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Homotopy invariant presheaves with framed transfers*

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In memory of Andrei Suslin

The category of framed correspondences $Fr_*(k)$, framed presheaves and framed sheaves were invented by Voevodsky in his unpublished notes [20]. Based on the notes [20] a new approach to the classical Morel–Voevodsky motivic stable homotopy theory was developed in [8]. This approach converts the classical motivic stable homotopy theory into an equivalent local theory of framed bispectra. The main result of the paper is the core of the theory of framed bispectra. It states that for any homotopy invariant quasi-stable radditive framed presheaf of Abelian groups $\mathcal{F}$, the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ is strictly homotopy invariant and quasi-stable whenever the base field k is infinite perfect of characteristic different from 2.

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1. Introduction

The main goal of the Voevodsky theory on framed correspondences (see [20, Introduction]) is to suggest a new approach to the stable motivic homotopy theory $SH(k)$ over a field k. This approach is more amenable to explicit calculations. Recall that Voevodsky [20, Section 2] invented a category of framed correspondences $Fr_*(k)$ whose objects are those of Sm/k and morphisms sets $Fr_*(X,Y) = \sqcup_{n\geqslant 0} Fr_n(X,Y)$ are defined by means of certain geometric data. The elements of $Fr_n(X,Y)$ are called framed correspondences of level n. Definitions of $Fr_*(k)$ and stable framed correspondences $Fr(X,Y)$ are given in Section 2. In [20] framed presheaves of sets (respectively Nisnevich framed sheaves) are defined and their basic properties are proved. Based on the notes [20] the theory of big framed motives of bispectra is introduced and studied in [8]. The big framed motive functor of [8] converts the classical motivic stable homotopy theory into an equivalent local theory of framed bispectra. Thus it gives a new approach to the classical Morel–Voevodsky stable motivic homotopy theory $SH(k)$ over an infinite perfect field k. It also has several important computational applications (see [8]). Particularly, an explicit computation of the suspension spectra/bispectra of smooth algebraic varieties (or, more generally, of simplicial smooth schemes $Y^\bullet$) in terms of motivic spaces with framed correspondences of the form $Fr(\Delta^\bullet \times -, Y^\bullet)$ is given in [8]. If the motivic space

$Fr(\Delta^\bullet \times -, Y^\bullet)$ is locally connected in the Nisnevich topology, then it is isomorphic in $H_{\mathbb{A}^1}(k)$ to the motivic space $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(Y^\bullet_+)$ (see [8]). Moreover, the motivic space $Fr(\Delta^\bullet \times -, Y^\bullet)$ is $\mathbb{A}^1$-local. This result can be regarded as a motivic counterpart of the Segal theorem [17, 3.5].

Theorem 1.1 stated below is the core of the theory of big framed motives of [8]. The main goal of this paper is to prove the theorem. Its equivalent form, Theorem 2.9, states that for any $\mathbb{A}^1$-invariant quasi-stable radditive framed presheaf of Abelian groups $\mathcal{F}$, the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ is strictly $\mathbb{A}^1$-invariant and quasi-stable whenever the base field k is infinite perfect of characteristic different from 2. The fact that Theorem 1.1 and Theorem 2.9 are equivalent is explained in Remark 2.18.

We should stress that the original Voevodsky theorem [22, Theorem 3.1.12] as well as similar results from [4, 11] are not suitable for the theory of big framed motives. The main reason for that is this: bigraded presheaves of $\mathbb{A}^1$-homotopy groups of a bispectrum $E \in SH(k)$ are naturally $Fr_*(k)$-presheaves (i.e. framed presheaves), however they are in no reasonable way presheaves with transfers in the sense of [22] or [4, 11]. It follows from [2] that bigraded sheaves of stable $\mathbb{A}^1$-homotopy groups of a bispectrum $E \in SH(k)$ are naturally $\widetilde{Cor}$-sheaves. But even this is not sufficient to develop the big framed motives theory. To prove [22, Theorem 3.1.12], Voevodsky used the standard triple machinery [21] developed by him as well as [22, Proposition 3.1.11]. We should also stress that the standard triple machinery of Voevodsky does not work at all to prove Theorem 1.1. However, the present paper is definitely inspired by Voevodsky's paper [21].

In the rest of the introduction we state Theorem 1.1. To this end, we choose a field k and write Sm/k for the category of smooth schemes over k. By Definition 2.11, for any pair $X, Y \in Sm/k$ each element $a \in Fr_n(X,Y)$ has its support Z_a. It is a closed subset in $X \times \mathbb{A}^n$ which is finite over X and determined by a uniquely. If the support Z_a of an element $a \in Fr_n(X,Y)$ is a disjoint union of Z_1 and Z_2, then the element a determines uniquely two elements a_1 and a_2 in $Fr_n(X,Y)$ such that the support of a_i is Z_i. (this is explained in Definition 2.11). Therefore one can form the subgroup $A(X,Y)$ of the free abelian group $\mathbb{Z}[Fr_n(X,Y)]$ generated by elements of the form $1 \cdot a - 1 \cdot a_1 - 1 \cdot a_2$, where $a \in Fr_n(X,Y)$ runs over those elements whose support Z_a is a disjoint union of Z_1 and Z_2, and a_1, a_2 are the elements as above determined by a.

The main result, Theorem 1.1, is stated in terms of $\mathbb{Z}F_*$-presheaves of abelian groups on smooth algebraic varieties Sm/k. Recall that $\mathbb{Z}F_*(k)$ is defined in [8, Definition 8.3] as an additive category whose objects are those of Sm/k and Hom-groups are defined as follows (see Definition 2.11). We set for every $n \geqslant 0$ and $X, Y \in Sm/k$,

$$\mathbb{Z}F_n(X,Y) := \mathbb{Z}[Fr_n(X,Y)]/A(X,Y).$$

In other words, $\mathbb{Z}F_n(X,Y)$ is a free abelian group generated by the framed correspondences of level n with connected supports. We then set

$$\mathrm{Hom}_{\mathbb{Z}F_*(k)}(X,Y) := \bigoplus_{n\geqslant 0} \mathbb{Z}F_n(X,Y).$$

By a presheaf of Abelian groups on $\mathbb{Z}F_*(k)$ we mean an additive functor $\mathbb{Z}F_*(k)^{\mathrm{op}} \to \mathrm{Ab}$.

By definition, a $\mathbb{Z}F_*(k)$-presheaf $\mathcal{F}$ of Abelian groups is stable if for any k-smooth variety the pullback map $\sigma_X^* : \mathcal{F}(X) \to \mathcal{F}(X)$ equals the identity map, where $\sigma_X = (X \times 0, X \times \mathbb{A}^1, t; pr_X) \in \mathbb{Z}F_1(X,X)$. In turn, $\mathcal{F}$ is quasi-stable if for any k-smooth variety the pullback map $\sigma_X^* : \mathcal{F}(X) \to \mathcal{F}(X)$ is an isomorphism.

The main result of the paper is as follows.

Theorem 1.1. *For any $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*(k)$-presheaf of Abelian groups $\mathcal{F}$, the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ is $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*(k)$-presheaf of Abelian groups whenever the base field k is infinite of characteristic different from 2. Moreover, if the base field k is infinite perfect of characteristic different from 2, then all Nisnevich cohomology presheaves $H^n_{\mathrm{nis}}(X, \mathcal{F}_{\mathrm{nis}})$ are canonically $\mathbb{Z}F_*(k)$-presheaves. All these $\mathbb{Z}F_*(k)$-presheaves are $\mathbb{A}^1$-invariant and quasi-stable. Furthermore, the same statements are true in characteristic 2 if we also suppose that the framed presheaf of abelian groups $\mathcal{F}$ is a presheaf of $\mathbb{Z}[1/2]$-modules.*

Throughout the paper the base field k is supposed to be infinite. We also employ the following notation:

- all schemes are separated Noetherian k-schemes, all morphisms of schemes are k-morphisms;
- Sm/k is the category of smooth k-schemes of finite type;
- we refer to the objects of Sm/k as k-smooth schemes or smooth k-schemes;
- Sm'/k is the category of essentially smooth k-schemes. Following [10], by an essentially smooth k-scheme we mean a Noetherian k-scheme X which is the inverse limit of a left filtering system $(X_i)_{i\in I}$ with each transition morphism $X_i \to X_j$ being an étale affine morphism between smooth k-schemes;

- by an affine essentially smooth k-scheme we mean a k-scheme of the form $Spec(A_M)$, where A is a smooth k-algebra of finite type and $M \subset A$ is a multiplicative system;
- by a quasi-affine essentially smooth k-scheme we mean an open subscheme of an affine essentially smooth k-scheme;
- $EssSm/k$ is the category of quasi-affine essentially smooth k-schemes.

2. Recollections on Voevodsky's framed correspondences

In this section we collect basic facts for framed correspondences and framed presheaves in the sense of Voevodsky [20]. We start with preparations.

Let S be a scheme and Z be a closed subscheme. Recall that an *étale neighborhood of Z in S* is a triple $(W, \pi : W \to S, s : Z \to W)$ satisfying the conditions:

(i) π is an étale morphism;
(ii) $\pi \circ s$ coincides with the inclusion $Z \hookrightarrow S$ (thus s is a closed embedding);
(iii) $(\pi)^{-1}(Z) = s(Z)$.

A morphism between two étale neighborhoods $(W, \pi, s) \to (V, \tau, t)$ of Z in S is a morphism $\rho : W \to V$ such that $\tau \circ \rho = \pi$ and $\rho \circ s = t$. Note that such ρ is automatically étale.

Definition 2.1 (Voevodsky [20])**.** For k-smooth schemes Y, X and $n \geqslant 0$ an *explicit framed correspondence* Φ *of level* n consists of the following data:

1. a closed subset Z in $\mathbb{A}^n_Y$ which is finite over Y;
2. an etale neighborhood $p : U \to \mathbb{A}^n_Y$ of Z in $\mathbb{A}^n_Y$;
3. a collection of regular functions $\varphi = (\varphi_1, \ldots, \varphi_n)$ on U such that $\cap_{i=1}^n \{\varphi_i = 0\} = Z$;
4. a morphism $g : U \to X$.

The subset Z will be referred to as the *support* of the correspondence. We shall also write triples $\Phi = (U, \varphi; g)$ or quadruples $\Phi = (Z, U, \varphi; g)$ to denote explicit framed correspondences.

Two explicit framed correspondences Φ and Φ' of level n are said to be *equivalent* if they have the same support and there exists an open neighborhood V of Z in $U \times_{\mathbb{A}^n_Y} U'$ such that on V, the morphism $g \circ pr$ agrees with $g' \circ pr'$ and $\varphi \circ pr$ agrees with $\varphi' \circ pr'$. A *framed correspondence of level* n is an equivalence class of explicit framed correspondences of level n.

We let $Fr_n(Y, X)$ denote the set of framed correspondences of level n from Y to X. We consider it as a pointed set with the basepoint being the class 0_n of the explicit correspondence with $U = \emptyset$.

As an example, the set $Fr_0(Y,X)$ coincides with the set of pointed morphisms $Y_+ \to X_+$. In particular, for a connected scheme X one has

$$Fr_0(Y,X) = \mathrm{Hom}_{Sm/k}(Y,X) \sqcup \{0_0\}.$$

If $f : Y' \to Y$ is a morphism in Sm/k and $\Phi = (U,\varphi;g)$ an explicit framed correspondence of level n from Y to X then

$$f^*(\Phi) := (U' = U \times_Y Y', \varphi \circ pr; g \circ pr)$$

is an explicit framed correspondence of level n from Y' to X.

Remark 2.2. Let $\Phi = (Z, \mathbb{A}^n_Y \xleftarrow{p} U, \varphi : U \to \mathbb{A}^n_k, g : U \to X) \in Fr_n(Y,X)$ be an *explicit framed correspondence of level n*. It can more precisely be written in the form

$$((\alpha_1,\alpha_2,\dots,\alpha_n), f, Z, U, (\varphi_1,\varphi_2,\dots,\varphi_n), g) \in Fr_n(Y,X),$$

where

- ◇ $Z \subset \mathbb{A}^n_Y$ is a closed subset finite over Y,
- ◇ an etale neighborhood $((\alpha_1,\alpha_2,\dots,\alpha_n), f) = p : U \to \mathbb{A}^n_k \times Y$ of Z,
- ◇ a collection of regular functions $\varphi = (\varphi_1,\dots,\varphi_n)$ on U such that $\cap_{i=1}^n \{\varphi_i = 0\} = Z$;
- ◇ a morphism $g : U \to X$.

We shall usually drop $((\alpha_1,\alpha_2,\dots,\alpha_n), f)$ from notation and just write

$$(Z, U, (\varphi_1,\varphi_2,\dots,\varphi_n), g) = ((\alpha_1,\alpha_2,\dots,\alpha_n), f, Z, U, (\varphi_1,\varphi_2,\dots,\varphi_n), g).$$

The following definition is to describe compositions of framed correspondences.

Definition 2.3. Suppose Y, X and S are k-smooth schemes. Let

$$a = ((\alpha_1,\alpha_2,\dots,\alpha_n), f, Z, U, (\varphi_1,\varphi_2,\dots,\varphi_n), g)$$

be an explicit correspondence of level n from Y to X and let

$$b = ((\beta_1,\beta_2,\dots,\beta_m), f', Z', U', (\psi_1,\psi_2,\dots,\psi_m), g') \in Fr_m(X,S)$$

be an explicit correspondence of level m from X to S. We define their composition as an explicit correspondence of level $n+m$ from Y to S by

$$((\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m), f \circ p_1, Z \times_X Z', U \times_X U', \\ (\varphi_1, \varphi_2, \ldots, \varphi_n, \psi_1, \psi_2, \ldots, \psi_m), g' \circ p_2).$$

Clearly, the composition of explicit correspondences respects the equivalence relation on them and defines associative maps

$$Fr_n(Y, X) \times Fr_m(X, S) \to Fr_{n+m}(Y, S).$$

Given $Y, X \in Sm/k$, denote by $Fr_+(Y, X)$ the pointed set $\bigvee_n Fr_n(Y, X)$. The composition of framed correspondences defined above gives a category $Fr_+(k)$. Its objects are those of Sm/k and the morphisms are given by the pointed sets $Fr_+(Y, X)$, $Y, X \in Sm/k$. Since the naive morphisms of schemes can be identified with certain framed correspondences of level zero, we get a canonical functor

$$Sm/k \to Fr_+(k).$$

The category $Fr_+(k)$ has the empty scheme as zero object. One can easily see that for a framed correspondence $\Phi : Y \to X$ and a morphism $f : Y' \to Y$, one has $f^*(\Phi) = \Phi \circ f$.

There is also a subcategory $Fr_0(k)$ of the category $Fr_+(k)$. Its objects are those of Sm/k and the morphisms are given by the sets $Fr_0(Y, X)$, $Y, X \in Sm/k$.

Definition 2.4 (Voevodsky [20])**.** Define $\mathbb{A}^1$ to be $\operatorname{Spec}(k[t])$. Given any k-smooth scheme X, there is a distinguished morphism $\sigma_X = (X \times \mathbb{A}^1, t, pr_X) \in Fr_1(X, X)$. It is worth to mention that for any $f \in Fr_0(Y, X)$ one has $\sigma_X \circ f = f \circ \sigma_Y$.

Voevodsky defined a category $Fr_*(k)$ in [20] whose objects are those of Sm/k and Hom-sets are given by $Fr_*(Y, X) = \sqcup_{n \geqslant 0} Fr_n(X, Y)$. There is an obvious functor $p : Fr_*(k) \to Fr_+(k)$, which is the identity on objects. We prefer to work with the category $Fr_+(k)$ since it has a zero object.

Definition 2.5. A *framed presheaf* $\mathcal{F}$ on Sm/k is a contravariant functor from $Fr_+(k)$ to the category of sets. A *pointed framed presheaf* $\mathcal{F}$ on Sm/k is a contravariant functor from $Fr_+(k)$ to the category of pointed sets.

A framed presheaf $\mathcal{F}$ on Sm/k is called *radditive* if $\mathcal{F}(\emptyset) = *$ and $\mathcal{F}(X_1 \sqcup X_2) = \mathcal{F}(X_1) \times \mathcal{F}(X_2)$. A *framed Nisnevich sheaf* on Sm/k is a framed presheaf $\mathcal{F}$ such that its restriction to Sm/k is a Nisnevich sheaf.

A framed presheaf $\mathcal{F}$ of Abelian groups on Sm/k is a contravariant functor from $Fr_+(k)$ to the category of Abelian groups. A framed presheaf $\mathcal{F}$ of Abelian groups on Sm/k is *radditive* if $\mathcal{F}(\emptyset) = 0$ and $\mathcal{F}(X_1 \sqcup X_2) = \mathcal{F}(X_1) \times \mathcal{F}(X_2)$. A framed Nisnevich sheaf of Abelian groups on Sm/k is a framed presheaf of Abelian groups $\mathcal{F}$ such that its restriction to Sm/k is a Nisnevich sheaf.

Finally, a framed presheaf $\mathcal{F}$ is *homotopy invariant* or $\mathbb{A}^1$*-invariant* if for any $X \in Sm/k$ and the projection $pr_X : \mathbb{A}^1 \times X \to X$ the map $p_X^* : \mathcal{F}(X) \to \mathcal{F}(\mathbb{A}^1 \times X)$ is a bijection. A framed Nisnevich sheaf is $\mathbb{A}^1$-invariant if it is $\mathbb{A}^1$-invariant as a framed presheaf.

Remark 2.6. The category of presheaves $\mathcal{F}$ of Abelian groups on $Fr_*(k)$ such that $\mathcal{F}(\emptyset) = 0$ is equivalent to the category of presheaves of abelian groups G on $Fr_+(k)$ with the property $G(\emptyset) = 0$. The equivalence is given by the functor $G \mapsto G \circ p$. Particularly, this comment is applicable to radditive framed presheaves of Abelian groups. By Corollary 2.16 below the category of radditive framed presheaves of Abelian groups is a Grothendieck category. Therefore we can apply the standard homological algebra to it.

Voevodsky uses in [20] the term "global framed functors" for our radditive framed presheaves of sets. Note that the representable presheaves on $Fr_+(k)$ are not radditive.

Definition 2.7 (Voevodsky [20])**.** A framed presheaf $\mathcal{F}$ is *stable* if for any k-smooth scheme the pullback map $\sigma_X^* : \mathcal{F}(X) \to \mathcal{F}(X)$ equals the identity map, where $\sigma_X = (X \times 0, X \times \mathbb{A}^1, t; pr_X)$. In turn, $\mathcal{F}$ is *quasi-stable* if for any k-smooth scheme the pull-back map $\sigma_X^* : \mathcal{F}(X) \to \mathcal{F}(X)$ is an isomorphism. Stable and quasi-stable framed presheaves of Abelian groups are defined in a similar fashion.

Lemma 2.8 (Voevodsky [20], Lemma 4.5)**.** *For every radditive framed presheaf of Abelian groups $\mathcal{F}$ the associated sheaf in the Nisnevich topology has a unique structure of a framed presheaf of Abelian groups such that the map $\mathcal{F} \to \mathcal{F}_{\mathrm{nis}}$ is a map of framed presheaves of Abelian groups.*

It is useful to have the following equivalent formulation of Theorem 1.1 in terms of framed raddive presheaves.

Theorem 2.9. *For any $\mathbb{A}^1$-invariant quasi-stable framed radditive presheaf of Abelian groups $\mathcal{F}$, the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ is $\mathbb{A}^1$-invariant quasi-stable framed presheaf of Abelian groups whenever the base field k is infinite of characteristic different from 2. Moreover, if the base field k is infinite perfect of characteristic different from 2, then all Nisnevich cohomology presheaves $X \mapsto H^n_{\mathrm{nis}}(X, \mathcal{F}_{\mathrm{nis}})$ are canonically framed radditive presheaves.*

Furthermore, all these cohomology framed presheaves are $\mathbb{A}^1$-invariant and quasi-stable. Also, the same statements are true in characteristic 2 if we suppose that the framed presheaf of abelian groups $\mathcal{F}$ is a presheaf of $\mathbb{Z}[1/2]$-modules.

Remark 2.10. The fact that Theorems 2.9 and 1.1 are equivalent is discussed in Remark 2.18.

The nearest aim is to define all notions related to Theorem 1.1. If the support Z of a framed correspondence $a = (Z, U, \varphi; g) \in Fr_n(Y, X)$ is a disjoint union $Z' \sqcup Z''$, then α gives two framed correspondences

$$a' = (Z', U \setminus Z'', \varphi|_{U \setminus Z''}; g|_{U \setminus Z''}) \quad \text{and} \quad a'' = (Z'', U \setminus Z', \varphi|_{U \setminus Z'}; g|_{U \setminus Z'})$$

in $Fr_n(Y, X)$. Based on this observation, recall the definition of the category of linear framed correspondences $\mathbb{Z}F_*(k)$ introduced in [8, Definition. 8.3].

Definition 2.11. Let Y and X be k-smooth schemes. Let $\mathbb{Z}[Fr_n(Y, X)]$ be the free abelian group generated by the set $Fr_n(Y, X)$. Denote by A its subgroup generated by elements of the form $(Z \sqcup Z', U, \varphi; g) - (Z, U \setminus Z', \varphi|_{U \setminus Z'}; g|_{U \setminus Z'}) - (Z', U \setminus Z, \varphi|_{U \setminus Z}, g|_{U \setminus Z})$. Set,

$$\mathbb{Z}F_n(X, Y) := \mathbb{Z}[Fr_n(X, Y)]/A.$$

We shall also refer to the latter relation as the *additivity property for supports*. In other words, it says that for a framed correspondence a in $Fr_n(Y, X)$ whose support is a disjoint union $Z' \sqcup Z''$ the element $1 \cdot a$ in $\mathbb{Z}F_n(Y, X)$ equals the sum $1 \cdot a' + 1 \cdot a''$ of the elements with supports Z' and Z'' respectively.

The elements of $\mathbb{Z}F_n(Y, X)$ are called *linear framed correspondences of level n* or just *linear framed correspondences*. It is worth mentioning that $\mathbb{Z}F_n(Y, X)$ is a free abelian group generated by the elements of $Fr_n(Y, X)$ with connected support.

Denote by $\mathbb{Z}F_*(k)$ the additive category whose objects are those of Sm/k with Hom-groups defined as

$$\mathrm{Hom}_{\mathbb{Z}F_*(k)}(Y, X) = \bigoplus_{n \geqslant 0} \mathbb{Z}F_n(Y, X).$$

The composition is induced by the composition in the category $Fr_+(k)$. The direct sum of X and X' is the disjoin union $X \sqcup X'$. There is a canonical functor $Sm/k \to \mathbb{Z}F_*(k)$ which is the identity on objects and which takes a regular morphism $f : Y \to X$ to the linear framed correspondence $1 \cdot (Y, Y \times \mathbb{A}^0, pr_{\mathbb{A}^0}, f \circ pr_Y) \in \mathbb{Z}F_0(k)$.

Remark 2.12. We will often write $\mathbb{Z}F_*$ for $\mathbb{Z}F_*(k)$ dropping (k) from notation. For any X, Y in Sm/k one has the equality $\mathbb{Z}F_*(-, X \sqcup Y) = \mathbb{Z}F_*(-, X) \oplus \mathbb{Z}F_*(-, Y)$.

Definition 2.13. By a *presheaf of Abelian groups on* $\mathbb{Z}F_*(k)$ we shall mean an additive contravariant functor from $\mathbb{Z}F_*(k)$ to the category of Abelian groups Ab.

A $\mathbb{Z}F_*(k)$*-Nisnevich sheaf of Abelian groups* is a $\mathbb{Z}F_*(k)$-presheaf of Abelian groups such that its restriction to Sm/k is an ordinary Nisnevich sheaf.

A $\mathbb{Z}F_*(k)$-presheaf $\mathcal{F}$ of Abelian groups is *homotopy invariant* or $\mathbb{A}^1$*-invariant* if for any $X \in Sm/k$ the projection $pr_X : \mathbb{A}^1 \times X \to X$ induces an isomorphism $p_X^* : \mathcal{F}(X) \to \mathcal{F}(\mathbb{A}^1 \times X)$. A Nisnevich $\mathbb{Z}F_*(k)$-sheaf is $\mathbb{A}^1$-invariant if it is $\mathbb{A}^1$-invariant as a $\mathbb{Z}F_*(k)$-presheaf.

The canonical maps $Fr_+(X, Y) \to \operatorname{Hom}_{\mathbb{Z}F_*(k)}(X, Y)$ define a functor $R : Fr_+(k) \to \mathbb{Z}F_*(k)$, which is the identity on objects. We often write σ_X for $1 \cdot \sigma_X \in \mathbb{Z}F_1(X, X)$.

Definition 2.14. A $\mathbb{Z}F_*(k)$-presheaf $\mathcal{F}$ of Abelian groups is called *stable* (respectively *quasi-stable*), if the framed presheaf $\mathcal{F} \circ R$ is stable (respectively quasi-stable). It is worth mentioning that for any $\mathbb{Z}F_*(k)$-presheaf $\mathcal{F}$ of Abelian groups the framed presheaf $\mathcal{F} \circ R$ is radditive.

Lemma 2.15. *The functor $G \mapsto G \circ R$ is an equivalence between the category of radditive framed presheaves of Abelian groups on Sm/k and the category of $\mathbb{Z}F_*(k)$-presheaves of Abelian groups.*

Proof. Let $\mathcal{F}$ be a radditive framed presheaf of Abelian groups on Sm/k. Let us show that there is a unique $\mathbb{Z}F_*(k)$-presheaf of Abelian groups $\mathcal{F}'$ such that $\mathcal{F} = \mathcal{F}' \circ R$. Consider k-smooth schemes V_1, V_2 and their disjoint union $V = V_1 \sqcup V_2$ as objects of the category $Fr_0(k)$. Let $i_r : V_r \to V$ be the inclusion for $r = 1, 2$ and $p_r : V \to V_r$ be the projection. So, $p_r|_{V_r} = id$ and $p_r|_{V_s}$ is the zero morphism for $s \neq r$. The radditivity of $\mathcal{F}$ guarantees that $(i_1 \circ p_1)^* + (i_2 \circ p_2)^* = id_V^* : \mathcal{F}(V) \to \mathcal{F}(V)$.

Take $(V, \varphi; g) \in Fr_n(Y, X)$ with $V = V_1 \sqcup V_2$. Then $(V, \varphi; g) = g \circ (V, \varphi; id_V)$ and $(V_r, \varphi|_{V_r}; id_{V_r}) = p_r \circ (V, \varphi; id_V)$ for $r = 1, 2$. Therefore, $(V, \varphi; id_V)^* = (V_1, \varphi|_{V_1}; id_{V_1})^* \circ i_1^* + (V_2, \varphi|_{V_1}; id_{V_1})^* \circ i_2^*$. Hence $(V, \varphi; g)^* = (V_1, \varphi|_{V_1}; id_{V_1})^* \circ g_1^* + (V_2, \varphi|_{V_1}; id_{V_1})^* \circ g_2^* = (V_1, \varphi|_{V_1}; g_1)^* + (V_2, \varphi|_{V_2}; g_2)^*$, where $g_r = g \circ i_r$ for $r = 1, 2$.

If the support Z of an element $(W, \varphi; g) \in Fr_n(Y, X)$ is a disjoint union $Z_1 \sqcup Z_2$, then $(W, \varphi; g) = (V, \varphi|_V; g|_V) \in Fr_n(Y, X)$, where $V = (W \setminus Z_2) \sqcup (W \setminus Z_1)$. The computations above show that for the group A from Definition

2.11 and any element $a \in A$ the map $a^* : \mathcal{F}(X) \to \mathcal{F}(Y)$ is the zero map. Hence $\mathcal{F} = \mathcal{F}' \circ R$ for a unique $\mathbb{Z}F_*(k)$-presheaf $\mathcal{F}'$ of Abelian groups.

If G is a $\mathbb{Z}F_*(k)$-presheaf of Abelian groups, then $G \circ R$ is a radditive framed presheaf of Abelian groups. The proof is completed. □

Corollary 2.16. *The category of radditive framed presheaves of Abelian groups is a Grothendieck category.*

Proof. The category of $\mathbb{Z}F_*(k)$-presheaves of Abelian groups is plainly a Grothendieck category. Lemma 2.15 completes the proof. □

The comment in Definition 2.14 together with Lemmas 2.8 and 2.15 imply the following

Corollary 2.17. *For every $\mathbb{Z}F_*(k)$-presheaf of Abelian groups $\mathcal{F}$ the associated sheaf in the Nisnevich topology has a unique structure of a $\mathbb{Z}F_*(k)$-presheaf of Abelian groups such that the map $\mathcal{F} \to \mathcal{F}_{\mathrm{nis}}$ is a map of $\mathbb{Z}F_*(k)$-presheaves of Abelian groups.*

Remark 2.18. Lemma 2.15 together with Lemma 2.8 show that Theorem 2.9 is equivalent to Theorem 1.1.

In the rest of this section we extend Definition 2.1 to make it suitable for lots of our computations.

Definition 2.19 (Voevodsky [20])**.** For any $Y \in Sm'/k$, $X \in Sm/k$ and $n \geqslant 0$, an *explicit framed correspondence* Φ *of level* n *from* Y *to* X consists of the following data:

1. a closed subset Z in $\mathbb{A}^n_Y$ which is finite over Y;
2. an etale neighborhood $p : U \to \mathbb{A}^n_Y$ of Z in $\mathbb{A}^n_Y$;
3. a collection of regular functions $\varphi = (\varphi_1, \ldots, \varphi_n)$ on U such that $\cap_{i=1}^n \{\varphi_i = 0\} = Z$;
4. a morphism $g : U \to X$.

The subset Z will be referred to as the *support* of the correspondence. We shall also write triples $\Phi = (U, \varphi; g)$ or quadruples $\Phi = (Z, U, \varphi; g)$ to denote explicit framed correspondences.

Two explicit framed correspondences Φ and Φ' of level n are said to be *equivalent* if they have the same support and there exists an open neighborhood V of Z in $U \times_{\mathbb{A}^n_Y} U'$ such that on V, the morphism $g \circ pr$ agrees with $g' \circ pr'$ and $\varphi \circ pr$ agrees with $\varphi' \circ pr'$. A *framed correspondence of level* n is an equivalence class of explicit framed correspondences of level n.

We let $Fr_n(Y,X)$ denote the set of framed correspondences of level n from Y to X. We consider it as a pointed set with the basepoint being the class 0_n of the explicit correspondence with $U=\emptyset$. As an example, the sets $Fr_0(Y,X)$ coincide with the set of pointed k-morphisms $Y_+\to X_+$. If $f: Y_1\to Y$ is a k-morphism of essentially k-smooth schemes and $\Phi=(U,\varphi;g)$ an explicit framed correspondence of level n from Y to X then

$$f^*(\Phi):=(U_1=U\times_Y Y_1,\varphi\circ pr;g\circ pr)$$

is an explicit framed correspondence of level n from Y_1 to X. The assignment $\Phi\mapsto f^*(\Phi)$ respects the equivalence relation and defines a map $f^*: Fr_n(Y,X)\to Fr_n(Y_1,X)$. We will write $\Phi\circ f$ for $f^*(\Phi)$.

