.3) implies that the integral of $|F_A|^2$ over the radius r ball centered at p is no greater than $c_0 (c_m E^{c_m})^3 \mathit{r}^{-1-3/8m}$. Note in particular that the latter L^2 norm bound violates the bound in (5.1) when r is large. If the lower bound $\mathit{r} \ge c_0 (c_m E^{c_m})^{2m}$ is obeyed, then Lemma 5.4's assertion follows directly from (5.37) and (5.38).

Part 2: The second bullet of Proposition 5.1 is an immediate corollary of Lemma 5.4 and the subsequent lemma.

Lemma 5.5. *Given* $m > 1$*, there exists* $\kappa \ge 100$ *with the following significance: Suppose that* $E \ge 1$*,* $\mathit{r} > \kappa E^{\kappa}$ *and that* $(A, \hat{\mathfrak{a}}) \in \mathrm{Conn}(P) \times C^\infty(M; (P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M)$ *obeys the conditions in (3.2). Fix* $p \in M$ *and suppose that* $r_\dagger \in [r_\diamond, \frac{1}{2} r_0]$ *is such that* $h(r) \ge r^{3-1/m}$ *when* $r \in [r_\dagger, r_0]$*. Then* $N(\cdot) \le \mathit{r}^{1/2m}$ *on the interval* $[r_\dagger, \frac{1}{2} r_0]$*.*

Proof of Lemma 5.5. Let κ_m denote the version of κ that is given by Lemma 5.4, and suppose that $\mathit{r} \ge \kappa_m E^{\kappa_m}$ so as to invoke Lemma 5.4. Suppose that $r_1 \in [r_\dagger, \frac{1}{2} r_0]$ is a point where $N(r_1) \ge \mathit{r}^{1/2m}$. It follows as a consequence of Lemma 5.4 that $N(\cdot) \ge \frac{1}{2} \kappa_m^{-1} \mathit{r}^{1/2m}$ on the interval $[r_1, r_0]$. Use this fact with the first bullet of Proposition 5.1 to see that

$$\text{(5.39)} \qquad \frac{d}{dr} h \ge c_*^{-1} \mathit{r}^{1/2m} r^{-1} h - c_0 E(\mathit{r}^{-2} + \mathit{r}^{-1-1/2m}) \quad \textit{when } r \in [r_1, r_0],$$

where c_* denotes a number that is greater than 1 and depends only on m. Introduce by way of shorthand z denote $c_*^{-1} \mathit{r}^{1/2m}$. The differential inequality in (5.39) implies that

$$\text{(5.40)} \quad h(r_0) \ge 2^z (h(r) - c_m E(\mathit{r}^{-2-1/2m} + \mathit{r}^{-1-1/m})) \quad \textit{for any } r \in \left[r_1, \tfrac{1}{2} r_0\right],$$

where c_m denotes here and in what follows a number that is greater than 1 and depends only on m. The assigned value for c_m can be assumed to increase between successive appearances.

The fact that $|\hat{\mathfrak{a}}| \le c_0 E$ means that $h(r_0) \le c_0 E r_0^2$. This runs afoul of the $r = \frac{1}{2} r_0$ version of (5.40) when $\mathit{r} \ge c_m$ unless $h(\frac{1}{2} r_0) < c_m E \mathit{r}^{-1-1/m}$. But the latter bound is not allowed if $\mathit{r} > c_m E^{c_m}$ because $h(\frac{1}{2} r_0)$ is no smaller than $r_0^{3-1/m}$.

5.e. Proof of the third and fourth bullets of Proposition 5.1

The fourth bullet follows directly from the third. The proof of the proposition's third bullet has three parts.

Part 1: Fix $\delta \in (0,1)$ for the moment. With δ chosen define the function χ on $[r_\diamondsuit, \frac{1}{2}r_0]$ by the rule $r \to \chi(r) = h(r) - r^{2+2\delta}$. The first bullet of Proposition 5.1 implies that χ obeys the differential inequality

$$\frac{d}{dr}\chi \leq 2r^{-1}(1+N)\chi + 2(N-\delta)r^{1+2\delta} + c_0E(r^{-2} + r^{-1/2}h^{1/2} + r^{-1}H^{1/2}). \tag{5.41}$$

Write $H^{1/2}$ as $(r^{-1}Nh)^{1/2}$ in (5.41) to see that (5.41) leads to the bound

$$\frac{d}{dr}\chi \leq 2r^{-1}(1+N)\chi + 2(N-\delta+r^{-1/8})r^{1+2\delta} + c_0E(1+\delta^{-1})r^{-7/8} + c_0Er^{-1/2}h^{1/2}. \tag{5.42}$$

Let $r_\delta \in [r_\diamondsuit, \frac{1}{2}r_0]$ denote a zero of χ, thus a value of r where the function h obeys $h(r) = r^{2+2\delta}$. It follows from (5.42) that χ is decreasing at r_δ unless

$$N \geq \delta - r^{-1/8} - c_0E(1+\delta^{-1})r^{-2}r^{-1-2\delta} - c_0Er^{-1/2}r^{-\delta}. \tag{5.43}$$

In particular, if $\delta \leq \frac{1}{4}$, then χ is decreasing at r_δ unless $N(r_\delta) > \frac{1}{4}\delta$ or else $r \leq c_0^{-1}\delta^4E^{-4}r^{-4/3}$. The latter option is of no concern if $r > c_0E^{c_0}$ because it runs afoul of the $r \geq r_\diamondsuit$ assumption. This understood, assume henceforth this lower bound for r so as to conclude that χ is decreasing at any zero in $[r_\diamondsuit, \frac{1}{2}r_0]$ where $N \leq \frac{1}{4}\delta$.

Part 2: Take $\delta = \frac{1}{16}$. Suppose that $r_* \in [r_\diamondsuit, \frac{1}{2}r_0]$ is a zero of χ with the property that $\chi(r_* + t) > 0$ if t is positive and sufficiently small. Let $r_1 \in (r_*, c_0^{-1}r_0]$ denote the largest value of r such that $h(s) \geq s^{2+3/4}$ when $s \in [r_*, r]$. Note in particular that r_1 is strictly larger than r_* because $h(r_*) = r_*^{2+1/16}$. It follows from the second bullet of Proposition 5.1 that

$$N(r) \geq \frac{1}{16} - r^{-1/8} - c_E(s^{1/c_0} - r_*^{1/c_0}) \quad \textit{for all } s \in [r_*, r_1]. \tag{5.44}$$

Keeping this in mind, use the first bullet of Proposition 5.1 with to see that

$$\frac{d}{dr}h \geq 2r^{-1}\big(1 + \frac{1}{16} - r^{-1/8} - c_E(r^{1/c_0} - r_*^{1/c_0})\big)h - c_E(r^{-2} + r^{-7/8}r) \tag{5.45}$$

on $[r_*, r_1]$. Since $r_* \geq c_0^{-1}r^{-1}$, this inequality implies that

$$\frac{d}{dr}h \geq 2r^{-1}\big(1 + \frac{1}{16} - r^{-1/8} - c_E(r^{1/c_0} - r_*^{1/c_0})\big)h - c_Er^{-3/4}r^{9/8} \tag{5.46}$$

on $[r_*, r_1]$. Integrating (5.46) finds that

$$h(r) \geq e^{-v(r)+v(r_*)} r^{2+1/16} \quad \textit{when } r \in [r_*, r_1], \tag{5.47}$$

where v is a non-negative function on $[0, c_0^{-1} r_0]$ that obeys $|v|(r) \leq c_0 r^{1/c_0}$. The lower bound in (5.47) implies that $h(r_1) \geq r_1^{2+3/4}$ if $r_1 \leq c_0^{-1} r_0$. This being the case, it follows that r_1 must equal $c_0^{-1} r_0$ and so both (5.44) and (5.46) hold on $[r_*, c_0^{-1} r_0]$.

Part 3: Fix $r \in [r_*, c_0^{-1} r_0]$ and use r_{0*} for the moment to denote the upper end point of this interval. Integrate (5.46) from r to r_{0*} to see that

$$h(r_{0*}) \geq c_0^{-1} \left(\frac{r_{0*}}{r} \right)^{2+1/16} h(r). \tag{5.48}$$

As noted previously, $h(r_0)$ is at most $c_0 E^2 r_{0*}^2$. Use the latter bound in (5.47) to see that $h(r) \leq c_0 r^{2+1/16}$ when $r \in [r_*, c_0^{-1} r_0]$. This last conclusion is the assertion of the third bullet of Proposition 5.1

6. Continuity of the limit

This section uses the results from Sections 3–5 to prove that Proposition 2.2's limit function $|\hat{a}_\diamond|$ is continuous. The proposition below makes a formal statement to this effect.

Proposition 6.1. *Fix a subsequence* $\Lambda \subset \{1, 2, \ldots\}$ *so that* $\{(A_n, \hat{a}_n)\}_{n \in \Lambda}$ *is described by Proposition 2.2. The limit function* $|\hat{a}_\diamond|$ *given by the second bullet of Proposition 2.2 is continuous. This function is also Hölder continuous with exponent* $\frac{1}{4}$ *where it is positive; and if* $p \in M$ *and if* $|\hat{a}_\diamond|(p) = 0$, *then there exists* $\kappa > 4$ *such that* $|\hat{a}_\diamond| \leq \kappa \mathrm{dist}(p, \cdot)^{1/\kappa}$ *on the radius* κ^{-1} *ball centered at* p.

The proof of Proposition 6.1 occupies the remainder of this section. Section 6.a proves that $|\hat{a}_\diamond|$ is continuous where it is positive and it proves the Hölder continuity assertion for the points where $|\hat{a}_\diamond|$ is positive. The arguments in Section 6.a assume Lemma 6.2, this being the crucial input. Section 6.b contains the proof of Lemma 6.2. Section 6.c proves that $|\hat{a}_\diamond|$ is continuous near its zeros, and it proves that $|\hat{a}_\diamond|$ has the asserted Hölder continuity property at each zero.

6.a. Continuity where $|\hat{a}_\diamond| > 0$

The assertion in Proposition 6.1 to the effect that $|\hat{a}_\diamond|$ where positive is Hölder continuous with exponent $\frac{1}{4}$ is seen momentarily to be a consequence of the upcoming Lemma 6.2. The notation for Lemma 6.2 and for the subsequent subsections refers back to notation that was introduced in Section 3.b. To say more, fix for the moment $p \in M$. Each $n \in \Lambda$ version of $(A_n, \hat{a}_n)$ has a corresponding version of what is denoted in Section 3.b as $r_\diamond$. The $(A_n, \hat{a}_n)$ version is denoted in what follows by $r_{\diamond n}$. Section 3.b describes a map that it denotes by ϕ from the radius $c_0^{-1} r_\diamond^{-1}$ ball in $\mathbb{R}^3$ to the radius c_0^{-1} ball in M centered at p. The (A_n, a_n) version of this map is denoted by ϕ_n. The corresponding pair $(\phi_n{}^* A_n, r_{\diamond n}^{-1} \hat{a}_n)$ is denoted by $(A_{\diamond n}, \hat{a}_{\diamond n})$, this being a pair of connection on $\phi_n{}^* P$ and $\phi_n{}^* P \times_{SO(3)} \mathfrak{su}(2)$ valued 1-form.

Lemma 6.2. *Let* p *denote a point in* M *where* $|\hat{a}_\diamond|(p) > 0$. *Choose a subsequence in* Λ *such that the corresponding sequence with* n*'th term* $|\hat{a}_n|(p)$ *converges to* $|\hat{a}_\diamond|(p)$. *This chosen subsequence has a subsequence, to be denoted by* Λ_p, *such that* $\liminf_{n \in \Lambda_p} r_{\diamond n} > 0$. *Moreover, there exists data consisting of*

- *A sequence* $\{g_n\}_{n \in \Lambda_p}$ *with* n*'th member being an isomorphism from the product principal* SO(3) *bundle over the* $|x| < 1$ *ball in* $\mathbb{R}^3$ *to* $\phi_n{}^* P$.
- *A pair* $(A_\diamond, \hat{a}_\diamond)$ *of* $L^2_{1;loc}$ *connection on the product principal* SU(2) *bundle over the* $|x| < \kappa^{-1}$ *ball in* $\mathbb{R}^3$, *and an* $\mathfrak{su}(2)$*-valued,* $L^2_{2;loc}$ *1-form on this ball with* $|a_\diamond| = |\hat{a}_\diamond|(p)$. *These are such that the sequence* $\{g_n{}^* A_{\diamond n}\}_{n \in \Lambda_p}$ *converges weakly in the* $L^2_{1;loc}$ *topology on the* $|x| < 1$ *ball to* $A_\diamond$ *and the sequence* $\{g_n{}^* a_{\diamond n}\}_{n \in \Lambda_\diamond}$ *converges weakly in the* $L^2_{2;loc}$ *topology on the* $|x| < 1$ *ball to* $a_\diamond$.

The proof of Lemma 6.2 is in the Section 6.b.

The lemma that follows asserts the parts of Proposition 6.1 that concern the points in M where $|\hat{a}_\diamond|$ is greater than zero.

Lemma 6.3. *Fix a subsequence of* $\Lambda \subset \{1, 2, \dots\}$ *so that* $\{(A_n, \hat{a}_n)\}_{n \in \Lambda}$ *is described by the six bullets in Proposition 2.2. Let* $|\hat{a}_\diamond|$ *denote the limit function given by the second bullet of Proposition 2.2. Then the* $|\hat{a}_\diamond| > 0$ *subset of* M *is open and* $|\hat{a}_\diamond|$ *on this subset is Hölder continuous with exponent* $\frac{1}{4}$.

Proof of Lemma 6.3. The proof has two parts.

Part 1: Fix $p \in M$ where $|\hat{a}_\diamond|(p) > 0$ and let Λ_p denote the corresponding subsequence from Lemma 6.2. Let $r_* > 0$ denote a lower bound for

$\{r_{\lozenge n}\}_{n\in\Lambda_p}$. The convergence of $\{g_n{}^*\hat{a}_{\lozenge n}\}_{n\in\Lambda_p}$ that is asserted by Lemma 6.2 with the fact that each $n \in \Lambda_p$ version of $r_{\lozenge n}$ is greater than r_* imply that the sequence $\{|\hat{a}_n|\}_{n\in\Lambda_p}$ converges strongly in the exponent $\upsilon = \frac{1}{4}$ Hölder topology in the radius $\frac{1}{2}r_*$ ball in M centered at p. That this is so follows from a 3-dimensional Sobolev inequality that asserts in part that L^2_2 functions in are Hölder continuous for any Hölder exponent less than $\frac{1}{2}$.

Since $|\hat{a}_\lozenge(p)| > 0$, so $|a_n|(p) > \frac{1}{2}|\hat{a}_\lozenge|(p)$ for all $n \in \Lambda_p$ if n is sufficiently large. This last observation with the Hölder topology convergence implies the following: There exists $r_p \in (0, r_*)$ such that $|\hat{a}_n| > 0$ on the radius r_p ball about p if $n \in \Lambda$ is sufficiently large. Let p' denote a point in this radius r_p ball. Then $|\hat{a}_\lozenge|(p') > 0$ because the second bullet of Proposition 2.2 has $|\hat{a}_\lozenge|(p') = \limsup_{n\to\infty} |\hat{a}_n|(p')$ and this lim sup is no smaller than $\lim_{n\in\Lambda_p} |\hat{a}_n|(p')$. This proves that the $|\hat{a}_\lozenge| > 0$ part of M is an open set.

Part 2: To see about the continuity and Hölder continuity assertions where $|\hat{a}_\lozenge|$ is positive, fix $p \in M$ with $|\hat{a}_\lozenge|(p) > 0$ and let Λ_p again denote the subsequence from Lemma 6.2. Let $B_p \subset M$ denote the radius r_p ball centered on p. For each $n \in \Lambda_p$, let u_n denote the automorphism of P's restriction to B_p given by $(\phi_n^{-1})^*(g_1^{-1}g_n)$. The convergence asserted by Lemma 6.2 for the sequence $\{g_n{}^*\hat{a}_{\lozenge n}\}_{n\in\Lambda_p}$ on the unit ball in $\mathbb{R}^3$ implies that the sequence $\{u_n{}^*A_n\}_{n\in\Lambda_p}$ converges weakly in the $L^2_{1;loc}$ topology on B_p and that $\{u_n{}^*\hat{a}_n\}_{n\in\Lambda_p}$ converges weakly in the $L^2_{2;loc}$ topology on B_p. A standard Sobolev inequality implies that convergence occurs in the exponent $\frac{1}{4}$ Hölder topology on compact subsets of this ball. Let $\mathfrak{a}_p$ denote the limit section over B_p of the bundle $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. Note that $|\mathfrak{a}_p| = \lim_{n\in\Lambda_p} |(\phi_n^{-1})^*\hat{a}_{\lozenge n}|$. It follows in particular from Proposition 2.2 that $|\hat{a}_\lozenge| \geq |\mathfrak{a}_{p'}|$ on B_p with equality at p and on a set of full measure, the measure being full because $\{|\hat{a}_n|\}_{n=1,2,\ldots}$ converges in the L^2 topology on B_p. Let p' denote another point in B_p. The point p' has a corresponding Hölder continuous $|\mathfrak{a}_{p'}|$ that is defined on a ball $B_{p'}$ about p' and is such that $|\hat{a}_\lozenge| \geq |\mathfrak{a}_{p'}|$ with equality at p' and on a set of full measure. Both $|\mathfrak{a}_{p'}|$ and $|\mathfrak{a}_{p'}|$ are Hölder continuous functions defined on $B_p \cap B_{p'}$ that are equal on a set of full measure. This can happen only if they are equal at all points in $B_p \cap B_{p'}$. This last observation implies that $|\hat{a}_\lozenge| = |\mathfrak{a}_p|$ on B_p and so $|\hat{a}_\lozenge|$ is Hölder continuous on B_p with exponent $\frac{1}{4}$.

6.b. Proof of Lemma 6.2

Item c) of Proposition 2.2 has the following consequence: There exists $\mathrm{E} > 1$ such that each $n \in \Lambda$ version of $(A_n, \hat{a}_n)$ is described by the E and $r = r_n$

version of (3.2). With this understood, invoke Proposition 3.1 for each $n \in \Lambda$ version of $(A_n, \hat{a}_n)$. Proposition 3.1 supplies corresponding $(A_n, \hat{a}_n)$ versions of what it denotes by g and $\hat{a}_{A_\Diamond}$. These versions are denoted in what follows by g_n and $\hat{a}_{A_{\Diamond n}}$.

It follows from what is said in Proposition 3.1 that $\{\hat{a}_{A_{\Diamond n}}\}_{n\in\Lambda}$ is bounded in the L^2_1 topology on the $|x| \le 1$ ball. Proposition 3.1 implies that the $\{{g_n}^*\hat{a}_n\}_{n\in\Lambda}$ is also bounded in the L^2_1 on the $|x| \le 1$ ball and, if $r < 1$, then it is also bounded in the L^2_2 topology on each $|x| \le r$ ball in $\mathbb{R}^3$. It follows as a consequence that there is a subsequence in Λ, this to be denoted by $\Lambda_\Diamond$, such that $\{\hat{a}_{A_{\Diamond n}}\}_{n\in\Lambda_\Diamond}$ converges weakly in the L^2_1 toplogy on the $|x| \le 1$ ball in $\mathbb{R}^3$ as does $\{{g_n}^*\hat{a}_{\Diamond n}\}_{n\in\Lambda_\Diamond}$. The latter sequence also converges weakly in the $L^2_{2;loc}$ topology on the $|x| \le 1$ ball in $\mathbb{R}^3$.

Granted what is said in the preceding paragraph, the proof of Lemma 6.2 needs only a proof of the following assertion: There exists a subsequence $\Lambda_p \subset \Lambda_\Diamond$ with the property that $\{r_{\Diamond n}\}_{n\in\Lambda_p}$ is bounded away from zero. This assertion is proved by assuming it false so as to derive nonsense. The derivation of this nonsense has seven parts.

To set the notation, suppose that $n \in \Lambda_\Diamond$. The various parts of the proof use h_n to denote the version of the function h in (5.2) that is defined using p and the pair $(A_n, \hat{a}_n)$. The corresponding versions of the functions H and N are denoted by H_n and N_n. The arguments that follow also use D to denote the assumed nonzero value of $|\hat{a}_\Diamond|(p)$. No generality is lost by assuming that $\frac{1}{2}\mathrm{D} \le |\hat{a}_n|(p) \le 2\mathrm{D}$ for all $n \in \Lambda$.

Part 1: Let p for the moment denote a given point in M. Fix $r \in (0, c_0^{-1})$ and let $\chi_{p,r}$ denote the function on M given by $\chi(r^{-1}\mathrm{dist}(p,\cdot) - 1)$. This function equals 1 where $\mathrm{dist}(p,\cdot) < r$ and it equals zero where the $\mathrm{dist}(p,\cdot) > 2r$. Let G_p denote the Green's function for the operator $d^\dagger d + 1$ on M with pole at p. Fix $n \in \{1, 2, \dots\}$ and integrate the function $\chi_{p,r} G_p\langle \hat{a}_n \wedge * q_{A_n}(\hat{a}_n)\rangle$ over M, then integrate by parts so as to derive the following local version of (2.40):

$$\frac{1}{2}|\hat{a}_n|^2(p) + \int_M \chi_{p,r} G_p(|\nabla_A \hat{a}_n|^2 + r_n^2|\hat{a}_n \wedge \hat{a}_n|^2) = \mathfrak{e}_{p0} \tag{6.1}$$

where $|\mathfrak{e}_{p0}|$ is bounded by

$$c_0 r^{1/2} + c_0 r^{-3}\int_{r\le \mathrm{dist}(p,\cdot)\le 2r} |\hat{a}_n|^2. \tag{6.2}$$

By way of an explanation, the bound on $|\mathfrak{e}_{p0}|$ follows from two facts, the first being that the L^2 norm of $\mathcal{q}_{A_n}(\hat{a}_n)$ is bounded by c_0 and those of both $d_{A_n}\hat{a}_n$ and $d_{A_n}{*}\hat{a}_n$ are bounded by $c_0 r_n^{-1}$; these are the bounds asserted by Items c) and d) of Proposition 2.2. The third fact is that the L^2 norm of G_p over B_{2r} is less than $c_0 r^{1/2}$. Proposition 2.2 asserts in part that $|\hat{a}_\diamond|(p) = \limsup_{n\in\Lambda} |a_n|(p)$ and that $\{\hat{a}_n\}_{n\in\Lambda}$ converges strongly in the L^2 topology on M to $|\hat{a}_\diamond|$. These facts with (6.1) and (6.2) imply that

$$(6.3)\qquad \tfrac{1}{2}|\hat{a}_\diamond|^2(p) \le c_0 r^{1/2} + c_0 r^{-3} \lim_{n\in\Lambda} \int_{r\le \mathrm{dist}(p,\cdot)\le 2r} |\hat{a}_n|^2.$$

Note that the sequence whose n'th term is the $n \in \Lambda$ version of the integral on the right hand side of (6.3) converges; and the limit is the integral of $|\hat{a}_\diamond|^2$ over the indicated domain. This is so because $\{|\hat{a}_n|\}_{n\in\Lambda}$ converges strongly in the L^2 topology on M to $|\hat{a}_\diamond|$.

Part 2: As explained momentarily, the following two assertions must hold:

(6.4)
- *If* $n \in \Lambda$ *is sufficiently large, then* $h_n(r) \ge c_0^{-1} D^2 r^2$ *for all* $r \in [\frac{1}{2} r_{\diamond n}, 9r_{\diamond n}]$.
- $\limsup_{r\in[\frac{1}{2}r_{\diamond n}, 9r_{\diamond n}]} N(r) = 0$.

Given (6.4), nothing is lost by assuming that the $n \in \Lambda$ versions of h_n and N_n are such that $h_n(r) \ge c_0^{-2} D^2 r^2$ and $N_n(r) \le 1$ for all $r \in [\frac{1}{2} r_{\diamond n}, 9r_{\diamond n}]$. Moreover, the second bullet of (6.1) plus the assumption that $\lim_{n\in\Lambda} r_{\diamond n} = 0$ implies the following: Given $\delta \in (0, \frac{1}{16})$, there exists an integer N_δ such that

$$(6.5)\qquad \sup_{r\in[\frac{1}{2}r_{\diamond n}, 9r_{\diamond n}]} N_n(r) < \delta \qquad \textit{and} \qquad r_{\diamond n} < e^{-1/\delta} \quad \textit{when } n \ge N_\delta$$

The subsequent steps generate the required nonsense when $\delta < c_0^{-1}$.

The proof of (6.4) follows directly. To prove the top bullet, use p's version of (6.3) to conclude that

$$(6.6)\qquad D^2 \le c_0 r^{1/2} + c_0 r^{-2} \sup_{s\in[r,2r]} h_n(s)$$

when $r \in (0, c_0^{-1}]$. Fix $R > \frac{1}{2}$ for the moment. If n is sufficiently large, then h_n is defined at $r = Rr_{\diamond n}$. Assuming this to be the case, then (6.6) demands a point $s \in [Rr_{\diamond n}, 2Rr_{\diamond n}]$ with $h_n(s) \ge c_0^{-1} D^2 R^2 r_{\diamond n}^2$. Fix such a point and integrate the first bullet of the p and $(A_n, \hat{a}_n)$ version of Proposition 5.1 using the fact that $h(t) \le c_0 t^2$ for any $t \in (0, c_0^{-1})$ to conclude that $h(Rr_\diamond) \ge$

$c_0^{-1}D^2R^2r_{\Diamond n}^2 - c_0(r_n^{-2}Rr_{\Diamond n} + r_n^{-1/2}R^2 r_{\Diamond n}^2)$. Since $r_{\Diamond n} \geq c_0^{-1} r_n$ in any event, this last inequality leads directly to the top bullet (6.4) when n is large.

The proof of the second bullet of (6.4) assumes to the contrary that the lim sup in question is positive and derives nonsense. To start the derivation, fix $m > 16$ so that the lim sup in the second bullet of (6.4) is greater than m^{-1}. Let Λ' denote a corresponding subsequence of Λ with the following property: If $n \in \Lambda'$, then there exists $r_{1n} \in [\frac{1}{2}r_{\Diamond n}, 9r_{\Diamond n}]$ with $N_n(r_{1n}) \geq m^{-1}$.

Fix $n \in \Lambda'$ and let κ denote the constant that is supplied by the p and (A_n, a_n) version of Proposition 5.1. Note in particular that κ is independent of n. The top bullet in (6.4) implies that $h_n(r) \geq r^{3-1/\kappa}$ for $r \in [\frac{1}{2}r_{\Diamond n}, 9r_{\Diamond n}]$ when n is large. This being the case, there exists a maximal $r_1 \in [9r_{\Diamond n}, c_0^{-1}r_0]$ such that $h_n(r) > r^{3-1/\kappa}$ for all $r \in [r_{\Diamond n}, r_1]$. Invoke the first bullet of Proposition 5.1 to conclude that

$$(6.7) \qquad \frac{d}{dr}h_n \geq 2r^{-1}(1 + m^{-1} - r^{-1/8} - c_E(r^{1/c_0} - r_*^{1/c_0}))h_n - c_E r^{-1/4} r^{5/4}$$

for all $r \in [r_{\Diamond n}, r_1]$. Integrate this equation to see that

$$(6.8) \qquad h_n(r) \geq c_0^{-1}D^2\left(\frac{r}{r_{\Diamond n}}\right)^{1/m} r^2.$$

for all $r \in [r_{\Diamond n}, r_1]$. It follows from this that r_1 must be equal to $c_0^{-1}r_0$.

Since $h_n(c_0^{-1}r_0) \leq c_0 r_0^2$ in any event, the $r = c_0^{-1}r_0$ version of (6.8) can hold only if $r_{\Diamond n} \geq c_0^{-1}r_0 D^{-2m}$. The latter conclusion constitutes the desired nonsense because it runs afoul of the assumption at the outset that $\limsup_{n\in\Lambda_\Diamond} r_{\Diamond n} = 0$.

Part 3: To set the notation that is used below, fix $n \in \Lambda$ and $p_{*n} \in M$. The notation uses $r_{*\Diamond n}$ to denote the p_{*n} version of $r_{\Diamond n}$ as defined using the pair $(A_n, \hat{a}_n)$. The notation also has h_{*n}, $\hbar_{*n}$ and N_{*n} denoting the p_{*n} versions of the functions h, $\hbar$ and N. The function that measures the distance to p_{*n} is denoted by r_{*n}.

Suppose that $n \in \Lambda$ and that p_{*n} is a point in M with $\text{dist}(p, p_{*n}) \leq 3r_{\Diamond n}$. As proved directly, the following must be true:

(6.9)
- $h_{*n}(r_{*n}) \geq r_{*n}^{2+1/16}$ *for all* $r_{*n} \in [\frac{1}{2}r_{*\Diamond n}, 4r_{\Diamond n}]$.
- $N_{*n}(r_{*n}) < c_0 D^{-2}\delta$ *for all* $r_{*n} \in [\frac{1}{2}r_{*\Diamond n}, 4r_{\Diamond n}]$ *if* $n \geq N_\delta$.

If the top bullet in (6.9) were false for some r_{*n} in the indicated range, then the fourth bullet in the p_{*n} and $(A_n, \hat{a}_n)$ version of Proposition 5.1 would apply with r_* being the relevant value of r_{*n}. In particular, this bullet would find $h_{*n} \le c_0 r_{*n}^{2+1/16}$ for all r_{*n} between $4r_{\Diamond n}$ and c_0^{-1}. This would imply that the integral of $|\hat{a}_n|^2$ over the spherical annulus centered at p_{*n} where $4r_{\Diamond n} \le \mathrm{dist}(p_{*n}, \cdot) \le 12r_{\Diamond n}$ is no greater than $c_0 r_{\Diamond n}^{3+1/16}$. But this is not possible when $n \ge c_0^{-1}$ because the latter spherical annulus contains the spherical annulus centered at p where $7r_{\Diamond n} \le \mathrm{dist}(p, \cdot) \le 9r_{\Diamond n}$ and it follows from (6.4) that the integral of $|\hat{a}_n|^2$ over the $7r_{\Diamond n} \le \mathrm{dist}(p, \cdot) \le 9r_{\Diamond n}$ annulus is greater than $c_0^{-1} D^2 r_{\Diamond n}^2$.

Granted that the top bullet is true, suppose for the sake of argument that the lower bullet is not true. The three steps that follow generate nonsense from this assumption. To set the stage for what is to come, fix for the moment $m > 1$ and suppose that there exists $n \in \Lambda$ with a corresponding $r_* \in [\frac{1}{2} r_{*\Diamond n}, 4r_{\Diamond n}]$ where $N_{*n}(r_*) \ge 2m\delta$.

Step 1. If $N_{*n}(r_*) \ge \frac{1}{16}$, then the argument in Part 2 of Section 5.e can be repeated to see that (5.48) must hold. The latter conclusion is untenable when n is large because it implies that $h_n(r_{\Diamond n}) < c_0 r_{\Diamond n}^{2+1/16}$. This understood, assume that $m\delta \le N_*(r_{*n}) < \frac{1}{16}$.

Step 2. There exists in any case $r > r_*$ such that $h_{*n}(s) > s^{2+3/4}$ for all $s \in [r_*, r]$. It then follows from the second bullet of Proposition 5.1 that

$$N_{*n}(s) \ge m\delta - c_0(s^{1/c_0} - r_*^{1/c_0}) \quad \textit{for all } s \in [r_*, r]. \tag{6.10}$$

This inequality is untenable when n is large if $r = 4r_{\Diamond n}$ for the following reason: The ball of radius $4r_{\Diamond n}$ centered on p_{*n} is contained in the ball of radius $7r_{\Diamond n}$ centered on p. This implies that the integral of $|\nabla \hat{a}_n|^2$ over the ball of radius $7r_{\Diamond n}$ is greater than $c_0^{-1} m\delta r_{\Diamond n}$ when n is large. But the latter integral can be no larger than $c_0 D^{-2} \delta r_{\Diamond n}$, this being a consequence of (6.4). These two bounds can not hold simultaneously if $m > c_0 D^{-2}$.

Step 3. If $m = c_0 D^{-2}$, then it follows from what was said in Step 2 that there exists $r_\ddagger \in [r_*, 4r_{\Diamond n}]$ such that $h_{*n}(r_\ddagger) \le r_\ddagger^{2+3/4}$. This understood, then the fourth bullet of Proposition 5.1 can be invoked using $r_\ddagger$ in lieu of r_* to conclude that $h_{*n}(r_{*n}) \le c_0 r_{*n}^{2+1/16}$ when $r_{*n} \in [4r_{\Diamond n}, c_0^{-1}]$ and n is large. Repeat verbatim the argument for the first bullet in (6.9) to see that this is a nonsensical conclusion.

Part 4: Fix $n \ge N_\delta$ and $p_{*n} \in M$ with $\mathrm{dist}(p, p_{*n}) \le 3r_{\Diamond n}$. Introduce by way of notation B_{*n} to denote the ball of radius $r_{*\Diamond n}$ centered on p_{*n}. Let ϕ_{*n} denote the p_{*n} and $(A_n, \hat{a}_n)$ version of the map ϕ that is described at the outset of Section 3.b, this to be viewed as a map from the radius 4 ball about the origin in $\mathbb{R}^3$ to the radius $4r_{*\diamondsuit n}$ ball centered on p_{*n}. Use this map to define the p_{*n} version of the pair that is denoted in Section 3.a by $(A_\diamondsuit, \hat{a}_\diamondsuit)$. The p_{*n} version is denoted by $(A_{*\diamondsuit n}, a_{*\diamondsuit n})$; it is a pair whose left hand member is connection on the ϕ_{*n} pull-back of P and whose right hand member is a section of the ϕ_{*n} pull-back of the bundle $(P\times_{SO(3)}\mathfrak{su}(2))\otimes T^*M$. Denote by $\hat{a}_{**n}$ the p_{*n} version of what is denoted in Section 3.b by $\hat{a}_*$, this having the form $z_{*n}^{-1}\hat{a}_{*\diamondsuit n}$ with z_{*n} denoting the L^2 norm of $\hat{a}_{*n}$ on B_{*n}. Thus, $\hat{a}_{**n}$ has L^2 norm 1 on the $|x| \leq 1$ ball in $\mathbb{R}^3$. Given the definition of N, the bound in the second bullet of (6.9) implies that

$$\int_{|x|\leq 1} |\nabla_{A_{*\diamondsuit n}} \hat{a}_{**n}|^2 \leq c_D\delta, \tag{6.11}$$

with c_D denoting here and in what follows a number that is greater than 1 that depends only on D. The value of c_D can be assumed to increase between successive appearances.

Part 5: Fix $n \geq N_\delta$ and introduce next $\tilde{a}_{**n}$ to denote p_{*n} version of what is denoted by $\tilde{a}_*$ in Section 4.a. The latter obeys the version of Proposition 4.1 that uses $(A_{*\diamondsuit n}, \hat{a}_{**n})$, $r_{*\diamondsuit n}$ and $r_{z_{*n}}$ in lieu of $(A_{\diamondsuit n}, \hat{a}_{*n})$, $r_{\diamondsuit n}$ and r_{z_n}. Given that $r_{z_{*n}} \geq c_D^{-1}, r_{*\diamondsuit n} \leq c_0 r_{\diamondsuit n}$ and $r_{\diamondsuit n} \leq e^{-1/\delta}$, the second bullet of Proposition 4.1 implies that $\|\tilde{a}_{**n}\|_2$ differs from 1 by at most $c_0 e^{-1/\delta}$. Meanwhile, the third bullet of Proposition 4.1 and (6.11) imply that

$$\int_{|x|\leq 1} |\nabla_{A_{*\diamondsuit n}} \tilde{a}_{**n}|^2 \leq c_D\delta. \tag{6.12}$$

With an appeal to Proposition 3.2 in mind, fix $\varepsilon \in (0,1)$ and $r \in (0,1)$ and let $\kappa_{\varepsilon,r}$ denote the ε and r version of what is denoted by $\kappa_{E,r,\varepsilon}$ in Proposition 3.2. If $\delta < c_D^{-1}\kappa_{\varepsilon,r}^{-1}$ and also $r_{\diamondsuit n} < c_D^{-1}\kappa_{\varepsilon,r}^{-1}$ then Proposition 3.2 says that the square of the L^2 norm of $F_{A_{*\diamondsuit n}}$ on the $|x| < r$ ball in $\mathbb{R}^3$ is less than ε.

Part 6: Fix an integer $n > N_\delta$. This part of the proof defines an iterative procedure that starts with p and generates from p a finite sequence of points. The k'th point in the sequence is denoted by $p_{*n,k}$. It is such that $\text{dist}(p, p_{*n,k}) < 3r_{\diamondsuit n}$. The description that follows of the iteration step uses $p_{*n,0}$ to denote p. If some $k \geq 0$ version of $p_{*n,k}$ has been defined, the notation has $r_{*\diamondsuit n,k}$ denoting the $p_{*n,k}$ version of $r_{*\diamondsuit n}$.

THE ITERATION PROCEDURE: Fix $k \geq 0$ and suppose that that the point $p_{*n,k}$ has been defined. Let $p_{*n} \in M$ denote any given point with

$\mathrm{dist}(p_{*n,k}, p_{*n}) \le 2r_{*\Diamond n,k}$. The assignment of $r_{*\Diamond n}$ to p_{*n} defines a continuous function on the radius $2r_{*\Diamond n,k}$ ball about $p_{*n,k}$. If the minimum value is less than or equal to $\frac{1}{256}r_{*\Diamond n,k}$, take $p_{*n,k+1}$ to be a point with distance $2r_{*\Diamond n,k}$ or less from $p_{*n,k}$ where the minimum value is achieved. If the minimum value of this function is greater than $\frac{1}{256}r_{*\Diamond n,k}$, then there is no $(k+1)$'st point and $p_{*n,k}$ is the last point in the sequence. Note that the iteration must stop after a finite number of runs because each p_{*n} version of $r_{*\Diamond n}$ is, in any event, greater than $c_0^{-1}r_n^{-1}$.

