On tidal energy in Newtonian two-body motion

Shuang Miao and Sohrab Shahshahani

According to the classical analysis of Newton the trajectory of two gravitating point masses is described by a conic curve. This conic curve is a hyperbola if the mechanical energy of the system is positive, and an ellipse if the mechanical energy is negative. The mechanical energy is a conserved quantity in an isolated two-body point mass system. If the point masses are initially very far, then the orbital energy is positive, corresponding to hyperbolic motion.

In this work we consider the situation when the point masses are replaced by gravitating incompressible fluid balls with free boundaries obeying the Euler-Poisson equations. In this case the conserved total energy $\tilde{\mathscr{E}}$ can be decomposed as $\tilde{\mathscr{E}} =: \widetilde{\mathscr{E}_{\text{orbital}}} + \widetilde{\mathscr{E}_{\text{tidal}}}$, where $\widetilde{\mathscr{E}_{\text{tidal}}}$ is the energy used in deforming the boundaries, and $\widetilde{\mathscr{E}_{\text{orbital}}}$ takes the role of the mechanical energy in the point mass system and is such that if $\widetilde{\mathscr{E}_{\text{orbital}}} < -c < 0$ for some absolute constant $c > 0$, then the orbit of the bodies must be bounded. In analogy with the point mass picture we consider the scenario where initially the two fluid balls are very far and their boundaries are unperturbed. In this case the initial orbital energy is equal to the total energy and is positive. However, in the motion of fluid bodies the orbital energy is no longer conserved because part of the conserved energy is used in deforming the boundaries of the bodies. In [4], based on a linear calculation Christodoulou conjectured that under appropriate conditions on the initial configuration of the system, the tidal energy $\widetilde{\mathscr{E}_{\text{tidal}}}$ can become larger than the total energy $\tilde{\mathscr{E}}$ during the evolution. In particular under these conditions $\widetilde{\mathscr{E}_{\text{orbital}}}$, which is initially positive, becomes negative before the point of the first closest approach. In this work we prove Christodoulou's conjecture in [4] for the full nonlinear system. That is, we prove that for a family of initial configurations the fluid boundaries and velocity remain regular up to the point of the first closest approach in the orbit, and that the tidal energy $\widetilde{\mathscr{E}_{\text{tidal}}}$ can be made arbitrarily large relative to the total energy $\tilde{\mathscr{E}}$. This reveals the possibility that the center of mass orbit, which is unbounded initially, may become bounded during the evolution. Since the initial distance of the bodies can be arbitrarily large relative to their distance at closest approach, the a-priori estimates, which as in the framework

of [16] are carried out in the language of Clifford analysis, are independent of the initial time and separation of the bodies.

1. Introduction

Consider two point masses of equal mass M approaching each other from a far distance with initial velocities v_0 and $-v_0$. According to Newton's laws, the masses accelerate due to the gravitational force between them. The mechanical energy (per unit mass) of the each point mass is

$$(1) \qquad \mathscr{E}_1 := \frac{1}{2}|v|^2 - \frac{GM}{4r_1},$$

where $r_1 = r_1(t)$ is the distance of each point mass to the center of mass of the system, $|v| = |v(t)|$ is the speed of each point mass, and G is the gravitational constant

$$G \approx 6.67 \times 10^{-11} \frac{(\text{meters})^3}{(\text{seconds})^2 (\text{kilograms})}.$$

The mechanical energy $\mathscr{E}_1$ is conserved in time. As shown by Newton, the orbit of the point masses is described by a conic curve whose shape is determined by the sign of $\mathscr{E}_1$: If $\mathscr{E}_1 < 0$, then the orbit is an ellipse, if $\mathscr{E}_1 > 0$, then the orbit is a hyperbola, and if $\mathscr{E}_1 = 0$, then the orbit is a parabola. Consider now the limiting case when the two point masses were infinitely far at time minus infinity, that is, $\lim_{t\to-\infty} r_1(t) = \infty$. When discussing the point mass system we will informally refer to $t \to -\infty$ as the *initial time*. Note that the initial velocities v_0 and $-v_0$ above now refer to the limiting initial velocities as $t \to -\infty$. Then initially $\lim_{t\to-\infty} \mathscr{E}_1 > 0$ and, since $\mathscr{E}_1$ is conserved, the orbit of the two point masses is hyperbolic.

In this work we are interested in the situation where the point masses are replaced by two incompressible fluid bodies, $\mathcal{B}_1$ and $\mathcal{B}_2$, of equal mass and density which are initially round spheres of radius R, with centers of mass $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. We consider this configuration as a model for the motion of two isolated astronomical fluid bodies whose motion is subject only to the mutual gravitational force between them. More precisely, suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are fluid bodies of constant density ρ and volume $|\mathcal{B}_j| = \frac{4\pi}{3}R^3$,

with free boundaries, which satisfy the Euler-Poisson equations

$$
(2) \quad
\begin{cases}
\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \mathbf{p} - \nabla(\psi_1 + \psi_2), & \text{in } \mathcal{B}_j(t) \\
\nabla \cdot \mathbf{v} = 0, \quad \nabla \times \mathbf{v} = 0, & \text{in } \mathcal{B}_j(t) \\
\mathbf{p} = 0, & \text{on } \partial\mathcal{B}_j(t) \\
(1, \mathbf{v}) \in T(t, \partial\mathcal{B}_j(t)), & \text{on } \partial\mathcal{B}_j(t)
\end{cases}
,
$$

$j = 1, 2$. Here $\mathbf{v}$ denotes the fluid velocity, $\mathbf{p}$ denotes the ratio of fluid pressure to density, and ψ_j, $j = 1, 2$, are the gravitational potentials

$$
\psi_j(t, \mathbf{x}) := -G\rho \int_{\mathcal{B}_j(t)} \frac{d\mathbf{y}}{|\mathbf{x} - \mathbf{y}|},
$$

so that with $\chi_{\mathcal{B}_j(t)}$ denoting the characteristic function of $\mathcal{B}_j(t)$

$$
\Delta \psi_j(t, \mathbf{x}) = 4\pi G\rho \chi_{\mathcal{B}_j(t)}.
$$

The relevant initial configuration is when the bodies are initially at equilibrium moving at constant speed and zero acceleration and their centers of mass are as in the point mass system described above. Here *equilibrium* refers to the assumption that initially the bodies are perfect spheres, so that the gravitational force of each body does not generate deformations of its own boundary. As for point masses we are interested in the limiting scenario where the bodies have infinite distance at time minus infinity. Mathematically, we take the perfect spherical bodies to be at distance $R_1 \gg 1$ at time T_0, a negative number such that $|T_0| \gg 1$. We then carry out the analysis with all quantitative estimates independent of R_1 and T_0. The equation $\nabla \times \mathbf{v} = 0$ in the second line of (2) holds because it holds initially, due to the fact that the bodies are initially at equilibrium. The vanishing of surface tension on the third line of (2) is imposed because surface tension should be neglected in the scale of astronomical bodies. The last condition in (2) is the free boundary condition stating that particles on the fluid boundary at a given time remain on the boundary at all times.

For this system the *total energy* of each body,

$$
(3) \quad \mathscr{E}(t) := \frac{1}{2} \int_{\mathcal{B}_1(t)} |\mathbf{v}(t, \mathbf{x})|^2 d\mathbf{x} + \frac{1}{2} \int_{\mathcal{B}_1(t)} \psi_1(t, \mathbf{x}) d\mathbf{x} + \frac{1}{2} \int_{\mathcal{B}_1(t)} \psi_2(t, \mathbf{x}) d\mathbf{x},
$$

is conserved during the evolution. Here we have defined the energy only relative to the first body because by symmetry this is also equal to the

energy relative to the other body. Note that since the bodies are no longer point masses, $\mathscr{E}$ would not agree with $\mathscr{E}_1$ above even if the bodies were perfect balls moving at speed $|v|$, because the contribution of $\boldsymbol{\psi}_1$ to the energy is now a nonzero constant. Indeed, when $\mathcal{B}_1$ is a ball of radius R centered at the origin,

$$\frac{1}{2}\int_{\mathcal{B}_1(t)}\boldsymbol{\psi}_1(t,\mathbf{x})d\mathbf{x} = -\frac{G\rho}{2}\int_{B_R(0)}\int_{B_R(0)}\frac{d\mathbf{x}\,d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} = -\frac{3GM|B_R(0)|}{5R}.$$

To account for this difference we renormalize the energy $\mathscr{E}$ above and define the *modified total energy* $\tilde{\mathscr{E}}$ as

$$\tilde{\mathscr{E}} := \frac{1}{|\mathcal{B}_1|}\mathscr{E} + \frac{3GM}{5R}.$$

When there is no risk of confusion we simply refer to $\tilde{\mathscr{E}}$ as the modified energy. Note that incompressibility implies that the volume $|\mathcal{B}_1|$ is conserved during the evolution, so the modified total energy $\tilde{\mathscr{E}}$ is also a conserved quantity. Now as the bodies approach each other their boundaries deform from their initial spherical shape and it is conceivable that the energy required for such deformation is so large that the energy left in the center of mass motion becomes negative. More significantly, it is conceivable that the orbit of the two bodies eventually becomes bounded, a phenomenon which is impossible in the point mass case, and to which we refer as *tidal capture*. To better understand the situation we decompose the modified total energy as

$$(4) \qquad \tilde{\mathscr{E}} = \widetilde{\mathscr{E}_{\text{orbital}}} + \widetilde{\mathscr{E}_{\text{tidal}}},$$

where

$$(5)$$
$$\widetilde{\mathscr{E}_{\text{tidal}}} = \widetilde{\mathscr{E}_{\text{tidal}}}(t)$$
$$:= \frac{1}{2|\mathcal{B}_1|}\int_{\mathcal{B}_1}|\mathbf{v}(t,\mathbf{x})|^2 d\mathbf{x} - \frac{1}{2}|\mathbf{x}_1'|^2 + \frac{1}{2|\mathcal{B}_1|}\int_{\mathcal{B}_1}\boldsymbol{\psi}_1(t,\mathbf{x})d\mathbf{x} + \frac{3GM}{5R},$$

and

$$(6) \qquad \widetilde{\mathscr{E}_{\text{orbital}}} = \widetilde{\mathscr{E}_{\text{orbital}}}(t)$$
$$:= \frac{1}{|\mathcal{B}_1|}\mathscr{E} + \frac{3GM}{5R} - \widetilde{\mathscr{E}_{\text{tidal}}} = \frac{1}{2}|\mathbf{x}_1'|^2 + \frac{1}{2|\mathcal{B}_1|}\int_{\mathcal{B}_1}\boldsymbol{\psi}_2(t,\mathbf{x})d\mathbf{x}.$$

Here, and in the rest of this work, the notation f' is used to denote the derivative of a function f of a single variable. We refer to $\widetilde{\mathscr{E}_{\text{tidal}}}$ as the tidal energy and to $\widetilde{\mathscr{E}_{\text{orbital}}}$ as the orbital energy. Note that the modified total energy is defined such that *initially* when the bodies are perfect balls, the tidal energy $\widetilde{\mathscr{E}_{\text{tidal}}}$ is zero and $\tilde{\mathscr{E}} = \widetilde{\mathscr{E}_{\text{orbital}}}$. Moreover, if the initial distance between the two bodies is sufficiently large, *then initially*

$$(7) \qquad \frac{1}{|\mathcal{B}_1|}\mathscr{E} + \frac{3GM}{5R} = \widetilde{\mathscr{E}_{\text{orbital}}} =: e_0 > 0, \quad \widetilde{\mathscr{E}_{\text{tidal}}} = 0.$$

We will see that, as in the case of point masses, the two bodies get closer until at some point they reach their minimum distance[1] and then start to move apart. The question of interest for us is if the bodies will continue to get arbitrarily far as in the point mass case, or if by contrast their orbit will become bounded and their distance will start to decrease again at some later time. Suppose now that we have a lower bound

$$(8) \qquad \widetilde{\mathscr{E}_{\text{tidal}}}(t) \geq m_0 > e_0,$$

for all times $t > T_1$ for some $T_1 > T_0$. This would imply that for $t > T_1$

$$
\begin{aligned}
(9) \quad & \frac{1}{2}|\mathbf{x}_1'(t)|^2 + \frac{1}{2|\mathcal{B}_1|}\int_{\mathcal{B}_1}\psi_2(t,\mathbf{x})d\mathbf{x} \\
& = \widetilde{\mathscr{E}_{\text{orbital}}}(t) \\
& = \frac{1}{|\mathcal{B}_1|}\mathscr{E}(t) + \frac{3GM}{5R} - \widetilde{\mathscr{E}_{\text{tidal}}}(t) \leq e_0 - m_0 < 0.
\end{aligned}
$$

Since

$$\lim_{|\mathbf{x}_1|\to\infty}\frac{1}{2|\mathcal{B}_1|}\int_{\mathcal{B}_1}\psi_2(t,\mathbf{x})d\mathbf{x} = 0,$$

and the first term on the left hand side in (9) is non-negative, this would imply that the orbit cannot be unbounded. Therefore, a uniform lower bound of the form (8) for all times $t > T_1$ for some $T_1 > T_0$, implies that tidal

---

[1]To be precise, this is the *first* local minimum of the distance between the bodies. However, for brevity, and when there is no risk of confusion, we will often refer to this as the minimum of their distance. Similarly we will often refer to the corresponding point in the orbit as the point of closest approach rather than the point of *first* closest approach.

capture occurs. This explains our choice of terminology "orbital energy" for $\widetilde{\mathscr{E}_{\text{orbital}}}$. The relation of the tidal energy $\widetilde{\mathscr{E}_{\text{tidal}}}$ to the surface deformation of the fluid bodies can also be made precise, but since this discussion requires introducing some more parameters, we postpone it to further down in the introduction; see (14) and Remark 1.4. In this work we will prove that if certain relations are satisfied in the initial configuration of the system, then (8) holds near and up to the point of the closest approach, where m_0 can be made arbitrarily large relative to the initial energy e_0. To state our result more precisely, we need to describe the setup in more detail.

Consider again the case of point masses. Suppose the two point masses $\mathbf{x}_1$ and $\mathbf{x}_2$ have equal mass M, and satisfy

$$\lim_{t \to -\infty} \mathbf{x}_1(t) = -\lim_{t \to -\infty} \mathbf{x}_2(t) = (-b, \infty, 0) \in \mathbb{R}^3,$$

$$\lim_{t \to -\infty} \mathbf{x}_1'(t) = -\lim_{t \to -\infty} \mathbf{x}_2'(t) = (0, -v_0, 0), \qquad v_0 > 0.$$

The following dimensionless parameter plays a crucial role in the analysis in this paper:

$$(10) \qquad\qquad p := \frac{GM}{bv_0^2}.$$

The differential equation describing the motion of the point masses is given by Newton's second law

$$\mathbf{x}_1''(t) = -\frac{GM\mathbf{x}_1(t)}{4|\mathbf{x}_1(t)|^3}.$$

By direct differentiation we see that the energy,

$$\mathscr{E}_1 := \frac{1}{2}|\mathbf{x}_1'(t)|^2 - \frac{GM}{4|\mathbf{x}_1(t)|},$$

and the angular momentum,

$$\mathbf{J}(t) = \mathbf{x}_1(t) \times \mathbf{x}_1'(t),$$

are constant in time. Conservation of angular momentum implies that the orbit of the point masses is confined to the x-y coordinate plane, and we will suppress the z-coordinate in the remainder of the discussion for point

masses[2]. As mentioned earlier, when $\mathscr{E}_1 > 0$ the orbit of the two point masses is a hyperbola. Indeed, a direct calculation shows that

$$(11) \qquad \mathbf{e} := \frac{4}{GM} \left(\mathbf{x}_1' \times \mathbf{J} - \frac{GM}{4|\mathbf{x}_1|} \mathbf{x}_1 \right),$$

is a constant vector, that is $\frac{d}{dt}\mathbf{e} = 0$. The constant scalar $e := |\mathbf{e}|$ is known as the *eccentricity* of the system, and is given by

$$e = \frac{4}{GM} \sqrt{\left(\frac{1}{4}GM\right)^2 + 2\mathscr{E}_1 |\mathbf{J}|^2} = \sqrt{1 + \frac{32\mathscr{E}_1 |\mathbf{J}|^2}{(GM)^2}}.$$

In particular,

$$\mathscr{E}_1 < 0 \iff e < 1 \quad \text{and} \quad \mathscr{E}_1 > 0 \iff e > 1.$$

Let $r := |\mathbf{x}_1|$ and θ be the angle between $\mathbf{x}_1$ and $\mathbf{e}$. It follows from (11) that

$$r = \frac{4|\mathbf{J}|^2}{GM} \frac{1}{1 + e \cos\theta}.$$

Since $e > 1$ in the configuration described above, this is the polar representation of a hyperbola. By construction, one asymptote of this hyperbola, corresponding to $t \to -\infty$, is the y-axis. The other asymptote, L, corresponding to $t \to \infty$ is determined by the choice of the parameters b, v_0, and M. Let α be the angle between L and the negative y-axis so that $\pi + \alpha$ is the angle L makes with the positive y-axis, measured counterclockwisely from the positive y-axis. See Figure 1. We call α the *scattering angle* of $\mathbf{x}_1$. If $\varphi(t)$ is the angle from the positive y-axis to $\mathbf{x}_1$, then

$$(12) \qquad \pi + \alpha = \int_{-\infty}^{\infty} \frac{d\varphi}{dt} dt.$$

Since $|\mathbf{x}_1'|^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\varphi}{dt}\right)^2$ and $|\mathbf{J}| = bv_0 = r^2 \frac{d\varphi}{dt}$,

$$\left(\frac{dr}{dt}\right)^2 = 2(\mathscr{E}_1 - \mathscr{U}_1),$$

---

[2]Here we use the standard convention that the x, y, and z axes correspond to the directions $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ in $\mathbb{R}^3$, respectively.

where

$$\mathscr{U}_1 := -\frac{GM}{4r} + \frac{|\mathbf{J}|^2}{2r^2}.$$

Since the orbit of $\mathbf{x}_1$ is a hyperbola, r will decrease until it reaches a minimum value r_+, which we call the *distance at closest approach*, and then increase again to infinity. If t_+ denotes the time at which $r(t_+) = r_+$, then

$$\frac{dr}{dt} = -\sqrt{2(\mathscr{E}_1 - \mathscr{U}_1)}, \qquad t \leq t_+,$$

$$\frac{dr}{dt} = \sqrt{2(\mathscr{E}_1 - \mathscr{U}_1)}, \qquad t \geq t_+.$$

Using the change of variables $\lambda = b^{-1}r$, with $\lambda_+ := b^{-1}r_+$, and recalling the definition of the parameter p from (10), we can rewrite (12) as

$$
\begin{aligned}
\alpha + \pi &= \int_{-\infty}^{\infty} \frac{d\varphi}{dt} dt \\
&= -\int_{\infty}^{r_+} \frac{|\mathbf{J}|}{bv_0 r\sqrt{\lambda^2 + \frac{p}{2}\lambda - 1}} dr + \int_{r_+}^{\infty} \frac{|\mathbf{J}|}{bv_0 r\sqrt{\lambda^2 + \frac{p}{2}\lambda - 1}} dr \\
&= 2|\mathbf{J}| \int_{r_+}^{\infty} \frac{dr}{bv_0 r\sqrt{\lambda^2 + \frac{p}{2}\lambda - 1}} = 2\int_{\lambda_+}^{\infty} \frac{d\lambda}{\lambda\sqrt{\lambda^2 + \frac{p}{2}\lambda - 1}}.
\end{aligned}
$$

The quadratic polynomial $\lambda^2 + \frac{p}{2}\lambda - 1$ factorizes as $(\lambda - \lambda_+)(\lambda - \lambda_-)$, where $\lambda_\pm = -\frac{p}{4} \pm \sqrt{\frac{p^2}{16} + 1}$. Finally, denoting the speed at closest approach by $v_+ := |\mathbf{x}_1'(t_+)|$, by conservation of the energy $\mathscr{E}_1$ we have $v_+^2 = \frac{GM}{2r_+}\left(1 + \frac{2v_0^2 r_+}{GM}\right)$. Summarizing we get

$$\lambda_+ = \frac{1}{\frac{p}{4} + \sqrt{\frac{p^2}{16} + 1}}, \quad r_+ = b\lambda_+, \quad \alpha = 2\int_{\lambda_+}^{\infty} \frac{d\lambda}{\lambda\sqrt{\lambda^2 + \frac{p}{2}\lambda - 1}} - \pi,$$

$$v_+^2 = \frac{GM}{2r_+}\left(1 + \frac{2v_0^2 r_+}{GM}\right).$$

Now consider the two limit cases where p is very small and very large. First, when $p \to 0$ (e.g., $bv_0^2 \gg GM$) it is not surprising that the gravitational force will almost not affect the trajectories of the masses. Indeed, in this case $\lambda_+ \to 1$ so $r_+ \to b$ and $\alpha \to 0$ and the motion of the two masses is almost along a straight line. The more interesting case is when $p \to \infty$ (e.g.,

Figure 1:

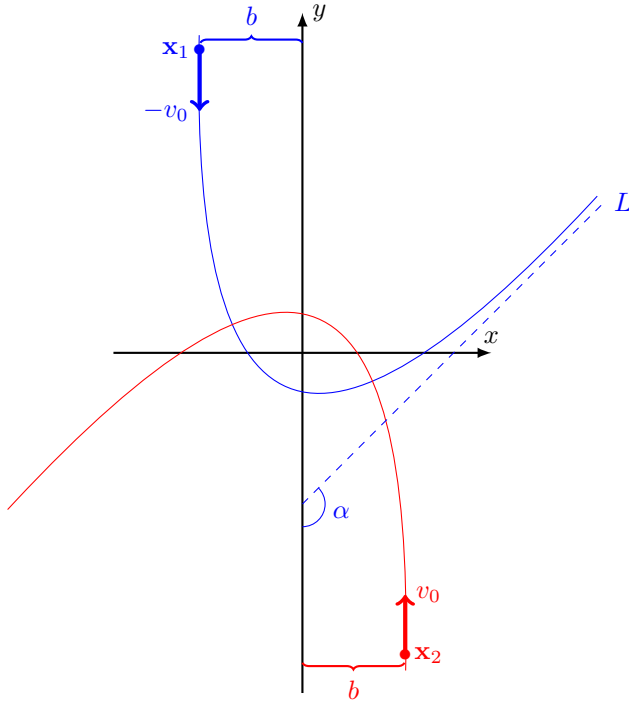$GM \gg bv_0^2$) and in this case we have

$$p\lambda_+ \to 2, \quad \frac{pr_+}{b} \to 2, \quad \alpha \to \pi, \quad \frac{2v_+^2 r_+}{GM} \to 1.$$

These limiting behaviors will be important guiding principles in choosing the initial parameters in the case of fluid bodies. Suppose now that the point masses are replaced by fluid bodies obeying (2). Assume that at time $t = T_0$ the two bodies $\mathcal{B}_1(T_0)$ and $\mathcal{B}_2(T_0)$ are balls of radius R centered at $(x, y, z) = (-b, R_0, 0)$ and $(x, y, z) = (b, -R_0, 0)$, respectively, with $R_0 \gg b \gg R$. For future reference let $R_1 := \sqrt{b^2 + R_0^2}$ be the initial distance of the center of mass of the entire system to the initial position of the center of mass of each body. Recall that we denote the center of masses of the two bodies by $\mathbf{x}_1$ and $\mathbf{x}_2 = -\mathbf{x}_1$ and suppose that the initial velocities of $\mathbf{x}_1$ and $\mathbf{x}_2$ are

$$\mathbf{x}_1'(T_0) = \mathbf{v}_0 := (0, -v_0, 0), \qquad \mathbf{x}_2'(T_0) = -\mathbf{v}_0 = (0, v_0, 0), \qquad v_0 > 0.$$

Note that, by symmetry, the center of mass of the entire system is always at the origin. We will discuss the local existence of a solution to equation (2) momentarily, but let us assume for now that there exists a classical solution on some interval $I := [T_0, T_*)$. As mentioned earlier, a direct computation shows that the total energy $\mathscr{E}$ defined in (3) is conserved in I. Since the center of mass of each body is given by

$$\mathbf{x}_j = \frac{\rho}{M} \int_{\mathcal{B}_j} \mathbf{x} d\mathbf{x},$$

we have

$$\mathbf{x}_j'' = \frac{\rho}{M} \int_{\mathcal{B}_j} (\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) d\mathbf{x} = -\frac{\rho}{M} \int_{\mathcal{B}_j} (\nabla \psi_1 + \nabla \psi_2) d\mathbf{x}.$$

Recall the following points from the discussion above. The modified total energy

(13) $$\tilde{\mathscr{E}} = \frac{1}{|\mathcal{B}_1|} \mathscr{E} + \frac{3GM}{5R}$$

admits the decomposition

$$\tilde{\mathscr{E}} = \widetilde{\mathscr{E}_{\text{tidal}}} + \widetilde{\mathscr{E}_{\text{orbital}}},$$

where $\widetilde{\mathscr{E}_{\text{tidal}}}$ and $\widetilde{\mathscr{E}_{\text{orbital}}}$ are defined in (5) and (6). As in the point mass case, if the initial separation, $2R_1$, of the two bodies is sufficiently large then $\tilde{\mathscr{E}}$ is positive. On the other hand, since the two bodies are initially unperturbed round balls, *initially* $\tilde{\mathscr{E}} = \widetilde{\mathscr{E}_{\text{orbital}}}$ and $\widetilde{\mathscr{E}_{\text{orbital}}}$ is positive, corresponding to hyperbolic motion in the point mass case. As discussed at the beginning of the introduction, our goal in this paper is to prove that the parameters of the problem can be set up so that $\widetilde{\mathscr{E}_{\text{tidal}}}$ becomes arbitrarily large relative to the modified total energy $\tilde{\mathscr{E}}$ at the first point of closest approach, that is, when the bodies reach their minimum distance. To show this, we also need to prove that the solution of (2) does not develop singularities until the bodies get sufficiently close for $\widetilde{\mathscr{E}_{\text{tidal}}}$ to become large. To state the precise result we need to introduce some more notation. First, motivated by the more detailed discussion of the point mass system above we define

$$p := \frac{GM}{bv_0^2}, \qquad r_+ := \frac{2b}{p}.$$

In addition we define the following dimensionless parameters which take into account the size of the fluid balls:

$$\eta = \eta(t) := \frac{R}{|\mathbf{x}_1(t)|}, \qquad \eta_+ := \frac{R}{r_+}, \qquad \beta := \frac{b}{R}.$$

The analysis in the point mass case shows that, in that case, if $p \gg 1$ the scattering angle approaches π. It is important to note that in the case of fluid bodies the condition $p \gg 1$ can be achieved while keeping r_+ arbitrarily large relative to R. For instance, one can take $v_0 \sim \sqrt{\frac{GM}{R}}\beta^{-9/10}$, $p \sim \beta^{4/5}$, and $\beta \gg 1$. It follows that $r_+ \sim R\beta^{1/5} \gg R$. It is in the context of the limiting scenario $p \gg 1$ and $r_+ \gg R$ that we will study the evolution of fluid bodies. We will choose M, b, and v_0 such that $b \gg r_+$ and $r_+ \gg R$, which in particular imply $\beta \gg 1$. As was just mentioned, the significance of the condition $b \gg r_+$, or equivalently $p \gg 1$, is that the scattering angle is close to π in the point mass analysis. The significance of the condition $r_+ \gg R$ is that it will allow us to treat the deformation of the bodies perturbatively in the proof of a-priori estimates. Indeed, note that we need the solution to remain regular starting at the initial time when the distance of the bodies is $2R_1$ until the time of closest approach. Since, as explained above, for the configuration of interest R_1 can be arbitrarily large relative to the distance at closest approach this amounts to controlling appropriate norms of the solution for *infinite* time, and the small parameter η together with sharp decay bootstrap assumptions are what allow us to achieve this. We refer the reader to Subsection 1.1.1 below for a more detailed discussion of this aspect of the analysis.

To relate the energy $\widetilde{\mathscr{E}}_{\text{tidal}}$ to the surface deformation, we introduce the Lagrangian parametrization of the surface. Let $\xi : \mathbb{R} \times S_R \to \mathcal{B}_1$ be the Lagrangian parametrization of $\mathcal{B}_1$ satisfying $\xi(T_0, p) = p$ for all p in S_R, that is,

$$\xi_t(t, p) = \mathbf{v}(t, \xi(t, p)).$$

Here S_R is the round sphere of radius R. We define the *height function* $h : \mathbb{R} \times S_R \to \mathbb{R}$ as

$$h(t, p) = |\xi(t, p) - \mathbf{x}_1(t)| - R.$$

The following theorem is the main result of this paper.

Theorem 1.1. *Suppose $r_+ \geq CR$ where $C > 0$ is sufficiently large. Then $|\mathbf{x}_1(t)|$ is decreasing on any time interval $I_T := [T_0, T)$ such that $|\mathbf{x}_1(t)| \geq \frac{3}{2} r_+$ for all $t \in I_T$, and a classical solution to (2) exists on the longest time interval on which $|\mathbf{x}_1|$ is decreasing. Moreover, there exist universal constants c_1 and c_2 such that, with r_0 denoting the first local minimum of $|\mathbf{x}_1|$, and for $|\mathbf{x}_1(t)| \in (r_0, 2r_0)$,*

$$c_1 \frac{GM}{R} \eta^6 \leq \widetilde{\mathscr{E}_{\text{tidal}}} \leq c_2 \frac{GM}{R} \eta^6.$$

Furthermore, $\widetilde{\mathscr{E}_{\text{tidal}}}$ is related to the height function h as [3]

$$(14) \qquad \widetilde{\mathscr{E}_{\text{tidal}}} \approx \frac{GM}{R^5} \|h\|^2_{L^2(S_R)} + \frac{1}{R^2} \|\partial_t h\|^2_{L^2(S_R)}.$$

Here the constants c_1 and c_2 as well as the implicit constants in (14) are independent of the initial time T_0 and the initial separation R_1. In particular, if $\eta_+^5 p^2 \gtrsim 1$, then for some $m > 2$, and when $|\mathbf{x}_1(t)| \in (r_0, 2r_0)$,

$$(15) \qquad \qquad \widetilde{\mathscr{E}_{\text{tidal}}} \geq m \, \tilde{\mathscr{E}}.$$

Remark 1.2. Condition (15) in Theorem 1.1 shows that by choosing the parameters of the problem appropriately we can guarantee that the orbital energy $\widetilde{\mathscr{E}_{\text{orbital}}}$, which is initially positive, becomes negative, which as described above is a significant step in understanding the phenomenon of tidal capture. In fact, we can make m in (15) arbitrarily large. To see this, suppose we choose $v_0 \sim \sqrt{\frac{GM}{R}} \beta^{-\alpha}$, which implies that $r_+ \sim \beta^{2-2\alpha} R$, and $\eta_+^5 p^2 \sim \beta^{14\alpha - 12}$. Then, for both of the conditions $r_+ \gg R$ and $\eta_+^5 p^2 \gtrsim 1$ to be satisfied, we should choose α in the range $[\frac{6}{7}, 1]$. In particular choosing $v_0 = \sqrt{\frac{\mu GM}{R}} \beta^{-1}$ with $\mu \gg 1$, we get $r_+ \sim \mu R \gg R$, and

$$\widetilde{\mathscr{E}_{\text{tidal}}} \sim \frac{GM}{R} \eta_+^6 \sim \mu^{-6} \frac{GM}{R}, \qquad \tilde{\mathscr{E}} \sim |v_0|^2 \sim \mu \beta^{-2} \frac{GM}{R}.$$

Therefore choosing β sufficiently large relative to μ we can make m in (15) arbitrarily large.

Remark 1.3. The significance of the comparison (14) can be explained as follows. As explained in Remark 1.2 above, Theorem 1.1 implies that the

---

[3]The notation $f \approx g$ means $K^{-1}|f| \leq |g| \leq K|f|$ for some constant $K > 0$.

tidal energy $\widetilde{\mathscr{E}_{\text{tidal}}}$ can be made arbitrarily large relative to the total energy $\tilde{\mathscr{E}}$ at the time of closest approach. By (14) the tidal energy is comparable to the $H^1 \times L^2$ norm of $(h, \partial_t h)$ (and this remains true as long as the tides remain small). Therefore, if $\|(h, \partial_t h)\|_{L^2 \times L^2}$ retains a nontrivial portion of its size at closest approach as the bodies move away from each other, we can argue as in the beginning of this introduction to conclude that tidal capture will indeed happen.

Remark 1.4. It is important to have constants that do not depend on the initial time T_0 and the initial separation R_1 in Theorem 1.1. The reason is that it makes sense for the bodies to be perfect balls only when their distance is infinite. The fact that for us the constants are independent of T_0 and R_1 allows us to take the limit $|T_0|, R_1 \to \infty$ in Theorem 1.1. It follows from our analysis that in this case the estimate $c_1 \frac{GM}{R} \eta^6 \leq \widetilde{\mathscr{E}_{\text{tidal}}} \leq c_2 \frac{GM}{R} \eta^6$ is valid for all times before the first closest approach, that is, before $|\mathbf{x}_1(t)| = r_0$.

1.1. Discussion of the Proof

We divide the proof of Theorem 1.1 into two parts. The first part consists of proving a-priori estimates, which in particular show that the solution remains regular as long as $|\mathbf{x}_1|$ remains larger than $\frac{3}{2} r_+$. More importantly we derive a precise description of the evolution of the bodies up to closest approach. The second part of the proof is an analysis of the tidal energy $\widetilde{\mathscr{E}_{\text{tidal}}}$. Here we use the precise description and a-priori estimates from the first part of the proof to derive a lower bound for the tidal energy. This lower bound depends on the distance of the two bodies, and comparing it with the conserved total energy we will see that if $|\mathbf{x}_1|$ is sufficiently close to r_+ and $p^2 \eta_+^5 \gtrsim 1$, then (15) holds.

1.1.1. A-priori estimates. Note that since each component of the fluid velocity is a harmonic function inside $\mathcal{B}_1$ and $\mathcal{B}_2$, to prove regularity of the solution it suffices to prove regularity of the boundary and the velocity on the boundary. Let $\xi : \mathbb{R} \times S_R \to \partial \mathcal{B}_1(t)$ be the Lagrangian parametrization of the boundary of the first body, that is, $\xi_t = \mathbf{v}(t, \xi)$, with $\xi(T_0, \cdot) = \text{Id}$, and let $\zeta := \xi - \mathbf{x}_1$ and $u := \zeta_t$. ζ is the renormalized Lagrangian parametrization as the motion of the center of mass is subtracted, and u is the Lagrangian fluid velocity relative to the center of mass velocity. Local existence of a solution starting at $t = T_0$ follows from the local well-posedness of the

system[4]. However, as we are interested in understanding the dynamics of the motion up to the point of closest approach, local existence is far from sufficient for our purposes. To prove the necessary long time existence result, we derive a-priori bounds using energy estimates to which we now turn. The first step is to derive a quasilinear equation for u. Since $\mathbf{p} \equiv 0$ on $\partial\mathcal{B}_1$, the gradient $\nabla\mathbf{p}|_{\partial\mathcal{B}_1}$ points in the direction of the normal to the boundary. With n denoting the exterior normal (in Lagrangian coordinates), let a be defined by the relation

$$-\nabla\mathbf{p}(t,\xi) = an.$$

Note that the positivity of a corresponds to the Taylor sign condition (cf. [13]), and can be verified using the maximum principle after applying the divergence operator to (2). With this notation, we write the quasilinear equation for u, which is derived in Section 2, in the schematic form

$$(16) \qquad \partial_t^2 u + an \times \nabla u + \frac{GM}{R^3}u - \frac{3GM}{R^3}\mathcal{P}u = \mathcal{F} + \mathcal{N}.$$

Let us explain the notation. On the right hand side, the term $\mathcal{F}$ consists of the principal contribution of the gravity from the second body $\mathcal{B}_2$, and represents the contribution to the *tidal acceleration* from the body $\mathcal{B}_2$ acting on $\mathcal{B}_1$. The error from considering only the principal contribution of the gravity of the second body as well as the genuinely nonlinear terms are contained in $\mathcal{N}$. The operator $n \times \nabla$, is intrinsic to the surface $\partial\mathcal{B}_1$. Indeed, for a scalar function f defined on $\mathcal{B}_1$, $n \times \nabla f$ depends only on the restriction of f to $\partial\mathcal{B}_1$ as the cross product with n annihilates the normal component of ∇f. For a vectorfield u, which is the restriction to $\partial\mathcal{B}_1$ of a divergence-curl free vectorfield in $\mathcal{B}_1$, $n \times \nabla u$ is given by

$$n \times \nabla u = \frac{1}{|N|}(\xi_\beta \times u_\alpha - \xi_\alpha \times u_\beta),$$

---

[4]Local well-posedness in Sobolev spaces for the free boundary problem of the incompressible Euler equation with constant gravity was proved in [15, 16]. In particular, Clifford analysis was used in [16]. Later, the result in [16] was generalized to the case with non-zero vorticity, again using Clifford analysis, in [19]. For the particular model at hand with only one body this was established in [11, 12], and the presence of a second body does not affect the local well-posedness of the system, since it acts as a lower order source term in the equation. We refer the interested reader to the works above and the references therein for a more detailed historical account of these developments.

in any (orientation preserving) local coordinates (α, β) on S_R, with $N = \xi_\alpha \times \xi_\beta$. The operator $\mathcal{P}$ is a non-local operator which maps u to the projection of $(u \cdot n)n$ into the space of curl and divergence free vectorfields on $\partial \mathcal{B}_1$[5]. For the reader who is familiar with Clifford analysis, the projection $\mathcal{P}$ admits the explicit representation

$$\mathcal{P}u = \frac{1}{2}(I + H_{\partial \mathcal{B}_1})\left((u \cdot n)n\right),$$

where $H_{\partial \mathcal{B}_1}$ denotes the Hilbert transform (see Appendix A)

$$(17) \qquad H_{\partial \mathcal{B}_1} f(\xi) = -\frac{\text{p.v.}}{2\pi} \int_{\partial \mathcal{B}_1} \frac{\xi' - \xi}{|\xi' - \xi|^3} n(\xi') f(\xi') dS(\xi').$$

We now discuss a novel part of our analysis which is the most delicate point in the proof of a-priori estimates, and is related to the correct bootstrap assumptions for the energy and the unknowns. Recall that for the usual lifespan estimates for a small data quasilinear system, one assumes a bootstrap bound of the form $\mathcal{E} \leq 2C\epsilon$ on an appropriate energy functional $\mathcal{E}$, where $C > 0$ and $\epsilon \ll 1$ are determined by the initial data, and tries to improve this to $\mathcal{E} \leq C\epsilon$. Instead, here we impose a bootstrap assumption of the form $\mathcal{E} \leq 2C(t)$ where $C(t)$ is a function that decays to zero as $t \to -\infty$, and improve this to $\mathcal{E} \leq C(t)$. We emphasize that here the energy functional $\mathcal{E}$ is an energy functional in the mathematical sense of the term as used to prove a-priori estimates and is different from the physical energies $\mathscr{E}$, $\tilde{\mathscr{E}}$, etc. above. The reason for this choice is that the initial data for the problem are trivial, so for the appropriate choice of $\mathcal{E}$, initially $\mathcal{E}(T_0) = 0$ and we need to prove the existence of a solution for *infinite time* rather than on a time interval depending on the size of the data. In fact, it is more natural to think of the decay in terms of the distance of the bodies from each other rather than the time of evolution. To be able to close this type of bootstrap assumption, we need to determine the correct decay rate for the energy functional. This is based on a careful analysis of the source term F and the decay behavior of various *small quantities*[6] appearing in the nonlinearity N. More precisely, to define the energy functional above at the level of u (that is, before commuting derivatives with the equation), which we call $\mathcal{E}_u$, we take the inner

---

[5]More precisely, vectorfields on $\partial \mathcal{B}_1$ which can be written as the restriction of a curl and divergence free vectorfield in $\mathcal{B}_1$. We will often simply refer to such vector fields as curl and divergence free.

[6]That is, quantities that would be zero if $\mathcal{B}_1$ were a non-accelerating ball of radius R.

product of (16) with $\frac{u_t}{a}$, which leads to the following definition:

$$(18) \qquad \mathcal{E}_u := \frac{1}{2} \int_{\partial \mathcal{B}_1} \frac{|u_t|^2}{a} dS + \frac{1}{2} \int_{\partial \mathcal{B}_1} (n \times \nabla u) \cdot u \, dS$$
$$+ \frac{GM}{2R^3} \int_{\partial \mathcal{B}_1} \frac{|u|^2}{a} dS - \frac{3GM}{2R^3} \int_{\partial \mathcal{B}_1} \frac{(n \cdot u)^2}{a} dS.$$

By analyzing the source term in the equations satisfied by u and the height function h (see Subsection 1.1.2) we pose the bootstrap assumptions

$$(19) \qquad \|h(t)\|_{L^2(S_R)} \leq CR^2\eta^3(t), \qquad \|u(t)\|_{L^2(S_R)} \leq CR\eta^4(t)|\mathbf{x}_1'(t)|.$$

According to (19) the various terms appearing in the nonlinearity will have different decay rates, an observation which is crucial in controlling the contribution of the nonlinearity in the energy estimates. Moreover, as shown in the final section of this article where the tidal energy is analyzed, the decay rates above for u and h are in fact sharp. The assumptions (19) then lead to the following bootstrap assumption for the energy:

$$(20) \qquad \mathcal{E}_u(t) \leq CR\eta^8(t)|\mathbf{x}_1'(t)|^2.$$

Recall that $\eta(t) := \frac{R}{r_1(t)}$, so (19) and (20) are formulated in terms of the position and speed of the centers of mass of the bodies. Therefore, to be able to prove a-priori estimates we need to control the center of mass motion of the bodies to obtain bounds on $|\mathbf{x}_1(t)|$ and $|\mathbf{x}_1'(t)|$ when the bodies are sufficiently far. This requires analyzing the point mass system and the error resulting from approximating the motion of the bodies by that of point masses up to the point of closest approach.

The energy functional $\mathcal{E}_u$ satisfies

$$\frac{d\mathcal{E}_u}{dt} = \int_{\partial \mathcal{B}_1} (\mathcal{F} + \mathcal{N}) \frac{u_t}{a} dS + \frac{1}{2} \int_{\partial \mathcal{B}_1} \frac{1}{|N|} \partial_t \left(\frac{|N|}{a} \right) |u_t|^2 dS + \dots .$$

We have not written out all the error terms on the right hand side as they are not relevant for our discussion of the main challenges in this introduction, but the precise statement can be found in Proposition 3.3. The definition of our energy and the energy identity are similar to the ones used in [16, 17] to study water waves, but here several new ingredients are needed in using them to prove a-priori estimates for equation (16). These are needed to deal with the different geometry of the domain as well as the new linear and

nonlinear contributions from the gravitational force which, in contrast to the water wave problem, is not a constant. The first issue to discuss is the coercivity of $\mathcal{E}_u$. It is not hard to see that for a curl and divergence free vectorfield f defined on $\mathcal{B}_1$,

$$n \times \nabla f = \nabla_n f$$

where ∇_n denotes the Dirichlet-Neumann operator on $\mathcal{B}_1$. This shows that in terms of the Eulerian fluid velocity relative to the center of mass, $\mathbf{u} := \mathbf{v} - \mathbf{x}_1'$,

$$\int_{\partial \mathcal{B}_1} (n \times \nabla u) \cdot u \, dS = \int_{\mathcal{B}_1} |\nabla \mathbf{u}|^2 d\mathbf{x}.$$

However, in view of the negative sign of the last term on the right hand side of (18), it is not clear that the energy functional $\mathcal{E}_u$ is positive in general. To show that this negative term can be controlled by the other positive terms in the definition of the energy functional, first note that since $\int_{\mathcal{B}_1} \mathbf{u} \, d\mathbf{x} = 0$, by the Poincaré estimate

$$\int_{\mathcal{B}_1} |\nabla \mathbf{u}|^2 d\mathbf{x} \geq C \int_{\mathcal{B}_1} |\mathbf{u}|^2 d\mathbf{x}.$$

Combining this with a careful computation using the trace embedding $H^1(\mathcal{B}_1) \hookrightarrow L^2(\partial \mathcal{B}_1)$, we are able to show the coercivity of the energy functional, and that in particular

$$\mathcal{E}_u \gtrsim \int_{\partial \mathcal{B}_1} (n \times \nabla u) \cdot u \, dS + \frac{GM}{R^3} \int_{\partial \mathcal{B}_1} \frac{|u|^2}{a} dS.$$

To derive this estimate we need to have bounds on the constants in the trace embedding and Poincaré estimates, but such bounds are available so long as the surface $\partial \mathcal{B}_1$ is close to S_R in the appropriate sense. These statements are made precise in Lemmas 3.5, 3.7, and 3.11.