If $f_1: Y_2\to Y_1$ is another k-morphism of essentially k-smooth schemes, then

$$(\Phi\circ f)\circ f_1=\Phi\circ(f\circ f_1).$$

If Y is an essentially k-smooth scheme, X,S are Sm/k, then repeating literally Definition 2.3 we get a pairing $Fr_n(Y,X)\times Fr_m(X,S)\to Fr_{n+m}(Y,S)$. If X,S,T are k-smooth schemes, $a\in Fr_n(Y,X)$, $b\in Fr_n(X,S)$, $c\in Fr_n(S,T)$, then $(a\circ b)\circ c=a\circ(b\circ c)$. Note that if $Y\in Sm/k$, then Definitions 2.1 and 2.19 coincide.

Definition 2.20. Let Y be in Sm'/k and X be in Sm/k. Set

- ⋄ $\mathbb{Z}F_n(Y,X):=\mathbb{Z}[Fr_n(Y,X)]/A$, where A is a subgroup generated by the elements
  $(Z\sqcup Z',U,\varphi;g)-(Z,U\setminus Z',\varphi|_{U\setminus Z'};g|_{U\setminus Z'})-(Z',U\setminus Z,\varphi|_{U\setminus Z},g|_{U\setminus Z})$.

The groups have the same functorial properties as the pointed sets $Fr_n(Y,X)$.

Note that if $Y\in Sm/k$, then Definitions 2.11 and 2.20 coincide.

For an affine k-smooth scheme Y and a multiplicative set $M\subset k[Y]$ set $Y_M=Spec(k[Y]_M)$. For any $m\in M$ let $f_m: Y_M\to Y_m$ be the canonical map. Let X be in Sm/k. Then the family of maps $f_m^*: Fr_n(Y_m,X)\to Fr_n(Y_M,X)$ defines a map $can_M:\operatorname{colim}_{m\in M}Fr_n(Y_m,X)\to Fr_n(Y_M,X)$. Let $Y_0\subset Y$ be an open subset and let $Y_{0,M}=Y_M\cap Y_0$.

Lemma 2.21. *Let $Y,X\in Sm/k$, Y be an affine k-variety and M be a multiplicative system. Then the map $can_M:\operatorname{colim}_{m\in M}Fr_n(Y_m,X)\to Fr_n(Y_M,X)$ is a bijection of pointed sets. The same is true if we replace Y_M by $Y_{0,M}$, where $Y_0\subset Y$ is an open subset.*

Proof. We first prove that the map in question is injective. Suppose Y is an affine k-variety. Let m be in M and $(Z_m, V_m, \varphi_m; g_m), (Z'_m, V'_m, \varphi'_m; g'_m) \in Fr_n(Y_m, X)$ be such that their images in $Fr_n(Y_M, X)$ coincide. Firstly, this yields that $(Z_m)_M = (Z'_m)_M$ in $Y_M \times \mathbb{A}^n$. Thus enlarging m, we may assume that $Z_m = Z'_m$. In this case V_m and V'_m are both étale neighborhoods of Z_m in $Y_m \times \mathbb{A}^n$. Refining V_m and V'_m, we may assume that $V_m = V'_m$. Set $V = (V_m)_M$ and $Z = (Z_m)_M$. Then we know that $(Z, V, \varphi|_V; g|_V) = (Z, V, \varphi'|_V; g'|_V)$ in $Fr_n(Y_M, X)$. Thus there is a refinement $\pi : W \to V$ of the neighborhood V of Z such that $\pi^*(\varphi|_V) = \pi^*(\varphi'|_V)$ and $\pi^*(g|_V) = \pi^*(g'|_V)$. Since $\pi^* : \Gamma(V, \mathcal{O}_V) \to \Gamma(W, \mathcal{O}_W)$ and $\pi^* : Mor_k(V, X) \to Mor_k(W, X)$ are injective, we see that $\varphi|_V = \varphi'|_V$ and $g|_V = g'|_V$. Enlarging $m \in M$ we may assume that the maps $k[V_m] \to \Gamma(V, \mathcal{O}_V)$ and $Mor_k(V_m, X) \to Mor_k(V, X)$ are injective. We see that $\varphi = \varphi'$ and $g = g'$. This completes the proof of the injectivity in the case of affine Y. The proof of the injectivity for the case of an open $Y_0 \subset Y$ is similar.

To prove surjectivity, we need some preparations. Let $d > 0$ be an integer and $Hilb_d := Hilb_d(\mathbb{P}^n)$ be the Hilbert scheme of closed subschemes in $\mathbb{P}^n$ of degree d over k. By [9, Theorem 3.2] it is a projective k-scheme. Let $\mathcal{Z}_{un} \subset Hilb_d \times \mathbb{P}^n$ be the universal closed subscheme which is flat, finite and of degree d over $Hilb_d$. If T is a Noetherian k-scheme and $f : T \to Hilb_d$ is a morphism, then $T \times_{Hilb_d} \mathcal{Z}_{un}$ is a closed subscheme in $T \times \mathbb{P}^n$ which is finite, flat of degree d over T. Vice versa, for any T as above, any closed subscheme S in $T \times \mathbb{P}^n$ which is finite, flat of degree d over T, there is a unique morphism $f_S : T \to Hilb_d$ such that the closed subschemes S and $T \times_{Hilb_d} \mathcal{Z}_{un}$ coincide in $T \times \mathbb{P}^n$.

For a point $s \in Hilb_d$ let $\mathcal{Z}_s = s \times_{Hilb_d} \mathcal{Z}_{un}$ be the fibre of $\mathcal{Z}_{un}$ over s. It is a closed subscheme in $\mathbb{P}^n_{k(s)}$ of degree d over $k(s)$. Let $\mathbb{P}^{n-1} = \mathbb{P}^n - \mathbb{A}^n$. Let $Hilb_d(\mathbb{A}^n) = \{s \in Hilb_d : \mathcal{Z}_s \cap \mathbb{P}^{n-1} = \emptyset\}$. Then $Hilb_d(\mathbb{A}^n)$ is an open subset in $Hilb_d$. Set $\mathcal{Z}(\mathbb{A}^n) = \mathcal{Z}_{un} \cap Hilb_d(\mathbb{A}^n) \times \mathbb{P}^n$. Clearly, $\mathcal{Z}(\mathbb{A}^n) = \mathcal{Z}_{un} \cap Hilb_d(\mathbb{A}^n) \times \mathbb{A}^n$. Write $in : Hilb_d(\mathbb{A}^n) \hookrightarrow Hilb_d$ for the open embedding.

Suppose Y is an affine k-variety. Without loss of generality we may assume that Y is irreducible. Let $(Z, \pi : V \to Y_M \times \mathbb{A}^n, \varphi; g)$ be in $Fr_n(Y_M, X)$. We need to find $(Z_m, V_m, \psi_m; g_m) \in Fr_n(Y_m, X)$ for some $m \in M$ which is a lift of $(Z, V, \varphi; g) \in Fr_n(Y_M, X)$.

We may assume that V is an affine Y_M-scheme. Since V is an étale neighborhood of Z in $Y_M \times \mathbb{A}^n$ we are given with a closed embedding $s : Z \to V$. Let I be the ideal in $\Gamma(V, \mathcal{O}_V)$ generated by $\varphi_1, ..., \varphi_n$. Since $(Z, \pi : V \to Y_M \times \mathbb{A}^n, \varphi; g)$ is in $Fr_n(Y_M, X)$ we know that the closed subsets $s(Z)$ and $\{\varphi_1 = ... = \varphi_n = 0\}$ of V coincide. Write $\mathcal{Z}$ for the closed subscheme of V defined by I. It is easy to check that $\pi|_{\mathcal{Z}} : \mathcal{Z} \to Y_M \times \mathbb{A}^n$ is a closed embedding. We will write S for the closed subscheme $\pi(\mathcal{Z})$ of $Y_M \times \mathbb{A}^n$. We will also write $\tilde{s} : S \to V$ for the scheme morphism $S \xrightarrow{(\pi|_{\mathcal{Z}})^{-1}} \mathcal{Z} \hookrightarrow V$. Clearly, $\tilde{s}|_Z = s$, $(V, \pi, \tilde{s})$ is an étale neighborhood of S in $Y_M \times \mathbb{A}^n$ and Z (regarded as a scheme with the reduced scheme structure) is the maximal closed reduced subscheme of S.

Since Z is finite over Y_M the schemes $\mathcal{Z}$ and S are also finite over Y_M. Since the scheme $\mathcal{Z}$ is a locally complete intersection over Y_M, it is flat over Y_M, and hence so is S. Since S is finite over Y_M it remains closed if we regard it as a subscheme in $Y_M \times \mathbb{P}^n$. So, the closed subscheme S of $Y_M \times \mathbb{P}^n$ is finite, flat of some degree $d > 0$ over Y_M. As explained above there is a unique morphism $f_S : Y_M \to Hilb_d$ such that the closed subschemes $Y_M \times_{Hilb_d} \mathcal{Z}_{un}$ and S of the scheme $Y_M \times \mathbb{P}^n$ coincide.

The inclusion $S \subset Y_M \times \mathbb{A}^n$ yields that $f_S = in \circ f$ for a unique morphism $f : Y_M \to Hilb_d(\mathbb{A}^n)$ and $S = Y_M \times_{Hilb_d(\mathbb{A}^n)} \mathcal{Z}(\mathbb{A}^n)$ as closed subschemes in $Y_M \times \mathbb{A}^n$. Since the k-scheme $Hilb_d(\mathbb{A}^n)$ is of finite type, by [19, Appendix C.5.1] there is an $m \in M$ and a morphism $f_m : Y_m \to Hilb_d(\mathbb{A}^n)$ such that $f = f_m|_{Y_M}$. Set $S_m = Y_m \times_{Hilb_d(\mathbb{A}^n)} \mathcal{Z}(\mathbb{A}^n) \subset Y_m \times \mathbb{A}^n$. Then $S = (S_m)_M$. For any $m' \in M$ such that $m' = m \cdot m_1$ with $m_1 \in M$, set $S_{m'} = (S_m)_{m'}$. Enlarging $m \in M$ we may assume that there is an étale neighborhood $(V_m, \pi_m, \tilde{s}_m)$ of S_m in $Y_m \times \mathbb{A}^n$ such that V_m is an affine $Y_m \times \mathbb{A}^n$-scheme and $(V_m, \pi_m, \tilde{s}_m)_M = (V, \pi, \tilde{s})$. For any $m' \in M$ such that $m' = m \cdot m_1$ with $m_1 \in M$ set $(V_{m'}, \pi_{m'}, \tilde{s}_{m'}) = (V_m, \pi_m, \tilde{s}_m)_{m'}$. Then $(V_{m'}, \pi_{m'}, \tilde{s}_{m'})$ is an étale neighborhood of $S_{m'}$ in $Y_{m'} \times \mathbb{A}^n$.

Let $I_{m'} \subset k[V_{m'}]$ be the ideal defining the closed subscheme $\tilde{s}_{m'}(S_{m'})$ of the scheme $V_{m'}$. Enlarging $m \in M$ once again we can find $\psi_1, \dots \psi_n \in I_m$ such that their restrictions to V coincide with $\varphi_1, \dots, \varphi_n$ respectively. Since $V = (V_m)_M$, $S = (S_m)_M$ and $\tilde{s} = (\tilde{s}_m)_M$ we have the equality $I = (I_m)_M$. Thus enlarging $m \in M$, we may assume that the ideal $I_m \subset k[V_m]$ is generated by the functions $\psi_1|_{V_m}, \dots \psi_n|_{V_m}$. Since X is a k-scheme of finite type we can enlarge $m \in M$ once again and find a morphism $g_m : V_m \to X$ such that $g_m|_V : V \to X$ coincides with g. Let Z_m be the maximal closed reduced subscheme of the scheme S_m and $s_m := \tilde{s}_m|_{Z_m} : Z_m \to V_m$. Then (V_m, π_m, s_m) is an étale neighborhood of Z_m in $Y_m \times \mathbb{A}^n$. We now see that $(Z_m, V_m, \psi_m; g_m) \in Fr_n(Y_m, X)$ is a lift of $(Z, V, \varphi; g)$. This completes the proof of surjectivity for an affine Y.

The proof of surjectivity for the case of an open $Y_0 \subset Y$ is a bit more technical, but it is shown in the same fashion. The key is to use the Hilbert scheme $Hilb_d(\mathbb{A}^n)$. We leave this part of the proof to the reader. □

The following fact immediately follows from Lemma 2.21.

Corollary 2.22. *Under the assumptions of Lemma 2.21 the following map is an isomorphism*

$$can_M : \mathrm{colim}_{m\in M}\, \mathbb{Z}F_n(Y_m, X) \to \mathbb{Z}F_n(Y_M, X).$$

The same is true if we replace Y_M with $Y_{0,M}$, where $Y_0 \subset Y$ is an open subset.

Let Y, X be in Sm/k and $y \in Y$ be a point. Consider the henselization Y^h_y at y of the local scheme $Y_y := Spec\mathcal{O}_{Y,y}$. For an étale neighborhood $(V, \pi : V \to Y_y, s : y \to V)$ let $f_V : Y^h_y \to V$ be the canonical map and $f^*_V : Fr_n(V, X) \to Fr_n(Y^h_y, X)$ be the induced map. Then the family of pointed sets maps f^*_V defines a map

$$can_{Y,y} : \mathrm{colim}_{(V,\pi,s)} Fr_n(V, X) \to Fr_n(Y^h_y, X),$$

where the colimit is taken over the co-filtered category of étale neighborhoods of y in Y_y. Arguing as in the proof of Lemma 2.21 we get the following

Lemma 2.23. *Let Y, X be in Sm/k and $y \in Y$ be a point. Then the map $can_{Y,y}$ is an isomorphism.*

Corollary 2.24. *Under the assumptions of Lemma 2.23 the following map is an isomorphism*

$$can_{Y,y} : \mathrm{colim}_{(V,\pi,s)}\, \mathbb{Z}F_n(V, X) \to \mathbb{Z}F_n(Y^h_y, X).$$

Remark 2.25. Lemma 2.23 and Corollary 2.24 show that the pointed set $Fr_n(Y^h_y, X)$ (respectively the group $\mathbb{Z}F_n(Y^h_y, X)$) coincides with the Nisnevich stalk at the point $y \in Y$ of the presheaf $Fr_n(-, X)$ on Sm/k (respectively of the presheaf $\mathbb{Z}F_n(-, X)$ on Sm/k).

3. A few theorems

The main goal of this section is to state a few theorems on preshaves with framed transfers. As an application, we deduce the following result (which is the first assertion of Theorem 1.1).

Theorem 3.1. *For any $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-presheaf of abelian groups $\mathcal{F}$, the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ is $\mathbb{A}^1$-invariant and quasi-stable $\mathbb{Z}F_*$-presheaf if the characteristic of the base field k is different from 2. If the characteristic of k equals 2 and $\mathcal{F}$ is an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-presheaf of $\mathbb{Z}[1/2]$-modules, then the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ is $\mathbb{A}^1$-invariant and quasi-stable $\mathbb{Z}F_*$-presheaf of $\mathbb{Z}[1/2]$-modules.*

We need some definitions. We will write $(V,\varphi;g)$ for an element a in $Fr_n(Y,X)$. We also write Z_a to denote the support of $(V,\varphi;g)$. It is a closed subset in $Y\times\mathbb{A}^n$ which is finite over Y and which coincides with the common vanishing locus of the functions $\varphi_1,...,\varphi_n$ in V. Next, by $\langle V,\varphi;g\rangle$ we denote the image of the element $1\cdot(V,\varphi;g)$ in $\mathbb{Z}F_n(Y,X)$.

In what follows by $SmOp/k$ we mean a category whose objects are pairs (X,V), where $X\in Sm/k$ and V is an open subset of X. For (Y,W) and (X,V) in $SmOp/k$ a morphism between them is a morphism $f:Y\to X$ in Sm/k such that $f(W)\subset V$. By $Sm'Op/k$ we mean a category whose objects are pairs (Y,W), where $Y\in Sm'/k$ and W is an open subset of Y. Morphisms in $Sm'Op/k$ are defined similar to morphisms in $SmOp/k$. The category $SmOp/k$ is a full subcategory of the category $Sm'Op/k$.

Definition 3.2. Define $\mathbb{Z}F^{pr}_*(k)$ as an additive category whose objects are those of $SmOp/k$ and Hom-groups are defined as follows. We set for every $n\geqslant 0$ and $(Y,W),(X,V)\in SmOp/k$:

$$\mathbb{Z}F^{pr}_*((Y,W),(X,V))=\ker[\mathbb{Z}F_*(Y,X)\oplus\mathbb{Z}F_*(W,V)\xrightarrow{i_Y^*-i_{X,*}}\mathbb{Z}F_*(W,X)],$$

where $i_Y:W\to Y$ is the embedding and $i_X:V\to X$ is the embedding. In other words, each group $\mathbb{Z}F^{pr}_n((Y,W),(X,V))$ consists of pairs $(a,b)\in\mathbb{Z}F_n(Y,X)\oplus\mathbb{Z}F_n(W,V)$ such that $i_X\circ b=a\circ i_Y$. By definition, the composite $(a,b)\circ(a',b')$ is the pair $((a\circ a'),(b\circ b'))$.

We define $\overline{\mathbb{Z}F}_*(k)$ as an additive category whose objects are those of Sm/k and Hom-groups are defined as follows. We set for every $n\geqslant 0$ and $X,Y\in Sm/k$:

$$\overline{\mathbb{Z}F}_*(Y,X)=\operatorname{Coker}[\mathbb{Z}F_*(\mathbb{A}^1\times Y,X)\xrightarrow{i_0^*-i_1^*}\mathbb{Z}F_*(Y,X)].$$

Next, one defines $\overline{\mathbb{Z}F}^{pr}_*(k)$ as an additive category whose objects are those of $SmOp/k$ and Hom-groups are defined as follows. We set for every $n\geqslant 0$ and $(X,V),(Y,W)\in SmOp/k$:

$$\overline{\mathbb{Z}F}^{pr}_*((Y,W),(X,V))=$$
$$=\operatorname{Coker}[\mathbb{Z}F^{pr}_*(\mathbb{A}^1\times(Y,W),(X,V))\xrightarrow{i_0^*-i_1^*}\mathbb{Z}F^{pr}_*((Y,W),(X,V)].$$

Definition 3.3. Using Definition 2.20 and literally repeating Definition 3.2, define groups

$$\overline{\mathbb{Z}F}_*(Y,X),\mathbb{Z}F^{pr}_*((Y,W),(X,V)),\overline{\mathbb{Z}F}^{pr}_*((Y,W),(X,V))$$

with $(Y,W) \in Sm'Op/k$ and $(X,V) \in SmOp/k$. These groups are covariantly functorial in morphisms of $\overline{\mathbb{Z}F}_*(k)$, $\mathbb{Z}F^{pr}_*(k)$ and $\overline{\mathbb{Z}F}^{pr}_*(k)$ respectively. These groups are also contravariantly functorial in morphisms of Sm'/k and in $Sm'Op/k$ respectively.

Notation 3.4. Let (Y,Y^0) be $Sm'Op/k$ and (X,X^0) be in $SmOp/k$. Given $a \in \mathbb{Z}F_*(Y,X)$, denote by $[a]$ its class in $\overline{\mathbb{Z}F}_*(Y,X)$.

Similarly, if $r=(a,b) \in \mathbb{Z}F^{pr}_*((Y,Y^0),(X,X^0))$, then we will write $[[r]]$ to denote its class in $\overline{\mathbb{Z}F}^{pr}_*((Y,Y^0),(X,X^0))$.

Let $(V,\varphi;g)$ be in $Fr_n(Y,X)$. If $i: Y^0 \hookrightarrow Y$ is open in Y, X^0 is open in X and $g(Z^0) \subset X^0$ with Z^0 the support of $(V,\varphi;g)\circ i$, then $\langle\langle V,\varphi;g\rangle\rangle$ will stand for the element

$$(\langle V,\varphi;g\rangle, \langle V^0,\varphi^0;g^0\rangle) \in \mathbb{Z}F_n((Y,Y^0),(X,X^0)),$$

where $V^0 := (Y^0 \times_Y V) \cap g^{-1}(X^0)$, $\varphi^0 = \varphi|_{V^0}$, $g^0 = g|_{V^0}$.

We will also write $[V,\varphi;g]$ to denote the class of $\langle V,\varphi;g\rangle$ in $\overline{\mathbb{Z}F}_n(Y,X)$. In turn, $[[V,\varphi;g]]$ will stand for the class of $\langle\langle V,\varphi;g\rangle\rangle$ of $\overline{\mathbb{Z}F}^{pr}_n((Y,Y^0),(X,X^0))$.

Remark 3.5. Clearly, the category $\mathbb{Z}F_*(k)$ is a full subcategory of $\mathbb{Z}F^{pr}_*(k)$ via the assignment $X \mapsto (X,\emptyset)$. Similarly, the category $\overline{\mathbb{Z}F}_*(k)$ is a full subcategory of $\overline{\mathbb{Z}F}^{pr}_*(k)$ via the assignment $X \mapsto (X,\emptyset)$.

In what follows we will also use the following groups.

Definition 3.6. Let (X,X^0) be in $SmOp/k$. Let $Y \in Sm'/k$ and $Y^0 \subset Y$ be an open subset. Let $j:(X^0,X^0) \hookrightarrow (X,X^0)$ be the open embedding. For any integer $n \geqslant 0$ set

$$\begin{aligned}&\overline{\overline{\mathbb{Z}F}}_n((Y,Y^0),(X,X^0)) = \\ &\qquad = \operatorname{Coker}[j_*: \overline{\mathbb{Z}F}_n((Y,Y^0),(X^0,X^0)) \to \overline{\mathbb{Z}F}_n((Y,Y^0),(X,X^0))],\end{aligned}$$

where j_* takes r to $j\circ r$.

These groups are contravariantly functorial with respect to morphisms $f:(Y_1,Y_1^0)\to(Y,Y^0)$ in $Sm'Op/k$. They are also covariantly functorial with respect to morphisms in $\overline{\mathbb{Z}F}^{pr}_*(k)$. Namely, if $s \in \overline{\mathbb{Z}F}^{pr}_n((X,X^0),(S,S^0))$, then the rule $r\mapsto s\circ r$ induces a homomorphism

$$s_*: \overline{\overline{\mathbb{Z}F}}_m((Y,Y^0),(X,X^0)) \to \overline{\overline{\mathbb{Z}F}}_{m+n}((Y,Y^0),(S,S^0)).$$

Notation 3.7. Let (Y,Y^0) be in $Sm'Op/k$ and (X,X^0) be in $SmOp/k$. If $r=(a,b)$ belongs to $\mathbb{Z}F^{pr}_m((Y,Y^0),(X,X^0))$, then we will write $\overline{[[r]]}$ for its class in $\overline{\overline{\overline{\mathbb{Z}F}}}_m((Y,Y^0),(X,X^0))$. For any morphism $f:(Y_1,Y_1^0)\to(Y,Y^0)$ in $Sm'Op/k$ and any $s\in\overline{\mathbb{Z}F}_n^{pr}((X,X^0),(S,S^0))$ set

$$\overline{[[r]]}\circ\overline{[[f]]}:=f^*(\overline{[[r]]}),\quad \overline{[[s]]}\circ\overline{[[r]]}:=s_*(\overline{[[r]]})$$

in $\overline{\overline{\overline{\mathbb{Z}F}}}_n((Y_1,Y_1^0),(X,X^0))$ and $\overline{\overline{\overline{\mathbb{Z}F}}}_{m+n}((Y,Y^0),(S,S^0))$ respectively.

Given (X,X^0) in $SmOp/k$, we will write $\langle\langle\sigma_X\rangle\rangle$ for the morphism $(1\cdot\sigma_X,1\cdot\sigma_{X^0})$ in $\mathbb{Z}F_1((X,X^0),(X,X^0))$, $[[\sigma_X]]$ will denote the class of $\langle\langle\sigma_X\rangle\rangle$ in $\overline{\mathbb{Z}F}_1((X,X^0),(X,X^0))$, and $\overline{[[\sigma_X]]}$ is its class in $\overline{\overline{\overline{\mathbb{Z}F}}}_n((X,X^0),(X,X^0))$.

The class of the element $[[V,\varphi;g]]$ in $\overline{\overline{\overline{\mathbb{Z}F}}}_n((Y,Y^0),(X,X^0))$ will be denoted by $\overline{[[V,\varphi;g]]}$.

Construction 3.8. Let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant $\mathbb{Z}F_*(k)$-presheaf of abelian groups. Then the assignments $(X,V)\mapsto\mathcal{F}(X,V):=\mathcal{F}(V)/Im(\mathcal{F}(X))$ and

$$(a,b)\mapsto[(a,b)^*=b^*:\mathcal{F}(V)/Im(\mathcal{F}(X))\to\mathcal{F}(W)/Im(\mathcal{F}(Y))],$$

for any $(a,b)\in\mathbb{Z}F_*((Y,W),(X,V))$ define a presheaf $\mathcal{F}^{pr}$ on the category $\overline{\mathbb{Z}F}_*^{pr}(k)$.

The nearest aim is to formulate a series of theorems (each of which is of independent interest), which are crucial for the proof of Theorem 1.1. To formulate these theorems, we use notation and definitions from this section.

Theorem 3.9 (Injectivity on the affine line)**.** *Let $U\subset\mathbb{A}^1_k$ be an open subset and let $i:V\hookrightarrow U$ be a non-empty open subset. Then there is a morphism $r\in\mathbb{Z}F_1(U,V)$ such that $[i]\circ[r]=[\sigma_U]$ in $\overline{\mathbb{Z}F}_1(U,U)$.*

Theorem 3.10 (Excision on the affine line)**.** *Let $U\subset\mathbb{A}^1_k$ be an open subset. Let $i:V\hookrightarrow U$ be an open inclusion with V non-empty. Let $S\subset V$ be a proper closed subset. Then there are morphisms $r\in\mathbb{Z}F_1((U,U-S),(V,V-S))$ and $l\in\mathbb{Z}F_1((U,U-S),(V,V-S))$ such that*

$$\overline{[[i]]}\circ\overline{[[r]]}=\overline{[[\sigma_U]]}\quad\textit{and}\quad\overline{[[l]]}\circ\overline{[[i]]}=\overline{[[\sigma_V]]}$$

in $\overline{\overline{\overline{\mathbb{Z}F}}}_1((U,U-S),(U,U-S))$ and $\overline{\overline{\overline{\mathbb{Z}F}}}_1((V,V-S),(V,V-S))$ respectively.

Theorem 3.11 (Injectivity for local schemes)**.** *Let $X\in Sm/k$ be irreducible, $x\in X$ be a point, $U=Spec(\mathcal{O}_{X,x})$, $D\subsetneq X$ be a closed subset. Then there exists an integer N and an element $r\in\mathbb{Z}F_N(U,X-D)$ such that*

$$[j]\circ[r]=[\sigma_X^N]\circ[can]$$

in $\overline{\mathbb{Z}F}_N(U,X)$ *with* $j : X - D \hookrightarrow X$ *the open inclusion and* $can : U \to X$ *the canonical morphism.*

Theorem 3.12 (Excision on the relative affine line)**.** *Let* $W \in Sm/k$ *be an affine variety. Let* $i : V = (\mathbb{A}^1_W)_f \subset \mathbb{A}^1_W$ *be an affine open subset, where* $f \in k[W][t]$ *is a monic polynomial such that* $f(0) \in k[W]^\times$*. Then there are morphisms*

$$r \in \mathbb{Z}F_1((\mathbb{A}^1_W, \mathbb{A}^1_W - 0 \times W), (V, V - 0 \times W)) \quad \textit{and} \quad l \in \mathbb{Z}F_1((\mathbb{A}^1_W, \mathbb{A}^1_W - 0 \times W), (V, V - 0 \times W))$$

such that

$$\overline{[[i]]} \circ \overline{[[r]]} = \overline{[[\sigma_{\mathbb{A}^1_W}]]} \quad \textit{and} \quad \overline{[[l]]} \circ \overline{[[i]]} = \overline{[[\sigma_V]]}$$

in $\overline{\overline{\mathbb{Z}F}}_1((\mathbb{A}^1_W, \mathbb{A}^1_W - 0 \times W), (\mathbb{A}^1_W, \mathbb{A}^1_W - 0 \times W))$ *and* $\overline{\overline{\mathbb{Z}F}}_1((V, V - 0 \times W), (V, V - 0 \times W))$ *respectively.*

To formulate two further theorems concering étale excision properties, we need some preparations. Let X, X' be in Sm/k and let both be irreducible. Suppose $V \subset X$ and $V' \subset X'$ are open subschemes. Let

$$\begin{array}{ccc} V' & \longrightarrow & X' \\ \downarrow & & \downarrow{\scriptstyle \Pi} \\ V & \longrightarrow & X \end{array}$$

be an elementary distinguished square in the sense of [13, Definition 3.1.3]. This means that Π is etale, the square is cartesian and, moreover, if $S = X - V$ and $S' = X' - V'$ are closed subschemes equipped with reduced structures, then Π induces a scheme isomorphism $S' \to S$. Let $x \in S$ and $x' \in S'$ be two points such that $\Pi(x') = x$. Let $U = Spec(\mathcal{O}_{X,x})$ and $U' = Spec(\mathcal{O}_{X',x'})$. Let $\pi : U' \to U$ be the morphism induced by Π.

Theorem 3.13 (Injective étale excision)**.** *Under the notation above there is an integer* N *and an element* $r \in \mathbb{Z}F_N((U, U - S), (X', X' - S'))$ *such that*

$$\overline{[[\Pi]]} \circ \overline{[[r]]} = \overline{[[\sigma_X^N]]} \circ \overline{[[can]]}$$

in $\overline{\overline{\mathbb{Z}F}}_N((U, U - S), (X, X - S))$, *where* $can : U \to X$ *is the canonical morphism.*

The statements of the next theorem depend on the characteristic of the base field k.

Theorem 3.14 (Surjective étale excision)**.** *Under the above notations suppose in addition that S is k-smooth and k is of characteristic different from 2. Then there is an integer N and an element $l \in \mathbb{Z}F_N((U, U-S),(X', X'-S'))$ such that*

$$\overline{[[l]]} \circ \overline{[[\pi]]} = \overline{[[\sigma^N_{X'}]]} \circ \overline{[[can']]}$$

in $\overline{\overline{\mathbb{Z}F}}_N((U', U'-S'),(X', X'-S'))$ with $can' : U' \to X'$ the canonical morphism.

If the charcteristic of k is 2, then there is an integer N and an element $l \in \mathbb{Z}F_N((U, U-S),(X', X'-S'))$ such that

$$2 \cdot \overline{[[l]]} \circ \overline{[[\pi]]} = 2 \cdot \overline{[[\sigma^N_{X'}]]} \circ \overline{[[can']]}$$

in $\overline{\overline{\mathbb{Z}F}}_N((U', U'-S'),(X', X'-S'))$.

To formulate Theorem 3.15 it is convenient to give the following comments. Consider the inclusion of categories $inc : Sm/k \to Sm'/k$, where Sm'/k is the category of essentially smooth schemes over k. Then for any presheaf G of Abelian groups on Sm/k the restriction of the presheaf $inc_*(G)$ on Sm'/k to Sm/k equals G (that is $inc^*(inc_*(G)) = G$ on Sm/k). For any essentially smooth scheme Y over k we will use notation $G(Y)$ instead of $inc_*(G)(Y)$. Any $\mathbb{Z}F_*(k)$-presheaf $\mathcal{F}$ can be regarded as a Sm/k-presheaf. So this comment is applicable to the presheaf $\mathcal{F}|_{Sm/k}$ and allows to state items (3) and (3') of Theorem 3.15.