The sequence $\{p_{*n,k}\}_{k=1,2,\ldots}$ lies in the radius $(2+\frac{1}{2})r_{\Diamond n}$ ball centered on p. To understand why this is, keep in mind that

$$\mathrm{dist}(p_{*n,k}, p_{*n,k+1}) \le \tfrac{1}{128}r_{*\Diamond n,k}. \tag{6.13}$$

Meanwhile, $r_{*\Diamond n,k} < \frac{1}{256}r_{*\Diamond n,k-1}$ and so $r_{*\Diamond n,k} \le (\frac{1}{256})^k r_{\Diamond n}$.
Thus, $\mathrm{dist}(p_{*n,k}, p_{*n,k+1}) \le 4(\frac{1}{256})^k r_{\Diamond n}$ and so $\mathrm{dist}(p, p_{*\Diamond n,k+1}) \le (2 + 4(\sum_{m=1,2,\ldots k}(\frac{1}{256})^m))r_{\Diamond n}$ which is no greater than $(2+\frac{1}{32})r_{\Diamond n}$.

Part 7: Let p_{*k} now denote the final point in the iteration sequence $\{p, p_{*n,1}, \ldots\}$. It follows from the definition that the radius $2r_{*\Diamond n,k}$ ball centered on $p_{*n,k}$ is contained in the radius $3r_{\Diamond n}$ ball centered around p. If p_{*n} is a given point in the radius $2r_{*\Diamond n,k}$ ball centered $p_{*n,k}$, then its corresponding $r_{*\Diamond n}$ is no less than $\frac{1}{256}r_{*\Diamond n,k}$. This understood, it follows that the radius $2r_{*\Diamond n,k}$ ball centered on $p_{*n,k}$ has a cover, $\mathfrak{U}$, with the following properties:

(6.14)
- $\mathfrak{U}$ consist of at most c_0^{-1} balls with centers in the radius $2r_{*\Diamond n,k}$ ball about $p_{*n,k}$.
- Let p_{*n} denote a center point of a ball from $\mathfrak{U}$. The radius of its ball is $\frac{1}{2}r_{*\Diamond n}$.

Let p_{*n} denote the center point of a given ball from $\mathfrak{U}$ and let B_{*n} denote the corresponding ball from $\mathfrak{U}$. Use the $r = \frac{3}{4}$ version of what is said at the end of Part 5 to see that the square of the L^2 norm of F_{A_n} over B_{*n} is at most $\varepsilon^2 r_{*\Diamond n}^{-1}$ and thus at most $256\varepsilon^2 r_{*\Diamond n,k}^{-1}$. Since there are at most c_0^{-1} balls in $\mathfrak{U}$, this bound implies in turn that

$$\int_{\mathrm{dist}(p_{*n,k},\cdot)\le 2r_{*\Diamond n,k}} |F_{A_n}|^2 \le c_D\varepsilon^2 r_{*\Diamond n,k}^{-1}. \tag{6.15}$$

The latter bound is nonsensical if $\varepsilon < c_D^{-1}$ because it runs afoul of the definition of $r_{*\Diamond n,k}$. This is the promised nonsense from the small δ versions of (6.5).

6.c. $|\hat{a}_\diamond|$ near its zero locus

This subsection proves the assertions in Proposition 6.1 that concern $|\hat{a}_\diamond|$ on its zero locus. The proof has two parts.

Part 1: The lemma that follows plays a central role in the subsequent arguments. To set the notation, suppose that $p \in M$. Given a positive integer n, the lemma uses h_n to denote the version of the function h in (5.2) that is defined using p and the pair $(A_n, \hat{\mathfrak{a}}_n)$.

Lemma 6.4. *Fix* $p \in M$ *and* $c > 2$. *Suppose there exists a subsequence in Proposition 6.1's sequence* Λ, *this denoted by* Λ_p, *such that* $\lim_{n\in\Lambda_p} h_n(r) \leq c r^{1/c} r^2$ *when* $r \leq c^{-1}$. *Then* $|\hat{a}_\diamond|(p) = 0$. *Moreover,* $|\hat{a}_\diamond|$ *is continuous at* p *and there exists a constant* $\kappa > 1$ *such that* $|\hat{a}_\diamond|(\cdot) \leq \kappa \mathrm{dist}(p, \cdot)^{1/\kappa}$ *on the ball of radius* κ^{-1} *centered at* p.

Proof of Lemma 6.4. Fix $r \in (0, c_0^{-1})$ and it follows from (6.3) that $|\hat{a}_\diamond|^2(p)$ is bounded by $c_0(r^{1/2} + r^{1/c})$. It follows as a consequence that $|\hat{a}_\diamond|(p) = 0$. Let q denote some other point in M with $\mathrm{dist}(p, q) < c_0^{-1}$. Take r to equal $4\mathrm{dist}(p, q)$ and use q's version of (6.3) to see that

$$|\hat{a}_\diamond|^2(q) \leq c_0 r^{1/2} + c_0 r^{-3} \lim_{n\in\Lambda} \int_{r/2\leq \mathrm{dist}(p,\cdot)\leq 4r} |\hat{\mathfrak{a}}_n|^2 \tag{6.16}$$

The final arguments for Lemma 6.4 exploit (6.16). To this end, fix $n \in \Lambda$ for the moment and let $h_n(\cdot)$ denote the version of the function h in (5.2) that is defined by the point p and the pair $(A_n, \hat{\mathfrak{a}}_n)$. If $\mathrm{dist}(p, q) < c_0^{-1}$, then h_n is defined on the interval $[0, 4r]$. Granted this is the case, use the definition of h_n to conclude that

$$\int_{B_{4r}-B_{r/2}} |\hat{\mathfrak{a}}_n|^2 \leq c_0 r \sup_{s\in[r/2,4r]} h_n(s). \tag{6.17}$$

Let c and Λ_p be as described by Lemma 6.4. If 4r is less than c^{-1}, then the right hand side of (6.17) is no greater than $c_0 c r^{1/c} r^3$ when $n \in \Lambda_p$ is large. This implies that the right hand side of (6.16) is no greater than $c_0(r^{1/2} + c r^{1/c})$ because the limit of a convergent sequence is equal to the limit of any its subsequences. Thus $|\hat{a}_\diamond|^2(q) \leq c_0(r^{1/2} + c r^{1/c})$. This is the assertion made by Lemma 6.4 because $r = 4\mathrm{dist}(p, q)$.

Part 2: The assertion in Proposition 6.1 to the effect that $|\hat{a}_\diamondsuit|$ is continuous across its zero locus follows from the assertion in the proposition to the effect that $|\hat{a}_\diamondsuit|$ is Hölder continuous at each of its zeros. The proof of the latter assertion is given in the three steps that follow. These steps employ the following terminology: The *local Hölder property* is said to hold at a given point p if there exists $\kappa > 1$ such that $|\hat{a}_\diamondsuit(q)| < \kappa \mathrm{dist}(p, q)^{1/\kappa}$ for all $q \in M$ with $\mathrm{dist}(p, q) < \kappa^{-1}$.

Step 1. Fix $p \in M$ with $|\hat{a}_\diamondsuit|(p) = 0$. Since $|\hat{a}_\diamondsuit|(p) = 0$, so $\lim_{n\to\infty} |\hat{a}_n|(p) = 0$ also, this a consequence of the second bullet in Proposition 2.2. There are now two cases to consider, the first being where p's version of the sequence $\{r_{\diamondsuit n}\}_{n\in\Lambda}$ has a subsequence that is bounded away from zero. If such is the case, let $r_* > 0$ denote a lower bound for this subsequence. The corresponding subsequence of $\{\hat{a}_n\}_{n\in\Lambda}$ is bounded in the L^2_2 topology on the radius r_* ball in M centered at p, and so it has a subsequence that converges strongly in the exponent $\frac{1}{4}$ Hölder topology on this ball. Let $\Lambda_p \subset \Lambda$ denote the indexing set for the latter subsequence. The Hölder convergence of $\{\hat{a}_n\}_{n\in\Lambda_p}$ on the radius r_* ball centered at p has the following consequence: Given $\varepsilon > 0$, there exists N_ε such that if $n \in \Lambda_p$ and $n > N_\varepsilon$, then

$$(6.18)\qquad |a_n| \le \varepsilon + \mathrm{dist}(p, \cdot)^{1/4} \quad \textit{on the radius } r_* \textit{ ball centered at } p.$$

Fix $n \in \Lambda_p$ with $n > N_\varepsilon$. The bound in (6.18) implies that the p and $(A_n, \hat{a}_n)$ version of the function $h_n(\cdot)$ obeys $h_n(r) \le c_0(\varepsilon + r^{1/4})r^2$ when $r \in (0, r_*)$. Granted this last bound, invoke Lemma 6.4 to see that the local Hölder property holds at p.

Step 2. Assume here and in the subsequent steps that $\{(A_n, a_n)\}_{n\in\Lambda}$ is such that $\lim_{n\in\Lambda} |a_n|(p) = 0$ and $\lim_{n\in\Lambda} r_{\diamondsuit n} = 0$. Granted this assumption, then at least one of the three cases in the subsequent list describes $\{(A_n, \hat{a}_n)\}_{n\in\Lambda}$. Step 3 contains the proof that the list is inclusive.

CASE 1. This case occurs if there is a subsequence $\Lambda_p \subset \Lambda$ with two properties, the first being the following: If $n \in \Lambda_p$, then there exists $r_{\ddagger n} \in [\frac{1}{2} r_{\diamondsuit n}, c_0^{-1}]$ which is such that $h_n(r_{\ddagger n}) \le r_{\ddagger n}^{2+1/16}$. The second property requires that $\lim_{n\in\Lambda_p} r_{\ddagger n} = 0$. Fix $n \in \Lambda_p$. Let $r_{1n} \in [r_{\ddagger n}, c_0^{-1}]$ denote the maximal value for r such that $h_n(s) \le s^{2+1/16}$ for all $s \in [r_{\ddagger n}, r_{1n}]$. It follows from the fourth bullet of Proposition 5.1 that $h_n(r) \le c_0 r^{2+1/16}$ for all $r \in [r_{1n}, c_0^{-1}]$; it follows from the definitions of r_{1n} and $r_{\ddagger n}$ that $h_n(r) < r^{2+1/16}$ for all $r \in [r_{\ddagger n}, r_{1n}]$. Since $\lim_{n\in\Lambda_p} r_{\ddagger n} = 0$, the subsequence Λ_p with any $c > c_0$ can be used as input to Lemma 6.4 to prove that the local Hölder property holds at p.

CASE 2. This case occurs if there exists $\delta > 0$ and a subsequence $\Lambda' \subset \Lambda$ with the following property: If $n \in \Lambda'$, then $h_n(r) > r^{2+1/16}$ for all $r \in [\frac{1}{2}r_{\diamond n}, 9r_{\diamond n}]$ and there exists $r \in [\frac{1}{2}r_{\diamond n}, 9r_{\diamond n}]$ with $N_n(r) \geq \delta$. Let $r_{1n} \in (9r_{\diamond n}, c_0^{-1}]$ denote the maximal r which is such that $h_n(s) \geq s^{2+1/16}$ for all $s \in [r_{\diamond n}, r_{1n}]$.

Suppose first that $\liminf_{n\in\Lambda'} r_{1n} = 0$. Fix $n \in \Lambda'$. The fourth bullet of Proposition 5.1 implies that $h_n(r) \leq c_0 r^{2+1/16}$ for all $r \in [r_{1n}, c_0^{-1}]$. Fix a subsequence $\Lambda_p \subset \Lambda'$ such that $\lim_{n\in\Lambda_p} r_{1n} = 0$. The fact that $\lim_{n\in\Lambda_p} r_{1n} = 0$ implies that Λ_p and any $c > c_0$ version of Lemma 6.4 can again be used to prove that the local Hölder property holds at p.

Suppose on the other hand that there exists $r_0 < c_0^{-1}$ such that $\liminf_{n\in\Lambda'} r_{1n} > 2r_0$. Fix $n \in \Lambda'$ such that $r_{1n} > r_0$. Then $h_n(r) > r^{2+1/16}$ on $[r_{\diamond n}, r_0]$. This being the case, the second bullet of Proposition 5.1 can be invoked to see that $N_n(r) \geq \frac{1}{2}\delta$ if $r \in [\frac{1}{2}r_{\diamond n},\ c_0^{-1}r_0]$. With this understood, invoke the first bullet of Proposition 5.1 to obtain the inequality

$$\frac{d}{dr}h_n \geq 2r^{-1}\big(1 + \tfrac{1}{2}\delta\big)h_n - c_0\delta^{-1}r_n^{-1}r \tag{6.19}$$

where $r \in [\frac{1}{2}r_{\diamond n}, c_0^{-1}r_0]$. Fix r in this range, and integrate (6.19) from r to $c_0^{-1}r_0$ and use the fact that $h_n(c_0^{-1}r_0) \leq c_0 r_0^2$ to conclude that $h_n(r) \leq c_0(r_0^{-1/(2\delta)}r^{2+1/(2\delta)} + r_n^{-1}r_0^2)$. Let $r_{\ddagger n}$ denote the number $r_n^{-1/(2+1/(2\delta))}r_0$ and let c denote $c_0(\delta^{-1} + c_0 r_0^{-1/2\delta})$. Then the preceding bound on $h_n(r)$ implies that $h_n(r) \leq c r^{2+1/c}$ when $r \in [r_{\ddagger n}, c_0^{-1}r_0]$. Noting that $\lim_{n\in\Lambda'} r_{\ddagger n} = 0$, Lemma 6.4 can be invoked using as input $\Lambda_p = \Lambda'$ and c to prove that the local Hölder property assertion holds at p.

The statement of the third case reintroduces notation from Part 3 of Section 6.b.

CASE 3. This case occurs when three conditions are met. The first condition requires that $h_n(r) > r^{2+1/16}$ for all $r \in [\frac{1}{2}r_{\diamond n}, 9r_{\diamond n}]$ when $n \in \Lambda$ is sufficiently large; and the second condition requires that $\lim_{n\in\Lambda}\sup_{r\in[\frac{1}{2}r_{\diamond n}, 9r_{\diamond n}]} N_n(r) = 0$. The third condition requires there be a subsequence $\Lambda' \in \{1, 2, \dots\}$ and an associated sequence $\{p_{*n}\}_{n\in\Lambda'} \subset M$ with the following properties.

(6.20)
- *Each* $n \in \Lambda'$ *version of* p_{*n} *has distance less than* $3r_{\diamond n}$ *from* p.
- *Either or both of the following statements are true.*
  - i) *If* $n \in \Lambda'$, *then there exists* $r_{*\ddagger n} \in [\frac{1}{2}r_{*\diamond n}, 9r_{*\diamond n}]$ *such that* $h_{*n}(r_{*\ddagger n}) \leq r_{*\ddagger n}^{2+1/16}$.
  - ii) $\sup_{r\in[\frac{1}{2}r_{*\diamond n}, 9r_{*\diamond n}]} N_{*n}(r) \geq \delta$

Suppose there is a subsequence $\Lambda'' \subset \Lambda'$ such that Item i) in the second bullet of (6.20) holds for all $n \in \Lambda''$. Fix $n \in \Lambda''$ and let h_{*n} denote the p_{*n} version of h. But for cosmetic changes, the argument in Case 1 can be used with p_{*n} replacing p to see that $h_{*n}(s) \leq c_0 s^{2+1/16}$ for all $s \in [9r_{*\Diamond n}, c_0^{-1}]$. Keeping this in mind, use an integration by parts with the fact that $\|\nabla_{A_n}\hat{a}_n\|_2 \leq c_0$ and $|a_n| \leq c_0$ to see that

$$h_n(s) \leq h_{*n}(s + 4r_{\Diamond n}) + c_0 r_{\Diamond n}^{3/2}, \tag{6.21}$$

when $s > 10r_{\Diamond n}$. Fix $r > 0$ and (6.21) implies that $\lim_{n\in\Lambda'} h_n(r) \leq c_0 r^{2+1/16}$. This being the case, then Lemma 6.4 can be invoked using as input $\Lambda_p = \Lambda''$ and any $c \geq c_0$ to prove that p has the local Hölder property.

Suppose next that Item i) of (6.20) is not true if $n \in \Lambda'$ is large. This understood, throw out the finite set of integer where Item i) is true and use Λ' now to denote the remaining set. Each $n \in \Lambda'$ obeys the condition in Item ii) of (6.20). But for cosmetic changes, the argument in Case 2 can be used with each $n \in \Lambda'$ version of p_{*n} replacing p to obtain the following data: A number $c > 1$ and a sequence $\{r_{*\ddagger n}\}_{n\in\Lambda'} \subset (0, c^{-1})$ with limit zero and with the following additional property: Each $n \in \Lambda'$ version of $r_{*\ddagger n}$ is such that $h_{*n}(s) \leq c s^{2+1/c}$ when $s \in [r_{*\ddagger n}, c^{-1}]$. Granted this data, use (6.21) to conclude that $\lim_{n\in\Lambda'} h_n(r) \leq c_0 c r^{2+1/c}$ for each $r \in (0, c^{-1})$. It follows from the latter bound that the sequence $\Lambda_p = \Lambda'$ and the given value of c can be used as input to Lemma 6.4 to prove that p has the local Hölder property.

Step 3. Assume that $\{(A_n, \hat{a}_n)\}_{n\in\Lambda}$ is such that $\lim_{n\in\Lambda} |\hat{a}_n|(p) = 0$ and $\lim_{n\to\infty} r_{\Diamond n} = 0$. The paragraphs that follow proves that at least one of the three cases in Step 2 apply. To this end, assume to the contrary that none of these cases. The existence of such a sequence is shown below to lead to nonsense.

After discarding a finite set of terms and then relabling the result as Λ, the sequence $\{(A_n, \hat{a}_n)\}_{n\in\Lambda}$ must have the following property: The first bullet in (6.9) is obeyed for all $n \in \Lambda$; and given $\delta > 0$, there exists $N_\delta > 1$ such that the second bullet in (6.9) is obeyed when $n \geq N_\delta$. Indeed, the first bullet of (6.9) must be obeyed to avoid a Case 1 label and the second bullet of (6.9) must be obeyed to avoid a Case 2 label. Given the preceding, then what is said in Parts 4 and 5 of Section 6.b applies to each point p_{*n} in the radius $3r_{\Diamond n}$ ball centered at p with the number c_D in (6.11) replaced by c_0. This replacement is allowed by virtue of the fact that Item ii) of (6.20) is violated. Granted that Parts 4 and 5 of Section 6.b apply, then an essentially verbatim repetition of the arguments in Parts 6 and 7 of Section 6.b prove that $\{r_{\Diamond n}\}_{n\in\Lambda}$ has a strictly positive lower bound. This last conclusion is nonsense because it runs afoul of the assumptions made at the outset.

7. The data Z, $\mathcal{I}$ and ν

The forthcoming proposition is used in subsequent sections to characterize the zero locus of $|\hat{a}_\Diamond|$. By way of notation, the proposition denotes this zero locus by Z. The following notation is also used: Fix $p \in M$ and $r > 0$. The proposition uses B_r to denote the radius r ball centered at p and it uses ∂B_r to denote the boundary of the closure of B_r. Keep in mind that the term *real line bundle* is used here to describe the associated line bundle to a principal $\mathbb{Z}/2\mathbb{Z}$ bundle.

Proposition 7.1. *The set* $Z \subset M$ *is a closed set and so* $M{-}Z$ *is open. There is a real line bundle over* $M{-}Z$*, this denoted by* $\mathcal{I}$*, and a harmonic section,* ν*, of* $T^*(M{-}Z) \otimes \mathcal{I}$ *with the properties listed below.*

- $|\nu| = |\hat{a}_\Diamond|$.
- $|\nabla\nu|$ *extends from* $M{-}Z$ *as an* L^2 *function on* $M{-}Z$.
- *For any point* p *in* M*, the function* $|\nabla\nu|\mathrm{dist}(\cdot, p)^{-1/2}$ *is an* L^2 *function on* $M{-}Z$ *and there is a* p*-independent bound on its* L^2 *norm.*
- *There exists* $\kappa \geq 1$ *with the following significance. Fix* $p \in M$ *to define functions* h *and* H *on* $(0, \kappa^{-1})$ *by the rules*

$$r \to h(r) = \int_{\partial B_r} |\nu|^2 \quad \text{and} \quad r \to H(r) = \int_{B_r - Z} |\nabla\nu|^2.$$

   a) *The function* h *is strictly positive on* $(0, \kappa^{-1})$.

   b) *Define the function* N *on* $(0, \kappa^{-1})$ *by the rule* $r \to N(r) = rH(r)/h(r)$. *The function* h *is differentiable on* $[0, \kappa^{-1})$. *Moreover, its derivative on* $(0, \kappa^{-1})$ *can be written as* $\frac{d}{dr}h = 2r^{-1}(1 + N + \mathfrak{e})h$ *with* $\mathfrak{e}$ *such that* $|\mathfrak{e}| \leq \kappa r^2$.

   c) *If* $s > r > 0$*, then* $N(s) \geq e^{-\kappa(s^2 - r^2)}N(r) - \kappa(s^2 - r^2)$.

With regards to the Item b) of the fourth bullet, keep in mind that Nh is rH and so r^{-1}Nh has limit zero as r limits to 0. This is also the case for $r^{-1}h$ because $h \leq c_0 r^2$.

The function N plays the role here of the frequency function introduced by [Al] and used by [HHL] and [Han]. It is perhaps needless to say that Proposition 7.1's version of N is the analog for ν of the function in (5.3).

Granted that Z is closed and granted the second and third bullets of the proposition, the convention in what follows is to extend $|\nabla\nu|$ and any $p \in M$ version of $\mathrm{dist}(p, \cdot)^{-1}|\nabla\nu|$ as an L^2 functions on the whole of M by declaring them to be zero on Z. For example, this convention writes H(r) as $\int_{B_r} |\nabla\nu|^2$.

Sections 7.a and 7.b contain the proof of Proposition 7.1. Section 7.c contains a lemma that concern the behavior of the $r \to 0$ limit of Proposition 7.1's function N as the point p is varied in M.

7.a. The construction of $\mathcal{I}$ and ν

Proposition 6.1 asserts in part that $|\hat{a}_\diamondsuit|$ is continuous, and this implies directly that its zero locus, Z, is a closed set. The subsequent lemma provides what is needed to define $\mathcal{I}$ and ν. The notation is that used by Proposition 6.1.

Lemma 7.2. *Let $\Lambda \subset \{1, 2, \dots\}$ denote the subsequence from Proposition 6.1. There exists a subsequence $\Lambda_\diamondsuit \subset \Lambda$ such that the corresponding sequence $\{(A_n, \hat{\mathfrak{a}}_n)\}_{n\in\Lambda_*}$ has the properties listed below.*

- *The sequence of $\{F_{A_n}\}_{n\in\Lambda_*}$ has bounded L^2 norm on compact subsets of $M{-}Z$ and $\{\nabla_{A_n}\hat{\mathfrak{a}}_n\}_{n\in\Lambda_*}$ has bounded L^2 norm on M.*
- *There exists a sequence $\{h_n\}_{n\in\Lambda_*} \subset \mathrm{Aut}(P)$ such that each $n \in \Lambda_*$ version of ${h_n}^*A_n$ can be written on $M{-}Z$ as $A_0 + \hat{a}_{h^*A_n}$ with $\{\hat{a}_{h^*A_n}\}_{n\in\Lambda_*}$ having bounded L^2_1 norm on any given compact subset of $M{-}Z$ and converging in the $L^2_{1;loc}$ topology on $M{-}Z$. Meanwhile $\{{h_n}^*\mathfrak{a}_n\}_{n\in\Lambda_*}$ has bounded L^2_2 norm on any given compact set in $M{-}Z$ and it converges weakly in the $L^2_{2;loc}$ topology on $M{-}Z$.*
- *Let $(A_\diamondsuit, \mathfrak{a}_\diamondsuit)$ denote the limit pair of $L^2_{1;loc}$ connection on $P|_{M-Z}$ and $L^2_{2;loc}$ section over $M{-}Z$ of $(P \times_{SO(3)} \mathfrak{su}(2)) \otimes T^*M$. These are such that*

  a) $d_{A_\diamondsuit}\mathfrak{a}_\diamondsuit = 0$, $d_{A_\diamondsuit}{*}\mathfrak{a}_\diamondsuit = 0$ *and* $\mathfrak{a}_\diamondsuit \wedge \mathfrak{a}_\diamondsuit = 0$.

  b) $|\mathfrak{a}_\diamondsuit| = |\hat{a}_\diamondsuit|$.

  c) $|\nabla_{A_\diamondsuit}\mathfrak{a}_\diamondsuit| = \lim_{n\in\Lambda}|\nabla_{A_n}\hat{\mathfrak{a}}_n|$ *with the convergence being in the $L^2_{1;loc}$ topology on the set where $|\hat{a}_\diamondsuit| > 0$.*

This lemma is proved momentarily. Assume it to be true for the time being.

Parts 1 of what follows directly use Lemma 7.2 to define $\mathcal{I}$, and Part 2 of what follows defines ν and verifies the first bullet of Proposition 7.1. Part 3 of what follows uses Lemma 7.2 to prove the second and third bullets of Proposition 7.1.

Part 1: Let SM denote the unit sphere bundle TM. Since $|\hat{a}_\diamond| < c_0$, Item b) of the third bullet in Lemma 7.2 finds $|\mathfrak{a}_\diamond| < c_0$. This understood, define a quadratic map from the SM/{±1} to $[0, c_0)$ by the rule $v \to |\mathfrak{a}_\diamond(v)|^2$. This map has a unique maximum at each point of M−Z, this being one consequence of the fact that $\mathfrak{a}_\diamond \wedge \mathfrak{a}_\diamond = 0$. The corresponding line in $TM|_{M-Z}$ defines a real line subbundle of TM over M−Z. This real line bundle subbundle is denoted by $I_\ddagger$. It is associated in a canonical way to the principal $\mathbb{Z}/2\mathbb{Z}$ that is defined by the points where $I_\ddagger$ intersects the unit sphere bundle in TM. The bundle I is the dual to $I_\ddagger$.

Part 2: Let $\mathfrak{U}$ denote a countable, locally finite open cover of M−Z by balls, and let B denote a given ball from this cover. The fact that $\mathfrak{a}_\diamond \wedge \mathfrak{a}_\diamond = 0$ implies that $\mathfrak{a}_\diamond$ can be written on B as $\sigma_B \nu_B$ with σ_B being a Sobolev class L^2_2 map from B to the unit sphere in $\mathfrak{su}(2)$ and with ν_B being an $\mathbb{R}$ valued 1-form on B that annihilates the orthogonal complement in $TM|_B$ of the line subbundle $I_\ddagger|_B$.

The equations $[\sigma_B, d_{A_\diamond} \mathfrak{a}_\diamond] = 0$ and $[\sigma_B, d_{A_\diamond} {*\mathfrak{a}_\diamond}] = 0$ are equivalent to the assertions $\nabla_{A_\diamond} \sigma_B \wedge \nu_B = 0$ and $\nabla_{A_\diamond} \sigma_B \wedge {*\nu_B} = 0$. Since $|\nu_B| = |\hat{a}_\diamond| \neq 0$ on B, these together assert that $\nabla_{A_\diamond} \sigma_B = 0$. Thus, σ_B is $A_\diamond$-covariantly constant. Meanwhile, the two equations $\langle \sigma_B d_{A_\diamond} \mathfrak{a}_\diamond \rangle = 0$ and $\langle \sigma_B d_{A_\diamond} {*\mathfrak{a}_\diamond} \rangle = 0$ say that ν_B is a harmonic 1-form on B. As a parenthetical remark, the fact that σ_B is $A_\diamond$ covariantly constant implies that $F_{A_\diamond}$ can be written on B as $\sigma_B w_B$ with w_B denoting a closed square integrable 2-form on B.

The following turns out to be a crucial observation about this writing of $\mathfrak{a}_\diamond$ The definition of ν_B has a sign ambiguity because there are automorphisms of B × SU(2) that pull σ_B back as $-\sigma_B$. This sign ambiguity disappears if and only if $I_\ddagger$ is the product line bundle.

What follows gives a second view of this sign ambiguity. Fix a length 1 element, $\sigma \in \mathfrak{su}(2)$. There exists a Sobolev class L^2_2 automorphism of the bundle B × SU(2) that writes $A_\diamond$ as $\theta_0 + \sigma A_B$ and writes $\mathfrak{a}_\diamond$ on B as $\sigma \nu_B$ with A_B denoting a 1-form on B of Sobolev class L^2_1. Now let $\tau \in \mathfrak{su}(2)$ denote a given element with $\langle \sigma\tau \rangle = 0$ and length $|\tau| = \pi$. View e^τ as an automorphism The latter pulls back $A_\diamond$ as $\theta_0 - \sigma A_B$ and it pulls back $\mathfrak{a}_\diamond$ as $-\sigma\nu_B$. The sign ambiguity is due to the fact that $\theta_0 - \sigma A_B$ and $-\sigma\nu_B$ can be written respectively as $\theta_0 + \sigma A'_B$ and $\sigma \nu'_B$ with $A'_B = -A_B$ and $\nu'_B = -\nu_B$.

Let B and B' denote intersecting balls from $\mathfrak{U}$. Then ν_B can be written on $B \cap B'$ as $z_{BB'} \nu_{B'}$ with $z_{BB'}$ being either 1 or −1. The collection $\{z_{BB'}\}_{B,B' \in \mathfrak{U}}$ defines the transition functions for the line bundle I. Meanwhile, the collection $\{\nu_B\}_{B \in \mathfrak{U}}$ defines a smooth, harmonic section over M−Z of $T^*(M{-}Z) \otimes I$ that vanishes on Z, this being Proposition 7.1's section ν.

Part 3: The fact that $|\nu| = |\hat{a}_\diamondsuit|$ follows from Item b) of Lemma 7.2's third bullet because $|\nu| = |\mathfrak{a}_\diamondsuit|$. The fact that $|\nabla\nu|$ is square integrable follows from the first bullet of Lemma 7.2 because $|\nabla\nu| = |\nabla_{A_\diamondsuit}\mathfrak{a}_\diamondsuit|$. To elaborate, fix $\rho > 0$ and introduce by way of notation $Z_\rho \subset M$ to denote the set where $|\hat{a}_\diamondsuit| < \rho$. Lemma 7.2 implies directly that

$$\int_{M-Z_\rho} |\nabla\nu|^2 = \lim_{n\in\Lambda} \int_{M-Z_\rho} |\nabla_{A_n}\hat{\mathfrak{a}}_n|^2. \tag{7.1}$$

With (7.1) in mind, define a function on $(0, c_0^{-1})$ by the rule that assigns a given number ρ in this integral the value of the left hand integral in (7.1). The first bullet of Lemma 7.2 asserts in part that this function is bounded on $(0, c_0^{-1})$ and since it is a decreasing function of ρ, so the dominated convergence theorem says that it has a unique $\rho \to 0$ limit. This limit is the integral of $|\nabla\nu|^2$.

Proposition 7.1's third bullet follows using an identical argument after invoking what is said in the sixth bullet of Proposition 2.2 about the sequence whose n'th term is the integral over M of the function $G_p|\nabla_{A_n}\hat{\mathfrak{a}}_n|^2$.

Proof of Lemma 7.2. The proof has three steps.

Step 1. Fix $\rho > 0$. This step proves the following assertion:

$$\textit{The sequence } \left\{\int_{M-Z_\rho} |F_{A_n}|^2\right\}_{n\in\Lambda} \textit{ is bounded.} \tag{7.2}$$

To see why this is, suppose to the contrary that (7.2) is false. Given a point p in the $M-Z_\rho$, and $n \in \Lambda$, let $r_{\diamondsuit n,p}$ denote p's version of the number $r_\diamondsuit$ that is defined in (3.5) by the connection A_n. If (7.2) is false, then it must be that $\liminf_{p\in M-Z_\rho}(\liminf_{n\in\Lambda} r_{\diamondsuit n,p}) = 0$. This understood, there exists a point $p \in M-Z_\rho$ a subsequence $\Lambda' \subset \Lambda$ and a sequence $\{p_n\}_{n\in\Lambda'} \subset M-Z_\rho$ that converges to p and is such that $\lim_{n\in\Lambda'} r_{\diamondsuit n,p_n} = 0$. Granted this, then it must be true that $\lim_{n\in\Lambda'} r_{\diamondsuit n,p} = 0$ also.

The sequence $\{(A_n, \hat{\mathfrak{a}}_n)\}_{n\in\Lambda'}$ can be used as input for Proposition 2.2. Let $|\hat{a}'_\diamondsuit|$ denote the Λ' version of what is denoted in Proposition 2.2 by $|\hat{a}_\diamondsuit|$. Use $|\hat{a}_\diamondsuit|$ to denote the original version that is supplied by Λ. Proposition 6.1 asserts that $|\hat{a}'_\diamondsuit|$ is also a Hölder continuous function on the complement of its zero locus, and locally Hölder continuous on its zero locus. It then follows that $|\hat{a}'_\diamondsuit| = |\hat{a}_\diamondsuit|$ because both are Hölder continuous and because they define the same L^2 function. Since $p \in M-Z$, it follows that $|\hat{a}'_\diamondsuit|(p) > 0$ and so Λ' has a subsequence, this denoted by Λ_p, with $\lim_{n\in\Lambda_p} |\hat{\mathfrak{a}}_n|(p) = |\hat{a}'_\diamondsuit|(p)$. Invoke Lemma 6.2 using the point p and Λ_p to conclude that $\{r_{\diamondsuit n,p}\}_{n\in\Lambda_p}$ is bounded away from zero. But this is nonsense because $\Lambda_p \subset \Lambda'$ and Λ' was chosen so $\lim_{n\in\Lambda'} r_{\diamondsuit n,p} = 0$.

Step 2. Granted (7.2), then the constructions of Uhlenbeck in [U] can be employed to obtain a subsequence $\Lambda' \subset \Lambda$ and a sequence $\{h_n\}_{n\in\Lambda'}$ of automorphisms of $P|_{M-Z}$ such that the sequence $\{{h_n}^*A_n\}_{n\in\Lambda'}$ has the properties asserted by the second bullet of Lemma 7.3. The $L^2_{1;loc}$ limit connection can be taken to be $A_\diamondsuit$.

Proposition 2.2 implies in part that the sequence $\{\nabla_{A_n}\hat{a}_n\}_{n\in\Lambda'}$ is bounded, and this with the first bullet of (3.10) implies that the sequence $\{{h_n}^*\hat{a}_n\}_{n\in\Lambda'}$ is bounded in the L^2_1 topology on compact subsets of M−Z. As the sequence $\{\mathfrak{q}_{A_n}(\hat{a}_n)\}_{n\in\Lambda}$ is also bounded, (7.2) with the same sort of integration by parts argument that is used to prove the fourth bullet of Proposition 4.1 proves that $\{\nabla_{A_n}(\nabla_{A_n}\hat{a}_n)\}_{n\in\Lambda'}$ has bounded L^2 norm on compact subsets of M−Z. It follows that (3.10) can be used again to see that $\{{h_n}^*\hat{a}_n\}_{n\in\Lambda'}$ has bounded L^2_2 norm on compact subsets of M−Z. This implies in particular that Λ' has a subsequence, this being Λ_*, such that $\{{h_n}^*\hat{a}_n\}_{n\in\Lambda_*}$ converges weakly in the $L^2_{2;loc}$ topology on M−Z. Use $\mathfrak{a}_\diamondsuit$ to denote the limit.

Step 3. What is said in (7.2) has the following implication: Suppose that U is a compact set in M−Z. Then there exists $r_U > 0$ such that $\inf_{p\in U}\{\inf_{n\in\Lambda'}\{r_{\diamondsuit n,p}\}\} > r_U$. This being the case, the argument in Part 2 of the proof of Lemma 6.3 can be used to prove that $|\mathfrak{a}_\diamondsuit| = |\hat{a}_\diamondsuit|$. Item c) of Lemma 7.2 with what is said in Step 2 about convergence implies that $d_{A_\diamondsuit}\hat{a}_\diamondsuit = 0$ and $d_{A_\diamondsuit}{*}\hat{a}_\diamondsuit = 0$ and $\hat{a}_\diamondsuit \wedge \hat{a}_\diamondsuit = 0$.

7.b. The fourth bullet of Proposition 7.1

The eight parts of this subsection prove the fourth bullet of Proposition 7.1. The three items are proved in more or less reverse order.

Part 1: Fix $\rho > 0$ and let $Z_\rho \subset M$ again denote the set where $|\nu| < \rho$. The following observation is invoked repeatedly in subsequent arguments in this section and in later sections.

$$\lim_{\rho\to 0}\int_{Z_\rho} |\nabla\nu|^2 = 0. \tag{7.3}$$

By way of an explanation, the integrand is a measurable function with support on Z_ρ−Z. Meanwhile, Z is a closed set and so the function of ρ given by the volume of Z_ρ−Z has limit zero as ρ limits to zero.

With $\rho > 0$ given, the subsequent arguments refer to the function $\chi_\rho = \chi(2-\rho^{-1}|\nu|)$, this being a function on M that equals 0 on Z_ρ and 1 on $\mathrm{M}-\mathrm{Z}_{2\rho}$. It is introduced to avoid certain delicate issues with regard to derivatives of ν near Z.