The last point to discuss regarding energy estimates is commuting derivatives with equation (16) to estimate higher order derivatives of u. Since the spatial domain of the Lagrangian variables is S_R, to prove higher order regularity for u we need to estimate $\Omega^k u$, where Ω^k denotes k differentiations using any combination of the restriction of the three rotational vectorfields

$$\Omega_{ij} = \mathbf{x}_i \partial_j - \mathbf{x}_j \partial_i, \quad 1 \leq i < j \leq 3,$$

to S_R. However, to preserve the structure of the equation we need the derivative we commute to have a small commutator with the operator $\mathcal{P}$ in (16), and for the derivative of u to be approximately curl and divergence free. For this purpose we introduce a modified Lie derivative $\mathfrak{D}u$ which satisfies these properties, and such that control of u and $\mathfrak{D}u$ give us control of Ωu. To motivate our definition of the modified Lie derivative, suppose for the moment that $\partial \mathcal{B}_1$ is a round sphere S_R centered at the origin, and that u is the restriction to S_R of a curl and divergence free vectorfield, $\mathbf{u}$, defined in $B_R(0)$. It is then easy to see that $\mathcal{L}_\Omega \mathbf{u}$ is also curl and divergence free, where $\mathcal{L}_\Omega$ denotes the Lie derivative with respect to Ω. Moreover, since the normal vector n to S_R is the radial vectorfield ∂_r, we have $\mathcal{L}_\Omega n = 0$. Recalling the definition of $\mathcal{P}$ as the projection of $(n \cdot u)n$ into the space of curl and divergence free vectorfields, we see that $\mathcal{L}_\Omega$ commutes with $\mathcal{P}$. Now in general, when $\mathcal{B}_1$ is not S_R, the Lie derivative $\mathcal{L}_\Omega$ is not well-defined. Instead, we observe that in $\mathbb{R}^3$, if $\mathbf{e}$ is the axis of rotation for Ω, we have the simple relation

$$\mathcal{L}_\Omega f = \Omega f - \mathbf{e} \times f,$$

for any vectorfield f. Motivated by this we define the differential operator

$$\mathfrak{D}_\Omega = \Omega - \mathbf{e} \times .$$

This operator will then have a small commutator with $\mathcal{P}$, and $\mathfrak{D}^k u$ is almost curl and divergence free, in a sense that is made precise in Section 2. Higher order energy functionals are then defined by replacing u in (18) by the projection of $\mathfrak{D}^k u$ into the space of curl and divergence vectorfields[7]. For the precise definition see Definition 3.12.

1.1.2. The Tidal Energy. Having proved the existence of a solution to (2) and estimates on the velocity and height function, $h = |\zeta| - R$, we can turn to the study of the dynamics of the equation and the proof of (15). Using the already-established a-priori estimates, it is not hard to show that

$$\int_{\mathcal{B}_1} |\mathbf{v} - \mathbf{x}_1'|^2 d\mathbf{x} \approx R\|\partial_t h\|_{L^2(S_R)}^2,$$

$$\int_{\mathcal{B}_1} \psi_1 d\mathbf{x} + \frac{3GM|\mathcal{B}_1|}{5R} \approx \frac{GM}{R^2}\|h\|_{L^2(S_R)}^2,$$

---

[7]More precisely by $\frac{1}{2}(I + H_{\partial \mathcal{B}_1})\mathfrak{D}^k u$, which has a scalar component as well. See Definition 3.12

proving (14). Therefore, to prove (15) we need to obtain a *lower bound* on

$$GMR^{-2}\|h\|^2_{L^2(S_R)} + R\|h_t\|^2_{L^2(S_R)}.$$

To obtain this lower bound, we study the linearized equation for the height function h (see equation (22) below) to derive lower bounds for $\|h\|_{L^2(S_R)}$ and $\|\partial_t h\|_{L^2(S_R)}$ in the linearized setting. We then use the a-priori estimates to show that these lower bounds remain valid even after considering nonlinear effects. To derive the desired linearized equation we first derive an equation for h in Lagrangian coordinates which we schematically write as

$$(21) \qquad \partial_t^2 h + \nabla_n(I - 3K_{\partial\mathcal{B}_1})h = \mathcal{F} + \mathcal{N},$$

where the source term $\mathcal{F}$ contains the contribution of the gravitational force from $\mathcal{B}_2$ and $\mathcal{N}$ contains the genuinely nonlinear terms. Here $\mathcal{F}$ and $\mathcal{N}$ are not the same as in (16). ∇_n denotes the Dirichlet-Neumann map of $\mathcal{B}_1$ and the non-local operator $K_{\partial\mathcal{B}_1}$ is the double-layered potential for $\mathcal{B}_1$, defined as

$$K_{\partial\mathcal{B}_1} f(\xi) = \frac{\text{p.v.}}{2\pi} \int_{\partial\mathcal{B}_1} \frac{(\xi' - \xi)}{|\xi' - \xi|^3} \cdot n(\xi')f(\xi')dS(\xi')$$

for any real-valued function f. To derive a lower bound for $\|\partial_t h\|_{L^2(S_R)}$ we do not rely on energy estimates for (21) but use the fundamental solution for this equation instead. More precisely, we first transfer the equation to an equation on $\mathbb{R} \times S_R$, by replacing ∇_n and $K_{\partial\mathcal{B}_1}$ by $\mathbf{D}$ and $\mathbf{K}$, respectively, where $\mathbf{D}$ and $\mathbf{K}$ are the Dirichlet-Neumann map and double-layered potential for S_R. The resulting equation is

$$(22) \qquad \partial_t^2 h + \mathbf{D}(I - 3\mathbf{K})h = \mathcal{F} + \tilde{\mathcal{N}},$$

where $\tilde{\mathcal{N}}$ contains the new error terms which arise in passing to the equation on $\mathbb{R}\times S_R$. Note that some of these error terms (in fact, also some of the error terms in $\mathcal{N}$) are of highest order in terms of regularity, that is, equation (22) is fully nonlinear. However, since we have already established higher order regularity and a-priori estimates, regularity is not relevant in the analysis of (22). We can now use the fundamental solution of (22) by decomposing h, $\mathcal{F}$, and $\tilde{\mathcal{N}}$ into spherical harmonics h_ℓ, $\mathcal{F}_\ell$, and $\tilde{\mathcal{N}}_\ell$. Studying the source term $\mathcal{F}$, we find that the main contribution to the equation comes from the

second harmonic $\mathcal{F}_2$, where h_2 satisfies

$$(23) \qquad \begin{aligned} \partial_t^2 h_2 + a_2 h_2 &= \mathcal{F}_2 + \tilde{\mathcal{N}}_2, \\ \lim_{t \to T_0} h_2(t) &= \lim_{t \to T_0} \partial_t h_2(t) = 0, \end{aligned}$$

for some constant coefficient $a_2 > 0$. The analysis of the system (23), using the explicit representation of $\mathcal{F}_2$, is carried out in Section 4. This analysis relies heavily on the fact that the frequency of oscillation of the source term $\mathcal{F}_2$ is much smaller than the natural frequency of the system (23). Using the explicit representation of $\mathcal{F}_2$, we then derive the lower bounds

$$\|\partial_t h_2\|_{L^2(S_R)} \gtrsim R\eta^4 |\mathbf{x}_1'|, \qquad \|h_2\|_{L^2(S_R)} \gtrsim R^2 \eta^3.$$

Moreover, we show that the contribution of the other harmonics, h_ℓ and $\partial_t h_\ell$, $\ell \neq 2$, are smaller and do not affect these lower bounds, so

$$\|\partial_t h\|_{L^2(S_R)} \gtrsim R\eta^4 |\mathbf{x}_1'|, \qquad \|h\|_{L^2(S_R)} \gtrsim R^2 \eta^3.$$

Note that these lower bounds are consistent with the bootstrap assumptions on h and $\partial_t h$ in the discussion of a-priori estimates above, proving the sharpness of our bootstrap assumptions on the decay rates of h and u. Now since the total energy is conserved during the evolution, a comparison of the implied lower bound on the tidal energy with the initial total energy allows us to conclude the proof of Theorem 1.1. In fact, it follows from the estimates above that the main contribution to the tidal energy up to the point of closest approach is from the the potential energy which contributes $GMR^{-5}\|h\|_{L^2(S_R)}^2$.

1.1.3. Christodoulou's conjecture and analysis. In [4], Christodoulou showed that *at the linearized level* under the initial configuration $v_0 \sim \sqrt{\frac{GM}{R}}\beta^{-\frac{9}{10}}, p \sim \beta^{\frac{4}{5}}$, the kinetic part $\frac{1}{R^2}\|\partial_t h\|_{L^2(S_R)}$ of the tidal energy $\widetilde{\mathcal{E}_{\text{tidal}}}$ can be made larger than the total energy $\widetilde{\mathcal{E}}$ near the point of closest approach, assuming that a smooth solution to the free boundary problem (2) exists up to this point. He conjectured that the same conclusion holds in the nonlinear setting. In this work we give a complete proof of Christodoulou's conjecture in the following sense: For a larger set of initial configurations than the one proposed in [4], we prove the existence of smooth solution to (2) up to the closest approach. Then based on the a-priori estimates derived in this analysis, we use the fully nonlinear equation (22) to prove that the tidal energy $\widetilde{\mathcal{E}_{\text{tidal}}}$ can be made larger than the total energy $\widetilde{\mathcal{E}}$.

1.2. Organization of the Paper

In Section 2 we derive the quasilinear system for $u = \zeta_t$. Higher order derivatives $\mathfrak{D}^k u$ and the equations they satisfy are discussed in Subsection 2.2. The energy estimates are carried out in Section 3. In Subsection 3.1 we define the energy functional and derive the basic energy identity. Here we also discuss the coercivity of the energy functional as discussed above. Subsection 3.2 is devoted to the analysis of the motion of the center of mass $\mathbf{x}_1$ of $\mathcal{B}_1$, and estimates on $\frac{d|\mathbf{x}_1|}{dt}$ and $|\mathbf{x}_1'|$ are derived. In Subsection 3.3 we introduce the bootstrap assumptions and show how the various terms appearing in the nonlinearity can be controlled in terms of the energy under the bootstrap assumptions. Finally, we close the energy estimates in Subsection 3.4. Section 4 is devoted to the analysis of the heigh function h and the proof of Theorem 1.1. The equation for h is derived in Subsection 4.1, where we also write the equation in spherical harmonics and obtain the key lower bounds on h and $\partial_t h$. This analysis is then used in Subsection 4.2 to complete the proof of Theorem 1.1. There are two appendices to this article where several technical ingredients are discussed. In Appendix A we recall a few basic definitions from Clifford analysis as well as the definition of layered-potentials. The notation and terminology introduced in this appendix are used throughout the paper. We also carry out some spherical harmonic decompositions which are used in Section 4. Appendix B contains classical important singular integral estimates which are used regularly in the proof of a-priori estimates.

2. The Equations in Lagrangian Coordinates

In this section we derive the main equations of motion in Lagrangian coordinates, starting with the system

$$
(24) \quad
\begin{cases}
\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \mathbf{p} - \nabla \psi, & \text{in } \mathcal{B}(t) \\
\nabla \cdot \mathbf{v} = 0, \quad \nabla \times \mathbf{v} = 0, & \text{in } \mathcal{B}(t) \\
\mathbf{p} = 0, & \text{on } \partial\mathcal{B}(t) \\
(1, \mathbf{v}) \in T(t, \partial\mathcal{B}(t)), & \text{on } \partial\mathcal{B}(t)
\end{cases} \quad .
$$

For this we work with the time differentiated equation, taking the Lagrangian velocity as our main unknown. In deriving the desired equation, we will freely use the basic notation and concepts from Clifford analysis reviewed in Appendix A. In particular see this appendix for the definition of the operators $H_{\partial\mathcal{B}_1}, K_{\partial\mathcal{B}_1}$ and $H^*_{\partial\mathcal{B}_1}, K^*_{\partial\mathcal{B}_1}$ etc. Also the convention introduced in Remark A.1 is in use in the rest of the paper.

2.1. The Equation for u

We start by introducing the notation

$$r_1 := |\mathbf{x}_1|, \quad \boldsymbol{\xi}_1 = \frac{\mathbf{x}_1}{|\mathbf{x}_1|}, \quad \eta = \frac{R}{|\mathbf{x}_1|}.$$

Let $\xi : \mathbb{R} \times S_R \to \partial\mathcal{B}_1(t)$ be the Lagrangian parametrization of $\partial\mathcal{B}_1 = \partial\mathcal{B}_1(t)$, such that $\xi(T_0, \cdot) - \mathbf{x}_1(T_0)$ is the identity map of $S_R \subseteq \mathbb{R}^3$, and define

$$\zeta : \mathbb{R} \times S_R \to \partial\mathcal{B}_1 - \mathbf{x}_1, \qquad \zeta(t, p) = \xi(t, p) - \mathbf{x}_1.$$

Occasionally we write

$$\zeta = \sum_{i=1}^{3} \zeta^i \mathbf{e}_i,$$

where $\zeta^i = \mathbf{x}^i \circ \zeta = \mathbf{e}_i \cdot \zeta$. We will denote the exterior unit normal to $\partial\mathcal{B}_1$ by $\mathbf{n}$ and let

$$n(t, p) = \mathbf{n}(t, \xi(t, p)).$$

In arbitrary (orientation preserving) local coordinates (α, β) on S_R we have

$$n = \frac{N}{|N|}, \quad \text{where} \quad N = \xi_\alpha \times \xi_\beta = \zeta_\alpha \times \zeta_\beta.$$

If $\mathbf{f} : \mathcal{B}_1 \to \mathbb{R}$ is a (possibly time-dependent) differentiable function, and $f = \mathbf{f} \circ \xi$, then by a slight abuse of notation we write

$$\nabla f = (\nabla\mathbf{f}) \circ \xi, \quad df = (d\mathbf{f}) \circ \xi,$$

where d denotes the exterior differentiation operators on $\partial\mathcal{B}_1$. With this notation, and using the fact that $N = \zeta_\alpha \times \zeta_\beta$

$$n \times \nabla f := (\mathbf{n} \times \nabla\mathbf{f}) \circ \xi = \frac{\xi_\beta f_\alpha - \xi_\alpha f_\beta}{|N|} = \frac{\zeta_\beta f_\alpha - \zeta_\alpha f_\beta}{|N|}.$$

Note that this definition is independent of the extension $\mathbf{f}$ of $\mathbf{f}|_{\partial\mathcal{B}_1}$ to the interior of $\mathcal{B}_1$.

The fluid velocity in Lagrangian coordinates will be denoted by v, that is,

$$v(t,p) = \mathbf{v}(t,\xi(t,p)) = \xi_t(t,p).$$

We also let

$$u(t,p) = \mathbf{v}(t,\xi(t,p)) - \mathbf{x}_1' = \zeta_t(t,p).$$

The gravity potential $\boldsymbol{\psi}$ is written as $\boldsymbol{\psi} = \boldsymbol{\psi}_1 + \boldsymbol{\psi}_2$ with

$$\boldsymbol{\psi}_i(t,\mathbf{x}) = -G\rho \int_{\mathcal{B}_j(t)} \frac{d\mathbf{y}}{|\mathbf{x} - \mathbf{y}|},$$

and in Lagrangian coordinates we set

$$\psi_j(t,p) := \boldsymbol{\psi}_j(t,\xi(t,p)).$$

Since $\mathbf{p} = 0$ on $\partial \mathcal{B}_1$,

$$-\nabla \mathbf{p} = \mathbf{a}\mathbf{n}, \qquad \text{where} \qquad \mathbf{a} := -\nabla \mathbf{p} \cdot \mathbf{n},$$

and we let

$$a(t,p) = \mathbf{a}(t,\xi(t,p)).$$

Remark 2.1. Suppose $\mathbf{f} : \partial \mathcal{B}_1 \to \mathbb{R}$ is a function defined on the boundary of the fluid domain, and let $f = \mathbf{f} \circ \xi$. By a slight abuse of notation, we often write integrals on the boundary $\partial \mathcal{B}_1$ in terms of f instead of $\mathbf{f}$. For instance we write $\int_{\partial \mathcal{B}_1} f \, dS$ to mean $\int_{\partial \mathcal{B}_1} \mathbf{f} \, dS$, even though f is a function with domain S_R.

With the notation above the first equation in (24) becomes

(25) $$\zeta_{tt} = an - \nabla\psi_1 - (\nabla\psi_2 + \mathbf{x}_1'').$$

To state the main result of this section we need to introduce some more notation. Let

(26) $\qquad h(t,p) = |\zeta(t,p)| - R, \qquad$ and $\qquad \tilde{h}(t,p) = |\zeta(t,p)|^2 - R^2.$

Both h and $\tilde{h}$ vanish when $\partial \mathcal{B}_1$ is a round sphere, and in general we expect them to be small during the evolution. In computations it is often convenient

to replace ζ by $R^3|\zeta|^{-3}\zeta$. The reason for this is that the function $\frac{\mathbf{x}-\mathbf{x}_1}{|\mathbf{x}-\mathbf{x}_1|^3}$ has zero curl and divergence outside $\mathcal{B}_1$ so $H_{\partial\mathcal{B}_1}(|\zeta|^{-3}\zeta) = -|\zeta|^{-3}\zeta$. Here $H_{\partial\mathcal{B}_1}$ denotes the Hilbert transform introduced in (17). The error that is generated from replacing ζ by $R^3|\zeta|^{-3}\zeta$ is encoded in

$$(27) \qquad \mu := 1 - \frac{R^3}{|\zeta|^3}.$$

Taking common denominators gives

$$(28) \qquad \mu = \frac{3}{2R^2}\tilde{h} + \nu\tilde{h},$$

where

$$(29) \qquad \nu = \frac{|\zeta|^2 + R|\zeta| + R^2}{|\zeta|^3(|\zeta|+R)} - \frac{3}{2R^2}.$$

Note that in view of the discussion above

$$(I + H_{\partial\mathcal{B}_1})\zeta = (I + H_{\partial\mathcal{B}_1})(\mu\zeta).$$

We now state the main result of this section.

Proposition 2.2. *u satisfies*

$$(30) \qquad \begin{aligned} \partial_t^2 u + an\times\nabla u + \frac{GM}{R^3}u - \frac{3GM}{2R^3}(I + H_{\partial\mathcal{B}_1})((u\cdot n)n) \\ = -F_t - \partial_t E_1 + E_2 + \partial_t\left(\frac{a}{|N|}\right)N + \partial_t\left(\frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\mathbf{E}_1(t,\mathbf{x})d\mathbf{x}\right), \end{aligned}$$

where with $\boldsymbol{\zeta} = \mathbf{x}-\mathbf{x}_1$ and $\boldsymbol{\zeta}' = \mathbf{y}-\mathbf{x}_2$

$$(31) \qquad F := \frac{GM\eta^3}{8R^3}(\zeta - 3(\zeta\cdot\boldsymbol{\xi}_1)\boldsymbol{\xi}_1),$$

$$(32)$$
$$E_1 = G\rho\eta^2\int_{\mathcal{B}_2}\left[\frac{\eta(\zeta-(\mathbf{y}-\mathbf{x}_2))+2R\boldsymbol{\xi}_1}{|\eta(\zeta-(\mathbf{y}-\mathbf{x}_2))+2R\boldsymbol{\xi}_1|^3} - \frac{\boldsymbol{\xi}_1}{4R^2}\right.$$
$$\left. - \frac{\eta}{8R^3}\left(\zeta - (\mathbf{y}-\mathbf{x}_2) - 3((\zeta-(\mathbf{y}-\mathbf{x}_2))\cdot\boldsymbol{\xi}_1)\boldsymbol{\xi}_1\right)\right]d\mathbf{y},$$

$$\mathbf{E}_1(t, \mathbf{x}) = G\rho\,\eta^2 \int_{\mathcal{B}_2} \left[\frac{\eta(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2R\boldsymbol{\xi}_1}{|\eta(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2R\boldsymbol{\xi}_1|^3} - \frac{\boldsymbol{\xi}_1}{4R^2} \right.$$
$$\left. - \frac{\eta}{8R^3} \left(\boldsymbol{\zeta} - \boldsymbol{\zeta}' - 3((\boldsymbol{\zeta} - \boldsymbol{\zeta}') \cdot \boldsymbol{\xi}_1)\,\boldsymbol{\xi}_1 \right) \right] d\mathbf{y},$$

and

(33)
$$E_2 := \frac{3GM}{2R^5}(I + H_{\partial\mathcal{B}_1})((u \cdot (\zeta - Rn))\zeta) + \frac{3GM}{2R^4}(I + H_{\partial\mathcal{B}_1})((u \cdot n)(\zeta - Rn))$$
$$+ \frac{GM}{2R^3}(I + H_{\partial\mathcal{B}_1})(\mu u) + \frac{GM}{2R^3}(I + H_{\partial\mathcal{B}_1})((\tilde{h}_t \nu + \tilde{h}\nu_t)\zeta)$$
$$+ \frac{GM}{2R^3}[\partial_t, H_{\partial\mathcal{B}_1}](\mu\zeta).$$

Remark 2.3. Note that the left-hand side of equation (30) is a pure vector, that is $(I + H_{\partial\mathcal{B}_1})((n \cdot u)n)$ has no real part. Indeed (recall that the convention of Remark A.1 is in use)

$$H_{\partial\mathcal{B}_1}((n \cdot u)n) = \text{p.v.} \int_{\partial\mathcal{B}_1} Kn'n'(n' \cdot u')dS' = -\text{p.v.} \int_{\partial\mathcal{B}_1} K(n' \cdot u')dS'$$

is a pure vector.

To derive equation (30) we will differentiate equation (25) in time. To do this efficiently we need more convenient expressions for $\nabla\psi_1$ and $\nabla\psi_2$, which are derived in the following lemma.

Lemma 2.4.   *1. Let* $\nabla\psi_1(t, p) := \nabla\psi_1(t, \xi(t, p))$. *Then*

$$(34) \qquad \nabla\psi_1 = \frac{GM}{2R^3}(I - H_{\partial\mathcal{B}_1})\zeta = \frac{GM}{R^3}\zeta - \frac{GM}{2R^3}(I + H_{\partial\mathcal{B}_1})\zeta.$$

2. Let $\mathbf{x}$ *be any point in* $\mathcal{B}_2^c$. *Then*

$$(35) \quad \nabla\boldsymbol{\psi}_2(t, \mathbf{x}) = \frac{GM\eta^2}{4R^2}\boldsymbol{\xi}_1 + \frac{GM\eta^3}{8R^3}\left((\mathbf{x} - \mathbf{x}_1) - 3((\mathbf{x} - \mathbf{x}_1) \cdot \boldsymbol{\xi}_1)\,\boldsymbol{\xi}_1\right)$$
$$+ \mathbf{E}_1(t, \mathbf{x}),$$

where with $\boldsymbol{\zeta} = \mathbf{x} - \mathbf{x}_1$ and $\boldsymbol{\zeta}' = \mathbf{y} - \mathbf{x}_2$

(36)

$$\mathbf{E}_1(t,\mathbf{x}) = G\rho\,\eta^2 \int_{\mathcal{B}_2} \left[\frac{\eta(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2R\boldsymbol{\xi}_1}{|\eta(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2R\boldsymbol{\xi}_1|^3} - \frac{\boldsymbol{\xi}_1}{4R^2} \right.$$
$$\left. - \frac{\eta}{8R^3} \left(\boldsymbol{\zeta} - \boldsymbol{\zeta}' - 3((\boldsymbol{\zeta} - \boldsymbol{\zeta}') \cdot \boldsymbol{\xi}_1)\,\boldsymbol{\xi}_1 \right) \right] d\mathbf{y}.$$

3. *Let* $\nabla\psi_2(t,p) := \nabla\boldsymbol{\psi}_2(t, \xi(t,p))$ *and* $E_1(t,p) := \mathbf{E}_1(t, \xi(t,p))$. *Then*

(37) $$\nabla\psi_2 = \frac{GM\eta^2}{4R^2}\boldsymbol{\xi}_1 + \frac{GM\eta^3}{8R^3}\left(\boldsymbol{\zeta} - 3(\boldsymbol{\zeta} \cdot \boldsymbol{\xi}_1)\,\boldsymbol{\xi}_1\right) + E_1.$$

Remark 2.5. Note that the integrand in the definition (36) of $\mathbf{E}_1$ consists the $O(\eta^2)$ terms in the Taylor expansion of

$$\frac{\eta(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2R\boldsymbol{\xi}_1}{|\eta(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2R\boldsymbol{\xi}_1|^3}$$

in η, and it is for this reason that $\mathbf{E}_1$ is regarded as an error term.

Proof of Lemma 2.4. We start with the equation for $\nabla\psi_1$. Recall that $\boldsymbol{\psi}_1$ satisfies

$$\Delta\boldsymbol{\psi}_1 = 4\pi G\rho\chi_{\mathcal{B}_1}.$$

It follows that

$$\nabla \cdot \left(\nabla\boldsymbol{\psi}_1 - \frac{4\pi G\rho}{3}\mathbf{x} \right) = 0 \quad \text{and} \quad \nabla \times \left(\nabla\boldsymbol{\psi}_1 - \frac{4\pi G\rho}{3}\mathbf{x} \right) = 0$$

in $\mathcal{B}_1$ and hence

$$(I - H_{\partial\mathcal{B}_1})(\nabla\psi_1) = \frac{4\pi G\rho}{3}(I - H_{\partial\mathcal{B}_1})\xi.$$

Similarly since $\nabla\boldsymbol{\psi}_1$ is curl and divergence-free outside of $\mathcal{B}_1$,

$$(I + H_{\partial\mathcal{B}_1})\nabla\psi_1 = 0.$$

It follows that

$$\nabla\psi_1 = \frac{1}{2}(I - H_{\partial\mathcal{B}_1})\nabla\psi_1 + \frac{1}{2}(I + H_{\partial\mathcal{B}_1})\nabla\psi_1$$

$$
\begin{aligned}
&= \frac{2\pi G\rho}{3}(I - H_{\partial\mathcal{B}_1})\xi = \frac{GM}{2R^3}(I - H_{\partial\mathcal{B}_1})\xi \\
&= \frac{GM}{R^3}\xi - \frac{GM}{2R^3}(I + H_{\partial\mathcal{B}_1})\xi = \frac{GM}{R^3}(\xi - \mathbf{x}_1) - \frac{GM}{2R^3}(I + H_{\partial\mathcal{B}_1})(\xi - \mathbf{x}_1) \\
&= \frac{GM}{R^3}\zeta - \frac{GM}{2R^3}(I + H_{\partial\mathcal{B}_1})\zeta,
\end{aligned}
$$

proving (34). Next we turn to the gravity of the second body. The formula for $\nabla\psi_2$ follows directly from that of $\nabla\boldsymbol{\psi}_2$ so we concentrate on the latter. If $\mathbf{x} \in \mathcal{B}_2^c$ is an arbitrary point, then (suppressing the time variable from the notation)

$$
\boldsymbol{\psi}_2(\mathbf{x}) = -G\rho \int_{\mathcal{B}_2} \frac{d\mathbf{y}}{|\mathbf{y} - \mathbf{x}|}.
$$

Since $\mathbf{x} \in \mathcal{B}_2^c$, the integration kernel is nonsingular and we can differentiate inside the integral to get

$$
\nabla\boldsymbol{\psi}_2(\mathbf{x}) = G\rho \int_{\mathcal{B}_2} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} d\mathbf{y}.
$$

Introducing the notation $\boldsymbol{\zeta} = \mathbf{x} - \mathbf{x}_1$ and $\boldsymbol{\zeta}' = \mathbf{y} - \mathbf{x}_2$ and recalling that $\mathbf{x}_1 = -\mathbf{x}_2$ we can rewrite this as

$$
\begin{aligned}
(38) \qquad \nabla\boldsymbol{\psi}_2(\mathbf{x}) &= G\rho \int_{\mathcal{B}_2 - \mathbf{x}_2} \frac{(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2\mathbf{x}_1}{|(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2\mathbf{x}_1|^3} d\boldsymbol{\zeta}' \\
&= G\rho\eta^2 \int_{\mathcal{B}_2 - \mathbf{x}_2} \frac{\eta(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2R\boldsymbol{\xi}_1}{|\eta(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2R\boldsymbol{\xi}_1|^3} d\boldsymbol{\zeta}'.
\end{aligned}
$$

Now Taylor expansion in η gives

$$
\begin{aligned}
G\rho &\frac{\eta(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2R\boldsymbol{\xi}_1}{|\eta(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2R\boldsymbol{\xi}_1|^3} \\
&= \frac{G\rho}{4R^2}\boldsymbol{\xi}_1 + \frac{G\rho\,\eta}{8R^3}\left((\boldsymbol{\zeta} - \boldsymbol{\zeta}') - 3((\boldsymbol{\zeta} - \boldsymbol{\zeta}') \cdot \boldsymbol{\xi}_1)\boldsymbol{\xi}_1\right) + \widetilde{\mathbf{E}}_1,
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{\mathbf{E}}_1 = G\rho &\frac{\eta(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2R\boldsymbol{\xi}_1}{|\eta(\boldsymbol{\zeta} - \boldsymbol{\zeta}') + 2R\boldsymbol{\xi}_1|^3} - \frac{G\rho}{4R^2}\boldsymbol{\xi}_1 \\
&- \frac{G\rho\,\eta}{8R^3}\left((\boldsymbol{\zeta} - \boldsymbol{\zeta}') - 3((\boldsymbol{\zeta} - \boldsymbol{\zeta}') \cdot \boldsymbol{\xi}_1)\boldsymbol{\xi}_1\right)
\end{aligned}
$$

denotes the remainder of order η^2 in the Taylor expansion. Note that by definition,

$$\fint_{\mathcal{B}_2 - \mathbf{x}_2} \boldsymbol{\zeta}' d\boldsymbol{\zeta}' = 0, \qquad \text{and} \qquad \rho \int_{\mathcal{B}_2 - \mathbf{x}_2} d\boldsymbol{\zeta}' = M,$$

so plugging the expansion above into (38) gives the desired identity

$$\nabla \boldsymbol{\psi}_2(\mathbf{x}) = \frac{GM\eta^2}{4R^2} \boldsymbol{\xi}_1 + \frac{GM\eta^3}{8R^3} \left((\mathbf{x} - \mathbf{x}_1) - 3((\mathbf{x} - \mathbf{x}_1) \cdot \boldsymbol{\xi}_1) \boldsymbol{\xi}_1 \right) + \mathbf{E}_1. \qquad \square$$

The following simple lemma provides an expression for $\mathbf{x}_1''$.

Lemma 2.6. *The acceleration of $\mathbf{x}_1$ is given by the average of $-\nabla \boldsymbol{\psi}_2$ over $\mathcal{B}_1$. In other words,*

$$\mathbf{x}_1''(t) = -\frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \nabla \boldsymbol{\psi}_2(t, \mathbf{x}) d\mathbf{x}.$$

Proof. We start by computing $\mathbf{x}_1''(t)$:

$$\begin{aligned}
\mathbf{x}_1'' &= \frac{1}{|\mathcal{B}_1|} \frac{d^2}{dt^2} \left(\int_{\mathcal{B}_1} \mathbf{x} d\mathbf{x} \right) = \frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \left(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right) d\mathbf{x} \\
&= -\frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \left(\nabla \mathbf{p} + \nabla \boldsymbol{\psi}_1 + \nabla \boldsymbol{\psi}_2 \right) d\mathbf{x}.
\end{aligned}$$

Using divergence theorem, we have

$$\int_{\mathcal{B}_1} \nabla \mathbf{p} d\mathbf{x} = \int_{\partial \mathcal{B}_1} \mathbf{p} \mathbf{n} dS = 0.$$

For the contribution from $\nabla \boldsymbol{\psi}_1$, we have

$$\begin{aligned}
\int_{\mathcal{B}_1} \nabla \boldsymbol{\psi}_1 d\mathbf{x} &= G\rho \int_{\mathcal{B}_1} \int_{\mathcal{B}_1} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} d\mathbf{y} d\mathbf{x} = G\rho \int_{\mathcal{B}_1} \int_{\mathcal{B}_1} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} d\mathbf{x} d\mathbf{y} \\
&= -G\rho \int_{\mathcal{B}_1} \int_{\mathcal{B}_1} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} d\mathbf{y} d\mathbf{x} = 0. \qquad \square
\end{aligned}$$

We now turn to the proof of Proposition 2.2.

Proof of Proposition 2.2. The statement will follow from differentiating (25) in time. We start by differentiating $\nabla \psi_1$. Using equations (27), (28), and (34)

we have

(39)

$$
\begin{aligned}
-\partial_t \nabla \psi_1 &= -\frac{GM}{R^3} u + \frac{GM}{2R^3} \partial_t (I + H_{\partial \mathcal{B}_1})(\mu\zeta) \\
&= -\frac{GM}{R^3} u + \frac{GM}{2R^3} [\partial_t, H_{\partial \mathcal{B}_1}](\mu\zeta) \\
&\quad + \frac{GM}{2R^3}(I + H_{\partial \mathcal{B}_1})\partial_t(\mu\zeta) \\
&= -\frac{GM}{R^3} u + \frac{3GM}{4R^5}(I + H_{\partial \mathcal{B}_1})(\tilde{h}_t \zeta) \\
&\quad + \frac{GM}{2R^3}(I + H_{\partial \mathcal{B}_1})(\mu u) + \frac{GM}{2R^3}(I + H_{\partial \mathcal{B}_1})((\tilde{h}_t \nu + \tilde{h}\nu_t)\zeta) \\
&\quad + \frac{GM}{2R^3}[\partial_t, H_{\partial \mathcal{B}_1}](\mu\zeta) \\
&= -\frac{GM}{R^3} u + \frac{3GM}{2R^5}(I + H_{\partial \mathcal{B}_1})((u \cdot \zeta)\zeta) \\
&\quad + \frac{GM}{2R^3}(I + H_{\partial \mathcal{B}_1})(\mu u) + \frac{GM}{2R^3}(I + H_{\partial \mathcal{B}_1})((\tilde{h}_t \nu + \tilde{h}\nu_t)\zeta) \\
&\quad + \frac{GM}{2R^3}[\partial_t, H_{\partial \mathcal{B}_1}](\mu\zeta) \\
&= -\frac{GM}{R^3} u + \frac{3GM}{2R^3}(I + H_{\partial \mathcal{B}_1})((u \cdot n)n) \\
&\quad + \frac{3GM}{2R^5}(I + H_{\partial \mathcal{B}_1})((u \cdot (\zeta - Rn))\zeta) \\
&\quad + \frac{3GM}{2R^4}(I + H_{\partial \mathcal{B}_1})((u \cdot n)(\zeta - Rn)) + \frac{GM}{2R^3}(I + H_{\partial \mathcal{B}_1})(\mu u) \\
&\quad + \frac{GM}{2R^3}(I + H_{\partial \mathcal{B}_1})((\tilde{h}_t \nu + \tilde{h}\nu_t)\zeta) + \frac{GM}{2R^3}[\partial_t, H_{\partial \mathcal{B}_1}](\mu\zeta) \\
&= -\frac{GM}{R^3} u + \frac{3GM}{2R^3}(I + H_{\partial \mathcal{B}_1})((u \cdot n)n) + E_2.
\end{aligned}
$$

Next we differentiate an. Since u is curl and divergence free, by (162)

(40)
$$
N_t = -(\xi_\beta \times u_\alpha - \xi_\alpha \times u_\beta) = -|N| n \times \nabla u.
$$

It follows that

(41)
$$
\partial_t(an) = \partial_t \left(\frac{a}{|N|} \right) N - an \times \nabla u.
$$

Finally to compute $\partial_t(\nabla\psi_2 + \mathbf{x}_1'')$, we note that by Lemma 2.6 $\mathbf{x}_1''(t) = -\frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\nabla\boldsymbol{\psi}_2(t,\mathbf{x})d\mathbf{x}$, and therefore by Lemma 2.4

$$\nabla\psi_2 + \mathbf{x}_1'' = F + E_1 - \frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\mathbf{E}_1(t,\mathbf{x})d\mathbf{x},$$

so

$$(42) \qquad \partial_t(\nabla\psi_2 + \mathbf{x}_1'') = F_t + \partial_t E_1 - \partial_t\left(\frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\mathbf{E}_1(t,\mathbf{x})d\mathbf{x}\right).$$

Equation (30) now follows by combining (39), (41), and (42). □

Before deriving the equations for higher derivatives of u we clarify the structure of the nonlinearity in equation (30) a bit more. We start with deriving a formula for $\partial_t\left(\frac{a}{|N|}\right)N$ which is also of independent interest for the energy estimates.

Proposition 2.7. *There holds*

$$(43)$$
$$-(I + K_{\partial\mathcal{B}_1}^*)\left(|N|\partial_t\left(\frac{a}{|N|}\right)\right) = \operatorname{Re} n[\partial_t^2 + an\times\nabla, H_{\partial\mathcal{B}_1}]u$$
$$- \frac{GM}{2R^3}\operatorname{Re} n[\partial_t, H_{\partial\mathcal{B}_1}](I + H_{\partial\mathcal{B}_1})\zeta$$
$$+ \operatorname{Re} n[\partial_t, H_{\partial\mathcal{B}_1}](\nabla\psi_2 + \mathbf{x}_1'').$$

Proof. We go back to equation (25) which using (34) we rewrite as

$$\partial_t^2\zeta - an = -\frac{GM}{R^3}\zeta + \frac{GM}{2R^3}(I + H_{\partial\mathcal{B}_1})\zeta - (\nabla\psi_2 + \mathbf{x}_1'').$$

Differentiating in time and using $N_t = -N\times\nabla u$ we get

$$\partial_t\left(\frac{a}{|N|}\right)N$$
$$= (\partial_t^2 + an\times\nabla)u + \frac{GM}{R^3}u - \frac{GM}{2R^3}\partial_t(I + H_{\partial\mathcal{B}_1})\zeta + \partial_t(\nabla\psi_2 + \mathbf{x}_1'').$$

Since $(I - H_{\partial\mathcal{B}_1})u = (I - H_{\partial\mathcal{B}_1})\nabla\psi_2 = (I - H_{\partial\mathcal{B}_1})\mathbf{x}_1'' = 0$, applying $(I - H_{\partial\mathcal{B}_1})$ to this equation gives

$$(I - H_{\partial\mathcal{B}_1})\left(\partial_t\left(\frac{a}{|N|}\right)N\right)$$

$$= [\partial_t^2 + an \times \nabla, H_{\partial \mathcal{B}_1}]u - \frac{GM}{2R^3}[\partial_t, H_{\partial \mathcal{B}_1}](I + H_{\partial \mathcal{B}_1})\zeta$$
$$+ [\partial_t, H_{\partial \mathcal{B}_1}](\nabla \psi_2 + \mathbf{x}_1'').$$

Multiplying the equation by n on both sides we get

$$- (I + H_{\partial \mathcal{B}_1}^*) \left(|N|\partial_t \left(\frac{a}{|N|} \right) \right)$$
$$= n[\partial_t^2 + an \times \nabla, H_{\partial \mathcal{B}_1}]u - \frac{GMn}{2R^3}[\partial_t, H_{\partial \mathcal{B}_1}](I + H_{\partial \mathcal{B}_1})\zeta$$
$$+ n[\partial_t, H_{\partial \mathcal{B}_1}](\nabla \psi_2 + \mathbf{x}_1'').$$

The desired identity now follows from taking real parts. $\square$

Finally we use equation (25) to derive expressions for $a - \frac{GM}{R^2}$ and $Rn - \zeta$ which will allow us to estimate these terms in the energy estimates.

Lemma 2.8. *Let*

$$w := u_t - \frac{GM}{2R^3}(I + H_{\partial \mathcal{B}_1})\zeta + (\nabla \psi_2 + \mathbf{x}_1''),$$

and

$$b := |w|^2 + \frac{2GM}{R^3}\zeta \cdot w + \frac{G^2 M^2}{R^6} \left(|\zeta|^2 - R^2 \right).$$

Then

(44)
$$a - \frac{GM}{R^2} = \frac{b}{\frac{GM}{R^2} + \sqrt{\left(\frac{GM}{R^2} \right)^2 + b}},$$

and

(45)
$$Rn - \zeta = \frac{R^3}{GM}u_t - \frac{R^3}{GM} \left(a - \frac{GM}{R^2} \right) n - \frac{1}{2}(I + H_{\partial \mathcal{B}_1})\zeta + \frac{R^3}{GM}(\nabla \psi_2 + \mathbf{x}_1'').$$

Proof. First by the definition of $a = -\nabla \mathbf{p} \cdot \mathbf{n}$ we know $a \geq 0$. Using Lemma 2.4 we rearrange (25) to get

$$a = \left| u_t + \frac{GM}{R^3}\zeta - \frac{GM}{2R^3}(I + H_{\partial \mathcal{B}_1})\zeta + (\nabla \psi_2 + \mathbf{x}_1'') \right| = \left| \frac{GM}{R^3}\zeta + w \right|.$$

A direct computation shows that $\left|\frac{GM}{R^3}\zeta + w\right| = \sqrt{b + \left(\frac{GM}{R^2}\right)^2}$ so

$$a - \frac{GM}{R^2} = \sqrt{b + \left(\frac{GM}{R^2}\right)^2} - \sqrt{\left(\frac{GM}{R^2}\right)^2} = \frac{b}{\frac{GM}{R^2} + \sqrt{\left(\frac{GM}{R^2}\right)^2 + b}}.$$

This proves (44) and (45) follows directly from rearranging equation (25) and using Lemma 2.4. $\qquad\square$

Remark 2.9. Lemma 2.8 shows that under the bootstrap assumptions to be stated in Section 3, $a \sim \frac{GM}{R^2}$, which is more precise than $a \geq 0$.

2.2. The Equation for Derivatives of u

To obtain higher regularity, we need to commute spatial derivatives with equation (30). Let $f : \mathbb{R} \times S_R \to \mathbb{R}^3$ be a vectorfield (typically f is of the form $f(t, p) = \mathbf{f}(t, \xi(t, p))$ for some vectorfield $\mathbf{f} : \mathbb{R} \times \partial\mathcal{B}_1 \to \mathbb{R}^3$, which is not necessarily tangent to $\partial\mathcal{B}_1$ or S_R). Motivated by the discussion in the introduction we define

$$(46) \qquad\qquad \mathfrak{D}_i f := \Omega_i f - \mathbf{e}_i \times f.$$

Here Ω_1 is the rotational vectorfield about the $\mathbf{e}_i$ axis in $\mathbb{R}^3$ and $\Omega_i f = \sum_{j=1}^3 (\Omega_i f^j)\mathbf{e}_j$ is computed componentwisely. We also extend $\mathfrak{D}_i$ to real-valued functions as

$$(47) \qquad\qquad \mathfrak{D}_i f := \Omega_i f,$$

and if f is a general Clifford algebra-valued function the we let

$$\mathfrak{D}_i f := \mathfrak{D}_i \mathrm{Re} f + \mathfrak{D}_i \mathrm{Vec}\ f.$$

Here recall from Appendix A that for a Clifford valued function f the notation $\mathrm{Re} f$ denotes the scalar, or real, part of f and $\mathrm{Vec}\ f$ its vector part. Often, the choice of axis of symmetry $\mathbf{e}_i$ is irrelevant in our computations, so we simply write Ω instead of Ω_i, $\mathfrak{D}$ instead of $\mathfrak{D}_i$, and $\mathbf{e}$ instead of $\mathbf{e}_i$ so for instance if f is vector-valued, then

$$\mathfrak{D} f = \Omega f - \mathbf{e} \times f.$$

Finally, if α is any multi-index $\alpha = (i_1, \ldots, i_k)$ we let

$$\mathfrak{D}^\alpha = \mathfrak{D}_{i_1} \ldots \mathfrak{D}_{i_k}$$

and if only the size $|\alpha| = k$ is important we simply write $\mathfrak{D}^k$ instead of $\mathfrak{D}^\alpha$.

Before computing the equation satisfied by $\mathfrak{D}u$ we record a simple product rule for $\mathfrak{D}$ and an integration-by-parts formula for Ω.