Let $(X, V),(X', V') \in SmOp/k$ be the pairs from the elementary distinguished square above. The assignments $X_0 \mapsto \mathcal{F}^{pr}(X_0, X_0 \cap V)$, $X'_0 \mapsto \mathcal{F}^{pr}(X'_0, X'_0 \cap V')$ are Zariski presheaves on X and on X' respectively. So they have Zariski stalks at the points x and x' respectively. The morphism $\Pi : X' \to X$ induces the pull-back map between these stalks. It is written as $[[\pi]]^*$ in the item (5) of Theorem 3.15.

We are now in a position to state the following

Theorem 3.15. *For any $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-presheaf of abelian groups $\mathcal{F}$ the following statements are true:*

(1) under the assumptions of Theorem 3.9 the map $i^* : \mathcal{F}(U) \to \mathcal{F}(V)$ is injective;

(2) under the assumptions of Theorem 3.10 the map

$$[[i]]^* : \mathcal{F}(U-S)/\mathcal{F}(U) \to \mathcal{F}(V-S)/\mathcal{F}(V)$$

is an isomorphism;

(3) under the assumptions of Theorem 3.11 the map

$$\eta^* : \mathcal{F}(U) \to \mathcal{F}(Spec(k(X)))$$

is injective, where $\eta : Spec(k(X)) \to U$ is the canonical morphism;

(3') under the assumptions of Theorem 3.11 let U^h_x be the henselization of U at x and let $k(U^h_x)$ be the function field on U^h_x. Then the map

$$\eta^*_h : \mathcal{F}(U^h_x) \to \mathcal{F}(Spec(k(U^h_x)))$$

is injective, where $\eta_h : Spec(k(U^h_x)) \to U^h_x$ is the canonical morphism;

(4) under the assumptions of Theorem 3.12 the map

$$[[i]]^* : \mathcal{F}(\mathbb{A}^1_W - 0 \times W)/\mathcal{F}(\mathbb{A}^1_W) \to \mathcal{F}(V - 0 \times W)/Im(\mathcal{F}(V))$$

is an isomorphism;

(5) under the assumptions of Theorems 3.13 and 3.14 the map

$$[[\pi]]^* : \mathcal{F}(U - S)/\mathcal{F}(U) \to \mathcal{F}(U' - S')/\mathcal{F}(U')$$

is an isomorphism whenever the characteristic of k is different from 2.

If the characteristic of k is 2 and the presheaf $\mathcal{F}$ is a presheaf of $\mathbb{Z}[1/2]$-modules, then the map

$$[[\pi]]^* : \mathcal{F}(U - S)/\mathcal{F}(U) \to \mathcal{F}(U' - S')/\mathcal{F}(U')$$

is an isomorphism.

Remark 3.16. By [22, Proposition 3.1.11] any presheaf with transfers in the sense of [22] is a pretheory in the sense of [21, Definition 3.1]. This allows Voevodsky to conclude that all results from [21] are applicable to homotopy invariant presheaves with transfers. This is a reason to make a link between the preceding theorem and some results from [21].

The assertions (1) and (2) are similar to some assertions from the proof of [21, Theorem 4.15]. The assertion (3) is similar to [21, Corollary 4.18]. The assertion (4) (together with Corollary 3.19 below) is reminiscent of [21, Proposition 4.11]. The assertion (5) is similar to [21, Corollary 4.13].

Theorem 3.15 is derived from Theorems 3.9–3.14 as we will show below in this section. In turn, Theorems 3.9, 3.10 and 3.12 will be proved in Sections 6 and 7 respectively. Theorem 3.11 will be derived from Theorem

14.3 in Section 9. In turn, Theorem 14.3 will be proved in Section 14. In Section 11, Theorem 3.13 will be derived from Proposition 10.9 and Theorem 14.3 (Theorem 14.3 will be proved in Section 14). Proposition 10.9 will be proved in Section 15. In Section 13, Theorem 3.14 will be derived from Proposition 12.6 and Theorems 14.4–14.5. In turn, Theorems 14.4–14.5 will be proved in Section 14. Proposition 12.6 will be proved in Section 15.

To derive all assertions of Theorem 3.15 except (1), (2) and (4) we need a couple of lemmas.

Lemma 3.17. *Under the hypotheses of Lemma 2.21 the following map is an isomorphism*

$$can_M : \operatorname{colim}_{m\in M} \overline{\mathbb{Z}F}_n(Y_m, X) \to \overline{\mathbb{Z}F}_n(Y_M, X).$$

The same is true if we replace Y_M with $Y_{0,M}$, where $Y_0 \subset Y$ is an open subset.

Proof. By Corollary 2.22 one has $\mathbb{Z}F_n(\mathbb{A}^1 \times Y_M, X) = \operatorname{colim}_{m\in M} \mathbb{Z}F_n(\mathbb{A}^1 \times Y_m, X)$ and $\mathbb{Z}F_n(Y_M, X) = \operatorname{colim}_{m\in M}\mathbb{Z}F_n(Y_m, X)$. This proves the lemma. □

Lemma 3.18. *Under the hypotheses of Lemma 2.21 the following maps are isomorphisms:*

$$can_M : \operatorname{colim}_{m\in M} \mathbb{Z}F_n((Y_m, Y_{0,m}), (X, X_0)) \to \mathbb{Z}F_n((Y_M, Y_{0,M}), (X, X_0)),$$

$$can_M : \operatorname{colim}_{m\in M} \overline{\mathbb{Z}F}_n((Y_m, Y_{0,m}), (X, X_0)) \to \overline{\mathbb{Z}F}_n((Y_M, Y_{0,M}), (X, X_0)),$$

$$can_M : \operatorname{colim}_{m\in M} \overline{\overline{\mathbb{Z}F}}_n((Y_m, Y_{0,m}), (X, X_0)) \to \overline{\overline{\mathbb{Z}F}}_n((Y_M, Y_{0,M}), (X, X_0)).$$

Proof. Prove the first assertion. For any pair (S, S_0) in $EssSmOp/k$ one has $\mathbb{Z}F_n((S, S_0), (X, X_0)) = \ker[\mathbb{Z}F_n(S, X) \oplus \mathbb{Z}F_n(S_0, X_0) \xrightarrow{i_S^*-i_{Y,*}} \mathbb{Z}F_n(S_0, X)]$ by Definition 3.2. Therefore $\mathbb{Z}F_n((Y_M, Y_{0,M}), (X, X_0)) = \ker[\mathbb{Z}F_n(Y_M, X) \oplus \mathbb{Z}F_n(Y_{0,M}, X_0) \to \mathbb{Z}F_n(Y_{0,M}, X)]$ and for any $m \in M$, we have

$$\begin{aligned}\mathbb{Z}F_n((Y_m, Y_{0,m}), (X, X_0)) &= \\ &= \operatorname{Ker}[\mathbb{Z}F_n(Y_m, X) \oplus \mathbb{Z}F_n(Y_{0,m}, X_0) \to \mathbb{Z}F_n(Y_{0,m}, X)].\end{aligned}$$

Corollary 2.22 completes the proof of the first assertion.

The second assertion follows from the first assertion in the same fashion as Lemma 3.17 was derived from Corollary 2.22. To prove the third assertion, recall that for any $(S, S_0) \in EssSmOp/k$ one has $\overline{\overline{\mathbb{Z}F}}_n((S, S_0), (X, X_0)) =$

$\mathrm{Coker}[j_* : \overline{\mathbb{Z}F}_n((S,S_0),(X_0,X_0)) \to \overline{\mathbb{Z}F}_n((S,S_0),(X,X_0))]$ by Definition 3.6. Hence $\overline{\overline{\mathbb{Z}F}}_n((Y_M,Y_{0,M}),(X,X_0)) = \mathrm{Coker}[j_* : \overline{\mathbb{Z}F}_n((Y_M,Y_{0,M}),(X_0,X_0)) \to \overline{\mathbb{Z}F}_n((Y_M,Y_{0,M}),(X,X_0))]$ and for any $m \in M$, $\overline{\overline{\mathbb{Z}F}}_n((Y_m,Y_{0,m}),(X,X_0)) = \mathrm{Coker}[j_* : \overline{\mathbb{Z}F}_n((Y_m,Y_{0,m}),(X_0,X_0)) \to \overline{\mathbb{Z}F}_n((Y_m,Y_{0,m}),(X,X_0))]$. Applying the second part of the lemma, we complete the proof of the third assertion. □

Reducing Theorem 3.15 to Theorems 3.9–3.14. Theorem 3.9 implies the assertion (1). To prove the assertion (2), use Construction 3.8 and apply Theorem 3.10. To prove the assertion (4), use Construction 3.8 and apply Theorem 3.12.

Let us prove assertion (3). Let $a \in \mathcal{F}(U)$ be such that $\eta^*(a) = 0$. Shrinking X, one can find an element $a' \in \mathcal{F}(X)$ such that $a'|_U = a$. Since $a'|_{Spec(k(X))} = 0$, there is a closed subset $D \subsetneq X$ such that $a'|_{X-D} = 0$. By Theorem 3.11, Corollary 2.22 and Lemma 3.17 there are a Zariski open subset U_1 containing x and a morphism $r_1 \in \mathbb{Z}F_N(U_1, X-D)$ such that $[j] \circ [r_1] = [\sigma_X^N] \circ [in]$ in $\overline{\mathbb{Z}F}_N(U_1, X)$ (here $in : U_1 \hookrightarrow X$ is the embedding). Since $\sigma_X^N \circ in = in \circ \sigma_{U_1}^N$ and $\mathcal{F}$ is $\mathbb{A}^1$-invariant, we have $0 = r_1^*(j^*(a')) = (\sigma_{U_1}^N)^*(in^*(a'))$. Since $\mathcal{F}$ is quasi-stable, we have $in^*(a') = 0$ and $a = a'|_U = in^*(a'|_U) = 0$. The assertion (3) is proved. The assertion (3') is a simple consequence of the assertion (3).

We use Construction 3.8 in the rest of the proof. Prove the assertion (5). Prove its first part when $\mathrm{char}(k) \neq 2$. Let us verify injectivity of $[[\pi]]^*$. Let $a \in \mathcal{F}^{pr}(U, U-S)$ be such that $[[\pi]]^*(a) = 0$ in $\mathcal{F}^{pr}(U', U'-S')$. Replacing X with an open $X_1 \subset X$, we may assume that $a = a_1|_{(U,U-S)}$ for an element $a_1 \in \mathcal{F}^{pr}(X_1, X_1 - S)$. Set $X_1' = \Pi^{-1}(X_1)$ and $\Pi_1 = \Pi|_{X_1'} : X_1' \to X_1$. Then the square consisting of $X_1', X_1, X_1' - S', X_1 - S$ and the obvious morphisms including Π_1 is an elementary distinguished square. For any open X_2 in X_1 write a_2 for $a_1|_{(X_2, X_2-S)}$. Since Π is étale it is an open morphism. Replacing X_1' with a neighborhood X_2' of the point $x' \in X'$ and setting $X_2 = \Pi_1(X_2')$ we may assume that $[[\Pi_2]]^*(a_2) = 0$, where $\Pi_2 = \Pi|_{X_2'} : X_2' \to X_2$. One can check that the square consisting of $X_2', X_2, X_2' - S', X_2 - S$ and the obvious morphisms including Π_2 is an elementary distinguished square. Using Theorem 3.13 and Lemma 3.18 one can find a neighborhood $U_1 \subset X_2$ of the point $x \in X$, morphisms $r_1 \in \mathbb{Z}F_1(U_1, U_1 - S), (X_2', X_2' - S'))$, $b \in \mathbb{Z}F_1((U_1, U_1 - S), (X_2 - S, X_2 - S))$ such that

$$[[\Pi_2]] \circ [[r_1]] = [[\sigma_{X_2}]] \circ [[In]] + [[J]] \circ [[b]] \in \overline{\mathbb{Z}F}_1((U_1, U_1 - S), (X_2, X_2 - S)),$$

where $In : (U_1, U_1 - S) \hookrightarrow (X_2, X_2 - S)$, $J : (X_2 - S, X_2 - S) \hookrightarrow (X_2, X_2 - S)$ are embeddings. Clearly, $\mathcal{F}^{pr}(X_2 - S, X_2 - S) = 0$. Since $[[\sigma_{X_2}]] \circ [[In]] =$

$[[In]]\circ[[\sigma_{U_1}]]$ and $[[\Pi_2]]^*(a_2)=0$ in $\mathcal{F}^{pr}(X_2',X_2'-S')$, we have

$$0=[[r_1]]^*([[\Pi_2]]^*(a_2))=[[\sigma_{U_1}]]^*([[In]]^*(a_2)).$$

Since $\mathcal{F}$ is quasi-stable we get equalities $a_2|_{(U_1,U_1-S)}=[[In]]^*(a_2)=0$ in $\mathcal{F}^{pr}(U_1,U_1-S)$. Thus, $a=a_1|_{(U,U-S)}=a_2|_{(U,U-S)}=0$.

Let us verify that $[[\pi]]^*$ is surjective. Take an element $a\in\mathcal{F}^{pr}(U',U'-S')$. Replacing X' with an open $X_1'\subset X'$ we may assume that $a=a_1|_{(U',U'-S')}$ for some $a_1\in\mathcal{F}^{pr}(X_1',X_1'-S')$. Set $X_1=\Pi(X_1')$ and $\Pi_1=\Pi|_{X_1'}:X_1'\to X_1$. Then the square consisting of $X_1',X_1,X_1'-S',X_1-S$ and the obvious morphisms including Π_1 is an elementary distinguished square. Since $\mathcal{F}$ is quasi-stable there is an element $a_1'\in\mathcal{F}^{pr}(X_1',X_1'-S')$ such that $a_1=[[\sigma_{X_1'}]]^*(a_1')$. Using Theorem 3.14 and Lemma 3.18 one can find neighborhoods U_2, $U_2'\subset\Pi^{-1}(U_2)$ of points $x\in X$, $x'\in X'$ respectively and morphisms $l_2\in\mathbb{Z}F_1((U_2,U_2-S),(X_1',X_1'-S'))$, $c\in\mathbb{Z}F_1((U_2',U_2'-S'),(X_1'-S',X_1'-S'))$ such that

$$[[l_2]]\circ[[\pi_2]]=[[\sigma_{X_2'}]]\circ[[in]]+[[j]]\circ[[c]]\in\overline{\mathbb{Z}F}_1((U_2',U_2'-S'),(X_1',X_1'-S')),$$

where $in:(U_2',U_2'-S')\hookrightarrow(X_1',X_1'-S')$, $j:(X'-S',X'-S')\hookrightarrow(X_1',X_1'-S')$ are embeddings and $\pi_2=\Pi|_{U_2'}:U_2'\to U_2$. Clearly, $\mathcal{F}^{pr}(X_1'-S',X_1'-S')=0$. Thus we have

$$[[\pi_2]]^*([[l_2]]^*(a_1'))=[[in]]^*([[\sigma_{X_1'}]]^*(a_1'))=[[in]]^*(a_1)\in\mathcal{F}^{pr}(U_2',U_2'-S').$$

Set $\tilde a=[[l_2]]^*(a_1')$. Then,

$$a=a_1|_{(U',U'-S')}=([[in]]^*(a_1))|_{(U',U'-S')}=$$
$$=([[\pi_2]]^*(\tilde a))|_{(U',U'-S')}=[[\pi]]^*(\tilde a|_{(U,U-S)}).$$

The surjectivity of $[[\pi]]^*$ is proved. The case when $\mathrm{char}(k)\neq 2$ of the assertion (5) is proved. If $\mathrm{char}(k)=2$ the proof is similar. □

Reducing Theorem 3.1 to Theorem 3.15. We provide the reduction for fields of characteristic not 2 and leave the reader the case of characteristic 2. By Corollary 2.17 the sheaf $\mathcal{F}_{\mathrm{nis}}$ has a unique structure of a $\mathbb{Z}F_*(k)$-presheaf of Abelian groups such that the morphism $\mathcal{F}\to\mathcal{F}_{\mathrm{nis}}$ is a morphism of $\mathbb{Z}F_*(k)$-presheaves.

We now prove that $\mathcal{F}_{\mathrm{nis}}$ is $\mathbb{A}^1$-invariant. Firstly, (1) and (2) imply $\mathcal{F}|_{\mathbb{A}^1}$ is a Zariski sheaf. Using (5) applied to $X=\mathbb{A}^1$, one shows that for any open V in $\mathbb{A}^1$ one has $\mathcal{F}_{\mathrm{nis}}(V)=\mathcal{F}(V)$.

Now consider the following Cartesian square of schemes

$$\begin{array}{ccc} Spec(k(X)) & \xrightarrow{\eta} & X \\ {\scriptstyle i_{0,k(X)}}\downarrow & & \downarrow{\scriptstyle i_{0,X}} \\ \mathbb{A}^1_{k(X)} & \xrightarrow{\eta\times id} & X\times\mathbb{A}^1 \end{array}$$

Evaluating the Nisnevich sheaf $\mathcal{F}_{\text{nis}}$ on this square, we get a square of abelian groups

$$\begin{array}{ccc} \mathcal{F}_{\text{nis}}(Spec(k(X))) & \xleftarrow{\eta^*} & \mathcal{F}_{\text{nis}}(X) \\ {\scriptstyle i^*_{0,k(X)}}\uparrow & & \uparrow{\scriptstyle i^*_{0,X}} \\ \mathcal{F}_{\text{nis}}(\mathbb{A}^1_{k(X)}) & \xleftarrow{(\eta\times id)^*} & \mathcal{F}_{\text{nis}}(X\times\mathbb{A}^1) \end{array}$$

The map $i^*_{0,X}$ is plainly surjective. It remains to check its injectivity. The map $(\eta \times id)^*$ is injective (apply Theorem 3.15(3')). We have already proved above that $\mathcal{F}_{\text{nis}}(\mathbb{A}^1_{k(X)}) = \mathcal{F}(\mathbb{A}^1_{k(X)})$. Since $\mathcal{F}_{\text{nis}}(Spec(k(X)) = \mathcal{F}(Spec(k(X))$, we see that the map $i^*_{0,k(X)}$ is an isomorphism. Thus the map $i^*_{0,X}$ is injective.

Now prove that $\mathcal{F}_{\text{nis}}$ is quasi-stable. Let $\mathcal{F}$ be a $\mathbb{Z}F_*(k)$-presheaf. The property $\sigma_X \circ f = f \circ \sigma_X$ from Definition 2.4 yields the following: the assignment $X \mapsto (\sigma_X^* : \mathcal{F}(X) \to \mathcal{F}(X))$ is an endomorphism of the presheaf $\mathcal{F}|_{Sm/k}$. Denote it by σ. If $\mathcal{F}$ is quasi-stable, then σ acts as an isomorphism on Nisnevich stalks of the presheaf $\mathcal{F}$. We already know that $\mathcal{F}_{\text{nis}}$ is a $\mathbb{Z}F_*(k)$-presheaf of Abelian groups. Lemma 2.8 yields that σ acts as an isomorphism on Nisnevich stalks of $\mathcal{F}_{\text{nis}}$. Thus for any $X \in Sm/k$ the map $\sigma_X : \mathcal{F}_{\text{nis}}(X) \to \mathcal{F}_{\text{nis}}(X)$ is an isomorphism. Hence $\mathcal{F}_{\text{nis}}$ is quasi-stable as required. □

We finish the section by proving the following useful statement, which is a consequence of Theorem 3.15(4):

Corollary 3.19. *Let $X \in Sm/k$, $x \in X$ be a point, $W = Spec(\mathcal{O}_{X,x})$. Let $\mathcal{V} := Spec(\mathcal{O}_{W\times\mathbb{A}^1,(x,0)})$ and $can : \mathcal{V} \hookrightarrow W \times \mathbb{A}^1$ be the canonical embedding. Let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-presheaf of abelian groups. Then the pullback map*

$$[[can]]^* : \mathcal{F}(W \times (\mathbb{A}^1 - \{0\}))/\mathcal{F}(W \times \mathbb{A}^1) \to \mathcal{F}(\mathcal{V} - W \times \{0\})/\mathcal{F}(\mathcal{V})$$

is an isomorphism (both quotients make sense: the second quotient makes sense due to Theorem 3.15(3), the first one makes sense due to homotopy invariance of $\mathcal{F}$).

Proof. Consider the category, $\mathcal{C}$, of Zariski neighborhoods of $W\times 0$ in $W\times\mathbb{P}^1$ as well as the presheaf $V\mapsto \mathcal{F}(V-W\times 0)/Im(\mathcal{F}(V))$ on $\mathcal{C}$. Write $G(V)$ for $\mathcal{F}(V-W\times 0)/Im(\mathcal{F}(V))$ and $G(\mathcal{V})$ for $\mathcal{F}(\mathcal{V}-W\times 0)/\mathcal{F}(\mathcal{V})$. Clearly, the category $\mathcal{C}$ is cofiltered. By definition, one has

$$\mathcal{F}(\mathcal{V})=\operatorname{colim}\,\mathcal{F}(V)\quad\text{and}\quad \mathcal{F}(\mathcal{V}-W\times 0)=\operatorname{colim}\,\mathcal{F}(V-W\times 0),$$

where V runs over all Zariski neighborhoods of $W\times 0$ in $W\times\mathbb{P}^1$. Thus $G(\mathcal{V})=\operatorname{colim} G(V)$. For any closed subset $D\subset W\times\mathbb{P}^1$ of pure codimension one (including the empty D as well) and any section $H\subset W\times\mathbb{P}^1$ of the projection $p_W:W\times\mathbb{P}^1\to W$ with $H\cap D=\emptyset=(W\times 0)\cap(H\cup D)$ set

$$V_{H,D}=W\times\mathbb{P}^1-(H\cup D).$$

Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ consisting of objects of the form $V_{H,D}$. Since the base field k is infinite and W is regular local, then the subcategory $\mathcal{C}'$ is cofinal in $\mathcal{C}$. Thus $G(\mathcal{V})=\operatorname{colim}\,G(V_{H,D})$, where $V_{H,D}$ runs over the category $\mathcal{C}'$. Let $V_{H_1,D_1},V_{H_2,D_2}\in\mathcal{C}'$ be such that $V_{H_2,D_2}\subset V_{H_1,D_1}$ and let $\alpha:V_{H_2,D_2}\hookrightarrow V_{H_1,D_1}$ be the inclusion.

We claim that the pullback map $[[\alpha]]^*:G(V_{H_1,D_1})\to G(V_{H_2,D_2})$ is an isomorphism. To prove this claim, set $D=H_1\cup D_1\cup H_2\cup D_2$ and find a section $H\subset W\times\mathbb{P}^1$ of the projection p_W such that $H\cap D=\emptyset=(W\times 0)\cap H$. Using a projective change of coordinates on $W\times\mathbb{P}^1$, we may assume that $H=W\times\infty$ and $W\times 0$ remains the same. Consider the open inclusions

$$V_{H_2,D_2}\xrightarrow{\alpha}V_{H_1,D_1}\xrightarrow{\beta}W\times\mathbb{A}^1=V_{H,\emptyset}$$

and set $\gamma=\beta\circ\alpha$. By Theorem 3.15(4) the maps $[[\beta]]^*:G(V_{H,\emptyset})\to G(V_{H_1,D_1})$ and $[[\gamma]]^*:G(V_{H,\emptyset})\to G(V_{H_2,D_2})$ are isomorphisms. Thus the map $[[\alpha]]^*$ is an isomorphism in this case, too. This proves the claim.

Thus for any $V_{H,D}\in\mathcal{C}'$ the map $G(V_{H,D})\to G(\mathcal{V})$ is an isomorphism. Particularly, the map $[[can]]^*:G(W\times\mathbb{A}^1)\to G(\mathcal{V})$ is an isomorphism. This proves the corollary. □

4. Notation and agreements

In this section we follow definitions, notation and constructions from Sections 2 and 3. We suppose in this section that $Y\in Sm'/k$ and $X\in Sm/k$. Particularly, Y can be in Sm/k.

Notation 4.1. We are recalling earlier defined notation for the convenience of the reader. Given $a \in Fr_n(Y,X)$, we write $\langle a \rangle$ for the image of $1 \cdot a$ in $\mathbb{Z}F_n(Y,X)$ and write $[a]$ for the class of $\langle a \rangle$ in $\overline{\mathbb{Z}F}_n(Y,X)$. We will write Z_a for the support of a (it is a closed subset in $Y \times \mathbb{A}^n$ which is finite over Y and determined by the element a uniquely). Also, we will often write

$$(\mathcal{V}_a, \varphi_a : \mathcal{V}_a \to \mathbb{A}^n; g_a : \mathcal{V}_a \to X) \quad \text{or just} \quad (\mathcal{V}_a, \varphi_a; g_a)$$

for a representative of the morphism a (here $(\mathcal{V}_a, \rho : \mathcal{V}_a \to Y \times \mathbb{A}^n, s : Z_a \hookrightarrow \mathcal{V}_a)$ is an étale neighborhood of Z_a in $Y \times \mathbb{A}^n$).

Remark 4.2. If the support Z_a of an element $a = (\mathcal{V}, \varphi; g) \in Fr_n(Y,X)$ is a disjoint union of Z_1 and Z_2, then the element a determines two elements a_1 and a_2 in $Fr_n(Y,X)$. Namely, $a_1 = (\mathcal{V} - Z_2, \varphi|_{\mathcal{V}-Z_2}; g|_{\mathcal{V}-Z_2})$ and $a_2 = (\mathcal{V} - Z_1, \varphi|_{\mathcal{V}-Z_1}; g|_{\mathcal{V}-Z_1})$. Moreover, by definition of $\mathbb{Z}F_n(Y,X)$ one has the equality

$$\langle a \rangle = \langle a_1 \rangle + \langle a_2 \rangle$$

in $\mathbb{Z}F_n(Y,X)$.

Definition 4.3. Let $i_Y : Y' \hookrightarrow Y$ and $i_X : X' \hookrightarrow X$ be open embeddings. Let $a \in Fr_n(Y,X)$. We say that the *restriction* $a|_{Y'}$ *of* a *to* Y' *runs inside* X' if there is $a' \in Fr_n(Y',X')$ such that

$$i_X \circ a' = a \circ i_Y \tag{1}$$

in $Fr_n(Y',X)$.

It is easy to see that if there is an element a' satisfying condition (1), then it is unique. In this case the pair (a,a') is an element of $\mathbb{Z}F_n((Y,Y'),(X,X'))$. For brevity we will write $\langle\langle a \rangle\rangle$ for $(a,a') \in \mathbb{Z}F_n((Y,Y'),(X,X'))$ and write $[[a]]$ to denote the class of $\langle\langle a \rangle\rangle$ in $\overline{\mathbb{Z}F}_n((Y,Y'),(X,X'))$.

Lemma 4.4. *Let $i_Y : Y' \hookrightarrow Y$ and $i_X : X' \hookrightarrow X$ be open embeddings. Let $a \in Fr_n(Y,X)$. Let $Z_a \subset Y \times \mathbb{A}^n$ be the support of a. Set $Z'_a = Z_a \cap Y' \times \mathbb{A}^n$. Then the following are equivalent:*

(1) $g_a(Z'_a) \subset X'$;
(2) the element $a|_{Y'}$ runs inside X'.

Proof. (1) $\Rightarrow$ (2). Set $\mathcal{V}' = p_Y^{-1} \cap g^{-1}(X')$, where $p_Y = pr_Y \circ \rho_a : \mathcal{V} \to Y \times \mathbb{A}^n$. Then $a' := (\mathcal{V}', \varphi|_{\mathcal{V}'}); g|_{\mathcal{V}'}) \in Fr_n(Y',X')$ satisfies condition (1).

(2) $\Rightarrow$ (1). If $a|_{Y'}$ runs inside X', then for some $a' = (\mathcal{V}', \varphi'; g') \in Fr_n(Y',X'))$ equality (1) holds. In this case the support Z' of a' must coincide with $Z'_a = Z_a \cap Y' \times \mathbb{A}^n$ and $g_a|_{Z'} = g'|_{Z'}$. Since $g'(Z')$ is a subset of X', then $g_a(Z'_a) = g_a(Z') \subset X'$. ◻

Corollary 4.5. *Let $i_Y : Y' \hookrightarrow Y$ and $i_X : X' \hookrightarrow X$ be open embeddings. Let $h_\theta = (\mathcal{V}_\theta, \varphi_\theta; g_\theta) \in Fr_n(\mathbb{A}^1 \times Y, X)$. Suppose Z_θ, the support of h_θ, is such that for $Z'_\theta := Z_\theta \cap \mathbb{A}^1 \times Y' \times \mathbb{A}^n$ one has $g_\theta(Z'_\theta) \subset X'$. Let $i_0, i_1 : Y \to \mathbb{A}^1 \times Y$ be the 0- and 1-sections respectively and set $h_0 = h_\theta \circ i_0$, $h_1 = h_\theta \circ i_1$. Then h_θ, h_0, h_1 define elements $\langle\langle h_\theta \rangle\rangle \in \mathbb{Z}F_n(\mathbb{A}^1 \times (Y, Y'), (X, X'))$, $\langle\langle h_0 \rangle\rangle, \langle\langle h_1 \rangle\rangle \in \mathbb{Z}F_n((Y, Y'), (X, X'))$ with*

$$[[h_0] = [[h_1]]$$

in $\overline{\mathbb{Z}F}_n((Y, Y'), (X, X'))$.

Lemma 4.6 (A disconnected support case)**.** *Let $i_Y : Y' \hookrightarrow Y$ and $i_X : X' \hookrightarrow X$ be open embeddings. Let $a \in Fr_n(Y, X)$ and let $Z_a \subset Y \times \mathbb{A}^n$ be the support of a. Set $Z'_a = Z_a \cap Y' \times \mathbb{A}^n$. Suppose that $Z_a = Z_{a,1} \sqcup Z_{a,2}$. For $i = 1, 2$ set $\mathcal{V}_i = \mathcal{V}_a - Z_{a,j}$ with $j \in \{1, 2\}$ and $j \neq i$. Also, set $\varphi_i = \varphi_a|_{\mathcal{V}_i}$ and $g_i = g_a|_{\mathcal{V}_i}$. Suppose $a|_{Y'}$ runs inside X', then*

(1) for each $i = 1, 2$ the element $a_i := (\mathcal{V}_i, \varphi_i; g_i)$ is such that $a_i|_{Y'}$ runs inside X';
(2) $\langle\langle a \rangle\rangle = \langle\langle a_1 \rangle\rangle + \langle\langle a_2 \rangle\rangle$ in $\mathbb{Z}F_n((Y, Y'), (X, X'))$.

5. Some homotopies

Suppose $U, W \subset \mathbb{A}^1_k$ are open and non-empty.

Lemma 5.1. *Let $a_0 = (\mathcal{V}, \varphi; g_0) \in Fr_1(U, W)$, $a_1 = (\mathcal{V}, \varphi; g_1) \in Fr_1(U, W)$. Denote their common support by Z. If $g_0|_Z = g_1|_Z$, then $[a_0] = [a_1]$ in $\overline{\mathbb{Z}F}_1(U, W)$.*

Proof. Consider a function $g_\theta = (1 - \theta)g_0 + \theta g_1 : \mathbb{A}^1 \times \mathcal{V} \to \mathbb{A}^1$ and set $\mathcal{V}_\theta = g_\theta^{-1}(W)$, $\varphi_\theta = \varphi \circ pr_{\mathcal{V}} : \mathcal{V}_\theta \to \mathbb{A}^1_k$. Next, consider a homotopy

$$h_\theta = (\mathcal{V}_\theta, \varphi_\theta; g_\theta) \in Fr_1(\mathbb{A}^1 \times U, W). \tag{2}$$

The support of h_θ equals $\mathbb{A}^1 \times Z \subset \mathbb{A}^1 \times U \times \mathbb{A}^1$. Clearly, $h_0 = a_0$ and $h_1 = a_1$. ☐

Corollary 5.2. *Under the assumptions of Lemma 5.1 let $U' \subset U$ and $W' \subset W$ be open subsets. Suppose that $a_0|_{U'}$ runs inside W'. Then $a_1|_{U'}$ runs inside W', the restriction $h_\theta|_{\mathbb{A}^1 \times U'}$ of the homotopy h_θ runs inside W' and*

$$[[a_0]] = [[a_1]]$$

in $\overline{\mathbb{Z}F}_1((U, U'), (W, W'))$.