Define $\mathrm{h}_{(\rho)}$ and $\mathrm{H}_{(\rho)}$ to be the functions

$$\mathrm{h}_{(\rho)} = \int_{\partial \mathrm{B}_\mathrm{r}} \chi_\rho |\nu|^2 \quad \textit{and} \quad \mathrm{H}_{(\rho)} = \int_{\mathrm{B}_\mathrm{r}} \chi_\rho |\nabla \nu|^2 \tag{7.4}$$

It follows from the definitions that $\mathrm{h}_{(\rho)}(\mathrm{r}) \leq \mathrm{h}(\mathrm{r}) \leq \mathrm{h}_{(\rho)}(\mathrm{r}) + \mathrm{c}_0\mathrm{r}^2\rho^2$. Meanwhile, $\mathrm{H}_{(\rho)}(\mathrm{r})$ is no greater than $\mathrm{H}(\mathrm{r})$, and (7.3) implies that $\lim_{\rho\to 0} |\mathrm{H}(\mathrm{r}) - \mathrm{H}_{(\rho)}(\mathrm{r})| = 0$ with the limit being uniform in the following sense: Given $\varepsilon > 0$, there exists $\rho_\varepsilon > 0$ such that if $\mathrm{r} \in [0, \mathrm{c}_0^{-1}]$ and $\rho < \rho_\varepsilon$, then $|\mathrm{H}(\mathrm{r}) - \mathrm{H}_\rho(\mathrm{r})| < \varepsilon$.

Part 2: The lemma stated and then proved below asserts in part that h is non zero on $(0, \mathrm{c}_0^{-1}]$.

Lemma 7.3. *There exists $\kappa > 1$ with the following significance: Fix $\mathrm{p} \in \mathrm{M}$ so as to define the function* h. *If $\mathrm{r} \in (0, \kappa^{-1})$, then $\int_{\mathrm{B}_\mathrm{r}} |\nu|^2 \leq (1-\kappa\mathrm{r}^2)\mathrm{r}\mathrm{h}(\mathrm{r})$.*

Proof of Lemma 7.3. Use Q to denote the symmetric section of the tensor bundle $\otimes_2 \mathrm{T}^*\mathrm{M}$ given by $\mathrm{Q} = \nu \otimes \nu - \frac{1}{2}\mathfrak{m}|\nu|^2$. The fact that ν is closed and coclosed implies that Q is divergence free. This means that it has vanishing L^2 inner product with the covariant derivatives of 1-forms. Note that Q is an L^2_1 section of $\otimes_2 \mathrm{T}^*\mathrm{M}$ that is smooth on M−Z and continuous and locally Hölder continuous across Z. The fact that Q is smooth on M−Z follows from the fact that ν is harmonic on M−Z and thus smooth. The fact that Q is locally Hölder continuous across Z follows from what is said by Proposition 6.1.

Fix a Gaussian coordinate system centered at p, and use the coordinate differentials as a basis for $\mathrm{T}^*\mathrm{M}$ and the coordinate vector fields as a basis for TM. Integrate the inner product between $\nabla \mathrm{d}(|\mathrm{x}|^2)$ and Q over the ball B_r. Use the fact that Q is divergence free and that $|\nu|$ is continuous and zero on Z with an integrate by parts to identify the latter integral with a boundary term. The resulting identity can be written as

$$\mathrm{r}\int_{\partial \mathrm{B}_\mathrm{r}} \left(|\nu_\mathrm{r}|^2 - \tfrac{1}{2}|\nu|^2\right) + \tfrac{1}{2}\int_{\mathrm{B}_\mathrm{r}} |\nu|^2 = \mathfrak{d}(\mathrm{r}), \tag{7.5}$$

where ν_r denotes the inner product between $\mathrm{d}|\mathrm{x}|$ and ν and where $|\mathfrak{d}(\mathrm{r})| \leq \mathrm{c}_0\mathrm{r}^2 \int_{\mathrm{B}_\mathrm{r}} |\nu|^2$. Granted this bound on $|\mathfrak{d}|$, the identity in (7.5) implies what is asserted by Lemma 7.3.

Part 3: Lemma 7.3 has the following consequence: If $r \in (0, c_0^{-1})$ and if $h(r) = 0$, then $h(s) = 0$ for all $s \le r$. This understood, let $\mathrm{D} \in [0, c_0^{-1}]$ denote the maximum value of r where h(r) is zero. The function N is defined on (D, c_0^{-1}). The rest of this Part 3 and all of Parts 4–6 prove Item c) from Proposition 7.1's fourth bullet for $s > r$ with $r > \mathrm{D}$.

The proof of Item c) for when s and r are greater than D will appeal to the lemma that follows directly and the upcoming Lemma 7.5.

Lemma 7.4. *There exists $\kappa \ge 1$ with the following significance. Let Λ' denote the subsequence from Lemma 7.2. Fix $\rho > 0$. If $n \in \Lambda$ is sufficiently large, then*

$$\int_{Z_\rho} (|\nabla_{A_n} \hat{a}_n|^2 + r_n^2 |\hat{a}_n \wedge \hat{a}_n|^2) < \kappa \rho^{1/\kappa}.$$

Proof of Lemma 7.4. Fix for the moment $s \in (0, c_0^{-1})$ and introduce the function $\chi_{n,s}$ on M that is given by $\chi(s^{-1}|\hat{a}_n| - 1)$. This function is equal to 1 where $|\hat{a}_n| < \frac{5}{4}s$ and it is equal to zero where $|\hat{a}_n| \ge \frac{7}{4}s$. Take $f = \chi_{n,s}$ in the $(A_n, \hat{a}_n)$ version of (2.2) to obtain an equation with the form

$$(7.6) \qquad -\int_M \chi'_{n,s} |\hat{a}_n| |d|\hat{a}_n||^2 + \int_M \chi_{n,s} (|\nabla_{A_n} \hat{a}_n|^2 + r_n^2 |\hat{a}_n \wedge \hat{a}_n|^2) = \mathfrak{e}_n(s)$$

where $|\mathfrak{e}_n(s)| \le c_0(r_n^{-1} + s^2)$.

Fix $\rho < c_0^{-1}$ and let N_ρ denote the smallest integer with the following property: If $n \in \Lambda$ and $n \ge N_\rho$, then

$$(7.7) \qquad r_n > \rho^{-2} \quad \textit{and} \quad \tfrac{7}{8}|\nu| \le |a_n| \le \tfrac{9}{8}|\nu| \quad \textit{where } |\nu| > \tfrac{1}{8}\rho.$$

If $n \ge N_\rho$ and $s \ge \rho$, then (7.6) implies the inequality

$$(7.8) \quad \int_{Z_s} (|\nabla_{A_n} \hat{a}_n|^2 + r_n^2 |\hat{a}_n \wedge \hat{a}_n|^2) \le c_0 \int_{Z_{2s} - Z_s} (|\nabla_{A_n} \hat{a}_n|^2 + r_n^2 |\hat{a}_n \wedge \hat{a}_n|^2) + c_0 s^2).$$

The integral on the left in (7.8) is the difference between the respective integrals of its integrand over Z_{2s} and Z_s. Use this fact to rewrite (7.8) so as to read

(7.9)

$$\int_{Z_s} (|\nabla_{A_n} \hat{a}_n|^2 + r_n^2 |\hat{a}_n \wedge \hat{a}_n|^2) \le (1 - c_0^{-1}) \int_{Z_{2s}} (|\nabla_{A_n} \hat{a}_n|^2 + r_n^2 |\hat{a}_n \wedge \hat{a}_n|^2) + c_0 s^2.$$

Given $k \in \{0, 1, \dots\}$ but less than $\frac{1}{\ln 2} |\ln \rho| - c_0$ introduce χ_k to denote the $s = 2^{-k} c_0^{-1}$ version of the integral on the left hand side of (7.9). With this notation understood, then (7.9) asserts that

$$\varkappa_k \leq (1 - c_0^{-1})\varkappa_{k-1} + c_0 2^{-2k}. \tag{7.10}$$

Iterating this finds $\varkappa_k \leq (1 - c_0^{-1})^k(\varkappa_0 + c_0)$. This implies that

$$\int_{Z_\rho} (|\nabla_{A_n} \hat{a}_n|^2 + r_n^2 |\hat{a}_n \wedge \hat{a}_n|^2) \leq c_0 \rho^{k/c_0}. \tag{7.11}$$

Since (7.11) holds for all $n \geq N_\rho$, it leads directly to the claim made by Lemma 7.4.

The next lemma is also needed for the fourth bullet of the proposition:

Lemma 7.5. *Let Λ' denote the subsequence from Lemma 7.2. Given $\rho > 0$, then there exists $\kappa > 1$ with the following significance: If $n \in \Lambda'$, then $r_n^2 \int_{M - Z_\rho} |\hat{a}_n \wedge \hat{a}_n|^2 \leq \kappa r_n^{-2}$.*

Proof of Lemma 7.5. The integral of $|F_{A_n}|^2$ on $M - Z_\rho$ enjoys an n-independent upper bound and so this is also the case for the integral of $r_n^4 |\hat{a}_n \wedge \hat{a}_n|^2$ on $M - Z_\rho$.

Part 4: Given $n \in \Lambda'$, use h_n, $\mathcal{h}_n$ and N_n to denote the version of the functions $\mathcal{h}$ and N that are defined in (5.2) and (5.3) using the point p, $r = r_n$ and $(A, \hat{a}) = (A_n, \hat{a}_n)$. Suppose $\rho > 0$ has been specified. Fix $r > 0$ and let $h_{n(\rho)}(r)$ and $H_{n(\rho)}(r)$ denote the respective integrals

$$\begin{aligned} h_{n(\rho)}(r) &= \int_{\partial B_r} \chi_\rho |\hat{a}_n|^2 \quad \textit{and} \\ H_{n(\rho)}(r) &= \int_{B_r} \chi_\rho (|\nabla_{A_n} \hat{a}_n|^2 + r_n^2 |\hat{a}_n \wedge \hat{a}_n|^2). \end{aligned} \tag{7.12}$$

These are such that

(7.13)
- $h_{n(\rho)} \leq h_n$ *for all* $n \in \Lambda$ *and* $h_{n(\rho)} \geq h_n - c_0 r^2 \rho^2$ *if* $n \in \Lambda$ *is sufficiently large.*
- $H_{n(\rho)} \leq H_n$ *for all* $n \in \Lambda$ *and* $H_{n(\rho)} \geq H_n - c_0 \rho^{1/c_0}$ *if* $n \in \Lambda$ *is sufficiently large.*

By way of an explanation, the lower bound for $h_{n(\rho)}$ follows from the fact that $\{|\hat{a}_n|\}_{n \in \Lambda}$ converges to $|\nu|$, and that for $H_{n(\rho)}$ follows from Lemma 7.4.

Fix $r \in (D, c_0^{-1})$. It follows from what is said by (7.13) and Lemma 7.5 by taking limits first as $n \in \Lambda'$ gets ever larger and then as ρ limits to zero that

$$N(r) = \lim_{n\in\Lambda'} N_n(r), \tag{7.14}$$

and that this limit is uniform as r various on compact subsets of (D, c_0^{-1}).

Part 5: This part of the subsection invokes some of what is said in Sections 5.c and 5.d to derive the $r > D$ version of Item c) of the fourth bullet from a lemma that is proved in Part 6. The starting point is the $r = r_n$ and $(A, \hat{a}) = (A_n, \hat{a}_n)$ version of (5.30). Keep in mind that what is denoted in (5.30) by $\mathfrak{r}$ is discussed subsequent to (5.11). It is enough to know that $|\mathfrak{r}| < c_0 r$. What is denoted by $\mathfrak{z}$ is defined subsequent to (5.28). A bound for $|\mathfrak{z}|$ is supplied by (5.32), but the latter is not sufficient for the purposes at hand. The upcoming Lemma 7.6 says what is needed about $\mathfrak{z}$'s absolute value. There is also a term in (5.30) that is denoted by $\mathfrak{R}$. This term enters via the second bullet of Lemma 5.2. A bound for $|\mathfrak{R}|$ is supplied by (5.31), but a stronger bound is needed and Lemma 7.6 provides one.

Lemma 7.6. *There exists $\kappa > 1$, and given $\varepsilon \in (0, 1]$, there exists $\kappa_\varepsilon > \kappa$ with the following significance: Let Λ' denote the subsequence from Lemma 7.2. If $n \in \Lambda'$ is greater than κ_ε, then the absolute values of the $r = r_n$ and $(A, \hat{a}) = (A_n, \hat{a}_n)$ versions of $\mathfrak{z}$ and $\mathfrak{R}$ at points $r \in [\varepsilon, \kappa^{-1})$ are such that $|\mathfrak{z}| < \varepsilon + \kappa r h_n$ and $|\mathfrak{R}| < \varepsilon + \kappa(h_n + rH_n)$.*

This lemma is proved in Part 6.

Accept Lemma 7.6 for the moment to complete the proof of the $s > D$ version of Item c) from Proposition 7.1's fourth bullet. To do this, fix first $r_* > D$ and suppose that r is greater than r_*. It then follows from Lemma 7.3 that $h(r) > m_*^{-1}$ with $m_* > 1$ a number that depends on r_* but not on r. Fix ε positive but less than the smaller of $c_0^{-1} m_*^{-1}$ and r_*. If n is large, then $h_n(r)$ will differ by at most ε^2 from the integral of $|\nu|^2$ over ∂B_r, and so $h_n(r)$ will be greater than $c_0^{-1} m_*^{-1}$.

With the preceding in mind, use Lemma 7.6 with (5.30) to conclude that

$$\frac{d}{dr} N_n \geq -c_0\left(r + \frac{r}{h_n}\varepsilon\right)(1 + N_n) \text{ when } \max(D, \varepsilon) < r \leq c_0^{-1} \text{ and } n \text{ is large.} \tag{7.15}$$

Since $h_n(r) \geq c_0^{-1} m_*^{-1}$, the inequality (7.15) leads to the bound

$$\frac{d}{dr} N_n \geq -c_0(r + m_*\varepsilon)(1 + N_n) \text{ when } \max(D, \varepsilon) < r \leq c_0^{-1} \text{ and } n \text{ is large.} \tag{7.16}$$

Suppose now that ε is greater than m_*^{-2}. If both s and r are greater than $\max(\mathrm{D}, \varepsilon)$ and less than c_0^{-1}, and if $s > r$, then integrating (7.17) finds that

$$\text{(7.17)} \quad N_n(s) > e^{-c_0(s^2 - r^2 + \varepsilon^{1/2})} N_n(r) - c_0(s^2 - r^2 + \varepsilon^{1/2}) \text{ when n is large.}$$

Proposition 7.1's fourth bullet for $r > \mathrm{D}$ follows directly from (7.14) and (7.17).

Part 6: This part of the subsection contains the proof of Lemma 7.6.

Proof of Lemma 7.6. The proof has four steps.

Step 1. What is said about $\mathfrak{z}$ subsequent to (5.28) finds

$$\text{(7.18)} \qquad |\mathfrak{z}| \le c_E\left(|H - \hbar| + \int_{B_r} |\hat{a}|^2 + \left|\int_{B_r} \langle \hat{a} \wedge *q_A(\hat{a})\rangle\right|\right),$$

with $c_E > 1$ denoting here and in what follows, a number that depends only on the value of E in (3.2) and whose value can increase between successive appearances. The absolute value of $H - \hbar$ is no greater than $c_E r^{-1}$, this being the content of (5.12). Meanwhile, the integral that involves $q_A(\hat{a})$ is bounded by $c_E r^{-1/2}$, this being a consequence of (5.16). If the integer $n \in \Lambda'$ is large, then $r_n^{-1/2}$ will be less than $c_E^{-1}\varepsilon^2$ and in particular, the left most and right most terms in the $r = r_n$ and $(A, \hat{a}) = (A_n, \hat{a}_n)$ version of (7.16) will make a contribution to $|\mathfrak{z}|$ that is less than $\frac{1}{2}\varepsilon^2$. Meanwhile, the integral of $|\hat{a}_n|^2$ over any $r > \varepsilon$ version of B_r differs from that $|\nu|^2$ by at most $c_E^{-1}\varepsilon^2$ if n is large, and Lemma 7.5 asserts that the latter integral is no greater than $r(1 - c_0 r^2)$ times that of $|\nu|^2$ over the boundary of B_r. This integral of $|\nu|^2$ will differ from $r(1 - c_0 r^2)h_n(r)$ by at most $c_E^{-1}\varepsilon^2$ when $r \ge \varepsilon$ if n is large. It follows as a consequence that $|\mathfrak{z}| \le \varepsilon + c_E r h_n$ when $r \ge \varepsilon$ and n is large.

Step 2. Reintroduce the notation from Step 1 of the proof of Lemma 5.2. The idnentities in (5.6)–(5.8) lead to a bound on $|\mathfrak{R}|$ that can be written as

$$\text{(7.19)} \quad |\mathfrak{R}| \le c_0\left|\int_{B_r} \langle (\nabla_{A,i}\hat{a})_k q_A(\hat{a})_k\rangle \partial_i \Delta\right| + c_E\left(h + rH + \int_{B_r} |\hat{a}|^2 + r^{-1/2}\right).$$

What is denoted by Δ designates the function $\mathrm{dist}(\cdot, p)^2$ and $\partial_i\Delta$ designates the directional derivative of the function Δ along the i'th basis vector. Remarks in Step 1 imply the following: If $r \in (\varepsilon, c_0^{-1})$ and n is large, then the

$r = r_n$ and $(A, \hat{a}) = (A_n, \hat{a}_n)$ version of the right hand side of (7.19) is no greater than

$$c_0 \left| \int_{B_r} \langle (\nabla_{A,i}\hat{a})_k \mathfrak{q}_A(\hat{a})_k \rangle \partial_i \Delta \right| + c_E(h + rH) + c_E^{-1}\varepsilon^2. \tag{7.20}$$

Fix $\rho > 0$ and use the identity $1 = (1 - \chi_\rho) + \chi_\rho$ to break the integral in (7.20) into two parts, the first having the factor $(1 - \chi_\rho)$ in the integrand and the second having the factor χ_ρ. The next step supplies bounds for the absolute values of these two integrals.

Step 3. Use Lemma 7.4 with Item d) in Proposition 2.2 to see that the norm of

$$\int_{B_r} (1 - \chi_\rho) \left\langle (\nabla_{A_n,i}\hat{a}_n)_k \mathfrak{q}_{A_n}(\hat{a}_n)_k \right\rangle \partial_i \Delta \tag{7.21}$$

is no greater than $c_0 r \rho^{1/c_0}$ when n is sufficiently large.

The χ_ρ part of the integral in (7.20) is

$$\int_{B_r} \chi_\rho \left\langle (\nabla_{A_n,i}\hat{a}_n)_k \mathfrak{q}_{A_n}(\hat{a}_n)_k \right\rangle \partial_i \Delta. \tag{7.22}$$

To bound the absolute value of (7.22), fix $\mu \in (0, \frac{1}{64})$ for the moment and introduce by way of notation σ_μ to denote the function on B_r given by $\chi(2 - \mu^{-1}(1 - r^{-1}\mathrm{dist}(\cdot, p)))$. This function equals 0 where the distance to p is greater than $(1 - \mu)r$ and it equals 1 where the distance to p is less than $1 - 2\mu$. Insert the identity $1 = (1 - \sigma_\mu) + \sigma_\mu$ into the integrand in (7.22) to write it as a sum of two terms.

The absolute value of the term with the factor $1 - \sigma_\mu$ is no greater than

$$c_0 r \left(\int_{1-2\mu<\mathrm{dist}(\cdot,p)<1} \chi_\rho |\nabla_{A_n}\hat{a}_n|^2 \right)^{1/2}, \tag{7.23}$$

this being a consequence of Item d) of the second bullet of Proposition 2.2. Meanwhile, the integral in (7.23) is no greater than $c_0 r^{1/2}\mu^{1/4}$ times the L^4 norm of $\nabla_{A_n}\hat{a}_n$ over B_r. The latter norm has a ρ dependent but n independent upper bound, this being a consequence of the first bullet of (3.10) and the second bullet of Lemma 7.2. This understood, it then follows that the contribution to the absolute value of the integral in (7.22) from the term with $1 - \sigma_\mu$ is no greater than $c_0 r^{3/2}\mu K_\rho$ with K_ρ being a number that is determined by ρ but independent of μ and n.

The contribution to the integral of (7.22) from the term with σ_μ is the integral over B_r of $\sigma_\mu \chi_\rho \langle (\nabla_{A_n,i}\hat{a}_n)_j \mathcal{q}_{A_n}(\hat{a}_n)_j \rangle \partial_i \Delta$. A bound for the absolute value of this integral is obtained by writing

$$\mathcal{q}_{A_n}(\hat{a}_n) = *d_{A_n}(*d_{A_n}\hat{a}_n) - d_{A_n}(*d_{A_n}*\hat{a}_n), \tag{7.24}$$

and then integrating by parts to make an integrand which has terms that are linear in the components of $d_{A_n}\hat{a}_n$ and $d_{A_n}*\hat{a}_n$. Use Item c) of Proposition 2.2 and the L^2_2 bound in the second bullet of Lemma 7.2 to see from the resulting integral that the absolute value of the σ_μ contribution to (7.22) is bounded by $K_\rho \mu^{-1} r_n^{-1}$ with K_ρ denoting again a number that is determined by ρ but is independent of both μ and n.

Step 4. By way of a summary, what is said in Steps 1–3 bound the absolute value of the explicit integral in (7.20) by

$$c_0 \rho^{1/c_0} + K_\rho(\mu^{1/4} + \mu^{-1} r_n^{-1}) \tag{7.25}$$

With ε given, first choose $\rho < \varepsilon^{c_0}$ so that the left most term in (7.25) is less than $\frac{1}{3}\varepsilon^2$. With ρ so chosen, choose a postive value of μ, but sufficiently small so as to make the middle term in (7.25) less than $\frac{1}{3}\varepsilon^2$ also. With ρ and μ fixed, the right most term in (7.25) is less than $\frac{1}{3}\varepsilon^2$ when n is sufficiently large. Granted these bounds, then (7.20) leads directly to the bound asserted by Lemma 7.6.

Part 7: This part of the subsection derives a differential equation for h that is the same as that given in Item b) of Proposition 7.1 where $r > D$. To set the notation, let m denote the metric inner product on T^*M. Use ∂_r to denote the derivative along the radial geodesics from p in B_r. Fix $\rho \in (0, c_0^{-1})$ and differentiate to obtain the identity

$$\frac{d}{dr}\int_{\partial B_r} \chi_\rho |\nu|^2 = 2r^{-1}(1 + \mathfrak{e}_1)\int_{\partial B_r} \chi_\rho |\nu|^2 + 2\int_{\partial B_r} \chi_\rho \mathfrak{m}(\nu, \partial_r \nu) \tag{7.26}$$

with $\mathfrak{e}_1$ being bounded by $c_0 r^2$. Integrate by parts in the second integral using the fact that ν is harmonic to write it as

$$\int_{\partial B_r} \chi_\rho \mathfrak{m}(\nu, \partial_r \nu) = \int_{B_r} \chi_\rho(|\nabla \nu|^2 + \mathrm{Ric}(\nu, \nu)) + \mathfrak{e}_2. \tag{7.27}$$

where $\mathfrak{e}_2$ is a term with absolute value obeying

$$|\mathfrak{e}_2| \le c_0 \rho^{-1} \int_{B_r \cap Z_{2\rho}} |\nu||\nabla \nu|^2 \tag{7.28}$$

Given that $|\nu| < 2\rho$ on $Z_{2\rho}$, this bound for $|\mathfrak{e}_2|$ implies directly that

$$|\mathfrak{e}_2| \leq c_0 \int_{Z_{2\rho}} |\nabla\nu|^2. \tag{7.29}$$

As noted in (7.3), the $\rho \to 0$ limit in (7.29) is zero.

Fix $r > 0$ and $\varepsilon > 0$; then integrate (7.26) and use (7.27) to see that

$$\begin{aligned} &\int_{\partial B_{r+\varepsilon}} \chi_\rho|\nu|^2 - \int_{\partial B_r} \chi_\rho|\nu|^2 \\ &= \int_r^{r+\varepsilon} \left(s^{-1} \int_{\partial B_s} (1+\mathfrak{e}_1)\chi_\rho|\nu|^2 + \int_{B_s} \chi_\rho(|\nabla\nu|^2 + \mathrm{Ric}(\nu,\nu)) + \mathfrak{e}_2 \right) ds. \end{aligned} \tag{7.30}$$

Take ρ to zero on both sides of (7.30). What is said subsequent to (7.4) and what is said in (7.3) imply that the $\rho \to 0$ limit of (7.30) is the identity

$$h(r+\varepsilon) - h(r) = \int_r^{r+\varepsilon} \left(s^{-1} \int_{\partial B_s} (1+\mathfrak{e}_1)|\nu|^2 + \int_{B_s} (|\nabla\nu|^2 + \mathrm{Ric}(\nu,\nu)) \right) ds. \tag{7.31}$$

Divide both sides of (7.31) by ε and take the $\varepsilon \to 0$ limit. As the $\varepsilon \to 0$ limit of the right hand side exists, the result of taking the $\varepsilon \to 0$ limit is an identity for the derivative of h that reads

$$\tfrac{d}{dr}h = 2r^{-1}(1+\mathfrak{e}_*)h + H + \int_{B_r} \mathrm{Ric}(\nu,\nu) \tag{7.32}$$

with $\mathfrak{e}_*$ being a function of r obeying $|\mathfrak{e}_*| \leq c_0 r^2$.

The integral of $\mathrm{Ric}(\nu,\nu)$ that appears in (7.32) is bounded by c_0 times the integral of $|\nu|^2$ over B_r. This understood, use Lemma 7.3 to write (7.32) schematically as

$$\tfrac{d}{dr}h = 2r^{-1}(1+\mathfrak{e})h + H, \tag{7.33}$$

where $\mathfrak{e}$ is a function of r that obeys $|\mathfrak{e}| \leq c_0 r^2$. The equation in (7.33) is the equation in Item b) of Proposition 7.1's fourth bullet at values of $r > \mathrm{D}$, this because the definition of N can be invoked to write H as $H = r^{-1}\mathrm{N}h$.

Part 8: This step proves the assertion in Item a) of Proposition 7.1's fourth bullet to the effect that any given $p \in M$ version of h is positive on the whole of $(0, c_0^{-1})$. This is done by assuming that there exists $p \in M$ where the corresponding version of D is positive and deriving nonsense.

To set the stage for the derivation, note that if D is positive, then Z contains an open set. If such is the case, then there exists, for any $\varepsilon > 0$, a point $p \in Z$ with D positive but less than ε. In particular, there exists a point $p \in M$ with D positive, with h defined on $(0, r_0)$ with $r_0 > c_0^{-1}$ and with $h(\frac{1}{2}r_0) > 0$. Let $p \in M$ denote such a point.

Write H in (7.33) on the $r > \mathrm{D}$ part of $(0, r_0)$ as $r^{-1}\mathrm{N}h$. Fix $s > r > \mathrm{D}$ with both in $(0, r_0)$ and integrate this rewriting of (7.33) to see that

$$h(s) = (1 + \mathfrak{d})\left(\frac{s}{r}\right)^2 \exp\left(2\int_r^s \frac{1}{t}\mathrm{N}(t)dt\right)h(r), \tag{7.34}$$

where $\mathfrak{d}(s)$ is such that $|\mathfrak{d}| \le c_0 s^2$ and $|\frac{d}{ds}\mathfrak{d}| \le c_0 s$. As the $r \to \mathrm{D}$ limit of $h(r)$ is zero and as D is positive, the identity in (7.34) implies that the function N can not be a bounded function on $(\mathrm{D}, \frac{1}{2}r_0]$. This understood, fix for the moment $m > 1$ and some $r_m \in (\mathrm{D}, \frac{1}{2}r_0]$ with $\mathrm{N}(r_m) > m$. The $r > \mathrm{D}$ version of Item c) of the fourth bullet in Proposition 7.1 says that $\mathrm{N}(\frac{1}{2}r_0) \ge c_0^{-1}m$. Since m can be as large as desired, this constitutes the desired nonsense. The reason it is nonsense is as follows: By assumption, $h(\frac{1}{2}r_0) > 0$. Since H is bounded and since $\mathrm{N} = rH/h$, so $\mathrm{N}(\frac{1}{2}r_0)$ is finite, and thus less than m if m is sufficiently large.

7.c. The $r \to 0$ limit of N

Given a point $p \in M$, let $\mathrm{N}_{(p)}$ denote p's version of the function N. The upcoming Lemma 7.7 concerns the behavior of the $r \to 0$ limit of $\mathrm{N}_p(r)$ as p varies in Z.

Lemma 7.7. *There exists $\kappa \ge 1$ with the following significance: Any given $p \in M$ version of $\mathrm{N}_{(p)}$ extends to $[0, \kappa]$ as a continuous function. Moreover,*

- *If $\{q_k\}_{k=1,2,\ldots} \subset M$ converges to a given point $p \in M$ then $\lim_{k\to\infty} \mathrm{N}_{(q_k)}(0) \le \mathrm{N}_{(p)}(0)$.*
- *If $p \in M{-}Z$, then $\mathrm{N}_{(p)}(0) = 0$; and if $p \in Z$, then $\mathrm{N}_{(p)}(0) > \kappa^{-1}$.*

The remainder of this subsection contains the proof of this lemma.

Proof of Lemma 7.7. The proof has four parts.

Part 1: To see that a given $p \in M$ version of $\mathrm{N}_{(p)}$ is continuous on its original domain of definition, keep in mind that p's version of the function h is differentiable, this being an assertion of Proposition 7.1. This understood, it follows that $\mathrm{N}_{(p)}$ is continuous on $(0, c_0^{-1})$ if and only if p's version of the function H is continuous. The fact that H is continuous follows from what is said in Part 1 of Section 7.b about H and (7.4)'s function $H_{(\rho)}$. To say more, fix $\varepsilon > 0$ and use what is said at the end of Part 1 of Section 7.b to find $\rho_\varepsilon > 0$ such that $|H(\cdot) - H_{(\rho)}(\cdot)| < \varepsilon$ when $\rho < \rho_\varepsilon$. Fix such a value for ρ. Since ν is smooth on M−Z, the function $H_{(\rho)}$ is smooth. This understood fix $r \in (0, c_0^{-1}]$ and then fix $\Delta > 0$ so that $|H_{(\rho)}(r + \Delta) - H_{(\rho)}(r)| < \varepsilon$. It then follows that $|H(r + \Delta) - H(r)| < 3\varepsilon$ and so H is continuous.

The assertion that $N_{(p)}$ extends continuously to $[0, c_0^{-1}]$ follows from Item c) of the fourth bullet of Proposition 7.1 given the fact that $N_{(p)}$ is positive.

Part 2: The proof of the bulleted assertions requires a weak version of what is said by the lemma's first bullet, this being the following:

(7.35) *If* $p \in M{-}Z$, *then* $N_{(p)}(0) = 0$; *and if* $p \in Z$, *then* $N_{(p)}(0) > 0$.

The proof of (7.35) has two steps.

Step 1. If $p \in M{-}Z$, then $|\nu|^2$ is greater than zero at p and smooth in a neighborhood of p. This being the case, p's version of the function h must have the form $4\pi r^2 |\nu|^2(p) + \mathfrak{e}$ with $\mathfrak{e}$ being a function of r with absolute value bounded by $c_0 r^3$. With this in mind, write $N_{(p)}$ near $r = 0$ as $N_{(p)}(0) + \mathfrak{o}$ with $\lim_{r\to 0} \mathfrak{o}(r) = 0$. Suppose for the sake of argument that $\varepsilon > 0$ and that $N_{(p)}(0)$ is greater than ε so as to derive nonsense. To do this, fix $s > 0$ and use (7.34) to see that $h(r) \le c(s) r^{2+\varepsilon/2}$ when r is small.

Step 2. Suppose that $p \in Z$ and suppose for the sake of argument that $N_{(p)}(0) = 0$ so as to derive nonsense. To start, introduce by way of notation c to denote the version of the constant κ that is assigned to p by Proposition 6.1. It follows from Proposition 6.1 that p's version of the function h is such that $h(r) \le c r^{2+2/c}$ when $r \le c^{-1}$. Hold onto this bound for a moment.

Use the fact that $N_{(p)}$ is continuous on $[0, c_0^{-1}]$ and 0 at $r = 0$ to draw the following conclusion: Given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that $N_{(p)}(r) < \varepsilon$ when $r < 2r_\varepsilon$. Granted this bound, use the $s = r_\varepsilon$ version of (7.34) to see that $h(r) \ge c_\varepsilon^{-1} r^{2+2\varepsilon}$ when $r < r_\varepsilon$ with $c_\varepsilon > 1$ being a number that depends on ε but not on r.

The lower bound $h(r) \ge c_\varepsilon^{-1} r^{2+2\varepsilon}$ runs afoul of the upper bound $c r^{2+2/c}$ when $\varepsilon > 1/c$. The fact that the lower bound holds for all nonzero ε constitutes the desired nonsense.

Part 3: The three steps that follow prove Lemma 7.7's assertion about the behavior of the function $p \to N_{(p)}(0)$. The notation used below has $h_{(p)}$ and $H_{(p)}$ denoting a given $p \in M$ version of the functions h and H.

Step 1. Granted that $N_{(p)}(0) = 0$ when $p \in M{-}Z$, then what is said by Lemma 7.7 about the semi-continuity $N_{(\cdot)}(0)$ at a point $p \in M{-}Z$ follows directly from the fact that $M{-}Z$ is an open set.

Step 2. Fix $p \in Z$ and $D > 0$. Let $q \in Z$ denote a point with distance D or less from p. If $r \in (c_0 D, c_0^{-1})$, then the sphere of radius r centered at q is contained in the ball of radius $r + D$ centered at p and it contains the ball of radius $r - D$ centered at p. This understood, an application of the fundamental theorem of calculus finds

$$|h_{(q)}(r) - h_{(p)}(r)| \le c_0 (H_{(p)}(r + D))^{1/2} r D^{1/2}. \tag{7.36}$$

Use the third bullet of Proposition 7.1 to bound $H_{(p)}(r{+}D)$ by $c_0(r{+}D)^{1/2}$ and use this bound with (7.35) to conclude that $|h_{(q)}(r) - h_{(p)}(r)| \le c_0 r^{3/2} D^{1/2}$ when $r \in (10D, c_0^{-1})$.

Step 3. The fact that $N_{(p)}$ is continuous on $[0, c_0^{-1})$ has the following consequence: Fix $\varepsilon > 0$ and there exists $r_\varepsilon \in (0, \varepsilon)$ such that $N_{(p)}(0) - \varepsilon \le N_{(p)}(r) \le N_{(p)}(0) + \varepsilon$ if $r \in (0, 2r_\varepsilon]$. This being the case, then (7.34) implies that $h_{(p)}(r)$ for $r \in [0, 2r_\varepsilon]$ can be written as

$$h_{(p)}(r) = c_\varepsilon r^{2(1+N_{(p)}(0)+\mathfrak{e})}, \tag{7.37}$$

where $c_\varepsilon > 0$ and where the function $\mathfrak{e}$ is such that $|\mathfrak{e}| \le c_0\varepsilon$. If $D < c_0^{-1} r_\varepsilon$, then this equation for $h_{(p)}$ with (7.36) implies that the function $h_{(q)}(r)$ for $r \in (10D, 2r_\varepsilon]$ can be written as

$$h_{(q)}(r) = c_\varepsilon r^{2(1+N_{(p)}(0)+\mathfrak{e})} + \mathfrak{r} \tag{7.38}$$

where the function $\mathfrak{r}$ is such that $|\mathfrak{r}| \le c_0 r^{3/2} D^{1/2}$. If it is the case that $D < c_0^{-1} c_\varepsilon^2 r_\varepsilon^{c_0}$, then (7.38) holds for $r \in [r_\varepsilon, 2r_\varepsilon]$ with $r = 0$ but with a different version of $\mathfrak{e}$ such that $|\mathfrak{e}| \le c_0\varepsilon$.

Step 4. Suppose that D obeys the bound $D \le c_0^{-1} c_\varepsilon^2 r_\varepsilon^{c_0}$. It follows from what is said at the end of Step 2 that if $s > r$ with both from $[r_\varepsilon, 2r_\varepsilon]$, then

$$h_{(q)}(r)/h_{(q)}(s) \ge \left(\frac{r}{s}\right)^{2(1+N_{(p)}(0))+c_0\varepsilon}. \tag{7.39}$$

Meanwhile, the point q has its corresponding version of Proposition 7.1's fourth bullet. Item c) of the q version implies that $N_{(q)}(t) > N_{(q)}(0) - c_0 r_\varepsilon^2$

for $t \in [0, 2r_\varepsilon]$. This bound with q's version of (7.34) implies that

$$h_{(q)}(r)/h_{(q)}(s) \leq \left(\frac{r}{s}\right)^{2(1+N_{(q)}(0))-c_0\varepsilon}. \tag{7.40}$$

These upper and lower bounds are not compatible unless $N_{(q)}(0) \leq N_{(p)}(0) + c_0\varepsilon$. Since ε can be made as small as desired by taking D sufficiently small, this last inequality proves Lemma 7.7's claim about the semi-continuity of $N_{(\cdot)}(0)$ at p.

Part 4: To prove Lemma 7.7's second bullet, fix $n \in \{1, 2, \ldots\}$ for the moment and denote by V_n the set $\{p \in Z : N_{(p)}(0) \geq \frac{1}{n}\}$. The first bullet of Lemma 7.7 implies that V_n is closed. Use the topology from M to view Z as a topological space. The set V_n is a closed subspace and let U_n denote its interior. If not empty, then U_n is an open subset of Z. It follows from (7.33) that each point in Z is contained in some set from the collection $\{U_n\}_{n\in\{1,2,\ldots\}}$ and so this collection defines an open cover of Z. As Z is compact, there is a finite subcover consisting of sets from the collection $\{U_n\}_{n\in\{1,2,\ldots\}}$. As a consequence, there exists $n \geq 1$ such that each $p \in Z$ version of $N_{(p)}(0)$ is greater than $\frac{1}{n}$.