Lemma 2.10. *1. If f and g are Clifford algebra-valued functions then*

$$\mathfrak{D}(fg) = (\mathfrak{D}f)g + f(\mathfrak{D}g).$$

Moreover, if f and g are vector-valued, then

$$\mathfrak{D}(f \times g) = \mathfrak{D}f \times g + f \times \mathfrak{D}g,$$

and

$$\mathfrak{D}(f \cdot g) = \mathfrak{D}f \cdot g + f \cdot \mathfrak{D}g.$$

2. For any differentiable f and g and with $|\not{g}| = \sqrt{\det \not{g}}$ denoting the volume element on S_R,

$$\int_{\partial \mathcal{B}_1} f \Omega g \, dS = -\int_{\partial \mathcal{B}_1} (\Omega f) g \, dS - \int_{\partial \mathcal{B}_1} fg \frac{1}{|N||\not{g}|^{-1}} \Omega(|N||\not{g}|^{-1}) dS.$$

Here $\not{g}$ is induced Euclidean metric on S_R.

Proof. 1. The first statement follows from the usual product rule if either f or g are scalar-valued, so we assume that both f and g are vector-valued. Then

$$\begin{aligned}
\mathfrak{D}(fg) &= \mathfrak{D}(-f \cdot g + f \times g) \\
&= -\Omega f \cdot g - f \cdot \Omega g + \Omega f \times g + f \times \Omega g - \mathbf{e} \times (f \times g).
\end{aligned}$$

On the other hand

$$\begin{aligned}
(\mathfrak{D}f)g &+ f(\mathfrak{D}g) \\
&= -(\Omega f - \mathbf{e} \times f) \cdot g - f \cdot (\Omega g - \mathbf{e} \times g) + (\Omega f - \mathbf{e} \times f) \times g \\
&\quad + f \times (\Omega g - \mathbf{e} \times g) \\
&= -\Omega f \cdot g - f \cdot \Omega g + \Omega f \times g + f \times \Omega g - (\mathbf{e} \times f) \times g - f \times (\mathbf{e} \times g) \\
&= -\Omega f \cdot g - f \cdot \Omega g + \Omega f \times g + f \times \Omega g - \mathbf{e} \times (f \times g) = \mathfrak{D}(fg)
\end{aligned}$$

according to the previous computation. This proves the first statement of the lemma, and comparing the real and vector parts proves the last two statements.

2. This follows from the following computation, where $\overset{\star}{\nabla}$ and dS_R are the gradient and volume form on S_R respectively:

$$
\int_{\partial \mathcal{B}_1} f\Omega g\, dS = \int_{S_R} f(\Omega g)|N||\cancel{g}|^{-1} dS_R = \int_{S_R} f(\Omega \cdot \overset{\star}{\nabla} g)|N||\cancel{g}|^{-1} dS_R
$$
$$
= -\int_{\partial \mathcal{B}_1} (\Omega f)g\, dS - \int_{\partial \mathcal{B}_1} fg\frac{1}{|N||\cancel{g}|^{-1}}\Omega(|N||\cancel{g}|^{-1})\, dS. \quad \square
$$

We now derive the equation for the higher derivatives of u in a slightly more abstract setting.

Proposition 2.11. *Suppose* $f = \mathbf{f} \circ \xi$ *where* $\mathbf{f} : \mathbb{R} \times \partial \mathcal{B}_1 \to \mathbb{R}^3$ *is such that* $H_{\partial \mathcal{B}_1} f = f$. *If* f *satisfies*

$$
\partial_t^2 f + an \times \nabla f + \frac{GM}{R^3} f - \frac{3GM}{2R^3}(I + H_{\partial \mathcal{B}_1})((n \cdot f)n) = g_0.
$$

Then for any positive integer k, $\mathfrak{D}^k f$ *satisfies*

(48)
$$
\partial_t^2 \mathfrak{D}^k f + an \times \nabla \mathfrak{D}^k f + \frac{GM}{R^3}\mathfrak{D}^k f - \frac{3GM}{2R^3}(I + H_{\partial \mathcal{B}_1})((\mathfrak{D}^k f \cdot n)n) = g_k
$$

where

$$
g_k := \mathfrak{D}^k g_0 - \sum_{j=1}^{k}(\mathfrak{D}^j a)n \times \nabla \mathfrak{D}^{k-j} f - \sum_{j=1}^{k}(\mathfrak{D}^j a)[\mathfrak{D}^{k-j}, n \times \nabla]f
$$
(49)
$$
- a[\mathfrak{D}^k, n \times \nabla]f + \frac{3GM}{2R^3}[\mathfrak{D}^k, H_{\partial \mathcal{B}_1}]((n \cdot f)n)
$$
$$
+ \frac{3GM}{2R^3}\sum_{1 \le i+j \le k}(I + H_{\partial \mathcal{B}_1})((\mathfrak{D}^i n \cdot \mathfrak{D}^{k-i-j} f)\mathfrak{D}^j n).
$$

Proof. This follows from applying $\mathfrak{D}^k$ to the equation satisfied by f and using the product rules in Lemma 2.10. $\quad \square$

In the following lemma we derive formulas for the commutators $[\mathfrak{D}, n \times \nabla]$ and $[\mathfrak{D}, H_{\partial \mathcal{B}_1}]$ appearing on the right-hand side of (48).

Lemma 2.12.   *1. For any differentiable function f*

(50) $$[\mathfrak{D}, H_{\partial \mathcal{B}_1}]f = - \int_{\partial \mathcal{B}_1} K(\mathfrak{D}'\zeta' - \mathfrak{D}\zeta) \times (n' \times \nabla f') dS.$$

2. For any C^2 function f

(51)
$$\begin{aligned}[\mathfrak{D}, n \times \nabla]f &= - \frac{\Omega(|N||\not{g}|^{-1})}{|N||\not{g}|^{-1}} n \times \nabla f \\ &+ \frac{1}{|N|} \left((\partial_\beta (\mathfrak{D}\xi)) f_\alpha - (\partial_\alpha (\mathfrak{D}\xi)) f_\beta \right).\end{aligned}$$

Proof.   1. First note that $H_{\partial \mathcal{B}_1} 1 = 1$ which in particular implies

$$\mathfrak{D}\left(\text{p.v.} \int_{\partial \mathcal{B}_1} Kn' dS' \right) = 0.$$

Using this observation and two applications of Lemma B.4 we get (where some of the integrals below need to be interpreted in the principal value sense)

(52)
$$\begin{aligned}[\mathfrak{D}, H_{\partial \mathcal{B}_1}]f =& \mathfrak{D}\int_{\partial \mathcal{B}_1} Kn' f' dS' - \int_{\partial \mathcal{B}_1} K\mathfrak{D}'(n'f') dS' \\ &+ \int_{\partial \mathcal{B}_1} K(\mathfrak{D}'n') f' dS' \\ =& \int_{\partial \mathcal{B}_1} ((\Omega + \Omega')K - \mathbf{e} \times K)n' f' dS' + \int_{\partial \mathcal{B}_1} K(\mathfrak{D}'n') f' dS' \\ &+ \int_{\partial \mathcal{B}_1} \frac{Kn'f'}{|N'||\not{g}'|^{-1}} \Omega'(|N'||\not{g}'|^{-1}) dS' \\ =& \int_{\partial \mathcal{B}_1} ((\Omega + \Omega')K - \mathbf{e} \times K)n'(f' - f) dS' \\ &+ \int_{\partial \mathcal{B}_1} ((\Omega + \Omega')K - \mathbf{e} \times K)n' dS' f \\ &+ \int_{\partial \mathcal{B}_1} K(\mathfrak{D}'n') f' dS' + \int_{\partial \mathcal{B}_1} \frac{Kn'f'}{|N'||\not{g}'|^{-1}} \Omega'(|N'||\not{g}'|^{-1}) dS' \\ =& \int_{\partial \mathcal{B}_1} ((\Omega + \Omega')K - \mathbf{e} \times K)n'(f' - f) dS'\end{aligned}$$

$$+ \int_{\partial \mathcal{B}_1} K(\mathfrak{D}'n')(f' - f)dS'$$

$$+ \int_{\partial \mathcal{B}_1} \frac{Kn'(f' - f)}{|N'||\mathscr{g}'|^{-1}} \Omega'(|N'||\mathscr{g}'|^{-1})dS'$$

$$= \int_{\partial \mathcal{B}_1} ((\mathfrak{D}'\zeta' - \mathfrak{D}\zeta) \cdot \nabla K)n'(f' - f)dS'$$

$$+ \int_{\partial \mathcal{B}_1} \frac{Kn'(f' - f)}{|N'||\mathscr{g}'|^{-1}} \Omega'(|N'||\mathscr{g}'|^{-1})dS'$$

$$+ \int_{\partial \mathcal{B}_1} K(\mathfrak{D}'n')(f' - f)dS'$$

$$= \int_{\partial \mathcal{B}_1} ((\mathfrak{D}'\zeta' - \mathfrak{D}\zeta) \cdot \nabla K)n'(f' - f)dS'$$

$$+ \int_{\partial \mathcal{B}_1} \frac{Kn'(f' - f)}{|\mathscr{g}'|^{-1}} \Omega'(|\mathscr{g}'|^{-1})dS'$$

$$+ \iint K(\mathfrak{D}'(\zeta'_{\alpha'} \times \zeta'_{\beta'}))(f' - f)d\alpha'd\beta'.$$

Now using (167) we write (the integration by parts here can be justified by choosing specific coordinates on S_R or invariantly as in Lemma 3.2 below)

$$\int_{\partial \mathcal{B}_1} ((\mathfrak{D}'\zeta' - \mathfrak{D}\zeta) \cdot \nabla K)n'(f' - f)dS'$$

$$= \iint \partial_{\alpha'} K((\mathfrak{D}'\zeta' - \mathfrak{D}\zeta) \times \zeta'_{\beta'})(f' - f)d\alpha'd\beta'$$

$$+ \iint \partial_{\beta'} K(\zeta'_{\alpha'} \times (\mathfrak{D}'\zeta' - \mathfrak{D}\zeta))(f' - f)d\alpha'd\beta'$$

$$= - \int_{\partial \mathcal{B}_1} K(\mathfrak{D}'\zeta' - \mathfrak{D}\zeta) \times (n' \times \nabla f')dS'$$

$$+ \iint K(\zeta'_{\beta'} \times \partial_{\alpha'}\mathfrak{D}'\zeta' - \zeta'_{\alpha'} \times \partial_{\beta'}\mathfrak{D}'\zeta')(f' - f)d\alpha'd\beta'.$$

Plugging this back into (52) we get

$$(53) \qquad [\mathfrak{D}, H_{\partial \mathcal{B}_1}]f = - \int_{\partial \mathcal{B}_1} K(\mathfrak{D}'\zeta' - \mathfrak{D}\zeta) \times (n' \times \nabla f')dS'$$

$$+ \int_{\partial \mathcal{B}_1} \frac{Kn'(f' - f)}{|\mathscr{g}'|^{-1}} \Omega'(|\mathscr{g}'|^{-1})dS'$$

$$+ \iint K([\Omega', \partial_{\alpha'}]\zeta' \times \zeta'_{\beta'}$$
$$+ \zeta'_{\alpha'} \times [\Omega', \partial_{\beta'}]\zeta')(f' - f)d\alpha' d\beta'.$$

We claim that the last two lines cancel. To see this we write $\Omega = \Omega^\alpha \partial_\alpha + \Omega^\beta \partial_\beta$ so that

$$[\Omega, \partial_\alpha] = -\partial_\alpha \Omega^\alpha \partial_\alpha - \partial_\alpha \Omega^\beta \partial_\beta, \qquad \text{and}$$
$$[\Omega, \partial_\beta] = -\partial_\beta \Omega^\alpha \partial_\alpha - \partial_\beta \Omega^\beta \partial_\beta.$$

It follows that

$$[\Omega, \partial_\alpha]\zeta \times \zeta_\beta + \zeta_\alpha \times [\Omega, \partial_\beta]\zeta + |\not g|\Omega(|\not g|^{-1})N$$
$$= -\left(\sum_{\mu=1}^{2} |\not g|^{-1}\partial_\mu(|\not g|\Omega^\mu)\right) N$$
$$= -(\not\nabla \cdot \Omega)N = 0,$$

because Ω is divergence free. Going back to (53) we conclude that

$$[\mathfrak{D}, H_{\partial \mathcal{B}_1}]f = -\int_{\partial \mathcal{B}_1} K(\mathfrak{D}'\zeta' - \mathfrak{D}\zeta) \times (n' \times \nabla f')dS,$$

as desired.

2. Using the product rule for $\mathfrak{D}$ from Lemma 2.10 we write $\mathfrak{D}\,(n \times \nabla f)$ as

$$\mathfrak{D}\frac{1}{|N|}(\xi_\beta f_\alpha - \xi_\alpha f_\beta) = -\frac{\Omega|N|}{|N|}n \times \nabla f + \frac{1}{|N|}\mathfrak{D}(\xi_\beta f_\alpha - \xi_\alpha f_\beta).$$

We rewrite the second term as

$$\mathfrak{D}(\xi_\beta f_\alpha - \xi_\alpha f_\beta)$$
$$= (\xi_\beta \partial_\alpha \mathfrak{D}f - \xi_\alpha \partial_\beta \mathfrak{D}f) + ((\partial_\beta \mathfrak{D}\xi)f_\alpha - (\partial_\alpha \mathfrak{D}\xi)f_\beta)$$
$$+ ([\Omega, \partial_\beta]\xi)f_\alpha - ([\Omega, \partial_\alpha]\xi)f_\beta + \xi_\beta[\Omega, \partial_\alpha]f - \xi_\alpha[\Omega, \partial_\beta]f.$$

For the first term we have

$$\frac{1}{|N|}\Omega|N| = \frac{1}{|N||\not g|^{-1}}\Omega(|\not g|^{-1}|N|) - |\not g|\Omega|\not g|^{-1}.$$

Therefore

$$\mathfrak{D} n \times \nabla f$$

$$= n \times \nabla \mathfrak{D} f - \frac{\Omega(|N||\not{g}|^{-1})}{|N||\not{g}|^{-1}} n \times \nabla f$$

$$+ \frac{1}{|N|} \left((\partial_\beta(\mathfrak{D}\xi)) f_\alpha - (\partial_\alpha(\mathfrak{D}\xi)) f_\beta \right)$$

$$+ \frac{1}{|N|} \left(([\Omega, \partial_\beta]\xi) f_\alpha - ([\Omega, \partial_\alpha]\xi) f_\beta + \xi_\beta[\Omega, \partial_\alpha] f - \xi_\alpha[\Omega, \partial_\beta] f \right)$$

$$+ |\not{g}|\Omega|\not{g}|^{-1} n \times \nabla f.$$

Now using an argument similar to the one following (53) above we see that the last line is zero. □

Note that in view of Proposition 2.7 we will also need to commute $\mathfrak{D}$ with $(I + K^*_{\partial\mathcal{B}_1})^{-1}$ in order to estimate the higher derivatives of the time derivative of a. Since $K_{\partial\mathcal{B}_1} = \mathrm{Re} H_{\partial\mathcal{B}_1}$ and in the case where $\partial\mathcal{B}_1$ is a round sphere $H^*_{\partial\mathcal{B}_1} = H_{\partial\mathcal{B}_1}$, where $H^*_{\partial\mathcal{B}_1} = n H_{\partial\mathcal{B}_1} n$ is as defined in Appendix A, it suffices to compute the commutator between $\mathfrak{D}$ and $(I + K_{\partial\mathcal{B}_1})^{-1}$. This is an abstract computation which is presented in the next lemma.

Lemma 2.13. *Let f be a real-valued function. Then*

$$(54) \qquad\qquad [\mathfrak{D}, K_{\partial\mathcal{B}_1}] f = \mathrm{Re}[\mathfrak{D}, H_{\partial\mathcal{B}_1}] f$$

and

$$(55) \qquad [\mathfrak{D}, (I + K_{\partial\mathcal{B}_1})^{-1}] f = -(I + K_{\partial\mathcal{B}_1})^{-1}[\mathfrak{D}, K_{\partial\mathcal{B}_1}](I + K_{\partial\mathcal{B}_1})^{-1} f.$$

Proof. To prove (54) we first note that by definition $\mathfrak{D}F = \mathfrak{D}\mathring{F} + \mathfrak{D}\vec{F}$ (in the notation of Appendix A) for any Clifford algebra-valued function F, so $\mathfrak{D}\mathrm{Re}F = \mathrm{Re}\mathfrak{D}F$. It follows that

$$\mathfrak{D} K_{\partial\mathcal{B}_1} f = \mathfrak{D}\mathrm{Re} H_{\partial\mathcal{B}_1} f = \mathrm{Re}\mathfrak{D} H_{\partial\mathcal{B}_1} f = \mathrm{Re} H_{\partial\mathcal{B}_1}\mathfrak{D} f + \mathrm{Re}[\mathfrak{D}, H_{\partial\mathcal{B}_1}] f$$

$$= K_{\partial\mathcal{B}_1}\mathfrak{D} f + \mathrm{Re}[\mathfrak{D}, H_{\partial\mathcal{B}_1}] f,$$

proving (54). For (55) we let $g := (I + K_{\partial\mathcal{B}_1})^{-1} f$. Then

$$\mathfrak{D} g = \mathfrak{D}(I + K_{\partial\mathcal{B}_1})^{-1} f = (I + K_{\partial\mathcal{B}_1})^{-1}\mathfrak{D} f + [\mathfrak{D}, (I + K_{\partial\mathcal{B}_1})^{-1}] f.$$

So

$$[\mathfrak{D}, (I + K_{\partial \mathcal{B}_1})^{-1}]f = \mathfrak{D}g - (I + K_{\partial \mathcal{B}_1})^{-1}\mathfrak{D}(I + K_{\partial \mathcal{B}_1})g$$
$$= \mathfrak{D}g - (I + K_{\partial \mathcal{B}_1})^{-1}[\mathfrak{D}, K_{\partial \mathcal{B}_1}]g - \mathfrak{D}g$$
$$= -(I + K_{\partial \mathcal{B}_1})^{-1}[\mathfrak{D}, K_{\partial \mathcal{B}_1}](I + K_{\partial \mathcal{B}_1})^{-1}f. \qquad \square$$

Proposition 2.11 can be used to derive the equation for $\mathfrak{D}^k u$. However, as we will see in Section 3, for the purposes of the energy estimates it is more convenient to work with the unknown $\vec{u}_k$ where

$$u_k := \frac{1}{2}(I + H_{\partial \mathcal{B}_1})\mathfrak{D}^k u.$$

In view of Lemma 2.12, the difference between $\mathfrak{D}^k u$ and $\vec{u}_k$ is small, but $\vec{u}_k$ has the advantage that it is the vector part of a Clifford analytic function. The following proposition allows us to derive the equation satisfied by $\vec{u}_k$.

Proposition 2.14. *Suppose $f = \mathbf{f} \circ \xi$ where $\mathbf{f} : \mathbb{R} \times \partial \mathcal{B}_1 \to \mathbb{R}^3$ is such that $H_{\partial \mathcal{B}_1} f = f$, and f satisfies*

$$\partial_t^2 f + an \times \nabla f + \frac{GM}{R^3}f - \frac{3GM}{2R^3}(I + H_{\partial \mathcal{B}_1})((n \cdot f)n) = g_0.$$

For any positive integer k let $f_k := \frac{1}{2}(I + H_{\partial \mathcal{B}_1})\mathfrak{D}^k f$ and let g_k be as defined in (49). Then $\vec{f}_k$ satisfies

$$\partial_t^2 \vec{f}_k + an \times \nabla \vec{f}_k + \frac{GM}{R^3}\vec{f}_k - \frac{3GM}{2R^3}(I + H_{\partial \mathcal{B}_1})((n \cdot \vec{f}_k)n) = \tilde{g}_k,$$

where

(56)
$$\tilde{g}_k := g_k - \frac{1}{2}an \times \nabla \text{Vec}\,[\mathfrak{D}^k, H_{\partial \mathcal{B}_1}]f - \frac{GM}{2R^3}\text{Vec}\,[\mathfrak{D}^k, H_{\partial \mathcal{B}_1}]f$$
$$+ \frac{3GM}{4R^3}(I + H_{\partial \mathcal{B}_1})((n \cdot \text{Vec}\,[\mathfrak{D}^k, H_{\partial \mathcal{B}_1}]f)n) - \frac{\text{Vec}}{2}[\partial_t^2, [\mathfrak{D}^k, H_{\partial \mathcal{B}_1}]]f$$
$$- \frac{\text{Vec}}{2}[\mathfrak{D}^k, H_{\partial \mathcal{B}_1}]\left(g_0 - an \times \nabla f - \frac{GM}{R^3}f\right.$$
$$\left. + \frac{3GM}{2R^3}(I + H_{\partial \mathcal{B}_1})((n \cdot f)n)\right).$$

Proof. Let $\mathcal{P}$ denote the operator on the left-hand side of the equation for f, and let $\mathcal{P}_0$ be the spatial part of this operator, that is

$$\mathcal{P}f := \partial_t^2 f + \mathcal{P}_0 f := \partial_t^2 f + an \times \nabla f + \frac{GM}{R^3}f - \frac{3GM}{2R^3}(I + H_{\partial\mathcal{B}_1})((n \cdot f)n),$$

Then since $H_{\partial\mathcal{B}_1}f = f$ and $\mathrm{Re}f = 0$

$$\begin{aligned}
\mathcal{P}\vec{f_k} =& \mathcal{P}\mathfrak{D}^k f - \frac{1}{2}\mathcal{P}\mathrm{Vec}\,[\mathfrak{D}^k, H_{\partial\mathcal{B}_1}]f \\
=& g_k - \frac{1}{2}\mathcal{P}_0\mathrm{Vec}\,[\mathfrak{D}^k, H_{\partial\mathcal{B}_1}]f - \frac{\mathrm{Vec}}{2}[\mathfrak{D}^k, H_{\partial\mathcal{B}_1}]\partial_t^2 f \\
& - \frac{\mathrm{Vec}}{2}[\partial_t^2, [\mathfrak{D}^k, H_{\partial\mathcal{B}_1}]]f.
\end{aligned}$$

The desired identity now follows if we use the equation for f to solve for $\partial_t^2 f$. $\square$

We can now combine Propositions 2.2, 2.11, and 2.14 to derive the equation for higher derivatives of u. We record this equation below for future reference.

Corollary 2.15. *Let*

$$g_0 := -F_t - \partial_t E_1 + \partial_t\left(\frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\mathbf{E}_1(t,\mathbf{x})d\mathbf{x}\right) + E_2 + \partial_t\left(\frac{a}{|N|}\right)N,$$

where F, $\mathbf{E}_1$ and E_2 are defined as in (31)*,* (32)*, and* (33) *respectively. With this choice of g_0, let g_k and $\tilde{g}_k$ be defined as in* (49) *and* (56) *respectively, with $f = u$. Let $u_k := \frac{1}{2}(I + H_{\partial\mathcal{B}_1})\mathfrak{D}^k u$. Then $\vec{u}_k$ satisfies*

$$(57) \qquad \partial_t^2 \vec{u}_k + an \times \nabla\vec{u}_k + \frac{GM}{R^3}\vec{u}_k - \frac{3GM}{2R^3}(I + H_{\partial\mathcal{B}_1})((n \cdot \vec{u}_k)n) = \tilde{g}_k,$$

Proof. This follows directly from Propositions 2.2, 2.11, and 2.14. $\square$

3. Energy Estimates

3.1. General Setup

Given two Clifford algebra-valued functions f and g we define their "dot product" as

$$f \cdot g = \mathrm{Re}f\mathrm{Re}g + \mathrm{Vec}\,(f) \cdot \mathrm{Vec}\,(g).$$

Recall that

$$H^*_{\partial\mathcal{B}_1} := \mathbf{n} H_{\partial\mathcal{B}_1} \mathbf{n}$$

is the formal adjoint of $H_{\partial\mathcal{B}_1}$ with respect to the pairing

$$\langle f, g \rangle = \int_{\partial\mathcal{B}_1} f \cdot g \, dS,$$

where dS denotes the volume form on $\partial\mathcal{B}_1$. Recall also that sometimes we use the notation

$$\vec{f} := \mathrm{Vec}\, f \qquad \text{and} \qquad \mathring{f} = \mathrm{Re} f.$$

We consider the model equation

(58) $$\partial_t^2 f + an \times \nabla f + \frac{GM}{R^3} f - \frac{3GM}{2R^3}(I + H_{\partial\mathcal{B}_1})((n \cdot f)n) = g,$$

where $f = \mathrm{Vec}\, \tilde{f}$ and $\tilde{f}$ satisfies $\tilde{f} = H_{\partial\mathcal{B}_1}\tilde{f}$, and define the following associated energy

(59)
$$\begin{aligned}
\mathcal{E} := \mathcal{E}_f(t) \\
:= \frac{1}{2}\int_{\partial\mathcal{B}_1} \frac{|f_t|^2}{a} dS + \frac{1}{2}\int_{\partial\mathcal{B}_1}(n \times \nabla f) \cdot f \, dS + \frac{GM}{2R^3}\int_{\partial\mathcal{B}_1} \frac{|f|^2}{a} dS \\
- \frac{3GM}{2R^3}\int_{\partial\mathcal{B}_1} \frac{(n \cdot f)^2}{a} dS.
\end{aligned}$$

When $\tilde{f} = u$, then of course $\mathrm{Re}\tilde{f} = 0$ and $\tilde{f}$ is Clifford analytic, but after commuting derivatives with equation (30) the new unknowns $\mathfrak{D}^k u$ are not necessarily Clifford analytic, and we will instead work with $\vec{u}_k$ where $u_k = \frac{1}{2}(I + H_{\partial\mathcal{B}_1})\mathfrak{D}^k u$. It is for this reason that we have defined the energies above for $\mathrm{Vec}\,\tilde{f}$ where $\tilde{f}$ satisfies $\tilde{f} = H_{\partial\mathcal{B}_1}\tilde{f}$. Also note that in our applications, the domain of the function f is S_R rather than $\partial\mathcal{B}_1$ (for instance when $f = u$). In this context we understand the notation $\mathcal{E}_f$ as

$$\mathcal{E}_f := \mathcal{E}_{f \circ \xi}.$$

Remark 3.1. Note that in the definition above f was considered as a function on $\partial\mathcal{B}_1$, whereas $a = \mathbf{a} \circ \xi$ is a function on S_R. So to be precise, we should replace a by $\mathbf{a}$ in the definition of $\mathcal{E}_f$, but by abuse of notation we use a both as the function defined on S_R and as $\mathbf{a}|_{\partial\mathcal{B}_1}$.

Before stating the main energy identity we record some integration-by-parts identities. For any two Clifford algebra-valued functions f and g define

$$(60) \qquad Q(f,g) := \frac{1}{|N|}(f_\alpha g_\beta - f_\beta g_\alpha),$$

Here $|N| = |\zeta_\alpha \times \zeta_\beta|$ which makes $Q(f,g)$ coordinate-invariant. If f and g are vector-valued we also define

$$(61) \qquad \vec{Q}(f,g) := \frac{1}{|N|}(f_\alpha \times g_\beta - f_\beta \times g_\alpha),$$

Note that when f and g are both scalars-valued $Q(f,g) = -Q(g,f)$, and when they are both vector-valued $\vec{Q}(f,g) = \vec{Q}(g,f)$.

Lemma 3.2. *Let Q and $\vec{Q}$ be defined as in* (60) *and* (61).

1. *If f, g, and h are scalar-valued then*

$$\int_{\partial\mathcal{B}} Q(f,g)h\,dS = -\int_{\partial\mathcal{B}} fQ(h,g)\,dS.$$

2. *If f, g, and h are vector-valued then*

$$\int_{\partial\mathcal{B}} \vec{Q}(f,g) \cdot h\,dS = \int_{\partial\mathcal{B}} f \cdot \vec{Q}(h,g)\,dS.$$

Proof. In the scalar case the identity follows by writing

$$(62) \qquad Q(f,g)dS = (f_\alpha g_\beta - f_\beta g_\alpha)d\alpha \wedge d\beta = df \wedge dg,$$

where d denotes the exterior differentiation operator on $\partial\mathcal{B}_1$, and using Stokes' Theorem. In the vector case we write

$$\vec{Q}(f,g) \cdot h = \sum_{i,j,k=1}^{3} Q(f^i, g^j)h^k(\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k,$$

and then apply the scalar identity. $\qquad\qquad\qquad\qquad\qquad \square$

We are now ready to prove the main energy identity.

Proposition 3.3. *Suppose $\tilde{f} = H_{\partial\mathcal{B}_1}\tilde{f}$, and $f := \mathrm{Vec}\ \tilde{f}$ satisfies* (58). *Then with $\mathcal{E}$ as in* (59), *and Q as in* (61),

(63)
$$
\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \left\langle g, \frac{f_t}{a} \right\rangle + \frac{1}{2}\left\langle \frac{1}{|N|}\partial_t\left(\frac{|N|}{a}\right)f_t, f_t \right\rangle - \frac{1}{2}\langle \vec{Q}(u,f), f\rangle \\
&\quad + \frac{GM}{2R^3}\left\langle \frac{1}{|N|}\partial_t\left(\frac{|N|}{a}\right)f, f \right\rangle \\
&\quad - \frac{3GM}{2R^3}\left\langle \frac{1}{|N|}\partial_t\left(\frac{|N|}{a}\right)f \cdot n, f \cdot n \right\rangle - \frac{3GM}{R^3}\left\langle a^{-1}f \cdot n, f \cdot n_t \right\rangle \\
&\quad + \frac{3GM}{2R^3}\langle (n \cdot f)n, [H_{\partial\mathcal{B}_1}, a^{-1}\partial_t]f\rangle \\
&\quad + \frac{3GM}{2R^3}\langle (n \cdot f)n, (H^*_{\partial\mathcal{B}_1} - H_{\partial\mathcal{B}_1})(a^{-1}f_t)\rangle \\
&\quad - \frac{3GM}{2R^3}\langle (n \cdot f)n, a^{-1}\partial_t(I - H_{\partial\mathcal{B}_1})f\rangle.
\end{aligned}
$$

Proof. We take the inner product of (58) with $\frac{1}{a}f_t$ and study the terms on the left hand side one by one. The contributions of the first and third terms on the left hand side are clear, so we focus on the second and fourth terms. For the second term we have

$$
\begin{aligned}
\int_{\partial\mathcal{B}_1} & (n \times \nabla f) \cdot f_t dS \\
&= \iint (\zeta_\beta \times f_\alpha - \zeta_\alpha \times f_\beta) \cdot f_t \, d\alpha d\beta \\
&= \partial_t \int_{\partial\mathcal{B}_1} (n \times \nabla f) \cdot f dS - \iint (\zeta_\beta \times f_{t\alpha} - \zeta_\alpha \times f_{t\beta}) \cdot f d\alpha d\beta \\
&\quad - \iint (u_\beta \times f_\alpha - u_\alpha \times f_\beta) \cdot f d\alpha d\beta \\
&= \partial_t \int_{\partial\mathcal{B}_1} (n \times \nabla f) \cdot f dS + \int_{\partial\mathcal{B}_1} \vec{Q}(u,f) \cdot f dS \\
&\quad - \iint f_t \cdot (\zeta_\beta \times f_\alpha - \zeta_\alpha \times f_\beta) d\alpha d\beta,
\end{aligned}
$$

where in the last step we have used Lemma 3.2. It follows that

$$
\int_{\partial\mathcal{B}_1} (n \times \nabla f) \cdot f_t dS = \frac{1}{2}\partial_t \int_{\partial\mathcal{B}_1} (n \times \nabla f) \cdot f dS + \frac{1}{2}\int_{\partial\mathcal{B}_1} \vec{Q}(u,f) \cdot f dS.
$$

Finally we consider the contribution from $(I + H_{\partial\mathcal{B}_1})((n \cdot f)n)$. We have

$$
\begin{aligned}
&\langle (I + H_{\partial\mathcal{B}_1})((n \cdot f)n), a^{-1}f_t \rangle \\
&= \langle (n \cdot f)n, (I + H^*_{\partial\mathcal{B}_1})(a^{-1}f_t) \rangle \\
&= \langle (n \cdot f)n, (I + H_{\partial\mathcal{B}_1})(a^{-1}f_t) \rangle + \langle (n \cdot f)n, (H^*_{\partial\mathcal{B}_1} - H_{\partial\mathcal{B}_1})(a^{-1}f_t) \rangle \\
&= 2\langle (n \cdot f)n, a^{-1}f_t \rangle + \langle (n \cdot f)n, [H_{\partial\mathcal{B}_1}, a^{-1}\partial_t]f \rangle \\
&\quad - \langle (n \cdot f)n, a^{-1}\partial_t(I - H_{\partial\mathcal{B}_1})f \rangle + \langle (n \cdot f)n, (H^*_{\partial\mathcal{B}_1} - H_{\partial\mathcal{B}_1})(a^{-1}f_t) \rangle \\
&= \partial_t\langle a^{-1}f \cdot n, f \cdot n \rangle - \langle |N|^{-1}\partial_t(|N|a^{-1})f \cdot n, f \cdot n \rangle - 2\langle a^{-1}f \cdot n, f \cdot n_t \rangle \\
&\quad + \langle (n \cdot f)n, [H_{\partial\mathcal{B}_1}, a^{-1}\partial_t]f \rangle + \langle (n \cdot f)n, (H^*_{\partial\mathcal{B}_1} - H_{\partial\mathcal{B}_1})(a^{-1}f_t) \rangle \\
&\quad - \langle (n \cdot f)n, a^{-1}\partial_t(I - H_{\partial\mathcal{B}_1})f \rangle.
\end{aligned}
$$

The statement of the proposition now follows by combining the previous identities. $\qquad\square$

Remark 3.4. Suppose $f = \sum_{i=1}^3 f^i \mathbf{e}_i$ is the vector part of a Clifford analytic function $\tilde{f}$, $\tilde{f} = H_{\partial\mathcal{B}_1}\tilde{f}$, and let $\mathring{f} = \mathrm{Re}\tilde{f}$. Then from (160)

$$
n \times \nabla f = \sum_{i=1}^3 (\nabla_n f^i)\mathbf{e}_i + \nabla_n\mathring{f} - n \times \nabla\mathring{f},
$$

where ∇_n denotes the Dirichlet-Neumann map of $\partial\mathcal{B}_1$. It follows that in this case

$$
\int_{\partial\mathcal{B}_1} (n \times \nabla f) \cdot f\, dS \geq -\int_{\partial\mathcal{B}_1} (n \times \nabla\mathring{f}) \cdot f\, dS.
$$

The energy defined in (59) is the natural energy associated to the time symmetry of the equation, but unfortunately it is not clear that it is in general positive definite. A similar problem was encountered in [2]. As in [2] here we will be able to show that $\mathcal{E}_f$ is positive if $\mathcal{B}_1$ is a small perturbation of S_R (that is under our bootstrap assumptions) and for f of interest. The approach we take here is more direct than the one in [2]. The following two general results are the first steps in this direction. The first result provides a lower bound on the constant for the Poincaré inequality, or equivalently the first nonzero Neumann eigenvalue of the positive Laplacian.

Lemma 3.5. *[18] Let $D \subseteq \mathbb{R}^3$ be a simply connected bounded domain with diameter[8] d, such that the second fundamental form of ∂D with respect*

---

[8]The diameter d of D is by definition $d = \sup_{\mathbf{x},\mathbf{y}\in D}|\mathbf{x} - \mathbf{y}|$.

to the exterior normal is non-negative. Then for any function f satisfying $\int_D f(\mathbf{x})d\mathbf{x} = 0$ there holds

$$\|f\|_{L^2(D)} \leq \frac{d}{\pi}\|\nabla f\|_{L^2(D)}.$$

Remark 3.6. We will later prove that the hypotheses of Lemma 3.5 are satisfied under the bootstrap assumptions to be introduced.

The next result provides a lower bound for the constant of the trace embedding $H^1(D) \hookrightarrow L^2(\partial D)$.

Lemma 3.7. *[8] Let $D \subseteq \mathbb{R}^3$ be a simply connected bounded domain with C^1 boundary ∂D, and let $\boldsymbol{\nu}$ denote the exterior normal. Suppose $\boldsymbol{\mu}$ is a C^1 vectorfield defined in a neighborhood of D such that $\boldsymbol{\mu}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) \geq \mathfrak{b} > 0$ for all $\mathbf{x} \in \partial D$. Then for any function $\mathbf{f}$ in $H^1(D)$,*

$$\mathfrak{b} \int_{\partial D} |\mathbf{f}(\mathbf{x})|^2 dS(\mathbf{x}) \leq \sup_{\overline{D}} |\boldsymbol{\mu}|^2 \int_D |\nabla \mathbf{f}(\mathbf{x})|^2 d\mathbf{x} + (1 + \sup_{\overline{D}} |\nabla \cdot \boldsymbol{\mu}|) \int_D |\mathbf{f}(\mathbf{x})|^2 d\mathbf{x}.$$

Proof. Because the statement is slightly different from the one in [8], we provide the proof which is directly from [8]. By the divergence theorem

$$2\int_D \mathbf{f}\nabla \mathbf{f} \cdot \boldsymbol{\mu} d\mathbf{x} = \int_D \nabla(\mathbf{f}^2) \cdot \boldsymbol{\mu} d\mathbf{x} = -\int_D \mathbf{f}^2 \nabla \cdot \boldsymbol{\mu} d\mathbf{x} + \int_{\partial D} \mathbf{f}^2 \boldsymbol{\mu} \cdot \boldsymbol{\nu} dS.$$

The desired estimate follows by rearranging and applying Cauchy-Schwarz. □

Remark 3.8. In our applications ∂D will be close to S_R, in which case $\boldsymbol{\nu}(\mathbf{x}) = R^{-1}\mathbf{x}$. The choice $\boldsymbol{\mu}(\mathbf{x}) = \mathbf{x}$ then gives $\mathfrak{b} = R^{-1}$.

To use Lemma 3.5 to prove lower bounds for the energies for $\vec{u}_k$ we also need to show that the average of u_k is small for all k. We will use the following notation for a Clifford algebra-valued function $\mathbf{f}$ defined on $\mathcal{B}_1$:

$$\mathrm{AV}(\mathbf{f}) := \frac{1}{M}\int_{\mathcal{B}_1} \mathbf{f}d\mathbf{x}.$$

Lemma 3.9. *Let $u_k := \frac{1}{2}(I + H_{\partial \mathcal{B}_1})\mathfrak{D}^k u$, and let $\mathbf{u}_k$ be the Clifford analytic extension of u_k to $\mathcal{B}_1$, with $u_0 = u$ and $\mathbf{u}_0 = \mathbf{u}$. Then for $k \geq 0$*

(64)
$$\begin{aligned}
\mathrm{AV}(\mathbf{u}_k) = &\frac{R}{3M}\int_{\partial \mathcal{B}_1} u_k dS - \frac{1}{3M}\int_{\partial \mathcal{B}_1}(\zeta - Rn)nu_k dS \\
&+ \frac{1}{6M}\int_{\partial \mathcal{B}_1}((I + H_{\partial \mathcal{B}_1})\zeta)nu_k dS,
\end{aligned}$$

and for $k \geq 1$

(65)
$$\int_{\partial \mathcal{B}_1} u_k dS = \int_{\partial \mathcal{B}_1} u_{k-1} \frac{\Omega(|N||\not g|^{-1})}{|N||\not g|^{-1}} dS - \mathbf{e} \times \text{Vec} \int_{\partial \mathcal{B}_1} u_{k-1} dS$$
$$- \frac{1}{2} \int_{\partial \mathcal{B}_1} [\mathfrak{D}, H_{\partial \mathcal{B}_1}] \mathfrak{D}^{k-1} u \, dS.$$

Remark 3.10. This lemma can be used inductively as follows. First, since[9] $\text{AV}(\mathbf{u}) = 0$, we can use (64) to control $\int_{\partial \mathcal{B}_1} u_0 dS$. Then we use (65) to bound $\int_{\partial \mathcal{B}_1} u_1 dS$. We then use (64) to estimate $\text{AV}(\mathbf{u}_1)$ in terms of $\int_{\partial \mathcal{B}_1} u_1 dS$, which we in turn estimate in terms of $\int_{\partial \mathcal{B}_1} u_0 dS$ according to (65). This process can now be continued inductively to estimate $\text{AV}(\mathbf{u}_k)$ for all k. We summarize this process in the following chart:

$$0 = \text{AV}(\mathbf{u}_0) \quad \rightarrow \quad \int_{\partial \mathcal{B}_1} u_0 dS \quad \rightarrow \quad \int_{\partial \mathcal{B}_1} u_1 dS \quad \rightarrow \quad \int_{\partial \mathcal{B}_1} u_2 dS$$

$$\rightarrow \dots \rightarrow \quad \int_{\partial \mathcal{B}_1} u_k dS \quad \rightarrow \dots,$$

$$\int_{\partial \mathcal{B}_1} u_k dS \quad \rightarrow \quad \text{AV}(\mathbf{u}_k) \quad \text{for} \quad k \geq 1.$$

Proof of Lemma 3.9. Since u_k is Clifford analytic, by Theorem A.2, for any $\boldsymbol{\xi}' \in \mathcal{B}_1$

$$\mathbf{u}_k(\boldsymbol{\xi}') = \frac{1}{4\pi} \int_{\partial \mathcal{B}_1} \frac{\boldsymbol{\xi}' - \boldsymbol{\xi}}{|\boldsymbol{\xi}' - \boldsymbol{\xi}|^3} \mathbf{n}(\boldsymbol{\xi}) \mathbf{u}_k(\boldsymbol{\xi}) dS(\boldsymbol{\xi}),$$

so

$$\text{AV}(\mathbf{u}_k) = \frac{1}{4\pi M} \int_{\partial \mathcal{B}_1} \left(\int_{\mathcal{B}_1} \frac{\boldsymbol{\xi}' - \boldsymbol{\xi}}{|\boldsymbol{\xi}' - \boldsymbol{\xi}|^3} d\boldsymbol{\xi}' \right) \mathbf{n}(\boldsymbol{\xi}) \mathbf{u}_k(\boldsymbol{\xi}) dS(\boldsymbol{\xi}).$$

Notice that the inner integral is equal to $-\frac{1}{G\rho} \nabla \psi_1(\boldsymbol{\xi})$. So using Lemma 2.4,

$$\text{AV}(\mathbf{u}_k) = -\frac{1}{3M} \int_{\partial \mathcal{B}_1} \zeta n u_k dS + \frac{1}{6M} \int_{\partial \mathcal{B}_1} ((I + H_{\partial \mathcal{B}_1})\zeta) n u_k dS$$

$$= \frac{R}{3M} \int_{\partial \mathcal{B}_1} u_k dS - \frac{1}{3M} \int_{\partial \mathcal{B}_1} (\zeta - Rn) n u_k dS$$

---

[9]By definition, $\mathbf{u} = \mathbf{v} - \mathbf{x}_1'$. On the other hand, $\mathbf{x}_1' = \partial_t \left(\frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \mathbf{x} d\mathbf{x} \right) = \frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \mathbf{v} d\mathbf{x}$.

$$+ \frac{1}{6M} \int_{\partial \mathcal{B}_1} ((I + H_{\partial \mathcal{B}_1})\zeta) n u_k dS.$$

This proves (64), and (65) follows from the identity

$$u_k := \frac{1}{2}(I + H_{\partial \mathcal{B}_1})\mathfrak{D}^k u = \Omega u_{k-1} - \mathbf{e} \times \text{Vec } u_{k-1} - \frac{1}{2}[\mathfrak{D}, H_{\partial \mathcal{B}_1}]\mathfrak{D}^{k-1}u,$$
$$k \geq 1,$$

and an integration by parts according to Lemma 2.10.   □

Using Lemmas 3.5 and 3.7 we can prove the positivity of the energy to leading order. In fact we will show that $\mathcal{E}_f$ controls the L^2 norm of f.