Lemma 5.3. *Let* $a_0 = (\mathcal{V}, \varphi u_0; g) \in Fr_1(U,W)$ *and* $a_1 = (\mathcal{V}, \varphi u_1; g) \in Fr_1(U,W)$*, where* $u_0, u_1 \in k[\mathcal{V}]$ *are both invertible. In this case the supports of* a_0 *and* a_1 *coincide. Denote their common support by* Z*. Suppose* $u_0|_Z = u_1|_Z$*, then* $[a_0] = [a_1]$ *in* $\overline{\mathbb{Z}F}_1(U,W)$*.*

Proof. Set $u_\theta = (1-\theta)u_0 + \theta u_1 \in k[\mathbb{A}^1 \times \mathcal{V}]$. Clearly, $u_\theta|_{\mathbb{A}^1 \times Z} = pr_Z^*(u_0) = pr_Z^*(u_1) \in k[\mathbb{A}^1 \times Z]$. Let $\mathcal{V}_\theta = \{u_\theta \neq 0\} \subset \mathbb{A}^1 \times \mathcal{V}$. Set,

$$h_\theta = (\mathcal{V}_\theta, u_\theta \varphi; g \circ pr_{\mathcal{V}}) \in Fr_1(\mathbb{A}^1 \times U, W). \tag{3}$$

The support of h_θ equals $\mathbb{A}^1 \times Z \subset \mathbb{A}^1 \times U \times \mathbb{A}^1$. Clearly, $h_0 = a_0$ and $h_1 = a_1$. □

Corollary 5.4. *Under the assumptions of Lemma 5.3, let* $U' \subset U$ *and* $W' \subset W$ *be open subsets. Suppose* $a_0|_{U'}$ *runs inside* W'*. Then* $a_1|_{U'}$ *runs inside* W'*, the restriction* $h_\theta|_{\mathbb{A}^1 \times U'}$ *of the homotopy* h_θ *from the proof of Lemma 5.3 runs inside* W' *and*

$$[[a_0]] = [[a_1]]$$

in $\overline{\mathbb{Z}F}_1((U,U'),(W,W'))$*.*

Lemma 5.5. *Let* $U \subset \mathbb{A}^1_k$ *be non-empty open as above and* $F_0(Y), F_1(Y) \in k[U][Y]$*. Suppose* $deg_Y(F_0) = deg_Y(F_1) = d > 0$ *and their leading coefficients coincide and invertible in* $k[U]$*. Then,*

$$[U \times \mathbb{A}^1, F_0(Y), pr_U] = [U \times \mathbb{A}^1, F_1(Y), pr_U] \in \overline{\mathbb{Z}F}_1[U,U].$$

Proof. Set $F_\theta(Y) = (1-\theta)F_0(Y) + \theta F_1(Y) \in k[U][\theta, Y]$. Consider the morphism

$$h_\theta = (\mathbb{A}^1 \times U \times \mathbb{A}^1, F_\theta; pr_U) \in Fr_1(\mathbb{A}^1 \times U, U). \tag{4}$$

Clearly, $h_0 = (U \times \mathbb{A}^1, F_0(Y), pr_U)$ and $h_1 = (U \times \mathbb{A}^1, F_1(Y), pr_U)$. This proves the lemma. □

Corollary 5.6. *Under the assumptions of Lemma 5.5 let* $U' \subset U$ *be an open subset. Then*

$$(U \times \mathbb{A}^1, F_0(Y), pr_U)|_{U'}, \quad (U \times \mathbb{A}^1, F_1(Y), pr_U)|_{U'}$$

run inside U'*, the restriction* $h_\theta|_{\mathbb{A}^1 \times U'}$ *of the homotopy* h_θ *from the proof of Lemma 5.5 runs inside* $W' = \mathbb{A}^1 \times U'$ *and*

$$[[U \times \mathbb{A}^1, F_0(Y), pr_U]] = [[U \times \mathbb{A}^1, F_1(Y), pr_U]$$

in $\overline{\mathbb{Z}F}_1((U,U'),(U,U'))$.

Proposition 5.7. *Let $U \subset \mathbb{A}^1_k$ and $U' \subset U$ be open subsets. Let $t \in k[\mathbb{A}^1]$ be the standard parameter on $\mathbb{A}^1_k$. Set $X := (t \otimes 1)|_{U\times U} \in k[U \times U]$ and $Y := (1 \otimes t)|_{U\times U} \in k[U \times U]$. Then for any integer $n \geqslant 1$, one has an equality*

$$[[U \times U, (Y - X)^{2n+1}, p_2]] = [[U \times U, (Y - X)^{2n}, p_2]] + [[\sigma_U]]$$

in $\overline{\mathbb{Z}F}_1((U,U'),(U,U'))$.

Proof. Let $m \geqslant 1$ be an integer. Then

$$\begin{aligned}[[U \times U, (Y - X)^m, p_2]] &= [[U \times U, (Y - X)^m, p_1]] = \\ &= [[U \times \mathbb{A}^1, (Y - X)^m, p_1]] = [[U \times \mathbb{A}^1, Y^m, p_1]]\end{aligned} \tag{5}$$

in $\overline{\mathbb{Z}F}_1((U,U'),(U,U'))$. The first equality follows from Corollary 5.2, the third one follows from Corollary 5.6, the middle one is obvious.

There is a chain of equalities in $\overline{\mathbb{Z}F}_1((U,U'),(U,U'))$:

$$[[U \times \mathbb{A}^1, Y^{2n+1}; p_1]] = [[U \times \mathbb{A}^1, Y^{2n}(Y + 1); p_1]] =$$

$$= [[U \times (\mathbb{A}^1 - \{-1\}), Y^{2n}(Y + 1); p_1]] + [[U \times (\mathbb{A}^1 - \{0\}), Y^{2n}(Y + 1); p_1]] =$$

$$= [[\mathcal{V}_0, Y^{2n}; p_1]] + [[\mathcal{V}_1, (Y + 1); p_1]] =$$

$$= [[U \times \mathbb{A}^1, Y^{2n}; p_1]] + [[U \times \mathbb{A}^1, (Y + 1); p_1]].$$

Here the first equality holds by Corollary 5.6, the second one holds by Lemma 4.6, the third one holds by Corollary 5.4, the forth one is obvious (replacement of neighborhoods).

Continue the chain of equalities in $\overline{\mathbb{Z}F}_1((U,U'),(U,U'))$ as follows:

$$\begin{aligned}&[[U \times \mathbb{A}^1, Y^{2n}; p_1]] + [[U \times \mathbb{A}^1, (Y + 1); p_1]] = \\ &\quad = [[U \times \mathbb{A}^1, (Y - X)^{2n}; p_1]] + [[U \times \mathbb{A}^1, Y; p_1]] = \\ &= [[U \times \mathbb{A}^1, (Y - X)^{2n}; p_1]] + [[\sigma_U]] = [[U \times U, (Y - X)^{2n}; p_1]] + [[\sigma_U]] = \\ &\qquad = [[U \times U, (Y - X)^{2n}; p_2]] + [[\sigma_U]].\end{aligned}$$

Here the first equality holds by Corollary 5.6, the second one holds by the definition of σ_U (see Notation 3.7), the third one is obvious, the fouth one holds by Corollary 5.2. We have proved the equality

$$[[U \times \mathbb{A}^1, Y^{2n+1}; p_1]] = [[U \times U, (Y - X)^{2n}; p_2]] + [[\sigma_U]]. \tag{6}$$

Combining that with the equality (5) for $m = 2n + 1$ we get the desired equality

$$[[U \times U, (Y - X)^{2n+1}; p_2]] = [[U \times U, (Y - X)^{2n}; p_2]] + [[\sigma_U]]$$

in $\overline{\mathbb{Z}F}_1((U, U'), (U, U'))$. This proves the proposition. □

6. Injectivity and excision on affine line

The aim of this section is to prove Theorems 3.9 and 3.10.

Lemma 6.1. *Let $U \subset \mathbb{A}^1$ be open and non-empty. Let $A = \mathbb{A}^1_k - U$. Let $G_0(Y), G_1(Y) \in k[U][Y]$ be such that*

(1) $deg_Y(G_0) = deg_Y(G_1)$;
(2) both are monic in Y, that is the leading coefficients equal one;
(3) $G_0|_{U\times A}$, $G_1|_{U\times A}$ are both invertible and $G_0|_{U\times A} = G_1|_{U\times A} \in k[U \times A]^\times$.

Then

$$[U \times U, G_0; p_2] = [U \times U, G_1; p_2]$$

in $\overline{\mathbb{Z}F}_1(U, U)$.

Proof. One has a homotopy $h_\theta = (\mathbb{A}^1 \times U \times U, G_\theta, p_{2,U}) \in Fr_1(\mathbb{A}^1 \times U, U)$, where $G_\theta = (1 - \theta)G_0 + \theta G_1$ and $pr_{2,U} : \mathbb{A}^1 \times U \times U \to U$ is the projection onto the second copy of U. Its restriction to $0 \times U$ and to $1 \times U$ coincides with morphisms $(U \times U, G_0; p_2)$ and $(U \times U, G_1; p_2)$ respectively. This proves the lemma. □

Proof of Theorem 3.9. Under the assumptions of this theorem set $A = \mathbb{A}^1_k - U$ and $B = U - V$. For each big enough integer $m \geqslant 0$ find a polynomial $F_m(Y) \in k[U][Y]$ such that $F_m(Y)$ is of degree m with the leading coefficient equal 1 and such that

(i) $F_m(Y)|_{U\times A} = (Y - X)^m|_{U\times A} \in k[U \times A]^\times$;
(ii) $F_m(Y)|_{U\times B} = 1 \in k[U \times B]^\times$.

Take $n \gg 0$ and set $r = \langle U \times V, F_{2n+1}; pr_V\rangle - \langle U \times V, F_{2n}; pr_V\rangle \in \mathbb{Z}F_1(U, V)$. Then one has a chain equalities in $\overline{\mathbb{Z}F}_1(U, U)$:

$$[i] \circ [r] = [U \times U, F_{2n+1}; p_2] - [U \times U, F_{2n}; p_2] =$$

$$= [U \times U, (Y - X)^{2n+1}; p_2] - [U \times U, (Y - X)^{2n}; p_2] = [\sigma_U].$$

Here the first equality is obvious, the second one holds by Lemma 6.1, the third one holds by Proposition 5.7. This proves the theorem. □

Corollary 6.2 (of Lemma 6.1)**.** *Under the conditions and notation of Lemma 6.1 let $S \subset U$ be a proper closed subset. In addition to the conditions* $(1)-(3)$ *suppose that the following two conditions hold:*

(4) $G_0(Y)|_{U\times S} = G_1(Y)|_{U\times S}$,
(5) $G_0(Y)|_{(U-S)\times S}$ *is invertible.*

Then one has an equality

$$[[U\times U, G_0; p_2]] = [[U\times U, G_1; p_2]]$$

in $\overline{\mathbb{Z}F}_1((U,U-S),(U,U-S))$.

Proof of the corollary. The support Z_θ of the homotopy h_θ from the proof of Lemma 6.1 coincides with the vanishing locus of the polynomial G_θ. Since $G_\theta|_{\mathbb{A}^1\times(U-S)\times S}$ is invertible, then $Z_\theta \cap \mathbb{A}^1\times(U-S)\times S=\emptyset$. By Lemma 4.4 the homotopy $h_\theta|_{\mathbb{A}^1\times(U-S)}$ runs inside $U-S$. Hence

$$[[U\times U, G_0; p_2]] = [[h_0]] = [[h_1]] = [[U\times U, G_1; p_2]]$$

in $\overline{\mathbb{Z}F}_1((U,U-S),(U,U-S))$. In fact, the second equality here holds by Corollary 4.5. The first and the third equalities hold since for $i=0,1$ one has $h_i = (U\times U, G_i; p_2)$ in $Fr_1(U,U)$. □

Proof of Theorem 3.10. Firstly, we construct a morphism $r \in \mathbb{Z}F_1((U,U-S)),(V,V-S))$ such that for its class $[[r]]$ in $\overline{\mathbb{Z}F}_1((U,U-S)),(V,V-S))$ one has

$$[[i]]\circ[[r]] = [[\sigma_U]] \tag{7}$$

in $\overline{\mathbb{Z}F}_1((U,U-S)),(U,U-S))$.

To this end, set $A=\mathbb{A}^1_k-U$, $B=U-V$. Recall that $S\subset V$ is a proper closed subset. Take any big enough integer $m\geqslant 1$ and find a monic polynomial $F_m(Y)$ of degree m satisfying the following properties:

(i) $F_m(Y)|_{U\times A} = (Y-X)^m|_{U\times A} \in k[U\times A]^\times$;
(ii) $F_m(Y)|_{U\times B} = 1 \in k[U\times B]^\times$;
(iii) $F_m(Y)|_{U\times S} = (Y-X)^m|_{U\times S} \in k[U\times S]$.

Note that $F_m(Y)|_{(U-S)\times S}\in k[(U-S)\times S]^\times$. Hence by Lemma 4.4 the morphism $(U\times V, F_m; pr_V)\in Fr_1(U,V)$ being restricted to $U-S$ runs inside $V-S$. Thus using Definition 4.3 we get a morphism

$$\langle\langle U\times V, F_m; pr_V\rangle\rangle \in \mathbb{Z}F_1((U,U-S)),(V,V-S)).$$

For that morphism one has equalities

$$[[i]]\circ[[U\times V,F_m;pr_V]]=[[U\times U,F_m;p_2]]=[[U\times U,(Y-X)^m;p_2]]$$

in $\overline{\mathbb{Z}F}_1((U,U-S)),(U,U-S))$. Here the first equality is obvious, the second one follows from Corollary 6.2. Take a big enough integer n. Set

$$r=\langle\langle U\times V,F_{2n+1};pr_V\rangle\rangle-\langle\langle U\times V,F_{2n};pr_V\rangle\rangle\in\mathbb{Z}F_1((U,U-S)),(V,V-S)).$$

We claim that $[[i]]\circ[[r]]=[[\sigma_U]]$ in $\overline{\mathbb{Z}F}_1((U,U-S)),(U,U-S))$. In fact,

$$[[i]]\circ[[r]]=[[U\times U,(Y-X)^{2n+1};p_2]]-[[U\times U,(Y-X)^{2n};p_2]]=[[\sigma_U]].$$

The first equality is proven a few lines above and the second one follows from Proposition 5.7. We see that equality (7) holds.

We now find morphisms $l\in\mathbb{Z}F_1((U,U-S)),(V,V-S))$ and $g\in\mathbb{Z}F_1((V,V-S)),(V-S,V-S))$ such that

$$[[l]]\circ[[i]]-[[j]]\circ[[g]]=[[\sigma_V]] \tag{8}$$

in $\overline{\mathbb{Z}F}_1(V,V-S)),(V,V-S))$. Here $j:(V-S,V-S)\to(V,V-S)$ is the inclusion. Clearly, equality (8) yields $[[l]]\circ[[i]]=[[\sigma_V]]\in\overline{\overline{\mathbb{Z}F}}_1(V,V-S)),(V,V-S))$.

Set $A'=\mathbb{A}^1_k-U$, $B=U-V$ and recall that $S\subset V$ is a proper closed subset. Take an integer m big enough and find a polynomial $F_m(Y)\in k[U][Y]$ of degree m, monic in Y, such that

(i) $F_m(Y)|_{U\times A'}=(Y-X)|_{U\times A'}\in k[U\times A']^\times$;
(ii) $F_m(Y)|_{U\times B}=1\in k[U\times B]^\times$;
(iii) $F_m(Y)|_{U\times S}=(Y-X)|_{U\times S}\in k[U\times S]$.

Note that $F_m(Y)|_{(U-S)\times S}\in k[(U-S)\times S]^\times$. Hence by Lemma 4.4 the morphism $(U\times V,F_m;pr_V)\in Fr_1(U,V)$ being restricted to $U-S$ runs inside $V-S$. Thus, using Definition 4.3, we get a morphism

$$l=\langle\langle U\times V,F_m;pr_V\rangle\rangle\in\mathbb{Z}F_1((U,U-S)),(V,V-S)).$$

To construct the desired morphism g, find a polynomial $E_{m-1}\in k[V][Y]$ of degree $m-1$, monic in Y, such that

(i') $E_{m-1}(Y)|_{V\times A'}=1|_{U\times A'}\in k[V\times A']^\times$;
(ii') $E_{m-1}(Y)|_{V\times B}=(Y-X)^{-1}\in k[V\times B]^\times$;

(iii') $E_{m-1}(Y)|_{V\times S} = 1|_{V\times S} \in k[U\times S]$;
(iv') $E_{m-1}(Y)|_{\Delta(V)} = 1|_{\Delta(V)} \in k[\Delta(V)]$.

Let $G \subset V \times \mathbb{A}^1_k$ be a closed subset defined by $E_{m-1}(Y) = 0$. By conditions $(i') - (iv')$ one has $G \subset V \times (V - S)$ and $G \cap \Delta(V) = \emptyset$. Set $g' = \langle V \times (V-S) - \Delta(V), (Y-X)E_{m-1}(Y)\rangle; pr_{V-S}) \in \mathbb{Z}F_1(V, V-S)$. Since $g'|_{V-S} \in \mathbb{Z}F_1(V-S, V-S)$, we get a morphism

$$g = (g', g'|_{V-S}) \in \mathbb{Z}F_1((V, V-S)), (V-S, V-S)). \tag{9}$$

Claim 6.3. *Equality (8) holds for the morphisms l and g defined above.*

Note firstly that $l \circ \langle\langle i\rangle\rangle = \langle\langle V\times V, F_m(Y)|_{V\times V}; pr_2\rangle\rangle \in \mathbb{Z}F_1((V, V-S), (V, V-S))$. Applying Corollary 6.2 to the case $V \subset \mathbb{A}^1$, $S \subset V$ and $A := A' \cup B$, we get an equality

$$[[V\times V, F_m(Y)|_{V\times V}; pr_2]] = [[V\times V, (Y-X)E_{m-1}(Y); pr_2]]$$

in $\overline{\mathbb{Z}F}_1((V, V-S), (V, V-S))$. By Lemma 4.6 and the fact that $G\cap\Delta(V) = \emptyset$, one has

$$[[V\times V, (Y-X)E_{m-1}(Y); pr_2]] =$$

$$[[V\times V - G, E_{m-1}(Y-X); pr_2]] + [[V\times V - \Delta(V), (Y-X)E_{m-1}; pr_2]] =$$

$$= [[V\times V - G, E_{m-1}(Y-X); pr_2]] + [[j]] \circ [[g]]$$

in $\overline{\mathbb{Z}F}_1((V, V-S)), (V, V-S))$.

One has a chain of equalities

$$[[V\times V - G, E_{m-1}(Y-X); pr_2]] = [[V\times V - G, (Y-X); pr_2]] =$$
$$= [[V\times V, (Y-X); pr_2]] = [[V\times\mathbb{A}^1, Y; pr_1]] = [[\sigma_V]].$$

The first equality holds by condition (iv') and Corollary 5.4. The second one is obvious. The third one is equality (5) for $m = 1$ from the proof of Proposition 5.7. The forth one is the definition of $\langle\langle\sigma_V\rangle\rangle$ (see Definition 2.4 and Notation 3.7). Combining altogether, we get a chain of equalities

$$[[l]] \circ [[i]] = [[V\times V, F_m(Y)|_{V\times V}; pr_2]] =$$
$$= [[V\times V, (Y-X)E_{m-1}(Y); pr_2]] = [[\sigma_V]] + [[j]] \circ [[g]],$$

which proves the claim. The theorem now follows. □

7. Excision on relative affine line

Proof of Theorem 3.12. Let $U = \mathbb{A}^1_W$, let $V \subset U$ be the open V from Theorem 3.12. Let $S = 0 \times W$. Note that $S \subset V$. Set $A = \mathbb{A}^1_W - U = \emptyset$, $B = U - V = \{f = 0\}$. Then B is finite over W, since f is monic. Note that $B \cap (0 \times W) = \emptyset$.

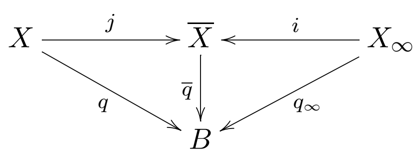
Repeat literally the proof of Theorem 3.10 replacing fibre products over $Spec(k)$ by fibre products over W. For instance, replace $\mathbb{A}^1_k \times \mathbb{A}^1_k$ from Section 6 by $\mathbb{A}^1_W \times_W \mathbb{A}^1_W$. □

8. Almost elementary fibrations

In this section we recall a modification of a result of M. Artin from [3] concerning the existence of nice neighborhoods. The following notion (see [16, Definition 2.1]) is a modification of that introduced by Artin in [3, Exp. XI, Déf. 3.1].

Definition 8.1. *([16])* An *almost elementary fibration* over a scheme B is a morphism of schemes $p : X \to B$ which can be included in a commutative diagram

(10)



of morphisms satisfying the following conditions:

(i) j is an open immersion dense at each fibre of $\overline{q}$, and $X = \overline{X} - X_\infty$;
(ii) $\overline{q}$ is smooth projective all of whose fibres are geometrically irreducible of dimension one;
(iii) q_∞ is a finite flat morphism all of whose fibres are non-empty;
(iv) the morphism i is a closed embedding and the ideal sheaf $I_{X_\infty} \subset \mathcal{O}_{\overline{X}}$ defining the closed subscheme X_∞ in $\overline{X}$ is locally principal.

Proposition 8.2 ([16])**.** *Let k be an infinite field, X be a smooth geometrically irreducible affine variety over k, $x_1, x_2, \ldots, x_n \in X$ be closed points. Then there exists a Zariski open neighborhood X^0 of the family $\{x_1, x_2, \ldots, x_n\}$ and an almost elementary fibration $p : X^0 \to S$, where S is an open subscheme of the projective space $\mathbb{P}^{\dim X - 1}_k$. If, moreover, Z is a closed codimension one subvariety in X, then one can choose X^0 and p in such a way that $p|_{Z \bigcap X^0} : Z \bigcap X^0 \to S$ is finite surjective.*

Proposition 8.3 ([16]). *Let $p : X \to S$ be an almost elementary fibration. If S is a regular semi-local irreducible scheme, then there exists a commutative diagram of S-schemes*

$$\begin{array}{ccccc} X & \xrightarrow{j} & \overline{X} & \xleftarrow{i} & Y \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \overline{\pi}} & & \downarrow \\ \mathbb{A}^1 \times S & \xrightarrow{\text{in}} & \mathbb{P}^1 \times S & \xleftarrow{i} & \{\infty\} \times S \end{array} \qquad (11)$$

with $\overline{\pi}$ a finite surjective morphism such that the left hand side square is Cartesian. Here j and i are the same as in Definition 8.1, while $pr_S \circ \pi = p$, where pr_S is the projection $\mathbb{A}^1 \times S \to S$.

In particular, $\pi : X \to \mathbb{A}^1 \times S$ is a finite surjective morphism of S-schemes, where X and $\mathbb{A}^1 \times S$ are regarded as S-schemes via the morphism p and the projection pr_S, respectively.

9. Reducing the injectivity for local schemes to Theorem 14.3

In this section we follow definitions, notation and constructions from Sections 2 and 3. In particular, we can work with pointed sets $Fr_n(Y, X)$ and abelian groups $\mathbb{Z}F_n(Y, X)$, $\overline{\mathbb{Z}F}_n(Y, X)$, with $Y \in EssSm/k$ and $X \in Sm/k$. The main aim of this section is to prove Theorem 3.11. Let $X \in Sm/k$ be irreducible, $x \in X$ be a point, $U = Spec(\mathcal{O}_{X,x})$, $i : D \hookrightarrow X$ be a proper closed subset. Let $j : X - D \hookrightarrow X$ be the open inclusion. Under the notation of Theorem 3.11 we will find an integer N and an element $r \in \mathbb{Z}F_N(U, X - D)$ such that

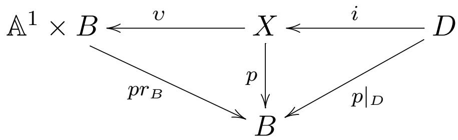
$$[j] \circ [r] = [\sigma_X^N] \circ [can]$$

in $\overline{\mathbb{Z}F}_N(U, X)$ (see Definition 9.8). For this we need some preparations as well as Theorem 14.3.

Let $X' \subset X$ be an open subset containing the point x and let $D' = X' \cap D$. Clearly, if we solve a similar problem for the triple U, X' and $X' - D'$, then we solve the problem for the given triple U, X and $X - D$. So, we may shrink X appropriately. In particular, we may assume that the canonical sheaf $\omega_{X/k}$ is trivial, i.e. is isomorphic to the sheaf $\mathcal{O}_X$. Let $d = \dim X$.

Shrinking X further (and replacing D with its trace) and using Propositions 8.2 and 8.3, we can find a commutative diagram of the form

(12)

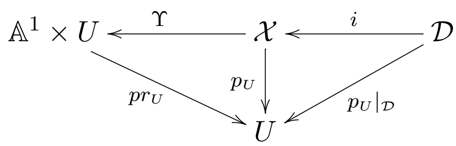


where $p : X \to B$ is an almost elementary fibration in the sense of [16], B is an affine open subset of the projective space $\mathbb{P}^{d-1}_k$, υ is a finite surjective morphism, $p|_D$ is a finite morphism.

The canonical sheaf $\omega_{X/k}$ remains trivial. Since p is an almost elementary fibration, then it is a smooth morphism such that for each point $b \in B$ the fibre $p^{-1}(b)$ is a $k(b)$-smooth absolutely irreducible affine curve. Since υ is finite, then the B-scheme X is affine.

Set $U = Spec(\mathcal{O}_{X,x})$, $\mathcal{X} = U \times_B X$, $\mathcal{D} = U \times_B D$. There is an obvious morphism $\Delta = (id, can) : U \to \mathcal{X}$. It is a section of the projection $p_U : \mathcal{X} \to U$. Let $p_X : \mathcal{X} \to X$ be the projection to X. The base change of diagram (12) gives a commutative diagram of the form

(13)



where $p_U : \mathcal{X} \to U$ is an almost elementary fibration over U in the sense of Definition 8.1, i is a closed embedding, Υ is a finite surjective morphism, $p_U|_{\mathcal{D}}$ is a finite morphism. Since $\omega_{X/k}$ is trivial and U is local and essentially k-smooth, the relative canonical sheaf $\omega_{\mathcal{X}/U}$ is trivial, i.e. isomorphic to the structure sheaf $\mathcal{O}_{\mathcal{X}}$.

Lemma 9.1 ([15], Lemma 10.1)**.** *Given the commutative diagram (13), there is a finite surjective morphism $H_\theta = (p_U, h_\theta) : \mathcal{X} \to \mathbb{A}^1 \times U$ of U-schemes such that for the closed subschemes $\mathcal{D}_1 := H_\theta^{-1}(1 \times U)$ and $\mathcal{D}_0 := H_\theta^{-1}(0 \times U)$ of $\mathcal{X}$ one has*

- *(i)* $\mathcal{D}_1 \subset \mathcal{X} - \mathcal{D}$;
- *(ii)* $\mathcal{D}_0 = \Delta(U) \sqcup \mathcal{D}'_0$ *(equality of schemes) and* $\mathcal{D}'_0 \subset \mathcal{X} - \mathcal{D}$.

Now regard $\mathcal{X}$ as an affine $\mathbb{A}^1 \times U$-scheme via the morphism H_θ. And also regard $\mathcal{X}$ as an X-scheme via p_X.

Lemma 9.2. *There is an integer $N \geqslant 0$, a closed embedding $\mathcal{X} \hookrightarrow \mathbb{A}^1 \times U \times \mathbb{A}^N$ of $\mathbb{A}^1 \times U$-schemes, an étale affine neighborhood $(\mathcal{V}, \rho : \mathcal{V} \to \mathbb{A}^1 \times$*

$U \times \mathbb{A}^N, s : \mathcal{X} \hookrightarrow \mathcal{V})$ of $\mathcal{X}$ in $\mathbb{A}^1 \times U \times \mathbb{A}^N$, functions $\varphi_1, ..., \varphi_N \in k[\mathcal{V}]$ and a morphism $r : \mathcal{V} \to \mathcal{X}$ such that:

(i) the functions $\varphi_1, ..., \varphi_N$ generate the ideal $I_{s(\mathcal{X})}$ in $k[\mathcal{V}]$ defining the closed subscheme $s(\mathcal{X})$ of $\mathcal{V}$;
(ii) $r \circ s = id_{\mathcal{X}}$;
(iii) the morphism r is a U-scheme morphism if $\mathcal{V}$ is regarded as a U-scheme via the morphism $pr_U \circ \rho$ and $\mathcal{X}$ is regarded as a U-scheme via the morphism p_U.

Proof. Since H_θ is a finite morphism, then for some integer $N \geqslant 1$ there is a closed embedding of U-schemes $in : \mathcal{X} \hookrightarrow \mathbb{A}^1 \times U \times \mathbb{A}^N$. Consider the short exact sequence of vector bundles on $\mathcal{X}$ defining the normal bundle $\mathcal{N} = N_{\mathbb{A}^1 \times U \times \mathbb{A}^N / \mathcal{X}}$:

$$\{0\} \to T_{\mathcal{X}/U} \to \mathbb{A}^1 \times \mathcal{X} \times \mathbb{A}^N = T_{(\mathbb{A}^1 \times U \times \mathbb{A}^N)/U}|_{\mathcal{X}} \xrightarrow{q} \mathcal{N} \to \{0\} \tag{14}$$

Since H_θ is finite, the scheme $\mathcal{X}$ is affine. As mentioned above the bundle $T_{\mathcal{X}/U}$ is trivial. Thus the bundle $\mathcal{N}$ is stably trivial. Increasing the integer N we may assume that the bundle $\mathcal{N}$ is trivial. Since the scheme $\mathcal{X}$ is affine, there is a linear section $t : \mathcal{N} \to \mathbb{A}^1 \times \mathcal{X} \times \mathbb{A}^N$ of the morphism q. Let $q_{\mathcal{X}} : \mathcal{N} \to \mathcal{X}$ be the projection on $\mathcal{X}$. There are two morphisms of U-schemes:

$$in \circ q_{\mathcal{X}} : \mathcal{N} \to \mathbb{A}^1 \times U \times \mathbb{A}^N \text{ and } (id \times p_U \times id) \circ t : \mathcal{N} \to \mathbb{A}^1 \times U \times \mathbb{A}^N.$$

Regarding $\mathbb{A}^1 \times U \times \mathbb{A}^N$ as a vector bundle over U we have a morphism

$$+ : (\mathbb{A}^1 \times U \times \mathbb{A}^N) \times_U (\mathbb{A}^1 \times U \times \mathbb{A}^N) \to \mathbb{A}^1 \times U \times \mathbb{A}^N.$$

Set $\rho' = in \circ q_{\mathcal{X}} + (id \times p_U \times id) \circ t : \mathcal{N} \to \mathbb{A}^1 \times U \times \mathbb{A}^N$. It is easy to check that ρ' is étale along $s_0(\mathcal{X})$, where $s_0 : \mathcal{X} \to \mathcal{N}$ is the zero section of $\mathcal{N}$. Hence ρ' is étale in an affine neighborhood $\mathcal{V}'$ of $s_0(\mathcal{X})$. Since $\rho' \circ s_0 = in : \mathcal{X} \hookrightarrow \mathbb{A}^1 \times U \times \mathbb{A}^N$, hence $(\rho')^{-1}(in(\mathcal{X})) = s_0(\mathcal{X}) \sqcup Y$. Hence there is an open affine subscheme $\mathcal{V}$ in $\mathcal{V}'$ containing $s_0(\mathcal{X})$ such that $(\rho'|_{\mathcal{V}})^{-1}(in(\mathcal{X})) = s_0(\mathcal{X})$. Set $\rho = \rho'|_{\mathcal{V}} : \mathcal{V} \to \mathbb{A}^1 \times U \times \mathbb{A}^N$. Set $s = s_0 : \mathcal{X} \to \mathcal{V}$.