8. Rescaling the 1-form ν

A data set $(Z, \mathcal{I}, \nu)$ with $Z \subset M$ being a closed set and $\mathcal{I} \to M{-}Z$ being a real line bundle and ν being a harmonic, $\mathcal{I}$-valued 1-form on $M{-}Z$ is said to be a *twisted harmonic form data set* when three conditions are met. The first condition requires that the function $|\nu|$ extend to M as a continuous function with $Z \subset |\nu|^{-1}(0)$; the second condition requires that the second, third and fourth bullets of Proposition 7.1 be obeyed; and the third condition requires that the conclusions of Lemma 7.7 hold. This section and Sections 9 and 10 assume that $(Z, \mathcal{I}, \nu)$ defines a twisted harmonic form. In particular, various lemmas and propositions that follow in Sections 8–10 do not ask that $(Z, \mathcal{I}, \nu)$ arise from a sequence as described in the second bullet of Proposition 2.2.

This section begins the analysis of $(Z, \mathcal{I}, \nu)$ by looking at ever smaller length scales on balls centered around a given point in Z. The results of this analysis are summarized by the upcoming Proposition 8.1. The proof of this proposition occupies the subsections of this section.

To set the stage and notation for Proposition 8.1, fix $R \in (0, \infty)$ for the moment and define the rescaling map $\psi_R : \mathbb{R}^3 \to \mathbb{R}^3$ by the rule $x \to \psi_R(x) = Rx$. A closed set in $\mathbb{R}^3$ is said to be scale invariant when it is mapped to itself by each $R > 0$ version of ψ_R. An example is the union of the origin with a finite set of rays from the origin. A real line bundle defined on the complement of a scale invariant set is canonically isomorphic to its pull-back by any $R > 0$ version of ψ_R.

Fix $r_0 > c_0^{-1}$ so that the radius $100r_0$ ball about any given point in M is well inside the domain of a Gaussian coordinate chart centered on the point in question. Given $p \in M$, a Gaussian coordinate chart centered at p and $\lambda \in (0, r_0]$, define ϕ_λ to be the map from the radius $\lambda^{-1}r_0$ ball about the origin in $\mathbb{R}^3$ to M that composes first ψ_λ and then the chosen Gaussian coordinate chart map.

Given $p \in Z$ and $\lambda \in (0, r_0]$, Proposition 8.1 use Z_λ to denote $\phi_\lambda^{-1}(Z)$ and it uses I_λ to denote $\phi_\lambda{}^* I$, these defined on the radius $\lambda^{-1}r_0$ ball centered on the origin in $\mathbb{R}^3$. Let $h_{(p)}$ denote p's version of the function h in Proposition 7.1. Proposition 8.1 uses ν_λ to denote $h_{(p)}(\lambda)^{-1/2}\phi_\lambda{}^*\nu$, this being an I_λ valued 1-form on the complement of Z_λ in $|x| < \lambda^{-1}r_0$ ball centered on the origin in $\mathbb{R}^3$ that extends over Z_λ as 0.

Proposition 8.1. *There exists $\kappa > 1$, and given $p \in Z$ plus a Gaussian coordinate chart centered at p, there exists a data set (Z_*, I_*, ν_*) with the properties listed below.*

- *Z_* is the union of the origin in $\mathbb{R}^3$ and a finite set of rays from the origin.*
- *I_* is a real line bundle defined on the complement of Z_* in $\mathbb{R}^3$.*
- *ν_* is an I_* valued, harmonic 1-form on $\mathbb{R}^3 - Z_*$ whose norm extends over Z_* as an $L^2_{1;loc}$ and exponent $\upsilon = \kappa^{-1}$ Hölder continuous function with zero locus Z_*.*
- *$\psi_R{}^*\nu_* = R^{1+N_{(p)}(0)}\nu_*$ if $R > 0$.*
- *The sequence of data sets $\{(Z_\lambda, I_\lambda, \nu_\lambda)\}_{\lambda\in(0,r_0]}$ converges to (Z_*, I_*, ν_*) in the following sense: Given $\varepsilon \in (0, 1)$, there exists $r_\varepsilon \in (0, 1]$ such that if $\lambda \in (0, r_\varepsilon)$, then*
  - *a) The functions $|\nu_\lambda|$ and $|\nu_*|$ differ where $|x| < \varepsilon^{-1}$ by a function with exponent $\upsilon = \kappa^{-1}$ Hölder norm less than ε.*
  - *b) Let $\mathcal{T}_\varepsilon \subset \mathbb{R}^3$ denote the radius ε tubular neighborhood of Z_*. Each $|x| < \varepsilon^{-1}$ point in Z_λ lies in $\mathcal{T}_\varepsilon$.*
  - *c) There exists an isomorphism between I_* and I_λ over the $|x| < \varepsilon^{-1}$ part of $\mathbb{R}^3 - \mathcal{T}_\varepsilon$ that identifies ν_λ with an I_* valued 1-form that differs from ν_* by an I_* valued 1-form with any $k < \varepsilon^{-1}$ version of the C^k norm being less than ε^{-1}.*

The upcoming Lemma 8.12 in Section 8.h says more about the behavior of ν_* near the rays in Z_*; it asserts in particular that $|\nu_*|^2$ vanishes to integer order along each of them.

Proof of Proposition 8.1. The assertions of the proposition are consequences of various lemmas that are stated and then proved in Sections 8.a–8.g. The definition of what turns out to be $|\nu_*|$ is supplied by Lemma 8.5 from Section 8.c. This lemma also gives the assertion in Proposition 8.1's third bullet that $|\nu_*|$ extends over Z_* as an $L^2_{1;loc}$ function. Lemma 8.6 in Section 8.c asserts that the extension of $|\nu_*|$ is Hölder continuous. Lemma 8.7 in Section 8.d gives a construction of Z_*, I_* and ν_*. Lemma 8.7 together with Lemma 8.8 in Section 8.e imply in part what is asserted by Proposition 8.1's fifth bullet. Lemma 8.9 in Section 8.f gives the assertion of the fourth bullet of Proposition 8.1. The proof of Proposition 8.1 is completed by Lemma 8.10 from Section 8.g. This lemma asserts that Z_* is the union of the origin and a finite set of rays from the origin.

Sections 8.a and 8.b prove various facts about $|\nu|$ that are used to prove the lemmas in the subsequent subsections.

8.a. Hölder continuity of $|\nu|$ along Z

This section supplies via Lemma 8.2 a result that is used as input in some of the forthcoming subsections. Keeping in mind that $|\nu| = |\hat{a}_\diamondsuit|$, this lemma refines a part of Proposition 6.1 by saying more about Hölder continuity of $|\hat{a}_\diamondsuit|$ on its zero locus.

Lemma 8.2. *There exists $\kappa > 1$ with the following significance: Fix a point $p \in Z$ and fix $s \in (0, r_0)$. Then*

- *There exists $z_s \in (0, \kappa)$ such that $h_{(p)}(s) = z_s^2 s^{2+2N_{(p)}(0)}$. Moreover, if z_s is such that $h_{(p)}(s) = z_s^2 s^{2+2N_{(p)}(0)}$, then $h_{(p)}(r) \leq (1+\kappa) z_s^2 r^{2+2N_{(p)}(0)}$ when $r \in (0, s]$.*
- *If $q \in M$ is such that $\mathrm{dist}(p, q) \leq \kappa^{-1} s$, then $|\nu|(q) \leq \kappa z_s (\mathrm{dist}(p,q))^{N_{(p)}(0)}$.*

The remainder of this subsection contains the proof of this lemma.

Proof of Lemma 8.2. The existence of a value of z_s in the indicated range that makes the first bullet true follows from (7.34) using two additional facts. The first is Lemma 7.7's observation that $N_{(p)}(0) > 0$; and the second is the inequality that is stated by Item c) of Proposition 7.1's fourth bullet.

To start the proof of the second bullet, use the fact that ν is closed and coclosed where non-zero to see that $|\nu|$ obeys the equation

$$\tfrac{1}{2}\mathrm{d}^{\dagger}\mathrm{d}|\nu|^2 + |\nabla\nu|^2 + \mathrm{Ric}(\nu\otimes\nu) = 0 \ \textit{on}\ \mathrm{M}{-}\mathrm{Z}. \tag{8.1}$$

To exploit this equation, fix for the moment $\rho > 0$ and use χ_ρ to once again denote the function on M given by $\chi(2-\rho^{-1}|\nu|)$. By way of a reminder, χ_ρ equals 0 where $|\nu| < \rho$ and it equals 1 where $|\nu| > 2\rho$. Let f for the moment denote a given, differentiable function on M. Multiply both sides of (8.1) by $f\chi_\rho$, integrate the result over M and integrate by parts to obtain the identity

$$\tfrac{1}{2}\int_{\mathrm{M}} \mathrm{d}^{\dagger}\mathrm{d}f|\nu|^2 + \int_{\mathrm{M}} f(|\nabla\nu|^2 + \mathrm{Ric}(\nu\otimes\nu)) + \mathfrak{e}_\rho(f) = 0, \tag{8.2}$$

where $|\mathfrak{e}_\rho(f)| \le \mathrm{c}_f \int_{\mathrm{M}-\mathrm{Z}_{2\rho}} |\nabla\nu|^2$ with $\mathrm{c}_f \ge 1$ depending only on f but not on ρ. Invoke (7.3) to see that $\lim_{\rho\to 0}|\mathfrak{e}_\rho(f)| = 0$. This understood, what is written in (8.2) implies that

$$\tfrac{1}{2}\int_{\mathrm{M}} \mathrm{d}^{\dagger}\mathrm{d}f|\nu|^2 + \int_{\mathrm{M}} f(|\nabla\nu|^2 + \mathrm{Ric}(\nu\otimes\nu)) = 0. \tag{8.3}$$

when f is a C^2 function on M.

Fix $\mathrm{q} \in \mathrm{M}$ and let $\mathrm{G_q}(\cdot)$ denote the Green's function for the operator $\mathrm{d}^{\dagger}\mathrm{d}+1$ with pole at q. Let σ denote a given smooth function on M. Use the sequence of functions $\{f_{\mathrm{q},\varepsilon}\}_{\varepsilon\in(0,1)}$ from Step 1 of the proof of Lemma 2.1 to see that (8.3) still holds when f is equal to $\sigma\mathrm{G_q}$. By way of a parenthetical remark, when (Z, I, ν) comes from a sequence that is described by Proposition 2.2, then a proof of this last assertion can also be had by using the first and third bullets of Proposition 7.1 and the local Hölder assertion in Proposition 6.1. In any event, the $f = \sigma\mathrm{G_q}$ version of (8.3) reads

$$\begin{aligned}\tfrac{1}{2}\sigma(\mathrm{q})|\nu|(\mathrm{q})^2 &+ \int_{\mathrm{M}} \sigma\mathrm{G_q}|\nabla\nu|^2 \\ &= \int_{\mathrm{M}} \sigma\mathrm{G_q}(|\nu|^2 - \mathrm{Ric}(\nu\otimes\nu)) - \int_{\mathrm{M}} \big(\tfrac{1}{2}\mathrm{d}^{\dagger}\mathrm{d}\sigma\mathrm{G_q} + \mathfrak{m}(\mathrm{d}\sigma, \mathrm{dG_q})\big)|\nu|^2,\end{aligned} \tag{8.4}$$

where $\mathfrak{m}(\cdot,\cdot)$ is used here to denote the metric inner product on $\mathrm{T^*M}$.

Fix $\mathrm{p} \in \mathrm{Z}$ and $\mathrm{r} \in (0, \mathrm{r}_0)$. Let q denote a point on the boundary of the ball of radius r centered at p. Fix $\mathrm{R} > 3$ and take the function σ in (8.4) to be the function on M given by $\chi((\mathrm{Rr})^{-1}\mathrm{dist}(\cdot,\mathrm{p}) - 1)$. This function is equal to 1 where the distance to p is less than Rr and it is equal to 0 where the dist$(\cdot, p) > 2R$. In particular, it is equal to 1 at q.

Keeping in mind that integrand for the right-most integral in (8.4) is supported in $B_{2Rr} - B_{Rr}$, the fact that

$$|G_p - G_q| \le c_0 \frac{\text{dist}(p, q)}{\text{dist}(\cdot, p)^2} \tag{8.5}$$

on the complement of the radius 3r ball centered at p leads to a bound on the absolute value of the right most integral by $c_0(Rr)^{-2}h_{(p)}(Rr)$. If Rr is less than the chosen value of s, then this last expression is no greater than $c_0 z_s (Rr)^{2N_p(0)}$.

The left most integral on the right hand side of (8.4) can be bounded using the fact that G_q near q can be written as $G_q = \frac{1}{4\pi}\text{dist}(q, \cdot)^{-1} + \mathfrak{r}$ with $|\mathfrak{r}| \le c_0 \text{dist}(q, \cdot)|\ln(\text{dist}(q, \cdot)|$. The contribution of $\mathfrak{r}$ to the integral is less than $c_0(Rr)^2|\ln(Rr)|h_{(p)}(Rr)$. This is less than $c_0(Rr)^4|\ln(Rr)|z_s(Rr)^{2N_p(0)}$ when Rr is less than s. The contribution to the left most integral on the right side of (8.4) from $\frac{1}{4\pi}\text{dist}(\cdot, q)^{-1}$ can be broken in to two parts, these being the contribution from the part where the distance to q is greater than $\frac{1}{2}Rr$ and that where the contribution is less than $\frac{1}{2}Rr$. The absolute value of the former is bounded by $c_0 h_{(p)}(Rr)$ and so it is no greater than $c_0(Rr)^2 z_s(Rr)^{2N_p(0)}$ when Rr is less than s. An integration by parts can be used to bound the absolute value of the latter by the integral of $c_0|\nu||\nabla\nu|$ over B_{Rr} plus that of $c_0(Rr)^{-1}|\nu|^2$ over the boundary of B_{Rr}. Both of these integrals are bounded by $c_0(Rr)^{-1}h_p(Rr)$, and thus by $c_0(Rr)z_s(Rr)^{2N_p(0)}$ when $Rr < s$.

Take $R = 4$ in the preceding two paragraphs to obtain the bound that is asserted by the second bullet in Lemma 8.2.

8.b. The r dependence of h and H near r = 0

Let p denote a given point in Z, thus a point where ν is zero. The three parts of this subsection prove that the functions $h_{(p)}$, $H_{(p)}$ and some related functions are nearly powers of r when r is small. Lemmas 8.3 and 8.4 makes precise what this means.

Part 1: The definition is such that $|\nu_\lambda|^2$ has integral 1 on the sphere of radius 1 about the origin in $\mathbb{R}^3$ when the inner product and area form are defined using the metric on the radius $\lambda^{-1}r_0$ ball that is obtained by multiplying the ϕ_λ pull back of the latter metric by λ^{-2}. This metric is denoted here by $\mathfrak{m}_\lambda$. It differs from the Euclidean metric by a term with norm bounded by $c_0\lambda^2$. The norms of the first and second derivative of $\mathfrak{m}_\lambda$ are also bounded by $c_0\lambda^2$. Derivatives of order $k \geq 3$ are bounded by a k dependent multiple of λ^k.

The convention for what follows in this subsection and henceforth with regards to metrics is as follows: All inner products and covariant derivatives of line bundle valued tensors on subsets of $\mathbb{R}^3$ are understood to be defined by the Euclidean metric unless explicitly stated to the contrary. The Hodge star is the Euclidean Hodge star unless stated to the contrary. Integration on submanifolds and domains in $\mathbb{R}^3$ is likewise defined using the Euclidean metric unless noted otherwise.

Part 2: The lemma that follows concerns the $\lambda \to 0$ limit of integrals that involve the 1-forms in the sequence $\{\nu_\lambda\}_{\lambda\in(0,r_0]}$. With λ given, the first lemma uses $\nu_{\lambda r}$ to denote the radial component of ν_λ.

Lemma 8.3. *This sequence has the properties that are listed in the bullets that follow. The four bullets refers to a chosen number* $R \in (0, \infty)$*.*

- $\{\int_{|x|=R} |\nu_\lambda|^2\}_{\lambda<r_0/R}$ *converges as* $\lambda \to 0$ *with limit* $R^{2+2N_{(p)}(0)}$.
- $\{\int_{|x|=R} |\nu_{\lambda r}|^2\}_{\lambda<r_0/R}$ *converges as* $\lambda \to 0$ *with limit* $\frac{1+N_{(p)}(0)}{3+N_{(p)}(0)}R^{2+2N_{(p)}(0)}$.
- $\{\int_{|x|\leq R} |\nu_\lambda|^2\}_{\lambda<r_0/R}$ *converges as* $\lambda \to 0$ *with limit* $\frac{1}{3+2N_{(p)}(0)}R^{3+2N_{(p)}(0)}$.
- $\{\int_{|x|\leq R} |\nabla\nu_\lambda|^2\}_{\lambda<r_0/R}$ *converges as* $\lambda \to 0$ *with limit* $N_{(p)}(0)R^{1+2N_{(p)}(0)}$.

Moreover, all four of the preceding assertions hold when the norms, covariant derivative and volume or area form are defined for each $\lambda \in (0, 1]$ *by the corresponding metric* $\mathfrak{m}_\lambda$*.*

Proof of Lemma 8.3. The proof has two steps.

Step 1. What is said in Part 1 about the difference between the Euclidean metric and a given $\lambda \in (0, \frac{1}{100}r_0]$ version of $\mathfrak{m}_\lambda$ implies that the Euclidean metric version of Lemma 8.3 holds if and only if the $\mathfrak{m}_\lambda$ metric version holds. Keep in mind that the definition of any ν_λ is such that the $\mathfrak{m}_\lambda$ version of the integral on the radius R sphere in $\mathbb{R}^3$ $\mathfrak{m}_\lambda$ version of $|\nu_\lambda|^2$ using $\mathfrak{m}_\lambda$'s area 2-form is $h(\lambda R)/h(\lambda)$.

Step 2. Lemma 7.7 has the following consequence: Given $\varepsilon > 0$, then there exists $r_\varepsilon \in (0, \frac{1}{100}r_0]$ such that $|N_{(p)}(r) - N_{(p)}(0)| \leq \varepsilon$ when $r \in (0, 2r_\varepsilon]$. With the preceding understood, fix $s > r$ from $(0, 2r_\varepsilon]$ and use (7.34) to write

$$h(s) = \left(\frac{s}{r}\right)^{2(1+N_{(p)}(0))+\mathfrak{e}} h(r), \tag{8.6}$$

where $\mathfrak{e}$ is such that $|\mathfrak{e}| \leq c_0\varepsilon$. This identity with what is said in Step 1 implies what is asserted by the first bullet of Lemma 8.3. The assertion in the third bullet of Lemma 8.3 is obtained from the first by integrating. Meanwhile, the assertion in the second bullet follows from the first and third via the identity in (7.5). The fourth bullet's assertion is obtained from the first bullet by writing p's version of the function H as $H = r^{-1}\mathrm{N}_{(p)}h$.

Part 3: The upcoming Lemma 8.4 concerns the $\lambda \to 0$ limit of integrals on constant $|x|$ spheres in $\mathbb{R}^3$ of $|\nabla\nu_\lambda|^2$ and of the square of the norm of ν_λ's radial derivative. There is some subtlety here by virtue of the fact that there is no assertion as yet that a given $\lambda \in (0, \frac{1}{100}r_0]$ version of the function $|\nabla\nu_\lambda|^2$ is square integrable on any given constant $|x|$ sphere. Even so, it is none-the-less a fact that the function given by the rule

$$R \to \int_{|x|=R} |\nabla\nu_\lambda|^2 \tag{8.7}$$

defines an L^1 function on $(0, \lambda^{-1}r_0)$. The proof that this is so starts with the following observation: Fix $\lambda \in (0, \frac{1}{100}r_0]$. Use $\mathrm{H}_\lambda : (0, \lambda^{-1}r_0) \to [0, \infty)$ to denote the function of R given by the rule

$$R \to \mathrm{H}_\lambda(R) = \int_{|x|\leq R} |\nabla\nu_\lambda|^2. \tag{8.8}$$

This function is bounded, nondecreasing and continuous. The continuity of H_λ follows from what is said in Part 1 of Lemma 7.7's proof to the effect that H is continuous. Since H_λ is continuous and nondecreasing, it is differentiable almost everywhere; see for example Theorem 3.23 in [R]. Its derivative is an L^1 function on $[0, \lambda^{-1}r_0)$, this being the function in (8.7).

Let ∇_r denote the directional covariant derivative in the radial direction. The same argument proves that the function defined by the rule $R \to \int_{|x|=R} |\nabla_r\nu_\lambda|^2$ is also L^1.

Lemma 8.4. *Fix* $p \in Z$ *so as to define the sequence* $\{\nu_\lambda\}_{\lambda\in(0,r_0]}$.

- *The sequence of* L^1 *functions* $\{R \to \int_{|x|=R} |\nabla\nu_\lambda|^2\}_{\lambda<r_0/R}$ *converges as* $\lambda \to 0$ *on compact subsets of* $[0, \infty)$ *with limit* $\mathrm{N}_{(p)}(0)(1 + 2\mathrm{N}_{(p)}(0))R^{2\mathrm{N}_{(p)}(0)}$.
- *The sequence of* L^1 *functions* $\{R \to \int_{|x|=R} |\nabla_r\nu_\lambda|^2\}_{\lambda<r_0/R}$ *converges as* $\lambda \to 0$ *on compact subsets of* $[0, \infty)$ *with limit* $\mathrm{N}_{(p)}(0)^2R^{2\mathrm{N}_{(p)}(0)}$.

Proof of Lemma 8.4. The proof has seven steps. The Steps 1 and 2 prove the top bullet and the subsquent steps prove the bottom bullet.

Step 1. Use the definition of $\mathrm{N}_{(\rho)}$ to write the derivative of H_λ as

$$\frac{d}{dR}\mathrm{H}_\lambda = \frac{d}{dR}\left(\frac{1}{R}\mathrm{N}_{(p)}(R\lambda)\frac{h_{(p)}(R\lambda)}{h_{(p)}(\lambda)}\right). \tag{8.9}$$

Now use (7.34) to rewrite the right hand side of (8.9) as $\frac{1}{R^2}\mathrm{Q}_\lambda(R)\int_{|x|=R}|\nu_\lambda|^2$ with

$$\mathrm{Q}_\lambda(R) = (1+2\mathrm{N}_{(p)}(R\lambda))\mathrm{N}_{(p)}(R\lambda) + \left(r\frac{d}{dr}\mathrm{N}_{(p)}(r)\right)\bigg|_{r=R\lambda}. \tag{8.10}$$

The next paragraph says precisely what is meant by the formula in (8.10)

To explain the right hand side of (8.10), note first that the function of R given by the rule $R \to (1+2\mathrm{N}_{(p)}(R\lambda))\mathrm{N}_{(p)}(R\lambda)$ is continuous on $[0, \lambda^{-1}r_0]$ and nothing more need be said about it at this point. To say what is meant by derivative of $\mathrm{N}_{(p)}$, fix $r \in (0, r_0)$ and then $\varepsilon \in (0, c_0^{-1}r)$. Having done so, use the fourth bullet of Proposition 7.1 to conclude that

$$\mathrm{N}_{(p)}(r+\varepsilon) - \mathrm{N}_{(p)}(r) \geq -c_0\varepsilon r. \tag{8.11}$$

Since $\mathrm{N}_{(p)}$ is continuous and also bounded on $(0, r_0]$, this lower bound implies that $\mathrm{N}_{(p)}$ is a function of *bounded variation* (see, e.g. [R]). It follows as a consequence that $\mathrm{N}_{(p)}$ is almost everywhere differentiable (see for example Theorem 8.19 in [R]) and its derivative is an L^1 function. This being the case, the right hand side of (8.10) defines an L^1 function on $[0, \lambda^{-1}r_0)$ and thus Q_λ is an L^1 function.

Step 2. Given what is said by the first bullet of Lemma 8.3, the assertion in the top bullet of Lemma 8.4 follows directly from a proof that the sequence of functions $\{R \to \mathrm{Q}_\lambda(R)\}_{\lambda\in(0,r_0)}$ has an appropriate $\lambda \to 0$ limit. With this understood, it is sufficient to prove that the sequence of functions given by $\{R \to (r\frac{d}{dr}\mathrm{N}_{(p)})|_{r=R\lambda}\}_{\lambda\in(0,r_0)}$ has a suitable limit, this because $\lim_{\lambda\to 0}(1 + 2\mathrm{N}_{(p)}(R\lambda))\mathrm{N}_{(p)}(R\lambda) = (1 + 2\mathrm{N}_{(p)}(0))\mathrm{N}_{(p)}(0)$ with the limit being uniform with respect to variations of R on compact subsets of $[0, \infty)$.

To see about the $\lambda \to 0$ limit of $\{R \to (r\frac{d}{dr}\mathrm{N}_{(p)})|_{r=R\lambda}\}_{\lambda\in(0,r_0)}$, first use (8.11) to conclude that $\frac{d}{dr}\mathrm{N}_{(p)}(r)) \geq -c_0 r$. This lower bound implies that

$$\int_0^t \frac{1}{s}\left|\left(r\frac{d}{dr}\mathrm{N}\right)\bigg|_{r=s}\right| ds \leq (\mathrm{N}_{(p)}(t) - \mathrm{N}_{(p)}(0) + c_0t^2) \quad \textit{for any given } t \in [0, r_0]. \tag{8.12}$$

Given that $N_{(p)}$ is continuous, the inequality in (8.12) leads directly to the following conclusion: The sequence of L^1 functions $\{R \to (r\frac{d}{dr}N_{(p)})|_{r=R\lambda}\}_{\lambda\in(0,r_0)}$ converges to zero in the L^1_{loc} topology on compact subsets of $[0, \infty)$. This being the case, it then follows that the sequence of L^1 functions $\{Q_\lambda\}_{\lambda\in(0,r_0)}$ converges on compact subsets of $[0, \infty)$ to the constant function $(1 + 2N_{(p)}(0))N_{(p)}(0)$. This last fact plus (8.9) imply what is asserted by the top bullet of Lemma 8.4.

Step 3. This step and Steps 4–7 prove Lemma 8.4's second bullet. The subsequent steps establish the identity

$$\lim_{\lambda\to 0}\int_{|x|\le R} |\nabla_r\nu_\lambda|^2 = \frac{1}{1 + 2N_{(p)}(0)} N_{(p)}(0)^2 R^{1+2N_{(p)}(0)}. \tag{8.13}$$

Granted (8.13), then arguments that differ only cosmetically from those in Step 2 can be repeated to prove that the derivative with respect to R of the limit on the left hand side of (8.13) is an L^1_{loc} function on $[0, \infty)$ whose restriction to bounded interval is equal to the function $R \to \lim_{\lambda\to 0}\int_{|x|=R} |\nabla_r\nu_\lambda|^2$. The second bullet then follows by differentiating (8.13).

Step 4. This step and the subsequent steps prove the assertion in (8.13). To set the notation, fix $\lambda \in (0, r_0)$ and reintroduce the metric $\mathfrak{m}_\lambda$ on the $|x| < \lambda^{-1}r_0$ part of $\mathbb{R}^3$. Denote the components of $\mathfrak{m}_\lambda$ as defined using the basis $\{dx_i\}_{i=1,2,3}$ as $\{\mathfrak{m}_{\lambda ik}\}_{1\le i,k\le 3}$ and denote those of the dual metric on $T^*\mathbb{R}^3$ by $\{\mathfrak{m}^{ik}_\lambda\}_{1\le i,k\le 1}$. The m_λ covariant derivative in the three Euclidean coordinate directions are denoted $\{\nabla^{(\lambda)}_i\}_{1\le i\le 3}$. As before, $\{\nabla_i\}_{1\le i\le 3}$ is used to denote the corresponding set of Euclidean metric covariant derivatives. Keep in mind that each $i \in \{1, 2, 3\}$ version of $\nabla^{(\lambda)}_i - \nabla_i$ is an endomorphism with norm bounded by $c_0\lambda^2|x|$. The pointwise norm on line bundle valued tensors that is defined using $\mathfrak{m}_\lambda$ is denoted by $|\cdot|_\lambda$. As before, $|\cdot|$ denotes the corresponding Euclidean metric norm.

Define the symmetric tensor $Q_\lambda = \nu_\lambda\otimes\nu_\lambda - \frac{1}{2}\mathfrak{m}_\lambda|\nu_\lambda|^2$ and let $\{Q_{\lambda ij}\}_{1\le i,j\le 3}$ denote its components when written with respect to the coordinate basis. The tensor Q_λ is coclosed in the sense that $\{\mathfrak{m}^{ik}_\lambda\nabla^{(\lambda)}_i Q_{(\lambda)kj} = 0\}_{1\le j\le 3}$. The fact that $Q_{(\lambda)}$ is coclosed follows from the fact that $\nu_{(\lambda)}$ is closed and m_λ-coclosed.

The fact that ν_λ is closed and coclosed can be used to see that

$$\tfrac{1}{2}(\mathfrak{m}^{mn}_\lambda\nabla^{(\lambda)}_m\nabla^{(\lambda)}_n Q_\lambda)_{ij} = \mathfrak{m}^{nm}\nabla_i\nu_{\lambda n}\nabla_j\nu_{\lambda m} - \tfrac{1}{2}\mathfrak{m}_{ij}|\nabla\nu_\lambda|^2_\lambda + \mathfrak{r}_{0\lambda} \tag{8.14}$$

where $\{\nu_{\lambda i}\}_{1\le i\le 3}$ denote the components of ν_λ, and where $|\mathfrak{r}_{0\lambda}| \le c_0\lambda^2|\nu|^2_\lambda$. The notation in (8.14) and subsequently uses the convention that repeated indices are summed.

Step 5. For $i \in \{1,2,3\}$, let $\hat{x}_i$ denote $\frac{1}{|x|}x_i$. Fix $\rho > 0$ and then multiply each $i, j \in \{1,2,3\}$ version of (8.14) by $\chi_\rho \hat{x}_i \hat{x}_j$, sum the resulting equalities and integrate both sides of the sum over the ball of radius R centered at the origin. Having done so, use an integration by parts and the fact that Q_λ is coclosed to derive the identity

$$(8.15)\quad \int_{|x|\le R} \left(|\nabla_r \nu_\lambda|^2 - \tfrac{1}{2}|\nabla \nu_\lambda|^2\right) = \tfrac{1}{2}R^{-2}\int_{|x|=R} x_i x_j \nabla_r Q_{\lambda ij} + \int_{|x|\le R} \frac{1}{|x|}\hat{x}_i\hat{x}_j\hat{x}_k \nabla_k Q_{\lambda ij} + \mathfrak{e}_{1\lambda} + \mathfrak{e}(\rho),$$

where $|\mathfrak{e}_{1\lambda}(R)| \le c_0\lambda^2 R^{1/c_p}$ with $c_p > 1$ depending only on p. Meanwhile, $\mathfrak{e}(\rho)$ is a term that has limit zero as $\rho \to 0$, this being a consequence of (7.3). By way of an explanation for the $\mathfrak{e}_{1\lambda}$ term, it has one contribution from the $\mathfrak{r}_\lambda$ term in (8.14) and a second contribution from the fact that $\nabla_i^{(\lambda)}$ differs from ∇_i by an endomorphism with norm at most $c_0\lambda^2|x|$. Granted this, the bound on the norm of the second contribution follows from the bounds in the third and fourth bullets of Lemma 8.3.

Step 6. Introduce by way of notation ∂_r to denote the unit length, radial vector field on $\mathbb{R}^3$ and introduce $Q_{\lambda rr}$ to denote the pairing between Q_λ and $\partial_r \otimes \partial_r$. Use the fact that $|\nabla_i^{(\lambda)} - \nabla_i| \le c_0\lambda^2|x|$ and Lemma 8.3 to see that the far right hand integral in the $\rho = 0$ version of (8.15) is equal to

$$(8.16)\quad \int_{|x|\le R}\left(\frac{1}{|x|^3}\nabla_r(|x|^2 Q_{\lambda rr}) - 2\frac{1}{|x|^2}Q_{\lambda rr}\right) + \mathfrak{e}_{2\lambda}(R),$$

where $|\mathfrak{e}_{2\lambda}(R)| \le c_0\lambda^2 R^{1/c_p}$. The identity $Q_{\lambda rr} = |\nu_{\lambda r}|^2 - \frac{1}{2}|\nu_\lambda|^2_\lambda$ with an integration by parts writes the integral in (8.16) as

$$(8.17)\quad R^{-1}\int_{|x|=R}\left(|\nu_{\lambda r}|^2 - \tfrac{1}{2}|\nu_\lambda|^2\right) - \int_{|x|\le R}\frac{1}{|x|^2}\left(2|\nu_{\lambda r}|^2 - |\nu_\lambda|^2\right) + \mathfrak{e}_{3\lambda}(R)$$

with $|\mathfrak{e}_{3\lambda}(R)|$ also bounded by $c_0\lambda^2 R^{1/c_p}$. Invoke Lemma 8.3 once again to conclude that the $\lambda \to 0$ limit of the expression in (8.16) is equal to $\frac{1}{2}\frac{1}{3+2N_{(p)}(0)}\frac{1-2N_{(p)}(0)}{1+2N_{(p)}(0)}R^{1+2N_{(p)}(0)}$.

Step 7. Use the bound $|\nabla_i^{(\lambda)} - \nabla_i| \le c_0\lambda^2|x|$ to rewrite the left most integral on the right hand side of the $\rho = 0$ version of (8.15) as

$$(8.18)\quad \int_{|x|=R} x_i x_j \nabla_r Q_{\lambda ij} = R^2\frac{d}{dR}\left(\int_{|x|=R}\left(|\nu_{\lambda r}|^2 - \tfrac{1}{2}|\nu_\lambda|^2\right)\right) - R\int_{|x|=R}\left(2|\nu_{\lambda r}|^2 - |\nu_\lambda|^2\right) + \mathfrak{e}_{3\lambda}(R),$$

where $|\mathfrak{e}_{3\lambda}(R)| \leq c_0\lambda^2 R^{c_0}$. Use Lemma 8.3 to conclude that the $\lambda \to 0$ limit of the right most integral on (8.18)'s right hand side is equal to $-\frac{2}{3+2N_{(p)}(0)}R^{3+2N_{(p)}(0)}$. Use the identity in (7.5) with Lemma 8.3 to prove that the $\lambda \to 0$ limit of the left most term on (8.18)'s right hand side is equal to $-\frac{2+N_{(p)}(0)}{3+2N_{(p)}(0)}R^{3+2N_{(p)}(0)}$. It then follows that the $\lambda \to 0$ limit of the whole of (8.18)'s right hand side is $-\frac{N_{(p)}(0)}{3+2N_{(p)}(0)}R^{3+2N_{(p)}(0)}$.

Use the conclusions of the preceding paragraph and those in Step 6 to see that the $\lambda \to 0$ limit of the right hand side of (8.15) is equal to $-\frac{1}{2}\frac{1}{1+2N_{(p)}(0)}N_{(p)}(0)R^{1+2N_{(p)}(0)}$. The claim in (8.13) follows from this last conclusion and Lemma 8.3's fourth bullet.

8.c. A definition of the function $|\nu_*|$

The $\lambda \to 0$ limit of the set $\{|\nu_\lambda|\}_{\lambda\in(0,r_0)}$ is analyzed in the two parts that comprise this subsection with the result being a definition of $|\nu_*|$.

Part 1: The lemma that follows asserts the existence of subsequences of the set $\{|\nu_\lambda|\}_{\lambda\in(0,r_0)}$ that converge strongly in the L^2 topology on any ball about the origin in $\mathbb{R}^3$.

Lemma 8.5. *There exists $\kappa > 0$ with the following signifance: Fix a decreasing sequence in $(0, r_0)$ with limit zero. There is a subsequence of this sequence, denoted here by Λ, such that the corresponding sequence of functions $\{|\nu_\lambda|\}_{\lambda\in\Lambda}$ converges as $\lambda \to 0$ weakly in the L^2_1 topology on every ball about the origin in $\mathbb{R}^3$. The limit function is an $L^2_{1;loc}$ and L^∞ function on each such ball whose pointwise norm is bounded by $\kappa R^{N_{(p)}(0)}$ on the radius R ball about the origin in $\mathbb{R}^3$. Meanwhile, the sequence $\{|\nabla\nu_\lambda|\}_{\lambda\in\Lambda}$ converges weakly in the L^2 topology on each such ball.*

The limit function of one of Lemma 8.5's sequence $\{|\nu_\lambda|\}_{\lambda\in\Lambda}$ is denoted in what follows by $|\nu_*|$. Although the notation indicates that this function is independent of Λ, an assertion to this effect is not proved until later.

Proof of Lemma 8.5. It follows directly from Lemma 8.3 that there exists Λ, this being a subsequence of the original sequence, such that $\{|\nu_\lambda|\}_{\lambda\in\Lambda}$ converges weakly in the L^2_1 topology on every ball about the origin in $\mathbb{R}^3$ and such that $\{|\nabla\nu_\lambda|\}_{\lambda\in\Lambda}$ converges weakly in the L^2 topology on every such ball. Use $|\nu_*|$ to denote the limit function of $\{|\nu_\lambda|\}_{\lambda\in\Lambda}$. To prove the pointwise bound, fix $\varepsilon \in (0, 1)$ and reintroduce r_ε from Step 2 of the proof of Lemma 8.3. Fix $R > 1$ and suppose that $\lambda \in (0, R^{-1}r_0)$ is less than r_ε. It follows from the $s = \lambda R$ and $z_s^2 = h_{(p)}(s)s^{-2-2N_{(p)}(0)}$ version of Lemma 8.2 using (8.6) that

$$|\nu_\lambda| \leq c_0(1 + e^{c_0\varepsilon R})R^{N_{(p)}(0)} \tag{8.19}$$

on the radius R ball about the origin in $\mathbb{R}^3$. The function $|\nu_*|$ must also obeys the bound in (8.19) and since ε has no positive lower bound, the function $|\nu_*|$ must obey the $\varepsilon = 0$ version of (8.19).