Lemma 3.11. *Suppose the second fundamental form of $\partial \mathcal{B}_1$ with respect to the exterior normal $\mathbf{n}$ is non-negative. Suppose further that there is $\delta \in [0,1)$ such that ζ satisfies $\zeta \cdot n \geq (1 - \delta)R$ and such that for all $\mathbf{x} \in \mathcal{B}_1$, $|\mathbf{x} - \mathbf{x}_1| \leq (1+\delta)R$. Let $\mathbf{f} = \text{Vec } \tilde{\mathbf{f}}$ and $\mathbf{f}_0 = \text{Re}\tilde{f}$, where $\tilde{\mathbf{f}}$ is a Clifford analytic function in $\mathcal{B}_1$, and let f and f_0 be the restrictions of $\mathbf{f}$ and $\mathbf{f}_0$ to $\partial \mathcal{B}_1$. If δ is sufficiently small and f satisfies the assumptions in Proposition 3.3, then there exist absolute constants $c, C > 0$ such that*

(66)
$$\frac{1}{2} \int_{\partial \mathcal{B}_1} (\mathbf{n} \times \nabla f) \cdot f dS + \frac{1}{2R} \int_{\partial \mathcal{B}_1} |f|^2 dS - \frac{3}{2R} \int_{\partial \mathcal{B}_1} (n \cdot f)^2 dS$$
$$\geq \frac{c}{4R} \int_{\partial \mathcal{B}_1} |f|^2 dS - \frac{3}{2R} \int_{\partial \mathcal{B}_1} (f \cdot n) f \cdot (n - R^{-1}\zeta) dS - \frac{CM\rho}{R^2}(\text{AV}(\mathbf{f}))^2$$
$$- \frac{3}{4R} \int_{\partial \mathcal{B}_1} |f|^2 (R^{-1}\zeta - n) \cdot n \, dS - \frac{3}{2R^2} \int_{\partial \mathcal{B}_1} (f \times \zeta) \cdot (n - R^{-1}\zeta) f_0 dS$$
$$- \frac{9}{4R^2} \int_{\mathcal{B}_1} f_0^2 d\mathbf{x} + \frac{3}{4R^2} \int_{\partial \mathcal{B}_1} (\zeta \cdot n) f_0^2 dS - CR \int_{\partial \mathcal{B}_1} |n \times \nabla f_0|^2 dS.$$

Moreover,

(67)
$$\int_{\mathcal{B}_1} |\mathbf{f}_0|^2 d\mathbf{x} \leq CR \int_{\partial \mathcal{B}_1} |f_0|^2 dS + CR^3 \int_{\partial \mathcal{B}_1} |\nabla_n f_0|^2 dS + CM\rho(\text{AV}(\mathbf{f}_0))^2.$$

Proof. We start by proving (66). Let $\mathbf{f} = \text{Vec } \tilde{\mathbf{f}}$, where $\tilde{f}$ is the Clifford analytic extension of $\tilde{f}$ to the interior of $\mathcal{B}_1$, and let $\boldsymbol{\zeta} = \mathbf{x} - \mathbf{x}_1$ be the

harmonic extension of ζ to the interior. First note that since $n \times \nabla f = \nabla_n f + \nabla_n f_0 - n \times \nabla f_0$ and $\mathbf{f}$ has harmonic components,

$$\frac{1}{2} \int_{\partial \mathcal{B}_1} (n \times \nabla f) \cdot f dS = \frac{1}{2} \int_{\mathcal{B}_1} |\nabla \mathbf{f}|^2 d\mathbf{x} - \frac{1}{2} \int_{\partial \mathcal{B}_1} (n \times \nabla f_0) \cdot f dS,$$

where $|\nabla \mathbf{f}|^2 = \sum_{i=1}^3 |\nabla \mathbf{f}^i|^2$. Next, by assumption $\partial_i \mathbf{f}^j - \partial_j \mathbf{f}^i = \epsilon_{ijk} \partial_k \mathbf{f}_0$, where $\epsilon_{ijk} = 1$ if (ijk) is an even permutation of (123), $\epsilon_{ijk} = -1$ if (ijk) is an odd permutation of (123), and $\epsilon_{ijk} = 0$ otherwise. Combining this with the facts that $\nabla \cdot \mathbf{f} = 0$ and $\nabla \cdot \boldsymbol{\zeta} = 3$, we write

(68)
$$\begin{aligned}
\int_{\partial \mathcal{B}_1} (n \cdot f)^2 dS &= \frac{1}{R} \int_{\partial \mathcal{B}_1} (f \cdot \zeta)(f \cdot n) dS + \int_{\partial \mathcal{B}_1} (f \cdot n) f \cdot (n - R^{-1}\zeta) dS \\
&= \frac{1}{R} \int_{\mathcal{B}_1} \nabla \cdot ((\mathbf{f} \cdot \boldsymbol{\zeta})\mathbf{f}) d\mathbf{x} + \int_{\partial \mathcal{B}_1} (f \cdot n) f \cdot (n - R^{-1}\zeta) dS \\
&= \frac{1}{R} \int_{\mathcal{B}_1} |\mathbf{f}|^2 d\mathbf{x} + \frac{1}{2R} \int_{\mathcal{B}_1} \boldsymbol{\zeta} \cdot \nabla |\mathbf{f}|^2 d\mathbf{x} + \int_{\partial \mathcal{B}_1} (f \cdot n) f \cdot (n - R^{-1}\zeta) dS \\
&\quad + \frac{1}{R} \int_{\mathcal{B}_1} \mathbf{f}^i \boldsymbol{\zeta}^j \epsilon_{ijk} \partial_k f_0 d\mathbf{x} \\
&= -\frac{1}{2R} \int_{\mathcal{B}_1} |\mathbf{f}|^2 d\mathbf{x} + \frac{1}{2R} \int_{\mathcal{B}_1} \nabla \cdot (|\mathbf{f}|^2 \boldsymbol{\zeta}) d\mathbf{x} + \int_{\partial \mathcal{B}_1} (f \cdot n) f \cdot (n - R^{-1}\zeta) dS \\
&\quad + \frac{1}{R} \int_{\mathcal{B}_1} \partial_k (\mathbf{f}^i \boldsymbol{\zeta}^j \epsilon_{ijk} f_0) d\mathbf{x} - \frac{1}{R} \int_{\mathcal{B}_1} \partial_k \mathbf{f}^i \boldsymbol{\zeta}^j \epsilon_{ijk} f_0 d\mathbf{x} \\
&= -\frac{1}{2R} \int_{\mathcal{B}_1} |\mathbf{f}|^2 d\mathbf{x} + \frac{1}{2R} \int_{\partial \mathcal{B}_1} |f|^2 \zeta \cdot n dS + \int_{\partial \mathcal{B}_1} (f \cdot n) f \cdot (n - R^{-1}\zeta) dS \\
&\quad + \frac{1}{R} \int_{\partial \mathcal{B}_1} (f \times \zeta) \cdot (n - R^{-1}\zeta) f_0 dS - \frac{1}{2R} \int_{\mathcal{B}_1} \epsilon_{ki\ell} \epsilon_{kij} \boldsymbol{\zeta}^j f_0 \partial_\ell f_0 d\mathbf{x} \\
&= -\frac{1}{2R} \int_{\mathcal{B}_1} |\mathbf{f}|^2 d\mathbf{x} + \frac{1}{2} \int_{\partial \mathcal{B}_1} |f|^2 dS \\
&\quad + \frac{1}{2} \int_{\partial \mathcal{B}_1} |f|^2 (R^{-1}\zeta - n) \cdot n dS + \int_{\partial \mathcal{B}_1} (f \cdot n) f \cdot (n - R^{-1}\zeta) dS \\
&\quad + \frac{1}{R} \int_{\partial \mathcal{B}_1} (f \times \zeta) \cdot (n - R^{-1}\zeta) f_0 dS + \frac{3}{2R} \int_{\mathcal{B}_1} f_0^2 d\mathbf{x} \\
&\quad - \frac{1}{2R} \int_{\partial \mathcal{B}_1} (\zeta \cdot n) f_0^2 dS.
\end{aligned}$$

To pass to the last equality, we have used the fact that

$$-\frac{1}{2}\epsilon_{ki\ell}\epsilon_{kij}\boldsymbol{\zeta}^j\partial_\ell\mathbf{f}_0^2 = -\frac{1}{2}\partial_\ell(\boldsymbol{\zeta}^j\mathbf{f}_0^2\epsilon_{ki\ell}\epsilon_{kij}) + 3\mathbf{f}_0^2 = -\nabla\cdot(\mathbf{f}_0^2\boldsymbol{\zeta}) + 3\mathbf{f}_0^2.$$

Combining this with the previous identity, and for any constant $c \in (0,1)$, we get

(69)
$$\frac{1}{2}\int_{\partial\mathcal{B}_1}(\mathbf{n}\times\nabla f)\cdot f\,dS + \frac{1}{2R}\int_{\partial\mathcal{B}_1}|f|^2dS - \frac{3}{2R}\int_{\partial\mathcal{B}_1}(n\cdot f)^2dS$$
$$= \frac{1}{2}\int_{\mathcal{B}_1}|\nabla\mathbf{f}|^2d\mathbf{x} + \frac{3}{4R^2}\int_{\mathcal{B}_1}|\mathbf{f}|^2d\mathbf{x} - \frac{1+c}{4R}\int_{\partial\mathcal{B}_1}|f|^2dS + \frac{c}{4R}\int_{\partial\mathcal{B}_1}|f|^2dS$$
$$- \frac{3}{4R}\int_{\partial\mathcal{B}_1}|f|^2(R^{-1}\zeta - n)\cdot n\,dS - \frac{3}{2R}\int_{\partial\mathcal{B}_1}(f\cdot n)f\cdot(n - R^{-1}\zeta)dS$$
$$- \frac{1}{2}\int_{\partial\mathcal{B}_1}(n\times\nabla f_0)\cdot f\,dS - \frac{3}{2R^2}\int_{\partial\mathcal{B}_1}(f\times\zeta)\cdot(n - R^{-1}\zeta)f_0dS$$
$$- \frac{9}{4R^2}\int_{\mathcal{B}_1}f_0^2d\mathbf{x} + \frac{3}{4R^2}\int_{\partial\mathcal{B}_1}(\zeta\cdot n)f_0^2dS.$$

Now applying Lemma 3.7 with $D = \mathcal{B}_1$ and $\boldsymbol{\mu} = \boldsymbol{\zeta}$ we get

$$(1-\delta)R\int_{\partial\mathcal{B}_1}|f|^2dS \leq \sup_{\mathcal{B}_1}|\boldsymbol{\zeta}|^2\int_{\mathcal{B}_1}|\nabla\mathbf{f}|^2d\mathbf{x} + 4\int_{\mathcal{B}_1}|\mathbf{f}|^2d\mathbf{x}$$
$$\leq (1+\delta)^2R^2\int_{\mathcal{B}_1}|\nabla\mathbf{f}|^2d\mathbf{x} + 4\int_{\mathcal{B}_1}|\mathbf{f}|^2d\mathbf{x}.$$

It follows that

$$\frac{1}{2}\int_{\mathcal{B}_1}|\nabla\mathbf{f}|^2d\mathbf{x} + \frac{3}{4R^2}\int_{\mathcal{B}_1}|\mathbf{f}|^2d\mathbf{x} - \frac{1+c}{4R}\int_{\partial\mathcal{B}_1}|f|^2dS$$
$$\geq \frac{2 - (1+\delta)^2(1+c)(1-\delta)^{-1}}{4}\int_{\mathcal{B}_1}|\nabla\mathbf{f}|^2d\mathbf{x}$$
$$- \frac{(4(1+c)(1-\delta)^{-1} - 3)(1+\varepsilon)}{4R^2}\int_{\mathcal{B}_1}|\mathbf{f} - \rho\mathrm{AV}(\mathbf{f})|^2d\mathbf{x}$$
$$- \frac{CM\rho}{R^2}(\mathrm{AV}(\mathbf{f}))^2,$$

where $\varepsilon > 0$ is a small constant to be chosen, and $C > 0$ is an absolute constant depending only on c, ε, and δ. Now since by assumption $|\mathbf{x} - \mathbf{x}_1| <$

$(1+\delta)R$ for all $\mathbf{x} \in \mathcal{B}_1$, the diameter d of $\mathcal{B}_1$ satisfies $d \leq 2(1+\delta)R$. Therefore applying the Poincaré estimate in Lemma 3.5 to the right-hand side of the estimate above, and choosing δ, ε, and c sufficiently small we get

$$\frac{1}{2} \int_{\mathcal{B}_1} |\nabla \mathbf{f}|^2 d\mathbf{x} + \frac{3}{4R^2} \int_{\mathcal{B}_1} |\mathbf{f}|^2 d\mathbf{x} - \frac{1+c}{4R} \int_{\partial \mathcal{B}_1} |f|^2 dS \geq -\frac{CM\rho}{R^2}(\mathrm{AV}(\mathbf{f}))^2.$$

Identity (66) follows from plugging this back into (69) and applying Cauchy-Schwarz to $\int_{\partial \mathcal{B}_1} (n \times \nabla f_0) \cdot f dS$.

We next prove (67). Note that by the Poincaré estimate in Lemma 3.5

$$\int_{\mathcal{B}_1} |\mathbf{f}_0|^2 d\mathbf{x} = \int_{\mathcal{B}_1} |\mathbf{f}_0 - \rho \mathrm{AV}(\mathbf{f}_0) + \rho \mathrm{AV}(\mathbf{f}_0)|^2 d\mathbf{x}$$

$$\leq 2 \int_{\mathcal{B}_1} |\mathbf{f}_0 - \rho \mathrm{AV}(\mathbf{f}_0)|^2 d\mathbf{x} + 2\rho^2 \int_{\mathcal{B}_1} |\mathrm{AV}(\mathbf{f}_0)|^2 d\mathbf{x}$$

$$\lesssim R^2 \int_{\mathcal{B}_1} |\nabla \mathbf{f}_0|^2 d\mathbf{x} + M\rho(\mathrm{AV}(\mathbf{f}_0))^2.$$

But since $\mathbf{f}_0$ is harmonic in $\mathcal{B}_1$,

$$\int_{\mathcal{B}_1} |\nabla \mathbf{f}_0|^2 d\mathbf{x} = \int_{\partial \mathcal{B}_1} f_0 \nabla_n f_0 dS \leq \frac{1}{2R} \int_{\partial \mathcal{B}_1} |f_0|^2 dS + \frac{R}{2} \int_{\partial \mathcal{B}_1} |\nabla_n f_0|^2 dS,$$

proving (67). $\qquad\square$

Since a is equal to $\frac{GM}{R^2}$ to leading order (see Lemma 2.8), and $\mathbf{u}$ has zero average in $\mathcal{B}_1$, we can use Lemma 3.11 to prove positivity of the energy $\mathcal{E}_f$ for $f = u$.

We close this section by defining the main energies, motivated by the discussion above.

Definition 3.12. Let $u_0 := u$ and $u_\alpha := \frac{1}{2}(I + H_{\partial \mathcal{B}_1})\mathfrak{D}^\alpha u$ for any multi-index α with $|\alpha| \geq 1$. We then define

$$\mathcal{E}_j := \sum_{|\alpha|=j} \mathcal{E}_{\vec{u}_\alpha}, \quad \text{and} \quad \mathcal{E}_{\leq k} = \sum_{j=1}^{k} \mathcal{E}_j,$$

where $\mathcal{E}_f$ is defined in (59).

3.2. Estimates on the Center of Mass x₁ and Angular Momentum J

In the next section we pose our bootstrap assumptions in terms of the distance $|\mathbf{x}_1|$ between the center of mass of each body to the origin, which is the center of mass of the entire system. We expect that as long as this distance is large, the size of $\mathbf{x}_1$ and its time derivatives can be approximated by the corresponding quantities in the motion of point masses. Our goal in this section is to make this claim rigorous by providing quantitative estimates. Recall once more the definitions

$$r_1 := |\mathbf{x}_1|, \quad \boldsymbol{\xi}_1 = \frac{\mathbf{x}_1}{|\mathbf{x}_1|}, \quad \eta = \frac{R}{|\mathbf{x}_1|}, \quad \beta = \frac{b}{R}.$$

We also define the angular momentum vector $\mathbf{J}$ as

$$\mathbf{J} := \mathbf{x}_1 \times \mathbf{x}_1',$$

and the velocity v_1 as $v_1 = \mathbf{x}_1'$ so that

$$|v_1|^2 = |\mathbf{x}_1'|^2 = |r_1'|^2 + \frac{J^2}{r_1^2}, \tag{70}$$

where $J = |\mathbf{J}|$. Recall from Lemma 2.6 that $\mathbf{x}_1$ satisfies the ODE

$$\mathbf{x}_1'' = -\frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \nabla \psi_2(t, \mathbf{x}) d\mathbf{x} = -\frac{GM\eta^2}{4R^2} \boldsymbol{\xi}_1 - \frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \mathbf{E}_1(t, \mathbf{x}) d\mathbf{x} \tag{71}$$

$$= -\frac{GM\mathbf{x}_1}{4|\mathbf{x}_1|^3} - \frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \mathbf{E}_1(t, \mathbf{x}) d\mathbf{x}, \tag{72}$$

where $\mathbf{E}_1$ is defined in (36). In the remainder of this section we write

$$\widetilde{\mathrm{AV}}(\mathbf{E}_1) := \frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \mathbf{E}_1(t, \mathbf{x}) d\mathbf{x}.$$

It follows from a simple computation that

$$r_1'' = -\frac{GM}{4|\mathbf{x}_1|^2} + \frac{|\mathbf{x}_1 \times \mathbf{x}_1'|^2}{|\mathbf{x}_1|^3} - \boldsymbol{\xi}_1 \cdot \widetilde{\mathrm{AV}}(\mathbf{E}_1) = -\frac{GM}{4|\mathbf{x}_1|^2} + \frac{J^2}{|\mathbf{x}_1|^3} - \boldsymbol{\xi}_1 \cdot \widetilde{\mathrm{AV}}(\mathbf{E}_1). \tag{73}$$

The point mass energy $\mathscr{E}_1$ of $\mathbf{x}_1$ is by definition

$$\mathscr{E}_1 := \frac{|v_1|^2}{2} - \frac{GM}{4r_1},$$

so

(74)
$$|r_1'|^2 = 2\mathscr{E}_1 + \frac{GM}{2r_1} - \frac{J^2}{r_1^2}.$$

To be consistent with the assumptions of Theorem 1.1, we write the initial velocity as

$$v_0 = |\mathbf{x}_1'(T_0)| = \kappa\beta^{-\alpha}\sqrt{\frac{GM}{R}},$$

where $\alpha \in [\frac{6}{7}, 1]$, $\kappa^2\beta^{2-2\alpha} \gg 1$, and $\kappa^{-14}\beta^{14\alpha-12} \gg 1$ (see Remark 1.2)[10]. To unify the notation, we introduce the new parameter

$$c_0 := \kappa\beta^{\frac{6}{7}-\alpha},$$

so that $c_0 \ll 1$ for all choices of $\alpha \in [\frac{6}{7}, 1]$ and

$$v_0 = c_0\beta^{-\frac{6}{7}}\sqrt{\frac{GM}{R}}.$$

This implies that

(75) $$J^2(T_0) = c_0^2\beta^{\frac{2}{7}}GMR, \qquad 2\mathscr{E}_1(T_0) = c_0^2\beta^{-\frac{12}{7}}\frac{GM}{R} - \frac{GM}{2R_1}.$$

Unlike the point-mass case, the energy $\mathscr{E}_1$ and angular momentum $\mathbf{J}$ are not conserved and instead satisfy the evolution laws

(76)
$$\mathbf{J}' = \widetilde{\mathrm{AV}}(\mathbf{E}_1) \times \mathbf{x}_1,$$
$$\frac{d}{dt}J^2 = 2\left((\widetilde{\mathrm{AV}}(\mathbf{E}_1) \cdot \mathbf{x}_1)(\mathbf{x}_1 \cdot \mathbf{x}_1') - (\widetilde{\mathrm{AV}}(\mathbf{E}_1) \cdot \mathbf{x}_1')|\mathbf{x}_1|^2\right),$$

and

(77)
$$\mathscr{E}_1' = -\widetilde{\mathrm{AV}}(\mathbf{E}_1) \cdot \mathbf{x}_1'.$$

---

[10]In practice we first choose κ and then β depending on κ so that both conditions are satisfied. For instance, when $\alpha = 1$ we choose $\kappa \gg 1$ and β such that $\beta^2 \gg \kappa^{14}$. When $\alpha = \frac{6}{7}$ we choose $\kappa \ll 1$ and β such that $\beta^{\frac{2}{7}} \gg \kappa^{-2}$.

In the following proposition we derive estimates on the radial velocity $|r_1'|$. Since r_1' is initially negative, by continuity, it will remain negative for t close to T_0. It follows that there exists $\tilde{r}$ such that for any $T > T_0$

(78) $\qquad r(t) \geq \tilde{r} \quad \forall t \leq T, \qquad \text{implies} \qquad r_1'(t) < 0 \quad \forall t \leq T.$

We define r_0 to be the smallest such $\tilde{r}$, that is,

$$r_0 := \inf\{\tilde{r} \quad \text{s.t. (78) holds}\},$$

and let t_0, if it exists, be the first time such that $r_1(t_0) = r_0$. Then by finding the roots of the quadratic polynomial in r_1^{-1} in (74) we see that r_0, if it exists, satisfies,

(79) $$r_0 = \frac{4J^2(t_0)}{\sqrt{G^2M^2 + 32\mathscr{E}_1(t_0)J^2(t_0)} + GM},$$

whenever $G^2M^2 + 32\mathscr{E}_1(t_0)J^2(t_0) \geq 0$. We prove the estimates on r_1 and its derivative under the mild assumption that the diameter of the second body $\mathcal{B}_2$ is bound by a constant multiple of R. When we apply these estimates in the proof of energy estimates, the diameter of $\mathcal{B}_2$ will in fact be close to $2R$ under our bootstrap assumptions.

Proposition 3.13. *The following statements hold if a solution to* (24) *exists and the diameter of* $\mathcal{B}_2$ *is no larger than* $10R$.

1. *If* β *is sufficiently large, then for some universal constant* C *and all* $r \geq r_0$

   (80) $$\frac{GM}{C|r_1|^2} \leq |v_1'| \leq \frac{CGM}{|r_1|^2}.$$

2. *If* β *is sufficiently large, then* $\frac{3}{2}c_0^2R\beta^{\frac{2}{7}} \leq r_0 \leq \frac{5}{2}c_0^2R\beta^{\frac{2}{7}}$ *and* r_1' *satisfies the following estimates for some universal constant* C

   (81)
   $$\frac{1}{C}\sqrt{\frac{GM}{R}}c_0\beta^{-\frac{6}{7}} \leq |r_1'| \leq C\sqrt{\frac{GM}{R}}c_0\beta^{-\frac{6}{7}} \qquad \text{if} \qquad r_1 \geq Rc_0^{-2}\beta^{\frac{12}{7}},$$
   $$\frac{1}{C}\sqrt{\frac{GM}{R}}\eta^{\frac{1}{2}} \leq |r_1'| \leq C\sqrt{\frac{GM}{R}}\eta^{\frac{1}{2}} \qquad \text{if} \qquad 3Rc_0^2\beta^{\frac{2}{7}} \leq r_1 \leq Rc_0^{-2}\beta^{\frac{12}{7}},$$
   $$\frac{1}{C}\frac{\sqrt{GM}}{c_0^2R}\beta^{-\frac{2}{7}}\sqrt{r_1 - r_0} \leq |r_1'| \leq C\frac{\sqrt{GM}}{c_0^2R}\beta^{-\frac{2}{7}}\sqrt{r_1 - r_0}$$
   $$\text{if} \qquad r_0 < r_1 \leq 4c_0^2R\beta^{\frac{2}{7}}.$$

3. If β is sufficiently large, v_1 satisfies the following estimates

(82)

$$\frac{1}{C}\sqrt{\frac{GM}{R}}c_0\beta^{-\frac{6}{7}} \leq |v_1| \leq C\sqrt{\frac{GM}{R}}c_0\beta^{-\frac{6}{7}} \qquad \text{if} \qquad r_1 \geq Rc_0^{-2}\beta^{\frac{12}{7}},$$

$$\frac{1}{C}\sqrt{\frac{GM}{R}}\eta^{\frac{1}{2}} \leq |v_1| \leq C\sqrt{\frac{GM}{R}}\eta^{\frac{1}{2}} \qquad \text{if} \qquad 3c_0^2R\beta^{\frac{2}{7}} \leq r_1 \leq Rc_0^{-2}\beta^{\frac{12}{7}},$$

$$\frac{c_0^{-1}}{C}\sqrt{\frac{GM}{R}}\beta^{-\frac{1}{7}} \leq |v_1| \leq Cc_0^{-1}\sqrt{\frac{GM}{R}}\beta^{-\frac{1}{7}} \qquad \text{if} \qquad r_0 < r_1 \leq 4c_0^2R\beta^{\frac{2}{7}}.$$

Proof. Note that under the assumption on the diameter of $\mathcal{B}_2$ and by (36), there exists a universal constant c such that

(83) $$|\mathbf{E}_1| \leq \frac{cGM\eta^4}{R^2}, \quad \Rightarrow \quad \left|\frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\mathbf{E}_1(t,\mathbf{x})d\mathbf{x}\right| \leq \frac{cGM\eta^4}{R^2}.$$

Combined with (76) and (77) this implies that

(84) $$\left|\frac{d\mathscr{E}_1}{dt}\right| \leq \frac{cGM\eta^4|v_1|}{R^2} \qquad \text{and} \qquad \left|\frac{d}{dt}J^2\right| \leq \frac{cGM\eta^4r_1^2|v_1|}{R^2}.$$

The estimate on $|v_1'|$ is a direct consequence of (71) and (83). Next we prove the estimates on $|r_1'|$ and $|v_1|$ in the first two stages, that is, when $r \geq 4Rc_0^2\beta^{\frac{2}{7}}$. For this we rewrite (70) and (74) as

(85)

$$|r_1'(t)|^2 = 2\mathscr{E}_1(T_0) + \frac{GM}{2r_1(t)} - \frac{J^2(T_0)}{r_1^2(t)} + 2\int_{T_0}^t \frac{d\mathscr{E}_1}{ds}ds - \frac{1}{r_1^2(t)}\int_{T_0}^t \frac{d}{ds}J^2ds,$$

and

(86) $$|v_1(t)|^2 = |r_1'(t)|^2 + \frac{J^2(T_0)}{r_1^2(t)} + \frac{1}{r_1^2(t)}\int_{T_0}^t \frac{d}{ds}J^2ds.$$

If R_1 is sufficiently large, it follows from (75) that the desired estimates hold at $t = T_0$ with $C = 1 + 10^{-10}$. Therefore, it suffices to assume that the estimates on $|v_1|$ and $|r_1'|$ hold for $r_1 \geq c_0^{-2}\beta^{12/7}R$ with $C = 20$, and show that the same estimates hold with $C = 10$. To prove this we show that the contribution of the time integrals of $\left|\frac{d\mathscr{E}_1}{dt}\right|$ and $\left|\frac{d}{dt}J^2\right|$ to (85) and (86) can be estimated. If $r_1 \geq Rc_0^{-2}\beta^{\frac{12}{7}}$ it follows from (84) and the bootstrap

assumption on $|v_1|$ and $|r_1'|$ that

(87) $$\left|\frac{d\mathscr{E}_1}{dt}\right| \lesssim \frac{GM\eta^4|r_1'|}{R^2} \qquad \text{and} \qquad \left|\frac{d}{dt}J^2\right| \lesssim \frac{GM\eta^4 r_1^2|r_1'|}{R^2},$$

so

$$\int_{T_0}^t \left|\frac{d\mathscr{E}_1}{dt}\right| dt = \int_{r_1(t)}^{r_1(T_0)} \left|\frac{d\mathscr{E}_1}{dt}\right| \left|\frac{dt}{dr_1}\right| dr_1 \lesssim \frac{GM\eta^3}{R} \lesssim \frac{c_0^6 GM\beta^{-\frac{36}{7}}}{R},$$

and similarly

$$\int_{T_0}^t \left|\frac{d}{dt}J^2\right| dt \lesssim GMR\eta \lesssim c_0^2 GMR\beta^{-\frac{12}{7}}.$$

If β is sufficiently large these estimates allow us to close the bootstrap assumption in the region $r_1 \geq Rc_0^{-2}\beta^{\frac{12}{7}}$. The analysis in the second region $3Rc_0^2\beta^{\frac{2}{7}} \leq r_1 \leq Rc_0^{-2}\beta^{\frac{12}{7}}$ is similar. Indeed, the estimates at $r_1 = Rc_0^{-2}\beta^{\frac{12}{7}}$ are satisfied, because we have already proved the estimates in the first stage with $C = 10$. So as in the first stage, we assume that the estimates on $|v_1|$ and $|r_1'|$ hold with $C = 20$ and improve this to $C = 15$. Since (84) are still valid in the region region $3Rc_0^2\beta^{\frac{2}{7}} \leq r_1 \leq Rc_0^{-2}\beta^{\frac{12}{7}}$,

$$\int_{T_0}^t \left|\frac{d\mathscr{E}_1}{dt}\right| dt \lesssim \frac{GM\eta^3}{R} \qquad \text{and} \qquad \int_{T_0}^t \left|\frac{d}{dt}J^2\right| dt \lesssim GMR\eta.$$

It follows that if $3Rc_0^2\beta^{\frac{2}{7}} \leq r_1(t) \leq Rc_0^{-2}\beta^{\frac{12}{7}}$

(88) $$\left|2\mathscr{E}_1(t) - \frac{GMc_0^2\beta^{-\frac{9}{5}}}{R}\right| \lesssim \frac{GM\eta^3}{R},$$

$$c_0^2 GMR\beta^{\frac{2}{7}} - CGMR\eta \leq J^2(t) \leq c_0^2 GMR\beta^{\frac{2}{7}} + CGMR\eta.$$

Combining this with (70) and (74) we can close the bootstrap assumption in the second region. Here we have used the fact that in the second stage $GM(2r_1)^{-1} - J^2(T_0)r_1^{-2}$ is positive. For the last part we argue a bit differently. Since we have already closed the bootstrap assumptions in the second region, the estimates on $|r_1'|$ and $|v_1|$ hold at the starting point of the last region where $r_1 = 4Rc_0^2\beta^{\frac{2}{7}}$. Let t_2 be the time at which $r_1(t_2) = 4c_0^2\beta^{\frac{2}{7}}R$. We still use a continuity argument by assuming the estimates for $|v_1|$ and

$|r_1'|$ in the last region hold with a larger constant. Since the estimates (84) still hold, for $r(t) > r_0$,

$$\int_{t_2}^{t} \left| \frac{d\mathscr{E}_1}{dt} \right| dt = \int_{r_1(t)}^{r_1(t_2)} \left| \frac{d\mathscr{E}_1}{dt} \right| \left| \frac{dt}{dr_1} \right| dr_1 \lesssim \frac{GMc_0^{-7}}{R^{\frac{3}{2}}} \int_{r_1(t)}^{r_1(t_2)} \frac{\beta^{-1}}{\sqrt{r' - r_0}} dr'$$

$$\lesssim \frac{GMc_0^{-6}}{R} \beta^{-\frac{6}{7}} \lesssim \frac{GM\eta^3}{R}.$$

It follows that the estimate on $\mathscr{E}_1$ in (88) still holds in the last stage. A similar argument shows that the estimate on J in (88) also holds in the last stage. Now we can argue that r_0 exists as follows. From (88) and (73), we see that r_1'' is positive for $r_1 \in [c_0^2 \beta^{2/7} R, 3c_0^2 \beta^{2/7} R]$. Assume for contradiction that r_1' does not become zero in finite time. We first show that in this case r_1 must get as small as $\frac{3}{2} c_0^2 \beta^{2/7} R$. If not, then since r_1' is negative, it follows that r_1 will remain in the interval $(\frac{3}{2} c_0^2 \beta^{2/7} R, 3c_0^2 \beta^{2/7} R]$. But then using (88) and (73) we conclude that r_1'' has a non-trivial lower bound, contradicting the fact that r_1' does not vanish in finite time. Let t_* be the time at which $r_1(t_*) = \frac{3}{2} c_0^2 \beta^{2/7} R$, and t_3 the time at which $r_1(t_3) = 3c_0^2 \beta^{2/7} R$. Integrating the identity

$$\frac{d}{dt} |r_1'|^2 = 2 r_1' r_1'',$$

from t_3 to t_* and using (88) and (73) we conclude that r_1' becomes positive, which is the desired contradiction. Having proved that r_1' vanishes in finite time, the formula (79) together with (88) give the desired range for r_0.

Next, we rewrite (74) as

$$|r_1'(t)|^2 = 2\mathscr{E}_1(t_0) + \frac{GM}{2r_1(t)} - \frac{J^2(t_0)}{r_1^2(t)} + 2(\mathscr{E}_1(t) - \mathscr{E}_1(t_0))$$

$$- \frac{1}{r_1^2(t)}(J^2(t) - J^2(t_0)) = I + II,$$

where $I := 2\mathscr{E}_1(t_0) + \frac{GM}{2r_1} - \frac{J^2(t_0)}{r_1^2(t)}$ and $II := 2(\mathscr{E}_1(t) - \mathscr{E}_1(t_0)) - \frac{1}{r_1^2(t)}(J^2(t) - J^2(t_0))$. A simple calculation using (74) and (79) shows that

$$I = \left(\frac{8\mathscr{E}_1(t_0)r_1 + GM + \sqrt{G^2M^2 + 32\mathscr{E}_1(t_0)J^2(t_0)}}{4r_1^2} \right) |r_1 - r_0|.$$

It follows that

$$\frac{2}{Cc_0^4}\frac{GM}{R^2}\beta^{-\frac{4}{7}}|r-r_0| \leq I \leq \frac{C}{2c_0^4}\frac{GM}{R^2}\beta^{-\frac{4}{7}}|r-r_0|,$$

so it suffices to show that the contribution of II can be estimated using the bootstrap assumptions. For this we use (77) and (76) to write

$$\text{(89)}\qquad\begin{aligned}II &= \frac{2}{r_1^2(t)}\int_{t_0}^t \widetilde{\text{AV}}(\mathbf{E}_1)(s)\cdot v_1(s)(r_1^2(s)-r_1^2(t))ds\\ &\quad -\frac{2}{r_1^2(t)}\int_{t_0}^t \mathbf{x}_1(s)\cdot\widetilde{\text{AV}}(\mathbf{E}_1)(s)r_1(s)r_1'(s)ds.\end{aligned}$$

For the second term above we use (83) to estimate

$$\begin{aligned}\left|\frac{2}{r_1^2(t)}\int_{t_0}^t \mathbf{x}_1(s)\cdot\widetilde{\text{AV}}(\mathbf{E}_1)(s)r_1(s)r_1'(s)ds\right| &\lesssim \frac{GM\eta^4}{R^2}|r-r_0|\\ &\lesssim \frac{GM\beta^{-\frac{8}{7}}}{c_0^8 R^2}|r-r_0|.\end{aligned}$$

To estimate the first term on the right-hand side in (89) we note that since $r_1(t)$ is decreasing for $t \geq t_0$, we have

$$|r_1^2(t)-r_1^2(s)| = |r_1(t)-r_1(s)||r_1(t)+r_1(s)| \leq 2r_1(t)|r_1(t)-r_0|.$$

It follows that under the bootstrap assumptions

$$\begin{aligned}&\left|\frac{2}{r_1^2(t)}\int_{t_0}^t \widetilde{\text{AV}}(\mathbf{E}_1)(s)\cdot v_1(s)(r_1^2(s)-r_1^2(t))ds\right|\\ &\qquad\lesssim \frac{GM\eta^4\beta^{-\frac{1}{7}}}{c_0 R^{\frac{5}{2}}}\left(\int_{r_0}^{r_1}\frac{d\tilde{r}_1}{\sqrt{\tilde{r}_1-r_0}}\right)|r_1-r_0|\\ &\qquad\lesssim \frac{GM\beta^{-\frac{8}{7}}}{c_0^8 R^2}|r_1-r_0|,\end{aligned}$$

which can be controlled if β is sufficiently large. This proves the desired estimates for $|r_1'|$. The estimates on $|v_1|$ are simpler and similar to the arguments in the first two stages. We omit the details. □

Finally for future use we record the formulas for v_1'' and r_1''' in the following lemma.

Lemma 3.14. $\mathbf{x}_1'''$ and r_1''' satisfy

$$(90) \qquad \mathbf{x}_1''' = -\frac{GMv_1}{4|\mathbf{x}_1|^3} + \frac{3GM\mathbf{x}_1}{4|\mathbf{x}_1|^4}r_1' - \frac{d}{dt}\left(\frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\mathbf{E}_1(t,\mathbf{x})d\mathbf{x}\right),$$

and

$$(91)$$
$$r_1''' = \frac{GMr_1'}{2|\mathbf{x}_1|^3} - \frac{3J^2 r_1'}{|\mathbf{x}_1|^4}$$
$$+ 2\left(\left(\frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\mathbf{E}_1(t,\mathbf{x})d\mathbf{x}\right)\cdot\boldsymbol{\xi}_1\right)\frac{r_1'}{|\mathbf{x}_1|} - 2\left(\frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\mathbf{E}_1(t,\mathbf{x})d\mathbf{x}\right)\cdot\frac{\mathbf{x}_1'}{|\mathbf{x}_1|}$$
$$- \boldsymbol{\xi}_1 \cdot \frac{d}{dt}\left(\frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\mathbf{E}_1(t,\mathbf{x})d\mathbf{x}\right) - \boldsymbol{\xi}_1' \cdot \left(\frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\mathbf{E}_1(t,\mathbf{x})d\mathbf{x}\right).$$

Proof. These formulas follow by differentiating (71) and (73) and using (76). $\qquad\square$

We end this subsection with the following simple consequence of Proposition 3.13 which will be used many times in the remainder of the paper.

Lemma 3.15. *Under the assumptions of Proposition 3.13 and for $r_1 \geq r_0$, the following estimate holds for any $m \geq 1$, and some universal constant $C = C(m)$:*

$$\int_{T_0}^{t} \eta^{m+1}(s)|v_1(s)|ds \leq CR\eta^m(t).$$

Proof. We write

$$\int_{T_0}^{t} \eta^{m+1}(s)|v_1(s)|ds = \int_{r_1(t)}^{R_1} \eta^{m+1}\left|\frac{ds}{dr_1}\right||v_1|dr_1.$$

According to Proposition 3.13 in the first two stages of the evolution, that is, when $r_1 \geq 3c_0^2\beta^{2/7}R$, $|v_1|$ and $|r_1'|$ are comparable and the statement of the lemma follows from the identity above. In the final stage of the evolution when r_1 is between r_0 and $3c_0^2\beta^{2/7}R$ we get

$$\int_{T_0}^{t} \eta^{m+1}(s)|v_1(s)|ds$$
$$\lesssim \int_{3c_0^2\beta^{2/7}R}^{R_1} \eta^{m+1}dr_1 + \eta^m(t)c_0^{-1}\beta^{-\frac{1}{7}}R^{\frac{1}{2}}\int_{r_1(t)}^{3c_0^2\beta^{2/7}R}\frac{dr_1}{\sqrt{r_1 - r_0}} \lesssim R\eta^m(t). \quad\square$$

3.3. Bootstrap Assumptions and Estimates on the Error Terms

In this subsection we estimate various error terms appearing in the equations for u_j, $j \geq 0$, in terms of the energies in Definition 3.12. We will do this under smallness bootstrap assumptions on u. The precise bootstrap assumptions, which are motivated by the analysis in Section 3.2 and the ODE (21), are as follows. Suppose $T > 0$ is such that $u(t,p)$ is a solution of (30) for $t \leq T$ and $r_1(t) \geq r_0$ for $t \leq T$, and let $\ell \geq 5$ be a fixed integer. We assume that for some fixed constants C_1, all $p, q \in S_R$, and all $t \leq T$

$$
(92) \quad
\begin{cases}
R^{-1} \|\mathfrak{D}^\alpha u(t)\|_{L^2(S_R)} + \sqrt{\frac{R}{GM}} \|\partial_t \mathfrak{D}^\alpha u(t)\|_{L^2(S_R)} \leq C_1 \eta^4(t) |v_1(t)|, \\
\quad |\alpha| \leq \ell \\[4pt]
\frac{1}{2} \leq \frac{|\xi(t,p) - \xi(t,q)|}{|p-q|} \leq 2, \quad |\xi(t,p) - \mathbf{x}_1(t)| \leq 5R, \quad \frac{1}{2} \leq \frac{|N|}{|\not{g}|} \leq 2 \\[4pt]
R^{-1} \|\mathfrak{D}^\alpha (|N||\not{g}|^{-1})\|_{L^2(S_R)} \leq C_1 \eta^3, \quad 1 \leq |\alpha| \leq \ell \\[4pt]
R^{-2} \|\mathfrak{D}^\alpha (\zeta - Rn)\|_{L^2(S_R)} \leq C_1 \eta^3, \quad |\alpha| \leq \ell \\[4pt]
\frac{R}{GM} \left\| \mathfrak{D}^\alpha \left(a - \frac{GM}{R^2} \right) \right\|_{L^2(S_R)} \leq C_1 \eta^3, \quad |\alpha| \leq \ell \\[4pt]
R^{-2} \|\mathfrak{D}^\alpha \zeta\|_{L^2(S_R)} + R^{-1} \|\mathfrak{D}^\gamma n\|_{L^2(S_R)} \leq C_1 \eta^3, \quad 1 \leq |\alpha| \leq \ell + 1, \\
\quad 1 \leq |\gamma| \leq \ell \\[4pt]
R^{-2} \|\mathfrak{D}^\alpha h\|_{L^2(S_R)} + R^{-1} \|\mathfrak{D}^\alpha \mu\|_{L^2(S_R)} + R \|\mathfrak{D}^\alpha \nu\|_{L^2(S_R)} \leq C_1 \eta^3, \\
\quad |\alpha| \leq \ell
\end{cases} .
$$

Here we recall from (27) and (29) that

$$
\mu = 1 - \frac{R^3}{|\zeta|^3} \quad \text{and} \quad \nu = \frac{|\zeta|^2 + R|\zeta| + R^2}{|\zeta|^3(|\zeta| + R)} - \frac{3}{2R^2}.
$$

Before proceeding to the estimates, we observe that by the Sobolev embedding, the bootstrap assumptions (92) imply the simple pointwise bound

$$
(93) \qquad \|\mathfrak{D}^\alpha u(t)\|_{L^\infty(S_R)} \lesssim C_1 \eta^4(t) |v_1(t)|, \quad |\alpha| \leq \ell - 2.
$$

We start with the following simple estimate on the expression $n \times \nabla f$.

Lemma 3.16. *Suppose the bootstrap assumptions* (92) *hold. Then*

$$
|n \times \nabla f| \lesssim |\not{d}\xi||\not{d}f| \lesssim \frac{1}{R} \sum_{i=1}^{3} |\Omega_i f|.
$$

528 Shuang Miao and Sohrab Shahshahani

Proof. The first estimate is a direct consequence of (62), and the second a consequence of the first and the fact that, by (92), $|\not{d}\xi| \lesssim 1$. □

More generally, we have the following estimate on the quadratic form Q.

Lemma 3.17. *Suppose the bootstrap assumptions* (92) *hold. Then*

$$|Q(f,g)| \lesssim |\not{d}f||\not{d}g| \lesssim \frac{1}{R^2}\left(\sum_{i=1}^{3}|\Omega_i f|\right)\left(\sum_{i=1}^{3}|\Omega_i g|\right).$$

Proof. This follows from the same argument as in the proof of Lemma 3.16. □

Using Lemma 3.16 we are able to compare L^2 norms on S_R with L^2 norms on $\partial\mathcal{B}_1$. For this, we first recall that by equation (40)

$$(94) \qquad \partial_t \frac{|N|}{|\not{g}|} = -\frac{|N|}{|\not{g}|}n\cdot(n\times\nabla u).$$

Since $\lim_{t\to T_0}\frac{|N|}{|\not{g}|} = 1$, we conclude that

$$
\begin{aligned}
(95) \qquad \frac{|N|}{|\not{g}|} - 1 &= -\int_{T_0}^{t}\frac{|N(s)|}{|\not{g}|}n(s)\cdot(n(s)\times\nabla u(s))ds \\
&= -\int_{T_0}^{t}\left(\frac{|N(s)|}{|\not{g}|} - 1\right)n(s)\cdot(n(s)\times\nabla u(s))ds \\
&\quad - \int_{T_0}^{t}n(s)\cdot(n(s)\times\nabla u(s))ds.
\end{aligned}
$$

Lemma 3.18. *Suppose the bootstrap assumptions* (92) *hold and β is sufficiently large. Then*

$$\left|\frac{|N|}{|\not{g}|} - 1\right| \lesssim \eta^3.$$

Moreover, if f is a function defined on $\partial\mathcal{B}_1$ then

$$\frac{3}{4}\|f\circ\xi\|_{L^2(S_R)} \leq \|f\|_{L^2(\partial\mathcal{B}_1)} \leq \frac{5}{4}\|f\circ\xi\|_{L^2(S_R)}.$$

Proof. The first estimate is a consequence of (93), Lemmas 3.15 and 3.16, and (95). The second statement follows from writing

$$\|f\|_{L^2(\partial\mathcal{B}_1)}^2 = \int_{S_R} |f \circ \xi(p)|^2 \frac{|N(p)|}{|\not{g}(p)|} dS(p)$$
$$= \|f \circ \xi\|_{L^2(S_R)}^2 + \int_{S_R} |f \circ \xi(p)|^2 \left(\frac{|N(p)|}{|\not{g}(p)|} - 1 \right) dS(p),$$

and using the bound on $\left| \frac{|N|}{|\not{g}|} - 1 \right|$. $\qquad\qquad\qquad\qquad\qquad\square$

We now state the following important result which allows us to estimate error terms involving $H_{\partial\mathcal{B}_1}$ and $K_{\partial\mathcal{B}_1}$.