Clearly, $(\mathcal{V}, \rho : \mathcal{V} \to \mathbb{A}^1 \times U \times \mathbb{A}^N, s : \mathcal{X} \hookrightarrow \mathcal{V})$ is an étale neighborhood of $in(\mathcal{X})$ in $\mathbb{A}^1 \times U \times \mathbb{A}^N$. We will write in this proof $\mathcal{X}$ for $in(\mathcal{X})$.

Set $r = (q_{\mathcal{X}})|_{\mathcal{V}} : \mathcal{V} \to \mathcal{X}$. Since the bundle $\mathcal{N}$ is trivial we can choose a trivialization $\mathcal{N} \cong \mathcal{X} \times \mathbb{A}^N$. The trivialization gives functions $\varphi_1, ..., \varphi_N$ which generate the ideal $I_{s(\mathcal{X})}$ in $k[\mathcal{V}]$ defining the closed subscheme $s_0(\mathcal{X})$ of $\mathcal{V}$.

Clearly, $r \circ s = id_{\mathcal{X}}$. Also, the morphism r is a U-scheme morphism if $\mathcal{V}$ is regarded as a U-scheme via the morphism $pr_U \circ \rho$ and $\mathcal{X}$ is regarded as a U-scheme via the morphism p_U. Whence follows the lemma. □

By Lemma 9.1, $\mathcal{D}_0 = \Delta(U) \sqcup \mathcal{D}'_0$. Set $\mathcal{V}_0 = \rho^{-1}(0 \times U \times \mathbb{A}^N)$. For a suitable affine open neighborhood $\mathcal{W}_0$ of $s(\Delta(U))$ in $\mathcal{V}_0$ the triple $(\mathcal{W}_0, \rho|_{\mathcal{W}_0} : \mathcal{W}_0 \to U \times \mathbb{A}^N, s|_{\Delta(U)} : \Delta(U) \hookrightarrow \mathcal{W}_0)$ is an étale neighborhood of $\Delta(U)$ in $U \times \mathbb{A}^N$.

Remark 9.3. By Lemma 9.2 the functions $\varphi_1|_{\mathcal{W}_0}, ..., \varphi_N|_{\mathcal{W}_0}$ generate the ideal I defining the closed subscheme $s(\Delta(U))$ of the scheme $\mathcal{W}_0$. In particular, the family

$$\overline{(\varphi_1|_{\mathcal{W}_0})}, ..., \overline{(\varphi_N|_{\mathcal{W}_0})} \in I/I^2$$

is a free basis of the free $k[U]$-module I/I^2. Another basis of the $k[U]$-module I/I^2 is the family

$$\overline{(t_1 - \Delta^*(t_1))|_{\mathcal{W}_0}}, ..., \overline{(t_N - \Delta^*(t_N))|_{\mathcal{W}_0}} \in I/I^2.$$

Let $A \in GL_N(k[U])$ be a unique matrix which converts the second free basis to the first one and let $J := \det(A)$ be its determinant. Replacing φ_1 by $J^{-1}\varphi_1$, we may and will assume below in this section that $J = 1 \in k[U]$. This is useful to apply Theorem 14.3 below.

Set $\mathcal{V}_1 = \rho^{-1}(1 \times U \times \mathbb{A}^N) \cap r^{-1}(\mathcal{X} - \mathcal{D})$. Then $s(\mathcal{D}_1) \subset \mathcal{V}_1$. In fact, $(r \circ s)(\mathcal{D}_1) = \mathcal{D}_1 \subset \mathcal{X} - \mathcal{D}$ and $\rho(\mathcal{D}_1) \subset 1 \times U \times \mathbb{A}^N$. Thus $\mathcal{V}_1 \neq \emptyset$.

Construction 9.4 (Étale neighborhood of $\mathcal{D}_1$)**.** *The morphism $\rho|_{1\times U\times\mathbb{A}^N} : \rho^{-1}(1 \times U \times \mathbb{A}^N) \to 1 \times U \times \mathbb{A}^N$ is étale and the inclusion $i_1 : \mathcal{V}_1 \hookrightarrow \rho^{-1}(1 \times U \times \mathbb{A}^N)$ is open. Set $\rho_1 = (\rho|_{1\times U\times\mathbb{A}^N}) \circ i_1$. Then the triple*

$$(\mathcal{V}_1, \rho_1 : \mathcal{V}_1 \to 1 \times U \times \mathbb{A}^N, s_1 = s|_{\mathcal{D}_1} : \mathcal{D}_1 \to \mathcal{V}_1)$$

is an étale neighborhood of $\mathcal{D}_1$ in $1 \times U \times \mathbb{A}^N$. Let $r_1 = r|_{\mathcal{V}_1} : \mathcal{V}_1 \to \mathcal{X} - \mathcal{D}$.

Definition 9.5. *We set $a_1 = (\mathcal{D}_1, \mathcal{V}_1, \varphi_1|_{\mathcal{V}_1}, ..., \varphi_N|_{\mathcal{V}_1}; (p_X)|_{\mathcal{X}-\mathcal{D}} \circ r_1) \in Fr_N(U, X - D)$.*

Set $\mathcal{V}'_0 = \rho^{-1}(0 \times U \times \mathbb{A}^N) \cap r^{-1}(\mathcal{X} - \mathcal{D})$. Then $s(\mathcal{D}'_0) \subset \mathcal{V}'_0$. In fact, $(r \circ s)(\mathcal{D}'_0) = \mathcal{D}'_0 \subset \mathcal{X} - \mathcal{D}$ and $\rho(\mathcal{D}'_0) \subset 0 \times U \times \mathbb{A}^N$. Thus $\mathcal{V}'_0 \neq \emptyset$. The functions $\varphi_1, ..., \varphi_N$ define $s(\mathcal{X})$ in $\mathcal{V}$, so their restriction to $\mathcal{V}'_0$ define $s(\mathcal{X} \cap 0 \times U \times \mathbb{A}^N) = s(\Delta(U) \sqcup \mathcal{D}'_0)$. Set $\mathcal{V}''_0 = \mathcal{V}'_0 - s(\Delta(U))$.

Construction 9.6. *The morphism* $\rho|_{0\times U\times\mathbb{A}^N}:\rho^{-1}(0\times U\times\mathbb{A}^N)\to 0\times U\times\mathbb{A}^N$ *is étale and the inclusion* $i_0:\mathcal{V}_0''\hookrightarrow\rho^{-1}(0\times U\times\mathbb{A}^N)$ *is open. Set* $\rho_0=(\rho|_{0\times U\times\mathbb{A}^N})\circ i_0$. *Then the triple*

$$(\mathcal{V}_0'',\rho_0:\mathcal{V}_0''\to 0\times U\times\mathbb{A}^N, s_0=s|_{\mathcal{D}_0'}:\mathcal{D}_0'\to\mathcal{V}_0'')$$

is an étale neighborhood of $\mathcal{D}_0'$ *in* $0\times U\times\mathbb{A}^N$. *Let* $r_0=r|_{\mathcal{V}_0''}:\mathcal{V}_0''\to\mathcal{X}-\mathcal{D}$.

Definition 9.7. *We set* $a_0=(\mathcal{D}_0',\mathcal{V}_0'',\varphi_1|_{\mathcal{V}_0''},...,\varphi_N|_{\mathcal{V}_0''};(p_X)|_{\mathcal{X}-\mathcal{D}}\circ r_0)\in Fr_N(U,X-D)$.

Definition 9.8. *Set* $r=\langle a_1\rangle-\langle a_0\rangle\in\mathbb{Z}F_N(U,X-D)$.

Claim 9.9. *One has an equality* $[j]\circ[r]=[\sigma_X^N]\circ[can]\in\overline{\mathbb{Z}F}_N(U,X)$.

In fact, take the element $h_\theta=(\mathcal{X},\mathcal{V},\varphi_1,...,\varphi_N;p_X\circ r)\in Fr_N(\mathbb{A}^1\times U,X)$. By Lemma 9.1 the support of h_0 is the closed subset $\Delta(U)\sqcup\mathcal{D}_0'$. Thus by Lemma 4.2 $\langle h_0\rangle$ is the sum of two summands. Namely,

$$\langle h_0\rangle=j\circ\langle a_0\rangle+\langle\Delta(U),\mathcal{W}_0,\varphi_1|_{\mathcal{W}_0},...,\varphi_N|_{\mathcal{W}_0};p_X\circ(r|_{\mathcal{W}_0})\rangle$$

in $\mathbb{Z}F_N(U,X)$. By Remark 9.3 and Theorem 14.3 for the second summand one has

$$[\Delta(U),\mathcal{W}_0,\varphi_1|_{\mathcal{W}_0},...,\varphi_N|_{\mathcal{W}_0};p_X\circ(r|_{\mathcal{W}_0})]=[\sigma_X^N]\circ[p_X\circ r|_{\mathcal{W}_0}\circ(s|_{\Delta(U)}\circ\Delta)]=$$

$$=[\sigma_X^N]\circ[p_X\circ\Delta]=[\sigma_X^N]\circ[can]$$

in $\overline{\mathbb{Z}F}_N(U,X)$. Clearly, $h_1=j\circ a_1$ in $Fr_N(U,X)$. Thus one has a chain of equalities

$$[j]\circ[a_1]=[h_1]=[h_0]=[j]\circ[a_0]+[\sigma_X^N]\circ[can]$$

in $\overline{\mathbb{Z}F}_N(U,X)$. This reduces the claim to Theorem 14.3. Thus we have derived Theorem 3.11 from Theorem 14.3.

10. Preliminaries for the injective part of the étale excision

In this section we follow definitions, notation and constructions from Sections 2 and 3. In particular, we can work with pointed sets $Fr_n(Y,X)$ and abelian groups like $\mathbb{Z}F_n((Y,Y^0),(X,Y^0))$, $\overline{\mathbb{Z}F}_n((Y,Y^0),(X,X^0))$, where

$(Y, Y^0) \in EssSm/k$ and $(X, X^0) \in Sm/k$. Let X, X' be irreducible smooth k-schemes. Let $V \subset X$ and $V' \subset X'$ be open subschemes. Suppose

$$\begin{array}{ccc} V' & \longrightarrow & X' \\ \downarrow & & \downarrow \Pi \\ V & \longrightarrow & X \end{array}$$

is an elementary distinguished square in the sense of [13, Definition 3.1.3]. Let $S = X - V$ and $S' = X' - V'$ be closed subschemes equipped with reduced structures. Let $x \in S$ and $x' \in S'$ be two points such that $\Pi(x') = x$. Let $U = Spec(\mathcal{O}_{X,x})$ and $U' = Spec(\mathcal{O}_{X',x'})$. Let $\pi : U' \to U$ be the morphism induced by Π.

To prove Theorem 3.13, it suffices to find elements $a \in \mathbb{Z}F_N((U, U - S)), (X', X' - S'))$ and $b_G \in \mathbb{Z}F_N((U, U - S)), (X - S, X - S))$ such that

$$[[\Pi]] \circ [[a]] - [[j]] \circ [[b_G]] = [[\sigma_X^N]] \circ [[can]] \tag{15}$$

in $\overline{\mathbb{Z}F}_N(U, U - S)), (X, X - S))$. Here $j : (X - S, X - S) \to (X, X - S)$ and $can : (U, U - S) \to (X, X - S)$ are inclusions. In this section we do some preparations to construct the desired elements $a \in \mathbb{Z}F_N((U, U - S)), (X', X' - S'))$ and $b_G \in \mathbb{Z}F_N((U, U - S)), (X - S, X - S))$ in Section 11 satisfying (15).

Let $in : X^\circ \hookrightarrow X$ and $in' : (X')^\circ \hookrightarrow X'$ be open such that

(1) $x \in X^\circ$,
(2) $x' \in (X')^\circ$,
(3) $\Pi((X')^\circ) \subset X^\circ$,
(4) the square

$$\begin{array}{ccc} V' \cap (X')^\circ & \longrightarrow & (X')^\circ \\ \downarrow & & \downarrow \Pi|_{(X')^\circ} \\ V \cap X^\circ & \longrightarrow & X^\circ \end{array}$$

is an elementary distinguished square.

Suppose $a^\circ \in \mathbb{Z}F_N((U, U - S)), ((X')^\circ, (X')^\circ - S'))$, $b_G^\circ \in \mathbb{Z}F_N((U, U - S)), (X^\circ - S, X^\circ - S))$ are such that for the inclusions $j^\circ : (X^\circ - S, X^\circ - S) \to (X^\circ, X^\circ - S)$ and $can_{X^\circ} : (U, U - S) \to (X^\circ, X^\circ - S)$ one has

$$[[\Pi|_{(X')^\circ}]] \circ [[a^\circ]] - [[j^\circ]] \circ [[b_G^\circ]] = [[\sigma_{X^\circ}^N]] \circ [[can_{X^\circ}]]. \tag{16}$$

Then the elements $a = in' \circ a^\circ$ and $b_G = in \circ b_G^\circ$ satisfy (15). Thus if we shrink X and X' in such a way that properties $(1)-(4)$ are fulfilled and find appropriate elements a° and b_G°, then we find a and b_G satisfying (15).

Remark 10.1. One way of shrinking X and X' such that properties $(1)-(4)$ are fulfilled is as follows. Replace X by an affine open X° containing x and then replace X' by $(X')^\circ = \Pi^{-1}(X^\circ)$.

Let X'_n be the normalization of X in $Spec(k(X'))$. Let $\Pi_n : X'_n \to X$ be the corresponding finite morphism. Since X' is k-smooth it is an open subscheme of X'_n. Let $Y'' = X'_n - X'$. It is a closed subset in X'_n. Since $\Pi|_{S'} : S' \to S$ is an isomorphism of schemes, then S' is closed in X'_n. Thus $S' \cap Y'' = \emptyset$. Hence there is a function $f \in k[X'_n]$ such that $f|_{Y''} = 0$ and $f|_{S'} = 1$.

Definition 10.2. Set $X'_{new} = (X'_n)_f$, $Y' = \{f = 0\}$, $Y = \Pi_n(Y'_{red}) \subset X$. Note that X'_{new} is an affine k-variety as a principal open subset of the affine k-variety X'_n. We regard Y' as an effective Cartier divisor of X'_n. The subset Y is closed in X, because Π_n is finite. Set $\Pi_{new} = \Pi|_{X'_{new}}$.

Remark 10.3. We note that $\Pi^{-1}_{new}(S) = S'$ and the open subsets $X^\circ = X$, $(X')^\circ = X'_{new} \subset X'$ satisfy the properties $(1)-(4)$. Thus, we may change notation and write X' for X'_{new}.

Remark 10.4. Shrinking X and X' as described in Remark 10.1, changing notation again, and using Proposition 8.3, one can find an almost elementary fibration $q : X \to B$ in the sense of Definition 8.1 (here B is affine open in $\mathbb{P}^{n-1}$) such that $q|_{Y\cup S} : Y \cup S \to B$ is finite, $\omega_{B/k} \cong \mathcal{O}_B$, $\omega_{X/k} \cong \mathcal{O}_X$.

The scheme X' will be regarded below as a B-scheme via the morphism $q \circ \Pi$.

Remark 10.5. If $q : X \to B$ is the almost elementary fibration from Remark 10.4, then $\Omega^1_{X/B} \cong \mathcal{O}_X$. In fact, $\omega_{X/k} \cong q^*(\omega_{B/k}) \otimes \omega_{X/B}$. Thus $\omega_{X/B} \cong \mathcal{O}_X$. Since X/B is a smooth relative curve, then $\Omega^1_{X/B} = \omega_{X/B} \cong \mathcal{O}_X$.

If, furthermore, $j : X \hookrightarrow B \times \mathbb{A}^N$ is a closed embedding of B-schemes, then one has $[\mathcal{N}(j)] = (N-1)[\mathcal{O}_X]$ in $K_0(X)$, where $\mathcal{N}(j)$ is the normal bundle to X for the imbedding j.

Thus by increasing the integer N, we may assume that the normal bundle $\mathcal{N}(j)$ is isomorphic to the trivial bundle $\mathcal{O}_X^{N-1}$.

Repeating arguments from the proof of Lemma 9.2 we get the following

Proposition 10.6. *Let $q : X \to B$ be the almost elementary fibration from Remark 10.4. Then there are an integer $N \geqslant 0$, a closed embedding $X \hookrightarrow$*

$B \times \mathbb{A}^N$ of B-schemes, an étale affine neighborhood $(\mathcal{V}, \rho : \mathcal{V} \to B \times \mathbb{A}^N, s : X \hookrightarrow \mathcal{V})$ of X in $B \times \mathbb{A}^N$, functions $\varphi_1, ..., \varphi_{N-1} \in k[\mathcal{V}]$ and a morphism $r : \mathcal{V} \to X$ such that:

(i) the functions $\varphi_1, ..., \varphi_{N-1}$ generate the ideal $I_{s(X)}$ in $k[\mathcal{V}]$ defining the closed subscheme $s(X)$ of $\mathcal{V}$;
(ii) $r \circ s = id_X$;
(iii) the morphism r is a B-scheme morphism if $\mathcal{V}$ is regarded as a B-scheme via the morphism $pr_U \circ \rho$, and X is regarded as a B-scheme via the morphism q.

Definition 10.7. Let $x \in S$, $x' \in S'$ be such that $\Pi(x') = x$. Set $U = Spec(\mathcal{O}_{X,x})$. There is an obvious morphism $\Delta = (id, can) : U \to U \times_B X$. It is a section of the projection $p_U : U \times_B X \to U$. Let $p_X : U \times_B X \to X$ be the projection onto X. Let $\pi : U' \to U$ be the restriction of Π to U'.

Notation 10.8. In what follows we will write $U \times X$ to denote $U \times_B X$, $U \times X'$ to denote $U \times_B X'$, $U' \times X'$ to denote $U' \times_B X'$, etc. Here X' is regarded as a B-scheme via the morphism $q \circ \Pi$.

The following proposition will be proved in Section 15.

Proposition 10.9. *Under the conditions of Remark 10.4 and Notation 10.8 there is a function $h_\theta \in k[\mathbb{A}^1 \times U \times X]$ (θ is the parameter on the left factor $\mathbb{A}^1$) such that the following properties hold for the functions h_θ, $h_1 := h_\theta|_{1 \times U \times X}$ and $h_0 := h_\theta|_{0 \times U \times X}$:*

(a) the morphism $(pr, h_\theta) : \mathbb{A}^1 \times U \times X \to \mathbb{A}^1 \times U \times \mathbb{A}^1$ is finite surjective, and hence the closed subscheme $Z_\theta := h_\theta^{-1}(0) \subset \mathbb{A}^1 \times U \times X$ is finite flat and surjective over $\mathbb{A}^1 \times U$;
(b) for the closed subscheme $Z_0 := h_0^{-1}(0)$ one has $Z_0 = \Delta(U) \sqcup G$ (an equality of closed subschemes) and $G \subset U \times (X - S)$;
(c) the closed subscheme $(id_U \times \Pi)^*(h_1) = 0$ is a disjoint union of the form $Z_1' \sqcup Z_2'$ and $m := (id_U \times \Pi)|_{Z_1'}$ identifies Z_1' with the closed subscheme $Z_1 := \{h_1 = 0\}$;
(d) $Z_\theta \cap \mathbb{A}^1 \times (U - S) \times S = \emptyset$ or, equivalently, $Z_\theta \cap \mathbb{A}^1 \times (U - S) \times X \subset \mathbb{A}^1 \times (U - S) \times (X - S)$.

Remark 10.10. Item (d) yields the following inclusions: $Z_\theta \cap \mathbb{A}^1 \times (U - S) \times X \subset \mathbb{A}^1 \times (U - S) \times (X - S)$, $Z_0 \cap (U - S) \times X \subset (U - S) \times (X - S)$, and $Z_1 \cap (U - S) \times X \subset (U - S) \times (X - S)$. Applying item (c), we get another inclusion: $Z_1' \cap (U - S) \times X' \subset (U - S) \times (X' - S')$.

11. Reducing Theorem 3.13 to Proposition 10.9

As ususal we follow definitions, notation and constructions from Sections 2 and 3. In this section we construct the desired elements $a \in \mathbb{Z}F_N((U, U - S)), (X', X' - S'))$ and $b_G \in \mathbb{Z}F_N((U, U - S)), (X - S, X - S))$ satisfying the relation (15). To construct $b \in Fr_N(U, X)$, we first construct its support in $U \times \mathbb{A}^N$ for some integer N, then we construct an étale neighborhood of the support in $U \times \mathbb{A}^N$, then one constructs a framing of the support in the neighborhood, and finally one constructs b itself. In the same manner we construct $a \in Fr_N(U, X')$ and a homotopy $H \in Fr_N(\mathbb{A}^1 \times U, X)$ between $\Pi \circ a$ and b. Using the fact that the support Z_0 of b is of the form $\Delta(U) \sqcup G$ with $G \subset U \times (X - S)$, we get an equality

$$\langle b \rangle = \langle b_1 \rangle + \langle b_2 \rangle$$

in $\mathbb{Z}F_N(U, X)$. Then we prove that $[b_1] = [\sigma_X^N] \circ [can]$ and $[b_2]$ factors through $X - S$. Moreover, we are able to work with elements of pairs. These will end up with the equality (15) and will complete the proof of Theorem 3.13 at the very end of the section. We will use systematically the data from Proposition 10.6 in this section (the details are given below).

Under the assumptions and notation of Proposition 10.6, Lemma 10.6 and Remark 10.3, set $\mathcal{V}' = X' \times_B \mathcal{V}$. So we have a Cartesian square

$$\begin{array}{ccc} \mathcal{V}' & \xrightarrow{\Pi'} & \mathcal{V} \\ \downarrow{\scriptstyle r'} & & \downarrow{\scriptstyle r} \\ X' & \xrightarrow{\Pi} & X, \end{array}$$

where r' and Π' are the projections to the first and second factors respectively. The section $s : X \to \mathcal{V}$ defines a section $s' = (id, s) : X' \to \mathcal{V}'$ of r'. For brevity, we will write below $U \times \mathcal{V}$ to denote $U \times_B \mathcal{V}$, $U \times \mathcal{V}'$ for $U \times_B \mathcal{V}'$, and $id \times \rho$ for $id \times_B \rho : U \times_B \mathcal{V} \to U \times_B (B \times \mathbb{A}^n) = U \times \mathbb{A}^N$. Let $p_{\mathcal{V}} : U \times \mathcal{V} \to \mathcal{V}$ be the projection.

Let $X \subset B \times \mathbb{A}^N$ be the closed inclusion from Proposition 10.6. Taking the base change of the latter inclusion by means of the morphism $U \to B$, we get a closed inclusion $U \times X \subset U \times \mathbb{A}^N$.

Under the notation from Proposition 10.6 and Proposition 10.9, we now construct an element $b \in Fr_N(U, X)$. Let $Z_0 \subset U \times X$ be the closed subset from Proposition 10.9. Then one has the closed inclusions

$$\Delta(U) \sqcup G = Z_0 \subset U \times X \subset U \times \mathbb{A}^N.$$

Let $in_0 : Z_0 \subset U \times X$ be the closed inclusion. Define an étale neighborhood of Z_0 in $U \times \mathbb{A}^N$ as follows:

$$(17) \qquad (U \times \mathcal{V}, id \times \rho : U \times \mathcal{V} \to U \times \mathbb{A}^N, (id \times s) \circ in_0 : Z_0 \to U \times \mathcal{V}).$$

We will write $\Delta(U) \sqcup G = Z_0 \subset U \times \mathcal{V}$ for $((id \times s) \circ in_0)(Z_0) \subset U \times \mathcal{V}$. Let $f \in k[U \times \mathcal{V}]$ be a function such that $f|_G = 0$ and $f|_{\Delta(U)} = 1$. Then $\Delta(U)$ is a closed subset of the affine scheme $(U \times \mathcal{V})_f$.

Definition 11.1. Under the notation from Proposition 10.6 and Proposition 10.9, set

$$b' = (Z_0, U\times\mathcal{V}, p^*_{\mathcal{V}}(\varphi_1), ..., p^*_{\mathcal{V}}(\varphi_{N-1}), (id\times r)^*(h_0); pr_X\circ(id\times r))\in Fr_N(U, X).$$

We will sometimes write below $(Z_0, U\times\mathcal{V}, p^*_{\mathcal{V}}(\varphi), (id\times r)^*(h_0); pr_X\circ(id\times r))$ to denote the element b'.

To construct the desired element $b \in Fr_N(U, X)$, we need to modify slightly the function $p^*_{\mathcal{V}}(\varphi_1)$ in the framing of Z_0. By Proposition 10.6 and item (b) of Proposition 10.9, the functions

$$p^*_{\mathcal{V}}(\varphi_1), ..., p^*_{\mathcal{V}}(\varphi_{N-1}), (id \times r)^*(h_0)$$

generate the ideal $I_{(id\times s)(\Delta(U))}$ in $k[(U \times \mathcal{V})_f]$ defining the closed subscheme $\Delta(U)$ of the scheme $(U\times\mathcal{V})_f$. Let $t_1, t_2, \dots, t_N \in k[U\times\mathbb{A}^N]$ be the coordinate functions. For any $i = 1, 2, \dots, N$, set $t'_i = t_i - (t_i|_{\Delta(U)}) \in k[U \times \mathbb{A}^N]$. Then the family

$$(t''_1, t''_2, \dots, t''_N) = (id \times \rho)^*(t'_1), (id \times \rho)^*(t'_2), \dots, (id \times \rho)^*(t'_N)$$

also generates the ideal $I = I_{(id\times s)(\Delta(U))}$ in $k[(U \times \mathcal{V})_f]$. This holds, because (17) is an étale neighborhood of Z_0 in $U \times \mathbb{A}^N$. By Remark 10.5 the $k[U] = k[(id \times s)(\Delta(U))]$-module I/I^2 is free of rank N. Thus the families $(\bar{t}''_1, \bar{t}''_2, \dots, \bar{t}''_N)$ and $(\overline{p^*_{\mathcal{V}}(\varphi_1)}, ..., \overline{p^*_{\mathcal{V}}(\varphi_{N-1})}, \overline{(id \times r)^*(h_0)})$ are two bases of the free $k[((id \times s) \circ \Delta)(U))]$-module I/I^2. Let $J \in k[U]^\times$ be the Jacobian of a unique matrix $A \in M_N(k[U])$ which transforms the first free basis to the second one. Set,

$$\varphi_1^{new} = q^*_U(J^{-1})\varphi_1 \in k[\mathcal{V}],$$

where $q_U = pr_U \circ (id \times \rho) : \mathcal{V} \to U$. Let $A^{new} \in M_N(k[U])$ be a unique matrix changing the first free basis to the basis

$$(\overline{p^*_{\mathcal{V}}(\varphi_1^{new})}, \overline{p^*_{\mathcal{V}}(\varphi_2)}, ..., \overline{p^*_{\mathcal{V}}(\varphi_{N-1})}, \overline{(id \times r)^*(h_0)}).$$

Then the Jacobian J^{new} of A^{new} is equal to 1:

$$J^{new} = 1 \in k[U]^{\times}. \tag{18}$$

We will write

$$(\psi_1, \psi_2, \dots, \psi_{N-1}) \text{ for } (p^*_{\mathcal{V}}(\varphi^{new}_1), p^*_{\mathcal{V}}(\varphi_2), ..., p^*_{\mathcal{V}}(\varphi_{N-1})).$$

Definition 11.2. Under the notation from Proposition 10.6 and Proposition 10.9 set

$$b = (Z_0, U \times \mathcal{V}, \psi_1, ..., \psi_{N-1}, (id \times r)^*(h_0); pr_X \circ (id \times r)) \in Fr_N(U, X).$$

For brevity, we will sometimes write

$$b = (Z_0, U \times \mathcal{V}, p^*_{\mathcal{V}}(\psi), (id \times r)^*(h_0); pr_X \circ (id \times r)).$$

Under the notation from Proposition 10.6 and Proposition 10.9 we now construct an element $a \in Fr_N(U, X)$. Let $Z_1 \subset U \times X$ be the closed subset from Proposition 10.9. Then one has closed inclusions

$$Z_1 \subset U \times X \subset U \times \mathbb{A}^N.$$

Set $(U \times X')_\circ = (U \times X') - Z''_2$ and $(U \times \mathcal{V}')_\circ = (id \times r')^{-1}((U \times X')_\circ)$. Let $in_1 : Z_1 \subset U \times X$ and $in'_1 : Z'_1 \subset (U \times X')_\circ$ be closed inclusions. Set,

$$r_\circ = (id \times r')|_{(U \times \mathcal{V}')_\circ} : (U \times \mathcal{V}')_\circ \to (U \times X')_\circ.$$

Using the notation of Proposition 10.6 and Proposition 10.9 (item (c)), define an étale neighborhood of Z_1 in $U \times \mathbb{A}^N$ as follows:

$$\begin{aligned} ((U \times \mathcal{V}')_\circ, &(id \times \rho) \circ (id \times \Pi') : (U \times \mathcal{V}')_\circ \to U \times \mathbb{A}^N, \\ &(id \times s') \circ in'_1 \circ m^{-1} : Z_1 \to (U \times \mathcal{V}')_\circ). \end{aligned} \tag{19}$$

Definition 11.3. Under the notation of Proposition 10.6 and Proposition 10.9 set

$$a := (Z_1, (U \times \mathcal{V}')_\circ, (id \times \Pi')^*(\psi_1), ..., (id \times \Pi')^*(\psi_{N-1}), r^*_\circ(id \times \Pi)^*(h_1); pr_{X'} \circ r_\circ) \in Fr_N(U, X').$$

For brevity, we will sometimes write

$$a = (Z_1, (U \times \mathcal{V}')_\circ, (id \times \Pi')^*(\psi), r_\circ^*(id \times \Pi)^*(h_1); pr_{X'} \circ r_\circ).$$

Under the notation of Proposition 10.6 and Proposition 10.9, let us construct now a element $H_\theta \in Fr_N(\mathbb{A}^1 \times U, X)$. Let $Z_\theta \subset \mathbb{A}^1 \times U \times X$ be the closed subset from Proposition 10.9. Then one has closed inclusions

$$Z_\theta \subset \mathbb{A}^1 \times U \times X \subset \mathbb{A}^1 \times U \times \mathbb{A}^N.$$

Let $in_\theta : Z_\theta \subset \mathbb{A}^1 \times U \times X$ be the closed inclusion. Define an étale neighborhood of Z_θ in $\mathbb{A}^1 \times U \times \mathbb{A}^N$ as follows:

$$(\mathbb{A}^1 \times U \times \mathcal{V}, id \times id \times \rho : \mathbb{A}^1 \times U \times \mathcal{V} \to \mathbb{A}^1 \times U \times \mathbb{A}^N, (id \times id \times s) \circ in_\theta : Z_\theta \to \mathbb{A}^1 \times U \times \mathcal{V}). \tag{20}$$

Definition 11.4. Under the notation of Propositions 10.6 and 10.9 we set

$$H_\theta = (Z_\theta, \mathbb{A}^1 \times U \times \mathcal{V}, \psi_1, ..., \psi_{N-1}, (id \times id \times r)^*(h_\theta); pr_X \circ (id \times id \times r)) \in Fr_N(\mathbb{A}^1 \times U, X).$$

We will sometimes write below $(Z_\theta, \mathbb{A}^1 \times U \times \mathcal{V}, \psi, (id \times id \times r)^*(h_\theta); pr_X \circ (id \times id \times r))$ to denote the element H_θ.