Part 2: The assertion in the next lemma implies in part that Lemma 8.5's sequence $\{\nu_\lambda\}_{\lambda\in\Lambda}$ converges uniformly in the C^0 topology on compact subsets of $\mathbb{R}^3$.

Lemma 8.6. *There exists a* p*-independent number* $\kappa > 10$ *with the following significance: Lemma 8.5's limit function* $|\nu_*|$ *is Hölder continuous with exponent* $\upsilon = \kappa^{-1}$ *and Lemma 8.5's sequence* $\{|\nu_\lambda|\}_{\lambda\in\Lambda}$ *converges to* $|\nu_*|$ *in the* $\upsilon = \kappa^{-1}$ *Hölder topology on compact subsets of* $\mathbb{R}^3$*.*

Proof of Lemma 8.6. The proof has five steps. By way of notation, these steps use N_0 as shorthand for $N_{(p)}(0)$.

Step 1. Fix $\lambda \in (0, r_0)$ and define z_λ by writing $h_{(p)}(\lambda)$ as $z_\lambda^2\lambda^{2+2N_0}$. Let Z_λ denote the zero locus of $|\nu_\lambda|$ in the radius $\lambda^{-1}r_0$ ball about the origin in $\mathbb{R}^3$. Let $z \in Z_\lambda$ denote a given point. This point corresponds to a point, p_z, on the zero locus of ν in the radius r_0 ball centered at p. Use $N_{z,0}$ as shorthand for $N_{(p_z)}(0)$. Suppose that $R > 16$ has been specified, that $\lambda < R^{-1}r_0$ and that the point z is in the radius $\frac{1}{8}R$ ball centered at the origin in $\mathbb{R}^3$. Define $z_{p,\lambda} \in (0, c_0]$ to be the infimum of numbers z such that $h_{(p_z)}(r) \leq z^2r^{2+2N_{z,0}}$ for all $r \in (0, \frac{1}{2}R\lambda]$. As explained momentarily, there exist a purely R-dependent constant c_R which is greater than R and has the following significance: If $\lambda \leq c_R^{-1}r_0$ then

$$z_{p,\lambda}z_\lambda^{-1} \leq c_0(R\lambda)^{N_0-N_{z,0}}. \tag{8.20}$$

To prove this claim, first use the p_z version of (7.34) and Item c) from the fourth bullets of the p_z version of Proposition 7.1 to see that

$$z_{p,\lambda}^2 \leq c_0\big(\tfrac{1}{2}R\lambda\big)^{-2-2N_{z,0}}h_{(p_z)}\big(\tfrac{1}{2}R\lambda\big) \tag{8.21}$$

Meanwhile, Stoke's theorem with the definition of $H_{(p)}$ can be used to bound the right hand side of (8.21) by c_0 times the value of $r^{-2-2N_0}h_{(p)}(r)$ at $r = R\lambda$. Granted this last bound, then (8.20) follows from (8.6).

Step 2. Fix $R > 16$, and then $\lambda \in (0, c_R^{-1} r_0)$ with c_R as described in Step 1. Lemma 8.2 with (8.20) have the following implication: Let z denote a point in Z_λ that lies in the radius $(16+c_0)^{-1}R$ ball about the origin; and let $x \in \mathbb{R}^3$ denote a point in the concentric ball of twice this radius. Then

$$|\nu_\lambda|(x) \leq c_0 R^{c_0} |z - x|^{N_{z,0}}. \tag{8.22}$$

As Lemma 7.7 finds $c_0 > 1$ such that $N_{z,0} > c_0^{-1}$, this last bound implies in particular that $|\nu_\lambda|$ is Hölder continuous with exponent $\upsilon > c_0^{-1}$ along its zero locus in any given radius ball about the origin in $\mathbb{R}^3$.

Step 3. Fix $\lambda \in (0, c_R^{-1} r_0)$. Given $\delta \in (0, c_0^{-1}]$, let $U_{R,\lambda,\delta}$ denote the set of points in the $|x| \leq R$ ball with distance δ or more from Z_λ. Being closed and coclosed, the 1-form ν obeys the equation $\nabla^\dagger \nabla \nu + \mathrm{Ric}((\cdot) \otimes \nu) = 0$. Use this equation with Lemma 8.3 to see that the second derivatives of ν_λ obey

$$\int_{U_{R,\lambda,\delta}} |\nabla\nabla\nu_\lambda|^2 \leq c_0 \delta^{-2} R^{1+2N_0}. \tag{8.23}$$

This last inequality with a standard Sobolev inequality can be used to prove the following: Suppose that x and y are points in $U_{R,\lambda,\delta}$. Then

$$|\nu_\lambda(x) - \nu_\lambda(y)| \leq c_0 \delta^{-1} R^{(1+2N_0)/2} |x - y|^{1/2}. \tag{8.24}$$

What is said by (8.24) implies that $||\nu_\lambda|(x) - |\nu_\lambda|(y)|$ is also bounded by the right hand side of (8.24) when x and y are both in $U_{R,\lambda,\delta}$.

Step 4. Suppose that x and y are both in the radius R ball about the origin. If either x or y is in Z_λ, then (8.22) implies that

$$||\nu_\lambda|(x) - |\nu_\lambda|(y)| \leq c_0 R^{c_0} |x - y|^{N_{z,0}} \tag{8.25}$$

for some $z \in Z_\lambda$ because the distance between x and y at most that between either and Z_λ.

Now assume that neither x nor y is in Z_λ. Set δ to equal the minimum of $\mathrm{dist}(x, Z_\lambda)^{1/5}$ and $\mathrm{dist}(y, Z_\lambda)^{1/5}$. If $|x - y| \leq \delta^5$, then what is said in Step 3 implies that

$$||\nu_\lambda|(x) - |\nu_\lambda|(y)| \leq c_0 R^{c_0} |x - y|^{1/20}. \tag{8.26}$$

Suppose next that $|x - y| \geq \delta^5$ and suppose for argument's sake that $\delta = \mathrm{dist}(x, Z_\lambda)$. Then there exists $z \in Z_\lambda$ with $|x - z| \leq 2\delta$. It then follows that

$|y - z| \le 2\delta + \delta^5$ and because $||\nu_\lambda|(x) - |\nu_\lambda|(y)| \le \max(|\nu_\lambda|(x), |\nu_\lambda|(y))$, it follows from (8.22) that

$$||\nu_\lambda|(x) - |\nu_\lambda|(y)| \le c_0 R^{c_0} |x - y|^{N_{z,0}/10} \tag{8.27}$$

for some $z \in Z_\lambda$.

Step 5. It follows from what is said in Step 4 using Lemma 7.7's lower bound $N_{(\cdot)} > c_0^{-1}$ that the $\lambda < c_R^{-1} r_0$ part of the sequence $\{\nu_\lambda\}_{\lambda\in\Lambda}$ is bounded in any $\upsilon < c_0^{-1}$ Hölder topology on the radius R ball. It follows as a consequence that this sequence is equi-Hölder continuous, and as it converges in the L^2 topology on the $|x| \le R$ ball, the same sort of argument that proves the Arzela-Arcoli theorem proves that $\{|\nu_\lambda|\}_{\lambda\in\Lambda}$ converges in any given exponent $\upsilon < c_0^{-1}$ Hölder topology in the $|x| \le R$ ball. This implies in particular that $|\nu_*|$ is Hölder continuous for any given exponent $\upsilon < c_0^{-1}$.

8.d. A construction of $(Z_*, \mathcal{I}_*, \nu_*)$

The next lemma supplies Proposition 8.1's data (Z_*, λ_*, ν_*).

Lemma 8.7. *There exists a* p*-independent number* $\kappa > 1$ *such that the following is true: Let* $|\nu_*|$ *denote the limit function of Lemma 8.5's sequence* $\{|\nu_\lambda|\}_{\lambda\in\Lambda}$ *and let* $Z_* \subset \mathbb{R}^3$ *denote the zero locus of* $|\nu_*|$.

- *The set* Z_* *is closed.*
- *There exists a real line bundle* $\mathcal{I}_* \to \mathbb{R}^3 - Z_*$ *and a harmonic,* $\mathcal{I}_*$ *valued 1-form on* $\mathbb{R}^3 - Z_*$ *with norm equal to* $|\nu_*|$. *The latter is denoted by* ν_*.
- *Fix* $R > 1$, $\varepsilon \in (0, 1]$ *and* $k \in \{0, 1, \dots\}$. *Let* B_R *denote the radius* R *ball centered on the origin in* $\mathbb{R}^3$ *and introduce* $Z_{R,\varepsilon}$ *to denote the radius* ε *tubular neighborhood* $Z_* \cap B_R$. *There exists* $\lambda_{R,\varepsilon,k} \in (0, R^{-1} r_0)$ *with the following significance: Suppose that* $\lambda \in \Lambda$ *with* $\lambda < \lambda_{R,\varepsilon,k}$. *Then* $Z_\lambda \cap B_R \subset Z_{R,\varepsilon}$ *and there is an isomorphism between* $\mathcal{I}_*$ *and* $\mathcal{I}_\lambda$ *over* $B_R - Z_{R,\varepsilon}$ *that identifies* ν_λ *as an* $\mathcal{I}_*$ *valued 1-form that differs from* ν_* *by an* $\mathcal{I}_*$ *valued 1-form whose* C^k *norm on* $B_R - Z_{R,\varepsilon}$ *is less than* ε.
- *The 1-form* ν_* *is such that for each* $R > 0$,
  - *i)* $\int_{|x|=R} |\nu_*|^2 = R^{2+2N_0}$.
  - *ii)* $\int_{|x|\le R} |\nu_*|^2 = \frac{1}{3+2N_0} R^{3+2N_0}$.
  - *iii)* $\int_{|x|\le R} |\nabla\nu_*|^2 = N_0 R^{1+2N_0}$.
  - *iv)* $\int_{|x|=R} |\nabla\nu_*|^2 = N_0(1 + 2N_0) R^{2N_0}$ *if* R *is not in a certain measure zero set.*

Proof of Lemma 8.7. The assertion in the first bullet to the effect that Z_* is closed follows from the fact that $|\nu_*|$ is continuous, the latter being a consequence of Lemma 8.6. To prove the second bullet, use the Hölder convergence given by Lemma 8.6 to see that

$$\lim_{\lambda\in\Lambda} \sup_{\{z\in Z_\lambda\cap B_R\}} \operatorname{dist}(z, Z_*) = 0. \tag{8.28}$$

It follows as a consequence that there exists $\lambda_{R,\varepsilon} \in (0, R^{-1}r_0)$ such that if $\lambda \in \Lambda$ is less than $\lambda_{R,\varepsilon}$, then each point in $Z_\lambda \cap B_R$ has distance ε^2 or less from $Z \cap B_R$. Fix pairs λ and λ' that obey these conditions. The convergence assertion of Lemma 8.6 can be used to construct an isomorphism between I_λ and $I_{\lambda'}$ over $B_R - Z_{R,\varepsilon}$. A directed limit of these isomorphisms defines the line bundle I_* over $\mathbb{R}^3 - Z$.

What follows is a tautological consequence of this construction: Fix $R > 1$ and then $\varepsilon \in (0, 1)$. There exists an isomorphism between each $\lambda < \lambda_{R,\varepsilon}$ version of I_λ and I_* over $B_R - Z_{R,\varepsilon}$. Fix an isomorphism of this sort for each such λ so as to view ν_λ as an I_* valued 1-form on $B_R - Z_{R,\varepsilon}$. Since each $\lambda < \lambda_{2R,2\varepsilon}$ version of ν_λ is closed on $B_{2R} - Z_{2R,2\varepsilon}$ and coclosed with Hodge star defined by $\mathfrak{m}_\lambda$, and as Lemma 8.3 gives a λ, R and ε independent bound for its L^2_1 norm, there exists an a priori pointwise bound for the derivatives of ν_λ to any given order on $B_R - Z_{R,\varepsilon}$.

Granted the conclusions of the preceding paragraph, a standard argument using the Arzela-Ascoli theorem and the fact that $\{\nu_\lambda\}_{\lambda\in\Lambda}$ converges in the L^2 topology on bounded domains in $\mathbb{R}^3$ supplies the I_* valued, harmonic 1-form ν_*. The convergence of $\{\nu_\lambda\}_{\lambda\in\Lambda}$ to ν_* is in any given $k \in \{0, 1, \dots\}$ version of the C^k topology on sets with compact closure in $\mathbb{R}^3 - Z$.

The first and second items of the fourth bullet follow from Lemma 8.3. To prove Item iii), first use the fourth bullet of Lemma 8.3 with the fact that ν_* is the limit of the sequence $\{\nu_\lambda\}_{\lambda\in\Lambda_*}$ to see that $\nabla\nu_*$ is square integrable on any $R > 0$ version of B_R and that the square of its L^2 norm is no greater than $\mathrm{N}_0 R^{1+2\mathrm{N}_0}$. This fact, what is said in the fourth bullet of Lemma 8.3 and the fact that $\{\nu_\lambda\}_{\lambda\in\Lambda}$ converges in the C^1 topology to ν_* on compact subsets of $\mathbb{R}^3 - Z$ implies that the L^2 norm of $\nabla\nu_*$ on the $|x| < R$ ball is in fact equal to $\mathrm{N}_0 R^{1+2\mathrm{N}_0}$. But for one change, the argument for Item iv) of the fourth bullet is identical to that just given for Item iii). This change replaces each reference to the fourth bullet of Lemma 8.3 in the latter argument by a reference to the first bullet of Lemma 8.4.

8.e. Uniqueness of the limit data set $(\mathrm{Z}_*, \mathcal{I}_*, \nu_*)$

This subsection proves that any two subsequences of $\{\nu_\lambda\}_{\lambda<\mathrm{r}_0}$ that converge as λ limits to zero as described in Lemma 8.5 have the same limit 1-form ν_*. The lemma that follows makes the formal statement to this effect.

Lemma 8.8. *Let Λ and Λ' denote sequences in $(0, \mathrm{r}_0)$ such that the corresponding sequences $\{\nu_\lambda\}_{\lambda\in\Lambda}$ and $\{\nu_\lambda\}_{\lambda\in\Lambda'}$ converge as described by Lemma 8.5. Let $|\nu_*|$ and $|\nu'_*|$ denote the respective $\{\nu_\lambda\}_{\lambda\in\Lambda}$ and $\{\nu_\lambda\}_{\lambda\in\Lambda'}$ limit functions supplied by Lemma 8.5. Let $\mathcal{I}_*$ and $\mathcal{I}'_*$ denote the corresponding real lines bundles over the complements of the zero loci of $|\nu_*|$ and $|\nu'_*|$ that are described in Lemma 8.7, and let ν_* and ν'_* denote the associated $\mathcal{I}_*$ and $\mathcal{I}'_*$ valued 1-forms from Lemma 8.7. The functions $|\nu_*|$ and $|\nu'_*|$ are equal and so $\mathcal{I}_*$ and $\mathcal{I}'_*$ are defined over the same domain in $\mathbb{R}^3$. As such, they are isomorphic, and there is an isomorphism from $\mathcal{I}_*$ to $\mathcal{I}'_*$ that identifies ν_* with ν'_*.*

Proof of Lemma 8.8. There are three steps in the proof.

Step 1. If $\mathrm{s} \in (0,1]$ and x is such that $\nu_{\lambda\mathrm{s}}(\mathrm{x}) \neq 0$, then the definition of $\nu_{\lambda\mathrm{s}}(\mathrm{x})$ as $\lambda\mathrm{s}\mathrm{h}(\lambda\mathrm{s})^{-1/2}\nu(\lambda\mathrm{s}\mathrm{x})$ with what is said in Item b) of Proposition 7.1's fourth bullet can be used to write

$$\left|\tfrac{\partial}{\partial \mathrm{s}}\nu_{\lambda\mathrm{s}}\right|(\mathrm{x}) = \left|(\mathrm{N}_{(\mathrm{p})}(\lambda\mathrm{s}) + \mathfrak{e}_{\lambda\mathrm{s}})\nu_{\lambda\mathrm{s}}(\mathrm{x}) - (\mathrm{r}\nabla_{\mathrm{r}}\nu_{\lambda\mathrm{s}})(\mathrm{x})\right|, \tag{8.29}$$

where $\mathfrak{e}_{\lambda\mathrm{s}}$ is such that $|\mathfrak{e}_{\lambda\mathrm{s}}| \leq \mathrm{c}_0\lambda^2\mathrm{s}^2$. The right hand side of (8.29) defines an L^2 function on the $|\mathrm{x}| = 1$ sphere for almost all $\mathrm{s} \in (0,1]$, this being a consequence of the first bullet of Lemma 8.3 and what is said in Part 3 of Section 8.b just prior to the statement of Lemma 8.4. Choose $\mathrm{s} \in (0,1]$ so that this so and the equality in (8.29) gives the identity

$$\begin{aligned} &\int_{|\mathrm{x}|=1} \left|\tfrac{\partial}{\partial \mathrm{s}}\nu_{\lambda\mathrm{s}}\right|^2 \\ &= \mathrm{N}_{(\mathrm{p})}(\lambda\mathrm{s})^2 \int_{|\mathrm{x}|=\mathrm{s}} |\nu_\lambda|^2 - 2\mathrm{N}_{(\mathrm{p})}(\lambda\mathrm{s})\mathrm{s} \int_{|\mathrm{x}|\leq\mathrm{s}} |\nabla\nu_\lambda|^2 + \mathrm{s}^2 \int_{|\mathrm{x}|=\mathrm{s}} |\nabla_{\mathrm{r}}\nu_\lambda|^2 + \mathfrak{n}_\lambda(\mathrm{s}), \end{aligned} \tag{8.30}$$

with $\mathfrak{n}_\lambda(\mathrm{s})$ obeying $|\mathfrak{n}_\lambda(\mathrm{s})| \leq \mathrm{c}_0\lambda^2\mathrm{s}^2$. By way of an explanation for the middle integral on the right hand side of (8.30), note that the latter is equal to the integral over the $|\mathrm{x}| = 1$ sphere of the inner product of $\nu_{\lambda\mathrm{s}}$ and $\nabla_{\mathrm{r}}\nu_{\lambda\mathrm{s}}$. This equality follows formally via an integration by parts from the fact that $\nabla^\dagger\nabla\nu + \mathrm{Ric}(\nu) = 0$. As in Section 7, the functions from the set $\{\chi_\rho\}_{\rho>0}$ with

(7.3) can be used to prove that the formal integration by parts identity is not compromised by contributions from neighborhoods of Z_λ.

Fix $\lambda \in (0, r_0)$ so as to view the right hand side of (8.30) as a function of the coordinate s for the interval $(0, 1]$. It follows from what is said just prior to Lemma 8.4 that the right hand side of (8.30) defines an L^1 function on this interval. This is therefore the case for the function of $s \in (0, 1]$ given by the integral on the left hand side of (8.30). With the preceding understood, then (8.30) with the first and fourth bullets of Lemma 8.3, the second bullet of Lemma 8.4 and Lemma 7.7 lead directly to the following conclusion: Given $\varepsilon > 0$, there exists $\lambda_\varepsilon \in (0, r_0)$ such that

$$\int_0^1 \Big(\int_{|x|=1} \big|\tfrac{\partial}{\partial s} \nu_{\lambda s}(\cdot)\big|^2 \Big) ds < \varepsilon \tag{8.31}$$

when $\lambda \in (0, \lambda_\varepsilon)$.

Step 2. Fix $\lambda \in (0, r_0)$ and define the function L_λ on $(0, 1]$ using the rule

$$t \to L_\lambda(t) = \int_{|x|=1} (|\nu_\lambda|(\cdot) - |\nu_{\lambda t}|(\cdot)|)^2. \tag{8.32}$$

This function is no greater than

$$\int_{|x|=1} \Big(\int_t^1 \big|\tfrac{\partial}{\partial s} \nu_{\lambda s}(\cdot)\big| ds \Big) \Big(\int_t^1 \big|\tfrac{\partial}{\partial s'} \nu_{\lambda s'}(\cdot)\big| ds' \Big) \tag{8.33}$$

which in turn is bounded by the integral on the left hand side of (8.31). This being the case, it follows directly from (8.31) that the sequence of functions $\{L_\lambda\}_{\lambda \in (0, r_0)}$ converges to zero in $L^1((0, 1])$ as λ limits to zero. This last observation implies that $|\nu_*| = |\nu'_*|$.

Let v denote a unit length vector in $\mathbb{R}^3$ and let $(\nu_\lambda)_v$ denote the $\mathcal{I}$ valued function on the radius λ^{-1} ball about the origin in $\mathbb{R}^3$ that is obtained by viewing v as a section of $T\mathbb{R}^3$ and ν_λ as a section of $\mathrm{Hom}(T^*\mathbb{R}^3; \mathcal{I})$ over the complement of the $|\nu_\lambda| = 0$ locus. Define the function $L_{\lambda,v}$ on $(0, 1]$ by replacing ν_λ in (8.32) by $(\nu_\lambda)_v$. The latter is also bounded by the integral on the left hand side of (8.31). As a consequence, the sequence of functions $\{L_{\lambda,v}\}_{\lambda \in (0, r_0)}$ converges to zero as in $L^1((0, 1])$ as λ limits to zero. This last observation implies that $|(\nu_*)_v| = |(\nu'_*)_v|$.

Step 3. The fact that $|\nu_*| = |\nu'_*|$ implies that $Z_* = Z'_*$. This step constructs an isomorphism between $\mathcal{I}_*$ and $\mathcal{I}'_*$ that identifies ν_* and ν'_*. To this end, introduce by way of notation ι to denote the section of $I_* \otimes I'_*$ that is obtained by multiplying the metric inner product of ν_* with ν'_* by $|\nu_*|^{-2}$. The observation at the end of Step 2 to the effect that $|(\nu_*)_{\mathrm{v}}| = |(\nu'_*)_{\mathrm{v}}|$ for any $\mathrm{v} \in T\mathbb{R}^3$ implies that ι has norm 1 at each point in $\mathbb{R}^3 - \mathrm{Z}_*$. This being the case, ι defines an isomorphism between I_* and I'_* that identifies ν_* with ν'_*.

8.f. Scaling behavior of ν_*

The lemma that follows asserts that ν_* has a very simple dependence on the radial coordinate $\mathbb{R}^3$.

Lemma 8.9. *Fix* $\mathrm{R} > 0$ *so as to define the map* $\psi_{\mathrm{R}} : \mathbb{R}^3 \to \mathbb{R}^3$ *by the rule* $\mathrm{x} \to \psi_{\mathrm{R}}(\mathrm{x}) = \mathrm{R}\mathrm{x}$. *Then* $\psi_{\mathrm{R}}(\mathrm{Z}_*) = \mathrm{Z}_*$ *and there exists an isometric isomorphism between* I_* *and* ${\psi_{\mathrm{R}}}^* I_*$ *that identifies* ${\psi_{\mathrm{R}}}^* \nu_*$ *with* $\mathrm{R}^{1+\mathrm{N}_0} \nu_*$.

This lemma is proved momentarily.

What follows directly describes an important consequence, this being that ν_* can be viewed using the scaling relation ${\psi_{\mathrm{R}}}^*\nu_* = \mathrm{R}^{1+\mathrm{N}_0}\nu_*$ as a pair $(\nu_{*\mathrm{r}}, \nu_{*\perp})$ with the first entry being an I_* valued function on the complement of Z_* in the $|\mathrm{x}| = 1$ sphere and with the second being an I_* valued 1-form on this same complement. This is done by writing ν_* as $\nu_{*\mathrm{r}}\mathrm{dr} + \nu_{*\perp}$ on $\mathbb{R}^3 - \mathrm{Z}_*$ with dr denoting the metric dual to the radial vector field ∂_{r} on $\mathbb{R}^3$ and with $\nu_{*\perp}$ annihilating this same vector field. The scaling formula ${\psi_{\mathrm{R}}}^*\nu_* = \mathrm{R}^{1+\mathrm{N}_0}\nu_*$ implies that ν_* is determined by the restrictions of $\nu_{*\mathrm{r}}$ and $\nu_{*\perp}$ to the $|\mathrm{x}| = 1$ sphere. This being the case, $(\nu_{*\mathrm{r}}, \nu_{*\perp})$ are henceforth viewed as a pair of I_* valued function and 1-form that are defined on the complement in the $|\mathrm{x}| = 1$ sphere of the of the $|\mathrm{x}| = 1$ points in Z_*.

The scaling relation ${\psi_{\mathrm{R}}}^*\nu_* = \mathrm{R}^{1+\mathrm{N}_0}\nu_*$ can be used to write the equations $d\nu_* = 0$ and $d{*}\nu_* = 0$ as equations on the $|\mathrm{x}| = 1$ sphere for $(\nu_{*\mathrm{r}}, \nu_{*\perp})$, these being the equations

$$(8.34)\qquad d\nu_{*\perp} = 0, \quad {*}d{*}\nu_* = -(\mathrm{N}_0 + 2)\nu_{*\mathrm{r}} \quad \textit{and} \quad d\nu_{*\mathrm{r}} = \mathrm{N}_0\nu_{*\perp}.$$

To explain the notation, the exterior derivatives in (8.34) are exterior derivatives on the $|\mathrm{x}| = 1$ sphere and the Hodge dual in (8.34) refers to the version on the $|\mathrm{x}| = 1$ sphere that is defined by its round metric. These equations play a central role in Sections 8.g and 8.h.

Proof of Lemma 8.9. Fix a decreasing sequence $\Lambda \subset (0, \mathrm{r}_0)$ with limit zero of the sort described in Lemma 8.5. As explained in Lemmas 8.5–8.8, the sequence $\{|\nu_\lambda|\}_{\lambda\in\Lambda}$ converges in the $\mathrm{L}^2_{1;\mathrm{loc}}$ topology on $\mathbb{R}^3$ to $|\nu_*|$ and this convergence is in an exponent $\upsilon > 0$ Hölder topology on compact subsets, and in the C^∞ topology on compact subsets of $\mathbb{R}^3 - Z_*$. The sequence $\{\nu_\lambda\}_{\lambda\in\Lambda}$ converges in the C^∞ topology on compact subsets of $\mathbb{R}^3 - Z_*$ to ν_* in the following sense: Fix $R > 0$. The sequence $\{\nu_{R\lambda}\}_{\lambda\in\Lambda}$ also converges in the manner dictated by Lemma 8.5, and so its limit is also ν_*.

It follows from the definitions that $|\nu_{R\lambda}| = h(R\lambda)^{-1/2} h(\lambda)^{1/2} |\psi_R{}^* \nu_\lambda|$. Meanwhile, it follows from (7.34) and from the continuity of the function N that $h(R\lambda) = R^{2+2N_0+\mathfrak{e}_\lambda} h(\lambda)$ with the collection $\{\mathfrak{e}_\lambda\}_{\lambda\in\Lambda}$ having the following property: Given $\varepsilon > 0$ and R, there exists $\lambda_{R,\varepsilon} \in (0, r_0)$ such that $|\mathfrak{e}_\lambda| < \varepsilon$ when $\lambda \in \Lambda$ is less than $\lambda_{R,\varepsilon}$. Granted these observations, take $\lambda \in \Lambda$ ever smaller and invoke what is said in the first paragraph to conclude that $|\nu_*| = R^{1+N_0} |\psi_R{}^* \nu_*|$. This implies that $\psi_R(Z_*) = Z_*$ and the latter implies that there is an isomorphism from $\mathcal{I}_*$ to $\psi_R{}^* \mathcal{I}_*$ and that the $R \in (0, \infty)$ of such isomorphisms is suitably continuous.

Fix $\varepsilon > 0$ and $T > 1$ and let $Z_{T,\varepsilon}$ denote the radius ε tubular neighborhood of the intersection between Z_* and the $|x| < T$ ball in $\mathbb{R}^3$. Fix $R < T$ and $x \in \mathbb{R}^3 - Z_{T,\varepsilon}$ with $|x|$ being less than the maximum of T and RT. If $\lambda \in \Lambda$ is small then Z_λ and $Z_{R\lambda}$ will intersect the $|x| < T$ ball inside $Z_{T,\varepsilon}$. Assuming this to be the case, then the bundles $\mathcal{I}_\lambda$ and $\mathcal{I}_{R\lambda}$ can be identified with $\mathcal{I}_*$ near x. Granted this identification, then $\nu_{R\lambda}$ and ν_λ near x can be viewed as sections of the same vector bundle. When viewed as such, it follows from the definitions that $\nu_{R\lambda}(x) = h(R\lambda)^{-1/2} h(\lambda)^{1/2} (\psi_R{}^* \nu_\lambda)(x)$. Granted this fact, take $\lambda \in \Lambda$ ever smaller and invoke what is said in the first paragraph to see that $(\psi_R{}^* \nu_*)(x) = R^{1+N_0} \nu_*(x)$.

8.g. The structure of Z_*

Lemma 8.9 implies that Z_* is a cone on its intersection with the $|x| = 1$ sphere in $\mathbb{R}^3$. The next lemma describes this cone.

Lemma 8.10. *The set Z_* contains a finite number of $|x| = 1$ points. Thus, Z_* consists of the union of the origin in $\mathbb{R}^3$ and a finite set of rays based at the origin in $\mathbb{R}^3$.*

Proof of Lemma 8.10. Suppose to the contrary that ν_* has an infinite set of zeros on the $|x| = 1$ sphere so as to generate nonsense. This is done in the five steps that follow.

Step 1. Fix a countable set of $|x| = 1$ zeros of ν_* that converge. Use stereographic projection based at the antipodal point to the limit point to view the limit point as the origin in $\mathbb{R}^2$ and use the standard complex structure on $\mathbb{R}^2$ to view $\mathbb{R}^2$ as $\mathbb{C}$. Use the stereographic projection map to view this countable set of zeros as a sequence of points in $\mathbb{C}$ that converge to the origin. Denote this set by $\{w_k\}_{k=1,2,\dots}$. No generality is lost by assuming that $\{w_k\}_{k=1,2,\dots}$ does not contain the origin, that $|w_1| < 1$ and that $|w_{k+1}| < \frac{1}{8}|w_k|$ for all $k \in \{1, 2, \dots\}$.

Use the stereographic projection to view the part of Z_* in its domain as a subset of $\mathbb{C}$. Denote this subset by Z_* also. The pull-back of I_* by the inverse map is a real line bundle over $\mathbb{C}-Z_*$, this denoted by I_* also. The corresponding pull-backs of $\nu_{*\perp}$ and ν_{*r} via the inverse to stereographic projection are an I_* valued 1-form and I_* valued function on $\mathbb{C}-Z_*$. As is the case with Z_* and I_*, the notation used below does not distinguish between $\nu_{*\perp}$ and ν_{*r} and their incarnations on $\mathbb{C}$.

Let z denote the standard complex coordinate on $\mathbb{C}$. Denote the z-derivative by ∂ and the $\bar{z}$ derivative by $\bar{\partial}$. Write $\nu_{*\perp}$ as $\nu_{*\perp} = \frac{2}{(1+|z|^2)}(\alpha d\bar{z} + \bar{\alpha} dz)$. The equations in (8.34) can be written schematically using this notation as

$$\partial\alpha = -\tfrac{1}{2}(1 + N_0)(1 + \mathfrak{r}_0)\nu_{*r} \quad \textit{and} \quad \bar{\partial}\nu_{*r} = 2N_0(1 + \mathfrak{r}_1)\alpha. \tag{8.35}$$

with $\mathfrak{r}_0$ and $\mathfrak{r}_1$ denoting smooth $\mathbb{C}$-valued functions that obey $|\mathfrak{r}_0| + |\mathfrak{r}_1| \le c_0|z|^2$.

Step 2. Given $m \in \{1, 2, \dots\}$, let $u_m = \prod_{m<k\le 2m} \frac{1}{(z-w_k)}$. Define Q_m to be $u_m/\bar{u}_m$. This is a $\mathbb{C}$ valued function on $\mathbb{C}-\{w_k\}_{m<k\le 2m}$ with norm 1. Fix for the moment $\varepsilon \in (0, 1)$ and for each $k \in \{m + 1, \dots, 2m\}$, set $\sigma_{m,k}$ to denote the function $\chi(2-\varepsilon^{-1}|(\cdot)-w_k|)$, this being a function on $\mathbb{C}$ that is equal to 1 where $|z - w_k| > 2\varepsilon$ and equal to 0 where $|z - w_k| < \varepsilon$. Fix $\delta \in (0, 1)$ and let σ_δ denote the function on $\mathbb{C}$ that is given by the rule $z \to \chi(\delta^{-1}|z| - 1)$. The function σ_δ is equal to 1 where $|z| < \delta$ and it is equal to 0 where $|z| > 2\delta$. Define σ to be $\sigma_\delta \prod_{m<k\le 2m} \sigma_{m,k}$.

With the preceding understood, define α_m to be $\sigma\bar{u}_m\alpha$ and β_m to be $\sigma u_m \nu_{*r}$. The equations in (8.36) imply that the pair (α_m, β_m) obey an equation of the form

$$\begin{aligned} \partial\alpha_m &= -\tfrac{1}{2}(1 + N_0)(1 + \mathfrak{r}_0)\bar{Q}_m\beta_m + \mathfrak{z}_\alpha + \mathfrak{u}_{\alpha,\varepsilon} \quad \textit{and} \\ \bar{\partial}\beta_m &= 2N_0 Q_m(1 + \mathfrak{r}_1)\alpha_m + \mathfrak{z}_\beta + \mathfrak{u}_{\beta,\varepsilon}. \end{aligned} \tag{8.36}$$

where $\mathfrak{z} = (\mathfrak{z}_\alpha, \mathfrak{z}_\beta)$ has the derivatives of σ_0 and thus has compact support where $\delta \le |z| \le 2\delta$; and where $\mathfrak{u}_\varepsilon = (\mathfrak{u}_{\alpha,\varepsilon}, \mathfrak{u}_{\beta,\varepsilon})$ has the derivatives of $\prod_{m<k\le 2m} \sigma_{m,k}$ and is therefore zero except where $\varepsilon \le |z-w_k| \le 2\varepsilon$ for some $k \in \{m + 1, \dots, 2m\}$.

Step 3. The lemma that follows says what else is needed with regards to $\mathfrak{z}$ and $\mathfrak{u}_\varepsilon$.

Lemma 8.11. *The* L^2 *norm of* $\mathfrak{z}$ *is bounded by* $\kappa\delta^{-m}$ *with* κ *being independent of* m. *Meanwhile,* $\limsup_{\varepsilon\to 0}\int_{\mathbb{C}}|\mathfrak{u}_\varepsilon|^2$ *exists and it is equal to zero.*

Proof of Lemma 8.11. The assertion about $\mathfrak{z}$ follows from the fact that $|\mathfrak{u}_m| \le c_0|z|^{-m}$ where $|z| \ge 2|w_m|$. To prove the assertion about $\mathfrak{u}_\varepsilon$, invoke Lemma 8.6 to conclude that the integral of $|\mathfrak{u}_\varepsilon|^2$ is no greater than $c_0 m|w_{2m}|^{-m+1}\varepsilon^{1/c_0}$.

Step 4. Introduce the radial coordinates $R \in (0,\infty)$ and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ by writing z as $Re^{i\theta}$. Write ∂ and $\bar\partial$ using these coordinates. Having done so, square the norm of both sides of both equations in (8.36) and multiply by R to obtain the inequalities that follow.

(8.37)

- $R|\partial_R\alpha_m|^2 + i\partial_R(\alpha_m\partial_\theta\bar\alpha_m) - i\partial_\theta(\alpha_m\partial_R\bar\alpha_m) + R^{-1}|\partial_\theta\alpha_m|^2 \le c_0R(|\alpha_m|^2 + |\beta_m|^2 + |\mathfrak{z}_\alpha|^2 + |\mathfrak{u}_{\alpha,\varepsilon}|^2)$.
- $R|\partial_R\beta_m|^2 - i\partial_R(\beta_m\partial_\theta\bar\beta_m) + i\partial_\theta(\beta_m\partial_R\bar\beta_m) + R^{-1}|\partial_\theta\beta_m|^2 \le c_0R(|\alpha_m|^2 + |\mathfrak{z}_\beta|^2 + |\mathfrak{u}_{\beta,\varepsilon}|^2)$.