Lemma 3.19. *Suppose the bootstrap assumptions* (92) *hold and that* β *is sufficiently large. Then the following estimates hold for any function* f *defined on* $\partial\mathcal{B}_1$ *and any* $k \leq \ell$:

$$\|\mathfrak{D}^k H_{\partial\mathcal{B}_1} f\|_{L^2(S_R)} \lesssim \sum_{j\leq k} \|\mathfrak{D}^j f\|_{L^2(S_R)},$$

$$\|\mathfrak{D}^k K_{\partial\mathcal{B}_1} f\|_{L^2(S_R)} + \|\mathfrak{D}^k (I + K_{\partial\mathcal{B}_1})^{-1} f\|_{L^2(S_R)} \lesssim \sum_{j\leq k} \|\mathfrak{D}^j f\|_{L^2(S_R)},$$

$$\|\mathfrak{D}^{k-1}[\mathfrak{D}, H_{\partial\mathcal{B}_1}] f\|_{L^2(S_R)} + \|[\mathfrak{D}^k, H_{\partial\mathcal{B}_1}] f\|_{L^2(S_R)} \lesssim \eta^3 \sum_{j\leq k-1} \|\mathfrak{D}^j f\|_{L^2(S_R)},$$

$$\|\mathfrak{D}^{k-1}[\mathfrak{D}, n \times \nabla] f\|_{L^2(S_R)} + \|[\mathfrak{D}^k, n \times \nabla] f\|_{L^2(S_R)} \lesssim R^{-1} \eta^3 \sum_{j\leq k} \|\mathfrak{D}^j f\|_{L^2(S_R)},$$

$$\|\mathfrak{D}^{k-1}[n \times \nabla, H_{\partial\mathcal{B}_1}] f\|_{L^2(S_R)} \lesssim R^{-1} \eta^3 \sum_{j\leq k-1} \|\mathfrak{D}^j f\|_{L^2(S_R)},$$

$$\|\mathfrak{D}^k [\partial_t, H_{\partial\mathcal{B}_1}] f\|_{L^2(S_R)} \lesssim R^{-1} \eta^4 |v_1| \sum_{j\leq k} \|\mathfrak{D}^j f\|_{L^2(S_R)},$$

$$\|\mathfrak{D}^k [\partial_t^2, H_{\partial\mathcal{B}_1}] f\|_{L^2(S_R)} \lesssim \sqrt{\frac{GM}{R^5}} \eta^4 |v_1| \sum_{j\leq k} \|\mathfrak{D}^j f\|_{L^2(S_R)}$$
$$+ R^{-1} \eta^4 |v_1| \sum_{j\leq k} \|\partial_t \mathfrak{D}^j f\|_{L^2(S_R)},$$

$$\|\nabla_n f\|_{L^2(S_R)} \lesssim \|n \times \nabla f\|_{L^2(S_R)} \lesssim R^{-1} \sum_{j\leq 1} \|\mathfrak{D}^j f\|_{L^2(S_R)}.$$

Proof. This is a corollary of Lemmas 2.12, 2.13, 3.16, 3.18, A.4, and A.7, Corollary B.6, equation (168), Propositions B.2 and B.3, and Theorem A.3.

Note that the estimate on $[\mathfrak{D}^k, H_{\partial \mathcal{B}_1}]f$ follows from the estimate on $\mathfrak{D}^{k-1}[\mathfrak{D}, H_{\partial \mathcal{B}_1}]f$ by writing

$$[\mathfrak{D}^k, H_{\partial \mathcal{B}_1}]f$$
$$= [\mathfrak{D}, H_{\partial \mathcal{B}_1}]\mathfrak{D}^{k-1}f + \mathfrak{D}[\mathfrak{D}, H_{\partial \mathcal{B}_1}]\mathfrak{D}^{k-2}f + \cdots + \mathfrak{D}^{k-1}[\mathfrak{D}, H_{\partial \mathcal{B}_1}]f,$$

and a similar argument can be used to estimate $[\mathfrak{D}^k, n \times \nabla]f$. $\qquad\square$

Recall the definition $u_\alpha = \frac{1}{2}(I + H_{\partial \mathcal{B}_1})\mathfrak{D}^\alpha u$. As a corollary of Lemmas 3.18 and 3.19 we can control the $L^2(S_R)$ norms of $\mathfrak{D}^\alpha u$ in terms of the $L^2(\partial \mathcal{B}_1)$ norms of $\vec{u}_\alpha$ which appear in the energies.

Corollary 3.20. *Suppose the bootstrap assumptions* (92) *hold and that β is sufficiently large. Then for all $|\alpha| \leq \ell$*

(96)
$$\|\mathfrak{D}^\alpha u\|_{L^2(S_R)} \lesssim \sum_{|\gamma| \leq |\alpha|} \|\vec{u}_\gamma\|_{L^2(\partial \mathcal{B}_1)}, \qquad and$$
$$\|\partial_t \mathfrak{D}^\alpha u\|_{L^2(S_R)} \lesssim \sum_{|\gamma| \leq |\alpha|} \|\partial_t \vec{u}_\gamma\|_{L^2(\partial \mathcal{B}_1)}.$$

Proof. We prove (96) inductively on $|\alpha|$. If $|\alpha| = 0$, this follows from the fact that $H_{\partial \mathcal{B}_1} u = u$ and u is a vector as well as Lemma 3.18. Assume (96) holds for all $|\alpha| \leq k \leq \ell - 1$. We write $\mathfrak{D}^{k+1}u$

$$\mathfrak{D}^{k+1}u = \frac{1}{2}(I + H_{\partial \mathcal{B}_1})\mathfrak{D}^{k+1}u + \frac{1}{2}[\mathfrak{D}^{k+1}, H_{\partial \mathcal{B}_1}]u$$
$$= \vec{u}_{k+1} + \frac{1}{2}\text{Re}(I + H)\mathfrak{D}^{k+1}u + \frac{1}{2}[\mathfrak{D}^{k+1}, H_{\partial \mathcal{B}_1}]u$$
$$= \vec{u}_{k+1} - \frac{1}{2}\text{Re}[\mathfrak{D}^{k+1}, H_{\partial \mathcal{B}_1}]u + \frac{1}{2}[\mathfrak{D}^{k+1}, H_{\partial \mathcal{B}_1}]u$$
$$= \vec{u}_{k+1} + \frac{1}{2}\text{Vec }[\mathfrak{D}^{k+1}, H_{\partial \mathcal{B}_1}]u.$$

Similarly $\partial_t \mathfrak{D}^{k+1}u = \partial_t \vec{u}_{k+1} + \frac{1}{2}\text{Vec } \partial_t[\mathfrak{D}^{k+1}, H_{\partial \mathcal{B}_1}]u$. Estimate (96) for $|\alpha| = k + 1$ now follows from Lemmas 3.18 and 3.19 and the induction hypothesis. $\qquad\square$

To be able to use Lemma 3.11 to replace the L^2 norms on the right-hand side of (96) by the energies defined in Definition 3.12, we need to show that the second fundamental form of $\partial \mathcal{B}_1$ is positive. This is an easy consequence of the bootstrap assumptions.

Lemma 3.21. *Suppose the bootstrap assumptions* (92) *hold and that β is sufficiently large. Then the second fundamental form of $\partial\mathcal{B}_1$ is positive.*

Proof. In arbitrary orientation-preserving local coordinates, we need to show that the eigenvalues of the matrix

$$M(t) = -\begin{pmatrix} \frac{1}{|\zeta_\alpha|^2}\zeta_{\alpha\alpha}(t)\cdot n(t) & \frac{1}{|\zeta_\alpha||\zeta_\beta|}\zeta_{\alpha\beta}(t)\cdot n(t) \\ \frac{1}{|\zeta_\alpha||\zeta_\beta|}\zeta_{\beta\alpha}(t)\cdot n(t) & \frac{1}{|\zeta_\beta|^2}\zeta_{\beta\beta}(t)\cdot n(t) \end{pmatrix}$$

are positive. Note that since $\partial\mathcal{B}_1$ at time $t = T_0$ is the round sphere S_R, the eigenvalues of $M(T_0)$ are both equal to R^{-1}. Now the positivity of the eigenvalues of $M(t)$ follow from writing

$$M(t) = M(T_0) + \int_{T_0}^t \frac{dM(s)}{ds}ds,$$

and using the bootstrap assumptions and equation (40). $\qquad\square$

We are now in the position to estimate the L^2 norms of $\mathfrak{D}^k u$ and $\partial_t \mathfrak{D}^k u$ in terms of the energies in Definition 3.12.

Proposition 3.22. *Suppose the bootstrap assumptions* (92) *hold and that β is sufficiently large. Let $\mathcal{E}_{\leq k}$ be as defined in Definition 3.12. Then for for all $3 \leq k \leq \ell$*

$$\sum_{j \leq k}\left(R^{-1}\|\mathfrak{D}^j u\|_{L^2(S_R)}^2 + (GM)^{-1}R^2\|\partial_t\mathfrak{D}^j u\|_{L^2(S_R)}^2\right) \lesssim \mathcal{E}_{\leq k}.$$

Proof. Note that since under the bootstrap assumptions $a \lesssim \frac{GM}{R^2}$ from Lemma 3.18 we have

$$\sum_{j\leq k}\|\partial_t\mathfrak{D}^j u\|_{L^2(S_R)}^2 \lesssim \sum_{j\leq k}\int_{\partial\mathcal{B}_1}|\partial_t\vec{u}_j|^2 dS \lesssim \frac{GM}{R^2}\sum_{j\leq k}\int_{\partial\mathcal{B}_1}\frac{|\partial_t\vec{u}_j|^2}{a}dS.$$

Therefore to prove the proposition it suffices to show that

$$(97) \qquad\qquad \frac{1}{R}\sum_{j\leq k}\|\mathfrak{D}^j u\|_{L^2(S_R)}^2 \lesssim \sum_{j\leq k}\mathcal{E}_j^0,$$

where

$$\mathcal{E}_j^0 := \frac{1}{2}\int_{\partial\mathcal{B}_1}(n\times\nabla\vec{u}_j)\cdot\vec{u}_j dS + \frac{GM}{2R^3}\int_{\partial\mathcal{B}_1}\frac{|\vec{u}_j|^2}{a}dS - \frac{3GM}{2R^3}\int_{\partial\mathcal{B}_1}\frac{(n\cdot\vec{u}_j)^2}{a}dS.$$

By the bootstrap assumptions (92) it suffices to prove (97) with $\mathcal{E}_j^0$ replaced by $\mathcal{E}_j^1$, where

$$\mathcal{E}_j^1 := \frac{1}{2}\int_{\partial\mathcal{B}_1}(n\times\nabla\vec{u}_j)\cdot\vec{u}_j dS + \frac{1}{2R}\int_{\partial\mathcal{B}_1}|\vec{u}_j|^2 dS - \frac{3}{2R}\int_{\partial\mathcal{B}_1}(n\cdot\vec{u}_j)^2 dS.$$

Indeed, using the bootstrap assumptions on $a - \frac{GM}{R^2}$ the difference between $\mathcal{E}_j^0$ and $\mathcal{E}_j^1$ can be absorbed into the left-hand side of (97). By Lemma 3.21 the second fundamental form of $\partial\mathcal{B}_1$ is positive. Therefore by Lemma 3.11 and with the same notation,

$$\begin{aligned}
\mathcal{E}_j^1 \geq{}& \frac{c}{4R}\int_{\partial\mathcal{B}_1}|\vec{u}_j|^2 dS - \frac{C}{R}\int_{\partial\mathcal{B}_1}(\vec{u}_j\cdot n)\vec{u}_j\cdot(n - R^{-1}\zeta)dS \\
& - \frac{C}{R}\int_{\partial\mathcal{B}_1}|\vec{u}_j|^2(R^{-1}\zeta - n)\cdot n\, dS - \frac{C}{R^2}\int_{\partial\mathcal{B}_1}(\vec{u}_j\times\zeta)\cdot(n - R^{-1}\zeta)\mathring{u}_j dS \\
& - \frac{C}{R^2}\int_{\mathcal{B}_1}|\mathring{\mathbf{u}}_j|^2 d\mathbf{x} - \frac{CM\rho}{R^2}|\mathrm{AV}(\vec{u}_j)|^2 + \frac{C}{R^2}\int_{\partial\mathcal{B}_1}|\zeta\cdot n||\mathring{u}_j|^2 dS \\
& - CR\int_{\partial\mathcal{B}_1}|n\times\nabla\mathring{u}_j|^2 dS.
\end{aligned}$$

Using the bootstrap assumptions we conclude that

$$\begin{aligned}
\frac{1}{R}\sum_{j\leq k}\|\mathfrak{D}^j u\|_{L^2(S_R)}^2 \lesssim{}& \sum_{j\leq k}\mathcal{E}_j^1 + \sum_{j\leq k}\left(\frac{C}{R^2}\int_{\mathcal{B}_1}|\mathring{\mathbf{u}}_j|^2 d\mathbf{x} + \frac{CM\rho}{R^2}|\mathrm{AV}(\vec{u}_j)|^2\right) \\
& + \sum_{j\leq k}\frac{C}{R}\int_{\partial\mathcal{B}_1}|\mathring{u}_j|^2 dS + \sum_{j\leq k}CR\int_{\partial\mathcal{B}_1}|n\times\nabla\mathring{u}_j|^2 dS.
\end{aligned}$$

Now by equation (67), we have

$$\int_{\mathcal{B}_1}|\mathring{\mathbf{u}}_j|^2 d\mathbf{x} \leq CR\int_{\partial\mathcal{B}_1}|\mathring{u}_j|^2 dS + CR^3\int_{\partial\mathcal{B}_1}|\nabla_n\mathring{u}_j|^2 dS + CM\rho(\mathrm{AV}(\mathring{\mathbf{u}}_j))^2.$$

For the average $\mathrm{AV}(\mathring{\mathbf{u}}_j)^2$, we use Lemma 3.9. Taking real parts on both sides of (64) and using the bootstrap assumptions (92) we have

$$\begin{aligned}
(\mathrm{AV}(\mathring{\mathbf{u}}_j))^2 &\lesssim \frac{R^4}{M^2}\int_{\partial\mathcal{B}_1}|\mathring{u}_j|^2 dS + \frac{\eta^6 R^4}{M^2}\sum_{j\leq k}\|\mathfrak{D}^j u\|_{L^2(S_R)}^2, \\
\frac{CM\rho}{R^2}(\mathrm{AV}(\mathring{\mathbf{u}}_j))^2 &\lesssim \frac{1}{R}\int_{\partial\mathcal{B}_1}|\mathring{u}_j|^2 dS + \frac{\eta^6}{R}\sum_{j\leq k}\|\mathfrak{D}^j u\|_{L^2(S_R)}^2.
\end{aligned}$$

Taking vector parts on both sides of (64) and using bootstrap assumptions (92) as well as (65) inductively, we have

$$\frac{CM\rho}{R^2}\left(\mathrm{AV}(\vec{\mathbf{u}}_j)\right)^2 \lesssim \frac{C\rho}{M}\left(\oint_{\partial\mathcal{B}_1}\vec{u}_j dS\right)^2$$
$$\lesssim \frac{\eta(t)^6}{R}\sum_{l\le j}\int_{\partial\mathcal{B}_1}|\vec{u}_l|^2 dS + \frac{\eta^6}{R}\sum_{j\le k}\|\mathfrak{D}^j u\|^2_{L^2(S_R)},$$

which, by the bootstrap assumptions and the argument in Remark 3.10, can be absorbed by the left hand side $\frac{1}{R}\sum_{j\le k}\|\mathfrak{D}^j u\|^2_{L^2(S_R)}$. So we get

$$\frac{1}{R}\sum_{j\le k}\|\mathfrak{D}^j u\|^2_{L^2(S_R)}$$
$$\lesssim \sum_{j\le k}\mathcal{E}^1_j$$
$$+ \sum_{j\le k}\left(\frac{C}{R}\int_{\partial\mathcal{B}_1}|\mathring{u}_j|^2 dS + CR\int_{\partial\mathcal{B}_1}|\nabla_n\mathring{u}_j|^2 dS + CR\int_{\partial\mathcal{B}_1}|n\times\nabla\mathring{u}_j|^2 dS\right).$$

Since $\mathfrak{D}^j u$ is a vector, $\mathring{u}_j = -\frac{1}{2}\mathrm{Re}[\mathfrak{D}^j, H_{\partial\mathcal{B}_1}]u$. So using Lemma 3.19 we conclude that

$$\frac{1}{R}\sum_{j\le k}\|\mathfrak{D}^j u\|^2_{L^2(S_R)} \lesssim \sum_{j\le k}\mathcal{E}^1_j. \qquad \square$$

Using Proposition 3.22, we can now estimate the quantities h, μ, ν, $\mathfrak{D}\zeta$, $\mathfrak{D}n$, $\mathfrak{D}(|N||\mathring{g}|^{-1}-1)$, $Rn-\zeta$, and $a-\frac{GM}{R^2}$ appearing in the bootstrap assumptions (92) in terms of the energies $\mathcal{E}_j$. We will also derive a simple estimate for n_t and $\partial_t(I-H_{\partial\mathcal{B}_1})\vec{u}_k$, where $u_k := \frac{1}{2}(I+H_{\partial\mathcal{B}_1})\mathfrak{D}^k u$, which appear in the energy identity (63). Note that since n_t appears only in the energy identity (63) and not in the nonlinearity in the equation (30) for u, we do not need to estimates the higher derivatives of n_t. Also note that in (99) and (103) below we do not estimate the top order derivatives yet. We will consider the top order derivatives later, after estimating $|N|\partial_t(a|N|^{-1})$ in Proposition 3.25.

Proposition 3.23. *Suppose that the bootstrap assumptions* (92) *hold and that β is sufficiently large. Let $u_k := \frac{1}{2}(I+H_{\partial\mathcal{B}_1})\mathfrak{D}^k u$. Then for all $3 \le k \le \ell$*

534   Shuang Miao and Sohrab Shahshahani

and some universal constant C

(98)
$$\sum_{j\leq k}\|\mathfrak{D}^j(\zeta-Rn)\|_{L^2(S_R)}\leq CR^{\frac12}\int_{T_0}^t\mathcal{E}_{\leq k}^{\frac12}(s)ds+C\frac{R^2}{\sqrt{GM}}\mathcal{E}_{\leq k}^{\frac12}+20R^2\eta^3,$$

(99)
$$\sum_{1\leq j\leq k}\|\mathfrak{D}^j\zeta\|_{L^2(S_R)}\leq CR^{\frac12}\int_{T_0}^t\mathcal{E}_{\leq k}^{\frac12}(s)ds,$$

(100)
$$\sum_{1\leq j\leq k}\|\mathfrak{D}^jn\|_{L^2(S_R)}\leq CR^{-\frac12}\int_{T_0}^t\mathcal{E}_{\leq k}^{\frac12}(s)ds+C\frac{R}{\sqrt{GM}}\mathcal{E}_{\leq k}^{\frac12}+30R\eta^3,$$

(101)
$$\sum_{j\leq k}\|\mathfrak{D}^jh\|_{L^2(S_R)}+R\sum_{j\leq k}\|\mathfrak{D}^j\mu\|_{L^2(S_R)}+R^3\sum_{j\leq k}\|\mathfrak{D}^j\nu\|_{L^2(S_R)}$$
$$\leq CR^{\frac12}\int_{T_0}^t\mathcal{E}_{\leq k}^{\frac12}(s)ds,$$

(102)
$$\sum_{j\leq k}\|\mathfrak{D}^j\left(a-GMR^{-2}\right)\|_{L^2(S_R)}$$
$$\leq C\frac{GM}{R^{\frac52}}\int_{T_0}^t\mathcal{E}_{\leq k}^{\frac12}(s)ds+C\frac{\sqrt{GM}}{R}\mathcal{E}_{\leq k}^{\frac12}+\frac{10\,GM\eta^3}{R},$$

(103)
$$\sum_{j\leq k-1}\|\mathfrak{D}^j(|N||\not g|^{-1}-1)\|_{L^2(S_R)}\leq CR^{-\frac12}\int_{T_0}^t\mathcal{E}_{\leq k}^{\frac12}(s)ds,$$

(104)
$$\|n_t\|_{L^2(S_R)}\leq CR^{-\frac12}\mathcal{E}_{\leq 3}^{\frac12},$$

(105)
$$\|\partial_t(I-H_{\partial\mathcal{B}_1})\vec{u}_k\|_{L^2(S_R)}\leq C\left(\frac{\sqrt{GM}}{R}\eta^3+R^{-\frac12}\eta^4|v_1|\right)\mathcal{E}_{\leq k}^{\frac12}.$$

Proof. We start with the estimates for h. Note that $\partial_t h=\frac{u\cdot\zeta}{|\zeta|}$ so $|\partial_t h|=|u|$. Since $h(T_0)=0$, we have

$$|h(t)|\leq\int_{T_0}^t|u(s)|ds.$$

It follows that

$$\|h\|_{L^2(S_R)} \leq \int_{T_0}^t \|u(s)\|_{L^2(S_R)} ds.$$

The desired estimate for $\|h\|_{L^2(S_R)}$ now follows from Proposition 3.22 and Lemma 3.15. The estimates for the derivatives of h follow similarly by differentiating $\partial_t h = \frac{u \cdot \zeta}{|\zeta|}$ and using the bootstrap assumptions. For μ and ν we argue exactly the same way using the fact that $\mu(T_0) = \nu(T_0) = 0$. To estimate the higher derivatives of ζ, first note that $\mathfrak{D}^\alpha \zeta(T_0) = 0$ for all $|\alpha| \geq 1$. It then follows that

$$\mathfrak{D}^\alpha \zeta(t) = \int_{T_0}^t \mathfrak{D}^\alpha u(s) ds,$$

and the estimates on $\|\mathfrak{D}^\alpha \zeta\|_{L^2(S_R)}$ follow from the assumptions (92) and Proposition 3.22. We next turn to $a - \frac{GM}{R^2}$ for which we use equation (44) from Lemma 2.8. First note that in view of (35) and Lemma 2.6, under the bootstrap assumptions (92) and if β is sufficiently large, we have

$$(106) \qquad \sum_{j \leq k} \|\mathfrak{D}^j(\nabla \psi_2 + \mathbf{x}_1'')\|_{L^2(S_R)} \leq \frac{GM\eta^3}{R}.$$

Since we have already proved (99) and (101), we can use (44) together with Proposition 3.22, the bootstrap assumptions (92), and Lemmas 2.4 and 2.6 to establish (102). Here $(I + H_{\partial \mathcal{B}_1})\zeta$ can be estimated by observing that $(I + H_{\partial \mathcal{B}_1})\zeta = (I + H_{\partial \mathcal{B}_1})(\zeta\mu)$ and using equation (101) and Lemma 3.19. We can now use (102) to prove (98). The argument is similar to the one used to prove (102), but we now use (45) instead of (44), and use (102) to estimate the term involving $a - \frac{GM}{R^2}$ in (45). Estimate (100) now follows from (98) and (99). Estimate (103) follows from differentiating equation (95) and using Lemma 3.16 and Proposition 3.22. For (104) we use (40) to get

$$n_t = -n \times \nabla u + (n \cdot (n \times \nabla u))n,$$

and use Proposition 3.22 and the bootstrap assumptions (92). Finally for (105) we note that since $\mathrm{Re}\,\mathfrak{D}^k u = 0$

$$\partial_t(I - H_{\partial \mathcal{B}_1})\vec{u}_k$$
$$= -\frac{1}{2}\partial_t(I - H_{\partial \mathcal{B}_1})\mathrm{Re}(I + H_{\partial \mathcal{B}_1})\mathfrak{D}^k u = \frac{1}{2}\partial_t(I - H_{\partial \mathcal{B}_1})\mathrm{Re}[\mathfrak{D}^k, H_{\partial \mathcal{B}_1}]u$$

$$= \frac{1}{2}(I - H_{\partial\mathcal{B}_1})\mathrm{Re}\partial_t[\mathfrak{D}^k, H_{\partial\mathcal{B}_1}]u - \frac{1}{2}[\partial_t, H_{\partial\mathcal{B}_1}]\mathrm{Re}[\mathfrak{D}^k, H_{\partial\mathcal{B}_1}]u$$

$$= -\frac{1}{2}(I - H_{\partial\mathcal{B}_1})\mathrm{Re}\left\{[\partial_t, H_{\partial\mathcal{B}_1}]\mathfrak{D}^k u + [H_{\partial\mathcal{B}_1}, \mathfrak{D}^k]\partial_t u + \mathfrak{D}^k[H_{\partial\mathcal{B}_1}, \partial_t]u\right\}$$

$$- \frac{1}{2}[\partial_t, H_{\partial\mathcal{B}_1}]\mathrm{Re}[\mathfrak{D}^k, H_{\partial\mathcal{B}_1}]u.$$

The estimate (105) now follows from the bootstrap assumptions (92), Lemma 3.19, and Proposition 3.22. $\qquad\square$

To estimate the nonlinearity in equation (30) we still need to estimate $|N|\partial_t(a|N|^{-1})$ in terms of the energies. For this we will use the expression (43) derived for $|N|\partial_t(a|N|^{-1})$ in Proposition 2.7. However, to estimate the derivatives of this expression we need to compute the commutator of $(I + K_{\partial\mathcal{B}_1}^*)^{-1}$ and $\mathfrak{D}$. The same argument used in Lemma 2.13 to derive (55) gives

$$[\mathfrak{D}, (I + K_{\partial\mathcal{B}_1}^*)^{-1}]f = -(I + K_{\partial\mathcal{B}_1}^*)^{-1}[\mathfrak{D}, K_{\partial\mathcal{B}_1}^*](I + K_{\partial\mathcal{B}_1}^*)^{-1}f.$$

The commutator $[\mathfrak{D}, K_{\partial\mathcal{B}_1}^*]$ can now be computed using the fact that $K_{\partial\mathcal{B}_1} - K_{\partial\mathcal{B}_1}^*$ is small. That is, we rewrite the identity above as

(107)
$$[\mathfrak{D}, (I + K_{\partial\mathcal{B}_1}^*)^{-1}]f = -(I + K_{\partial\mathcal{B}_1}^*)^{-1}[\mathfrak{D}, K_{\partial\mathcal{B}_1}](I + K_{\partial\mathcal{B}_1}^*)^{-1}f$$
$$- (I + K_{\partial\mathcal{B}_1}^*)^{-1}[\mathfrak{D}, K_{\partial\mathcal{B}_1}^* - K_{\partial\mathcal{B}_1}](I + K_{\partial\mathcal{B}_1}^*)^{-1}f.$$

A precise expression for the difference $K_{\partial\mathcal{B}_1}^* - K_{\partial\mathcal{B}_1}$ can be obtained by taking the real part of identity (108) in the following lemma, where we compute $H_{\partial\mathcal{B}_1} - H_{\partial\mathcal{B}_1}^*$ which also appears in the energy identity (63).

Lemma 3.24. *For any Clifford-algebra valued function f we have*

$$(H_{\partial\mathcal{B}_1}^* - H_{\partial\mathcal{B}_1})f = \frac{\mathrm{p.v.}}{2\pi R}\int_{\partial\mathcal{B}_1}\frac{(|\zeta'|^2 - R^2) - (|\zeta|^2 - R^2)}{|\xi' - \xi|^3}f'dS'$$

(108)
$$- (n - R^{-1}\zeta)\frac{\mathrm{p.v.}}{2\pi}\int_{\partial\mathcal{B}_1}K(\xi' - \xi)f'dS'$$

$$- \frac{\mathrm{p.v.}}{2\pi}\int_{\partial\mathcal{B}_1}K(\xi' - \xi)(n' - R^{-1}\zeta')f'dS'.$$

Moreover, if the bootstrap assumptions (92) hold and β is sufficiently large

then for any $k \leq \ell$

$$(109) \qquad \|\mathfrak{D}^k(H_{\partial \mathcal{B}_1} - H^*_{\partial \mathcal{B}_1})f\|_{L^2(S_R)} \lesssim \eta^3 \sum_{j \leq k} \|\mathfrak{D}^j f\|_{L^2(S_R)}$$

Proof. By definition $H^*_{\partial \mathcal{B}_1} = n H_{\partial \mathcal{B}_1} n$ so with $K = K(\xi' - \xi)$

$$H^*_{\partial \mathcal{B}_1} f = n \, \mathrm{p.v.} \int_{\partial \mathcal{B}_1} K n' n' f' dS' = -\mathrm{p.v.} \int_{\partial \mathcal{B}_1} n K f' dS'$$

$$= H_{\partial \mathcal{B}_1} f - \mathrm{p.v.} \int_{\partial \mathcal{B}_1} (nK + Kn') f' dS'.$$

Identity (108) follows by observing that

$$nK + Kn' = (n - R^{-1}\zeta)K + K(n' - R^{-1}\zeta') + R^{-1}(\zeta K + K\zeta')$$

$$= (n - R^{-1}\zeta)K + K(n' - R^{-1}\zeta') + \frac{1}{2\pi R} \frac{|\zeta|^2 - |\zeta'|^2}{|\zeta' - \zeta|^3}.$$

The proof of estimate (109) using (108) uses Propositions B.2 and B.3 and Corollary B.6 as in the proof of Lemma 3.19. □

We can now estimate $|N|\partial_t(a|N|^{-1})$.

Proposition 3.25. *Suppose the bootstrap assumptions* (92) *hold and that β is sufficiently large. Then for any $3 \leq k \leq \ell$ (the implicit constant below depends on C_1 in* (92))

$$\sum_{j \leq k} \|\mathfrak{D}^j(|N|\partial_t(a|N|^{-1}))\|_{L^2(S_R)} \lesssim \frac{GM}{R^{\frac{5}{2}}} \eta^3 \mathcal{E}_{\leq k}^{\frac{1}{2}}.$$

Proof. This proposition follows from differentiating equation (43) and applying the estimates in Lemma 3.19, Propositions 3.22, Propositions B.2, and B.3, Corollary B.6, and the bootstrap assumptions (92). To estimate the contribution of $\nabla \psi_2 + \mathbf{x}_1''$ we use (106), and to commute derivatives with $(I + K^*_{\partial \mathcal{B}_1})^{-1}$ we also use (107) and Lemma 3.24. □

Finally we use Proposition 3.25 to estimate one more derivative in (99) and (103).

Corollary 3.26. *Under the assumptions of Propositions 3.23 and 3.25,*

$$\|\mathfrak{D}^{\ell+1}\zeta\|_{L^2(S_R)} + R\|\mathfrak{D}^\ell(|N|^{-1}|\mathcal{g}| - 1)\|_{L^2(S_R)} \leq CR^{\frac{1}{2}} \int_{T_0}^t \mathcal{E}_{\leq \ell}^{\frac{1}{2}}(s)ds.$$

Proof. We first estimate $\mathfrak{D}^\ell(|g||N|^{-1})$. Note that in view of (92) and (102) it suffices to show that

$$\|\mathfrak{D}^\ell(a|g||N|^{-1})\|_{L^2(S_R)} \leq C \frac{GM}{R^{\frac{5}{2}}} \int_{T_0}^t \mathcal{E}_{\leq\ell}^{\frac{1}{2}}(s)ds.$$

But this follows from the identity

$$\partial_t \mathfrak{D}^\ell(a|g||N|^{-1}) = \mathfrak{D}^\ell \left(|N|^{-1}|g| \, |N|\partial_t(a|N|^{-1}) \right)$$
$$= \sum_{k\leq\ell} c_{k,\ell} \, \mathfrak{D}^k(|g||N|^{-1})\mathfrak{D}^{\ell-k}(|N|\partial_t(a|N|^{-1})),$$

where $c_{k,\ell}$ are some constants, combined with (92) and Propositions 3.23 and 3.25. Next, to estimate $\mathfrak{D}^{\ell+1}\zeta$ we first note that

$$n \times \nabla\zeta = \frac{1}{|N|}(\zeta_\beta \zeta_\alpha - \zeta_\alpha \zeta_\beta) = -2n.$$

It follows that

(110) $$n \times \nabla\mathfrak{D}^\ell\zeta = -[\mathfrak{D}^\ell, n\times\nabla]\zeta - 2\mathfrak{D}^\ell n.$$

Now using (51) we write

$$[\mathfrak{D}^\ell, n\times\nabla]\zeta$$
$$= [\mathfrak{D}^{\ell-1}, n\times\nabla]\mathfrak{D}\zeta + \mathfrak{D}^{\ell-1}[\mathfrak{D}, n\times\nabla]\zeta$$
$$= [\mathfrak{D}^{\ell-1}, n\times\nabla]\mathfrak{D}^\zeta$$
$$\quad + \mathfrak{D}^{\ell-1}\left(\frac{1}{|N|}((\partial_\beta(\mathfrak{D}\zeta))\zeta_\alpha - (\partial_\alpha(\mathfrak{D}\zeta))\zeta_\beta) - \frac{\Omega(|N||g|^{-1})}{|N||g|^{-1}} n\times\nabla\zeta \right)$$
$$=: \frac{1}{|N|}\left((\partial_\beta(\mathfrak{D}^\ell\zeta))\zeta_\alpha - (\partial_\alpha(\mathfrak{D}^\ell\zeta))\zeta_\beta \right) + I,$$

where I consists of terms which are lower order in regularity, or which can be absorbed using the bootstrap assumptions (92), and the fact that we have already estimated $\mathfrak{D}^\ell(|N||g|^{-1})$. Combining with (110) and using the coordinate expression for $n\times\nabla\mathfrak{D}^\ell\zeta$ we get

(111) $$\frac{2}{|N|}\left(\zeta_\beta \times \partial_\alpha(\mathfrak{D}^\ell\zeta) - \zeta_\alpha \times \partial_\beta(\mathfrak{D}^\ell\zeta) \right) = -I - 2\mathfrak{D}^\ell n,$$

Note that $\mathfrak{D}^\ell \zeta = \frac{1}{2}\text{Vec}\ (I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta + \frac{1}{2}\text{Vec}\ (I + H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta$. Since

$$\zeta_\alpha \times \partial_\beta \text{Vec}\ (I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta - \zeta_\beta \times \partial_\alpha \text{Vec}\ (I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta$$
$$= \zeta_\alpha \partial_\beta \text{Vec}\ (I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta - \zeta_\beta \partial_\alpha \text{Vec}\ (I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta$$
$$+ \zeta_\alpha \cdot \partial_\beta \text{Vec}\ (I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta - \zeta_\beta \cdot \partial_\alpha \text{Vec}\ (I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta,$$

and using a similar computation for $(I + H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta$, we get

$$\frac{1}{|N|}(\zeta_\beta \times \partial_\alpha\mathfrak{D}^\ell\zeta - \zeta_\alpha \times \partial_\beta\mathfrak{D}^\ell\zeta)$$
$$= n \times \nabla\mathfrak{D}^\ell\zeta$$
$$+ \frac{1}{2|N|}\left(\zeta_\alpha \cdot \partial_\beta\text{Vec}\ (I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta - \zeta_\beta \cdot \partial_\alpha\text{Vec}\ (I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta\right)$$
$$+ \frac{1}{2|N|}\left(\zeta_\alpha \cdot \partial_\beta\text{Vec}\ (I + H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta - \zeta_\beta \cdot \partial_\alpha\text{Vec}\ (I + H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta\right).$$

Now by (162) applied to $\mathcal{B}_1$ and $\mathcal{B}_1^c$,

$$\frac{1}{2|N|}\left(\zeta_\alpha \cdot \partial_\beta\text{Vec}\ (I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta - \zeta_\beta \cdot \partial_\alpha\text{Vec}\ (I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta\right)$$
$$= \nabla_n^{\text{out}}\frac{\text{Re}}{2}(I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta,$$
$$\frac{1}{2|N|}\left(\zeta_\alpha \cdot \partial_\beta\text{Vec}\ (I + H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta - \zeta_\beta \cdot \partial_\alpha\text{Vec}\ (I + H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta\right)$$
$$= \nabla_n^{\text{in}}\frac{\text{Re}}{2}(I + H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta,$$

where ∇_n^{out} and ∇_n^{in} are the Dirichlet-Neumann maps for $\mathcal{B}_1^c$ and $\mathcal{B}_1$, respectively, both with respect to the exterior normal n to $\mathcal{B}_1$. Moreover, since $H_{\partial\mathcal{B}_1}\zeta$ is a pure vector (see (34)),

$$\text{Re}(I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta = \text{Re}[\mathfrak{D}^\ell, H_{\partial\mathcal{B}_1}]\zeta \quad \text{and}$$
$$\text{Re}(I + H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta = -\text{Re}[\mathfrak{D}^\ell, H_{\partial\mathcal{B}_1}]\zeta,$$

which shows that the real parts $\text{Re}(I - H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta$ and $\text{Re}(I + H_{\partial\mathcal{B}_1})\mathfrak{D}^\ell\zeta$ are lower order with respect to regularity compared to $\mathfrak{D}^\ell\zeta$, so

$$(112) \qquad \frac{1}{|N|}(\zeta_\beta \times \partial_\alpha\mathfrak{D}^\ell\zeta - \zeta_\alpha \times \partial_\beta\mathfrak{D}^\ell\zeta) = n \times \nabla\mathfrak{D}^\ell\zeta + II.$$

where II is lower order with respect to regularity compared to $\mathfrak{D}^{\ell+1}\zeta$. Next we write

$$n \times \nabla \mathfrak{D}^{\ell}\zeta = \frac{1}{|\mathfrak{g}|}\left(\zeta_\beta(0)\partial_\alpha\mathfrak{D}^\ell\zeta - \zeta_\alpha(0)\partial_\beta\mathfrak{D}^\ell\zeta\right) + III,$$

where III denotes the error which can be absorbed using the bootstrap assumptions (92). Since $\zeta(0) : S_R \to S_R$ is the identity map, combined with (112) this gives us control of $\overset{\shortmid}{\nabla}\mathfrak{D}^\ell\zeta$ where $\overset{\shortmid}{\nabla}$ is the intrinsic covariant differentiation operator on S_R, which in turn gives us control of $\mathfrak{D}^{\ell+1}\zeta$. □

3.4. Closing the Energy Estimates

In this section we use the results in Sections 3.1, 3.2, and 3.3 to close the energy estimates for equation (30). More precisely, our goal is to prove the following result, where we use the notation introduced in Sections 3.1, 3.2, and 3.3.

Proposition 3.27. *Let $\ell \geq 5$ be a fixed integer, and suppose β is sufficiently large. There exists a constant C_0 such that if u is a solution to (30) on $[T_0, T)$ and $r_1(t) \geq r_0$ for all $t \in [T_0, T)$, then*

$$(113) \qquad \mathcal{E}_{\leq\ell}(t) \leq C_0 R\eta^8(t)|v_1(t)|^2, \qquad t \in [T_0, T).$$

Moreover, there exists a constant $C_1 > 0$ such that the estimates (92) are satisfied for all $t \in [T_0, T)$.

The existence of a solution up to the point of closest approach r_0 is an immediate corollary of Proposition 3.27 and the local well-posendess result in [12].

Corollary 3.28. *If t_0 is the time at which $r(t_0) = r_0$ then the solution to (24) remains regular for $t \leq t_0$.*

Proof. By the local well-posedness result of [11, 12][11] the solution exists locally in time starting at $t = T_0$. By Proposition 3.27 the higher order derivatives of the solution and the Lipschitz constant of the boundary remain bounded as long as $r < r_0$, so referring back to [11, 12] the solution can be extended to $t = t_0$. □

---

[11]See also [2] where the local well-posedness in dimension two and in the irrotational case was derived in a set up more similar to the one used here. The local well-posedness proof of [2] can also be extended to three dimensions as in [16].

The proof of Proposition 3.27 uses the energy identity (63), where we will treat the contributions from the source term $\partial_t F$ and the rest of the terms, to which we refer as "error terms", separately. To understand the contribution of the source term we use the analysis from Section 3.2, and for the error terms we use the estimates in Section 3.3.

Proof of Proposition 3.27. We divide the proof into several steps.

Step 1. Since $\mathcal{E}_j(T_0) = 0$ for all j, the estimate (113) is trivially satisfied at $t = T_0$ for any choice of C_0. Given C_0 (to be chosen later) we let T_1 be the largest time in $[T_0, T]$ such that

$$(114) \qquad \mathcal{E}_{\leq \ell}(t) \leq \frac{1}{2} C_0 R \eta^8(t) |v_1(t)|^2, \qquad t \in [T_0, T_1).$$

For another constant C_1 to be fixed later, we let T_2 be the largest time in $[T_0, T]$ such that the bootstrap assumptions (92) are satisfied for all $t \in [T_0, T_2)$. Note that since the assumptions (92) are trivially satisfied at $t = T_0$, we have $T_2 > T_0$.

Step 2. We claim that if C_1 and β are chosen sufficiently large, then $T_1 \leq T_2$. Assume for contradiction that $T_2 < T_1$. Let β be so large that the results in Section 3.3 hold. By Propositions 3.22, if C_1 is chosen sufficiently large relative to C_0, then the estimates on $\|\mathfrak{D}^k u\|_{L^2(S_R)}$ and $\|\partial_t \mathfrak{D}^k u\|_{L^2(S_R)}$ in (92) hold with C_1 replaced by $C_1/2$ for $t < T_2$. We will now use Proposition 3.23 and Corollary 3.26 to show that the remaining conditions in (92) also hold with C_1 replaced by $C_1/2$ for $t < T_2$, which is a contradiction. Let β so large that the conclusions of Proposition 3.13 hold. Then by (114) and Lemma 3.15,

$$\int_{T_0}^t \mathcal{E}_{\leq \ell}^{\frac{1}{2}}(s) ds \lesssim C_0^{\frac{1}{2}} R^{\frac{3}{2}} \eta^3(t).$$

It follows from Proposition 3.23 and Corollary 3.26 that if C_1 is chosen sufficiently large relative to C_0 then the bootstrap assumption on h, μ, ν, $\mathfrak{D}\zeta$, $\mathfrak{D}n$, $\zeta - Rn$, $a - \frac{GM}{R^2}$, and $|N||\mathcal{g}|^{-1} - 1$ are satisfied with C_1 replaced by $C_1/2$ for $t < T_2$. Note that the estimate on $|\zeta| = |\xi - \mathbf{x}_1|$ with $5R$ replaced by $3R$ follows from the estimate on h if β is sufficiently large. Similarly if β is sufficiently large the estimates on $|N||\mathcal{g}|^{-1} - 1$ imply the estimate $\frac{2}{3} \leq \frac{|N|}{|\mathcal{g}|^{-1}} \leq \frac{3}{2}$. Finally to show that $\frac{2}{3} \leq \frac{|\xi(p) - \xi(q)|}{|p - q|} \leq \frac{3}{2}$, we consider the map $\zeta : S_R \to \partial \mathcal{B}_1$ and derive a pointwise estimate on the differential of ζ with respect to $p \in S_R$, which we denote by $d\zeta$. To estimate $d\zeta$ we view ζ as a

map to all of $\mathbb{R}^3$ and identify the tangent space of $\mathbb{R}^3$ with $\mathbb{R}^3$. Note that

$$\zeta(t,p) = \zeta(0,p) + \int_{T_0}^t \zeta_t(t',p)dt' = p + \int_{T_0}^t \zeta_t(t',p)dt',$$

$$\Rightarrow \quad d\zeta(t,p) = \iota + \int_{T_0}^t d\zeta_t(t',p)dt',$$

where ι is the inclusion of the tangent space $T_p S_R$ into $\mathbb{R}^3$. Using the fact that $|d\zeta_t| \lesssim R^{-1} \left(\sum_{i=1}^3 |\Omega_i \zeta_t|^2 \right)^{\frac{1}{2}}$, Proposition 3.22, and the Sobolev inequality, we have

$$\left| \int_{T_0}^t d\zeta_t(t',p)dt' \right| \lesssim \eta(t)^3.$$

So the differential $d\zeta$ is a small perturbation of the inclusion map. It follows that, viewed as a map from $T_p S_T$ to $T_{\zeta(p)} \partial \mathcal{B}_1$, $d\zeta(p)$ has norm close to one. By the Inverse Function Theorem, as a map from S_R to $\partial \mathcal{B}_1$, ζ has an inverse and the differential of its inverse also has norm close to one. Using the mean value theorem for both ζ and its inverse, the desired estimate for $\frac{|\xi(p)-\xi(q)|}{|p-q|} = \frac{|\zeta(p)-\zeta(q)|}{|p-q|}$ follows.