Lemma 11.5. *One has equalities* $H_0 = b$, $H_1 = \Pi \circ a$ *in* $Fr_N(U, X)$.

Proof. The first equality is obvious. To check the second one, consider

$$H_1 = (Z_1, U \times \mathcal{V}, \psi, (id \times r)^*(h_1); pr_X \circ (id \times r)) \in Fr_N(U, X).$$

Here we use $(U \times \mathcal{V}, id \times \rho : U \times \mathcal{V} \to U \times \mathbb{A}^N, (id \times s) \circ in_1 : Z_1 \to U \times \mathcal{V})$ as an étale neighborhood of Z_1 in $U \times \mathbb{A}^N$. Take another étale neighborhood of Z_1 in $U \times \mathbb{A}^N$

$$((U \times \mathcal{V}')_\circ, (id \times \rho) \circ (id \times \Pi') : (U \times \mathcal{V}')_\circ \to U \times \mathbb{A}^N, (id \times s') \circ in_1' \circ m^{-1} : Z_1 \to (U \times \mathcal{V}')_\circ)$$

and the morphism $id \times \Pi' : (U \times \mathcal{V}')_\circ \to U \times \mathcal{V}$ regarded as a morphism of étale neighborhoods. Refining the étale neighborhood of Z_1 in the definition of H_1 by means of that morphism, we get a framed correspondence $H_1' = H_1$ of level N, which has the form

$$(Z_1, (U \times \mathcal{V}')_\circ, (id \times \Pi')^*(\psi), (id \times \Pi')^*(id \times r)^*(h_1); pr_X \circ (id \times r) \circ (id \times \Pi')).$$

Note that

$$(id \times \Pi')^*(id \times r)^*(h_1) = r_\circ^*(id \times \Pi)^*(h_1)$$

and

$$pr_X \circ (id \times r) \circ (id \times \Pi') = \Pi \circ pr_{X'} \circ r_\circ.$$

Thus, $H_1 = H_1' = \Pi \circ a$ in $Fr_N(U, X)$. □

The following lemma follows from Lemma 4.4 and Remark 10.10.

Lemma 11.6. *The elements $a|_{U-S}$, $b|_{U-S}$, $H_\theta|_{\mathbb{A}^1 \times (U-S)}$ and $\Pi|_{X'-S'}$ run inside $X' - S'$, $X - S$, $X - S$ and $X - S$ respectively.*

By the preceding lemma and Definition 4.3 the elements a, b, H_θ and Π define elements

$$\langle\langle a \rangle\rangle \in \mathbb{Z}F_N((U, U - S), (X', X' - S')),$$

$$\langle\langle b \rangle\rangle \in \mathbb{Z}F_N((U, U - S), (X, X - S)),$$

$$\langle\langle H_\theta \rangle\rangle \in \mathbb{Z}F_N(\mathbb{A}^1 \times (U, U - S), (X, X - S)),$$

$$\langle\langle \Pi \rangle\rangle \in \mathbb{Z}F_N((X', X' - S'), (X, X - S)).$$

Lemma 11.5 and Definition 4.3 yield equalities

$$\langle\langle \Pi \rangle\rangle \circ \langle\langle a \rangle\rangle = \langle\langle H_1 \rangle\rangle \quad \text{and} \quad \langle\langle H_0 \rangle\rangle = \langle\langle b \rangle\rangle$$

in $\mathbb{Z}F_N((U, U - S), (X, X - S))$.

Corollary 11.7. *One has an equality $[[\Pi]] \circ [[a]] = [[b]]$ in $\overline{\mathbb{Z}F}_N((U, U - S), (X, X - S))$.*

Proof of Corollary 11.7. In fact, by Corollary 4.5 one has a chain of equalities

$$[[\Pi]] \circ [[a]] = [[H_1]] = [[H_0]] = [[b]]$$

in $\overline{\mathbb{Z}F}_N((U, U - S), (X, X - S))$. □

Reducing Theorem 3.13 to Proposition 10.9. The support Z_0 of b is the disjoint union $\Delta(U) \sqcup G$. Thus, by Lemma 4.6 one has an equality

$$\langle\langle b \rangle\rangle = \langle\langle b_1 \rangle\rangle + \langle\langle b_2 \rangle\rangle$$

in $\mathbb{Z}F_N((U, U - S), (X, X - S))$, where

$$b_1 = (\Delta(U), (U \times \mathcal{V})_f, \psi_1, ..., \psi_{N-1}, (id \times r)^*(h_0); pr_X \circ (id \times r)),$$

$$b_2 = (G, (U \times \mathcal{V} - \Delta(U), \psi_1, ..., \psi_{N-1}, (id \times r)^*(h_0); pr_X \circ (id \times r)).$$

By Proposition 10.9 one has $G \subset U \times (X - S)$. Thus $b_2 = j \circ b_G$ for an obvious element $b_G \in Fr_N(U, X - S)$. Also,

$$\langle\langle b_2 \rangle\rangle = \langle\langle j \rangle\rangle \circ \langle\langle b_G \rangle\rangle \in ZF_N((U, U - S), (X, X - S)),$$

where $j : (X - S, X - S) \hookrightarrow (X, X - S)$ is a natural inclusion. By the latter comments and Corollary 11.7 one gets an equality

$$[[\Pi]] \circ [[a]] - [[j]] \circ [[b_G]] = [[b_1]]$$

in $\overline{\mathbb{Z}F}_N((U, U - S), (X, X - S))$. To prove equality (15), and hence to prove Theorem 3.13, it remains to check that $[[b_1]] = [[\sigma_X^N]] \circ [[can]]$. Recall that one has equality (18). Thus the equality $[[b_1]] = [[\sigma_X^N]] \circ [[can]]$ holds by Theorem 14.3. This finishes the proof of Theorem 3.13. □

12. Preliminaries for the surjective part of the étale excision

As ususal we follow definitions, notation and constructions from Sections 2 and 3. Let X, X' be irreducible k-smooth schemes. Let $V \subset X$ and $V' \subset X'$ be open subsets. Let

$$\begin{array}{ccc} V' & \longrightarrow & X' \\ \downarrow & & \downarrow \Pi \\ V & \longrightarrow & X \end{array}$$

be an elementary distinguished square in the sense of [13, Definition 3.1.3]. Let $S = X - V$ and $S' = X' - V'$ be closed subschemes equipped with reduced structures. Let $x \in S$ and $x' \in S'$ be two points such that $\Pi(x') = x$. Let $U = Spec(\mathcal{O}_{X,x})$ and $U' = Spec(\mathcal{O}_{X',x'})$. Let $\pi : U' \to U$ be the morphism induced by Π.

To prove Theorem 3.14 it suffices to find elements $a \in \mathbb{Z}F_N((U, U - S)), (X', X' - S'))$ and $b_G \in \mathbb{Z}F_N((U', U' - S')), (X' - S', X' - S'))$ such that in the characteristic different from 2 the following equality holds:

$$[[a]] \circ [[\pi]] - [[j]] \circ [[b_G]] = [[\sigma_{X'}^N]] \circ [[can']]. \tag{21}$$

If the characteristic of k is 2 then the following equality holds in $\overline{\mathbb{Z}F}_N(U', U' - S')), (X', X' - S'))$:

$$2 \cdot [[a]] \circ [[\pi]] - 2 \cdot [[j]] \circ [[b_G]] = 2 \cdot [[\sigma_{X'}^N]] \circ [[can']]. \tag{22}$$

Here $j : (X' - S', X' - S') \to (X', X' - S')$ and $can' : (U', U' - S') \to (X', X' - S')$ are inclusions. In this section we do some preparations to construct the desired elements

$$a \in \mathbb{Z}F_N((U, U - S)), (X', X' - S'))$$

and

$$b_G \in \mathbb{Z}F_N((U', U' - S')), (X' - S', X' - S'))$$

in Section 13 satisfying (21) in the characteristic different from 2, and satisfying (22) if the characteristic of k is 2. Our preparations are independent of the characteristic of the base field k.

Replace X by an affine open neighborhood $in : X^\circ \hookrightarrow X$ of the point x. Replace X' by $(X')^\circ := \Pi^{-1}(X^\circ)$ and write $in' : (X')^\circ \hookrightarrow X'$ for the inclusion. Replace V by $V \cap X^\circ$ and V' with $V' \cap (X')^\circ$. Let $can'_\circ : U' \to (X')^\circ$ be the canonical inclusion. Let $j^\circ : ((X'^\circ) - S', (X')^\circ - S') \to ((X')^\circ, (X')^\circ - S')$ be an inclusion of pairs. If we find

$$a^\circ \in \mathbb{Z}F_N((U, U - S)), ((X')^\circ, (X')^\circ - S'))$$

and

$$b_G^\circ \in \mathbb{Z}F_N((U', U' - S')), ((X')^\circ - S', (X')^\circ - S'))$$

such that

$$[[a^\circ]] \circ [[\pi]] - [[j^\circ]] \circ [[b_G^\circ]] = [[\sigma^N_{(X')^\circ}]] \circ [[can'_\circ]], \tag{23}$$

then the elements $a = in' \circ a^\circ$ and $b_G = in' \circ b_G^\circ$ satisfy condition (21). Thus we may assume that X is an affine variety.

Let X'_n be the normalization of X in $Spec(k(X'))$. Let $\Pi_n : X'_n \to X$ be the corresponding finite morphism. Since X' is k-smooth it is an open subscheme of X'_n. Let $Y'' = X'_n - X'$. It is a closed subset in X'_n. Since $\Pi|_{S'} : S' \to S$ is a scheme isomorphism, then S' is closed in X'_n. Thus $S' \cap Y'' = \emptyset$. Hence there is a function $f \in k[X'_n]$ such that $f|_{Y''} = 0$ and $f|_{S'} = 1$.

Remark 12.1. In this section we use agreements and notation from Definition 10.2 and Remark 10.3. Particularly, we may change notation and write X' for X'_{new}.

Remark 12.2. Shrinking X and X' exactly as in Remark 10.4 and changing notation again, consider the almost elementary fibration $q : X \to B$ from

Remark 10.4. Then $\Omega^1_{X'/B} \cong \mathcal{O}_{X'}$. In fact, by Remark 10.5 $\Omega^1_{X/B} = \omega_{X/B} \cong \mathcal{O}_X$. The morphism $\Pi : X' \to X$ is étale. Thus $\Omega^1_{X'/B} \cong \mathcal{O}_{X'}$.

Since X' is an affine k-variety, there is a closed embedding $j : X' \hookrightarrow B \times \mathbb{A}^N$ of B-schemes. Choose and fix such an embedding j. Since X' is affine k-smooth, hence $[\mathcal{N}(j)] = (N-1)[\mathcal{O}_X]$ in $K_0(X')$, where $\mathcal{N}(j)$ is the normal bundle to X' associated with the imbedding j.

Thus by increasing the integer N, we may assume that the normal bundle $\mathcal{N}(j)$ is isomorphic to the trivial bundle $\mathcal{O}^{N-1}_{X'}$.

Repeating arguments from the proof of Lemma 9.2, we get the following

Proposition 12.3. *Let $q : X \to B$ be the almost elementary fibration from Remark 12.2 and let X' be as in Remark 12.2. Then there are an integer $N \geqslant 0$, a closed embedding $j : X' \hookrightarrow B \times \mathbb{A}^N$ of B-schemes, an étale affine neighborhood $(\mathcal{V}'', \rho'' : \mathcal{V}'' \to B \times \mathbb{A}^N, s'' : X' \hookrightarrow \mathcal{V}'')$ of X' in $B \times \mathbb{A}^N$, functions $\varphi'_1, ..., \varphi'_{N-1} \in k[\mathcal{V}'']$ and a morphism $r'' : \mathcal{V}'' \to X'$ such that:*

- *(i) the functions $\varphi'_1, ..., \varphi'_{N-1}$ generate the ideal $I_{s''(X')}$ in $k[\mathcal{V}'']$ defining the closed subscheme $s''(X')$ of $\mathcal{V}''$;*
- *(ii) $r'' \circ s'' = id_{X'}$;*
- *(iii) the morphism r'' is a B-scheme morphism if $\mathcal{V}''$ is regarded as a B-scheme via the morphism $pr_U \circ \rho''$ and X' is regarded as a B-scheme via the morphism $q \circ \Pi$ from Lemma 10.4.*

Definition 12.4. Let $x \in S$, $x' \in S'$ be such that $\Pi(x') = x$. We put $U = Spec(\mathcal{O}_{X,x})$. There is an obvious morphism $\Delta' = (id, can) : U' \to U' \times_B X'$. It is a section of the projection $p_{U'} : U' \times_B X' \to U'$. Let $p_{X'} : U' \times_B X' \to X'$ be the projection onto X'. Let $\pi : U' \to U$ be the restriction of Π to U'.

Notation 12.5. We regard X as a B-scheme via the morphism q and regard X' as a B-scheme via the morphism $q \circ \Pi$. In what follows we write $U \times X'$ for $U \times_B X'$, $U' \times X'$ for $U' \times_B X'$, $\mathbb{A}^1 \times U' \times X'$ for $\mathbb{A}^1 \times U' \times_B X'$ etc.

The following proposition will be proved in Section 15.

Proposition 12.6. *Under the conditions of Remark 12.2 and Notation 12.5 there are functions $F \in k[U \times X']$ and $h'_\theta \in k[\mathbb{A}^1 \times U' \times X']$ (θ is the parameter on the left factor $\mathbb{A}^1$) such that the following properties hold for the functions h'_θ, $h'_1 := h'_\theta|_{1 \times U' \times X'}$ and $h'_0 := h'_\theta|_{0 \times U' \times X'}$:*

- *(a) the morphism $(pr, h'_\theta) : \mathbb{A}^1 \times U' \times X' \to \mathbb{A}^1 \times U' \times \mathbb{A}^1$ is finite and surjective, hence the closed subscheme $Z'_\theta := (h'_\theta)^{-1}(0) \subset \mathbb{A}^1 \times U' \times X'$ is finite flat and surjective over $\mathbb{A}^1 \times U'$;*

(b) *for the closed subscheme* $Z'_0 := (h'_0)^{-1}(0)$ *one has* $Z'_0 = \Delta'(U') \sqcup G'$ *(an equality of closed subschemes) and* $G' \subset U' \times (X' - S')$*;*
(c) $h'_1 = (\pi \times id_{X'})^*(F)$ *(we write* Z'_1 *to denote the closed subscheme* $\{h'_1 = 0\}$*);*
(d) $Z'_\theta \cap \mathbb{A}^1 \times (U' - S') \times S' = \emptyset$ *or, equivalently,* $Z'_\theta \cap \mathbb{A}^1 \times (U' - S') \times X' \subset \mathbb{A}^1 \times (U' - S') \times (X' - S')$*;*
(e) *the morphism* $(pr_U, F) : U \times X' \to U \times \mathbb{A}^1$ *is finite surjective, and hence the closed subscheme* $Z_1 := F^{-1}(0) \subset U \times X'$ *is finite flat and surjective over* U*;*
(f) $Z_1 \cap (U - S) \times S' = \emptyset$ *or, equivalently,* $Z_1 \cap (U - S) \times X' \subset (U - S) \times (X' - S')$*.*

Remark 12.7. Item (d) yields the following inclusions:

- $Z'_\theta \cap \mathbb{A}^1 \times (U' - S') \times X' \subset \mathbb{A}^1 \times (U' - S') \times (X' - S')$;
- $Z'_0 \cap (U' - S') \times X' \subset (U' - S') \times (X' - S')$;
- $Z'_1 \cap (U' - S') \times X' \subset (U' - S') \times (X' - S')$.

Applying (f), we get another inclusion: $Z_1 \cap (U-S) \times X' \subset (U-S) \times (X'-S')$.

13. Reducing Theorem 3.14 to Propositions 12.3 and 12.6

We follow here definitions, notation and constructions from Sections 2 and 3. We suppose in this section that $S \subset X$ is k-smooth. In the present section we construct the desired elements $a \in \mathbb{Z}F_N((U, U - S)), (X', X' - S'))$ and $b_G \in \mathbb{Z}F_N((U', U' - S')), (X' - S', X' - S'))$ satisfying (21) in the characteristic different from 2, and satisfying (22) if the characteristic equals 2. This construction does not depend on the characteristic of the base field k.

To construct an element $a \in Fr_N(U, X')$, we first construct its support in $U \times \mathbb{A}^N$ for some integer N, then we construct an étale neighborhood of the support in $U \times \mathbb{A}^N$, then one constructs a framing of the support in the neighborhood and finally one constructs a itself. In the same fashion we construct an element $b \in Fr_N(U', X')$ and a homotopy $H \in Fr_N(\mathbb{A}^1 \times U', X')$ between $a \circ \pi$ and b. Using the fact that the support Z'_0 of b is of the form $\Delta'(U') \sqcup G'$ with $G' \subset U' \times (X' - S')$, we get a relation

$$\langle b \rangle = \langle b_1 \rangle + \langle b_2 \rangle$$

in $\mathbb{Z}F_N(U', X')$. Then we prove that $[\sigma^N_{X'}] \circ [b_1] = [can']$ if char $k \neq 2$ and $[b_2]$ factors through $X' - S'$. If char $k = 2$ we prove that $2 \cdot [b_1] = 2 \cdot ([\sigma^N_{X'}] \circ [can'])$ and $[b_2]$ factors through $X' - S'$. Moreover, we are able to work with elements of pairs. In this section we will use systematically Propositions 12.3 and 12.6

and Notation 12.5. These will end up with the equalities (21), (22) and will complete the proof of Theorem 3.14 at the very end of the section (details are given below).

Let $X' \subset B \times \mathbb{A}^N$ be the closed inclusion from Proposition 12.3. Taking the base change of the latter inclusion by means of the morphism $U \to B$, we get a closed inclusion $U \times_B X' \subset U \times_B (B \times \mathbb{A}^N) = U \times \mathbb{A}^N$. Recall (see Notation 12.5) that we regard X as a B-scheme via the morphism q and regard X' as a B-scheme via the morphism $q \circ \Pi$. In what follows we write $U \times X'$ for $U \times_B X'$, $U' \times X'$ for $U' \times_B X'$, $U \times \mathcal{V}''$ for $U \times_B \mathcal{V}''$, $U' \times \mathcal{V}''$ for $U' \times_B \mathcal{V}''$, and $id \times \rho$ for $id \times_B \rho : U \times_B \mathcal{V}'' \to U \times_B (B \times \mathbb{A}^n) = U \times \mathbb{A}^N$. Let $p_{\mathcal{V}} : U \times \mathcal{V} \to \mathcal{V}$ be the projection.

Under the notation from Proposition 12.3 and Proposition 12.6, construct now an element $b \in Fr_N(U', X')$. Let $Z'_0 \subset U' \times X'$ be the closed subset from Proposition 12.6. Then one has closed inclusions

$$\Delta'(U') \sqcup G' = Z'_0 \subset U' \times X' \subset U' \times \mathbb{A}^N.$$

Let $in_0 : Z'_0 \subset U' \times X'$ be a closed inclusion. Define an étale neighborhood of Z'_0 in $U' \times \mathbb{A}^N$ as follows:

$$(24)\quad (U' \times \mathcal{V}'', id \times \rho'' : U' \times \mathcal{V}'' \to U' \times \mathbb{A}^N, (id \times s'') \circ in_0 : Z'_0 \to U' \times \mathcal{V}'').$$

We will write $\Delta'(U') \sqcup G' = Z'_0 \subset U' \times \mathcal{V}''$ for $((id \times s'') \circ in_0)(Z'_0) \subset U' \times \mathcal{V}''$. Let $f \in k[U' \times \mathcal{V}'']$ be a function such that $f|_{G'} = 0$ and $f|_{\Delta'(U')} = 1$. Then $\Delta'(U')$ is a closed subset of the affine scheme $(U' \times \mathcal{V}'')_f$.

Definition 13.1. Under the notation from Proposition 10.6 and Proposition 10.9, set

$$b' := (Z'_0, U' \times \mathcal{V}'', (\pi \times id)^*(p^*_{\mathcal{V}''}(\varphi'_1), ..., p^*_{\mathcal{V}''}(\varphi'_{N-1})), (id \times r'')^*(h'_0); pr_{X'} \circ (id \times r'')) \in Fr_N(U', X').$$

Here $p_{\mathcal{V}''} : U \times \mathcal{V}'' \to \mathcal{V}''$ is the projection. Below we will sometimes write $(Z'_0, U' \times \mathcal{V}'', (\pi \times id)^*(p^*_{\mathcal{V}''}(\varphi')), (id \times r'')^*(h'_0); pr_{X'} \circ (id \times r''))$ to denote the element b'.

To construct the desired element $b \in Fr_N(U', X')$, we slightly modify the function $p^*_{\mathcal{V}''}(\varphi'_1)$ in the framing of Z'_0. By Proposition 12.3 and Proposition 12.6(b), the functions

$$(\pi \times id)^*(p^*_{\mathcal{V}''}(\varphi'_1)), ..., (\pi \times id)^*(p^*_{\mathcal{V}''}(\varphi'_{N-1})), (id \times r'')^*(h'_0)$$

generate an ideal $I_{(id\times s'')(\Delta'(U'))}$ in $k[(U'\times\mathcal{V}'')_f]$ defining the closed subscheme $\Delta'(U')$ of the scheme $(U'\times\mathcal{V}'')_f$. Let $t_1,t_2,\ldots,t_N\in k[U'\times\mathbb{A}^N]$ be the coordinate functions. For any $i=1,2,\ldots,N$, set $t'_i=t_i-(t_i|_{\Delta'(U')})\in k[U'\times\mathbb{A}^N]$. Then the family

$$(t''_1,t''_2,\ldots,t''_N)=(id\times\rho'')^*(t'_1),(id\times\rho'')^*(t'_2),\ldots,(id\times\rho'')^*(t'_N)$$

also generates the ideal $I=I_{(id\times s'')(\Delta'(U'))}$ in $k[(U'\times\mathcal{V}'')_f]$. This holds, because (24) is an étale neighborhood of Z'_0 in $U\times\mathbb{A}^N$. By Remark 12.2 the $k[U']=k[(id\times s'')(\Delta'(U'))]$-module I/I^2 is free of rank N. Thus the families

$$(\bar{t}''_1,\bar{t}''_2,\ldots,\bar{t}''_N)\ \text{ and }\ (\overline{p^*_{\mathcal{V}''}(\varphi'_1)},\ldots,\overline{p^*_{\mathcal{V}''}(\varphi'_{N-1})},\overline{(id\times r'')^*(h'_0)})$$

are two bases of the free $k[((id\times s'')\circ\Delta')(U'))]$-module I/I^2. Let $J\in k[U']^\times$ be the Jacobian of a unique matrix $A\in M_N(k[U'])$ converting the first basis to the second one. There is an element $\lambda\in k[U]$ such that $\lambda|_{S\cap U}=J|_{S'\cap U'}$ (we identify here $S'\cap U'$ with $S\cap U$ via the morphism $\pi|_{S'\cap U'}$). Clearly, $\lambda\in k[U]^\times$. Set,

$$(\varphi'_1)^{new}=q^*_U(J^{-1})(\varphi'_1)\in k[\mathcal{V}''],$$

where $q_U=pr_U\circ(id\times\rho''):\mathcal{V}''\to U$. Let $A^{new}\in M_N(k[U])$ be a unique matrix which converts the first basis to the basis

$$(\overline{p^*_{\mathcal{V}''}((\varphi'_1)^{new})},\ldots,\overline{p^*_{\mathcal{V}''}(\varphi'_{N-1})},\overline{(id\times r'')^*(h'_0)}).$$

Then the Jacobian $J^{new}\in k[U']^\times$ of A^{new} has the property:

$$J^{new}|_{S'\cap U'}=1\in k[S'\cap U']. \tag{25}$$

We will write $(\psi_1,\psi_2,\ldots,\psi_{N-1})$ for $(p^*_{\mathcal{V}''}((\varphi'_1)^{new}),\ldots,p^*_{\mathcal{V}''}(\varphi'_{N-1}))$.

Definition 13.2. Under the notation from Proposition 12.3 and Proposition 12.6, set

$$b:=(Z'_0,U'\times\mathcal{V}'',(\pi\times id)^*(\psi_1),\ldots,(\pi\times id)^*(\psi_{N-1}),(id\times r'')^*(h'_0);pr_{X'}\circ(id\times r''))\in Fr_N(U',X').$$

We often write for brevity $b=(Z'_0,U'\times\mathcal{V}'',(\pi\times id)^*(\psi),(id\times r'')^*(h'_0);pr_{X'}\circ(id\times r''))$.

Under the notation from Proposition 12.3 and Proposition 12.6 we now construct an element $a \in Fr_N(U, X')$. Let $Z_1 \subset U \times X'$ be the closed subset from Proposition 12.6. Then one has closed inclusions

$$Z_1 \subset U \times X' \subset U \times \mathbb{A}^N.$$

Let $in_1 : Z_1 \subset U \times X$ be the closed inclusion. Define an étale neighborhood of Z_1 in $U \times \mathbb{A}^N$ as follows:

$$(U \times \mathcal{V}'', id \times \rho'' : U \times \mathcal{V}'' \to U \times \mathbb{A}^N, (id \times s'') \circ in_1 : Z_1 \hookrightarrow U \times \mathcal{V}''). \tag{26}$$

Definition 13.3. Under the notation from Proposition 12.3 and Proposition 12.6 set

$$a = (Z_1, U \times \mathcal{V}'', \psi_1, ..., \psi_{N-1}, (id \times r'')^*(F); pr_{X'} \circ (id_U \times r'')) \in Fr_N(U, X')$$

We will sometimes write $(Z_1, U \times \mathcal{V}'', \psi, (id \times r'')^*(F); pr_{X'} \circ (id_U \times r''))$ to denote a.

Under the notation from Proposition 12.3 and Proposition 12.6 we now construct an element $H_\theta \in Fr_N(\mathbb{A}^1 \times U', X')$. Recall that under that notation we write $\mathbb{A}^1 \times U' \times X'$ for $\mathbb{A}^1 \times U' \times_B X'$. Let $Z'_\theta \subset \mathbb{A}^1 \times U' \times X'$ be the closed subset from Proposition 12.6. Then one has closed inclusions

$$Z'_\theta \subset \mathbb{A}^1 \times U' \times X' \subset \mathbb{A}^1 \times U' \times \mathbb{A}^N.$$

Let $in_\theta : Z'_\theta \subset \mathbb{A}^1 \times U' \times X'$ be the closed inclusion. Define an étale neighborhood of Z'_θ in $\mathbb{A}^1 \times U' \times \mathbb{A}^N$ as follows:

$$\begin{aligned}(\mathbb{A}^1 \times U' \times \mathcal{V}'', \mathbb{A}^1 \times U' \times \mathcal{V}'' \xrightarrow{id \times id \times \rho''} \mathbb{A}^1 \times U' \times \mathbb{A}^N,\\ (\mathrm{id} \times id \times s'') \circ in_\theta : Z'_\theta \hookrightarrow \mathbb{A}^1 \times U' \times \mathcal{V}'').\end{aligned} \tag{27}$$

Definition 13.4. Under the notation from Proposition 12.3 and Proposition 12.6, set H_θ to be equal to

$$(Z'_\theta, \mathbb{A}^1 \times U' \times \mathcal{V}'', pr^*((\pi \times id)^*(\psi)), (id \times id \times r'')^*(h'_\theta); pr_{X'} \circ (id \times id \times r''))$$

from $Fr_N(\mathbb{A}^1 \times U', X')$, where $pr : \mathbb{A}^1 \times U' \times \mathcal{V}'' \to U' \times \mathcal{V}''$ is the projection.

Lemma 13.5. *One has equalities $H_0 = b$, $H_1 = a \circ \pi$ in $Fr_N(U', X')$.*

Proof. The first equality is obvious. Let us prove the second one. By Proposition 12.6 one has $h'_1 = (\pi \times id_{X'})^*(F)$. Thus one has a chain of equalities in $Fr_N(U', X')$:

$$a \circ \pi =$$
$$(Z'_1, U' \times \mathcal{V}'', (\pi \times id_{\mathcal{V}''})^*(\psi), (\pi \times id_{\mathcal{V}''})^*((id_U \times r'')^*(F)); pr_{X'} \circ (id_U \times r'') \circ (\pi \times id_{\mathcal{V}''})) =$$
$$(Z'_1, U' \times \mathcal{V}'', (\pi \times id_{\mathcal{V}''})^*(\psi), (id_{U'} \times r'')^*((\pi \times id_{X'})^*(F)); pr_{X'} \circ (\pi \times id_{X'}) \circ (id_{U'} \times r''))$$
$$= (Z'_1, U' \times \mathcal{V}'', (\pi \times id_{\mathcal{V}''})^*(\psi), (id_{U'} \times r'')^*(h'_1); pr_{X'} \circ (id_{U'} \times r'')) = H_1,$$

as required. □

The following lemma follows from Lemma 4.4 and Remark 12.7.

Lemma 13.6. *The elements $a|_{U-S}$, $b|_{U'-S'}$, $H_\theta|_{\mathbb{A}^1 \times (U'-S')}$ and $\pi|_{U'-S'}$ run inside $X' - S'$, $X' - S'$, $X' - S'$ and $U - S$ respectively.*

By the preceding lemma and Definition 4.3 the elements a, b, H_θ and π define elements

$$\langle\langle a \rangle\rangle \in \mathbb{Z}F_N((U, U-S), (X', X'-S')),$$
$$\langle\langle b \rangle\rangle \in \mathbb{Z}F_N((U', U'-S'), (X', X-S')),$$
$$\langle\langle H_\theta \rangle\rangle \in \mathbb{Z}F_N(\mathbb{A}^1 \times (U', U'-S'), (X', X'-S')),$$
$$\langle\langle \pi \rangle\rangle \in \mathbb{Z}F_N((U', U'-S'), (U, U-S)).$$

Lemma 13.5 and Definition 4.3 yield equalities

$$\langle\langle a \rangle\rangle \circ \langle\langle \pi \rangle\rangle = \langle\langle H_1 \rangle\rangle \quad \text{and} \quad \langle\langle H_0 \rangle\rangle = \langle\langle b \rangle\rangle$$

in $\mathbb{Z}F_N((U', U'-S'), (X', X'-S'))$.