Fix $\rho > 0$ and multiply both sides of both inequalities by the function $\chi_\rho = \chi(2-\rho^{-1}|\nu_*|)$; then integrate both sides of the result over the disk of radius 2δ about the origin in $\mathbb{C}$ using the measure $dRd\theta$. Integrate by parts to rewrite the integrals of $i\chi_\rho\partial_R(\alpha_m\partial_\theta\bar\alpha_m)$, $i\chi_\rho\partial_\theta(\alpha_m\partial_R\bar\alpha_m)$, $-i\chi_\rho\partial_R(\beta_m\partial_\theta\bar\beta_m)$ and $i\chi_\rho\partial_\theta(\beta_m\partial_R\bar\beta_m)$ to obtain derivatives of χ_ρ. As in previous applications of χ_ρ, these terms have $\rho\to 0$ limit equal to zero. Take this limit to conclude that

$$\int_{|z|\le 2\delta}(|\partial_R\alpha_m|^2 + |\partial_R\beta_m|^2)R\,dR\,d\theta \le c_0\int_{|z|\le 2\delta}(|\alpha_m|^2+|\beta_m|^2)R\,dR\,d\theta + \int_{|z|\le 2\delta}(|\mathfrak{z}|^2+|\mathfrak{u}_\varepsilon|^2)R\,dR\,d\theta. \tag{8.38}$$

Step 5. Since α_m and β_m have compact support where $|z|\le\delta$, the integral on the left hand side of (8.38) is no less than $c_0^{-1}\delta^{-2}$ times the integral of $|\alpha_m|^2+|\beta_m|^2$. This being the case, then (8.38) leads directly to the inequality

$$(\delta^{-2}-c_0)\int_{|z|\le 2\delta}(|\alpha_m|^2+|\beta_m|^2)R\,dR\,d\theta \le c_0\int_{|z|\le 2\delta}(|\mathfrak{z}|^2+|\mathfrak{u}_\varepsilon|^2)R\,dR\,d\theta. \tag{8.39}$$

Take $\delta < c_0^{-1}$ so that the left hand side of (8.39) is positive. Then take the $\varepsilon\to 0$ limit and use Lemma 8.11 to see that the result implies the bound

$$\int_{|z|\le 2\delta}(|\alpha_m|^2+|\beta_m|^2)R\,dR\,d\theta \le c_0\delta^{-2m}. \tag{8.40}$$

The left hand side of (8.40) is no less than the integral of $|\alpha_m|^2 + |\beta_m|^2$ over the disk in $\mathbb{C}$ where $\frac{1}{16}\delta < |z| \le \frac{1}{8}\delta$. With $\delta = c_0^{-1}$ fixed, then all $k > c_0^{-1}$ points w_k will have norm less than $\frac{1}{100}\delta^2$, and as a consequence, $|\alpha_m|^2 + |\beta_m|^2$ where $\frac{1}{16}\delta < |z| < \frac{1}{8}\delta$ will be greater than $c_0^{-1}(4\delta)^{-2m}(|\alpha|^2 + |\nu_{*r}|^2)$. This being the case, then the $\delta = c_0^{-1}$ and $m > c_0^{-1}$ versions of (8.40) imply that

$$\int_{\delta/16 \le |z| \le \delta/8} (|\nu_{*\perp}|^2 + |\nu_{*r}|^2) R\, dR\, d\theta \le c_0 4^{-m}. \tag{8.41}$$

Take m ever larger in (8.41) to deduce that $|\nu_*|$ must be zero on an annulus about the origin in $\mathbb{C}$. But this implies that $|\nu_*|$ is zero on the cone over this annulus, which is nonsense as it runs afoul of Item a) of some $p' \ne p$ version Proposition 7.1's fourth bullet.

8.h. The behavior of ν_* near Z_*

This last subsection first states and then proves the upcoming Lemma 8.12 which implies in part that $|\nu_*|^2$ vanishes to integer order along each ray component of Z_*.

Lemma 8.12. *Let* x *denote an* $|x| = 1$ *point in* Z_*. *There exists* $k \in \{1, 2, \dots\}$ *with the following properties:*

- *The function* $|\nu_*|$ *near* x *can be written as* $|\nu_*| = c|x - (\cdot)|^{k/2} + \cdots$ *where* c *is a positive number and where the unwritten terms are bounded by a multiple of* $|x - (\cdot)|^{k/2+1}$.
- *The bundle* $\mathcal{I}_*$ *near* x *is isomorphic to the product bundle if and only if* k *is even.*

Proof of Lemma 8.12. Let x a point on Z_* with norm 1. Use the stereographic coordinates from Step 1 of the proof of Lemma 8.10 to view this point as the origin in $\mathbb{C}$ and to likewise view $\nu_{*\perp}$ and ν_{*r} on $\mathbb{C}$. With this view understood, write $\nu_{*\perp}$ again as $\frac{2}{1+|z|^2}(\alpha d\bar{z} + \bar{\alpha} dz)$. Fix a small radius disk about the origin whose closure has no point but the origin where $|\alpha|^2 + |\nu_{*r}|^2 = 0$. Let D denote this disk and let ρ denote its radius. If the bundle $\mathcal{I}_*$ on D is isomorphic to the product bundle, it then follows from (8.35) that α and ν_{*r} are real analytic on D. This being the case, Taylor's theorem with remainder finds exists an integer $k \in \{1, 2, \dots\}$ and non-zero complex number α_0 such that

$$\alpha = \alpha_0 \bar{z}^k + \cdots \quad \textit{and} \quad \nu_{*r} = \frac{1}{k+1}\frac{1}{2}N_0^{-1}(\alpha_0 \bar{z}^{k+1} + \bar{\alpha}_0 z^{k+1}) + \cdots. \tag{8.42}$$

with the unwritten terms in the equation for α having norm bounded by a multiple of $|z|^{k+1}$ and with those in the equation for ν_{*r} having norm bounded by $|z|^{k+2}$. This equation implies the assertion of the first bullet of Lemma 8.12 when $\mathcal{I}_*$ is isomorphic to the product bundle on D−0.

Now assume that $\mathcal{I}_*$ is not isomorphic to the product bundle on D−0. Introduce a second copy of $\mathbb{C}$ and let w denote the complex coordinate on this second copy of $\mathbb{C}$. Use D_w to denote the $|w| < \rho^{1/2}$ disk in this second copy of $\mathbb{C}$. Define the 2-fold branched cover $\varphi : D_w \to D$ by the rule whereby $\varphi^* z = w^2$. The pull-back bundle $\varphi^* \mathcal{I}_*$ is the product bundle on D_w-0. This implies that $\varphi^*\alpha$ and $\varphi^*\nu_{*r}$ are respective $\mathbb{C}$ valued and $\mathbb{R}$ valued functions on D_w. It follows from (8.35) that these pull-backs obey the equations

$$(8.43)\qquad \partial(\varphi^*\alpha) = -(1+N_0)w\varphi^*\nu_{*r} \quad \textit{and} \quad \bar\partial(\varphi^*\nu_{*r}) = 4N_0\bar w\varphi^*\alpha.$$

These equations imply that $\varphi^*\alpha$ and $\varphi^*\nu_{*r}$ are real analytic functions of w. Moreover, they imply there exists $\alpha_0 \in \mathbb{C}-0$ and $k \in \{0, 1, \dots\}$ such that

$$(8.44)\qquad \begin{aligned} &\varphi^*\alpha = \alpha_0\bar w^{2k+1} + \cdots \quad \textit{and} \\ &\varphi^*\nu_{*r} = \frac{1}{4(2k+3)}N_0^{-1}(\alpha_0\bar w^{2k+3} + \bar\alpha_0 w^{2k+3}) + \cdots . \end{aligned}$$

where the unwritten terms in the equation for $\varphi^*\alpha$ have norm bounded by a multiple of $|w|^{2k+3}$ and those in the equation for $\varphi^*\nu_{*r}$ have norm bounded by a multiple of $|w|^{2k+5}$. Note in particular that $\varphi^*\alpha$ must vanish to odd order as a function of $\bar w$; if not, then the components of ν_* on D−0 could be used to obtain a nowhere zero section of $\mathcal{I}_*$ on D−0. What is written in (8.44) implies what is said by the first bullet when $\mathcal{I}_*$ is not isomorphic to the product bundle on D−0. The fact that $\varphi^*\alpha$ must vanish to odd order with (8.42) implies the assertion of Lemma 8.12's second bullet

9. Weakly continuous points in Z

As in Section 8, the implicit assumption in what follows is that $(Z, \mathcal{I}, \nu)$ define a twisted harmonic form data set. A point $p \in Z$ is said to be *weakly continuous* when there exists a sequence $\{p_i\}_{i=1,2,\dots} \subset Z$ with limit p such that $\{N_{(p_i)}(0)\}_{i=1,2,\dots}$ converges with its limit being $N_{(p)}(0)$. Let $p \in Z$ denote such a point. The central proposition in this section describes an alternative contruction of p's version of the set Z_*. This central proposition is Proposition 9.2. Proposition 9.2 leads to the characterization p's version of Z_* that is given below by Proposition 9.1. The lemmas used to prove Proposition 9.2 also play a role in Section 10.

Proposition 9.1. *Suppose that* $\mathrm{p} \in \mathrm{Z}$ *is a weakly continuous point.*

- $\mathrm{N}_{(\mathrm{p})}(0)$ *is half a positive integer.*
- *The set* Z_* *from* p*'s version of Proposition 8.1 has precisely two* $|\mathrm{x}| = 1$ *points, these being antipodal. This is to say that* Z_* *is a line through the origin.*
- *There exists* $c \in (0, \infty)$ *such that if* x *is either of the points with norm 1 on* Z_**, then the function* $|\nu_*|$ *near* x *has the form* $|\nu_*| = c|\mathrm{x} - (\cdot)|^{\mathrm{N}_{(\mathrm{p})}(0)} + \cdots$ *where the norm of the unwritten terms is no greater than a constant multiple of* $|\mathrm{x} - (\cdot)|^{\mathrm{N}_{(\mathrm{p})}(0)+1}$.

This proposition is proved in Section 9.c

To set the stage for Proposition 9.2, suppose that p is weakly continuous. Fix a sequence $\{\mathrm{p_i}\}_{\mathrm{i}=\{1,2,\ldots\}} \subset \mathrm{Z}$ converging to p such that $\{\mathrm{N}_{(\mathrm{p_i})}(0)\}_{\mathrm{i}=1,2,\ldots}$ converges with its limit being $\mathrm{N}_{(\mathrm{p})}(0)$. Discard a finite number of elements from the sequence $\{\mathrm{p_i}\}_{\mathrm{i}=1,2,\ldots}$ so that the resulting sequence lies in the radius $\frac{1}{100}\mathrm{r}_0$ ball centered at p; then renumber the resulting sequence consecutively from 1. Fix $\mathrm{i} \in \{1, 2, \ldots\}$ for the moment and select a number λ_i which is positive but less than $\frac{1}{100}\mathrm{r}_0$. Choose Gaussian coordinates centered at p to define an orthonormal basis for $TM|_\mathrm{p}$. Parallel transport this basis along the short geodesic ray from p to $\mathrm{p_i}$ to define an orthonormal basis for $TM|_{\mathrm{p_i}}$. Use the latter to define Gaussian coordinates centered at $\mathrm{p_i}$. Having done so, let $\phi_{(\mathrm{p_i})\lambda_\mathrm{i}}$ denote the map from the radius $\lambda_\mathrm{i}^{-1}\mathrm{r}_0$ ball centered at the origin in $\mathbb{R}^3$ to the radius r_0 ball centered at $\mathrm{p_i}$ that is obtained by composing first the rescaling map $\mathrm{x} \to \lambda_\mathrm{i}\mathrm{x}$ and then the Gaussian coordinate chart map. Define $\mathrm{Z_i}$ to be the inverse image via $\phi_{(\mathrm{p_i})\lambda_\mathrm{i}}$of Z, define $\mathcal{I}_\mathrm{i}$ to be the pull-back via $\phi_{(\mathrm{p_i})\lambda_\mathrm{i}}$ of $\mathcal{I}$ and define ν_i to be the product of $\mathrm{h}_{(\mathrm{p_i})}(\lambda_\mathrm{i})^{-1/2}$ with the $\phi_{(\mathrm{p_i})\lambda_\mathrm{i}}$ pull-back of ν. The set $\mathrm{Z_i}$ is a closed subset of the $|\mathrm{x}| < \lambda_\mathrm{i}^{-1}\mathrm{r}_0$ ball in $\mathbb{R}^3$, what is denoted by $\mathcal{I}_\mathrm{i}$ is a real line bundle over the complement of $\mathrm{Z_i}$ in this ball, and what is denoted by ν_i is an $\mathcal{I}_\mathrm{i}$ valued 1-form on the complement of $\mathrm{Z_i}$ with $\mathrm{Z_i}$ being the zero locus of $|\nu_\mathrm{i}|$.

Proposition 9.2. *Fix* $\mathrm{p} \in \mathrm{Z}$ *and suppose that* $\{\mathrm{p_i}\}_{\mathrm{i}=1,2,\ldots} \subset \mathrm{Z}$ *is a sequence with limit* p *such that* $\{\mathrm{N}_{(\mathrm{p_i})}(0)\}_{\mathrm{i}=1,2,\ldots}$ *converges with its limit being* $\mathrm{N}_{(\mathrm{p})}(0)$*. Let* $\mathrm{Z}_* \subset \mathbb{R}^3$ *denote the set that is supplied by* p*'s version of Proposition 8.1 and let* $\mathcal{I}_*$ *denote the corresponding real line bundle over* $\mathbb{R}^3 - \mathrm{Z}_*$*. There exists an* $\mathcal{I}_*$ *valued, harmonic 1-form on* $\mathbb{R}^3 - \mathrm{Z}_*$ *with the properties listed below. The list use* $\nu_\diamondsuit$ *to denote this* $\mathcal{I}_*$ *valued 1-form.*

- $|\nu_\diamondsuit|$ *extends to* $\mathbb{R}^3$ *as an* $\mathrm{L}^2_{1;\mathrm{loc}}$ *and Hölder continuous function with zero locus* Z_**.*
- *Fix* $\mathrm{R} > 0$*. Then* $\psi_\mathrm{R}{}^*\nu_\diamondsuit = \mathrm{R}^{1+\mathrm{N}_{(\mathrm{p})}(0)}\nu_\diamondsuit$*.*

- *Fix a decreasing sequence* $\{\lambda_i\}_{i=1,2,\ldots}$ *in* $(0, \frac{1}{100}r_0)$ *with limit zero; then construct the sequence* $\{Z_i, \mathcal{I}_i, \nu_i\}_{i=1,2,\ldots}$ *as instructed above. There exists a subsequence in* $\{1, 2, \ldots\}$ *to be denoted by* $\Lambda_\diamondsuit$ *with the following properties: Given* $\varepsilon \in (0, 1]$*, there exists* I_ε *such that if* $i \in \Lambda_\diamondsuit$ *and* $i > I_\varepsilon$*, then*
  - *a) The functions* $|\nu_i|$ *and* $|\nu_\diamondsuit|$ *differ where* $|x| < \varepsilon^{-1}$ *by a function with* L^2_1 *norm and Hölder norm less than* ε *with the Hölder exponent being independent of the data set* $\{(p_i, \lambda_i)\}_{i=1,2,\ldots}$ *and the point* p.
  - *b) Let* $\mathcal{T}_\varepsilon \subset \mathbb{R}^3$ *denote the radius* ε *tubular neighborhood of* Z_**. Each* $|x| < \varepsilon^{-1}$ *point in* Z_i *lies in* $\mathcal{T}_\varepsilon$.
  - *c) There exists an isometric identification between* $\mathcal{I}_i$ *and* $\mathcal{I}_*$ *over the* $|x| < \varepsilon^{-1}$ *part of* $\mathbb{R}^3 - \mathcal{T}_\varepsilon$ *that identifies* ν_i *as an* $\mathcal{I}_*$ *valued 1-form that differs from* $\nu_\diamondsuit$ *by an* $\mathcal{I}_\diamondsuit$ *valued 1-form with any* $k < \varepsilon^{-1}$ *version of the* C^k *norm being less than* ε^{-1}.

The proof of this proposition is contained in Sections 9.a and 9.b.

9.a. The data set $(Z_\diamondsuit, \mathcal{I}_\diamondsuit, \nu_\diamondsuit)$

The upcoming Lemma 9.3 states a weak analog of Proposition 9.2 that has Z_* replaced everywhere by a finite union of rays from the origin in $\mathbb{R}^3$ and $\mathcal{I}_*$ replaced by a real line bundle over the complement in $\mathbb{R}^3$ of this same finite set of rays.

Lemma 9.3. *Fix* $p \in Z$ *and suppose that* $\{p_i\}_{i=1,2,\ldots} \subset Z$ *is a sequence with limit* p *such that* $\{N_{(p_i)}(0)\}_{i=1,2,\ldots}$ *converges with its limit being* $N_{(p)}(0)$*. There exists a triple* $(Z_\diamondsuit, \mathcal{I}_\diamondsuit, \nu_\diamondsuit)$ *with the properties listed below.*

- $Z_\diamondsuit$ *is the union of the origin in* $\mathbb{R}^3$ *with a finite union of rays from the origin in* $\mathbb{R}^3$ *and* $\mathcal{I}_\diamondsuit$ *is a real line bundle on* $\mathbb{R}^3 - Z_\diamondsuit$.
- $\nu_\diamondsuit$ *is an* $\mathcal{I}_\diamondsuit$ *valued, harmonic 1-form on* $\mathbb{R}^3 - Z_\diamondsuit$ *with* $|\nu_\diamondsuit|$ *extending to* $\mathbb{R}^3$ *as an* $L^2_{1;loc}$ *and Hölder continuous function whose zero locus is* $Z_\diamondsuit$*. Moreover, this Hölder norm is independent of the data set* $\{(p_i, \lambda_i)\}_{i=1,2,\ldots}$ *and independent of* p.
- *Fix* $R > 0$ *and there exists an isometric isomorphism between* $\psi_R{}^*\mathcal{I}_\diamondsuit$ *and* $\mathcal{I}_\diamondsuit$ *that identifies* $\psi_R{}^*\nu_\diamondsuit$ *with* $R^{1+N_{(p)}(0)}\nu_\diamondsuit$.
- *Fix a decreasing sequence* $\{\lambda_i\}_{i=1,2,\ldots}$ *in* $(0, \frac{1}{100}r_0)$ *with limit zero; then construct the sequence* $\{Z_i, \mathcal{I}_i, \nu_i\}_{i=1,2,\ldots}$ *as just instructed. There exists a subsequence in* $\{1, 2, \ldots\}$ *denoted by* $\Lambda_\diamondsuit$ *with the following properties: Given* $\varepsilon \in (0, 1]$*, there exists* I_ε *such that if* $i \in \Lambda_\diamondsuit$ *and* $i > I_\varepsilon$*, then*

a) The functions $|\nu_{\mathrm{i}}|$ *and* $|\nu_{\Diamond}|$ *differ where* $|x| < \varepsilon^{-1}$ *by a function with* L^2_1 *norm and Hölder norm less than* ε *with the Hölder exponent being independent of the data set* $\{(p_i, \lambda_i)\}_{i=1,2,\ldots}$ *and the point* p.

b) Let $\mathcal{T}_\varepsilon \subset \mathbb{R}^3$ *denote the radius* ε *tubular neighborhood of* $Z_\Diamond$. *Each* $|x| < \varepsilon^{-1}$ *point in* Z_i *lies in* $\mathcal{T}_\varepsilon$.

c) There exists an isometric identification between $\mathcal{I}_i$ *and* $\mathcal{I}_\Diamond$ *over the* $|x| < \varepsilon^{-1}$ *part of* $\mathbb{R}^3 - \mathcal{T}_\varepsilon$ *that identifies* ν_i *as an* $\mathcal{I}_\Diamond$ *valued 1-form that differs from* $\nu_\Diamond$ *by an* $\mathcal{I}_\Diamond$ *valued 1-form with any* $k < \varepsilon^{-1}$ *version of the* C^k *norm being less than* ε^{-1}.

Proof of Lemma 9.3. The proof has seven parts. Part 4 constructs a closed set $Z_\Diamond \subset \mathbb{R}^3$ and data $(\mathcal{I}_\Diamond, \nu_\Diamond)$ from a given sequence $\{\lambda_i\}_{i=1,2,\ldots}$ with $\mathcal{I}_\Diamond$ being a real line bundle on $\mathbb{R}^3 - Z_\Diamond$ and with $\nu_\Diamond$ being an $\mathcal{I}_\Diamond$ valued 1-form on $\mathbb{R}^3 - Z_\Diamond$. Part 4 also verifies that $(Z_\Diamond, \mathcal{I}_\Diamond, \nu_\Diamond)$ is described by the second and fourth bullets of Lemma 9.3. Part 5 proves that the data set $(Z_\Diamond, \mathcal{I}_\Diamond, \nu_\Diamond)$ depends only on the sequence $\{p_i\}_{i=1,2,\ldots}$. Part 6 of the lemma proves the scaling relation that is asserted by the third bullet of Lemma 9.3 and Part 7 proves that $Z_\Diamond$ has the form that is asserted by the first bullet of the lemma.

By way of notation, the proof denotes each $i \in \{1, 2, \ldots\}$ version of $\mathrm{dist}(p_i, p)$ by D_i. The proof uses N_0 to denote $\mathrm{N}_{(p)}(0)$; it uses other notation from Section 8 as well.

Part 1: Fix $\lambda \in (0, \frac{1}{100} r_0)$ and suppose that $i \in \{1, 2, \ldots\}$ is such that $\mathrm{D}_i < \lambda$. Reintroduce p's version of the map ϕ_λ from Section 8. The point p_i corresponds via ϕ_λ to a point in $\mathbb{R}^3$ with norm $\lambda^{-1}\mathrm{D}_i$. Denote this point by x_i. Fix $R > 1$ and let $B_{\lambda R}(p_i) \subset M$ denote the ball of radius $R\lambda$ centered at p_i. This ball corresponds to a domain in $\mathbb{R}^3$ via the map ϕ_λ. Denote this domain by $U_{i,R}$. If i is large, then $U_{i,R}$ is very nearly a ball of radius R centered about x_i. In particular, $U_{i,R}$ will be a convex set containing the origin whose boundary is an embedded sphere that is very nearly the sphere where $|x - x_i| = R$. What follows defines what is meant by the term 'very nearly': Fix $k \geq 0$ and $\delta > 0$. Then there exists a λ-independent number $\iota_{k,\delta}$ such that if $i > \iota_{k,\delta}$, then the boundary of $U_{i,R}$ will differ by at most δ in the C^k topology from the round sphere in $\mathbb{R}^3$ with radius R and center x_i. This is because the composition of first the Gaussian coordinate chart map centered at p_i with the inverse of that centered at p has the schematic form $x \to x + \lambda x_i + \mathfrak{r}(\mathrm{D}_i, x)$ where $\mathfrak{r}$ has C^{100k} norm bounded by a k-dependent multiple of D_i and is such that $|\mathfrak{r}(\mathrm{D}_I, x)| \leq c_0 \mathrm{D}_i |x|$.

It follows from what was just said that if $\lambda < c_0^{-1} r_0$, and if the index i is large, then $U_{i,R}$ contains the $|x| < R - 2\lambda^{-1} D_i$ ball and it, in turn, is contained in the $|x| < R + 2\lambda^{-1} D_i$ ball. Use this last observation with Lemma 8.3 to write

$$\int_{\partial U_{i,R}} |\nu_\lambda|^2 = (1 + \mathfrak{e}_{i,R} + \mathfrak{e}_{\lambda,R}) R^{2+2N_0}, \tag{9.1}$$

where $\mathfrak{e}_{i,R}$ accounts for the fact that the boundary of $U_{i,R}$ is not the $|x| = R$ sphere and where $\mathfrak{e}_{\lambda,R}$ accounts for the fact that $N_{(p)}(\lambda R)$ is not exactly N_0. In particular $|\mathfrak{e}_{i,R}|$ is at most $c_0 R^{-1} \lambda^{-1} D_i$. Meanwhile, if $\delta > 0$ has been specified, then $|\mathfrak{e}_{\lambda,R}|$ will be less than δ if λR is sufficiently small because $N_{(p)}(\cdot)$ is continuous.

Part 2: Fix $\delta > 0$ for the moment. Having done so, take $\lambda < \delta^3$ and also small enough so as to guarantee that any $R \le \delta^{-1}$ version of $|\mathfrak{e}_{\lambda,R}|$ is less than δ. As explained in the next paragraph, with λ so chosen, then the following are true:

(9.2)
- $|N_{(p_i)}(\lambda) - N_0| \le c_0(\lambda^{-1} D_i + \delta + \lambda^2)$
- $N_{(p_i)}(r) \le N_0 + c_0(\lambda^{-1} D_i + \delta + \lambda^2 R^2)$ *if* $r \in (0, R\lambda)$ *and* $R < \delta^{-1}$.
- $N_{(p_i)}(r) \ge N_0 - \delta - c_0 r^2$ *if* i *is large*, $r \in (0, R\lambda)$ *and* $R < \delta^{-1}$.

To prove the assertion in the top bullet, first take $s = 2\lambda$ and $r = \lambda$ in p_i's version of (7.34) use the resulting equation with Item c) of the fourth bullet from the p_i version of Proposition 7.1 to see that $h_{(p_i)}(2\lambda)/h_{(p_i)}(\lambda) \ge (1 - c_0\lambda^2) 2^{2(1+N_{(p_i)}(\lambda))}$. Indeed, Item c) of the fourth bullet of p_i's version of Proposition 7.1 asserts that $N_{(p_i)}(r) \ge N_{(p_i)}(\lambda)$ up to a small error on the interval $[\lambda, 2\lambda]$; and this fact with p_i's version of (7.34) leads to the asserted lower bound for $h_{(p_i)}(2\lambda)/h_{(p_i)}(\lambda)$. Invoke p_i's version of (7.34) a second time with $s = \lambda$ and $r = \frac{1}{2}\lambda$ and use Item c) in the fourth bullet from the p_i version of Proposition 7.1 to see that $h_{(p_i)}(\lambda)/h_{(p_i)}(\frac{1}{2}\lambda) \le (1 + c_0\lambda^2) 2^{2(1+N_{(p_i)}(\lambda))}$. The proof of this inequality uses Item c) of the fourth bullet from p_i's version of Proposition 7.1 to conclude that $N_{(p_i)}(r) \le N_{(p_i)}(\lambda)$ but for a small error for $r \in [\frac{1}{2}\lambda, \lambda]$. This fact plus (7.34) leads to the asserted upper bound for $h_{(p_i)}(\lambda)/h_{(p_i)}(\frac{1}{2}\lambda)$. A comparison of these two inequalities with the respective $R = 2\lambda$, $R = \lambda$ and $R = \frac{1}{2}\lambda$ versions of (9.1) leads directly to the assertion in (9.2)'s top bullet. The $r < \lambda$ version of the middle bullet in (9.2) follows from the top bullet using Item c) from the fourth bullet of p_i's version of Proposition 7.1. The $r \in (\lambda, R\lambda)$ version of the middle bullet in (9.2) follows from (9.1) by invoking p_i's version of (7.34) with Item c) of the fourth bullet in p_i's version Proposition 7.1 to see that $h_{(p_i)}(R\lambda)/h_{(p_i)}(r) \geq (1 - c_0R^2\lambda^2)R^{2(1+N_{(p_i)}(r))}$. The assertion in the lower bullet follows from Item c) of the fourth bullet in the p_i version of Proposition 7.1 given the assumption that $\lim_{i\to\infty} N_{(p_i)}(0) = N_0$.

Part 3: Choose $\delta \in (0,1)$ and then $\lambda < \delta$ and in particular, so that $|\mathfrak{e}_{\lambda,1/\delta}| < \delta$. With λ so chosen, fix $i \in \{1,2,\dots\}$ so that $\lambda^{-1}D_i < \delta^2$ and so $\lambda_i^{-1}\lambda > \delta^{-2}$. The Gaussian coordinates centered at p_i that are used to define ν_i identify the ball of radius λ centered at p_i with the ball of radius $\lambda_i^{-1}\lambda$ centered at the origin in $\mathbb{R}^3$. Use this identification to view $|\nu_i|$ as a continuous function on the $|x| < \delta^{-1}$ ball in $\mathbb{R}^3$. By way of a reminder, the zero locus of $|\nu_i|$ on this ball is the $|x| < \delta^{-1}$ part of Z_i. The line bundle $\mathcal{I}_i$ on this ball is the pull-back of $\mathcal{I}$ to the complement of Z_i via the map $\phi_{(p_i)\lambda_i}$; and the 1-form ν_i on this ball is an $\mathcal{I}_i$ valued 1-form on the complement of Z_i.

The integral of $|\nu_i|^2$ over the $|x| = 1$ sphere in $\mathbb{R}^3$ is equal to 1, this by definition. Fix $R \in (0,\delta^{-1})$ and use what is said by the second and third bullets of (9.2) with the p_i version (7.34) to write

$$\int_{|x|=R} |\nu_i|^2 = (1 + \mathfrak{r}_{i,R})R^{2+2N_0}, \tag{9.3}$$

with the absolute value of $\mathfrak{r}_{i,R}$ obeying the bound

$$|\mathfrak{r}_{i,R}| \leq c_0(\lambda_i^2R^2 + R^{c_0\delta}). \tag{9.4}$$

Introduce ν_{ir} to denote the radial component of ν_i and $\nabla_r\nu_i$ to denote the covariant derivative of ν_i in the radial direction on $\mathbb{R}^3$. Keeping in mind that δ in (9.2) and (9.4) be chosen as small as desired, the arguments that prove Lemmas 8.3 and 8.4 and (8.19) can be invoked using the middle and lower bullets of (9.2), (9.3) and (9.4) to see that

(9.5)
- $\{\int_{|x|=R} |\nu_{ir}|^2\}_{i=1,2,\dots}$ *converges with limit* $\frac{1+N_0}{3+2N_0}R^{2+2N_0}$,
- $\{\int_{|x|\leq R} |\nu_i|^2\}_{i=1,2,\dots}$ *converges with limit* $\frac{1}{3+2N_0}R^{3+2N_0}$,
- $\{\int_{|x|\leq R} |\nabla\nu_i|^2\}_{i=1,2,\dots}$ *converges with limit* $N_0R^{1+2N_0}$,
- $\{\int_{|x|=R} |\nabla\nu_i|^2\}_{i=1,2,\dots}$ *converges with limit* $N_0(1+2N_0)R^{2N_0}$,
- $\{\int_{|x|=R} |\nabla_r\nu_i|^2\}_{i=1,2,\dots}$ *converges with limit* $N_0^2R^{2N_0}$.
- $\limsup_{i\to\infty}(\sup_{|x|\leq R} |\nu_i|) \leq c_0R^{N_0}$.

Part 4: The bounds given in (9.3) and (9.4) with those in the first three bullets and the last bullet of (9.5) lead directly to the following conclusion: There is a subsequence $\Lambda_0 \subset \{1, 2, \dots\}$ such that the corresponding sequence of functions $\{|\nu_i|\}_{i\in\Lambda}$ converges weakly in the L^2_1 topology on every ball about the origin in $\mathbb{R}^3$. The limit function defines an $L^2_{1;loc}$ function on $\mathbb{R}^3$ with norm bounded by κR^{N_0} on the radius R ball about the origin in $\mathbb{R}^3$. Meanwhile, the sequence $\{|\nabla\nu_\lambda|\}_{\lambda\in\Lambda}$ converges weakly in the L^2 topology on each such ball. Let $|\nu_\Diamond|$ denote the limit function. An almost verbatim repetition of the arguments that prove Lemma 8.6 proves that $|\nu_\Diamond|$ is Hölder continuous with exponent $\upsilon = c_0^{-1}$ and that $\{|\nu_i|\}_{i\in\Lambda_\Diamond}$ converges as $i \to \infty$ to $|\nu_\Diamond|$ in the exponent υ Hölder topology on every ball about the origin in $\mathbb{R}^3$.

Let $Z_\Diamond \subset \mathbb{R}^3$ denote the zero locus of $|\nu_\Diamond|$. This is a closed set because $|\nu_\Diamond|$ is continuous. The fact that $\{|\nu|\}_{i\in\Lambda_0}$ converves to $|\nu_\Diamond|$ on compact subsets of $\mathbb{R}^3$ in a Hölder norm has the following consequence: Fix $\varepsilon > 0$. Then the $|x| < \varepsilon^{-1}$ part of all sufficiently large i versions of Z_i lie in the radius ε tubular neighborhood of $Z_\Diamond$. Granted that this is the case, then the arguments used to prove the first three bullets of Lemma 8.7 can be reused with only cosmetic changes to define the line bundle $\mathcal{I}_\Diamond$ over $\mathbb{R}^3 - Z_\Diamond$ and the desired $\mathcal{I}_\Diamond$ valued harmonic 1-form $\nu_\Diamond$. This is to say that $\mathcal{I}_\Diamond$ is defined to be isomorphic to suitably large i versions of $\mathcal{I}_i$ over compact subsets of $\mathbb{R}^3 - Z_\Diamond$ and $\nu_\Diamond$ is then the C^∞ Frêchet space limit on any given compact subsets in $\mathbb{R}^3 - Z_\Diamond$ of the large i part of the sequence $\{\nu_i\}_{i\in\Lambda_\Diamond}$. Convergence of $\{\nu_i\}_{i\in\Lambda_\Diamond}$ in the C^∞ Frêchet space topology occurs on compact subsets for the reason given in the proof of Lemma 8.7 for the analogous convergence of $\{\nu_\lambda\}_{\lambda\in\Lambda}$ to ν_*. The limit 1-form $\nu_\Diamond$ is harmonic for the same reason that ν_* is harmonic.

Note that the assertions of Items i)–iv) of the fourth bullet from Lemma 8.7 hold with $\nu_\Diamond$ replacing ν_*, this being a consequence of (9.5) and what was said in the preceding paragraphs about convergence of $\{|\nu_i|\}_{i\in\Lambda_\Diamond}$ on compact subsets of $\mathbb{R}^3$ and of $\{\nu_i\}_{i\in\Lambda_\Diamond}$ on compact subsets of $\mathbb{R}^3 - Z_\Diamond$.

Part 5: This part of the proof explains why the data $(Z_\Diamond, \mathcal{I}_\Diamond, \nu_\Diamond)$ does not depend on the choice of the sequence $\{\lambda_i\}_{i=1,2,\dots}$. The argument has five steps.

Step 1. Let $\{\lambda'_i\}_{i=1,2,\dots}$ and $\{\lambda''_i\}_{i=1,2,\dots}$ denote decreasing sequences in $(0, \frac{1}{100}r_0)$ that converge to zero. Let $\Lambda'_\Diamond \subset \{\lambda'_i\}_{i=1,2,\dots}$ and $\Lambda''_\Diamond \subset \{\lambda''_i\}_{i=1,2,\dots}$ denote two subsequence with the properties that are described in Part 4. Relable the sequence $\Lambda'_\Diamond$ by the odd integers starting with 1 and relable $\Lambda''_\Diamond$

by the even integers starting from 2 so as to obtain a new sequence to be denoted now by $\{\lambda_i\}_{i=1,2,\ldots}$.

Step 2. Fix $i \in \{1, 2, \ldots\}$ and define

$$L_i = \int_{|x|=1} (|\nu_{2i}|(\cdot) - |\nu_{2i-1}|(\cdot)|)^2. \tag{9.6}$$

Given a contant, unit length vector v on $\mathbb{R}^3$, define $(\nu_{2i-1})_v$ to be the section of I_{2i-1} that is defined over the complement of Z_{2i-1} in the $|x| < 2$ ball by pairing ν_{2i} with v. The analogous rule defines $(\nu_{2i})_v$ as a section of I_{2i} over the complement of Z_{2i} in the $|x| < 2$ ball. Granted these definitions, define $L_{i,v}$ by replacing ν_{2i-1} in (9.6) by $(\nu_{2i-1})_v$ and replacing ν_{2i} by $(\nu_{2i})_v$. The next step explains why the sequences $\{L_i\}_{i=1,2,\ldots}$ and $\{L_{i,v}\}_{i=1,2,\ldots}$ converge as $i \to \infty$ with limit zero. Granted that such is the case, then the arguments used in Step 3 of the proof of Lemma 8.8 can be reused to prove that the $\{\lambda'_i\}_{i=1,2,\ldots}$ version of Part 4's data set $(Z_\diamondsuit, I_\diamondsuit, \nu_\diamondsuit)$ is the same as the $\{\lambda''_i\}_{i=1,2,\ldots}$ version.

Step 3. Fix $i \in \{1, 2, \ldots\}$. Assume first that $\lambda_{2i} < \lambda_{2i-1}$. Fix $s \in (0, 1]$ and define $\phi_{(p_i)s\lambda_{2i-1}}$ to be the map from the $|x| < 2$ ball in $\mathbb{R}^3$ to the radius $2s\lambda_{2i-1}$ ball centered at p_i that is obtained by composing first the rescaling map $x \to s\lambda_{2i-1}x$ and then the Gaussian coordinate chart map. Define $Z_{i,s}$ on the $|x| < 2$ ball in $\mathbb{R}^3$ to be the $\phi_{(p_i)s\lambda_{2i-1}}$ inverse image of Z, and define $I_{i,s}$ to be the real line bundle over the complement in $Z_{i,s}$ of the $\phi_{(p_i)s\lambda_{2i-1}}$ pull back of I. Then define $\nu_{i,s}$ to be the $I_{i,s}$ valued 1-form on the complement of the $|x| < 2$ ball in $\mathbb{R}^3$ of $Z_{i,s}$ by multiplying the $\phi_{(p_i)s\lambda_{2i-1}}$ pull-back of ν by $(h_{(p_i)}(s\lambda_{2i-1}))^{-1/2}$. Define the function on w_i on $[0, 1]$ by the rule

$$s \to w_i(s) = \int_{|x|=1} \left|\tfrac{\partial}{\partial s}\nu_{i,s}(\cdot)\right|^2 \tag{9.7}$$

If it is the case that $\lambda_{2i} \geq \lambda_{2i-1}$, then define w_i by repeating what was just said with λ_{2i} replacing λ_{2i-1} in all instances.

The argument used in Step 1 of the proof of Lemma 8.8 can be repeated to prove that each $i \in \{1, 2, \ldots\}$ version of w_i defines an L^1 function on $[0, 1]$. This understood, the arguments in Step 1 of the proof of Lemma 8.8 that lead to (8.31) can be repeated using the identities in (9.5) in lieu of their analogs from Lemmas 8.3 and 8.4 to prove that the sequence $\{w_i\}_{i=1,2,\ldots}$ converges to zero in $L^1([0, 1])$.

Meanwhile, an essentially verbatim version of the arguments in Step 2 of the proof of Lemma 8.8 can be used to draw the following conclusion: If

$i \in \{1, 2, \dots\}$, then both L_i and $L_{i,v}$ are bounded by the L^1 norm of w_i. This understood, the conclusions of the preceding paragraph imply that both $\limsup_{i\to\infty} L_i$ and $\limsup_{i\to\infty} L_{i,v}$ are zero.