Step 3. In this and the next step we show that if β and C_0 are sufficiently large, then $T_1 = T$. We will do this by showing that of β and C_0 are sufficiently large, then

(115) $$\mathcal{E}_{\leq \ell}(t) \leq \frac{1}{4} C_0 R \eta^8(t) |v_1(t)|^2, \qquad t \in [T_0, T_1).$$

Note that since we have already shown that $T_1 \leq T_2$, we may assume that the bootstrap assumptions (92) hold for all $t < T_1$. Moreover, by taking β sufficiently large we may assume that the conclusions of Propositions 3.22 and 3.23 hold for all $t \leq T_1$. We treat the contribution of the source term F_t to the energy identity (63) in this step, and leave the error terms to the following step. More precisely, we consider the contribution of

(116) $$I := \frac{R^2}{GM} \langle \partial_t \mathfrak{D}^k F, \partial_t \vec{u}_k \rangle$$

to the energy identity (63), where we have replaced a by its leading order contribution $\frac{GM}{R^2}$, and where the source term F is defined in (31). We will show that if β is large enough, then

(117) $$\left| \int_{T_0}^t I(s)ds \right| \lesssim C_0^{\frac{1}{2}} R \eta^8(t) |v_1(t)|^2.$$

Note that if C_0 is sufficiently large, then the right-hand side above is bounded by $\frac{C_0 \eta^8(t)|v_1(t)|^2}{100}$. Writing

$$I = \frac{R^2}{GM}\partial_t \langle \partial_t \mathfrak{D}^k F, \vec{u}_k \rangle - \frac{R^2}{GM}\langle \partial_t^2 \mathfrak{D}^k F, \vec{u}_k \rangle,$$

for any $t < T$ gives

$$(118) \quad \int_{T_0}^t I(s)ds = \frac{R^2}{GM}\langle \partial_t \mathfrak{D}^k F(t), \vec{u}_k(t) \rangle - \frac{R^2}{GM}\int_{T_0}^t \langle \partial_s^2 \mathfrak{D}^k F(s), \vec{u}_k(s) \rangle ds$$

$$=:I_1 + I_2.$$

Using the results from Sections 3.2 and 3.3, we will obtain estimates on $\partial_t F$ and $\partial_t^2 F$. Direct differentiation of (31) gives

$$\partial_t F = -\frac{3GM\eta^4 r_1'}{8R^4}(\zeta - 3(\zeta \cdot \boldsymbol{\xi}_1)\boldsymbol{\xi}_1) + \frac{GM\eta^3}{8R^3}(u - 3(u \cdot \boldsymbol{\xi}_1)\boldsymbol{\xi}_1 - 3(\zeta \cdot \boldsymbol{\xi}_1')\boldsymbol{\xi}_1$$
$$- 3(\zeta \cdot \boldsymbol{\xi}_1)\boldsymbol{\xi}_1'),$$

and

$$\partial_t^2 F = -\frac{3GM\eta^4 r_1'}{4R^4}(u - 3(u \cdot \boldsymbol{\xi}_1)\boldsymbol{\xi}_1 - 3(\zeta \cdot \boldsymbol{\xi}_1')\boldsymbol{\xi}_1 - 3(\zeta \cdot \boldsymbol{\xi}_1)\boldsymbol{\xi}_1')$$
$$+ \frac{GM\eta^3}{8R^3}\Big(u_t - 3(u_t \cdot \boldsymbol{\xi}_1)\boldsymbol{\xi}_1 - 3(\zeta \cdot \boldsymbol{\xi}_1'')\boldsymbol{\xi}_1 - 3(\zeta \cdot \boldsymbol{\xi}_1)\boldsymbol{\xi}_1''$$
$$- 6(u \cdot \boldsymbol{\xi}_1')\boldsymbol{\xi}_1 - 6(u \cdot \boldsymbol{\xi}_1)\boldsymbol{\xi}_1' - 6(\zeta \cdot \boldsymbol{\xi}_1')\boldsymbol{\xi}_1'\Big)$$
$$+ \frac{3GM\eta^5(r_1')^2}{2R^5}(\zeta - 3(\zeta \cdot \boldsymbol{\xi}_1)\boldsymbol{\xi}_1) - \frac{3GM\eta^4 r_1''}{8R^4}(\zeta - 3(\zeta \cdot \boldsymbol{\xi}_1)\boldsymbol{\xi}_1).$$

Also

$$\boldsymbol{\xi}_1' = \frac{v_1}{r_1} - \frac{r_1'}{r_1}\boldsymbol{\xi}_1,$$
$$\boldsymbol{\xi}_1'' = -\frac{2r_1' v_1}{r_1^2} + \frac{v_1'}{r_1} - \frac{r_1''\boldsymbol{\xi}_1}{r_1} + \frac{2(r_1')^2\boldsymbol{\xi}_1}{r_1^2},$$
$$r_1'' = v_1' \cdot \boldsymbol{\xi}_1 + \frac{|v_1|^2}{r_1} - \frac{(v_1 \cdot \boldsymbol{\xi}_1)^2}{r_1}.$$

Differentiating these relations using $\mathfrak{D}$ and combining with Proposition 3.13 and the bootstrap assumptions (92) we conclude that

(119)
$$\sum_{j \le k} \|\mathfrak{D}^j \partial_t F(t)\|_{L^2(S_R)}$$

$$\lesssim \frac{GM\eta^3(t)}{R^3} \sum_{j \le k} \|\mathfrak{D}^j u(t)\|_{L^2(S_R)} + \frac{GM\eta^4(t)|r_1'(t)|}{R^2} + \frac{GM\eta^4(t)|v_1(t)|}{R^2}$$

$$\lesssim \frac{GM\eta^3(t)}{R^3} \sum_{j \le k} \|\mathfrak{D}^j u(t)\|_{L^2(S_R)} + \frac{GM\eta^4(t)|v_1(t)|}{R^2},$$

and

(120)
$$\sum_{j \le k} \|\mathfrak{D}^j \partial_t^2 F(t)\|_{L^2(S_R)}$$

$$\lesssim \frac{GM\eta^3(t)}{R^3} \sum_{j \le k} \|\partial_t \mathfrak{D}^j u(t)\|_{L^2(S_R)}$$

$$+ \frac{GM\eta^4(t)(|r_1'(t)| + |v_1(t)|)}{R^4} \sum_{j \le k} \|\mathfrak{D}^j u(t)\|_{L^2(S_R)}$$

$$+ \frac{GM\eta^5(|v_1|^2 + |r_1'|^2)}{R^3} + \frac{GM\eta^4|v_1'|}{R^2}$$

$$\lesssim \frac{GM\eta^3(t)}{R^3} \sum_{j \le k} \|\partial_t \mathfrak{D}^j u(t)\|_{L^2(S_R)} + \frac{GM\eta^4(t)|v_1(t)|}{R^4} \sum_{j \le k} \|\mathfrak{D}^j u(t)\|_{L^2(S_R)}$$

$$+ \frac{GM\eta^5|v_1|^2}{R^3} + \frac{G^2 M^2 \eta^6}{R^4}.$$

Plugging (119) back into (118) and using (114) and Proposition 3.22 we get

$$|I_1| \lesssim \eta^3(t)R^{-1} \sum_{j \le k} \|\mathfrak{D}^j u(t)\|_{L^2(S_R)}^2 + \eta^4(t)|v_1(t)| \sum_{j \le k} \|\mathfrak{D}^j u(t)\|_{L^2(S_R)}$$

$$\lesssim C_0 R \eta^{11}(t)|v_1(t)|^2 + C_0^{\frac{1}{2}} R \eta^8(t)|v_1(t)|^2 \lesssim C_0^{\frac{1}{2}} R \eta^8(t)|v_1(t)|^2,$$

where the last step follows if β is sufficiently large. We next turn to I_2 in (118). By repeated applications of Proposition 3.22, (120), and (114) we get

(121)
$$\frac{R^2}{GM}\langle \partial_t^2 \mathfrak{D}^k F(s), \vec{u}_k(s)\rangle \lesssim C_0 \sqrt{\frac{GM}{R}} \eta^{11}(s)|v_1(s)|^2 + C_0 \eta^{12}(s)|v_1(s)|^3$$

$$+ C_0^{\frac{1}{2}} \eta^9(s)|v_1(s)|^3 + \frac{GMC_0^{\frac{1}{2}}}{R} \eta^{10}(s)|v_1(s)|.$$

Now using Proposition 3.13 and Lemma 3.15 we conclude that if β is sufficiently large, then

$$(122) \qquad |I_2(t)| \lesssim C_0^{\frac{1}{2}} R \eta^8(t) |v_1(t)|^2.$$

Since $|v_1|$ has higher powers in (121) compared to Lemma 3.15, we carry out this computation for the last two terms on the right hand side of (121), $C_0^{\frac{1}{2}} \eta^9(s) |v_1(s)|^3$ and $\frac{GMC_0^{\frac{1}{2}}}{R} \eta(s)^{10} |v_1(s)|$. Note that if $\beta > 0$ is large enough, the first term on the right had side of (121) is smaller than the last term and second term is smaller than the third term. Let

$$I_{2,3} := C_0^{\frac{1}{2}} \int_{r_1(t)}^{R_1} \eta^9(s) |v_1(s)|^3 \left| \frac{ds}{dr_1} \right| dr_1.$$

We consider three different cases according to Proposition 3.13 depending on the value of $r_1(t)$. If $r_1 \geq c_0^{-2} R \beta^{\frac{12}{7}}$ then

$$I_{2,3} \lesssim \frac{GM c_0^2 C_0^{\frac{1}{2}} \beta^{-\frac{12}{7}}}{R} \int_{r_1(t)}^{\infty} \eta^9(r) dr \lesssim C_0^{\frac{1}{2}} R \eta^8(r_1) |v_1(r_1)|^2.$$

Next, using this estimate, if $r_1 \in [3c_0^2 R \beta^{\frac{2}{7}}, c_0^{-2} R \beta^{\frac{12}{7}}]$, then

$$I_{2,3} \lesssim C_0^{\frac{1}{2}} R \eta^8(c_0^{-2} R \beta^{\frac{12}{7}}) |v_1(c_0^{-2} R \beta^{\frac{12}{7}})|^2 + \frac{GMC_0^{\frac{1}{2}}}{R} \int_{r_1}^{c_0^{-2} R \beta^{\frac{12}{7}}} \eta^{10}(r) dr$$

$$\lesssim C_0^{\frac{1}{2}} R \eta^8(r_1) |v_1(r_1)|^2 + C_0^{\frac{1}{2}} R \eta^8(r_1) \frac{GM}{R} \eta(r_1) \lesssim C_0^{\frac{1}{2}} R \eta^8(r_1) |v_1(r_1)|^2.$$

Finally, using this estimate, if $r_1 \in (r_0, 4c_0^2 R \beta^{\frac{2}{7}}]$, then

$$I_{2,3} \lesssim C_0^{\frac{1}{2}} R \eta^8(4c_0^2 R \beta^{\frac{2}{7}}) |v_1(4c_0^2 R \beta^{\frac{2}{7}})|^2 + \frac{GM c_0^{-19} C_0^{\frac{1}{2}}}{R^{\frac{1}{2}}} \beta^{-\frac{19}{7}} \int_{r_1}^{4c_0^2 R \beta^{\frac{2}{7}}} \frac{dr}{\sqrt{r - r_0}}$$

$$\lesssim C_0^{\frac{1}{2}} R \eta^8(r_1) |v_1(r_1)|^2 + C_0^{\frac{1}{2}} R c_0^{-16} \beta^{-\frac{16}{7}} \frac{GM c_0^{-2}}{R} \beta^{-\frac{2}{7}}$$

$$\lesssim C_0^{\frac{1}{2}} R \eta^8(r_1) |v_1(r_1)|^2,$$

completing the estimate for $I_{2,3}$. The estimate for

$$I_{2,4} := \frac{GMC_0^{\frac{1}{2}}}{R} \int_{r_1(t)}^{r(T_0)} \eta^{10}(s) |v_1(s)| \left| \frac{ds}{dr_1} \right| dr_1.$$

follows from the one for $I_{2,3}$ and the observation that, in view of (82), $|\eta| \lesssim (GM)^{-1}R|v_1|^2$. This completes the proof of (117).

Step 4. Let I be as defined in (116), and let II be defined according to the relations (63) and

$$\frac{d\mathcal{E}_k}{dt} =: II - I.$$

We will show that if β is sufficiently large then

(123) $$\left| \int_{T_0}^t II(s)ds \right| \leq \frac{1}{100}C_0 R\eta^8(t)|v_1(t)|^2.$$

Combined with (117) this will complete the proof of (115) and hence of the proposition. We divide the terms in II into several groups. Let $\tilde{g}_k$ be as defined in Corollary 2.15, and set

$$II_1 := \langle \tilde{g}_k, \frac{\partial_t \vec{u}_k}{a} \rangle + I,$$

$$II_2 := \frac{1}{2} \left\langle \frac{1}{|N|}\partial_t \left(\frac{|N|}{a} \right) \partial_t \vec{u}_k, \partial_t \vec{u}_k \right\rangle + \frac{GM}{2R^3} \left\langle \frac{1}{|N|}\partial_t \left(\frac{|N|}{a} \right) \vec{u}_k, \vec{u}_k \right\rangle$$
$$+ \frac{3GM}{2R^3} \left\langle \frac{1}{|N|}\partial_t \left(\frac{|N|}{a} \right) \vec{u}_k \cdot n, \vec{u}_k \cdot n \right\rangle,$$

$$II_3 := \frac{3GM}{R^3} \left\langle a^{-1}\vec{u}_k \cdot n, \vec{u}_k \cdot n_t \right\rangle,$$

$$II_4 := -\frac{3GM}{2R^3} \langle (n \cdot \vec{u}_k)n, (H^*_{\partial \mathcal{B}_1} - H_{\partial \mathcal{B}_1})(a^{-1}\partial_t \vec{u}_k)\rangle,$$

$$II_5 := \frac{3GM}{2R^3} \langle (n \cdot \vec{u}_k)n, a^{-1}\partial_t(I - H_{\partial \mathcal{B}_1})\vec{u}_k\rangle,$$

$$II_6 := -\frac{3GM}{2R^3} \langle (n \cdot \vec{u}_k)n, [H_{\partial \mathcal{B}_1}, a^{-1}\partial_t]\vec{u}_k \rangle - \frac{1}{2}\langle \vec{Q}(u, \vec{u}_k), \vec{u}_k \rangle.$$

The first term II_1 contains the error terms from the nonlinearity in the equation (57) for $\vec{u}_k$ and the remaining terms $II_2, \ldots, II_6$ contain the other error terms arising in the energy identity (63). Using the bootstrap assumptions (92) and (114), Propositions 3.22, 3.23, and 3.25, Corollary 3.26, and Lemmas 3.24 and Lemma B.1 we get (with constants possibly depending on C_1)

$$|II_2| \lesssim R^{-1}\eta^7|v_1|\mathcal{E}_{\leq k} \lesssim C_0\eta^{15}|v_1|^3,$$

$$|II_3| \lesssim R^{-\frac{3}{2}}\mathcal{E}_{\leq 3}^{\frac{1}{2}}\mathcal{E}_{\leq k} \lesssim C_0^{\frac{3}{2}}\eta^{12}|v_1|^3,$$

$$|II_4| \lesssim \sqrt{\frac{GM}{R}} R^{-1} \eta^3 \mathcal{E}_{\leq k} \lesssim C_0 \sqrt{\frac{GM}{R}} \eta^{11} |v_1|^2,$$

$$|II_5| \lesssim \sqrt{\frac{GM}{R}} R^{-1} \eta^3 \mathcal{E}_{\leq k} + R^{-1} \eta^4 |v_1| \mathcal{E}_{\leq k} \lesssim C_0 \sqrt{\frac{GM}{R}} \eta^{11} |v_1|^2 + C_0 \eta^{12} |v_1|^3,$$

$$|II_6| \lesssim R^{-1} \eta^4 |v_1| \mathcal{E}_{\leq k} \lesssim C_0 \eta^{12} |v_1|^3.$$

Arguing as in the proof of (122) we conclude that if β is sufficiently large (depending on C_0 and C_1), then

$$(124) \qquad \sum_{j=2}^{6} \int_{T_0}^{t} |II_j(s)| ds \leq \frac{1}{300} C_0 R \eta^8(t) |v_1(t)|^2.$$

We now turn to the error terms form the nonlinearity in II_1. Let g_0 be as defined in Corollary 2.15, that is, in the notation of Proposition 2.2,

$$g_0 = -\partial_t F - \partial_t E_1 + \partial_t \left(\frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \mathbf{E}_1(t, \mathbf{x}) d\mathbf{x} \right) + E_2 + \partial_t \left(\frac{a}{|N|} \right) N.$$

Repeated applications of Lemma 3.19, Propositions 3.22 and 3.23, and Corollary 3.26 give

$$\sum_{0 \leq j \leq k} \|\tilde{g}_j - \mathfrak{D}^j g_0\|_{L^2(S_R)} \lesssim \frac{GM}{R^{\frac{5}{2}}} \eta^3 \mathcal{E}_{\leq k}^{\frac{1}{2}} + \frac{\sqrt{GM}}{R^2} \eta^4 |v_1| \mathcal{E}_{\leq k}^{\frac{1}{2}}$$

$$+ \frac{GM}{R^{\frac{7}{2}}} \eta^4 |v_1| \int_{T_0}^{t} \mathcal{E}_{\leq k}^{\frac{1}{2}}(s) ds + \frac{GM}{R^2} \eta^7 |v_1|$$

$$\lesssim \frac{C_0^{\frac{1}{2}} GM}{R^2} \eta^7 |v_1| + \frac{\sqrt{GM C_0}}{R^{\frac{3}{2}}} \eta^8 |v_1|^2.$$

Here to estimate the term $[\partial_t^2, [\mathfrak{D}^k, H_{\partial \mathcal{B}_1}]] u$ we have used the fact that

$$[\partial_t^2, [\mathfrak{D}^k, H_{\partial \mathcal{B}_1}]] u = \mathfrak{D}^k [\partial_t^2, H_{\partial \mathcal{B}_1}] u - [\partial_t^2, H_{\partial \mathcal{B}_1}] \mathfrak{D}^k u,$$

and applied Lemma 3.19. Use Lemma 3.15 as in the proof of (122) now shows that if β is sufficiently large, then

$$\int_{T_0}^{t} \left| \langle \tilde{g}_k(s) - \mathfrak{D}^k g_0(s), \frac{\partial_s \vec{u}_k(s)}{a(s)} \rangle \right| ds \leq \frac{1}{400} C_0 \eta^8(t) |v_1(t)|^2.$$

For the contribution of $\mathfrak{D}^k g_0$ note that

$$
\begin{aligned}
\langle \mathfrak{D}^k g_0, \frac{\partial_t \vec{u}_k}{a} \rangle + I =& \langle \partial_t \mathfrak{D}^k F, \left(\frac{R^2}{GM} - \frac{1}{a} \right) \partial_t \vec{u}_k \rangle - \langle \partial_t \mathfrak{D}^k E_1, \frac{\partial_t \vec{u}_k}{a} \rangle \\
& + \langle \partial_t \mathfrak{D}^k \left(\frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \mathbf{E}_1(t, \mathbf{x}) d\mathbf{x} \right), \frac{\partial_t \vec{u}_k}{a} \rangle \\
& + \langle \mathfrak{D}^k E_2, \frac{\partial_t \vec{u}_k}{a} \rangle + \langle \mathfrak{D}^k (\partial_t (a|N|^{-1}) N), \frac{\partial_t \vec{u}_k}{a} \rangle \\
=& : II_{1,1} + II_{1,2} + II_{1,3} + II_{1,4} + II_{1,5}.
\end{aligned}
$$

The contribution of $II_{1,1}$ is smaller than the contribution of I computed in Step 3. The contribution of $II_{1,2}$ and $II_{1,3}$ can also be estimated similarly. Indeed E_1 was obtained by subtracting the leading order term F from $\nabla \psi_2 + \mathbf{x}_1''$ so the contributions of E_1 and $\frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \mathbf{E}_1(t, \mathbf{x}) d\mathbf{x}$ are also smaller than that of F which we estimated in Step 3. Note that since the fluid is incompressible and irrotational, and $\mathbf{E}_1$ is a function of $\mathbf{x} - \mathbf{x}_1$, the time derivative of $\frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \mathbf{E}_1(t, \mathbf{x}) d\mathbf{x}$ can be computed as

(125)
$$
\begin{aligned}
\partial_t \left(\frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \mathbf{E}_1(t, \mathbf{x}) d\mathbf{x} \right) =& \frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} (\nabla \mathbf{E}_1)(t, \mathbf{x}) \cdot (\mathbf{v}(t, \mathbf{x}) - \mathbf{x}_1') d\mathbf{x} \\
& + \frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} (\partial_t \mathbf{E}_1)(t, \mathbf{x}) d\mathbf{x} \\
=& \frac{1}{|\mathcal{B}_1|} \int_{\partial \mathcal{B}_1} E_1 n \cdot \zeta_t dS + \frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} (\partial_t \mathbf{E}_1)(t, \mathbf{x}) d\mathbf{x}.
\end{aligned}
$$

The time derivative $(\partial_t \mathbf{E}_1)(t, \mathbf{x})$ can be computed similarly using the fact that the fluid is incompressible and irrotational in $\mathcal{B}_2$, and the fact that in the integral expression for $\mathbf{E}_1$ in (36) the integrand depends only on $\mathbf{y} - \mathbf{x}_2$, where $\mathbf{y} \in \mathcal{B}_2$ is the variable of integration. Next, recalling the definition of E_2 from (33), from repeated applications of the bootstrap assumptions (92) and (114), Proposition 3.22, and Lemma 3.19, we get

$$
\| \mathfrak{D}^k E_2 \|_{L^2(S_R)} \lesssim \frac{GM}{R^{\frac{5}{2}}} \eta^3 \mathcal{E}_{\leq k}^{\frac{1}{2}} + \frac{GM}{R^2} \eta^7 |v_1| \lesssim \frac{C_0^{\frac{1}{2}} GM}{R^2} \eta^7 |v_1|.
$$

Using Lemma 3.15 as in the proof of (122) we conclude that if β is sufficiently

large, then

$$\int_{T_0}^{t} \left| \langle II_{1,4}(s), \frac{\partial_s \vec{u}_k(s)}{a(s)} \rangle \right| ds \leq \frac{1}{800} C_0 \eta^8(t) |v_1(t)|^2.$$

Finally direct application of Proposition 3.25 gives

$$\|II_{1,5}\|_{L^2(S_R)} \lesssim \sqrt{\frac{GM}{R^3}} \eta^3 \mathcal{E}_{\leq k} \lesssim \sqrt{\frac{GM}{R^3}} C_0 \eta^{11} |v_1|^2.$$

It follows that if β is sufficiently large then

$$\int_{T_0}^{t} \left| \langle II_{1,5}(s), \frac{\partial_s \vec{u}_k(s)}{a(s)} \rangle \right| ds \leq \frac{1}{1000} C_0 \eta^8(t) |v_1(t)|^2,$$

completing the proof of the proposition. $\qquad\square$

4. Tidal Energy

In this final section we prove Theorem 1.1. For this we will express the tidal energy in terms of the height function $h = |\zeta| - R$ and its time derivative $\partial_t h$. The first subsection below is devoted to the analysis of the dynamics of h and $\partial_t h$. We then use this analysis in the second subsection to prove Theorem 1.1.

4.1. The height function

A crucial step in the proof of Theorem 1.1 is to analyze the behavior of the tidal energy in terms of the height function (see (26))

$$(126) \qquad h(t,\omega) := h(\zeta(t,\omega)) := |\zeta(t,\omega)| - R, \quad \omega \in \mathbb{S}^2.$$

Recall also the definition of $\tilde{h}$ (see (26)):

$$(127) \qquad \tilde{h}(t,\omega) := |\zeta(t,\omega)|^2 - R^2,$$

and the relation $\tilde{h} = h(h+2R) = 2Rh + h^2$. From now on, we will call $\tilde{h}$ the *modified height function*. Our goal in this section is to derive the equation satisfied by h. Let ϕ and $\mathring{\phi}$ denote the velocity potentials for $\mathbf{v}$ and $\mathbf{x}'_1$, respectively, that is,

$$\mathbf{v}(t,\mathbf{x}) = -\nabla\phi(t,\mathbf{x}),$$

$$\mathbf{x}_1'(t) = -\nabla \mathring{\phi}(t, \mathbf{x}), \qquad \mathring{\phi}(t, \mathbf{x}) := -\mathbf{x}_1' \cdot (\mathbf{x} - \mathbf{x}_1).$$

We will denote $\mathbf{x} - \mathbf{x}_1$ by $\boldsymbol{\zeta}$. In Lagrangian coordinates, let $\phi(t, p) = \boldsymbol{\phi}(t, \xi(t, p))$ and $\mathring{\phi}(t, p) := \mathring{\boldsymbol{\phi}}(t, \zeta(t, p))$. The modified height function satisfies,

$$\tilde{h}_t = 2\zeta_t \cdot \zeta = 2R\zeta_t \cdot n + 2\zeta_t \cdot (\zeta - Rn) = -2R\nabla_n(\phi - \mathring{\phi}) + 2\zeta_t \cdot (\zeta - Rn).$$

Differentiating this equation in time we get

$$
\begin{aligned}
(128) \quad \tilde{h}_{tt} = & -2R\nabla_n(\phi_t - \mathring{\phi}_t) - 2R[\partial_t, \nabla_n](\phi - \mathring{\phi}) + 2\zeta_{tt} \cdot (\zeta - Rn) \\
& + 2\zeta_t \cdot (\zeta_t - Rn') \\
= & : -2R\nabla_n(\phi_t - \mathring{\phi}_t) + \mathfrak{R}_1,
\end{aligned}
$$

with

$$(129) \qquad \mathfrak{R}_1 := -2R[\partial_t, \nabla_n](\phi - \mathring{\phi}) + 2u_t \cdot (\zeta - Rn) + 2u \cdot (u - Rn').$$

To express $\phi - \mathring{\phi}$ in terms of h, we use the Bernoulli equation

$$\partial_t \boldsymbol{\phi} = \boldsymbol{\psi}_1 + \boldsymbol{\psi}_2 + \mathbf{p} + \frac{1}{2}|\mathbf{v}|^2, \quad \Rightarrow \quad \phi_t = \psi_1 + \psi_2 - \frac{1}{2}|\zeta_t + \mathbf{x}_1'|^2.$$

Remark 4.1. Note that here $\partial_t \boldsymbol{\phi}$ is the Eulerian timte derivative of $\boldsymbol{\phi}$ in fluid domain, while ϕ_t is the restriction of the Lagrangian time derivative of ϕ on the boundary. Therefore the coefficient of $|\zeta_t + \mathbf{x}_1'|^2$ in the second term above is $-\frac{1}{2}$ instead of $\frac{1}{2}$.

On the other hand, by the definition of $\mathring{\phi}$,

$$\mathring{\phi}_t = -\zeta_t \cdot \mathbf{x}_1' - \zeta \cdot \mathbf{x}_1'',$$

so

$$\phi_t - \mathring{\phi}_t = \psi_1 + \psi_2 - \frac{1}{2}|\zeta_t|^2 - \frac{1}{2}|\mathbf{x}_1'|^2 + \zeta \cdot \mathbf{x}_1''.$$

Equation (128) is therefore equivalent to

$$(130) \qquad \tilde{h}_{tt} = -2R\nabla_n\psi_1 - 2R(\mathbf{x}_1'' \cdot n + \nabla_n\psi_2) + \mathfrak{R}_1 + \mathfrak{R}_2,$$

where

$$(131) \qquad\qquad\qquad \mathfrak{R}_2 = R\nabla_n|u|^2.$$

Next we derive an expression for ψ_1. Recall from Lemma 2.4 that

$$\nabla\psi_1 = \frac{GM}{R^3}\zeta - \frac{GM}{2R^3}(I + H_{\partial\mathcal{B}_1})\zeta.$$

This formula will be useful for analyzing $\mathfrak{R}_1$, but since $\boldsymbol{\psi}_1$ is not harmonic inside $\mathcal{B}_1$, $\nabla_n\psi_1$ cannot be computed by taking the inner product of this identity with n. Instead, to compute $\nabla_n\psi_1$ we argue as follows. First, since

$$\mathcal{D}\boldsymbol{\psi}_1 = \nabla\boldsymbol{\psi}_1 = \mathcal{D}((\nabla\boldsymbol{\psi}_1)\boldsymbol{\zeta})$$

outside of $\mathcal{B}_1$ (here $(\nabla\boldsymbol{\psi}_1)\zeta$ denotes the Clifford product),

$$(I + H_{\partial\mathcal{B}_1})\psi_1 = (I + H_{\partial\mathcal{B}_1})((\nabla\boldsymbol{\psi}_1)\zeta).$$

Using the formula $\nabla\psi_1 = \frac{GM}{R^3}\zeta - \frac{GM}{2R^3}(I + H_{\partial\mathcal{B}_1})\zeta$ above we get

(132)
$$(I + H_{\partial\mathcal{B}_1})\psi_1 = -\frac{GM}{R^3}(I + H_{\partial\mathcal{B}_1})(|\zeta|^2) - \frac{GM}{2R^3}(I + H_{\partial\mathcal{B}_1})(((I + H_{\partial\mathcal{B}_1})\zeta)\zeta).$$

Now recall from (27)–(29) that with $\mu = 1 - \frac{R^3}{|\zeta|^3} = \frac{3}{2R^2}\tilde{h} + \tilde{h}\nu$,

$$(I + H_{\partial\mathcal{B}_1})\zeta = (I + H_{\partial\mathcal{B}_1})(\zeta\mu).$$

Going back to equation (132) we get

(133) $$(I + H_{\partial\mathcal{B}_1})\psi_1 = -\frac{GM}{R^3}(I + H_{\partial\mathcal{B}_1})(\tilde{h} + R^2) + B + A_1,$$

where

$$B := -\frac{3GM}{4R^5}(I + H_{\partial\mathcal{B}_1})(((I + H_{\partial\mathcal{B}_1})(\tilde{h}\zeta))\zeta)$$

and

$$A_1 := -\frac{GM}{2R^3}(I + H_{\partial\mathcal{B}_1})(((I + H_{\partial\mathcal{B}_1})(\tilde{h}\nu\zeta))\zeta).$$

To compute B we keep in mind that $H_{\partial\mathcal{B}_1}nf + nH_{\partial\mathcal{B}_1}f$ and $\zeta - Rn$ are small quantities for any f, and write

$$B = -\frac{3GM}{4R^5}(I + H_{\partial\mathcal{B}_1})[(\zeta)(I - H_{\partial\mathcal{B}_1})(\tilde{h})(\zeta)]$$

$$\overbrace{-\frac{3GM}{4R^5}(I + H_{\partial\mathcal{B}_1})\left[\left(\zeta H_{\partial\mathcal{B}_1}\tilde{h} + H_{\partial\mathcal{B}_1}(\zeta\tilde{h})\right)\zeta\right]}^{A_2}$$

$$=A_2 - \frac{3GM}{4R^5}(I + H_{\partial\mathcal{B}_1})[(\zeta)^2(I + H_{\partial\mathcal{B}_1})\tilde{h} - 2(\zeta)^2 K_{\partial\mathcal{B}_1}\tilde{h}]$$

$$\overbrace{-\frac{3GM}{2R^5}(I + H_{\partial\mathcal{B}_1})[(\zeta)((\zeta)\cdot(H_{\partial\mathcal{B}_1}\tilde{h} - K_{\partial\mathcal{B}_1}\tilde{h}))]}^{A_3}$$

$$=A_2 + A_3 + \frac{3GM}{4R^3}(I + H_{\partial\mathcal{B}_1})[(I + H_{\partial\mathcal{B}_1})\tilde{h} - 2K_{\partial\mathcal{B}_1}\tilde{h}]$$

$$\overbrace{+\frac{3GM}{4R^5}(I + H_{\partial\mathcal{B}_1})[\tilde{h}(I + H_{\partial\mathcal{B}_1})\tilde{h} - 2\tilde{h}K_{\partial\mathcal{B}_1}\tilde{h}]}^{A_4}$$

$$=A_2 + A_3 + A_4 + \frac{3GM}{2R^3}(I + H_{\partial\mathcal{B}_1})(I - K_{\partial\mathcal{B}_1})\tilde{h}.$$

Plugging back into (133) we get

$$(I + H_{\partial\mathcal{B}_1})\psi_1 = -\frac{GM}{R^3}(I + H_{\partial\mathcal{B}_1})(\tilde{h} + R^2)$$
$$+ \frac{3GM}{2R^3}(I + H_{\partial\mathcal{B}_1})(I - K_{\partial\mathcal{B}_1})\tilde{h} + A_1 + A_2 + A_3 + A_4.$$

Taking real parts and applying $(I + K_{\partial\mathcal{B}_1})^{-1}$ gives

(134)
$$\psi_1 = -\frac{GM}{R} + \frac{\mathrm{g}}{2R}(I - 3K_{\partial\mathcal{B}_1})\tilde{h} + \tilde{\mathfrak{R}}_3 = -\frac{GM}{R} + \mathrm{g}(I - 3K_{\partial\mathcal{B}_1})h + \mathfrak{R}_3,$$

where $\mathrm{g} = \frac{GM}{R^2}$,

$$\tilde{\mathfrak{R}}_3 = (I + K_{\partial\mathcal{B}_1})^{-1}\left(\mathrm{Re}(A_1 + A_2 + A_3 + A_4)\right),$$

and

(135)
$$\mathfrak{R}_3 = \tilde{\mathfrak{R}}_3 + \frac{\mathrm{g}}{2R}(I - 3K_{\partial\mathcal{B}_1})h^2.$$

It follows that

(136)
$$\nabla_n\psi_1 = \mathrm{g}\nabla_n(I - 3K_{\partial\mathcal{B}_1})h + \nabla_n\mathfrak{R}_3.$$

The formula above gives us an expression for the first term on the right-hand side of (130). Since $\boldsymbol{\psi}_2$ is harmonic in $\mathcal{B}_1$, the second term on the righthand side of (130) can simply be computed by using the relation $\nabla_n \psi_2 + n \cdot \mathbf{x}_1'' = n \cdot (\nabla \psi_2 + \mathbf{x}_1'')$. Indeed, this identity combined with Lemma 2.4 and the fact that $\mathbf{x}_1'' = -\frac{\rho}{M} \int_{\mathcal{B}_1} \nabla \psi_2 d\mathbf{x}$, gives

$$
\begin{aligned}
\nabla_n \psi_2 + n \cdot \mathbf{x}_1'' = & \frac{GM\eta(t)^3}{8R^3}(\zeta \cdot n - 3(\zeta \cdot \boldsymbol{\xi}_1)(\boldsymbol{\xi}_1 \cdot n)) \\
& + n \cdot \left(E_1 - \frac{\rho}{M} \int_{\mathcal{B}_1} \mathbf{E}_1(t, \mathbf{x}) d\mathbf{x} \right) \\
= & \frac{GM\eta(t)^3}{8R^2} \left(1 - \frac{3(\boldsymbol{\xi}_1 \cdot \zeta)^2}{R^2} \right) + \mathfrak{R}_4,
\end{aligned}
$$
(137)

where

$$
\begin{aligned}
\mathfrak{R}_4 := & \frac{GM\eta^3}{8R^3} \left((\zeta - Rn) \cdot n - 3(\zeta \cdot \boldsymbol{\xi}_1)(\boldsymbol{\xi}_1 \cdot (n - R^{-1}\zeta)) \right) \\
& + n \cdot \left(E_1 - \frac{\rho}{M} \int_{\mathcal{B}_1} \mathbf{E}_1(t, \mathbf{x}) d\mathbf{x} \right).
\end{aligned}
$$
(138)

Inserting (136) and (137) back into (130), we get the desired equation for h which we summarize in the following proposition.

Proposition 4.2. *Let $\mathfrak{R}_1, \ldots, \mathfrak{R}_4$ be as defined in* (129), (131), (135), *and* (138), *respsectively. Then h satisfies*

(139)
$$
\begin{aligned}
& h_{tt} + \mathrm{g}\nabla_n(I - 3K_{\partial\mathcal{B}_1})h \\
& = -\frac{GM\eta^3}{8R^2} \left(1 - 3(\boldsymbol{\xi}_1 \cdot R^{-1}\zeta)^2 \right) + \frac{\mathfrak{R}_1}{2R} + \frac{\mathfrak{R}_2}{2R} - \nabla_n\mathfrak{R}_3 - \mathfrak{R}_4 - \frac{1}{R}h_t^2 - \frac{1}{R}hh_{tt}.
\end{aligned}
$$

Proof. This follows by combining equations (128)–(138). $\quad\square$

Except for the term $[\partial_t, \nabla_n](\phi - \mathring{\phi})$ in $\mathfrak{R}_1$, all the error terms in (139) can be estimated using the results in Section 3. The commutator $[\partial_t, \nabla_n]$ is also calculated in Lemma A.7. We next derive an expression for $\phi - \mathring{\phi}$ in terms of h which is of independent interest for future calculations. Note that $\phi - \mathring{\phi}$ is the solution to the Neumann problem

$$
\begin{aligned}
\Delta(\phi - \mathring{\phi}) &= 0, & \text{in} \quad \mathcal{B}_1, \\
\mathbf{n} \cdot \nabla(\phi - \mathring{\phi}) &= -\mathbf{n} \cdot (\mathbf{v} - \mathbf{x}_1'(t)) & \text{on} \quad \partial\mathcal{B}_1.
\end{aligned}
$$
(140)

Therefore by equation (166),

(141)
$$
\begin{aligned}
\phi - \mathring{\phi} &= 2S_{\partial\mathcal{B}_1}(I - K^*_{\partial\mathcal{B}_1})^{-1}(n \cdot u) \\
&= \frac{2}{R} S_{\partial\mathcal{B}_1}(I - K^*_{\partial\mathcal{B}_1})^{-1}(\zeta \cdot u + (Rn - \zeta) \cdot u) \\
&= \frac{1}{R} S_{\partial\mathcal{B}_1}(I - K^*_{\partial\mathcal{B}_1})^{-1}(\tilde{h}_t + 2(Rn - \zeta) \cdot u) \\
&= \frac{2}{R} S_{\partial\mathcal{B}_1}(I - K^*_{\partial\mathcal{B}_1})^{-1}(Rh_t + hh_t + (Rn - \zeta) \cdot u).
\end{aligned}
$$

To estimate $[\partial_t, \nabla_n](\phi - \mathring{\phi})$, we only need the expression $\phi - \mathring{\phi} = 2S_{\partial\mathcal{B}_1}(I - K^*_{\partial\mathcal{B}_1})^{-1}(n \cdot u)$ together with Lemmas A.7 and B.4. However, the entire expression will be needed in the next subsection.

To analyze equation (139) we will decompose h into spherical harmonics, but in order to do this effectively we need to replace the non-local terms ∇_n and $K_{\partial\mathcal{B}_1}$ on the left-hand side of the equation by the corresponding operators on S_R. In the following lemma we calculate the resulting errors.