Corollary 13.7. *There is a relation $[[a]] \circ [[\pi]] = [[b]]$ in $\overline{\mathbb{Z}F}_N((U', U' - S'), (X', X' - S'))$.*

Proof of Corollary 13.7. In fact, by Corollary 4.5 one has a chain of equalities

$$[[a]] \circ [[\pi]] = [[H_1]] = [[H_0]] = [[b]]$$

in $\overline{\mathbb{Z}F}_N((U', U'-S'), (X', X'-S'))$. □

Reducing Theorem 3.14 to Propositions 12.6. The support Z_0 of b is the disjoint union $\Delta'(U') \sqcup G'$. Thus, by Lemma 4.6 one has,

$$\langle\langle b \rangle\rangle = \langle\langle b_1 \rangle\rangle + \langle\langle b_2 \rangle\rangle$$

in $\mathbb{Z}F_N((U', U'-S'), (X', X'-S'))$, where

$$b_1 = (\Delta'(U'), (U' \times \mathcal{V}'')_f, \psi_1, ..., \psi_{N-1}, (id \times r'')^*(h'_0); pr_{X'} \circ (id \times r'')),$$
$$b_2 = (G', (U' \times \mathcal{V}'' - \Delta'(U'), \psi_1, ..., \psi_{N-1}, (id \times r'')^*(h'_0); pr_{X'} \circ (id \times r''))$$

and the function f is defined just above Definition 13.1. By Proposition 12.6 one has $G' \subset U' \times (X' - S')$. Thus $b_2 = j \circ b_{G'}$ for the obvious element $b_{G'} \in Fr_N(U', X' - S')$. Also,

$$\langle\langle b_2 \rangle\rangle = \langle\langle j \rangle\rangle \circ \langle\langle b_{G'} \rangle\rangle \in \mathbb{Z}F_N((U', U'-S'), (X', X'-S')),$$

where $j : (X'-S', X'-S') \hookrightarrow (X', X'-S')$ is a natural inclusion. By the latter comments and Corollary 13.7 one gets,

$$[[a]] \circ [[\pi]] - [[j]] \circ [[b_{G'}]] = [[b_1]] \tag{28}$$

in $\overline{\mathbb{Z}F}_N((U', U'-S'), (X', X'-S'))$. Suppose now $\operatorname{char} k = 2$. Then by Theorem 14.5 one has

$$2 \cdot [[b_1]] = 2 \cdot ([[\sigma^N_{X'}]] \circ [[can']]) \tag{29}$$

in $\overline{\mathbb{Z}F}_N((U', U'-S'), (X', X'-S'))$. Hence the equality (28) yields the relation (22). So Theorem 3.14 follows for the case when $\operatorname{char} k = 2$.

Suppose now $\operatorname{char} k \neq 2$. To prove the equality (21), and hence to prove Theorem 3.14 in this case, it remains to check that $[[b_1]] = [[\sigma^N_{X'}]] \circ [[can']]$.

Recall that one has equality (25). Let us consider the étale $k[U']$-algebra $k[U'][t]/(t^2 - J^{new})$. Set $\tilde{U}' = Spec(k[U'][t]/(t^2 - J^{new}))$. Since $x' \in S' \cap U'$, we have equality $J^{new}(x') = 1$. Thus there are exactly two points x'', x''_1 in $\tilde{U}'$, which are over the point x'. Set $U'' = Spec(\mathcal{O}_{\tilde{U}', x''})$. Let $\pi' : U'' \to U'$ be the canonical morphism. Set $S'' = (\pi')^{-1}(S' \cap U')$. The morphism $\pi'|_{S''} : S'' \to S' \cap U'$ is an isomorphism, since $J^{new}_{S' \cap U'} = 1$. Note that $(\pi')^*(J^{new})$ is a square in $k[U'']^\times$. Thus by Theorem 14.4 one has an equality $[[b_1]] \circ [[\pi']] = [[\sigma^N_{X'}]] \circ [[can']] \circ [[\pi']]$ in $\overline{\mathbb{Z}F}_N((U'', U''-S''), (X', X'-S'))$. Applying Theorem 3.13 to the morphism $\pi' : U'' \to U'$, we see that for an integer $M \geqslant 0$ one has an equality

$$[[\sigma^M_{X'}]] \circ [[b_1]] = [[\sigma^{M+N}_{X'}]] \circ [[can']] \in \overline{\mathbb{Z}F}_{M+N}((U', U'-S'), (X', X'-S')).$$

Thus,

$$[[\sigma^M_{X'}]] \circ [[a]] \circ [[\pi]] - [[\sigma^M_{X'}]] \circ [[j]] \circ [[b_{G'}]] =$$
$$= [[\sigma^{M+N}_{X'}]] \circ [[can']] \in \overline{\mathbb{Z}F}_{M+N}((U', U'-S'), (X', X'-S')).$$

Note that $\langle\langle\sigma^M_{X'}\rangle\rangle\circ\langle\langle j\rangle\rangle=\langle\langle j\rangle\rangle\circ\langle\langle\sigma^M_{X'-S'}\rangle\rangle\in\mathbb{Z}F_M((X'-S',X'-S'),(X',X'-S'))$. Set $a_{new}=[[\sigma^M_{X'}]]\circ[[a]]$, $b^{new}_{G'}=\sigma^M_{X'-S'}\circ b_{G'}$, $N(new)=M+N$. Having these the following equality holds:

$$[[a_{new}]]\circ[[\pi]]-[[j]]\circ[[b^{new}_{G'}]]=[[\sigma^{N(new)}_{X'}]]\circ[[can']]$$
$$\in\overline{\mathbb{Z}F}_{M+N}((U',U'-S'),(X',X'-S')).$$

The latter equality is of the form (21). Thus, Theorem 3.14 is proved in the characteristic not 2. □

14. Three useful theorems

We follow definitions, notation and constructions from Sections 2 and 3. Let $Y,Y_1\in EssSm/k$ be essentially k-smooth schemes, $Z\subset Y$, $Z_1\subset Y_1$ be closed subsets. Let $f:Y_1\to Y$ be a k-morphism such that $Z_1\subset f^{-1}(Z)$. For an étale neighborhood $(V,\pi:V\to Y,s:Z\to V)$ of Z in Y set $V_1=Y_1\times_Y V$. Let $\pi_1:V_1\to Y_1$ be the projection and let $s_1=(i_1,f|_{Z_1}):Z_1\to V_1$, where $i_1:Z_1\hookrightarrow Y_1$ is the inclusion. Then (V_1,π_1,s_1) is an étale neighborhood of Z_1 in Y_1. We often will write $(f^*(V),f^*(\rho),f^*(s))$ for (V_1,ρ_1,s_1). Denote by $f^{nb}:f^*(V)=V_1\to V$ the projection. The following properties of this construction are straightforward:

Lemma 14.1. (1) *If $f=id_Y$, then $(f^*(V),f^*(\pi),f^*(s))=(V,\pi,s)$ and $f^{nb}=id_V$.*

(2) *Given a morphism $f_1:Y_2\to Y_1$ in $EssSm/k$ and a closed subset $Z_2\subset Y_2$ with $Z_2\subset f_1^{-1}(Z_1)$ one has $(f\circ f_1)^{nb}=f^{nb}\circ f_1^{nb}:(f\circ f_1)^*(V)\to V$.*

(3) *If $i:Z\hookrightarrow Y$ is a closed subset, $Y_1=Z$, $f=i$, then $i^*(\pi):i^*(V)\to Z$ identifies $(i^*(V),i^*(\pi),i^*(s))$ with (Z,id_Z,id_Z). So, we write Z for $i^*(V)$.*

(4) *If Z from the previous item is also in $EssSm/k$, $p:Y\to Z$ is a morphism in $EssSm/k$, then $(p\circ i)^*(V)=p^*(i^*(V))=p^*(Z)=Y$.*

Let $W\in EssSm/k$ and let $(\mathcal{W},\rho_0:\mathcal{W}\to W\times\mathbb{A}^N,s_0:W\to\mathcal{W})$ be an étale neighborhood of $W\times 0$ in $W\times\mathbb{A}^N$. The nearest aim is to formulate and prove Lemma 14.2 below.

Let $f_\theta:\mathbb{A}^1\times W\times\mathbb{A}^N\to W\times\mathbb{A}^N$ be a morphism given by $(\theta,w,y)\mapsto(w,\theta\cdot y)$. Since $\mathbb{A}^1\times W\times 0\subset f_\theta^{-1}(W\times 0)$ we see that $f^*_\theta(\mathcal{W})$ is an étale neighborhood of $\mathbb{A}^1\times W\times 0$ in $\mathbb{A}^1\times W\times\mathbb{A}^N$. Consider the étale neighborhoods $\mathrm{id}\times\rho:\mathbb{A}^1\times\mathcal{W}\to\mathbb{A}^1\times W\times\mathbb{A}^N$ of $\mathbb{A}^1\times W\times 0$ in $\mathbb{A}^1\times W\times\mathbb{A}^N$. Then

$$\mathcal{W}_\theta:=f^*_\theta(\mathcal{W})\times_{\mathbb{A}^1\times W\times\mathbb{A}^N}(\mathbb{A}^1\times\mathcal{W})=f^*_\theta(\mathcal{W})\times_{W\times\mathbb{A}^N}\mathcal{W}$$

is an étale neighborhood of $\mathbb{A}^1 \times W \times 0$ in $\mathbb{A}^1 \times W \times \mathbb{A}^N$. There are two morphisms

$$H_\theta : \mathcal{W}_\theta \xrightarrow{pr_1} f_\theta^*(\mathcal{W}) \xrightarrow{f_\theta^{nb}} \mathcal{W} \quad \text{and} \quad pr_{\mathcal{W}} : \mathcal{W}_\theta \to \mathcal{W}.$$

Let $i_0 : W \times \mathbb{A}^n \hookrightarrow \mathbb{A}^1 \times W \times \mathbb{A}^N$ be the inclusion taking (w, z) to $(0, w, z)$. Note that $f_\theta \circ i_0 = in_0 \circ pr_W$, where $in_0 : W \hookrightarrow W \times \mathbb{A}^N$ takes w to $(w, 0)$ and pr_W is the projection. Thus, $i_0^*(f_\theta^*(\mathcal{W})) = pr_W^*(in_0^*(\mathcal{W})) = pr_W^*(W) = W \times \mathbb{A}^N$. Clearly, $i_0^*(\mathbb{A}^1 \times \mathcal{W}) = \mathcal{W}$. So,

$$\mathcal{W}_0 := i_0^*(\mathcal{W}_\theta) = (W \times \mathbb{A}^N) \times_{W \times \mathbb{A}^N} \mathcal{W} = \mathcal{W}.$$

It is easy to check that $H_0 = H_\theta|_{\mathcal{W}_0} : \mathcal{W} \to \mathcal{W}$ equals $\mathcal{W} \xrightarrow{pr_W \circ \rho_0} W \xrightarrow{s_0} \mathcal{W}$. Clearly, $pr_{\mathcal{W}}|_{\mathcal{W}_0} = id_{\mathcal{W}}$.

Let $i_1 : W \times \mathbb{A}^n \hookrightarrow \mathbb{A}^1 \times W \times \mathbb{A}^N$ be the inclusion taking (w, z) to $(1, w, z)$. Note that $f_\theta \circ i_1 = id_W$. Thus, $i_1^*(f_\theta^*(\mathcal{W})) = \mathcal{W}$. Clearly, $i_1^*(\mathbb{A}^1 \times \mathcal{W}) = \mathcal{W}$. So, $i_1^*(\mathcal{W}_\theta) = \mathcal{W} \times_{W \times \mathbb{A}^N} \mathcal{W}$. Let $\mathcal{W}_1 = \Delta(\mathcal{W})$ be the diagonal. It is a finer étale neighborhood of $W \times 0$ in $W \times \mathbb{A}^N$. We write $\mathcal{W}$ for $\mathcal{W}_1$. Clearly, $H_1 = H_\theta|_{\mathcal{W}_1} : \mathcal{W} \to \mathcal{W}$ and $pr_{\mathcal{W}}|_{\mathcal{W}_1} : \mathcal{W} \to \mathcal{W}$ are the identity maps. We have thus proved the following

Lemma 14.2. *Let $W \in EssSm/k$ and let $(\mathcal{W}, \rho_0 : \mathcal{W} \to W \times \mathbb{A}^N, s_0 : W \to \mathcal{W})$ be an étale neighborhood of $W \times 0$ in $W \times \mathbb{A}^N$ (particularly, $in_0 = \rho_0 \circ s_0$). Suppose X is a k-smooth scheme and $(W \times 0, \mathcal{W}, \psi; g : \mathcal{W} \to X) \in Fr_N(W, X)$. Set $h_\theta = (\mathbb{A}^1 \times W \times 0, \mathcal{W}_\theta, \psi \circ pr_{\mathcal{W}}; g \circ H_\theta) \in Fr_N(\mathbb{A}^1 \times W, X)$. Then one has:*

(a) $h_1 = (W \times 0, \mathcal{W}, \psi; g) \in Fr_N(W, X)$;

(b) $h_0 = (W \times 0, \mathcal{W}, \psi; g \circ s_0 \circ p_W) \in Fr_N(W, X)$, where $p_W = (\mathcal{W} \xrightarrow{\rho_0} W \times \mathbb{A}^N \xrightarrow{pr_W} W)$.

If $W^\circ \subset W$ is open, $X^\circ \subset X$ is open and $g(W^\circ \times 0) \subset X^\circ$, then $h_\theta|_{\mathbb{A}^1 \times W^\circ}$ runs inside X°.

Theorem 14.3. *Let $W \in EssSm/k$ be a local scheme and let $N \geqslant 1$ be an integer. Let $i : W \to W \times \mathbb{A}^N$ be a section of the projection $pr_W : W \times \mathbb{A}^N \to W$. Let*

$$(\mathcal{W}_0, \rho : \mathcal{W}_0 \to W \times \mathbb{A}^N, s : W \to \mathcal{W}_0)$$

be an étale neighborhood of $i(W)$ in $W \times \mathbb{A}^N$ (particularly, $i = \rho_0 \circ s$). Let X be a k-smooth scheme. Suppose $\mathcal{W}_0$ is an affine essentially k-smooth scheme. Let

$$\alpha = (i(W), \mathcal{W}_0, \varphi_1, \ldots, \varphi_N; f : \mathcal{W}_0 \to X) \in Fr_N(W, X),$$

be a N-framed correspondence such that the functions $(\varphi_1,\dots,\varphi_N)$ generate the ideal $I=I_{s(W)}$ of those functions in $k[\mathcal{W}_0]$, which vanish on the closed subset $s(W)$. Let $A\in M_N(k[W])$ be the unique matrix transforming the basis $(\overline{t_1-(t_1|_{i(W)})},\dots,\overline{t_N-(t_N|_{i(W)})})$ of the free $k[W]$-module I/I^2 to the basis $(\bar\varphi_1,\dots,\bar\varphi_N)$ of the same $k[W]$-module. Let $J:=\det(A)\in k[W]^\times$ be the determinant of A. Suppose that $J=1\in k[W]^\times$. Then,

$$[\alpha]=[\sigma_X^N]\circ[f\circ s]\in\overline{\mathbb{Z}F}_N(W,X). \tag{30}$$

If $W^\circ\subset W$ is Zariski open and $X^\circ\subset X$ is Zariski open and $f(s(W^\circ))\subset X^\circ$, then

$$[[\alpha]]=[[\sigma_X^N]]\circ[[f\circ s]]\in\overline{\mathbb{Z}F}_N((W,W^\circ),(X,X^\circ)). \tag{31}$$

Theorem 14.4. *Suppose $\operatorname{char}k\neq 2$. Let $W\in EssSm/k$ be a local scheme and $N\geqslant 1$ be an integer. Let X be a k-smooth scheme. Let $i:W\to W\times\mathbb{A}^N$, $\alpha\in Fr_N(W,X)$, $A\in M_N(k[W])$, $J:=\det(A)\in k[W]^\times$, $s:W\to\mathcal{W}_0$ be the same as in Theorem 14.3. Suppose that $J\in k[W]^\times$ is a square. Then,*

$$[\alpha]=[\sigma_X^N]\circ[f\circ s]\in\overline{\mathbb{Z}F}_N(W,X). \tag{32}$$

If $W^\circ\subset W$ is Zariski open and $X^\circ\subset X$ is Zariski open and $f(s(W^\circ))\subset X^\circ$, then

$$[[\alpha]]=[[\sigma_X^N]]\circ[[f\circ s]]\in\overline{\mathbb{Z}F}_N((W,W^\circ),(X,X^\circ)). \tag{33}$$

Theorem 14.5. *Suppose that $\operatorname{char}k=2$. Let $W\in EssSm/k$ be a local scheme and $N\geqslant 1$ be an integer. Let X be a k-smooth scheme. Let $i:W\to W\times\mathbb{A}^N$, $\alpha\in Fr_N(W,X)$, $A\in M_N(k[W])$, $J:=\det(A)\in k[W]^\times$, $s:W\to\mathcal{W}_0$ be the same as in Theorem 14.3. Then,*

$$2\cdot[\alpha]=2\cdot(\ [\sigma_X^N]\circ[f\circ s]\ )\in\overline{\mathbb{Z}F}_N(W,X). \tag{34}$$

If $W^\circ\subset W$ is Zariski open and $X^\circ\subset X$ is Zariski open and $f(s(W^\circ))\subset X^\circ$, then

$$2\cdot[[\alpha]]=2\cdot(\ [[\sigma_X^N]]\circ[[f\circ s]]\ )\in\overline{\mathbb{Z}F}_N((W,W^\circ),(X,X^\circ)). \tag{35}$$

To prove these three theorems, we need several elementary lemmas. Their proofs are left to the reader. Below in this section we assume that $W\in EssSm/k$ and $W=Spec(R)$ for a k-algebra R. Also, we consider an étale neighborhood $(\mathcal{W}, \rho_0, s_0)$ of $W \times 0$ in $W \times \mathbb{A}^N$ such that $\mathcal{W}$ is of the form $Spec(\mathcal{R})$ for an étale $R[t_1, ..., t_N]$-algebra $\mathcal{R}$. We write $k[W]$ for R and $k[\mathcal{W}]$ for $\mathcal{R}$.

Lemma 14.6. *Let $(\mathcal{W}, \rho_0 : \mathcal{W} \to W \times \mathbb{A}^N, s_0 : W \to \mathcal{W})$ be the étale neighborhood of $W \times 0$ in $W \times \mathbb{A}^N$. Let $(W \times 0, \mathcal{W}, \psi; g \circ s_0 \circ p_W) \in Fr_N(W, X)$, where g and $p_W : \mathcal{W} \to W$ are the morphisms from Lemma 14.2. Let $A_\theta \in GL_N(k[W][\theta])$ be a matrix such that $A_0 = id$. Set $A := A_1 \in GL_N(k[W])$. Let $pr : \mathbb{A}^1 \times \mathcal{W} \to \mathcal{W}$ be the projection. Take the row $(\psi', \dots, \psi'_N) := (\psi_1, \dots, \psi_N) \cdot p_W^*(A)$ in $k[\mathcal{W}]$ and take the N-framed correspondence*

$$h_\theta := (\mathbb{A}^1 \times W \times 0, \mathbb{A}^1 \times \mathcal{W}, \Psi_\theta; g \circ s_0 \circ p_W \circ pr) \in Fr_N(\mathbb{A}^1 \times W, X),$$

where Ψ_θ is the row $(pr^*(\psi_1), \dots, pr^*(\psi_N)) \cdot (id \times p_W)^*(A_\theta)$ in $k[\mathbb{A}^1 \times \mathcal{W}]$. Then one has:

(a) $h_0 = (W \times 0, \mathcal{W}, \psi; g \circ s_0 \circ p_W)$;
(b) $h_1 = (W \times 0, \mathcal{W}, \psi'; g \circ s_0 \circ p_W)$.

If $W^\circ \subset W$ is open, $X^\circ \subset X$ is open and $g(W^\circ \times 0) \subset X^\circ$, then $h_\theta|_{\mathbb{A}^1 \times W^\circ}$ runs inside X°.

Lemma 14.7. *Let $(\mathcal{W}, \rho_0 : \mathcal{W} \to W \times \mathbb{A}^N, s_0 : W \to \mathcal{W})$ be the étale neighborhood of $W \times 0$ in $W \times \mathbb{A}^N$. Let $(W \times 0, \mathcal{W}, \psi; g \circ s_0 \circ p_W) \in Fr_N(W, X)$ be as in Lemma 14.6. Suppose the functions $\psi_1, \dots, \psi_N$ generate the ideal $I \subset k[\mathcal{W}]$ consisting of all the functions vanishing on the closed subset $W \times 0$. Furthermore, suppose that for any $i = 1, \dots, N$ one has that $\bar{\psi}_i = \bar{t}_i$ in I/I^2. Set $\psi_{\theta,i} := (1-\theta)\psi_i + \theta t_i \in k[\mathbb{A}^1 \times \mathcal{W}]$ and $\psi_\theta := (\psi_{\theta,1}, \dots, \psi_{\theta,N})$. Set*

$$h_\theta := (\mathbb{A}^1 \times W \times 0, \mathbb{A}^1 \times \mathcal{W}, \psi_\theta, g \circ s_0 \circ p_W \circ pr) \in Fr_N(\mathbb{A}^1 \times W, X).$$

Then one has:

(a) $h_0 = (W \times 0, \mathcal{W}, \psi; g \circ s_0 \circ p_W)$;

(b) $h_1 = (W \times 0, W \times \mathbb{A}^N, t_1, \dots, t_N; g \circ s_0 \circ pr_W) = \sigma_X^N \circ g \circ s_0$.

If $W^\circ \subset W$ is open, $X^\circ \subset X$ is open and $g(W^\circ \times 0) \subset X^\circ$, then $h_\theta|_{\mathbb{A}^1 \times W^\circ}$ runs inside X°.

Let $i : W \to W \times \mathbb{A}^N$, $(\mathcal{W}_0, \rho, s)$ be as in Theorem 14.3 and let $T : W \times \mathbb{A}^N \to W \times \mathbb{A}^N$ be the morphism taking a point (w, v) to the point $(w, v + i(w))$. Then $(T^*(\mathcal{W}_0), T^*(\rho), T^*(s))$ is an étale neighborhood of $W \times 0$ in $W \times \mathbb{A}^N$. Write $\mathcal{W}$ for $T^*(\mathcal{W}_0)$, s_0 for $T^*(s)$ and ρ_0 for $T^*(\rho)$. If T^{nb} :

$\mathcal{W} = T^*(\mathcal{W}_0) \to \mathcal{W}_0$ is the projection as in the beginning of this section, then $s = T^{nb} \circ T^*(s) = T^{nb} \circ s_0$.

Lemma 14.8. *Suppose that* $i : W \to W \times \mathbb{A}^N$, $(\mathcal{W}_0, \rho, s)$, X *and* $\alpha = (i(W), \mathcal{W}_0, \varphi_1, \dots, \varphi_N; f) \in Fr_N(W, X)$ *are as in Theorem 14.3. Then one has,*

$$[\alpha] = [W \times 0, \mathcal{W}, \varphi_1 \circ T^{nb}, \dots, \varphi_N \circ T^{nb}; f \circ T^{nb}] \in \overline{\mathbb{Z}F}_N(W, X).$$

Moreover, if $W^\circ \subset W$ *is open and if* $X^\circ \subset X$ *is any open such that* $g(s(W^\circ)) \subset X^\circ$*, then one has,*

$$[[\alpha]] = [[W \times 0, \mathcal{W}, \varphi_1 \circ T^{nb}, \dots, \varphi_N \circ T^{nb}; f \circ T^{nb}]] \in \overline{\mathbb{Z}F}_N((W, W^\circ), (X, X^\circ)).$$

Proof of Theorem 14.3. By Lemma 14.8 one has an equality in $\overline{\mathbb{Z}F}_N(W, X)$

$$[\alpha] = [W \times 0, \mathcal{W}, \psi_1, \dots, \psi_N; f \circ T^{nb}],$$

where $\psi_i = \varphi_i \circ T^{nb}$ for $i = 1, \dots, N$. Set $g = f \circ T^{nb} : \mathcal{W} \to X$. By Lemma 14.2 one has an equality in $\overline{\mathbb{Z}F}_N(W, X)$

$$[W \times 0, \mathcal{W}, \psi_1, \dots, \psi_N; g] = [W \times 0, \mathcal{W}, \psi; g \circ s_0 \circ p_W].$$

Thus one has

$$[\alpha] = [W \times 0, \mathcal{W}, \psi; g \circ s_0 \circ p_W] \in \overline{\mathbb{Z}F}_N(W, X).$$

Clearly, the functions $(\psi_1, \dots, \psi_N)$ generate the ideal $I_0 = I_{W \times 0}$ of those functions in $k[\mathcal{W}]$ that vanish on the closed subset $W \times 0$. Let $A' \in M_N(k[W])$ be the unique matrix that transforms the basis $(\bar{t_1}, \dots, \bar{t_N})$ of the free $k[W]$-module I_0/I_0^2 to the basis $(\bar{\psi}_1, \dots, \bar{\psi}_N)$ of the same $k[W]$-module. Clearly, $\det(A') = \det(A)$. Thus $\det(A') = 1 \in k[W]$. The ring $k[W]$ is local. Thus A' belongs to the group of elementary $N \times N$ matrices over $k[W]$. Hence there is a matrix $A_\theta \in M_N(k[W][\theta])$ such that $A_0 = id$ and $A_1 = (A')^{-1} \in GL_N(k[W])$. By Lemma 14.6 one has an equality

$$[W \times 0, \mathcal{W}, \psi, g \circ s_0 \circ p_W] = [W \times 0, \mathcal{W}, \psi', g \circ s_0 \circ p_W] \in \overline{\mathbb{Z}F}_N(W, X)$$

with the row $\psi'_1, \dots, \psi'_N$ as in Lemma 14.6. By construction, for any $i = 1, \dots, N$ the function ψ'_i has the property: $\bar{\psi}'_i = \bar{t}_i$ in I_0/I_0^2. By Lemma 14.7 one has an equality

$$[W \times 0, \mathcal{W}, \psi', g \circ s_0 \circ p_W] = [\sigma_X^N] \circ [g \circ s_0] \in \overline{\mathbb{Z}F}_N(W, X).$$

Since $s = T^{nb} \circ s_0$ and $g = f \circ T^{nb}$, we have equalities $g \circ s_0 = f \circ s$ and

$$[\alpha] = [\sigma_X^N] \circ [g \circ s_0] = [\sigma_X^N] \circ [f \circ s] \in \overline{\mathbb{Z}F}_N(W, X).$$

If $W^\circ \subset W$ is Zariski open and $X^\circ \subset X$ is Zariski open and $f(s(W^\circ)) \subset X^\circ$, then the same arguments prove the relation

$$[[\alpha]] = [[\sigma_X^N]] \circ [[f \circ s]] \in \overline{\mathbb{Z}F}_N((W, W^\circ), (X, X^\circ)). \tag{36}$$

This proves Theorem 14.3. □

Theorems 14.4 and 14.5 are proved at the end of this section. Some preparations are necessary for them. Let A be a finitely generated k-smooth algebra, $S = Spec(A)$, $S^0 \subset S$ be its open subset, $M \subset A$ be a multiplicative system, $Y = Spec(A_M)$, $Y^0 = S^0 \cap Y$. Then $Y^0 \subset Y$ is an open subset. For the rest of the section we fix these essentially k-smooth schemes Y and Y^0. We also choose and fix a k-smooth scheme X, its open subset X^0 and a k-scheme morphism $h : Y \to X$ such that $h(Y^0) \subset X^0$. We write $k[Y]$ for the k-algebra A_M. We begin with the following obvious

Lemma 14.9. *Let Y, X and $h : Y \to X$ be as above. Let $k[Y]$ be the ring of regular functions on Y. Let $n > 0$ and $a \in k[Y]^\times$. Let $p(t), q(t) \in k[Y][t]$ be two polynomials of degree n with the leading coefficient a. Let*

$$(Z(p), Y \times \mathbb{A}^1, p(t), h \circ pr_Y) \text{ and } (Z(q), Y \times \mathbb{A}^1, q(t), h \circ pr_Y) \in Fr_1(Y, X)$$

be two framed correspondences. Let $Z_s \subset \mathbb{A}^1 \times Y \times \mathbb{A}^1$ be the vanishing locus of the polynomial $p(t) + s(q(t) - p(t)) \in k[Y][s, t]$ (here s is the homotopy parameter). Let

$$H_s := (Z_s, \mathbb{A}^1 \times Y \times \mathbb{A}^1, p(t) + s(q(t) - p(t)), h \circ pr_Y) \in Fr_1(\mathbb{A}^1 \times Y, X).$$

Then one has equalities in $Fr_1(Y, X)$:

$$H_0 = (Z(p), \mathbb{A}^1, p(t), h \circ pr_Y) \text{ and } H_1 = (Z(q), \mathbb{A}^1, q(t), h \circ pr_Y) \in Fr_1(Y, X).$$

Under the notation introduced above Lemma 14.9 and under the hypotheses of Lemma 14.9 the framed correspondences

$$(Z(p), Y \times \mathbb{A}^1, p(t), h \circ pr_Y)|_{Y^0}$$

and

$$(Z(q), Y \times \mathbb{A}^1, q(t), h \circ pr_Y)|_{Y^0} \in Fr_1(Y^0, X)$$

run inside X^0 in the sense of Definition 4.3. Therefore following notation from that definition, they define elements $\langle\langle Z(p), Y\times\mathbb{A}^1, p(t), h\circ pr_Y\rangle\rangle$, $\langle\langle Z(q), Y\times\mathbb{A}^1, q(t), h\circ pr_Y\rangle\rangle\in\mathbb{Z}F_1((Y,Y^0),(X,X^0))$. The framed correspondence $(H_s)|_{\mathbb{A}^1\times Y^0}$ runs inside X^0. Hence it defines an element

$$\langle\langle H_s\rangle\rangle\in\mathbb{Z}F_1(\mathbb{A}^1\times(Y,Y^0),(X,X^0)).$$

Clearly, $\langle\langle H_0\rangle\rangle=\langle\langle Z(p), Y\times\mathbb{A}^1, p(t), h\circ pr_Y\rangle\rangle$, $\langle\langle H_1\rangle\rangle=\langle\langle Z(q), Y\times\mathbb{A}^1, q(t), h\circ pr_Y\rangle\rangle$ in $\mathbb{Z}F_1((Y,Y^0),(X,X^0))$.

We have thus proved the following

Lemma 14.10. *Under the notation introduced above Lemma 14.9 and under the notation and the hypotheses of Lemma 14.9 and the notation from Definition 4.3 the following equality holds:*

$$[[Z(p), Y\times\mathbb{A}^1, p(t), h\circ pr_Y]]=[[Z(q), Y\times\mathbb{A}^1, q(t), h\circ pr_Y]]\in\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0)).$$

Let Y be as above Lemma 14.9. Let $k[Y]$ be the ring of regular functions on Y. Let $n>0$ and $p(t)=t^nR(t)$, where $R(t)=r_0+r_1t+\ldots+r_Nt^N\in k[Y][t]$ is a polynomial such that r_N and r_0 are both in $k[Y]^\times$. Let $U=(Y\times\mathbb{A}^1)_{R(t)}\subset Y\times\mathbb{A}^1$ be the principal open subset corresponding to $R(t)$. One has $R(t)=r_0+tR_1(t)$. Consider the polynomial

$$h(s,t)=sR(t)t^n+(1-s)r_0t^n\in k[s,t].$$

Then $h(s,t)=t^n\cdot(r_0+t\cdot R_1(t)\cdot s)$. If S is the vanishing locus of $r_0+t\cdot R_1(t)\cdot s$, then $S\cap\mathbb{A}^1\times Y\times 0=\emptyset$. Hence for the zero locus $Z(h)$ of h one has $Z(h)=(\mathbb{A}^1\times Y\times 0)\sqcup S$. Set,

$$H_s^R:=(\mathbb{A}^1\times Y\times\{0\},(\mathbb{A}^1\times U)\setminus S, sR(t)t^n+(1-s)r_0t^n, h\circ pr_Y\circ pr_U)\in Fr_1(\mathbb{A}^1\times Y,X).$$

The following lemma is inspired by [1, Lemma 4.13].