Part 6: The argument that proves the third bullet of Lemma 9.3 is essentially the same as that used to prove Lemma 8.9. By way of a quick summary, fix a decreasing sequence $\{\lambda_i\}_{i=1,2,\dots} \subset (0, \frac{1}{100}R^{-1}r_0]$ and construct the data $(Z_\Diamond, \mathcal{I}_\Diamond, \nu_\Diamond)$ first using the latter. Having done so, construct this data again using the sequence $\{R\lambda_i\}_{i=1,2,\dots}$. The analog of each $i \in \{1, 2, \dots\}$ version of ν_i for the latter sequence is denoted by $\nu_{i,R}$. To prove the scaling relation asserted by the third bullet of Lemma 9.3, first use the definitions of ν_i and $\nu_{i,R}$ imply the identity $\nu_{i,R} = (\frac{h_{(p_i)}(\lambda_i)}{h_{(p_i)}(R\lambda_i)})^{1/2}\psi_R{}^*\nu_i$. The desired scaling relation follows directly from the assertion that any fixed R version of the sequence $\{(\frac{h_{(p_i)}(\lambda_i)}{h_{(p_i)}(R\lambda_i)})^{1/2}\}_{i=1,2,\dots}$ converges to $R^{-1-N_{(p)}(0)}$. The two steps that follow prove this convergence assertion.

Step 1. Fix $\delta > 0$ for the moment but less than R^{-1} and then fix $\lambda < \delta^3$. Use the second and third bullets of (9.2) with p_i's version of (7.34) to see that $(\frac{h_{(p_i)}(\lambda_i)}{h_{(p_i)}(R\lambda_i)})^{1/2}$ can be written as $(1+\mathfrak{z}_{i,R})R^{-1-N_{(p)}(0)}$ with $|\mathfrak{z}_{i,R}| \le c_0(\delta + \lambda^{-1}D_i + \lambda^2R^2 + \lambda_i^2R^2)\ln R$.

Step 2. Keeping in mind that $R < \delta^{-1}$ and $\lambda < \delta^3$, the preceding bound implies that $|\mathfrak{z}_{i,R}| \le c_0\delta|\ln\delta| + c_0(\lambda^{-1}D_I + \lambda_i^2R^2)\ln R)$. The latter in turn implies that $|\mathfrak{z}_{i,R}| < c_0\delta|\ln\delta|$ when i is sufficiently large. As there is no positive lower bounds for the choice of δ, this bound for $|\mathfrak{z}_{i,R}|$ implies that the $i \to \infty$ limit of $(\frac{h_{(p_i)}(\lambda_i)}{h_{(p_i)}(R\lambda_i)})^{1/2}$ has the desired limit.

Part 7: Given the scaling property in Lemma 9.3's third bullet, and given that $\nu_\Diamond$ is harmonic, the arguments that prove $Z_\Diamond$ to be the union of the origin with a finite union of rays through the origin in $\mathbb{R}^3$ are identical to those used in Sections 8.g and 8.h to prove Lemma 8.10 and Lemma 8.12.

Note for future reference that the $\nu_\Diamond$ analogs of what are denoted by α and ν_{*r} in Sections 8.g and 8.h obey (8.42) and (8.43) as the case may be.

9.b. The proof of Proposition 9.2

Proposition 9.2 follows directly from Lemma 9.3 and the lemma given below.

Lemma 9.4. *Let $Z_\Diamond \subset \mathbb{R}^3$ denote the set given by Lemma 9.3 and let $\mathcal{I}_\Diamond$ denote Lemma 9.3's real line bundle over $\mathbb{R}^3 - Z_\Diamond$. Then Z_* is a line through the origin, $Z_\Diamond = Z_*$ and $\mathcal{I}_\Diamond$ is isomorphic to $\mathcal{I}_*$.*

Proof of Lemma 9.4. The proof has three steps.

Step 1. Fix $\mathrm{i} \in \{1, 2, \dots\}$ and let $\mathrm{D_i}$ again denote $\mathrm{dist}(\mathrm{p}, \mathrm{p_i})$. Define λ_i to be $\mathrm{D_i}$. Let $\mathfrak{p}_\mathrm{i} \in \mathbb{R}^3$ denote $\phi_{(\mathrm{p_i})\lambda_\mathrm{i}}$-inverse image of p, this being a point on the $|\mathrm{x}| = 1$ sphere. Keep in mind that $\mathfrak{p}_\mathrm{i}$ is in $\mathrm{Z_i}$. Let $\Lambda_\Diamond \subset \{1, 2, \dots\}$ denote a subsequence that is obtained from $\{\lambda_\mathrm{i} = \mathrm{D_i}\}_{\mathrm{i}=1,2,\dots}$ by invoking the fourth bullet of Lemma 9.3. It follows from this fourth bullet that the set of limit points of the sequence $\{\mathfrak{p}_\mathrm{i}\}_{\mathrm{i}\in\Lambda_\Diamond}$ is a subset of the finite set of $|\mathrm{x}| = 1$ points in $\mathrm{Z}_\Diamond$. Fix a subsequence $\Lambda' \subset \Lambda_\Diamond$ so that the corresponding sequence $\{\mathfrak{p}_\mathrm{i}\}_{\mathrm{i}\in\Lambda'}$ converges and let $\mathrm{p}_\Diamond$ denote the limit point. Keep in mind that $\mathrm{Z}_\Diamond$ intersects any radius $\mathrm{r} \in (0, 1)$ ball in $\mathbb{R}^3$ centered on $\mathrm{p}_\Diamond$ as the $1 - \mathrm{r} < |\mathrm{x}| < 1 + \mathrm{r}$ segment in the ray from the origin that contains $\mathrm{p}_\Diamond$.

Step 2. Return now to the point p's version of Lemma 8.5. This lemma requires at its very start the choice of a decreasing sequence in $(0, \mathrm{r_0})$ with limit zero. Take this sequence to be $\{\lambda_\mathrm{i} = \mathrm{D_i}\}_{\mathrm{i}\in\Lambda'}$ so that sequence Λ in Lemma 8.5's is a subsequence of Λ'. Fix $\mathrm{i} \in \Lambda$. The corresponding function $|\nu_{\lambda_\mathrm{i}}|$ is a function on the $|\mathrm{x}| < \mathrm{c}_0^{-1}\lambda^{-1}$ ball in $\mathbb{R}^3$. This is also the case for the function on $|\nu_\mathrm{i}|$. As explained momentarily, given $\mathrm{R} > 1$, there exists a sequence $\{\varepsilon_{\mathrm{i},\mathrm{R}}\}_{\mathrm{i}=1,2,\dots} \subset (0, \mathrm{c_0})$ with limit zero such that

$$(9.8) \qquad ||\nu_\mathrm{i}|(\mathrm{x} + \mathrm{p}_\Diamond) - |\nu_{\lambda_\mathrm{i}}|(\mathrm{x})| \le \varepsilon_{\mathrm{i},\mathrm{R}} \quad \textit{at all points } \mathrm{x} \textit{ with } |\mathrm{x}| \le 2\mathrm{R}.$$

Indeed, this follows from three facts. The first is that each $\mathrm{i} \in \Lambda$ version of $|\nu_\mathrm{i}|$ is the pull-back of $|\nu_{\lambda_\mathrm{i}}|$ by a diffeomorphism that sends x to $\mathrm{x} - \mathfrak{p}_\mathrm{i} + \mathfrak{e}_\mathrm{i}$ with the various $\mathrm{i} \in \Lambda$ versions of $\mathfrak{e}_\mathrm{i}$ such that $\lim_{\mathrm{i}\to\infty} |\mathfrak{e}_\mathrm{i}| = 0$ on compact subsets of $\mathbb{R}^3$. The second fact is that the sequence $\{\mathfrak{p}_\mathrm{i}\}_{\mathrm{i}\in\Lambda}$ converges to $\mathrm{p}_\Diamond$, and the third is that the sequences $\{|\nu_\mathrm{i}|\}_{\mathrm{i}\in\Lambda}$ and $\{|\nu_{\lambda_\mathrm{i}}|\}_{\mathrm{i}\in\Lambda}$ are uniformly bounded in an exponent $\upsilon = \mathrm{c}_0^{-1}$ Hölder space on compact subsets of $\mathbb{R}^3$.

Step 3. It follows from (9.8) that the functions $\mathrm{x} \to |\nu_\Diamond|(\mathrm{x} + \mathrm{p}_\Diamond)$ and $\mathrm{x} \to |\nu_*|(\mathrm{x})$ are the same. This understood, what is said at the end of Step 1 leads to the following conclusion: The version of Z_* that is define for p by the sequence $\{\lambda_\mathrm{i}\}_{\mathrm{i}\in\Lambda}$ is a line through the origin in $\mathbb{R}^3$. Given that Lemma 8.8 asserts in part that all versions of Z_* are one and the same subset in $\mathbb{R}^3$, the fact that $|\nu_\Diamond|(\mathrm{x} + \mathrm{p}_\Diamond)$ and $|\nu_*|(\mathrm{x})$ are equal at all $\mathrm{x} \in \mathbb{R}^3$ implies that $\mathrm{Z}_\Diamond = \mathrm{Z}_*$.

To see about Lemma 9.4's assertion about $\mathcal{I}_\Diamond$ and $\mathcal{I}_*$, let v denote a constant, unit length vector on $\mathbb{R}^3$. Given $\mathrm{i} \in \Lambda$, use $(\nu_\mathrm{i})_\mathrm{v}$ to denote the pairing between ν_i and v, this being a section of $\mathcal{I}_\mathrm{i}$ on $\mathbb{R}^3 - \mathrm{Z_i}$. Define $(\nu_{\lambda_\mathrm{i}})_\mathrm{v}$ to be the analogous pairing between the $\mathcal{I}_{\lambda_\mathrm{i}}$ valued 1-form ν_{λ_i} and v. The argument that leads to (9.8) can be repeated to prove that the analog of

(9.8) holds with $(\nu_i)_v$ replacing ν_i and with $(\nu_{\lambda_i})_v$ replacing ν_{λ_i}. This being the case, then the argument used in the preceding paragraph can be repeated with only notational changes to prove that the $|(\nu_\diamondsuit)_v|(x+p_\diamondsuit)$ and $|(\nu_*)_v|(x)$ are equal at all $x \in \mathbb{R}^3$. Granted this last assertion, then an almost verbatim repetition of what is said in Step 3 of the proof of Lemma 8.8 proves that $I_\diamondsuit$ and I_* are isomorphic.

9.c. Proof of Proposition 9.1

The assertion that Z_* is a line through the origin is part of what is asserted by Lemma 9.4. To see about $N_{(p)}(0)$, reintroduce $\nu_\diamondsuit$. Mimic what was done in Section 8.f with ν_* by writing $\nu_\diamondsuit$ as $\nu_{\diamondsuit r}dr + \nu_{\diamondsuit\perp}$ with $\nu_{\diamondsuit r}$ denoting a section of $I_\diamondsuit$ on the complement of $Z_\diamondsuit$ and with $\nu_{\diamondsuit\perp}$ being an $I_\diamondsuit$ valued 1-form on the complement of $Z_\diamondsuit$ that annihilates the radial vector field ∂_r. Restrict the latter to the $|x| = 1$ sphere and use the fact that $\nu_\diamondsuit$ is harmonic with the scaling relation $\psi_R{}^*\nu_\diamondsuit = R^{1+N_0}\nu_\diamondsuit$ to see that the pair $(\nu_{\diamondsuit r}, \nu_{\diamondsuit\perp})$ when viewed as a pair consisting of an $I_\diamondsuit$ valued function and $I_\diamondsuit$ valued 1-form obey

$$d\nu_{\diamondsuit\perp} = 0, \quad *d{*}\nu_\diamondsuit = -(N_0+2)\nu_{\diamondsuit r} \quad \textit{and} \quad d\nu_{*r} = N_0\nu_{\diamondsuit\perp}, \tag{9.9}$$

with it understod that the exterior derivatives differentiate only in directions tangent to the $|x| = 1$ sphere, and the Hodge star is defined by the round metric on the $|x| = 1$ sphere. This equation holds on the complement of the set $\{p_\diamondsuit, -p_\diamondsuit\}$. Use stereographic projection from $-p_\diamondsuit$ to view $\nu_{\diamondsuit r}$ on the complement of $-p_\diamondsuit$ as an $I_\diamondsuit$ valued function on $\mathbb{C}-0$ and to view $\nu_{\diamondsuit\perp}$ as an $I_\diamondsuit$ valued 1-form on $\mathbb{C}-0$. Write the latter as $\frac{2}{(1+|z|^2)}(\alpha_\diamondsuit d\bar{z} + \bar{\alpha}_\diamondsuit dz)$ to see that the pair $(\nu_{\diamondsuit r}, \alpha_\diamondsuit)$ obey the equation

$$\partial\alpha_\diamondsuit = -\tfrac{1}{2}(2+N_0)(1+r_0)\nu_{\diamondsuit r} \quad \textit{and} \quad \bar{\partial}\nu_{\diamondsuit r} = 2N_0(1+r_1)\alpha_\diamondsuit. \tag{9.10}$$

with r_0 and r_1 being the same here as their namesakes in (8.35).

The arguments in the proof of Lemma 8.12 can repeated to see that α and $\nu_{\diamondsuit r}$ near the origin in $\mathbb{C}$ have the same form as that given for ν_{*r} and α in either (8.42) or (8.43). These equations have the following consequence: There exists $c > 0$ and $k \in \{1, 2, \dots\}$ such that if x is a point on the unit sphere near $p_\diamondsuit$, then $|\nu_\diamondsuit|(x) = c\,\mathrm{dist}(p_\diamondsuit, x)^{k/2} + \cdots$ with the unwritten terms being higher powers of $\mathrm{dist}(p_\diamondsuit, x)$. Given (9.8) and the scaling relation $\psi_R{}^*\nu_* = R^{1+N_0}\nu_*$, this last fact implies that $N_{(p)}(0) = \frac{1}{2}k$.

To prove the final assertion of Proposition 9.1, introduce the Euclidean coordinates (t, z_1, z_2) for $\mathbb{R}^3$ where $p_\diamondsuit$ appears as the point $(1, 0, 0)$. The fact that $|\nu_\diamondsuit|(\mathrm{x}+\mathrm{p}_\diamondsuit) = |\nu_*|(\mathrm{x})$ for all x, and the fact that both $\nu_\diamondsuit$ and ν_* obey the same scaling relation $\psi_{\mathrm{R}}{}^*(\cdot) = \mathrm{R}^{1+\mathrm{N}_0}(\cdot)$ implies that $|\nu_*|$ when written as a function of $(\mathrm{t}, \mathrm{z}_1, \mathrm{z}_2)$ has no dependence on t. This last fact with what was said in the preceding paragraph implies that $|\nu_*| = c(\mathrm{z}_1^2 + \mathrm{z}_2^2)^{\mathrm{N}_0/2} + \cdots$ with the unwritten terms being higher powers of $(\mathrm{z}_1^2 + \mathrm{z}_2^2)$. The writing of $|\nu_*|$ in this way verifies what is asserted by Proposition 9.1's third bullet.

10. Lipschitz curves and the function $\mathrm{N}_{(\cdot)}(0)$

The upcoming Proposition 10.1 implies the assertion that is made by Item 1) of the second bullet in Theorem 1.1.

By way of a reminder from the start of Section 8, a twisted harmonic form data set $(\mathrm{Z}, \mathcal{I}, \nu)$ has $\mathrm{Z} \subset \mathrm{M}$ being a closed set and $\mathcal{I} \to \mathrm{M}{-}\mathrm{Z}$ being a real line bundle and ν being a harmonic, $\mathcal{I}$-valued 1-form on $\mathrm{M}{-}\mathrm{Z}$. In addition, $|\nu|$ extends to M as a continuous function with $\mathrm{Z} \subset |\nu|^{-1}(0)$; the second, third and fourth bullets of Proposition 7.1 are obeyed; and the conclusions of Lemma 7.7 hold.

Proposition 10.1. *The set* Z *from a twisted harmonic form data set* $(\mathrm{Z}, \mathcal{I}, \nu)$ *is contained in a countable union of embedded Lipshitz curves. Moreover, given* $\varepsilon > 0$*, there exists a finite set of balls with pairwise disjoint closure whose union has volume less than* ε *and such that* Z*'s intersection with the complement of this union of balls is a properly embedded, finite length Lipshitz curve with finitely many components.*

The proof of this proposition is in Section 10.d. The intervening subsections supply various facts about the function $\mathrm{N}_{(\cdot)}(0)$ that play central roles in the proof.

10.a. Quantization of $\mathrm{N}_{(\cdot)}(0)$

The second bullet of the upcoming Lemma 10.2 implies in part that the function $\mathrm{N}_{(\cdot)}(0)$ has but a countable set of possible values. This is what is meant by the term *quantization* in the subsection title. To set the stage for the lemma, fix $\mathrm{m} \in \{1, 2, \ldots\}$ and introduce $\mathfrak{Z}_{\mathrm{m}}$ to denote the subset of Z where $\mathrm{N}_{(\cdot)}(0) = \frac{1}{2}\mathrm{m}$. Define $\mathrm{Z}_{\mathrm{m}} \subset \mathrm{Z}$ to be the set of points where $\mathrm{N}_{(\cdot)}(0) \geq \frac{1}{2}\mathrm{m}$; and given $\varepsilon \in (0, \frac{1}{2})$, define $\mathrm{Z}_{\mathrm{m},\varepsilon} \subset \mathrm{Z}_{\mathrm{m}}$ to be the set of points where $\mathrm{N}_{(\cdot)}(0) > \frac{1}{2}\mathrm{m} + \varepsilon$.

Lemma 10.2 refers to the notion of a weakly continuous point in Z. By way of a reminder from Section 9, a point $\mathrm{p} \in \mathrm{Z}$ is weakly continuous if there is a subsequence $\{\mathrm{p}_{\mathrm{i}}\}_{\mathrm{i}=1,2,\ldots} \subset \mathrm{Z}$ with limit p such that $\{\mathrm{N}_{(\mathrm{p}_{\mathrm{i}})}(0)\}_{\mathrm{i}=1,2,\ldots}$ converges to $\mathrm{N}_{(\mathrm{p})}(0)$.

Lemma 10.2. *The set* Z *is such that the following are true:*

- *The function* $N_{(\cdot)}(0)$ *is no less than* $\frac{1}{2}$.
- *The complement in* Z *of* $\cup_{m=1,2,\dots}\mathfrak{Z}_m$ *is a countable set. The complement in* Z *of its weakly continuous points is also countable.*
- *Fix* $m \in \{1, 2, \dots\}$ *and* $\varepsilon \in (0, \frac{1}{2})$. *Then*
  - *a) Both* Z_m *and* $Z_{m,\varepsilon}$ *are closed subsets of* Z.
  - *b) All limit points of* $Z_{m,\varepsilon}-Z_{m+1}$ *are in* Z_{m+1}.
  - *c) The set of points in* $Z_{m,\varepsilon}$ *with distance greater than* ε *from* Z_{m+1} *is finite.*

Proof of Lemma 10.2. The proof of the lemma has three parts.

Part 1: This part proves the assertion of the second bullet. Proposition 9.1 asserts in part that the set of weakly continuous points is a subset of $\cup_{m\in\{1,2,\dots\}}\mathfrak{Z}_m$. This being the case, it is sufficient to prove that the complement in Z of the set of weakly continuous points is a countable set. Define $\mathfrak{X} \subset Z$ to be this complement. A point $p \in \mathfrak{X}$ has the following property: If $\{p_i\}_{i=1,2,\dots} \subset Z-p$ converges to p, then $\lim_{i\to\infty} N_{(p_i)}(0) < N_{(p)}(0)$.

Fix $m = \{1, 2, \dots\}$ and define the subset $\mathfrak{X}_m \subset \mathfrak{X}$ using the following criteria: Suppose that $p \in \mathfrak{X}_m$. Then $N_{(\cdot)}(0) < N_{(p)}(0)$ on the complement of p in the ball of radius $\frac{1}{m}$ centered at p. This a finite set because the distance between any pair of distinct points in this set is no less than $\frac{1}{m}$. Indeed, were p and p′ in $\mathfrak{X}_m$ with $p \neq p'$ and $\mathrm{dist}(p, p') < \frac{1}{m}$, then $N_{(p')}(0)$ would be less than $N_{(p)}(0)$ and vice versa. The sets $\{\mathfrak{X}_m\}_{m=1,2,\dots}$ are nested in the sense that $\mathfrak{X}_1 \subset \mathfrak{X}_2 \subset \cdots \subset \mathfrak{X}_m \subset \cdots$, and their union is the whole of $\mathfrak{X}$. This last observation implies that $\mathfrak{X}$ is countable.

Part 2: This part proves Item a) of the third bullet and it explains why Item c) of the third bullet follows from Items a) and b) of the third bullet. To see about Item a), the fact that Z_m is closed follows from the first bullet of Lemma 7.7. The fact that $Z_{m,\varepsilon}$ is closed follows from the first bullet of Lemma 7.7 and the first bullet of Proposition 9.1. The first bullet of Proposition 9.1 is needed to rule out limit points of sequences in $Z_{m,\varepsilon}$ where $N_{(\cdot)}(0) = \frac{1}{2}m + \varepsilon$.

The proof that follows of Item c) of the lemma's third bullet assumes that Item b) is true. Granted that such is the case, suppose that Item c) is false so as to derive nonsense. If Item c) is false, then $Z_{m,\varepsilon}-Z_m$ has a sequence $\{p_i\}_{i=1,2,\dots}$ of distinct points with distance ε or more from Z_{m+1} that converges to a point $p \in Z$. Item b) of the third bullet puts this point in Z_{m+1}. This is nonsense because $\mathrm{dist}(p_i, Z_{m+1}) \geq \varepsilon$ for all $i \in \{1, 2, \dots\}$.

Part 3: This part simultaneously proves Lemma 10.2's first bullet and Item c) of its third bullet. The proof has three steps. Step 3 of what follows proves the lemma's first bullet and Item c) of the third bullet. Steps 1 and 2 supply input for Step 3.

Step 1. Suppose that $p \in Z$ is a limit point of a sequence in Z. Fix a sequence $\{p_i\}_{i=1,2,\ldots} \subset Z$ that converges to p such that each member is in the radius $\frac{1}{100}r_0$ ball centered at p. With $i \in \{1,2,\ldots\}$ fixed, define λ_i to be $\mathrm{dist}(p, p_i)$. Lemma 8.5 supplies a subsequence $\Lambda \subset \{1,2,\ldots\}$ such that the corresponding sequence $\{\nu_{\lambda_i}\}_{i\in\Lambda}$ is described by Lemmas 8.5–8.7. Fix $i \in \{1,2,\ldots\}$ and let $q_i \in \mathbb{R}^3$ denote the inverse image of p_i via the map ϕ_{λ_i}. The point q_i is a zero of ν_{λ_i} and so it lies in Z_i.

It follows from what is said by the third bullet of Lemma 8.7 that there is a point $q_* \in Z_*$ with norm 1 and a subsequence $\Lambda' \subset \Lambda$ such that $\{q_i\}_{i\in\Lambda'}$ converges to q_*. Meanwhile, the point q_* has an associated positive integer and numbers $c > 0$ and $z > 1$ with following significance: If $q \in \mathbb{R}^3$ is such that $|q_* - q| \le z^{-1}$, then

(10.1)
- $|\nu_*|(q) = c(1+\mathfrak{e}_0)|q_* - q|^{k/2}$ *with* $|\mathfrak{e}_0| \le z|q_* - q|^{k/2+1}$.
- $|\nabla\nu_*|(q) = c(1+\mathfrak{e}_1)\frac{1}{2}k|q_* - q|^{k/2-1}$ *with* $|\mathfrak{e}_1| \le z|q_* - q|^{k/2}$.

This follows from scaling relation $\psi_R{}^*\nu_* = R^{1+N_0}\nu_*$ and the formulas in (8.42) and (8.44). The equations in (10.1) lead directly to the following observation: If $\rho \in (0, z^{-1})$, then

(10.2)
- $\int_{|x-q_*|=\rho} |\nu_*|^2 = 4\pi c^2(1+\mathfrak{z}_0)\rho^{2+k}$ *where* $|\mathfrak{z}_0| \le c_0\rho$.
- $\int_{|x-q_*|\le\partial\rho} |\nabla\nu_*|^2 = 2\pi c^2 k(1+\mathfrak{z}_1)\rho^{1+k}$ *where* $|\mathfrak{z}_1| \le c_0\rho$.

What is said in Lemma 8.5 about the convergence of $\{\nu_{\lambda_i}\}_{i\in\Lambda'}$ to ν_* has the following implications when the index $i \in \Lambda'$ is sufficiently large:

(10.3)
- $\int_{|x-q_i|=\rho} |\nu_{\lambda_i}|^2 = 4\pi c^2(1+\mathfrak{z}_{0i})\rho^{2+k}$ *where* $|\mathfrak{z}_{0i}| \le c_0\rho$.
- $\int_{|x-q_i|\le\rho} |\nabla\nu_{\lambda_i}|^2 = 2\pi c^2 k(1+\mathfrak{z}_{1i})\rho^{1+k}$ *where* $|\mathfrak{z}_{1i}| \le c_0\rho$.

The equations in (10.3) in turn lead to an estimate for the function $\mathrm{N}_{(p_i)}$ that appears in the p_i version of Proposition 7.1, this being the following: Fix $\rho \in (0, z^{-1})$. If the index $i \in \Lambda'$ is sufficiently large, then

$$\mathrm{N}_{(p_i)}(\lambda_i\rho) = (1+\mathfrak{z}_i)\frac{k}{2} \quad \textit{where } |\mathfrak{z}_i| \le c_0\rho. \tag{10.4}$$

To deduce this last equation from (10.3), reintroduce the metric $\mathfrak{m}_{\lambda_i}$on the $|x| \le \lambda_i^{-1}r_0$ ball in $\mathbb{R}^3$, this being the metric that is obtained by multiplying the ϕ_{λ_i} pull-back of the metric on M by λ_i^{-2}. The metric $\mathfrak{m}_{\lambda_i}$ differs from the Euclidean metric by $c_0\lambda_i^2|x|^2$ and its derivatives are bounded by $c_0\lambda_i^2|x|$. This fact with (10.3) can be used to estimate the functions $h_{(p_i)}$ and $rH_{(p_i)}$ at $r = \lambda_i\rho$; and the resulting estimates lead directly to (10.4).

Step 2. Fix $R \geq 100$ and $i \in \Lambda'$ such that $\lambda_i R < r_0$. Given that ν_* is harmonic, what is said in Items i) and ii) from the fourth bullet of Lemma 8.7 imply that

(10.5)
- $\int_{|x-q_*|=R} |\nu_*|^2 = (1+\mathfrak{r}_0)R^{2+2N_0}$ *where* $|\mathfrak{r}_0| \leq c_0R^{-1}$.
- $\int_{|x-q_*|=R} |\nabla\nu_*|^2 = (1+\mathfrak{r}_1)N_0R^{1+2N_0}$ *where* $|\mathfrak{r}_1| \leq c_0R^{-1}$.

Granted (10.4), what is said by Lemma 8.5 has the following implications: If i is greater than a purely R dependent constant, then

(10.6)
- $\int_{|x-q_i|=R} |\nu_{\lambda_i}|^2 = (1+\mathfrak{r}_{0i})R^{2+2N_0}$ *where* $|\mathfrak{r}_{0i}| \leq c_0R^{-1}$.
- $\int_{|x-q_i|=R} |\nabla\nu_{\lambda_i}|^2 = (1+\mathfrak{r}_{1i})N_0R^{1+2N_0}$ *where* $|\mathfrak{r}_{1i}| \leq c_0R^{-1}$.

The equations in (10.6) lead directly to the following conclusion: If i is greater than a purely R dependent constant, then

$$N_{p_i}(\lambda_i R) = (1+\mathfrak{r}_i)N_0 \quad \textit{where } |\mathfrak{r}_i| \leq c_0R^{-1}. \tag{10.7}$$

By way of an explanation, (10.7) is obtained from (10.6) by using (10.6) with what is said in Step 1 about the metric $\mathfrak{m}_{\lambda_i}$ to estimate the values of $h_{(p_i)}$ and $H_{(p_i)}$ at $r = \lambda_i R$.

Step 3. To see about Item c) of the third bullet of Lemma 10.2, fix $m \in \{1, 2, \ldots\}$ and $\varepsilon \in (0, \frac{1}{2})$. Let $\{p_i\}_{i=1,2,\ldots}$ denote a sequence in $Z_{m,\varepsilon}$ that converges, and let $p \in Z$ denote the limit point. Fix $\rho \in (0, z^{-1})$. What is said by (10.4) with Item c) from the fourth bullet in the p_i version of Proposition 7.1 implies that $N_{(p_i)}(0) \leq (1+c_0\rho)\frac{k}{2}$ when $i \in \Lambda'$ is sufficiently large. Since $N_{(p_i)}(0) > \frac{m}{2} + \varepsilon$, this requires that $k \geq (1-c_0\rho)(m+\varepsilon)$. Meanwhile, (10.7) and $R = \rho^{-1}$ version of (10.7) with Item c) from the fourth bullet in the p_i version of Proposition 7.1 implies that $N_0 \geq (1-c_0\rho)\frac{k}{2}$ when i is large. Since ρ can be as small as desired, the lower bound on k implies that k can be no less than $m+1$ and the lower bound on N_0 implies that N_0 can be no less than $\frac{1}{2}(m+1)$. Thus p is in Z_{m+1}.

To prove the first bullet of Lemma 10.2, let $\mathfrak{n}$ denote $\inf_{p\in M} N_{(p)}(0)$. The second bullet of Lemma 7.7 asserts that $\mathfrak{n}$ is greater than zero. Fix $\varepsilon \in (0, \frac{1}{2})$ and then fix a point $p \in Z$ with $N_{(p)}(0) < \mathfrak{n} + \varepsilon$. Suppose first that p is not an isolated point in Z. Then there is a sequence $\{p_i\}_{i\in\{1,2,\ldots\}} \subset Z$ that converges to p. Fix $\rho \in (0, z^{-1})$ and a very large $i \in \Lambda'$. Use (10.4), the $R = \rho^{-1}$ version (10.7) and Item c) from the fourth bullet of the p_i version of Proposition 7.1 to conclude that N_0 is no less than $(1 - c_0\rho)\frac{1}{2}k$. Since ρ can be as small desired, this implies that $\mathfrak{n} \geq \frac{1}{2}k - \varepsilon$. Since ε can be taken as small as desired, this last bound implies that $\mathfrak{n} \geq \frac{1}{2}$.

If p is an isolated point of Z, then the bundle $\mathcal{I}$ is necessarily isomorphic to the product bundle on the complement of p in a sufficiently small radius ball centered at p. This being the case, $\mathcal{I}$ extends to the whole of this ball as the product real line bundle and ν defines an $\mathbb{R}$-valued harmonic 1-form on this ball. It follows that ν is smooth on this ball and Aronzajn's theorem [Ar] can be invoked to prove that it vanishes to finite order at p. This fact with Taylor's theorem implies that there is a positive integer k and a positive integer c such that $|\nu| = c\,\mathrm{dist}(p, \cdot)^k + \cdots$ near p with the unwritten terms being bounded by a multiple of $\mathrm{dist}(p, \cdot)^{k+1}$. Given what is said by the second bullet of Lemma 8.2, and given (7.34) and Item b) of the fourth bullet of Proposition 7.1, this depiction of $|\nu|$ near p implies that $N_{(p)}(0) = k$ and thus $\mathfrak{n} > k - \varepsilon$. Since ε can be as small as desired, so $\mathfrak{n}$ can not be less than 1.

10.b. The angle inequality

To set the notation for the upcoming Lemma 10.3, introduce $\mathbb{R}*$ to denote the multiplicative group of non-zero real numbers and let $\mathbb{P}M$ denote the $\mathbb{RP}^2$ bundle $(TM{-}0)/\mathbb{R}*$. Suppose that q, q_1 and q_2 are points in Z with q_1 and q_2 being distinct from q but such that $\mathrm{dist}(q, q_1)$ and $\mathrm{dist}(q, q_2)$ are less than $\frac{1}{100}r_0$. The tangent vector at q to the short geodesic segment between q_0 and q_1 defines a point in $\mathbb{P}M|_q$. Meanwhile, the tangent vector at q to the short geodesic segment between q and q_2 defines a second point in $\mathbb{P}M|_q$. Define Δ to be the distance in $\mathbb{P}M|_q$ between these two points. The lemma that follows concerns the behavior of Δ near a given weakly continuous point.

Lemma 10.3. *Let* $p \in Z$ *denote a weakly continuous point. Given* $\varepsilon > 0$*, there exists* $\kappa_\varepsilon > 1$ *with the following significance: Suppose that* q*,* q_1 *and* q_2 *are points in* Z *with distance less than* κ_ε^{-1} *from* p *with* q_1 *and* q_2 *being distinct from* q*. Suppose in addition that* $|N_{(q)}(0) - N_{(p)}(0)| < \kappa_\varepsilon^{-1}$*. Then the* (q, q_1, q_2) *version of* Δ *is less than* ε*.*

Proof of Lemma 10.3. Suppose that the lemma is false so as to generate nonsense. This is done in three steps.

Step 1. If the lemma is false, then there exists $\varepsilon > 0$ and a sequence $\{\mathfrak{q}_i\}_{i=1,2,\ldots}$ of the following sort: Each $i \in \{1, 2, \ldots\}$ version of $\mathfrak{q}_i$ is a triple (q_i, q_{i1}, q_{i2}) of points in Z with $\Delta \geq \varepsilon$, such that each point in $\mathfrak{q}_i$ has distance less than r_0 from p. In addition, $\lim_{i\to 0} |N_{(q_i)}(0) - N_0| = 0$ and $\lim_{i\to\infty} \sup_{q\in\mathfrak{q}_i} \mathrm{dist}(q, p) = 0$.

Step 2. Suppose that there are but finitely many elements in $\{\mathfrak{q}_i\}_{i=1,2,\ldots}$ that have the form (p, q_1, q_2). Use Lemma 9.3 to construct $(Z_\diamondsuit, I_\diamondsuit, \nu_\diamondsuit)$ using the sequences $\{p_i = q_i\}_{i=1,2,\ldots}$ and $\{\lambda_i = \mathrm{dist}(q_i, q_{i1})\}_{i=1,2,\ldots}$ for the input data. Use $\Lambda_\diamondsuit$ to denote the subsequence that is supplied by Lemma 9.3. Invoke Lemma 9.3 a second time using the sequences $\{p_i = q_i\}_{i\in\Lambda_\diamondsuit}$ and $\{\lambda_i = \mathrm{dist}(q_i, q_{i2})\}_{i\in\Lambda_\diamondsuit}$ for the input data. In the first instance, the resulting version of $Z_\diamondsuit$ is a line through the origin in $\mathbb{R}^3$. In the second instance, the resulting version of $Z_\diamondsuit$ is also a line through the origin in $\mathbb{R}^3$. The assertion that these are lines is part of Lemma 9.4. These two lines define respective points in $\mathbb{P}M|_p$ with distance $c_0^{-1}\varepsilon$ or greater between them. The next paragraph explains why. This claim about the distance between the two points in $\mathbb{P}M|_p$ constitutes the desired nonsense because it contradicts what is said by Lemma 9.3 to the effect that $Z_\diamondsuit$ does not depend on the choice of $\{\lambda_i\}$.

To explain why the two versions of $Z_\diamondsuit$ define distinct points in $\mathbb{P}M|_p$, keep in mind that each $i \in \{1, 2, \ldots\}$ version of q_{1i} in the first instance defines a point on the $|x| = 1$ sphere where $|\nu_i| = 0$. This is the case in the second instance with each $i \in \{1, 2, \ldots\}$ version of q_{2i}. The lines from the origin to these respective points define points in $\mathbb{RP}^2$ with distance $c_0^{-1}\varepsilon$ or greater between them because the $\mathfrak{q}_i$ version of Δ is no less than ε. Granted this observation, the claim about the distance between the two points in $\mathbb{P}M|_p$ follows from Item b) from the fourth bullet of Lemma 9.3.

Step 3. If a subsequence in $\{\mathfrak{q}_i\}_{i=1,2,\ldots}$ has the form (p, q_1, q_2), then pass to this subsequence and renumber it consecutively from 1. Contruct Z_* as directed in Section 8 twice, the first taking Lemma 8.5's input subsequence from $(0, r_0)$ to be $\{\mathrm{dist}(p, q_{1i})\}_{i=1,2,\ldots}$. With Λ denoting the subsequence from this version of Lemma 8.5, the second appeal to Lemma 8.5 uses the sequence $\{\mathrm{dist}(p, q_{2i})\}_{i\in\Lambda}$. These two instances result in two versions of Z_* that define distinct points in $\mathbb{P}M|_p$. The argument for this is identical but for notation to what is said in the second paragraph of Step 2. The fact that the two versions of Z_* are distinct constitutes the desired nonsense as it contradicts what is said by Proposition 8.1.

10.c. Points where $\mathrm{N}_{(\cdot)}(0)$ is half of a positive integer

To set the notation for what is to come, fix $p \in Z$ and $r \in (0, r_0)$. A subset of the radius r ball centered at p is said to be a *1-dimensional Lipshitz graph* when a Gaussian coordinate chart centered at p identifies the subset with the $|x| < r$ part of a Lipschitz map from a 1-dimensional subspace in $\mathbb{R}^3$ to its orthogonal complement. A 1-dimensional Lipshitz graph is, by definition, a 1-dimensional rectifiable set. As such it has finite 1-dimensional Hausdorff measure and thus finite length. A graph of this sort is almost everywhere a C^1 curve.