Lemma 4.3. *Let $\mathbf{K}$, $\mathbf{S}$, and $\mathbf{D}$ denote the double-layered potential, single-layered potential, and Dirichlet-Neumann operators on S_R, respectively. Let $f : \mathbb{R} \times S_R \to \mathbb{R}$ and $F : \mathbb{R} \times \partial\mathcal{B}_1 \to \mathbb{R}$ be related as $f(t,p) = F(t, \xi(t,p))$. Then*

$$
\begin{aligned}
K_{\partial\mathcal{B}_1}&F - \mathbf{K}f \\
&= -\int_{T_0}^{t} \left(\mathrm{p.v.} \int_{S_R} \left(((u' - u) \cdot \nabla)K(\xi' - \xi) \cdot n' + K(\xi' - \xi) \cdot n'_t \right)(s) \right. \\
&\qquad\qquad \left. \times \frac{|N'|}{|\mathcal{J}'|} F'(t) dS(p') \right) ds \\
&\quad - \int_{T_0}^{t} \left(\mathrm{p.v.} \int_{S_R} \left(K(\xi' - \xi) \cdot n' \right)(s) \frac{\partial_t |N'|}{|\mathcal{J}'|} F'(t) dS(p') \right) ds,
\end{aligned}
$$

$$
\begin{aligned}
S_{\partial\mathcal{B}_1}F - \mathbf{S}f &= R(\mathbf{K}f - K_{\partial\mathcal{B}_1}F) \\
&\quad + \mathrm{p.v.} \int_{\partial\mathcal{B}_1} \left(\frac{1}{4\pi} \frac{h' - h}{|\xi' - \xi|^3} - K(\xi' - \xi) \cdot (Rn' - \zeta') \right) F' dS',
\end{aligned}
$$

$$
\begin{aligned}
K^*_{\partial\mathcal{B}_1}F - \mathbf{K}f &= K_{\partial\mathcal{B}_1}F - \mathbf{K}f \\
&\quad + \mathrm{p.v.} \int_{\partial\mathcal{B}_1} \left(\frac{1}{2\pi R} \frac{h - h'}{|\xi' - \xi|^3} + K(\xi' - \xi) \cdot ((n' - R^{-1}\zeta') \right. \\
&\qquad\qquad \left. + (n - R^{-1}\zeta)) \right) F' dS',
\end{aligned}
$$

$$\nabla_n F - \mathbf{D}f = \left((I + K_{\partial \mathcal{B}_1}^*)^{-1} - (I + \mathbf{K})^{-1} \right) \text{p.v.} \int_{\partial \mathcal{B}_1} (n \times K) \cdot (n' \times \nabla) F' dS'$$

$$+ (I + \mathbf{K})^{-1} \text{p.v.} \int_{S_R} (n \times K) \cdot (n' \times \nabla) F' \left(\frac{|N'|}{|\mathscr{g}'|} - 1 \right) dS'$$

$$+ (I + \mathbf{K})^{-1} \int_{T_0}^t \left(\text{p.v.} \int_{S_R} (n_s(s) \times K(s) \right.$$

$$\left. + n(s) \times ((u'(s) - u(s)) \cdot \nabla) K(s)) \cdot n'(s) \times \nabla F'(t) dS' \right) ds$$

$$+ (I + \mathbf{K})^{-1} \int_{T_0}^t \left(\text{p.v.} \int_{S_R} (n(s) \times K(s)) \right.$$

$$\left. \cdot \left(|N(s)|^{-1} (u_\beta(s) F_\alpha(t) - u_\alpha(s) F_\beta(t)) \right) dS' \right) ds$$

$$- (I + \mathbf{K})^{-1} \int_{T_0}^t \left(\text{p.v.} \int_{S_R} (n(s) \times K(s)) \right.$$

$$\left. \cdot \left(\frac{\partial_s |N(s)|}{|N(s)|^2} (\xi_\beta(s) F_\alpha(t) - \xi_\alpha(s) F_\beta(t)) \right) dS' \right) ds.$$

Proof. Recall that by assumption $\partial \mathcal{B}_1(T_0) = \mathbf{x}_1(T_0) + S_R$ and by definition $\zeta(T_0, p) = p$ for all $p \in S_R$. It follows that

$$K_{\partial \mathcal{B}_1} F(\xi)$$

$$= -\text{p.v.} \int_{S_R} K(\xi - \xi') \cdot n(p') \frac{|N(p')|}{|\mathscr{g}(p')|} F(\xi') dS(p')$$

$$= \mathbf{K}f(p) - \int_{T_0}^t \left(\text{p.v.} \int_{S_R} \frac{\partial}{\partial s} \left(K(\xi - \xi') \cdot n' \frac{|N'|}{|\mathscr{g}'|} \right) (s) F'(t) dS(p') \right) ds$$

$$= \mathbf{K}f(p)$$

$$- \int_{T_0}^t \left(\text{p.v.} \int_{S_R} \left(((u' - u) \cdot \nabla) K(\xi' - \xi) \cdot n' + K(\xi' - \xi) \cdot n'_t \right) (s) \right.$$

$$\times \frac{|N'|}{|\mathscr{g}'|} F'(t) dS(p') \right) ds$$

$$- \int_{T_0}^t \left(\text{p.v.} \int_{S_R} \left(K(\xi' - \xi) \cdot n' \right) (s) \frac{\partial_t |N'|}{|\mathscr{g}'|} F'(t) dS(p') \right) ds.$$

The second and third identities follow by inspection of (164) and (165), and

noting that

$$-K(\xi' - \xi) \cdot n' = \frac{1}{2\pi R} \frac{(\zeta' - \zeta) \cdot \zeta'}{|\zeta' - \zeta|^3} - K(\xi' - \xi) \cdot (n' - R^{-1}\zeta')$$

$$= \frac{1}{4\pi R} \frac{1}{|\xi' - \xi|} + \frac{1}{4\pi R} \frac{\tilde{h}' - \tilde{h}}{|\xi' - \xi|^3} - K(\xi' - \xi) \cdot (n' - R^{-1}\zeta'),$$

and similarly

$$(n + n') \cdot K(\xi' - \xi)$$
$$= \frac{1}{2\pi R} \frac{\tilde{h} - \tilde{h}'}{|\xi' - \xi|^3} + K(\xi' - \xi) \cdot ((n' - R^{-1}\zeta') + (n - R^{-1}\zeta)).$$

For the last identity, we use Lemma A.4 to write

$$\nabla_n F = (I + \mathbf{K})^{-1} \text{p.v.} \int_{S_R} (n \times K) \cdot (n' \times \nabla) F' dS'$$

$$+ \left((I + K^*_{\partial \mathcal{B}_1})^{-1} - (I + \mathbf{K})^{-1}\right) \text{p.v.} \int_{\partial \mathcal{B}_1} (n \times K) \cdot (n' \times \nabla) F' dS'$$

$$+ (I + \mathbf{K})^{-1} \text{p.v.} \int_{S_R} (n \times K) \cdot (n' \times \nabla) F' \left(\frac{|N'|}{|\mathscr{J}'|} - 1\right) dS'$$

$$= \mathbf{D} f + \left((I + K^*_{\partial \mathcal{B}_1})^{-1} - (I + \mathbf{K})^{-1}\right) \text{p.v.} \int_{\partial \mathcal{B}_1} (n \times K) \cdot (n' \times \nabla) F' dS'$$

$$+ (I + \mathbf{K})^{-1} \text{p.v.} \int_{S_R} (n \times K) \cdot (n' \times \nabla) F' \left(\frac{|N'|}{|\mathscr{J}'|} - 1\right) dS'$$

$$+ (I + \mathbf{K})^{-1} \int_{T_0}^{t} \left(\text{p.v.} \int_{S_R} \left(n_s(s) \times K(s)\right. \right.$$

$$\left. \left. + n(s) \times ((u'(s) - u(s)) \cdot \nabla) K(s)\right) \cdot \left(n'(s) \times \nabla\right) F'(t) dS' \right) ds$$

$$+ (I + \mathbf{K})^{-1} \int_{T_0}^{t} \left(\text{p.v.} \int_{S_R} (n(s) \times K(s))\right.$$

$$\left. \cdot \left(|N(s)|^{-1}(u_\beta(s) F_\alpha(t) - u_\alpha(s) F_\beta(t))\right) dS' \right) ds$$

$$- (I + \mathbf{K})^{-1} \int_{T_0}^{t} \left(\text{p.v.} \int_{S_R} (n(s) \times K(s))\right.$$

$$\cdot \left. (\partial_s |N(s)|) |N(s)|^{-2} (\xi_\beta(s) F_\alpha(t) - \xi_\alpha(s) F_\beta(t)) dS \right) ds. \qquad \square$$

Combining the previous lemma with Proposition 4.2 we arrive at our final equation for h which we record in the following corollary.

Corollary 4.4. *Let*

(142)
$$f(t,\omega) := -\frac{g\eta^3}{8} \left(1 - 3(\boldsymbol{\xi}_1 \cdot \omega)^2\right),$$

(143)
$$\mathfrak{R}_5 := -\frac{1}{R} h_t^2 - \frac{1}{R} h h_{tt} + g \left(\mathbf{D}(I - 3\mathbf{K}) - \nabla_n (I - 3K_{\partial \mathcal{B}_1})\right) h$$
$$+ \frac{3g\eta^3}{8R^2} \left(\boldsymbol{\xi}_1 \cdot \int_{T_0}^t u(s) ds\right)^2 + \frac{3g\eta^3}{4R^2} (\boldsymbol{\xi}_1 \cdot \omega) \left(\boldsymbol{\xi}_1 \cdot \int_{T_0}^t u(s) ds\right),$$

and $\mathfrak{R} := \frac{\mathfrak{R}_1}{2R} + \frac{\mathfrak{R}_2}{2R} - \nabla_n \mathfrak{R}_3 - \mathfrak{R}_4 + \mathfrak{R}_5$, where $\mathfrak{R}_1, \ldots, \mathfrak{R}_4$ are as in Proposition 4.2. Then $h : \mathbb{R} \times S_R \to \mathbb{R}$ satisfies

(144)
$$(\partial_t^2 + g\mathbf{D}(I - 3\mathbf{K}))h(t,\omega) = f(t,\omega) + \mathfrak{R},$$

and the remainder $\mathfrak{R}$ satisfies the estimates

$$\|\mathfrak{R}\|_{L^2(S_R)} \lesssim GMR^{-1}\eta^4,$$
$$\|\partial_t \mathfrak{R}\|_{L^2(S_R)} \lesssim GMR^{-2}\eta^5 |v_1|,$$
$$\|\partial_t^2 \mathfrak{R}\|_{L^2(S_R)} \lesssim (GM)^{\frac{3}{2}} R^{-\frac{7}{2}} \eta^6 |v_1|,$$
$$\|\partial_t^3 \mathfrak{R}\|_{L^2(S_R)} \lesssim (GM)^2 R^{-5} \eta^7 |v_1|.$$

Proof. Equation (144) is just a rewriting of (139). The estimates on the remainder $\mathfrak{R}$ can be proved much in the same way as the estimates in Section 3. We will use Proposition 3.27 to obtain the estimates. Using Propositions 3.22, 3.23, 3.27 and Lemmas A.4, A.7, and B.4 and equations (141) and (129) we get

$$\|\mathfrak{R}_1\|_{L^2(S_R)} \lesssim \sqrt{GMR}\, \eta^7 |v_1|.$$

For $\mathfrak{R}_2$, by Lemmas A.4 and B.4 and (131) we have

$$\|\mathfrak{R}_2\|_{L^2(S_R)} \lesssim R\eta^8 |v_1|^2.$$

Next, using (135) and Lemmas A.4 and B.4, we get

$$\|\nabla_n \mathfrak{R}_3\|_{L^2(S_R)} \lesssim \frac{GM}{R} \eta^6.$$

Here to estimate A_2 we have written

$$\zeta H_{\partial \mathcal{B}_1} \tilde{h} + H_{\partial \mathcal{B}_1}(\zeta \tilde{h})$$
$$= Rn(H_{\partial \mathcal{B}_1} - H^*_{\partial \mathcal{B}_1})\tilde{h} + (\zeta - Rn)H_{\partial \mathcal{B}_1}\tilde{h} + H_{\partial \mathcal{B}_1}((\zeta - Rn)\tilde{h}),$$

and applied Lemma 3.24. Similarly for A_3 we have used the expression $K = K(\xi' - \xi) = -\frac{1}{2\pi} \frac{\zeta' - \zeta}{|\zeta' - \zeta|^3}$ for the kernel of $H_{\partial \mathcal{B}_1}$ to write

$$\zeta \cdot (H_{\partial \mathcal{B}_1} - K_{\partial \mathcal{B}_1})\tilde{h}$$
$$= \zeta \cdot \text{p.v.} \int_{\partial \mathcal{B}_1} K \times (n' - R^{-1}\zeta')\tilde{h}' dS'$$
$$+ R^{-1}\zeta \cdot \text{p.v.} \int_{\partial \mathcal{B}_1} (\zeta \times K + K \times \zeta')\tilde{h}' dS'$$
$$= \zeta \cdot \text{p.v.} \int_{\partial \mathcal{B}_1} K \times (n' - R^{-1}\zeta')\tilde{h}' dS'.$$

For $\mathfrak{R}_4$ by (138) we have

$$\|\mathfrak{R}_4\|_{L^2(S_R)} \lesssim \frac{GM}{R}\eta^6 + \frac{GM}{R}\eta^4,$$

where the last term on the right is the contribution of E_1. Finally by (143) and Lemma 4.3

$$\|\mathfrak{R}_5\|_{L^2(S_R)} \lesssim \sqrt{\frac{GM}{R}} \eta^7 |v_1| + \frac{GM}{R}\eta^6,$$

finishing the proof of the estimate on $\|\mathfrak{R}\|_{L^2(S_R)}$. The estimates for the time derivatives are similar where we additionally use Proposition 3.13, Lemma 3.14, and Lemma B.4 and to estimate the time derivative of $\mathbf{E}_1$ we use the same argument as in (125). □

Equation (144) is our working equation for the remainder of this section. Our goal now is to use this equation to obtain a lower bound for $\|\partial_t h\|_{L^2(S_R)}$. For this, we decompose h and $\mathfrak{R}$ into spherical harmonics (see Appendix A)

$$h = \sum_{\ell=0}^{\infty} h_\ell, \qquad \Delta h_\ell = \frac{\ell(\ell+1)}{R^2} h_\ell,$$

$$\mathfrak{R} = \sum_{\ell=0}^{\infty} \mathcal{R}_\ell, \qquad \Delta\!\!\!\!/\,\mathfrak{R}_\ell = \frac{\ell(\ell+1)}{R^2}\mathcal{R}_\ell.$$

Using Proposition A.6 and the facts that $\mathbf{D}h_\ell = \frac{\ell}{R}h_\ell$ and that f belongs to the second eigenspace of the Laplacian, $\mathcal{Y}_2$, we can write equation (144) as

(145)
$$\begin{aligned}
&\partial_t^2 h_\ell + a_\ell h_\ell = f_\ell, \\
&f_2 = f + \mathcal{R}_2, \qquad f_\ell := \mathcal{R}_\ell, \quad \ell \neq 2, \\
&\lim_{t \to T_0} h_\ell(t) = \lim_{t \to T_0} \partial_t h_\ell(t) = 0, \qquad \forall \ell \geq 0,
\end{aligned}$$

where

$$a_\ell := \frac{\mathrm{g}}{R}\frac{2\ell(\ell-1)}{2\ell+1}.$$

The following is the main result of this section.

Proposition 4.5. *The following estimates hold for* $r \in [r_0, 10r_0]$:

(146)
$$\begin{aligned}
&R^2\eta^3 \lesssim \|h_2\|_{L^2(S_R)} \lesssim 2\eta^3, \\
&\|h - h_2\|_{L^2(S_R)} \lesssim R^2\eta^4,
\end{aligned}$$

and

(147)
$$\begin{aligned}
&R\eta^4|v_1| \lesssim \|\partial_t h_2\|_{L^2(S_R)} \lesssim R\eta^4|v_1|, \\
&\|\partial_t(h - h_2)\|_{L^2(S_R)} \lesssim R\eta^5|v_1|.
\end{aligned}$$

In particular $\|h(t)\|_{L^2(S_R)} \gtrsim R^2\eta^3(t)$ *and* $\|\partial_t h(t)\|_{L^2(S_R)} \gtrsim R\eta^4(t)|v_1(t)|$.

Proof. The proofs of (146) and (147) are almost identical, so we provide the details only for the latter. Solving the ODE (145) we get

$$h_\ell(t) = \frac{1}{\sqrt{a_\ell}}\int_{T_0}^t \sin(\sqrt{a_\ell}(t-s))f_\ell(s)ds.$$

Differentiating in time and using several integration-by-parts, for $\ell \geq 2$ we arrive at

(148)
$$\partial_t h_\ell(t) = \int_{T_0}^t \cos(\sqrt{a_\ell}(t-s))f_\ell(s)ds$$

$$= -\frac{1}{\sqrt{a_\ell}}\int_{T_0}^t \frac{d}{ds}(\sin(\sqrt{a_\ell}(t-s)))f_\ell(s)ds$$

$$= \frac{1}{\sqrt{a_\ell}}\sin\left(\sqrt{a_\ell}(t-T_0)\right)f_\ell(T_0) + \frac{1}{\sqrt{a_\ell}}\int_{T_0}^t \sin(\sqrt{a_\ell}(t-s))f_\ell'(s)ds$$

$$= \frac{1}{a_\ell}\int_{T_0}^t \frac{d}{ds}(\cos(\sqrt{a_\ell}(t-s)))f_\ell'(s)ds + \frac{1}{\sqrt{a_\ell}}\sin\left(\sqrt{a_\ell}(t-T_0)\right)f_\ell(T_0)$$

$$= \frac{f_\ell'(t)}{a_\ell} - \frac{1}{a_\ell}\int_{T_0}^t \cos(\sqrt{a_\ell}(t-s))f_\ell''(s)ds$$

$$+ \frac{1}{\sqrt{a_\ell}}\sin\left(\sqrt{a_\ell}(t-T_0)\right)f_\ell(T_0) - \frac{1}{a_\ell}\cos\left(\sqrt{a_\ell}(t-T_0)\right)f_\ell'(T_0)$$

$$= \frac{f_\ell'(t)}{a_\ell} + \frac{1}{a_\ell^{3/2}}\int_{T_0}^t \frac{d}{ds}(\sin(\sqrt{a_\ell}(t-s)))f_\ell''(s)ds$$

$$+ \frac{1}{\sqrt{a_\ell}}\sin\left(\sqrt{a_\ell}(t-T_0)\right)f_\ell(T_0) - \frac{1}{a_\ell}\cos\left(\sqrt{a_\ell}(t-T_0)\right)f_\ell'(T_0)$$

$$= \frac{f_\ell'(t)}{a_\ell} - \frac{1}{a_\ell^{3/2}}\int_{T_0}^t \sin(\sqrt{a_\ell}(t-s))f_\ell'''(s)ds$$

$$- \frac{1}{a_\ell^{3/2}}\sin\left(\sqrt{a_\ell}(t-T_0)\right)f_\ell''(T_0)$$

$$+ \frac{1}{\sqrt{a_\ell}}\sin\left(\sqrt{a_\ell}(t-T_0)\right)f_\ell(T_0) - \frac{1}{a_\ell}\cos\left(\sqrt{a_\ell}(t-T_0)\right)f_\ell'(T_0).$$

It follows from the estimates in Corollary 4.4, Proposition 3.13, and Lemma 3.14, that

$$\int_{S_R}|\partial_t h_2|^2 dS = \frac{1}{a_2^2}\int_{S_R}|f_2'|^2 dS + O\left(|v_1|^2\eta^{10}\right).$$

Here we want to show that the first term on the right hand side above has a lower bound of order $O\left(|v_1|^2\eta^8\right)$. Since $\partial_t f_2 = \partial_t f + \partial_t \mathcal{R}_2$ and $\|\partial_t \mathcal{R}_2\|^2_{L^2(S_R)} \lesssim |v_1|^2\eta^{10}$, the problem reduces to deriving a lower bound of order $O\left(|v_1|^2\eta^8\right)$ for $\int_{S_R}|\partial_t f|^2 dS$. According to (142) we can write f and $\partial_t f$ in the schematic forms

$$f(t,\omega) = c_1\eta^3(t)(1 - c_2(\boldsymbol{\xi}_1(t)\cdot\omega)^2),$$
$$\Rightarrow \quad \partial_t f = c_1\left(\eta^3(t)\right)'(1 - c_2(\boldsymbol{\xi}_1(t)\cdot\omega)^2) - c_1c_2\eta^3(t)\left((\boldsymbol{\xi}_1\cdot\omega)^2\right)',$$

where c_1, c_2 are constants depending on G, M, R. This gives the following

formula for $|\partial_t f|^2$:

(149)
$$\begin{aligned}
|\partial_t f|^2 = &c_1^2 \left(3\eta^2(t)\eta'(t)\right)^2 (1 - c_2(\boldsymbol{\xi}_1 \cdot \omega)^2)^2 + 4c_1^2 c_2^2 \eta^6(t)(\boldsymbol{\xi}_1 \cdot \omega)^2 (\xi_1' \cdot \omega)^2 \\
&- 2c_1^2 c_2 \left(\eta^3(t)\right)' \eta^3(t)(1 - c_2(\boldsymbol{\xi}_1 \cdot \omega)^2) \left((\boldsymbol{\xi}_1 \cdot \omega)^2\right)'
\end{aligned}$$

The integral of the second line above on S_R is zero. Indeed,

$$\begin{aligned}
&-2c_1^2 \left(\eta^3(t)\right)' \eta^3(t) \int_{S_R} (1 - c_2(\boldsymbol{\xi}_1 \cdot \omega)^2) c_2 \left((\boldsymbol{\xi}_1 \cdot \omega)^2\right)' \, dS \\
&= c_1^2 \left(\eta^3(t)\right)' \eta^3(t) \int_{S_R} \left((1 - c_2(\boldsymbol{\xi}_1 \cdot \omega)^2)^2\right)' \, dS \\
&= c_1^2 \left(\eta^3(t)\right)' \eta^3(t) \frac{d}{dt} \left(\int_{S_R} (1 - c_2(\boldsymbol{\xi}_1 \cdot \omega)^2)^2 \, dS \right).
\end{aligned}$$

But due to the symmetry of S_R, the integral $\int_{S_R} (1 - c_2(\boldsymbol{\xi}_1 \cdot \omega)^2)^2 dS$ does not depend on $\boldsymbol{\xi}_1$ and hence on time, so the last term above vanishes. Going back to (149) and using the symmetry of S_R again, and using the values of c_1 and c_2 from (142), we see conclude that

$$\begin{aligned}
\int_{S_R} |\partial_t f|^2 dS \gtrsim &\left(\frac{GM}{R}\right)^2 \eta^4(t)(\eta'(t))^2 + \left(\frac{GM}{R}\right)^2 \eta^6(t)|\xi_1'(t)|^2 \\
\gtrsim &\left(\frac{GM}{R}\right)^2 R^{-2} \eta^8(t) \left(|r_1'(t)|^2 + |v_1 - r_1' \boldsymbol{\xi}_1|^2\right) \\
\gtrsim &\left(\frac{GM}{R}\right)^2 R^{-2} \eta^8(t)|v_1|^2.
\end{aligned}$$

For the last inequality above we simply observe that the estimate is trivial if $|r_1'| \geq \frac{1}{4}|v_1|$, and if $|r_1'| \leq \frac{1}{4}|v_1|$, then $|v_1 - r_1' \boldsymbol{\xi}_1| \geq \frac{3}{4}|v_1|$. Returning to (148) we have proved that

(150)
$$\begin{aligned}
R\eta^4 |v_1| &\lesssim \|\partial_t h_2\|_{L^2(S_R)} \lesssim R\eta^4 |v_1|, \\
\|\partial_t h_\ell\|_{L^2(S_R)} &\lesssim \frac{R^3}{GM} \|\partial_t \mathcal{R}_\ell\|_{L^2(S_R)}, \quad \ell \geq 3.
\end{aligned}$$

Since $a_0 = a_1 = 0$ this argument does not apply to $\ell = 0, 1$. Instead, to

estimate h_0 and h_1 we argue as follows. First, since the fluid is incompressible

$$
\begin{aligned}
4\pi R^3 &= \int_{\partial \mathcal{B}_1} n \cdot \zeta dS = \int_{S_R} n \cdot \zeta \frac{|N|}{|\not{g}|} dS \\
&= \int_{S_R} n \cdot \zeta dS + \int_{S_R} n \cdot \zeta \left(\frac{|N|}{|\not{g}|} - 1 \right) dS \\
&= \frac{1}{R} \int_{S_R} \zeta \cdot \zeta dS + \int_{S_R} \left(n - \frac{1}{R}\zeta \right) \cdot \zeta dS + \int_{S_R} n \cdot \zeta \left(\frac{|N|}{|\not{g}|} - 1 \right) dS \\
&= 4\pi R^3 + \frac{1}{R} \int_{S_R} \tilde{h} dS + \int_{S_R} \left(n - \frac{1}{R}\zeta \right) \cdot \zeta dS + \int_{S_R} n \cdot \zeta \left(\frac{|N|}{|\not{g}|} - 1 \right) dS \\
&= 4\pi R^3 + 2 \int_{S_R} h dS + \frac{1}{R} \int_{S_R} h^2 dS + \int_{S_R} \left(n - \frac{1}{R}\zeta \right) \cdot \zeta dS \\
&\quad + \int_{S_R} n \cdot \zeta \left(\frac{|N|}{|\not{g}|} - 1 \right) dS.
\end{aligned}
$$

We can now solve for $h_0 = \frac{1}{4\pi} \int_{S_R} h dS$ from this equation to estimate h_0 and differentiating in time. Indeed the third term on the right hand side above is already quadratic in h. On the other hand, the time derivative of the sum of the last two terms is

$$
\begin{aligned}
\int_{S_R} (n - R^{-1}\zeta) \cdot \zeta_t dS &+ \int_{S_R} n \cdot \zeta_t \left(\frac{|N|}{|\not{g}|} - 1 \right) dS - R^{-1} \int_{S_R} \zeta \cdot \zeta_t dS \\
&+ \int_{S_R} n \cdot \zeta \partial_t \frac{|N|}{|\not{g}|} dS.
\end{aligned}
$$

The first two terms are quadratic and their time integration is bounded as desired. For the third term, using the harmonicity of $\mathbf{v}$, we write

$$
\begin{aligned}
R^{-1} \int_{S_R} \zeta \cdot \zeta_t dS \\
&= \int_{S_R} (R^{-1}\zeta - n) \cdot \zeta_t dS + \int_{S_R} n \cdot \zeta_t \left(1 - \frac{|N|}{|\not{g}|} \right) dS + \int_{\partial \mathcal{B}_1} n \cdot \zeta_t dS \\
&= \int_{S_R} (R^{-1}\zeta - n) \cdot \zeta_t dS + \int_{S_R} n \cdot \zeta_t \left(1 - \frac{|N|}{|\not{g}|} \right) dS,
\end{aligned}
$$

which is quadratic and can be bounded as before. Finally, using (94) and

Lemma 3.2, and the facts that $n \times \nabla \zeta = -2n$ and $\mathbf{v}$ is harmonic,

$$\int_{S_R} n \cdot \zeta \partial_t \frac{|N|}{|g|} dS$$

$$= -\int_{\partial \mathcal{B}_1} (\zeta \cdot n) n \cdot (n \times \nabla u) dS$$

$$= -R \int_{\partial \mathcal{B}_1} n \cdot (n \times \nabla u) dS + \int_{\partial \mathcal{B}_1} (R - \zeta \cdot n) n \cdot (n \times \nabla u) dS$$

$$= -\int_{\partial \mathcal{B}_1} \zeta \cdot (n \times \nabla u) dS - \int_{\partial \mathcal{B}_1} (Rn - \zeta) \cdot (n \times \nabla u) dS$$

$$+ \int_{\partial \mathcal{B}_1} (R - \zeta \cdot n) n \cdot (n \times \nabla u) dS$$

$$= \int_{\partial \mathcal{B}_1} (n \times \nabla \zeta) \cdot u dS - \int_{\partial \mathcal{B}_1} (Rn - \zeta) \cdot (n \times \nabla u) dS$$

$$+ \int_{\partial \mathcal{B}_1} (R - \zeta \cdot n) n \cdot (n \times \nabla u) dS$$

$$= -\int_{\partial \mathcal{B}_1} (Rn - \zeta) \cdot (n \times \nabla u) dS + \int_{\partial \mathcal{B}_1} (R - \zeta \cdot n) n \cdot (n \times \nabla u) dS,$$

which is again quadratic. This completes the proof of the desired estimate for both h_0 and $\partial_t h_0$. Similarly, for h_1, since $\int_{\mathcal{B}_1} (\mathbf{x} - \mathbf{x}_1) d\mathbf{x} = 0$,

$$0 = \int_{\partial \mathcal{B}_1} |\zeta|^2 n dS = \int_{\partial \mathcal{B}_1} \tilde{h} n dS$$

$$= 2 \int_{\partial \mathcal{B}_1} h \zeta dS + 2 \int_{\partial \mathcal{B}_1} h(Rn - \zeta) dS + \int_{\partial \mathcal{B}_1} h^2 n dS$$

$$= 2 \int_{\partial \mathcal{B}_1} h \omega dS + 2 \int_{\partial \mathcal{B}_1} h(\zeta(t) - \zeta(0)) dS + 2 \int_{\partial \mathcal{B}_1} h(Rn - \zeta) dS$$

$$+ \int_{\partial \mathcal{B}_1} h^2 n dS$$

$$= 2 \int_{S_R} h \omega dS + 2 \int_{S_R} h \omega \left(\frac{|N|}{|g|} - 1 \right) dS + 2 \int_{\partial \mathcal{B}_1} h(\zeta(t) - \zeta(0)) dS$$

$$+ 2 \int_{\partial \mathcal{B}_1} h(Rn - \zeta) dS + \int_{\partial \mathcal{B}_1} h^2 n dS.$$

Since $\{\omega^j\}_{j=1,2,3}$ form a basis for $\mathcal{Y}_1$, we can solve for h_1 from this relation, and differentiating in time we can estiamte $\partial_t h_1$. Combining with (150) we

get the desired bound

$$\|\partial_t(h - h_2)\|_{L^2(S_R)} \lesssim R\eta^5|v_1|. \qquad \square$$

4.2. Proof of Theorem 1.1

Recall that the total energy

$$\mathscr{E} := \frac{1}{2}\int_{\mathcal{B}_1}|\mathbf{v}(t,\mathbf{x})|^2 dx + \frac{1}{2}\int_{\mathcal{B}_1}\psi_1(t,\mathbf{x})dx + \frac{1}{2}\int_{\mathcal{B}_1}\psi_2(t,\mathbf{x})dx,$$

is conserved during the evolution. To see this we compute the time derivative of each of the terms in the definition of $\mathscr{E}$. First,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\int_{\mathcal{B}_1}|\mathbf{v}(t,\mathbf{x})|^2 dx\right) &= \int_{\mathcal{B}_1}\mathbf{v}\cdot(\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v})\,dx \\
&= -\int_{\mathcal{B}_1}\mathbf{v}\cdot(\nabla P + \nabla\psi_1 + \nabla\psi_2)\,dx \\
&= -\int_{\mathcal{B}_1}\mathbf{v}\cdot(\nabla\psi_1 + \nabla\psi_2)\,dx.
\end{aligned}
$$

(151)

For the contribution of ψ_1 we have

$$
\begin{aligned}
\frac{d}{dt}\left(\int_{\mathcal{B}_1}\psi_1(t,\mathbf{x})dx\right) &= G\rho\int_{\mathcal{B}_1}\int_{\mathcal{B}_1}\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\cdot(\mathbf{v}(\mathbf{x})-\mathbf{v}(\mathbf{y}))\,dydx \\
&= \int_{\mathcal{B}_1}\mathbf{v}(\mathbf{x})\cdot\nabla\psi_1(\mathbf{x})dx \\
&\quad - G\rho\int_{\mathcal{B}_1}\mathbf{v}(\mathbf{y})\cdot\left(\int_{\mathcal{B}_1}\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}dx\right)dy \\
&= 2\int_{\mathcal{B}_1}\mathbf{v}(\mathbf{x})\cdot\nabla\psi_1(\mathbf{x})dx.
\end{aligned}
$$

(152)

For ψ_2

(153)
$$
\begin{aligned}
\frac{d}{dt}\left(\int_{\mathcal{B}_1}\psi_2(t,\mathbf{x})dx\right) &= G\rho\int_{\mathcal{B}_1}\int_{\mathcal{B}_2}\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\cdot(\mathbf{v}(\mathbf{x})-\mathbf{v}(\mathbf{y}))\,dydx \\
&= \int_{\mathcal{B}_1}\mathbf{v}(\mathbf{x})\cdot\nabla\psi_2(\mathbf{x})dx + G\rho\int_{\mathcal{B}_2}\mathbf{v}(\mathbf{y})
\end{aligned}
$$

$$\cdot \left(\int_{\mathcal{B}_1} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3} d\mathbf{x} \right) d\mathbf{y}$$

$$= \int_{\mathcal{B}_1} \mathbf{v}(\mathbf{x}) \cdot \nabla \psi_2(\mathbf{x}) d\mathbf{x} + \int_{\mathcal{B}_2} \mathbf{v}(\mathbf{y}) \cdot \nabla \psi_1(\mathbf{y}) d\mathbf{y}$$

$$= \int_{\mathcal{B}_1} \mathbf{v}(\mathbf{x}) \cdot \nabla \psi_2(\mathbf{x}) d\mathbf{x} + \int_{\mathcal{B}_1} \mathbf{v}(-\mathbf{y}) \cdot \nabla \psi_1(-\mathbf{y}) d\mathbf{y}.$$

Note that $\mathbf{v}(-\mathbf{y}) = -\mathbf{v}(\mathbf{y})$ and

$$\nabla \psi_1(-\mathbf{y}) = G\rho \int_{\mathcal{B}_1} \frac{-\mathbf{y} - \mathbf{z}}{|-\mathbf{y} - \mathbf{z}|^3} d\mathbf{z} = G\rho \int_{\mathcal{B}_2} \frac{\mathbf{z} - \mathbf{y}}{|\mathbf{z} - \mathbf{y}|^3} d\mathbf{z} = -\nabla \psi_2(\mathbf{y}),$$

so (153) becomes

(154)
$$\frac{d}{dt} \left(\int_{\mathcal{B}_1} \psi_2(t, \mathbf{x}) d\mathbf{x} \right) = 2 \int_{\mathcal{B}_1} \mathbf{v}(\mathbf{x}) \cdot \nabla \psi_2(\mathbf{x}) d\mathbf{x}.$$

Multiplying (152) and (154) by 2^{-1} and adding them to (151), we see that $\frac{d\mathscr{E}}{dt} = 0$. Recall also that $\widetilde{\mathscr{E}_{\text{tidal}}}$ is defined as

$$\widetilde{\mathscr{E}_{\text{tidal}}}(t) := \frac{1}{2|\mathcal{B}_1|} \int_{\mathcal{B}_1} |\mathbf{v}|^2 d\mathbf{x} - \frac{1}{2}|\mathbf{x}_1'|^2 + \frac{1}{2|\mathcal{B}_1|} \int_{\mathcal{B}_1} \psi_1 d\mathbf{x} + \frac{3GM}{5R}.$$

We divide $\widetilde{\mathscr{E}_{\text{tidal}}}$ into its kinetic and potential parts $\widetilde{\mathscr{E}_{\text{tidal}}} := \mathscr{E}_{\text{tidal}}^0 + \mathscr{E}_{\text{tidal}}^1$ where

$$\mathscr{E}_{\text{tidal}}^0 := \frac{1}{2|\mathcal{B}_1|} \int_{\mathcal{B}_1} |\mathbf{v}|^2 d\mathbf{x} - \frac{1}{2}|\mathbf{x}_1'|^2,$$

$$\mathscr{E}_{\text{tidal}}^1 := \frac{1}{2|\mathcal{B}_1|} \int_{\mathcal{B}_1} \psi_1 d\mathbf{x} + \frac{3GM}{5R}.$$

Proof of Theorem 1.1. The existence part of the theorem was already proved in Corollary 3.28. We now prove (14). First,

(155)

$$|\mathcal{B}_1| \mathscr{E}_{\text{tidal}}^0 = \frac{1}{2} \int_{\mathcal{B}_1} \left(|\mathbf{v}|^2 - |\mathbf{x}_1'|^2 \right) d\mathbf{x} = \frac{1}{2} \int_{\mathcal{B}_1} \left(|\mathbf{v} - \mathbf{x}_1' + \mathbf{x}_1'|^2 - |\mathbf{x}_1'|^2 \right) d\mathbf{x}$$

$$= \frac{1}{2} \int_{\mathcal{B}_1} |\mathbf{v} - \mathbf{x}_1'|^2 d\mathbf{x} + \int_{\mathcal{B}_1} \mathbf{x}_1' \cdot (\mathbf{v} - \mathbf{x}_1') d\mathbf{x} = \frac{1}{2} \int_{\mathcal{B}_1} |\mathbf{v} - \mathbf{x}_1'|^2 d\mathbf{x}$$

$$= \frac{1}{2} \int_{\mathcal{B}_1} |\nabla(\phi - \mathring{\phi})|^2 d\mathbf{x} = \frac{1}{2} \int_{\partial \mathcal{B}_1} (\phi - \mathring{\phi}) \nabla_{\mathbf{n}} (\phi - \mathring{\phi}) dS$$

$$= -\frac{1}{2}\int_{\partial\mathcal{B}_1}(\phi - \mathring{\phi})\mathbf{n}\cdot(\mathbf{v} - \mathbf{x}_1')dS = -\frac{1}{2}\int_{\partial\mathcal{B}_1}(\phi - \mathring{\phi})(n\cdot\zeta_t)dS$$

$$= -\frac{1}{4R}\int_{\partial\mathcal{B}_1}(\phi - \mathring{\phi})\tilde{h}_t dS - \frac{1}{2}\int_{\partial\mathcal{B}_1}(\phi - \mathring{\phi})\left(\left(n - \frac{1}{R}\zeta\right)\cdot\zeta_t\right)dS$$

$$= -\frac{1}{2}\int_{S_R}(\phi - \mathring{\phi})h_t dS + I,$$

where

$$I := -\frac{1}{2}\int_{S_R}(\phi - \mathring{\phi})h_t\left(|N||\mathring{g}|^{-1} - 1\right)dS - \frac{1}{2R}\int_{\partial\mathcal{B}_1}(\phi - \mathring{\phi})h\,h_t dS$$

$$- \frac{1}{2}\int_{\partial\mathcal{B}_1}(\phi - \mathring{\phi})\left(\left(n - \frac{1}{R}\zeta\right)\cdot\zeta_t\right)dS.$$

To understand the contribution of the main term we let $\mathbf{P}_\ell$ denote the projection on the ℓth eigenspace, $\mathcal{Y}_\ell$, of $\mathring{\Delta}$ and use (141) to write

$$\mathbf{D}(\phi - \mathring{\phi}) = -h_t - \frac{1}{R}hh_t + (R^{-1}\zeta - n)\cdot u + (\mathbf{D} - \nabla_n)(\phi - \mathring{\phi}).$$

Therefore,

(156)
$$\phi - \mathring{\phi} = \frac{2}{R}\mathbf{S}(I - \mathbf{K})^{-1}(Rh_t + hh_t + (Rn - \zeta)\cdot u - R(\mathbf{D} - \nabla_n)(\phi - \mathring{\phi}))$$

$$= -\frac{1}{2}\mathbf{P}_2(Rh_t + hh_t + (Rn - \zeta)\cdot u - R(\mathbf{D} - \nabla_n)(\phi - \mathring{\phi}))$$

$$+ \frac{2}{R}\mathbf{S}(I - \mathbf{K})^{-1}(I - \mathbf{P}_2)(Rh_t + hh_t + (Rn - \zeta)\cdot u$$

$$- R(\mathbf{D} - \nabla_n)(\phi - \mathring{\phi}))$$

$$= -\frac{R}{2}\partial_t h_2 + II,$$

where

$$II := -\frac{1}{2}\mathbf{P}_2(hh_t + (Rn - \zeta)\cdot u - R(\mathbf{D} - \nabla_n)(\phi - \mathring{\phi}))$$

$$+ \frac{2}{R}\mathbf{S}(I - \mathbf{K})^{-1}(I - \mathbf{P}_2)(Rh_t + hh_t + (Rn - \zeta)\cdot u$$

$$- R(\mathbf{D} - \nabla_n)(\phi - \mathring{\phi})).$$

Combining with (155) we get

$$|\mathcal{B}_1|\mathscr{E}^0_{\text{tidal}} = \frac{R}{4}\int_{S_R}|\partial_t h_2|^2 dS + I - \frac{1}{2}\int_{S_R} II \times h_t dS.$$

It follows from Propositions 4.5, B.2, B.3, 3.27, Corollary B.6, Lemma 4.3, and (141), that for some universal constant $C_0 > 0$,

$$(157) \qquad \mathscr{E}^0_{\text{tidal}}(t) \approx R^{-2}\|\partial_t h\|^2_{L^2(S_R)} \geq C_0\eta^8(t)|v_1(t)|^2.$$

Here the last estimate is valid for $r \in (r_0, 10r_0)$. For $\mathscr{E}^1_{\text{tidal}}$ we argue differently. First note that since $\mathcal{B}_1$ is a ball of radius R at $t = T_0$,

$$(158) \qquad \mathscr{E}^1_{\text{tidal}}(t) = \int_{T_0}^{t}\frac{d}{ds}\left(\frac{1}{2|\mathcal{B}_1|}\int_{\mathcal{B}_1(s)}\boldsymbol{\psi}_1(s,\mathbf{x})d\mathbf{x}\right)ds.$$

Recall that acceleration of $\mathcal{B}_1$ due to the self-gravitational force from $\mathcal{B}_1$ is zero, that is,

$$\int_{\mathcal{B}_1}\nabla\boldsymbol{\psi}_1 d\mathbf{x} = 0.$$

Using this observation, (152), (134),and the curl and divergence free properties of $\mathbf{v}$ we compute

$$\frac{d}{dt}\left(\frac{1}{2|\mathcal{B}_1|}\int_{\mathcal{B}_1}\boldsymbol{\psi}_1 d\mathbf{x}\right)$$

$$= \frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\mathbf{v}\cdot\nabla\boldsymbol{\psi}_1 d\mathbf{x} = \frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}(\mathbf{v} - \mathbf{x}'_1)\cdot\nabla\boldsymbol{\psi}_1 d\mathbf{x}$$

$$= \frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\nabla\cdot(\boldsymbol{\psi}_1(\mathbf{v} - \mathbf{x}'_1))d\mathbf{x}$$

$$= \frac{1}{|\mathcal{B}_1|}\int_{\partial\mathcal{B}_1}\zeta_t\cdot n\psi_1 dS = \frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\zeta_t\cdot n\left(\psi_1 + \frac{GM}{R}\right)dS$$

$$= \frac{R^{-1}}{2|\mathcal{B}_1|}\int_{\mathcal{B}_1}\partial_t\tilde{h}\left(\psi_1 + \frac{GM}{R}\right)dS$$

$$\quad + \frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\zeta_t\cdot(n - R^{-1}\zeta)\left(\psi_1 + \frac{GM}{R}\right)dS$$

$$= \frac{1}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}\partial_t h\left(\psi_1 + \frac{GM}{R}\right)dS + \frac{R^{-1}}{|\mathcal{B}_1|}\int_{\mathcal{B}_1}h\,\partial_t h\left(\psi_1 + \frac{GM}{R}\right)dS$$

$$+ \frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \zeta_t \cdot (n - R^{-1}\zeta) \left(\psi_1 + \frac{GM}{R} \right) dS$$
$$= \frac{\mathrm{g}}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \partial_t h (I - 3K_{\partial \mathcal{B}_1}) h \, dS + III,$$

where

$$III := \frac{R^{-1}}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} h \, \partial_t h \left(\psi_1 + \frac{GM}{R} \right) dS$$
$$+ \frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \zeta_t \cdot (n - R^{-1}\zeta) \left(\psi_1 + \frac{GM}{R} \right) dS + \frac{1}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \mathfrak{R}_3 \partial_t h \, dS.$$

Letting

$$IV := \frac{\mathrm{g}}{|\mathcal{B}_1|} \int_{\mathcal{B}_1} \partial_t h (I - 3K_{\partial \mathcal{B}_1}) h \, dS - \frac{\mathrm{g}}{|\mathcal{B}_1|} \int_{S_R} \partial_t h_2 (I - 3\mathbf{K}) h_2 \, dS,$$

and using Proposition A.5 we arrive at

$$\frac{d}{dt} \left(\frac{1}{2|\mathcal{B}_1|} \int_{\mathcal{B}_1} \boldsymbol{\psi}_1 d\mathbf{x} \right) = \frac{\mathrm{g}}{5|\mathcal{B}_1|} \partial_t \|h_2\|_{L^2(S_R)}^2 + III + IV.$$

Inserting this into (158) and applying Proposition 4.5, it follows as in the proof of (157) that for some universal constant $C_0 > 0$

$$\mathscr{E}^1_{\mathrm{tidal}}(t) \approx \frac{GM}{R^5} \|h(t)\|_{L^2(S_R)}^2 \geq C_0 \frac{GM}{R} \eta^6.$$

This completes the proof of (14). On the other hand, using the notation introduced in Subsection 3.2,

$$\tilde{\mathscr{E}} = \lim_{t \to T_0} \tilde{\mathscr{E}} = \frac{1}{2} c_0^2 \frac{GM}{R} \beta^{-\frac{12}{7}}.$$

In view of Proposition 3.13, if t^* is such that $r_1(t^*) = 3c_0^2 R \beta^{\frac{2}{7}}$, and if c_0 is sufficiently small, then $\widetilde{\mathscr{E}_{\mathrm{orbital}}}(t^*) = \tilde{\mathscr{E}} - \widetilde{\mathscr{E}_{\mathrm{tidal}}}(t^*) \leq -c_1 \frac{GM}{R} \eta^6(t^*)$ for some positive constant $c_1 > 0$, completing the proof of the theorem. $\square$

Appendix A. Clifford Analysis and Layered Potentials

A.1. The Clifford Algebra $C\ell_{0,2}(\mathbb{R})$

In this appendix we recall basic algebraic properties of the Clifford algebra $\mathcal{C} := C\ell_{0,2}(\mathbb{R})$. We refer the reader to [7] for a much more complete treat-

ment, and to [16, 17] and the references therein for earlier use of Clifford analysis in study of incompressible free boundary problems. The algebra $\mathcal{C}$ is the associative algebra generated by the four basis elements $\{1, e_1, e_2, e_3\}$ over $\mathbb{R}$, satisfying the relations

$$
\text{(159)} \quad
\begin{aligned}
&1e_i = e_i, \quad e_i e_j = -e_j e_i, \quad i \neq j, \ i, j = 1, 2, 3, \quad e_1 e_2 = e_3, \\
&e_i^2 = -1, \ i = 1, 2, 3.
\end{aligned}
$$

Every element $c \in \mathcal{C}$ has a unique representation $\xi = \xi^0 + \sum_{i=1}^{3} \xi^i e_i$. Sometimes we also write e_0 for 1 so that $\xi = \sum_{i=0}^{3} \xi^i e_i$. We call ξ^0 the real part of ξ and denote it by $\mathrm{Re}\xi$, and $\sum_{i=1}^{3} \xi^i e_i$ the vector part of ξ and denote it by $\mathrm{Vec}\,\xi$. An element $\xi \in \mathcal{C}$ is referred to as a Clifford vector. If $\mathrm{Re}\xi = 0$ we say that ξ is a vector (or pure vector) and if $\mathrm{Vec}\,\xi = 0$ we say that ξ is a real number or a scalar (or pure scalar). We identify the real numbers $\mathbb{R}$ with Clifford numbers using the relation $a \mapsto a1$, and identify vectors in $\mathbb{R}^3$ with Clifford vectors using the relation $\mathbf{v}^i \mathbf{e}_i \mapsto \mathbf{v}^i e_i$. Here $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis for $\mathbb{R}^3$. The dot product of two Clifford vectors η and ξ is defined as

$$
\xi \cdot \eta := \mathrm{Re}\xi \mathrm{Re}\eta + \mathrm{Vec}\,\xi \cdot \mathrm{Vec}\,\eta = \sum_{i=0}^{3} \xi^i \eta^i.
$$

If ξ and η are vectors, then, in view of (159), the usual (or Clifford) product of ξ and η is

$$
\xi\eta = -\xi \cdot \eta + \xi \times \eta.
$$

A.2. Layered Potentials and the Hilbert Transform

The Clifford differentiation operator $\mathcal{D}$, acting on Clifford algebra-valued functions, is defines as

$$
\mathcal{D} = \sum_{i=1}^{3} \partial_{x^i} e_i,
$$

where $x = (x^1, x^2, x^3)$ are the usual rectangular coordinates in $\mathbb{R}^3$. If $\mathbf{f}$ is a Clifford algebra-valued function we denote the real part of $\mathbf{f}$ by $\mathring{\mathbf{f}} := \mathrm{Re}\mathbf{f}$, and the vector part of $\mathbf{f}$ by $\vec{\mathbf{f}} := \mathrm{Vec}\,\mathbf{f} = \sum_{i=1}^{3} \mathbf{f}^i e_i$. The Clifford derivative of $\mathbf{f}$ then satisfies

$$
\mathcal{D}\mathbf{f} = \nabla\mathring{\mathbf{f}} + \nabla \times \vec{\mathbf{f}} - \nabla \cdot \vec{\mathbf{f}}.
$$

Moreover, by direct computation we see that $\mathcal{D}^2\mathbf{f} = -\Delta\mathbf{f} = -\sum_{i=0}^{3}\Delta\mathbf{f}^i e_i$.

Let Ω be a C^2, bounded, and simply-connected domain in $\mathbb{R}^3$ with boundary Σ and complement Ω^c. We denote the exterior normal vector to Σ by n and the induced volume form on Σ by dS. We say that a function $\mathbf{f}$ defined on Ω is Clifford analytic, if $\mathcal{D}\mathbf{f} = 0$. Note that if $\mathbf{f}$ is vector-valued, this is equivalent to $\mathbf{f}$ being curl and divergence free. In general the computation above shows that the components of a Clifford analytic function are harmonic. The following simple observations are used many times in this work. For any function $\mathbf{f}$ defined in $\overline{\Omega}$,

$$n\mathcal{D}\mathbf{f} = -n\cdot\nabla\mathbf{f} + n\times\nabla\mathbf{f}.$$

Since the components of a Clifford analytic function are harmonic, it follows that if f is the restriction to Σ of a Clifford analytic function $\mathbf{f}$ in Ω, then

$$(160) \qquad \nabla_n f = n\times\nabla f,$$

where ∇_n denotes the Dirichlet-Neumann map of Ω. Moreover, writing $f = \vec{f} + \mathring{f}$, with $\mathring{f} = \mathrm{Re}f = \mathring{\mathbf{f}}|_\Sigma$ and $\vec{f} = \mathrm{Vec}\, f = \vec{\mathbf{f}}|_\Sigma$, since $\nabla\times\vec{\mathbf{f}} = \nabla\mathring{\mathbf{f}}$ in Ω, we have (the same identity holds if Ω is unbounded but $\mathbf{f}$ decays at infinity)

$$(161) \quad \frac{1}{|N|}(\xi_\beta\cdot\vec{f}_\alpha - \xi_\alpha\cdot\vec{f}_\beta) = \frac{1}{|N|}\xi_\alpha^i\xi_\beta^j(\partial_i\vec{f}^j - \partial_j\vec{f}^i) = n\cdot\nabla\mathring{f} = \nabla_n\mathring{f}.$$

In particular, when f is a vector-valued Clifford analytic function

$$(162) \qquad n\times\nabla f = \frac{1}{|N|}(\xi_\beta\times f_\alpha - \xi_\alpha\times f_\beta).$$

We next turn to the definition of the Hilbert transform and layered potentials. Let $\Gamma(x) := -\frac{1}{4\pi|x|}$, $x\in\mathbb{R}^3$, be the fundamental solution of the Laplace equation in $\mathbb{R}^3$, and let

$$K(x) := -2\mathcal{D}\Gamma(x) = -\frac{1}{2\pi}\frac{x}{|x|^3}, \qquad x\in\mathbb{R}^3.$$

For a Clifford algebra-valued function f defined on Σ we define the Hilbert transform of f as

$$(163) \qquad H_\Sigma f(\xi) = \mathrm{p.v.}\int_\Sigma K(\xi'-\xi)n(\xi')f(\xi')dS(\xi'), \qquad \xi\in\Sigma.$$

Remark A.1. We often use the shorthand $\int_\Sigma K n' f' dS'$ for the integral in (163). More generally, when considering integrals with variable of integration ξ', we write f' instead of $f(\xi')$ and dS' instead of $dS(\xi')$.

Similarly, the Cauchy integral of f is defined for interior points $\eta \in \Omega$ as

$$C_\Sigma f(\eta) := \frac{1}{2} \int_\Sigma K(\xi' - \eta) n(\xi') f(\xi') dS(\xi'), \qquad \eta \in \Omega.$$

The Hilbert transform satisfies $H_\Sigma^2 = \mathrm{Id}$ and $H_\Sigma 1 = 1$. The following theorem summarizes the relation between the Hilbert transform and Clifford analyticity.

Theorem A.2. *[See [7] Chapter 2 and [16] Remark 1] If f is the restriction to Σ of a Clifford analytic function $\mathbf{f}$ defined in a neighborhood of Ω, then $\mathbf{f}(\eta) = C_\Sigma f(\eta)$ for every $\eta \in \Omega$. Conversely, if $f \in C^1(\Sigma, \mathcal{C})$, then $C_\Sigma f$ is Clifford analytic in Ω and continuous on $\overline{\Omega}$. Moreover, if $f \in C^1(\Sigma, \mathcal{C})$, then*

$$C_\Sigma f(\xi) = \frac{1}{2} f(\xi) + \frac{1}{2} H_\Sigma f(\xi), \qquad \xi \in \Sigma.$$

Finally, f is the restriction to Σ of a Clifford analytic function $\mathbf{f} \in C^0(\overline{\Omega}, \mathcal{C})$, if and only if $f = H_\Sigma f$. Similarly, f is the restriction to Σ of a Clifford analytic function $\mathbf{f} \in C^0(\overline{\Omega^c}, \mathcal{C})$ with $\lim_{|x| \to \infty} \mathbf{f}(x) = 0$, if and only if $f = -H_\Sigma f$.

The Hilbert transform is also related to the classical layered potentials for the Laplace operator on Ω. Recall that the double-layered potential, K_Σ, and the single-layered potential, S_Σ, are defined as

(164)
$$K_\Sigma f(\xi) = -\mathrm{p.v.} \int_\Sigma K(\xi' - \xi) \cdot n(\xi') f(\xi') dS(\xi'), \qquad \xi \in \Sigma,$$
$$S_\Sigma f(\xi) = -\frac{1}{4\pi} \int_\Sigma \frac{f(\xi')}{|\xi' - \xi|} dS(\xi'), \qquad \xi \in \mathbb{R}^3,$$

where f is a real-valued function defined on Σ. By direct inspection, we have

$$K_\Sigma f = \mathrm{Re} H_\Sigma f \quad \text{and} \quad \mathcal{D} S_\Sigma f = -C_\Sigma (nf).$$

We define the formal $L^2(\Sigma, dS)$ adjoint of H_Σ by $H_\Sigma^* := n H_\Sigma n$. Then the $L^2(\Sigma, dS)$ adjoint of K_Σ satisfies

(165)
$$K_\Sigma^* f(\xi) = \mathrm{p.v.} \int_\Sigma n(\xi) \cdot K(\xi' - \xi) f(\xi') dS(\xi') = \mathrm{Re} H_\Sigma^* f(\xi), \qquad \xi \in \Sigma.$$

As shown in [9, 10, 14], the operator $I + K_\Sigma : L^2(\Sigma, dS) \to L^2(\Sigma, dS)$ and its adjoint $I + K_\Sigma^* : L^2(\Sigma, dS) \to L^2(\Sigma, dS)$ are bounded and invertible.

Theorem A.3. *For any bounded domain $\Omega \subseteq \mathbb{R}^3$ with Lipschitz boundary Σ, the operators $I + K_\Sigma$, $I + K_\Sigma^* : L^2(\Sigma, dS) \to L^2(\Sigma, dS)$ are bounded and invertible. Similarly, with $L_0^2(\Sigma, dS)$ denoting the space of L^2 functions with zero average on Σ, the operators $I - K_\Sigma$, $I - K_\Sigma^* : L_0^2(\Sigma, dS) \to L_0^2(\Sigma, dS)$ are bounded and invertible. Moreover the operator bounds $\|(I \pm K_\Sigma^*)^{-1}\|_{2,2}$ and $\|(I \pm K_\Sigma)^{-1}\|_{2,2}$ depend only on the Lipschitz constant for Σ.*

The double-layered potential K_Σ is related to the Dirichlet problem on Ω in the following way. If $f \in L^2(\Sigma, dS)$ then the unique solution to the Dirichlet problem

$$\Delta u = 0 \quad in \ \Omega, \qquad u|_\Sigma = f,$$

is given by

$$u(\eta) = \frac{1}{2\pi} \int_\Sigma \frac{(\xi - \eta) \cdot n(\xi)}{|\xi - \eta|^3} (I + K_\Sigma)^{-1} f(\xi) dS(\xi), \qquad \eta \in \Omega.$$

See [9], [10], or [14] for a proof of this fact. Similarly, if $f \in L^2(\Sigma, dS)$ then the unique solution to the Neumann problem

$$\Delta u = 0 \quad in \ \Omega, \qquad \nabla_n u|_\Sigma = f,$$

is given by

$$(166) \quad u(\eta) = \frac{1}{2\pi} \int_\Sigma \frac{1}{|\xi - \eta|} (I - K_\Sigma^*)^{-1} f(\xi) dS(\xi) = -2 S_\Sigma (I - K_\Sigma^*)^{-1} f(\eta),$$
$$\eta \in \Omega.$$

See [9], [10], or [14] for a proof. Note that since $f = \nabla_n u$, $\int_\Sigma f dS = \int_\Omega \Delta u dx = 0$, so f belongs to the domain of $(I - K_\Sigma^*)^{-1}$. In the following lemma we also provide an expression for the Dirichlet-Neumann map ∇_n of Σ in terms of layered potentials.

Lemma A.4. *For any differentiable function f on Σ, the Dirichlet-Neumann map, ∇_n, satisfies*

$$\nabla_n f = (I + K_\Sigma^*)^{-1} \, \mathrm{p.v.} \int_\Sigma (n \times K) \cdot (n' \times \nabla) f' dS'.$$

Moreover, if $\mathbf{f}$ *denotes the harmonic extension of* f *to the interior,* Ω, *and* $\mathcal{D}f$ *the restriction of* $\mathcal{D}\mathbf{f}$ *to* Σ, *then*

$$\mathcal{D}f = (I + H_\Sigma)(n\,(I - K_\Sigma^*)^{-1}\nabla_n f).$$

Proof. See [16], equation (3.13), for the proof of this statement. □

Finally, we restrict attention to the case where Ω is a ball in $\mathbb{R}^3$, $\Omega = B_1(0)$, with the standard sphere as boundary, $\partial\Omega = \mathbb{S}^2$, and present a formula for the double-layered potential $\mathbb{K} := K_{\mathbb{S}^2}$ in terms of spherical harmonics. We start by fixing our notation for spherical harmonics. Recall that $L^2(\mathbb{S}^2)$ admits a direct sum decomposition $L^2(\mathbb{S}^2) = \bigoplus_{\ell=0}^\infty \mathring{\mathcal{Y}}_\ell$ into the eigenspaces of the Laplacian $\mathring{\Delta}$ on $\mathbb{S}^2$. In other words, any smooth function $f \in L^2(\mathbb{S}^2)$ admits a unique decomposition $f = \sum_{\ell=0}^\infty f_\ell$, $f_\ell \in \mathring{\mathcal{Y}}_\ell$, such that

$$-\mathring{\Delta}f_\ell = \ell(\ell+1)f_\ell.$$

The ℓth eigenspace $\mathring{\mathcal{Y}}_\ell$ has dimension $2\ell + 1$, and using the usual polar coordinates $\xi = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$ for $\xi \in \mathbb{S}^2$, an orthonormal basis for $\mathring{\mathcal{Y}}_\ell$ is given by

$$Y_\ell^m(\xi) = Y_\ell^m(\theta, \varphi) = (-1)^m\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)e^{im\varphi},$$
$$m = -\ell, \dots, \ell,$$

where P_ℓ^m are the Legendre polynomials $P_\ell^m(x) = \frac{1}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2 - 1)^\ell$. Using this basis we can decompose a function $f \in L^2(\mathbb{S}^2)$ as

$$f(\xi) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell f_m^\ell Y_\ell^m(\xi), \qquad f_m^\ell := \int_{\mathbb{S}^2} f(\xi')\overline{Y_\ell^m}(\xi')dS(\xi').$$

Our interest is in understanding the action of the double-layered potential $\mathbb{K}$ on $\mathring{\mathcal{Y}}_\ell$.

Proposition A.5. *Let* $\mathbb{K}$ *and* $\mathbb{S}$ *be the double-layered and single-layered potentials on* $\mathbb{S}^2$, *respectively.* $\mathbb{K}$ *is self-adjoint, that is,* $\mathbb{K}^* = \mathbb{K}$, *and for any* $f_\ell \in \mathring{\mathcal{Y}}_\ell$

$$\mathbb{K}f_\ell = -\mathbb{S}f_\ell = (1 - 4\mathring{\Delta})^{-\frac{1}{2}}f_\ell := \frac{1}{2\ell+1}f_\ell.$$

Proof. The self-adjointness of $\mathbb{K}$ follows by inspection of formulas (164) and (165), and the observation that when $\Sigma = \mathbb{S}^2$ the exterior normal is given by $n(\xi) = \xi$. Using this same observation we get

$$\mathbb{K}f_\ell(\xi) = \frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{f_\ell(\xi')}{|\xi'-\xi|}dS(\xi') = -\mathbb{S}f_\ell(\xi).$$

Since for any $\xi,\xi' \in \mathbb{S}^2$, $\frac{d}{dr}|\xi'-r\xi|^{-1} = |\xi'-r\xi|^{-3}(\xi'\cdot\xi - r) \leq 0$, by the dominate convergence theorem

$$\mathbb{K}f_\ell(\xi) = \lim_{r\to 1^+}\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{f_\ell(\xi')}{|\xi'-r\xi|}dS(\xi').$$

We now use the following representation for $|\xi'-r\xi|$, valid for any $r > 1$:

$$\frac{1}{|\xi'-r\xi|} = \frac{4\pi}{r}\sum_{k=0}^\infty\frac{r^{-k}}{2k+1}\sum_{m=-k}^k Y_k^m(\xi)\overline{Y_k^m(\xi')}.$$

This relation can be verified by direct computation using properties of Legendre polynomials. See for instance [1]. Plugging this back into the representation for $\mathbb{K}$ above and using the fact that $f_\ell \in \mathring{\mathcal{Y}}_\ell$, with $f_\ell(\xi) = \sum_{m=-\ell}^\ell f_m^\ell Y_\ell^m(\xi)$, we get

$$\mathbb{K}f_\ell(\xi) = \lim_{r\to 1^+}\frac{r^{-\ell-1}}{2\ell+1}\sum_{m=-\ell}^\ell f_m^\ell Y_\ell^m(\xi) = \frac{1}{2\ell+1}f_\ell. \qquad \square$$

The analysis above can be extended to S_R, the sphere of radius R. We denote the ℓth eigenspace of the Laplacian $\mathring{\Delta}$ of S_R by $\mathcal{Y}_\ell$, so that $\mathring{\Delta}f_\ell = -R^{-2}\ell(\ell+1)f_\ell$ for $f_\ell \in \mathcal{Y}_\ell$. Let $\mathbf{K}$ and $\mathbf{S}$ be the double and single layered potentials on S_R, respectively. The following proposition is the analogue of Proposition A.5 for S_R.

Proposition A.6. *Let $\mathbf{K}$ and $\mathbf{S}$ be the double-layered and single-layered potentials on S_R, respectively. Then $\mathbf{K}$ and $\mathbf{S}$ are self-adjoint and $\mathbf{S} = -R\mathbf{K}$. Moreover, for any $f_\ell \in \mathcal{Y}_\ell$ we have*

$$\mathbf{K} = \frac{1}{2\ell+1}f_\ell.$$

Proof. The proof is the same as that of Proposition A.5. $\qquad \square$

A.3. Commutator Formulas

We present a number of commutator formulas that are frequently used in this paper. The proofs of these formulas are similar to the ones in the case where the domain is diffeomorphic to $\mathbb{R}^2$ where they were derived by Wu in [16, 17].

The following important identity is from [16] (see equation (3.5) on page 453): Suppose $\xi, \xi', \eta \in \mathbb{R}^3$ are arbitrary vectors with $\xi \neq \xi'$, and let $K = K(\xi' - \xi)$. Then in local coordinates (α, β) on Σ

(167)
$$-(\eta \cdot \nabla)K(\xi'_{\alpha'} \times \xi'_{\beta'}) + (\xi'_{\alpha'} \cdot \nabla)K(\eta \times \xi'_{\beta'}) + (\xi'_{\beta'} \cdot \nabla)K(\xi'_{\alpha'} \times \eta) = 0.$$

The proof of this identity is identical to the one in [16] so we omit it. Before stating the commutator formulas for the Hilbert transform H_Σ we recall the following notation from (60):

$$Q(f, g) = \frac{1}{|N|}(f_\alpha g_\beta - f_\beta g_\alpha),$$

where (α, β) are orientation preserving local coordinates on Σ.

Lemma A.7. *Let* $f \in C^1(\mathbb{R} \times \Sigma)$ *be a Clifford algebra-valued function, and let* $a \in C^0(\mathbb{R} \times \Sigma)$ *be real-valued. Then*

$$[\partial_t, H_\Sigma]f = \int_\Sigma K(\xi' - \xi)(\xi_t - \xi'_t) \times (n' \times \nabla f')dS',$$

$$[an \times \nabla, H_\Sigma]f = \int_\Sigma K(\xi' - \xi)(an - a'n') \times (n' \times \nabla f')dS',$$

$$[\partial_t^2, H_\Sigma]f = \int_\Sigma K(\xi' - \xi)(\xi_{tt} - \xi'_{tt}) \times (n' \times \nabla f')dS'$$

$$+ 2\int_\Sigma K(\xi' - \xi)(\xi_t - \xi'_t) \times (n' \times \nabla f'_t)dS'$$

$$+ \int_\Sigma K(\xi' - \xi)(\xi_t - \xi'_t)Q(f', \xi'_t)dS'$$

$$+ \int_\Sigma \partial_t K(\xi' - \xi)(\xi_t - \xi'_t) \times (n' \times \nabla f')dS.$$

$$[\partial_t, \nabla_n]f = (I + K_\Sigma^*)^{-1}\mathrm{Re}\left\{-n_t H_\Sigma \mathcal{D}f - n[\partial_t, H_\Sigma]\mathcal{D}f + nH_\Sigma(n_t n \,\mathcal{D}f)\right\}$$

$$+ (I + K_\Sigma^*)^{-1}\int_\Sigma n \times K(\xi' - \xi)$$

$$\cdot \left(Q(f', u') - \frac{\partial_t |N'|}{|N'|} n' \times \nabla f' \right) dS'.$$

Proof. The proof is the same as the proof of Lemma 1.2 in [17]. The last statement has the same proof as equation (3.16) in Lemma 3.2 of [16]. Note that the analogues of the integration by parts in the proof of Lemma 1.2 in [17] in our case can be carried out using the invariant formulation in Lemma 3.2. □

Remark A.8. Expressions of the form $\eta \times (n \times \nabla f)$ in the identities above should be understood componentwise. That is,

$$\eta \times (n \times \nabla f) = \frac{1}{|N|} \left((\eta \times \xi_\beta) f_\alpha - (\eta \times \xi_\alpha) f_\beta \right).$$

To show that the commutator between $n \times \nabla$ and H_Σ is small in the case where Σ is close to a round sphere, we need an additional identity. This identity which is from the proof of Proposition 2.2 in [17] states that for any differentiable function f

$$\int_\Sigma K(\xi' - \xi)(\xi' - \xi) \times (n' \times \nabla f') dS' = 0.$$

We omit the simple proof which uses (167) and integration by parts, and can be found in [17], Proposition 2.2. Note that when Σ is a round sphere the vector ξ points in the same direction as the normal vector, so comparing with the expression for $[n \times \nabla, H_\Sigma]f$ in Lemma A.7 we see that $[n \times \nabla, H_{S_R}]f = 0$. More generally, for arbitrary Σ we get

(168)
$$[n \times \nabla, H_\Sigma]f = \frac{1}{R} \int_\Sigma K(\xi' - \xi)((Rn - \xi) - (Rn' - \xi')) \times (n' \times \nabla f') dS'.$$

Appendix B. Analytic Preparations

In this appendix we collect some general estimates that are used in the paper. In this section $\overset{\circ}{\nabla}$ denotes the covariant differentiation operator with respect to the standard metric on S_R and Ω_i denotes the rotational vectorfield about the axis $\mathbf{e}_i$ in $\mathbb{R}^3$. Ω_i is tangent to S_R and in coordinates is given by $\mathbf{x}^k \partial_\ell - \mathbf{x}^\ell \partial_k$ for some k and ℓ. Since the three vectorfields Ω_1, Ω_2, and Ω_3 span the

tangent space to S_R at each point, for any function $f : S_R \to \mathbb{R}$ we have the pointwise estimate

$$|\not\nabla f| \lesssim \frac{1}{R} \sum_{i=1}^{3} |\Omega_i f|.$$

The following lemma is just the standard Sobolev estimate on S_R which we record for reference.

Lemma B.1. *For any $f \in H^2(S_R)$*

$$\|f\|_{L^\infty(S_R)} \lesssim \frac{1}{R} \|f\|_{L^2(S_R)} + \frac{1}{R} \sum_{i=1}^{3} \|\Omega_i f\|_{L^2(S_R)} + \frac{1}{R} \sum_{i,j=1}^{3} \|\Omega_i \Omega_j f\|_{L^2(S_R)}.$$

When the axis of rotation is not important we simply write Ωf instead of $\Omega_i f$, and $\Omega^2 f$ instead of $\Omega_i \Omega_j f$, etc. We next turn to estimates on singular integral operators, due to Calderon, Coifman, David, McIntosh, and Meyer (see [3], [5], and [6]). The general setup is as as follows. Let $J : S_R \to \mathbb{R}^k$, $F : \mathbb{R}^k \to \mathbb{R}$, and $A : S_R \to \mathbb{R}$ be smooth functions. We want to estimate singular and nonsingular integrals of the following forms:

(169)

$$C_1 f(p) := \text{p.v.} \int_{S_R} F\left(\frac{J(p) - J(q)}{|p - q|}\right) \frac{\prod_{i=1}^{N}(A_i(p) - A_i(q))}{|p - q|^{N+2}} f(q) dS(q),$$

where dS denotes the surface measure on S_R, and where we assume that the kernel

$$k_1(p,q) = F\left(\frac{J(p) - J(q)}{|p - q|}\right) \frac{\prod_{i=1}^{N}(A_i(p) - A_i(q))}{|p - q|^{N+2}}$$

is odd, that is, $k_1(p,q) = -k_1(q,p)$. Similarly,

(170)

$$C_2 f(p) := \text{p.v.} \int_{S_R} F\left(\frac{J(p) - J(q)}{|p - q|}\right) \frac{\prod_{i=1}^{N}(A_i(p) - A_i(q))}{|p - q|^{N+1}} \mathfrak{D} f(q) dS(q),$$

where we assume that the kernel

$$k_2(p,q) = F\left(\frac{J(p) - J(q)}{|p - q|}\right) \frac{\prod_{i=1}^{N}(A_i(p) - A_i(q))}{|p - q|^{N+1}}$$

is even, that is, $k_2(p, q) = k_2(q, p)$. Here $\mathfrak{D}f = \Omega f$ where Ω is a rotational vectorfield defined earlier, but we have chosen the notation $\mathfrak{D}f$ instead of Ωf for consistency with the main body of the article. To prove the desired estimates we fix a finite covering $\{U_\alpha\}$ of S_R with geodesic balls of radius $r \ll R$, such that if $U_\alpha \cap U_\beta = \emptyset$, then $\inf_{\{x \in U_\alpha,\ y \in U_\beta\}} |x - y| > cR$ for some absolute constant c. We let $\{\chi_\alpha\}$ be a smooth partition of unity with respect to the finite covering $\{U_\alpha\}$. Before stating the main estimates we remark that with $\mathbf{d}$ denoting the geodesic distance on S_R we have

$$|p - q| \leq \mathbf{d}(p, q) \leq C|p - q|, \qquad \forall p, q \in S_R,$$

and for some absolute constant $C > 0$. We now state the estimate on C_1. Recall that $\not\nabla$ denotes the standard covariant differentiation operator on S_R.

Proposition B.2. *With the same notation as* (169)

$$(171) \quad \|C_1 f\|_{L^2(S_R)} \leq C \prod_{i=1}^{N} \left(\|\not\nabla A_i\|_{L^\infty(S_R)} + R^{-1} \|A_i\|_{L^\infty(S_R)} \right) \|f\|_{L^2(S_R)},$$

and

$$\|C_1 f\|_{L^2(S_R)} \leq C \left(\|\not\nabla A_1\|_{L^2(S_R)} + R^{-1} \|A_1\|_{L^2(S_R)} \right)$$

$$(172) \qquad\qquad \times \prod_{i=2}^{N} \left(\|\not\nabla A_i\|_{L^\infty(S_R)} + R^{-1} \|A_i\|_{L^\infty(S_R)} \right) \|f\|_{L^\infty(S_R)},$$

where the constants depend on F *and* $\|\not\nabla J\|_{L^\infty}$.

Proof. For simplicity of notation we assume $N = 1$. Writing

$$C_1 f(p) = \sum_\beta \mathrm{p.v.} \int_{U_\beta} F\left(\frac{J(p) - J(q)}{|p - q|} \right) \frac{A(p) - A(q)}{|p - q|^3} \chi_\beta(q) f(q) dS(q),$$

and using the triangle inequality in $L^2(S_R)$ we estimate

$$\|C_1 f\|_{L^2(S_R)}^2$$

$$\lesssim \sum_{\alpha, \beta} \int_{U_\alpha} \chi_\alpha(p)$$

$$\times \left(\mathrm{p.v.} \int_{U_\beta} F\left(\frac{J(p) - J(q)}{|p - q|} \right) \frac{A(p) - A(q)}{|p - q|^3} \chi_\beta(q) f(q) dS(q) \right)^2 dS(p)$$

$$=: I_{\alpha\beta}.$$

We estimate $I_{\alpha\beta}$ differently according to whether U_α and U_β intersect non-trivially. If $U_\alpha \cap U_\beta = \emptyset$ then $|p - q| \gtrsim R$, so we simply have

$$I_{\alpha\beta} \lesssim R^{-4}\|A\|^2_{L^\infty(S_R)} \left(\int_{S_R} f(q)dS(q) \right)^2 \lesssim R^{-2}\|A\|^2_{L^\infty(S_R)}\|f\|^2_{L^2(S_R)},$$

in accordance with (171). Similarly

$$I_{\alpha\beta} \lesssim R^{-6} \int_{S_R} \|f\|^2_{L^\infty(S_R)} \left(\int_{S_R} A(q)dS(q) + R^2 A(p) \right)^2 dS(p)$$
$$\lesssim R^{-2}\|f\|^2_{L^\infty(S_R)}\|A\|^2_{L^2(S_R)},$$

in accordance with (172). Next, if $U_\alpha \cap U_\beta \neq \emptyset$ then we define U to be a slight enlargement of $U_\alpha \cup U_\beta$ and let $\varphi : \mathbb{B} \to U \subseteq \mathbb{R}^3$ be a diffeomorphism onto its image, satisfying

(173)
$$c_1 \leq \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \leq c_2, \quad \forall x, y \in \mathbb{B} \quad \text{and} \quad c_3 \leq |\varphi'(x)| \leq c_4, \quad \forall x \in \mathbb{B}.$$

for some absolute constants $c_1, \ldots, c_4$, where $\mathbb{B} = \{x \in \mathbb{R}^2 \text{ s.t. } |x| \leq r\}$, and $|\varphi'| := |\partial_1\varphi \times \partial_2\varphi|$. We can now use the coordinate function φ to write the integrals $I_{\alpha\beta}$ as integrals on $\mathbb{R}^2$. For this we first extend φ to a map from all of $\mathbb{R}^2$ to $\mathbb{R}^3$ in such a way that (173) holds on all of $\mathbb{R}^2$ and $\varphi(x) \notin U_\alpha \cup U_\beta$ for $x \notin \mathbb{B}$. We then have

$$I_{\alpha\beta} =$$
$$\int_{\mathbb{R}^2} \chi_\alpha(\varphi(x)) \left(\text{p.v.} \int_{\mathbb{R}^2} F\left(\frac{J(\varphi(x)) - J(\varphi(y))}{|\varphi(x) - \varphi(y)|} \right) \frac{A(\varphi(x)) - A(\varphi(y))}{|\varphi(x) - \varphi(y)|^3} \right.$$
$$\left. \chi_\beta(\varphi(y))f(\varphi(y))|\varphi'(y)|dy \right)^2 |\varphi'(x)|dx.$$

Let $\tilde{A} : \mathbb{R}^2 \to \mathbb{R}$, $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$, and $\tilde{J} : \mathbb{R}^2 \to \mathbb{R}^{k+1}$ be

$$\tilde{A} = A \circ \varphi, \quad \tilde{J} = (J \circ \varphi, \varphi), \quad \tilde{f} = |\varphi'|f \circ \varphi.$$

We also define $\tilde{F} \in C^\infty(\mathbb{R}^{k+1}, \mathbb{R})$ in such away that outside the interval $|z_{k+1}| \leq \delta$, $\delta \ll c_1$,

$$\tilde{F}(z_1, \ldots, z_{k+1}) = \frac{1}{|z_{k+1}|^3} F\left(\frac{z_1}{|z_{k+1}|}, \ldots, \frac{z_k}{|z_{k+1}|}\right).$$

It follows that

$$F\left(\frac{J(\varphi(x)) - J(\varphi(y))}{|\varphi(x) - \varphi(y)|}\right) \frac{A(\varphi(x)) - A(\varphi(y))}{|\varphi(x) - \varphi(y)|^3}$$
$$= \tilde{F}\left(\frac{\tilde{J}(x) - \tilde{J}(y)}{|x - y|}\right) \frac{\tilde{A}(x) - \tilde{A}(y)}{|z - y|^3}.$$

Since $|\varphi'|, |\chi_\alpha|, |\chi_\beta| \lesssim 1$, and $\|\nabla \tilde{A}\|_{L^p(\mathbb{R}^2)} \lesssim \|\slashed{\nabla} A\|_{L^p(S_R)}$ for $p = 2, \infty$, the contribution of $I_{\alpha\beta}$ can be bounded using Proposition 2.6 in [17]. $\qquad \square$

For $C_2 f$ we have the following estimate.

Proposition B.3. *With the same notation as* (170)

$$(174) \quad \|C_2 f\|_{L^2(S_R)} \leq C \prod_{i=1}^N \left(\|\slashed{\nabla} A_i\|_{L_\infty(S_R)} + R^{-1}\|A_i\|_{L^\infty(S_R)}\right) \|f\|_{L^2(S_R)},$$

and

$$\|C_2 f\|_{L^2(S_R)} \leq C \left(\|\slashed{\nabla} A_1\|_{L^2(S_R)} + R^{-1}\|A_1\|_{L^2(S_R)}\right)$$
$$(175) \qquad\qquad \times \prod_{i=2}^N \left(\|\slashed{\nabla} A_i\|_{L_\infty(S_R)} + R^{-1}\|A_i\|_{L^\infty(S_R)}\right) \|f\|_{L^\infty(S_R)},$$

where the constants depend on F *and* $\|\slashed{\nabla} J\|_{L^\infty}$.

Proof. This follows from Proposition B.2 and integration by parts. Note that since $\slashed{\nabla} \cdot \Omega = 0$, for any differentiable functions f and g defined on S_R we have $\int_{S_R} f\Omega g\, dS = -\int_{S_R} g\Omega f\, dS$. $\qquad \square$

To estimate the derivatives of integral operators such as C_1 and C_2 we need to find a convenient expression for these derivatives. We derive such an expression in the next lemma, both in the case where the domain is S_R and when the domain is the free surface boundary $\partial \mathcal{B}_1$. The latter is of separate interest in the paper.

Lemma B.4.   *1. Suppose $L = L(\xi, \xi')$ is a vector-valued kernel on $\partial\mathcal{B}_1 \times \partial\mathcal{B}_1$ such that $|\xi - \xi'|^2 L(\xi, \xi')$ is continuous and L is differentiable away from $\xi' = \xi$. Then for any Clifford algebra-valued differentiable function g on $\partial\mathcal{B}_1$,*

(176)
$$\mathfrak{D} \int_{\partial\mathcal{B}_1} Lg'dS' - \int_{\partial\mathcal{B}_1} L\mathfrak{D}'g'dS' = \int_{\partial\mathcal{B}_1} \left((\Omega + \Omega')L - \mathbf{e} \times L\right) g'dS'$$
$$+ \int_{\partial\mathcal{B}_1} \frac{Lg'}{|N'||g'|^{-1}}\Omega'(|N'||g'|^{-1})dS'.$$

2. Suppose $L = L(\xi, \xi')$ is a vector-valued kernel on $S_R \times S_R$ such that $|\xi - \xi'|^2 L(\xi, \xi')$ is continuous and L is differentiable away from $\xi' = \xi$. Then for any Clifford algebra-valued differentiable function g on S_R,

(177)
$$\mathfrak{D} \int_{S_R} Lg'dS' - \int_{S_R} L\mathfrak{D}'g'dS' = \int_{S_R} \left((\Omega + \Omega')L - \mathbf{e} \times L\right) g'dS'.$$

Proof. The second part is a consequence of the first in the special case where $\partial\mathcal{B}_1$ is a round sphere, so we concentrate on the first. If g' is scalar-valued, then by Lemma 2.10

$$\mathfrak{D} \int_{\partial\mathcal{B}_1} Lg'dS' = \int_{\partial\mathcal{B}'} (\Omega L - \mathbf{e} \times L)g'dS'$$
$$= \int_{\partial\mathcal{B}_1} ((\Omega + \Omega')L - \mathbf{e} \times L)g'dS' + \int_{\partial\mathcal{B}_1} L\Omega'g'dS'$$
$$+ \int_{\partial\mathcal{B}_1} \frac{Lg'}{|N'||g'|^{-1}}\Omega'(|N'||g'|^{-1})dS',$$

proving (176) when g' is scalar-valued. If g' is vector-valued, then

$$\mathfrak{D} \int_{\partial\mathcal{B}_1} Lg'dS'$$
$$= -\int_{\partial\mathcal{B}_1} (\Omega L) \cdot g'dS' + \int_{\partial\mathcal{B}_1} (\Omega L) \times g'dS' - \int_{\partial\mathcal{B}_1} \mathbf{e} \times (L \times g')dS'$$
$$= \int_{\partial\mathcal{B}_1} ((\Omega + \Omega')L)g'dS' - \int_{\partial\mathcal{B}_1} \mathbf{e} \times (L \times g')dS'$$
$$+ \int_{\partial\mathcal{B}_1} L\Omega'g'dS' + \int_{\partial\mathcal{B}_1} \frac{Lg'}{|N'||g'|^{-1}}\Omega'(|N'||g'|^{-1})dS'$$

$$= \int_{\partial\mathcal{B}_1} ((\Omega+\Omega')L)g'dS' - \int_{\partial\mathcal{B}_1} \mathbf{e}\times(L\times g')dS' + \int_{\partial\mathcal{B}_1} L(\mathbf{e}\times g')dS'$$
$$+ \int_{\partial\mathcal{B}_1} L\mathfrak{D}'g'dS' + \int_{\partial\mathcal{B}_1} \frac{Lg'}{|N'||\slashed{g}'|^{-1}}\Omega'(|N'||\slashed{g}'|^{-1})dS'$$
$$= \int_{\partial\mathcal{B}_1} ((\Omega+\Omega')L)g'dS' - \int_{\partial\mathcal{B}_1} \mathbf{e}\times(L\times g')dS' + \int_{\partial\mathcal{B}_1} g'\cdot(\mathbf{e}\times L)dS'$$
$$+ \int_{\partial\mathcal{B}_1} (L\times\mathbf{e})\times g'dS' - \int_{\partial\mathcal{B}_1} \mathbf{e}\times(g'\times L)dS'$$
$$+ \int_{\partial\mathcal{B}_1} L\mathfrak{D}'g'dS' + \int_{\partial\mathcal{B}_1} \frac{Lg'}{|N'||\slashed{g}'|^{-1}}\Omega'(|N'||\slashed{g}'|^{-1})dS'$$
$$= \int_{\partial\mathcal{B}_1} ((\Omega+\Omega')L)g'dS' - \int_{\partial\mathcal{B}_1} (\mathbf{e}\times L)g'dS'$$
$$+ \int_{\partial\mathcal{B}_1} L\mathfrak{D}'g'dS' + \int_{\partial\mathcal{B}_1} \frac{Lg'}{|N'||\slashed{g}'|^{-1}}\Omega'(|N'||\slashed{g}'|^{-1})dS',$$

proving (176) when g' is vector-valued. $\qquad\square$

Combining the previous lemma (in the case where $\partial\mathcal{B}_1$ is S_R and ξ is the identity mapping) with Sobolev estimates and Proposition B.2 we get the following L^∞ estimate.

Proposition B.5. *There is a constant* $C = C(F, \|\slashed{\nabla}J\|_{L^\infty}, \|\slashed{\nabla}^2 J\|_{L^\infty}, \|\slashed{\nabla}^3 J\|_{L^\infty})$ *such that*

$$\|C_1 f\|_{L^\infty(S_R)} \le CR^{-N}\prod_{i=1}^N\left(\sum_{k=0}^3 R^k\|\slashed{\nabla}^k A_i\|_{L^\infty(S_R)}\right)\left(\sum_{k=0}^2 R^k\|\slashed{\nabla}^k f\|_{L^\infty(S_R)}\right).$$

Proof. The proof is immediate from Lemmas B.1 and B.4 and Proposition B.2. Note that we have used the embedding $L^\infty(S_R)\hookrightarrow L^2(S_R)$ to replace the L^2 norms on the right-hand side in Proposition B.2 by L^∞ norms. $\qquad\square$

In applications we often encounter integrals similar to C_1 and C_2 which are defined on $\partial\mathcal{B}_1$ rather than S_R. The estimates in Propositions B.2–B.5 can be transferred to this case if we parameterize $\partial\mathcal{B}_1$ by ξ. More precisely, we have the following corollary.

Corollary B.6. *Suppose for some* $\ell\ge 5$, ξ *satisfies*

$$\sup_{p,q\in S_R}\frac{|p-q|}{|\xi(p)-\xi(q)|}\le c_0, \quad \sum_{|\alpha|\le\ell}\|\mathfrak{D}^\alpha(|N||\slashed{g}|^{-1})\|_{L^2(S_R)}\le c_1,$$

$$\sum_{|\alpha|\leq\ell}\|\mathfrak{D}^\alpha\xi\|_{L^2(S_R)}\leq c_2,$$

where $N=\xi_\alpha\times\xi_\beta$, for some constants c_0, c_1, and c_2.

$$h(\xi)=\int_{\partial\mathcal{B}_1}K(\xi'-\xi)(g(\xi')-g(\xi))f(\xi')dS(\xi')$$

for some given functions f and g. Then there is a constant $C=C(c_0,c_1,c_2)$ such that for all $k\leq\ell$

$$\|\mathfrak{D}^k h\|_{L^2(\partial\mathcal{B}_1)}\leq\frac{C}{R}\sum_{j\leq\max\{k,3\}}\|\mathfrak{D}^j g\|_{L^2(\partial\mathcal{B}_1)}\sum_{j\leq\max\{k-1,3\}}\|\mathfrak{D}^j f\|_{L^2(\partial\mathcal{B}_1)}.$$

Proof. To be able to use Propositions B.2 and B.3 we write F (identifying $h\circ\xi$ with h as usual) as

$$\begin{aligned}h(p)=&-\frac{1}{2\pi}\int_{S_R}\frac{|p'-p|^3}{|\xi(p')-\xi(p)|^3}\frac{(\xi(p')-\xi(p))(g(p')-g(p))}{|p'-p|^3}\frac{|N(p')|}{|\not{g}(p')|}\\&\times f(p')dS(p').\end{aligned}$$

Similarly for the $L^2(\partial\mathcal{B}_1)$ norms we write

$$\|\mathfrak{D}^j h\|^2_{L^2(\partial\mathcal{B}_1)}=\int_{S_R}|\mathfrak{D}^j h(p)|^2\frac{|N(p)|}{|\not{g}(p)|}dS(p).$$

The statement now follows from Propositions B.2 and B.3, together with Lemma B.4, and the Sobolev embedding. We refer the reader to Lemma 2.5 in [2] for a similar argument in dimension two. □

Acknowledgements

References

[1] G. B. Arfken and H. J. Weber. *Mathematical methods for physicists.* Harcourt/Academic Press, Burlington, MA, fifth edition, 2001.

[2] L. Bieri, S. Miao, S. Shahshahani, and S. Wu. On the Motion of a Self-Gravitating Incompressible Fluid with Free Boundary. *Comm. Math. Phys.*, 355(1):161–243, 2017.

[3] A.-P. Calderón. Commutators of singular integral operators. *Proc. Nat. Acad. Sci. U.S.A.*, 53:1092–1099, 1965.

[4] D. Christodoulou. Private communication, see also https://www.youtube.com/watch?v=FFMSteogq40.

[5] R. R. Coifman, G. David, and Y. Meyer. La solution des conjecture de Calderón. *Adv. in Math.*, 48(2):144–148, 1983.

[6] R. R. Coifman, A. McIntosh, and Y. Meyer. L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes. *Ann. of Math. (2)*, 116(2):361–387, 1982.

[7] J. E. Gilbert and M. A. M. Murray. *Clifford algebras and Dirac operators in harmonic analysis*, volume 26 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1991.

[8] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 69 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original [MR0775683], With a foreword by Susanne C. Brenner.

[9] C. E. Kenig. Elliptic boundary value problems on Lipschitz domains. In *Beijing lectures in harmonic analysis (Beijing, 1984)*, volume 112 of *Ann. of Math. Stud.*, pages 131–183. Princeton Univ. Press, Princeton, NJ, 1986.

[10] C. E. Kenig. *Harmonic analysis techniques for second order elliptic boundary value problems*, volume 83 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994.

[11] H. Lindblad and K. H. Nordgren. A priori estimates for the motion of a self-gravitating incompressible liquid with free surface boundary. *J. Hyperbolic Differ. Equ.*, 6(2):407–432, 2009.

[12] K. H. Nordgren. Well-posedness for the equations of motion of an inviscid, incompressible, self-gravitating fluid with free boundary. *J. Hyperbolic Differ. Equ.*, 7(3):581–604, 2010.

[13] G. Taylor. The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I. *Proc. Roy. Soc. London. Ser. A.*, 201:192–196, 1950.

[14] G. Verchota. Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. *J. Funct. Anal.*, 59(3):572–611, 1984.

[15] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.*, 130(1):39–72, 1997.

[16] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.*, 12(2):445–495, 1999.

[17] S. Wu. Global wellposedness of the 3-D full water wave problem. *Invent. Math.*, 184(1):125–220, 2011.

[18] D.G. Yang. Lower bound estimates of the first eigenvalue for compact manifolds with positive Ricci curvature. *Pacific J. Math.*, 190(2):383–398, 1999.

[19] P. Zhang and Z. Zhang. On the free boundary problem of three-dimensional incompressible Euler equations. *Comm. Pure Appl. Math.*, 61(7):877–940, 2008.

SHUANG MIAO
SCHOOL OF MATHEMATICS AND STATISTICS
WUHAN UNIVERSITY
WUHAN 430072, HUBEI, CHINA
HUBEI KEY LABORATORY OF COMPUTATIONAL SCIENCE
WUHAN UNIVERSITY
WUHAN 430072, HUBEI, CHINA
E-mail address: shuang.m@whu.edu.cn

SOHRAB SHAHSHAHANI
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
710 N. PLEASANT STREET
AMHERST, MA, 01003-9305, USA
E-mail address: sohrab@math.umass.edu