To continue setting the notation, suppose that k is a given positive integer and that $\varepsilon \in (0, 1)$. The next lemma introduces $\mathfrak{Z}_{k,\varepsilon}$ to denote the subset in Z where $|\mathrm{N}_{(\cdot)}(0) - \frac{1}{2}k| < \varepsilon$.

Lemma 10.4. *Fix* $k \in \{1, 2, \dots\}$ *and* $p \in \mathfrak{Z}_k$. *There exists* $r_p \in (0, r_0)$ *and* $\varepsilon_p \in (0, 1)$ *with the properties listed below.*

- *If* p *is not weakly continuous, then* $\mathfrak{Z}_{k,\varepsilon_p} \cap B_{r_p} = p$.
- *If* p *is weakly continuous, then the following are true:*
  - *a) The set* $\mathfrak{Z}_{k,\varepsilon_p} \cap B_{r_p}$ *is contained in a connected, 1-dimensional Lipshitz graph whose union has length at most* $4r_p$.
  - *b) If* I *is an open subset of Item a)'s graph and if* $(\mathfrak{Z}_{k,\varepsilon_p} \cap B_{r_p}) \cap I$ *is dense in* I, *then the whole of* I *is in* $Z_k \cap B_{r_p}$ *and so this part of* $Z_k \cap B_{r_p}$ *is an embedded, Lipshitz curve. Moreover, a neighborhood in* I *of each weakly continuous point in* $I \cap \mathfrak{Z}_k$ *is an open subset of* Z.

If $\mathcal{I}$ is isomorphic to the product bundle on a neighborhood of p, then the assertion is little more than a corollary to Theorem 3.1 in [HHL]. The proof given below is modeled on the proof of Lemma 2.3 in [Han].

Proof of Lemma 10.4. The proof has six steps.

Step 1. If p is not a weakly continuous point, then there exists a pair $r \in (0, r_0)$ and $\varepsilon \in (0, 1)$ such that $\mathfrak{Z}_{k,\varepsilon} \cap B_r = p$. This understood, suppose in what follows that p is weakly continuous. Proposition 9.1 asserts that p's version of Z_* is a line in $\mathbb{R}^3$. Fix an orientation for this line to define a coordinate function $\tau : Z_* \to \mathbb{R}$ with $|\tau|$ giving the distance from the origin. Fix $r \in (0, r_0)$ and let B_r denote ball of radius r centered at p. Define $\iota : Z \cap B_r \to \mathbb{R}$ to be the pull-back of the function τ by the composition of the map to the radius r_0 ball in $\mathbb{R}^3$ given by the inverse of a chosen Gaussian coordinate chart and then the orthogonal projection in $\mathbb{R}^3$ to the line Z_*. The map ι is a continuous map.

Step 2. The assertion that follows summarizes the conclusions of this step.

(10.8) *There exist* $r_1 \in (0, \frac{1}{4}r_0)$ *and* $\varepsilon \in (0,1)$ *with the following significance: If* $p' \in Z \cap B_{r_1}$ *and* $|N_{(p')}(0) - N_0| < \varepsilon$, *then* $\iota^{-1}(\iota(p')) = p'$.

To prove this, assume to the contrary that this assertion is false so as to generate nonsense. If this assertion is false, there exist sequences $\{p'_i\}_{i=1,2,\ldots}$ and $\{y_i\}_{i=1,2,\ldots}$ in $(Z{-}p) \cap B_{r_1}$ with the following properties: Both converge to p and $\lim_{i\to\infty} N_{(p'_i)}(0) = N_0$. Moreover, $y_i \neq p'_i$ but $\iota(y_i) = \iota(p'_i)$ for each $i \in \{1, 2, \ldots\}$. Fix a pair of sequences of the sort just described.

For each $i \in \{1, 2, \ldots\}$, define the ordered triple q_i to be $(q_i = p'_i, q_{1i} = y_i, q_{2i} = p)$. The corresponding version of Δ is greater than c_0^{-1}. The existence of the sequence $\{q_i\}_{i=1,2,\ldots}$ is the required nonsense that proves (10.8) because a sequence of this sort runs afoul of Lemma 10.3.

Step 3. Let ê denote the chosen, oriented unit vector along Z_* and let $Z_*^\perp$ denote the 2-dimensional subspace orthogonal to $\hat{e}_*$. Use Π: $\mathbb{R}^3 \to Z_*^\perp$ to denote the orthogonal projection map. This notation used below writes a given point $x \in \mathbb{R}^3$ as $\tau(x)\hat{e} + \Pi(x)$.

Lemma 10.3 has an additional consequence, this being the existence of $r_2 \in (0, r_1)$ whose significance is as follows: Fix $r \in (0, r_2)$ and let p_1 and p_2 denote two points in $\mathfrak{Z}_{k,\varepsilon_p} \cap B_r$. Use the Gaussian coordinate chart that defined ι to view p_1 and p_2 as points in $\mathbb{R}^3$, these denoted by x_1 and x_2. Then

$$|\Pi(x_1) - \Pi(x_2)| \le \tfrac{1}{100}|\tau(x_1) - \tau(x_2)|. \tag{10.9}$$

The inequality in (10.9) plays a central role in Step 5.

Step 4. This step and Step 5 construct the Lipshitz graph for Item a) of the lemma's second bullet. To start this task, fix $r \in (0, \frac{1}{2}r_1)$. Use the inverse of the chosen Gaussian coordinate chart map to view $\mathfrak{Z}_{k,\varepsilon_p} \cap B_{2r}$ as a subset of the $|x| < 2r$ ball in $\mathbb{R}^3$. The latter incarnation is also denoted by $\mathfrak{Z}_{k,\varepsilon_p} \cap B_{2r}$. An inductive construction is given momentarily that produces a countable collection of finite subsets in $\mathfrak{Z}_{k,\varepsilon_p} \cap B_{2r}$ with the properties listed below. This list writes this collection as $\{\Theta_m\}_{m=1,2,\ldots}$.

(10.10)
- *The collection of sets is nested in the sense that* $\Theta_m \subset \Theta_{m+1}$ *for each* $m \in \{1, 2, \ldots\}$.
- *Each point in* $\tau(\mathfrak{Z}_{k,\varepsilon_p} \cap B_{2r})$ *has distance at most* $\frac{1}{m}r$ *from some point in* $\tau(\Theta_m)$.

Fix $m \in \{1, 2, \ldots\}$ and let N_m denote the number of points in Θ_m. The points in Θ_m are labeled as $\{x_{m,1}, \ldots, x_{m,N_m}\}$ so that $\tau(x_{m,i}) < \tau(x_{m,i+1})$ for all $i \in \{1, \ldots, N_m - 1\}$. Introduce by way of notation $x_{m,0} = -2r\hat{e}$ and $x_{m,N_m+1} = 2r\hat{e}$.

Step 5. Fix $m \in \{0, 2, \ldots, N_m\}$ and define a piecewise linear map, this denoted by I_m, from $(-2r, 2r)$ to $\mathbb{R}^3$ by the following rules:

$$\text{(10.11)} \qquad \textit{If } \tau \in [\tau(x_{m,i}), \tau(x_{m,i+1})],$$
$$\textit{then } \mathrm{I}_m(\tau) = x_{m,i} + \frac{\tau - \tau(x_{m,i})}{\tau(x_{m,i+1}) - \tau(x_{m,i})}(x_{m,i+1} - x_{m,i}).$$

The following is a direct consequence of (10.9) and the nesting asserted by (10.10): If m and m′ are positive integers, and if $\tau, \tau' \in [-r, r]$, then $|\mathrm{I}_m(\tau) - \mathrm{I}_{m'}(\tau')| \leq \frac{1}{50}|\tau - \tau'|$. This last fact has the following implication: The sequence $\{\mathrm{I}_m\}_{m=1,2,\ldots}$ converges pointwise on the $|x| \leq r$ ball to a Lipshitz graph with total length less than 3r. The second bullet of (10.10) guarantees that this graph contains $\mathfrak{Z}_{k,\varepsilon_p} \cap B_r$.

Step 6. This step and Step 7 prove Item b) of Lemma 10.4's second bullet. To start, let I denote an open subset in Item a)'s Lipshitz graph with $(\mathfrak{Z}_{k,\varepsilon_p} \cap B_{r_p}) \cap I$ being dense in I. It follows from what is said by Item a) of the third bullet of Lemma 10.2 that the closure in I of $(\mathfrak{Z}_{k,\varepsilon_p} \cap B_{r_p}) \cap I$ is in $\mathfrak{Z}_{k,\varepsilon_p} \cup Z_k$. It follows as a consequence that the whole of I is an embedded Lipshitz curve in $\mathfrak{Z}_{k,\varepsilon_p} \cup Z_k$. If $p' \in I$ is not in Z_k, then p′ is not weakly continuous. This being the case, there are at most a countable set of points in I that are not in Z_k. With this fact in mind, suppose for the sake of argument that I has a point that is not in Z_k. This is to say that I has a point where the value of $N_{(\cdot)}(0)$ is less than $\frac{1}{2}k$. As $(\mathfrak{Z}_{k,\varepsilon_p} \cap B_{r_p}) \cap I$ is dense in I, there would be a sequence in $\mathfrak{Z}_k \cap I$ that converges to such a point. Since a sequence of this sort runs afoul of the first bullet in Lemma 7.7, all points in I must be in Z_k.

Step 7. Let $p' \in I \cap \mathfrak{Z}_k$ denote now a weakly continuous point and assume for the sake of argument that there is no neighborhood in I that is an open subset in Z. The subsequent three paragraphs derive a pair of assertions that can not both be true. The existence of such a pair proves Item b) of Lemma 10.4's second bullet.

To start the derivation, suppose that p′ is as just described. There is a sequence $\{y_i\}_{i=1,2,\ldots} \subset Z - I$ that converges to p′. Nothing is lost by assuming that this sequence lies in the radius r_0 ball centered at p′. For each $i \in \{1, 2, \ldots\}$, let λ_i denote $\mathrm{dist}(y_i, p')$. Use the sequence $\{\lambda_i\}_{i=1,2,\ldots}$ in the

p' versions of Lemmas 8.5–8.7 to construct the p' version of Z_*, $\mathcal{I}_*$ and ν_*. Being that p' is weakly continuous, its version of Z_* is a line through the origin. Fix $i \in \{1, 2, \dots\}$ and use the p' version of the map ϕ_{λ_i} to view y_i as a point on the $|x| = 1$ sphere in $\mathbb{R}^3$, this denoted by x_{y_i}.

Fix $\varepsilon \in (0, \frac{1}{1000})$. With $i \in \{1, 2, \dots\}$ chosen, the map ϕ_{λ_i} identifies the part of I with distance less than $10\lambda_i$ from p' with a Lipshitz graph in the $|x| < 10$ ball in $\mathbb{R}^3$. This graph contains the origin, and it follows from the p' version of (10.9) that there are points on this graph with distance from the origin between $1 - \varepsilon$ and $1 + \varepsilon$ and with distance less than $\frac{3}{2}$ from x_{y_i} when i is large. Since the set of weakly continuous points in $I \cap \mathfrak{Z}_k$ is dense in I, there are weakly continuous points in this part of I. Let q_i denote such a point, and let x_{q_i} denote the corresponding point in $\mathbb{R}^3$. Use δ_i to denote the angle between the ray from the origin to x_{q_i} and the ray from the origin to x_{y_i}, this being a number between 0 and π. The fact that $|x_{x_i}| = 1$ and $|x_{q_i}| \in (1 - \varepsilon, 1 + \varepsilon)$ and $|x_{y_i} - x_{q_i}| < \frac{3}{2}$, this angle δ_i must be greater than c_0^{-1} but less than $\pi - c_0^{-1}$.

Fix $i \in \{1, 2, \dots\}$ and denote by $\mathfrak{q}_i$ the ordered triple (q_i, y_i, p'). Use this triple to define Lemma 10.3's number Δ. The fact that $\delta_i \in (c_0^{-1}, \pi - c_0^{-1})$ implies that $\mathfrak{q}_i$'s version of Δ is greater than c^{-1} with c being less than c_0. Meanwhile, the $\mathfrak{q} = \mathfrak{q}_i$ version of Lemma 10.3 asserts that Δ is no greater than $\frac{1}{1000}c^{-1}$ when i is large. These last two assertions can not both be true.

10.d. Proof of Proposition 10.1

To set the notation, fix for the moment $k \in \{1, 2, \dots\}$. Given $p \in \mathfrak{Z}_k$, let r_p and ε_p denote p's version of the constants r and ε that are supplied by Lemma 10.4. The proof also uses $B_{r_p}(p)$ to denote the ball of radius r_p centered on p.

The proof of Proposition 10.1 has three parts.

Part 1: Fix $k \in \{1, 2, \dots\}$ and let p denote a weakly continuous point from $\mathfrak{Z}_k$. By way of short hand, let B denote the ball $B_{r_p}(p)$. Lemma 10.4 describes an embedded, Lipshitz graph that contains $\mathfrak{Z}_k \cap B$. Denote this graph by $\Gamma_{k,p}$. The map $\iota : B \to (-r_p, r_p)$ from Step 1 of the proof of Lemma 10.4 restricts to $\Gamma_{k,p}$ as a Lipshitz homeomorphism onto $(-r_p, r_p)$. It follows from Item b) of Lemma 10.4 that $\Gamma_{k,p}$ has an open subset, this denoted by $I_{k,p}$, with the following properties:

(10.12)
- $I_{k,p}$ *is an open subset of* Z *lying entirely in* Z_k.
- $Z_k \cap (\Gamma_{k,p} - I_{k,p})$ *is a closed, nowhere dense subset of* $\Gamma_{k,p}$.

It is a consequence of the second bullet in (10.12) that $\Gamma_{k,p}$ can be further decomposed as the disjoint union $\Gamma_{k,p} = I_{k,p} \cup G_{k,p} \cup J_{k,p}$ with $G_{k,p} = Z_k \cap (\Gamma_{k,p} - I_{k,p})$ and with $J_{k,p}$ being an open set.

Let $V \subset B$ denote a given closed set whose boundary is disjoint from $G_{k,p}$. Use $\mathfrak{G}$ to denote $G_{k,p} - (V \cap G_{k,p})$. Since $G_{k,p}$ is closed, there exists $\delta_{V,p} > 0$ such that each point in the boundary of V has distance no less than $\delta_{V,p}$ from $G_{k,p}$. With the preceding understood, fix $\delta \in (0, c_0^{-1})$ but less than $\frac{1}{1000} r_p$ and less than $\frac{1}{1000}\delta_{V,p}$. The observation that follows is used in Step 5.

(10.13) *The set* $\mathfrak{G}$ *has a finite cover by balls with radius at most* δ *and with closures that are pairwise disjoint, and disjoint from* V.

An inductive algorithm is given the subsequent paragraphs that constructs such a cover.

To start the algorithm, let $\mathcal{I}_1$ denote the set of component intervals of $I_{k,p} \cup J_{k,p}$ with at least one end point not in V. Choose such an interval, denote it by ι_1 and let q_1 denote an end point point of ι_1 that is not in V. Being that the set $I_{k,p} \cup J_{k,p}$ is open and dense, there exits $r_1 \in (\frac{1}{4}\delta, \frac{1}{2}\delta)$ such that the boundary of the ball of radius r_1 centered on q_1 is disjoint from $G_{k,p} \cup V$. Denote this ball by B_1.

Use (10.9) to see that if $r_p < c_0^{-1}$, then the boundary of B_1 will intersect $I_{k,p} \cup J_{k,p}$ twice, once in ι_1 and once in a second interval, this denoted by ι_2. One end point of ι_2 will lie in B_1 and the other will not. If the other end point lies in V, then choose an interval ι_3 from $\mathcal{I} - \{\iota_1, \iota_2\}$ with at least one end point not in V. Denote this second end point by q_3. If the other end point does not lie in V, denote it by q_2. This end point has distance greater than δ from V. If $r_p < c_0^{-1}$, then (10.9) has the following consequence: There exists a ball with center on $\Gamma_{k,p}$ and radius between $\frac{1}{4}\delta$ and $\frac{1}{2}\delta$ that is disjoint from B_1 whose boundary intersects $\Gamma_{k,p}$ in ι_2 and in a second interval from $I_{k,p} \cup J_{k,p}$. Denote this second interval by ι_3 and denote this ball by B_2. If $r_p < c_0^{-1}$, then (10.9) guarantees that there are only two intersections between $\Gamma_{k,p}$ and B_2's boundary. One end point of ι_3 will not be in B_2. Denote the latter by q_3.

If $q_3 \in V$, then choose an interval from $\mathcal{I} - \{\iota_1, \iota_2, \iota_3\}$ with at least one end point not in V. Denote this interval by ι_4 and use q_4 to denote the end point of ι_4 that is not V. If q_3 is not in V, then what follows is again a consequence of (10.9) when $r_p < c_0^{-1}$: There is a ball with center on $\Gamma_{k,p}$ and radius between $\frac{1}{4}\delta$ and $\frac{1}{2}\delta$ that is disjoint from B_1 and B_2 whose boundary intersects $\Gamma_{k,p}$ in ι_3 and in a second interval from $I_{k,p} \cup J_{k,p}$. Denote this second interval by ι_4 and this new ball by B_3. As was the case in the previous paragraph, (10.9) guarantees that there are only two intersections between $\Gamma_{k,p}$ and the boundary of B_3. One end point of I_4 will not be in B_2. Denote the latter by q_4.

Continuing in this vein defines a collection of balls, this denoted by $\mathcal{U}_{k,p,\delta}$ that are pairwise disjoint, disjoint from V and cover the set $\mathfrak{G}$. The collection has at most $c_0 r_p \delta^{-1}$ balls, this because the length of Γ is less than $c_0 r_p$ and each ball in this collection intersects $\Gamma_{k,p}$ as an arc with length greater than $c_0^{-1}\delta$. The volume of the union of the balls from $\mathcal{U}_{k,\delta,p}$ is at most $c_0 r_p \delta^3$, this because $\mathcal{U}_{k,\delta,p}$ consists of at most $c_0 r_p \delta^{-1}$ balls and each has volume at most $c_0 \delta^3$.

Part 2: Define $K \in \{1, 2, \dots\}$ as follows: If $k \in \{1, 2, \dots\}$ and $\mathfrak{Z}_k \neq \emptyset$, then $k \leq K$. Thus, K is the largest integer from the set of integers with non-empty version of $\mathfrak{Z}_{(\cdot)}$. The lemma that follows describes the corresponding set Z_K.

Lemma 10.5. *The set Z_K is contained in a countable union of Lipshitz curves. Moreover, given $\varepsilon \in (0, \frac{1}{100} r_0)$, there is an open set in M and a finite collection of balls in M that together cover Z_K and have the properties listed below. The list uses $\mathcal{V}_K$ to denote the open set and $\mathcal{U}_{K*}$ to denote the collection of balls.*

- *The balls from $\mathcal{U}_{K*}$ have closures are pairwise disjoint, and their union has volume less than ε.*
- *The intersection of Z_K with $\mathcal{V}_K$ is an open set in Z consisting of a properly embedded, finite length, Lipshitz curve with finitely many components.*

Proof of Lemma 10.5. The proof has five steps. Steps 1–4 prove the assertions given in the two bullets. The last step uses the two bullets to prove the assertion that the whole of Z_K is contained in a countable union of embedded Lipshitz curves. The notation uses $\mathfrak{I}$ to denote the set of weakly continuous points in Z_K.

Step 1. It follows from Item c) of Lemma 10.2's third bullet that there exists a subset $\Lambda \subset \{1, 2, \dots\}$ of consecutive integers starting at 1 and a labeling of $\mathfrak{I}$ as $\{p_i\}_{i\in\Lambda}$ with the property that $N_{(p_i)}(0) \geq N_{(p_{i+1})}(0)$ for all indices $i \in \Lambda$. Fix $r_1 \in (\frac{1}{4}\varepsilon, \frac{1}{2}\varepsilon)$ such that the sphere of radius r_1 centered at p_1 contains no point in $\mathfrak{I}$. There is a dense set of choices for r_1 because $\mathfrak{I}$ is a countable set. Let $B_{(1)}$ denote the ball of radius r_1 centered at p_1. Use $m_2 \in \{2, \dots\}$ to denote the smallest integer that labels a point in $\mathfrak{I} - B_{(1)}$. Fix $r_2 \in (0, \frac{1}{2}\varepsilon^2)$ such that the closure of the ball of radius r_2 centered on $p_{m_2}$is disjoint from the closure of $B_{(1)}$ and has no points from $\mathfrak{I}$ on the boundary of its closure. Continue in this vein to define in an inductive fashion an increasing sequence of integers $\{1, m_2, m_3, \dots\}$, and a corresponding set of balls,with pairwise disjoint closures that have the properties listed below. The list uses $\{B_{(m_k)}\}_{k=1,2,\dots}$ to denote the relevant set of balls.

(10.14)
- *For each* $k \in \{1, 2, \dots\}$, *the ball* $B_{(m_k)}$ *has center* p_{m_k}, *radius less than* $\frac{1}{2}\varepsilon^k$ *and no points from* $\mathfrak{I}$ *on its boundary.*
- *For each* $k \in \{1, 2, \dots\}$, *the set* $\{p_i\}_{1 \le i \le m_k} \subset \cup_{1 \le i \le k} B_{(m_i)}$.
- *If* $k \in \{1, 2, \dots\}$ *and if* $p_{m_k} \in \mathfrak{Z}_K$, *then* $B_{(m_k)} \subset B_{r_p}(p)$.

It is a consequence of the construction that $\cup_{i=1,2,\dots} B_{(m_i)}$ has volume less than $c_0\varepsilon^3$ and that it contains $\mathfrak{J}$. Let $\mathfrak{U}_{K+}$ denote the collection $\{B_{(m_i)}\}_{i=1,2,\dots}$.

Step 2. Let $\mathfrak{U}_K$ denote the collection of balls from $\{B_{r_p}(p)\}_{p \in \mathfrak{Z}_K}$ with a weakly continuous center point. This collection of balls defines an open cover of $\mathfrak{Z}_K - (\mathfrak{J} \cap \mathfrak{Z}_K)$ and so the joint collection $\mathfrak{U}_{K+} \cup \mathfrak{U}_K$ defines an open cover of Z_K. As noted by Item a) of the third bullet of Lemma 10.2 the set Z_K is closed, and thus it is compact. This being the case, there is a finite set of balls from $\mathfrak{U}_{K+}$ and a finite set from $\mathfrak{U}_K$ that together define a finite cover of Z_K. Denote the finite set from $\mathfrak{U}_{K+}$ by $\mathcal{U}_{K+}$ and that from $\mathfrak{U}_K$ by $\mathcal{U}_K$.

Step 3. Let p denote the center point of a ball from $\mathcal{U}_K$. Introduce from Part 1 the corresponding set $G_{K,p}$. Let B' denote a ball from $\mathcal{U}_{K+}$. The boundary of the closure of B' may or may not contain a point in $G_{K,p}$. If it does, then its radius can be increased by a factor greater than 1 but othewise as close to 1 as desired so that the boundary of the closure of resulting ball is disjoint from $G_{K,p}$ and such that the closure of the result is disjoint from the closures of all other balls in $\mathcal{U}_{K+}$. This understood, nothing is lost by assuming that the closures of all balls from $\mathcal{U}_{K+}$ are disjoint from the various versions of $G_{K,p}$ that are defined by the centers of the balls from $\mathcal{U}_K$.

Step 4. Let N denote the number of balls that comprise $\mathcal{U}_K$. Label the center points of these balls as $\{p_1, \dots, p_N\}$. Fix $\delta > 0$ for the moment and use what is said in Part 1 to construct the $p = p_1$ version of $\mathcal{U}_{K,p,\delta}$ with V being the union of the closures of the balls from the set $\mathcal{U}_{K+}$. If $\delta_2 \in (0, \delta]$ is sufficiently small, then the constructions in Part 1 supply the set $\mathcal{U}_{K,p_2,\delta_2}$ using for V the union of the closures of the balls from $\mathcal{U}_{K+} \cup \mathcal{U}_{K,p_1,\delta}$. Continue in an inductive fashion consecutively for $i = 3, \dots, N$ to construct sets $\mathcal{U}_{K,p_i,\delta_i}$ with $\delta_i < \delta$ chosen to be small and using for V the union of closures of the balls from $\mathcal{U}_{K+} \cup (\cup_{m=1,\dots,i} \mathcal{U}_{K,p_m,\delta_m})$.

The collection $\mathcal{U}_{K+} \cup (\cup_{m=1,\dots,N} \mathcal{U}_{K,p_m,\delta_m})$ consists of a finite number of balls with pairwise disjoint closure whose total volume is at most $c_0(\delta^2 + \varepsilon^3)$.

It follows as a consequence that the total volume of the balls in $\mathcal{U}_{K+} \cup (\cup_{m=1,\dots,N}\mathcal{U}_{K,p_m,\delta_m})$ is less than ε if δ is less than $c_0^{-1}\varepsilon$ and if ε is less than c_0^{-1}.

Choose $\delta < c_0^{-1}\varepsilon$ and $\varepsilon < c_0^{-1}$ so that the total volume of $\mathcal{U}_{K+} \cup (\cup_{m=1,\dots,N}\mathcal{U}_{K,p_m,\delta_m})$ is indeed less than ε. Use $\mathcal{U}_{K*}$ to denote the collection $\mathcal{U}_{K+} \cup (\cup_{m=1,\dots,N}\mathcal{U}_{K,p_m,\delta_m})$ and let $\mathcal{V}_K$ denote a very small radius tubular neighborhood in M of $\cup_{p\in\Theta}I_{K,p}$. It follows from (10.12) that this pair meets the requirements of the two bullets in Lemma 10.5.

Step 5. To prove that Z_K is contained in a countable union of embedded Lipshitz curves, fix for the moment $n \in \{1, 2, \dots\}$ and let $\Gamma_{K,n}$ denote the countable set of Lipshitz curves that is obtained from the $\varepsilon = \frac{1}{n}$ version of the second bullet of the lemma. The set $\cup_{n=1,2,\dots}\Gamma_{K,n}$ is a countable union of embedded Lipshitz curves that contains $Z_K-\mathfrak{J}$. Meanwhile, $\mathfrak{J}$ is a countable set of points and thus also contained in a countable union of embedded Lipshitz curves.

Part 3: Lemma 10.5 proves that the subset $Z_K \subset Z$ has the properties that Proposition 10.1 attributes to the whole of Z. The proof that the whole of Z has the desired properties proceeds by downward induction on $k \in \{2, 3, \dots, K\}$. This is to say that the argument proves the assertion for Z_{K-1}, then Z_{K-2}, and so on. The argument for the generic induction step to go from Z_k to Z_{k-1} for $k \in \{2, 3, \dots, K\}$ differs only in notation from the argument given below for the step from Z_K to Z_{K-1}. The argument for the latter step is presented in lieu of an argument for the generic step to minimize the introduction of new notation.

The lemma below states the Z_{K-1} analog of Lemma 10.5.

Lemma 10.6. *The set Z_{K-1} is contained in a countable union of Lipshitz curves. Moreover, given $\varepsilon \in (0, \frac{1}{100}r_0)$, there is an open set in* M *and a finite collection of balls in* M *that together cover Z_{K-1} and have the properties listed below. The list uses $\mathcal{V}_{K-1}$ to denote the open set and $\mathcal{U}_{(K-1)*}$ to denote the collection of balls.*

- *The balls from $\mathcal{U}_{(K-1)*}$ have closures are pairwise disjoint, and their union has volume less than ε.*
- *The intersection of Z_{K-1} with $\mathcal{V}_{K-1}$ is an open set in* Z *consisting of a properly embedded, finite length Lipshitz curve with finitely many components.*

Proof of Lemma 10.6. The proof has three steps. Steps 1 and 2 prove the first two bullets of the lemma and Step 3 proves the assertion that Z_{K-1} is contained in a countable union of embedded Lipshitz curves. To set the notation, first use ε^3 in lieu of ε to invoke Lemma 10.5 so as to define the sets $\mathcal{U}_{K*}$ and $\mathcal{V}_K$. Use α_K to denote the the intersection of Z with $\mathcal{V}_K$, this being a properly embedded Lipshitz curve. Use $\mathfrak{J}$ now to denote the set of points in $Z_{K-1}-Z_K$ that are not weakly continuous. This set contains $Z_{K-1}-(Z_K \cup \mathfrak{Z}_{K-1})$ and the complement in $\mathfrak{Z}_{K-1}$ of the set of weakly continuous points.

Step 1. Suppose that $\{p_i\}_{i=1,2,\ldots}$ is a sequence of non-weakly continuous points in $Z_{K-1}-Z_K$ that converges to a point $p \in M$. If p is not in Z_{K-1}, then if follows from Item c) from the third bullet of Lemma 10.2 that p is in Z_K. If p is in Z_K, then it can not be on α_K and so it must be in a set from $\mathcal{U}_{K*}$. This fact and the fact that the set of non-weakly continuous points is countable have the following implications:

(10.15)
- *The complement in* $Z_{K-1}-Z_K$ *of its intersection with* $\mathcal{U}_{K*}$ *is compact.*
- *Let* $\mathfrak{U} \subset M$ *denote a given open neighborhood of the set of weakly continuous points in* $\mathfrak{Z}_{K-1}$. *There are but a finite number of points in* $Z_{K-1}-Z_K$ *that are not weakly continuous, not in* $\mathfrak{U}$ *and not in* $\mathcal{U}_{K*}$.

Let $\mathfrak{U}_{K-1}$ denote the collection of balls from $\{B_{r_p}(p)\}_{p\in\mathfrak{Z}_{K-1}}$ with weakly continuous center point. Set $\mathfrak{U} = \cup_{B\subset\mathfrak{U}_{K-1}} B$. This set is an open cover of the set of weakly continuous points in $\mathfrak{Z}_{K-1}$. Let Θ denote the set of points in $Z_{K-1}-Z_K$ that are not weakly continous, not in $\mathfrak{U}$ and not in $\mathcal{U}_{K*}$. The second bullet of (10.15) says that Θ is finite. It follows as a consequence that the radii of the balls that comprise $\mathcal{U}_{K*}$ can be increased by a factor greater than 1 but otherwise as close to 1 as desired so that their closures are still pairwise disjoint, and so that the boundaries of their closures are disjoint from Θ. Make an adjustment of this sort and henceforth use $\mathcal{U}_{K*}$ to denote the resulting set of slightly larger balls. This can and should be done so that this new incarnation of $\mathcal{U}_{K*}$ with $\mathcal{V}_K$ obey all of the requirements of the version of Lemma 10.5 that uses ε^3 in lieu of ε.

Use what is said in Step 1 to find $r \in (0, \varepsilon)$ so that the set of balls with radius r and center on the points in Θ have the following properties: The balls from this set have pairwise disjoint closures and the closure of each ball is disjoint from the closure of each ball in $\mathcal{U}_{K*}$ and from the closure of α_K. Let $\mathcal{U}_{(K-1)+}$ denote this set of balls.

Step 2. The set of $\mathfrak{U}_{K-1} \cup \mathcal{U}_{(K-1)+} \cup \mathcal{U}_K \cup \mathcal{V}_K$ gives an open cover of Z_{K-1}. Item a) from the third bullet of Lemma 10.2 says that Z_K is closed. This being the case, it is compact and so there exists a finite set from $\mathfrak{U}_{K-1}$ that defines with $\mathcal{V}_K$ and the balls from $\mathcal{U}_{(K-1)+} \cup \mathcal{U}_K$ an open cover of Z_{K-1}.

Denote this set from $\mathfrak{U}_{K-1}$ by $\mathcal{U}_{K-1}$. Let N denote the number of balls that comprise $\mathcal{U}_{K-1}$. Label the center points of these balls as $\{p_1, \ldots, p_N\}$. Fix $\delta > 0$ again, and use the construction in Part 1 to obtain a $p = p_1$ version of $\mathcal{U}_{K-1,p,\delta}$ using for V the union of the closures of the balls from the set $\mathcal{U}_{(K-1)+} \cup \mathcal{U}_K$. If $\delta_2 \in (0, \delta]$ is sufficiently small, then the constructions Part 1 supply a set $\mathcal{U}_{K-1,p_2,\delta_2}$ using for V the union of the closures of the balls from $\mathcal{U}_{(K-1)+} \cup \mathcal{U}_K \cup \mathcal{U}_{K-1,p_1,\delta}$. Continue in an inductive fashion consecutively for $i = 3, \ldots, N$ to construct sets $\mathcal{U}_{K-,p_i,\delta_i}$ with $\delta_i < \delta$ chosen to be small and using the union of closures of the balls from $\mathcal{U}_{(K-1)+} \cup \mathcal{U}_K \cup (\cup_{m=1,\ldots,i} \mathcal{U}_{K-1,p_m,\delta_m})$ for the set V.

The collection $\mathcal{U}_{(K-1)+} \cup \mathcal{U}_K \cup (\cup_{m=1,\ldots,N} \mathcal{U}_{K-1,p_m,\delta_m})$ consists of a finite number of balls with pairwise disjoint closure whose total volume is at most $c_0(\delta^2 + \varepsilon^3)$. It follows as a consequence that the total volume of the balls in $\mathcal{U}_{(K-1)+} \cup \mathcal{U}_K \cup (\cup_{m=1,\ldots,N} \mathcal{U}_{K-1,p_m,\delta_m})$ is less than ε if δ is less than $c_0^{-1}\varepsilon$ and if ε is less than c_0^{-1}. Choose $\delta < c_0^{-1}\varepsilon$ and $\varepsilon < c_0^{-1}$ so that this is so. Use $\mathcal{U}_{(K-1)*}$ to denote the collection $\mathcal{U}_{(K-1)+} \cup \mathcal{U}_K \cup (\cup_{m=1,\ldots,N} \mathcal{U}_{K-1,p_m,\delta_m})$ and let $\mathcal{V}_{K-1}$ denote the union of a very small radius tubular neighborhood of α_K with a very small radius tubular neighborhood in M of $\cup_{p\in\Theta} I_{K,p}$. It follows from (10.12) that this pair meets the requirements of the two bullets in Lemma 10.6.

Step 3. This step proves that Z_{K-1} is contained in a countable union of Lipshitz graphs. To start, invoke Lemma 10.5 to conclude that Z_K is contained in a countable union of Lipshitz curves. Lemma 10.2's second bullet implies that the non-weakly continuous points in $Z_{K-1} - Z_K$ form a countable set and so this set is also contained in a countable union of Lipshitz curves. Meanwhile, the weakly continuous points in $Z_{K-1} - Z_K$ are in $\mathfrak{Z}_{K-1}$, this being an assertion of Proposition 9.1. Granted these observations, it is sufficient to prove that the set of weakly continous points in $\mathfrak{Z}_{K-1}$ is contained in a countable union of Lipshitz curves.

With the preceding understood, fix $n \in \{1, 2, \ldots\}$ and define $\mathfrak{Z}_{K-1,n}$ as follows: A point p is in $\mathfrak{Z}_{K-1}$ if and only if it is weakly continuous and such that $r_p \geq \frac{1}{n}$. Lemma 10.4 asserts that the intersection of $\mathfrak{Z}_{K-1}$ with any $p \in \mathfrak{Z}_{K-1,n}$ version of $B_{r_p}(p)$ is contained in a connected, Lipshitz graph.

As the closure of $\mathfrak{Z}_{K-1,n}$ is compact and so it has a finite cover by balls. The next paragraph proves the following assertion:

(10.16) *The closure of the set of weakly continuous points in* $\mathfrak{Z}_{K-1}$ *has a finite cover with all balls from the collection* $\{B_{r_p}(p)\}_{p\in\mathfrak{Z}_{K-1,n}}$.

Granted that (10.15) is true, it follows that $\mathfrak{Z}_{K-1,n}$ is contained in a finite set of Lipshitz graphs. The assertion that $\mathfrak{Z}_{K-1}$ is contained in a countable set of Lipshitz curves follows directly because $\mathfrak{Z}_{K-1} = \cup_{n=1,2,\ldots}\mathfrak{Z}_{K-1,n}$.

The proof of (10.15) starts with the observation that $\mathfrak{Z}_{K-1,n}$ is covered by the balls that comprise the union of two collections, the first being $\{B_{r_p}(p)\}_{p\in\mathfrak{Z}_{K-1,n}}$, and the second being the collection of balls with radius $\frac{1}{2n}$ centered on the points of the boundary of the closure of $\mathfrak{Z}_{k-1,n}$. This understood, let $\mathfrak{U}$ denote a finite cover of the closure of $\mathfrak{Z}_{K-1,n}$ by balls from these two collections. Write $\mathfrak{U}$ as $\mathfrak{U}_I \cup \mathfrak{U}_\partial$ with $\mathfrak{U}_I$ consisting of balls from $\{B_{r_p}(p)\}_{p\in\mathfrak{Z}_{K-1,n}}$ and with $\mathfrak{U}_\partial$ consisting of balls with radius $\frac{1}{2n}$ and center on the boundary of the closure of $\mathfrak{Z}_{K-1,n}$. Let $B \in \mathfrak{U}_\partial$ denote one of the latter and let p denote its center point. By definition, there is a sequence $\{p_i\}_{i=1,2,\ldots} \in \mathfrak{Z}_{k-1,n}$ that converges to p. If the index i is sufficiently large, then p will have distance less than $\frac{1}{4n}$ from p_i. This being the case, it follows that B is contained entirely in $B_{r_{p_i}}(p_i)$ and so B can be replaced by the latter ball. Replace each ball from the collection $\mathfrak{U}_\partial$ by a ball $\{B_{r_p}(p)\}_{p\in\mathfrak{Z}_{K-1,n}}$ to obtain the desired cover of $\mathfrak{Z}_{K-1,n}$.

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