Lemma 14.11. *Let Y, X and $f:Y\to X$ be as above Lemma 14.9. Let $k[Y]$ be the ring of regular functions on Y. Let $a\in k[Y]$, $n>0$ and $q(t)=(t-a)^nQ(t)$, where $Q(t)\in k[Y][t]$ is a polynomial such that its leading coefficient and $Q(a)$ are both in $k[Y]^\times$. Let $U=(Y\times\mathbb{A}^1)_{Q(t)}\subset Y\times\mathbb{A}^1$ be the principal* *open subset corresponding to* $Q(t)$*. Then there is a framed correspondence* $H^Q_s \in Fr_1(\mathbb{A}^1 \times Y, X)$ *such that in* $Fr_1(Y,X)$ *one has equalities*

$$H^Q_1 = (Z(t-a), U, q(t); h \circ pr_Y)$$

and

$$H^Q_0 = (Z(t-a), Y \times \mathbb{A}^1, Q(a)(t-a)^n; h \circ pr_Y).$$

Proof. We may assume that $a = 0$. Then take H^R_s just as above with $R(t) = Q(t+a)$. Clearly, $H^R_1 = (Y \times \{0\}, U, R(t)t^n, h \circ pr_Y)$ and $H^R_0 = (Y \times \{0\}, U, R(0)t^n, h \circ pr_Y)$ in $Fr_1(Y,X)$. Whence follows the lemma. □

Under the hypotheses of Lemma 14.11 let $Y^0 \subset Y$, $X^0 \subset X$ be open subsets as just above Lemma 14.9. Then the framed correspondences

$$(Z(t-a), U, q(t); h \circ pr_Y)|_{Y^0} \in Fr_1(Y^0, X)$$

and

$$(Z(t-a), Y \times \mathbb{A}^1, Q(a)(t-a)^n; h \circ pr_Y)|_{Y^0} \in Fr_1(Y^0, X)$$

run inside X^0 in the sense of Definition 4.3. Thus following notation from that definition, they define elements

$$\langle\langle Z(t-a), U, q(t); h \circ pr_Y\rangle\rangle, \langle\langle Z(t-a), Y \times \mathbb{A}^1, Q(a)(t-a)^n; h \circ pr_Y\rangle\rangle \\ \in \mathbb{Z}F_1((Y,Y^0),(X,X^0)).$$

Lemma 14.12. *Under the notation and hypotheses of Lemma 14.11 let* $Y^0 \subset Y$*,* $X^0 \subset X$ *be open subsets as above Lemma 14.9. Then under the notation from Definition 4.3 one has*

$$[[Z(t-a), U, q(t); h \circ pr_Y]] = [[Z(t-a), Y \times \mathbb{A}^1, Q(a)(t-a)^n; h \circ pr_Y]] \\ \in \overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0)).$$

Proof. We may assume that $a = 0$. Let $R(t) = Q(t+a) \in k[Y][t]$ be as in the proof of Lemma 14.11. Clearly, the element $H^R_s|_{\mathbb{A}^1 \times Y^0} : \mathbb{A}^1 \times Y^0 \to X$ runs inside X^0 in the sense of Definition 4.3. Hence following that definition, it defines an element $\langle\langle H^R_s\rangle\rangle \in \mathbb{Z}F_1(\mathbb{A}^1 \times (Y,Y^0),(X,X^0))$. One has equalities in $\mathbb{Z}F_1((Y,Y^0),(X,X^0))$:

$$\langle\langle H^R_0\rangle\rangle = \langle\langle Y \times 0, Y \times \mathbb{A}^1, R(0)t^n; h \circ pr_Y\rangle\rangle$$

and

$$\langle\langle H_1^R\rangle\rangle = \langle\langle Y\times 0, U, R(t)t^n; h\circ pr_Y\rangle\rangle.$$

This proves the lemma. □

Corollaries 14.13, 14.14, Proposition 14.15 and their proofs are inspired by [14, Lemma 7.3].

Corollary 14.13. *Suppose* char $k\neq 2$. *Under the notation and hypotheses of Lemma 14.11 let $Y^0\subset Y$, $X^0\subset X$ be open subsets as above Lemma 14.9. Let $\lambda\in k[Y]^\times$. Then under the notation from Definition 4.3*

$$[[(Y\times 0, Y\times\mathbb{A}^1, \lambda\cdot t^2; h\circ pr_Y)]] = [[(Y\times 0, Y\times\mathbb{A}^1, t^2; h\circ pr_Y)]]$$

in $\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$.

Proof. For brevity we drop $h\circ pr_Y$ from the notation. By Lemma 14.10 one has an equality $[[(Y\times 0, Y\times\mathbb{A}^1, t^2)]] = [[(Y\times 0, Y\times\mathbb{A}^1,(t-1)(t+1))]]$ in $\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$. By the additivity relation from Lemma 4.6 and Lemma 14.12 one has

$$[[(Y\times 0, Y\times\mathbb{A}^1,(t-1)(t+1))]] = [[(Y\times 0, Y\times\mathbb{A}^1, 2t)]]+[[(Y\times 0, Y\times\mathbb{A}^1, -2t)]]$$

in $\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$. Similarly, $[[(Y\times 0, Y\times\mathbb{A}^1, \lambda\cdot t^2)]] = [[(Y\times 0, Y\times\mathbb{A}^1, \lambda\cdot(t-\lambda^{-1})\cdot(t+\lambda^{-1}))]]$ and

$$\begin{aligned}&[[(Y\times 0, Y\times\mathbb{A}^1, \lambda\cdot(t-\lambda^{-1})\cdot(t+\lambda^{-1}))]] =\\ &\qquad = [[(Y\times 0, Y\times\mathbb{A}^1, 2t)] + [(Y\times 0, Y\times\mathbb{A}^1, -2t)]]\end{aligned}$$

in $\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$. This proves the corollary. □

Corollary 14.14. *Suppose* char $k\neq 2$. *Under the notation and hypotheses of Lemma 14.11 let $Y^0\subset Y$, $X^0\subset X$ be open subsets as above Lemma 14.9. Let $\lambda\in k[Y]^\times$. Then under the notation from Definition 4.3 one has an equality in $\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$*

$$\begin{aligned}&[[(Y\times 0, Y\times\mathbb{A}^1, \lambda^2\cdot t; h\circ pr_Y)]] + [[(Y\times 0, Y\times\mathbb{A}^1, \lambda\cdot t^2; h\circ pr_Y)]] =\\ &\qquad = [[(Y\times 0, Y\times\mathbb{A}^1, t^3; h\circ pr_Y)]].\end{aligned}$$

Proof. For brevity we drop $h\circ pr_Y$ from the notation. By Lemma 14.10 one has an equality $[[(Y\times 0, Y\times\mathbb{A}^1, t^3)]] = [[(Y\times 0, Y\times\mathbb{A}^1, t^3+\lambda\cdot t^2)]]$ in

$\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$. By the additivity relation from Lemma 4.6 one has an equality in $\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$:

$$[[(Y\times 0, Y\times\mathbb{A}^1, t^3+\lambda\cdot t^2)]] =$$
$$[[(Y\times 0,(Y\times\mathbb{A}^1)_{t+\lambda},(t+\lambda)\cdot t^2)]]+[[\{t+\lambda=0\},(Y\times\mathbb{A}^1)_{t^2},t^2(t+\lambda))]].$$

By Lemma 14.12 one has equalities in $\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$:

$$[[(Y\times 0,(Y\times\mathbb{A}^1)_{t+\lambda},(t+\lambda)\cdot t^2)]]=[[(Y\times 0,(Y\times\mathbb{A}^1),\lambda\cdot t^2)]]$$

and

$$[[\{t+\lambda=0\},(Y\times\mathbb{A}^1)_{t^2},t^2(t+\lambda))]]=[[Y\times 0,(Y\times\mathbb{A}^1),\lambda^2\cdot t)]].$$

Whence follows the corollary. □

Proposition 14.15. *Suppose* char $k\neq 2$. *Under the notation and hypotheses of Lemma 14.11 let* $Y^0\subset Y$, $X^0\subset X$ *be open subsets as above Lemma 14.9. Let* $\lambda\in k[Y]^\times$. *Then under the notation from Definition 4.3 one has an equality in* $\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$

$$[[(Y\times 0,Y\times\mathbb{A}^1,\lambda^2\cdot t;h\circ pr_Y)]]=[[(Y\times 0,Y\times\mathbb{A}^1,t;h\circ pr_Y)]]=[[\sigma_X]]\circ[[h]].$$

Proof. For brevity we drop $h\circ pr_Y$ from the notation. The second equality is obvious. Let us prove the first one. By Corollary 14.14 one has

$$[[(Y\times 0,Y\times\mathbb{A}^1,\lambda^2\cdot t)]]+[[(Y\times 0,Y\times\mathbb{A}^1,\lambda\cdot t^2)]]=$$
$$=[[(Y\times 0,Y\times\mathbb{A}^1,t)]]+[[(Y\times 0,Y\times\mathbb{A}^1,t^2)]]$$

in $\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$. Corollary 14.13 now completes the proof of the proposition. □

Proof of Theorem 14.4. Take $h=f\circ s$. Repeating literally the proof of Theorem 14.3, one gets an equality

(37)
$$[[\alpha]]=[[\sigma_X^{N-1}]]\circ[[W\times 0,W\times\mathbb{A}^1,J\cdot t;(f\circ s)\circ pr_W]]\in\overline{\mathbb{Z}F}_N((W,W^\circ),(X,X^\circ)).$$

By assumptions of the theorem $J=\lambda^2$ for a unit $\lambda\in k[W]^\times$. By Proposition 14.15 one has an equality $[[W\times 0,W\times\mathbb{A}^1,J\cdot t;(f\circ s)\circ pr_W]]=[[\sigma_X]]\circ[[f\circ s]]$ in $\overline{\mathbb{Z}F}_N((W,W^\circ),(X,X^\circ))$. This proves the theorem. □

Proposition 14.16. *Suppose* char $k = 2$. *Under the notation and hypotheses of Lemma 14.11 let $Y^0 \subset Y$, $X^0 \subset X$ be open subsets as above Lemma 14.9. Let $\mu \in k[Y]^\times$. Then under the notation from Definition 4.3 one has equalities in $\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$*

$$2\cdot[[(Y\times 0, Y\times\mathbb{A}^1, \mu\cdot t; h\circ pr_Y)]] = 2\cdot[[(Y\times 0, Y\times\mathbb{A}^1, t; h\circ pr_Y)]] = \\ = 2\cdot([[\sigma_X]]\circ[[h]]).$$

Proof. The second equality is obvious. Let us prove the first one. For brevity we drop $h\circ pr_Y$ from the notation. By Lemma 14.10 one has an equality $[[(Y\times 0, Y\times\mathbb{A}^1, t^2)]] = [[(Y\times 0, Y\times\mathbb{A}^1, t(t+\mu))]]$ in $\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$. By the additivity relation from Lemma 4.6 and Lemma 14.12 one has

$$[[(Y\times 0, Y\times\mathbb{A}^1, t(t+\mu))]] = [[(Y\times 0, Y\times\mathbb{A}^1, \mu t)]] + [[(Y\times 0, Y\times\mathbb{A}^1, \mu t)]] = \\ = 2\cdot[[(Y\times 0, Y\times\mathbb{A}^1, \mu t)]]$$

in $\overline{\mathbb{Z}F}_1((Y,Y^0),(X,X^0))$. Thus,

$$2\cdot[[(Y\times 0, Y\times\mathbb{A}^1, \mu t)]] = [[(Y\times 0, Y\times\mathbb{A}^1, t^2)]] = 2\cdot[[(Y\times 0, Y\times\mathbb{A}^1, t)]].$$

This proves the proposition. □

Proof of Theorem 14.5. Repeating literally the proof of Theorem 14.3, one gets an equality

(38)

$$[[\alpha]] = [[\sigma_X^{N-1}]]\circ[[W\times 0, W\times\mathbb{A}^1, J\cdot t; (f\circ s)\circ pr_W]] \in \overline{\mathbb{Z}F}_N((W,W^\circ),(X,X^\circ)).$$

By Proposition 14.16 one has $2\cdot[[W\times 0, W\times\mathbb{A}^1, J\cdot t; (f\circ s)\circ pr_W]] = 2\cdot([[\sigma_X]]\circ[[f\circ s]]) \in \overline{\mathbb{Z}F}_N((W,W^\circ),(X,X^\circ))$. This proves the theorem. □

15. Constructing h'_θ, F and h_θ from Propositions 12.6 and 10.9

In the first part of this section we construct the functions h'_θ, F from Proposition 12.6 and prove this proposition. In the second part of this section we construct the function h_θ from Proposition 10.9 and prove this proposition as well.

Let X and X' be as in Remark 10.4 and let $q : X \to B$ be the almost elementary fibration from Remark 10.4. Since $q : X \to B$ is an almost elementary fibration there is a commutative diagram of the form (see Definition

8.1)

(39)
$$\begin{array}{ccccc} X & \xrightarrow{\ j\ } & \overline{X} & \xleftarrow{\ i\ } & X_\infty \\ & {\scriptstyle q}\searrow & \downarrow{\scriptstyle \overline{q}} & \swarrow{\scriptstyle q_\infty} & \\ & & B & & \end{array}$$

with morphisms j, $\overline{q}$, i, q_∞ satisfying the conditions (i)–(iv) from Definition 8.1.

The composite morphism $X' \xrightarrow{\Pi} X \xrightarrow{j} \bar{X}$ is quasi-finite. Let $\bar{X}'$ be the normalization X'_n of $\bar{X}$ in $Spec(k(X'))$. Let $\bar{\Pi} : \bar{X}' \to \bar{X}$ be the canonical morphism (it is finite and surjective). Then $(\bar{\Pi})^{-1}(X)$ coincides with the normalization X'_n of X in $Spec(k(X'))$. Let $f \in k[X'_n]$ be from Definition 10.2. Let $Y' = \{f = 0\}$ be the corresponding effective Cartier divisor of X'_n from that definition. The morphism $(q \circ (\bar{\Pi}|_{\bar{\Pi}^{-1}(X)}))|_{Y'} : Y' \to B$ is finite, since $q|_Y : Y \to B$ is finite and $\bar{\Pi}$ is finite. Thus Y' is closed in $\bar{X}'$. Since Y' is in $(\bar{\Pi})^{-1}(X)$ it has the empty intersection with $(\bar{\Pi})^{-1}(X_\infty)$. Hence

$$X' = \bar{X}' - ((\bar{\Pi})^{-1}(X_\infty) \sqcup Y').$$

Both $(\bar{\Pi})^{-1}(X_\infty)$ and Y' are Cartier divisors in $\bar{X}'$. The Cartier divisor $(\bar{\Pi})^{-1}(X_\infty)$ is ample. Thus the Cartier divisor $D' := (\bar{\Pi})^{-1}(X_\infty) \sqcup Y'$ is ample as well and $(q \circ \bar{\Pi})|_{D'} : D' \to B$ is finite.

Set $\bar{\Gamma} = \bar{X}' \xrightarrow{(\bar{\Pi}, id)} \bar{X} \times_B \bar{X}'$ (the transpose of the graph of the B-morphism $\bar{\Pi}$). The projection $\bar{X} \times_B \bar{X}' \to \bar{X}'$ is a smooth morphism, since $\bar{q}$ is smooth. The morphism $(\bar{\Pi}, id)$ is a section of the projection. Hence $\bar{\Gamma}$ is a Cartier divisor in $\bar{X} \times_B \bar{X}'$.

Set $\Gamma = pr_{\bar{X}}^{-1}(U) \cap \bar{\Gamma} \subset U \times_B \bar{X}'$. Then $\Gamma \subset U \times_B \bar{X}'$ is a Cartier divisor. The scheme U' is contained in Γ as an open subscheme via the inclusion (π, can'), where $can' : U' \to X'$ is the canonical morphism. The composite morphism $pr_U \circ (\pi, can') : U' \to U$ coincides with $\pi : U' \to U$. Thus $pr_{\bar{X}}|_\Gamma : \Gamma \to U$ is étale at the points of U'.

Lemma 15.1. *Set* $\Gamma' = U' \times_U \Gamma \subset U' \times_U U \times \bar{X}' = U' \times_B \bar{X}'$. *Then* $\Gamma' \subset U' \times_B \bar{X}'$ *is a Cartier divisor. Moreover,*

$$\Gamma' = \Delta(U') \sqcup G'$$

and $G' \cap (U' \times_B S') = \emptyset$, *where* $S' \subset X'$ *is the closed subscheme from Section 10.*

Proof. Consider the diagonal morphism $\Delta : U' \to U' \times_B \bar{X}'$. It lands in $U' \times_U \Gamma = \Gamma'$ and it is a section of the projection $U' \times_U \Gamma \to U'$. The morphism $\Gamma \to U$ is étale at all points of U'. Hence the morphism $\Gamma' = U' \times_U \Gamma \to U'$ is étale at all points of $U' \times_U U'$. Particularly, it is étale along the diagonal $\Delta(U')$. Hence the morphism $\Delta : U' \to U' \times_U \Gamma$ is étale. Thus $\Delta(U')$ is an open subset of $U' \times_U \Gamma = \Gamma'$. Since $\Delta(U')$ is a closed subset of Γ', hence $\Gamma' = \Delta(U') \sqcup G'$.

We now prove that $G' \cap (U' \times_B S') = \emptyset$. For that consider the closed subscheme S of the scheme X from Section 10 and recall that S and S' are reduced schemes and the morphism $\Pi|_{S'} : S' \to S$ is a scheme isomorphism. There is a chain of inclusions of subsets

$$(\pi \times id)(G' \cap U' \times_B S') \subset (\pi \times id)(G') \cap U \times_B S' \subset \Gamma \cap (U \times_B S') \subset \Gamma_{(\pi|_{S' \cap U'})},$$

where $\Gamma_{(\pi|_{S' \cap U'})}$ is the graph of $\pi|_{S' \cap U'} : S' \cap U' \to U$. One has an equality

$$((\pi \times id)|_{U' \times_B S'})^{-1}(\Gamma_{(\pi|_{S' \cap U'})}) = \Delta(S' \cap U').$$

Thus $G' \cap (U' \times_B S') \subset \Delta(S' \cap U')$. Finally,

$$G' \cap (U' \times_B S') \subset G' \cap \Delta(S' \cap U') \subset G' \cap \Delta(U') = \emptyset. \qquad \square$$

Remark 15.2. It is easy to check that $\Gamma \cap U \times_B S' = \delta(S' \cap U')$, where $\delta(s') = (\pi(s'), s')$.

Definition 15.3. Set $\mathcal{D}' = U \times_B D'$ and $\mathcal{D}'' = U' \times_U \mathcal{D}' = U' \times_B D'$. They are Cartier divisors on $U \times_B \bar{X}'$ and $U' \times_B \bar{X}'$ respectively. Note that the scheme $\mathcal{D}'$ is finite over U and the scheme $\mathcal{D}''$ is finite over U'.

Let $s_0 \in \Gamma(U \times_B \bar{X}', \mathcal{L}(\mathcal{D}'))$ be the canonical section of the invertible sheaf $\mathcal{L}(\mathcal{D}')$ (its vanishing locus is $\mathcal{D}'$). Let $s_\Gamma \in \Gamma(U \times_B \bar{X}', \mathcal{L}(\Gamma))$ be the canonical section of the invertible sheaf $\mathcal{L}(\Gamma)$ (its vanishing locus is Γ). Let $s_{\Delta(U')} \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(\Delta(U'))$ be the canonical section of the invertible sheaf $\mathcal{L}(\Delta(U'))$ (its vanishing locus is $\Delta(U')$). Let $s_{G'} \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(G'))$ be the canonical section of the invertible sheaf $\mathcal{L}(G')$ (its vanishing locus is G').

Notation 15.4. Set $I' = \mathcal{L}(-D')$, $I'' = \mathcal{L}(-D'')$. They are the ideal sheaves defining the Cartier divisors D' and D'' respectively. Denote by J' the ideal sheaf defining the closed subscheme $U \times_B S'$ of $U \times_B \bar{X}'$. Denote by J'' the ideal sheaf defining the closed subscheme $U' \times_B S'$ of $U' \times_B \bar{X}'$.

By Serre's vanishing theorem there is an integer $n \gg 0$ such that the following cohomology groups vanish: $H^1(U \times_B \bar{X}', J' \otimes I' \otimes \mathcal{L}(n\mathcal{D}'))$ and $H^1(U' \times_B \bar{X}', J'' \otimes I'' \otimes \mathcal{L}(n\mathcal{D}'' - \Delta(U')))$.

The fact that these cohomology groups vanish guaranties the existence of sections s_1 and t_0' from Constructions 15.5 and 15.7 below.

Construction 15.5. *Find a section $s_1 \in \Gamma(U \times_B \bar{X}', \mathcal{L}(n\mathcal{D}'))$ such that:*

(1) $s_1|_{U\times_B S'} = r_1 \otimes (s_\Gamma|_{U\times_B S'})$, *where* $r_1 \in \Gamma(U \times_B S', \mathcal{L}(n\mathcal{D}' - \Gamma))|_{U\times_B S'})$ *has no zeros;*

(2) $s_1|_{\mathcal{D}'}$ *has no zeros.*

Construction 15.6. *Set $t_1 := (\pi \times id)^*(s_1) \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(n\mathcal{D}''))$. Then*

(1′) $t_1|_{U'\times_B S'} = r_1' \otimes (s_{\Delta(U')}|_{U'\times_B S'}) \otimes (s_{G'}|_{U'\times_B S'})$,

where r_1' equals $((\pi \times id)|_{U'\times_B S'})^*(r_1) \in \Gamma(U' \times_B S', \mathcal{L}(n\mathcal{D}'' - \Gamma')|_{U'\times_B S'})$;

(2′) $t_1|_{\mathcal{D}''}$ *has no zeros.*

The second property of t_1 is obvious. To prove the first one recall that $\Gamma' = \Delta(U') \sqcup G'$ by Lemma 15.1. Hence $(\pi \times id)^*(s_\Gamma) = s_{\Delta(U')} \otimes s_{G'}$.

Construction 15.7. *Construct a section $t_0 \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(n\mathcal{D}''))$ of the form $t_0 = t_0' \otimes s_{\Delta(U')}$, where $t_0' \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(n\mathcal{D}'' - \Delta(U')))$ satisfies the following conditions:*

(1″) $t_0'|_{\mathcal{D}''} = (t_1|_{\mathcal{D}''}) \otimes (s_{\Delta(U')}|_{\mathcal{D}''})^{-1}$;

(2″) $t_0'|_{U'\times_B S'} = r_1' \otimes (s_{G'}|_{U'\times_B S'})$, *where r_1' is from Construction 15.6.*

Lemma 15.8. *The following properties are true:*

(1‴) $t_0|_{\mathcal{D}''} = t_1|_{\mathcal{D}''}$ *and both sections have no zeros on* $\mathcal{D}''$;

(2‴) $t_0|_{U'\times_B S'} = t_1|_{U'\times_B S'}$ *and both sections have no zeros on* $(U' - S') \times_B S'$.

Indeed, the first equality is obvious. The second one follows from the chain of equalities

$$t_0|_{U'\times_B S'} = (t_0'|_{U'\times_B S'}) \otimes (s_{\Delta(U')}|_{U'\times_B S'}) = \\ = r_1' \otimes (s_{G'}|_{U'\times_B S'}) \otimes (s_{\Delta(U')}|_{U'\times_B S'}) = t_1|_{U'\times_B S'}.$$

Definition 15.9. Let $s_0' := (\pi \times id)^*(s_0) \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(\mathcal{D}''))$. Set,

$$h_\theta' = \frac{((1-\theta)t_0 + \theta t_1)|_{\mathbb{A}^1\times U'\times_B X'}}{(s_0')^{\otimes n}|_{\mathbb{A}^1\times U'\times_B X'}} \in k[\mathbb{A}^1 \times U' \times_B X']$$

and

$$F = \frac{s_1|_{U\times_B X'}}{(s_0)^{\otimes n}|_{U\times_B X'}} \in k[U \times_B X'].$$

Proof of Proposition 12.6. Let $pr_{23} : \mathbb{A}^1 \times U' \times_B \bar{X}' \to U' \times_B \bar{X}'$ be the projection. Consider two sections $(1-\theta)t_0 + \theta t_1$ and $(s_0')^{\otimes n}$ of the line bundle $pr_{23}^*(\mathcal{L}(n \cdot \mathcal{D}''))$ on $\mathbb{A}^1 \times U' \times_B \bar{X}'$. By Lemma 15.8 these two sections have no common zeros. Thus one has morphism

$$[pr, (1-\theta)t_0 + \theta t_1 : (s_0')^{\otimes n}] : \mathbb{A}^1 \times U' \times_B \bar{X}' \to \mathbb{A}^1 \times U' \times \mathbb{P}^1,$$

where $pr : \mathbb{A}^1 \times U' \times_B \bar{X}' \to \mathbb{A}^1 \times U'$ is the projection. This morphism is quasi-finite and projective. Hence it is finite and surjective. It follows that any of its base changes is finite and surjective. Particularly, the morphism $(pr, h_\theta') : \mathbb{A}^1 \times U' \times X' \to \mathbb{A}^1 \times U' \times \mathbb{A}^1$ is finite and surjective, because the closed subset $\{(s_0')^{\otimes n} = 0\}$ in $\mathbb{A}^1 \times U' \times_B \bar{X}'$ coincides with the one $\mathbb{A}^1 \times \mathcal{D}''$. This proves the assertion (a) of Proposition 12.6. The assertion (e) of Proposition 12.6 is proved in the same fashion. Lemma 15.1 yields the assertion (b). Lemma 15.8(2"') yields the assertion (d). The assertion (c) follows from the construction of F and h_1'. The property (1) of the section s_1 yields the assertion (f), whence follows the proposition. □

In the rest of the section under the hypotheses of Proposition 10.9 we will construct a function $h_\theta \in k[\mathbb{A}^1 \times U \times_B X]$ and prove Proposition 10.9.

Let X and X' be as in Remark 10.4 and let $q : X \to B$ be the almost elementary fibration from Remark 10.4. Let $\bar{X}$, $j : X \to \bar{X}$ and X_∞ and $i : X_\infty \to \bar{X}$ be as in the diagram (39). So, they satisfy the conditions (i)–(iv) from Definition 8.1.

The composite morphism $X' \xrightarrow{\Pi} X \xrightarrow{j} \bar{X}$ is quasi-finite. Let $\bar{X}'$ be the normalization of $\bar{X}$ in $Spec(k(X'))$. Let $\bar{\Pi} : \bar{X}' \to \bar{X}$ be the canonical morphism (it is finite and surjective). Let $X_\infty \subset \bar{X}$ be the Cartier divisor from diagram (39). Set $X'_\infty := (\bar{\Pi})^{-1}(X_\infty)$ (scheme-theoretically). Then X'_∞ is a Cartier divisor on $\bar{X}'$. Set

$$E := U \times_B X_\infty \quad \text{and} \quad E' := U \times_B X'_\infty.$$

These are Cartier divisors on $U \times_B \bar{X}$ and $U \times_B \bar{X}'$ respectively and $(id \times \bar{\Pi})^*(E) = E'$.

Choose an integer $n \gg 0$. Find a section $r_1(n) \in \Gamma(U \times_B S, \mathcal{L}(nE - \Delta(U)|_{U \times_B S})$ which has no zeros. Let $s_{\Delta(U)} \in \Gamma(U \times_B \bar{X}, \mathcal{L}(\Delta(U))$ be the canonical section of the invertible sheaf $\mathcal{L}(\Delta(U))$ (its vanishing locus is $\Delta(U)$). To define the desired function h_θ, we need the following

Construction 15.10. *For any integer $n \gg 0$ find a section $s_1'(n) \in \Gamma(U \times_B \bar{X}', \mathcal{L}(nE'))$ having properties as follows.*

(1) *The Cartier divisor* $Z'_1(n) := \{s'_1(n) = 0\}$ *has the following properties:*
($1a$) $Z'_1(n) \subset U \times_B X'$;
($1a'$) *the Cartier divisor* $Z'_1(n)$ *is finite and étale over* U;
($1b$) *the morphism* $i = (id \times \Pi)|_{Z'_1(n)} : Z'_1(n) \to U \times_B X$ *is a closed embedding. Denote by* $Z_1(n)$ *the closed subscheme* $i(Z'_1(n))$ *of the scheme* $U \times_B X$.
(2) $s'_1(n)|_{U\times_B S'} = (id \times \bar{\Pi})^*(s_{\Delta(U)})|_{U\times_B S'} \otimes ((\mathrm{id} \times \Pi)|_{U\times_B S'})^*(r_1(n))$.

Proof. The main difficulty is to achieve the property (1b). For any integer $n \gg 0$, set

$$v_{fin}(n) = (id \times \bar{\Pi})^*(s_{\Delta(U)})|_{U\times_B S'} \otimes ((\mathrm{id} \times \Pi)|_{U\times_B S'})^*(r_1(n)) \in \Gamma(U \times_B S', \mathcal{L}(nE')|_{U\times_B S'}).$$

For any $n > 0$ choose a section $v_\infty(n) \in \Gamma(E', \mathcal{L}(nE')|_{E'})$ having no zeros. This is possible, because E' is a semi-local scheme. Set $T' = U \times_B S' \sqcup E'$. Let $v(n) \in \Gamma(T', \mathcal{L}(nE')|_{T'})$ be the unique section such that $v(n)|_{U\times_B S'} = v_{fin}(n)$ and $v(n)|_{E'} = v_\infty(n)$. Set $\bar{v}(n) = v(n)|_{x\times_B T'}$.

Firstly, for any integer $n \gg 0$ we will find a section $\bar{s}(n) \in \Gamma(x \times_B \bar{X}', \mathcal{L}(nE')|_{x\times_B \bar{X}'})$ such that the Cartier divisor $\bar{Z}(n) := \{\bar{s}(n) = 0\}$ is finite and étale over x, $\bar{Z}(n) \subset x \times_B X'$, the morphism $(id \times \Pi)|_{\bar{Z}(n)} : \bar{Z}(n) \hookrightarrow x \times_B X$ is a closed embedding and $\bar{s}(n)|_{x\times_B T} = \bar{v}(n)$.

Then we patch $\bar{s}(n)$ and $v(n)$ to get a section $s_{ext}(n)$ of $\mathcal{L}(nE')|_{T'\cup x\times_B \bar{X}'}$ such that the restriction of $s_{ext}(n)$ to T' coincides with $v(n)$ and the restriction of $s_{ext}(n)$ to $x \times_B \bar{X}'$ coincides with $\bar{s}(n)$. Finally, using Serre's vanishing theorem for $n \gg 0$, we lift the section $s_{ext}(n)$ to a section $s'_1(n) \in \Gamma(U \times_B \bar{X}', \mathcal{L}(nE'))$. The latter section $s'_1(n)$ is the desired one as one can easily check.

The nearest aim is to find the desired section $\bar{s}(n)$. This requires some cohomological computations. Consider $k(x)$-schemes $Y = x \times_B X$, $Y' = x \times_B X'$, $\bar{Y}' = x \times_B \bar{X}'$, $T'_x = x \times_B T'$. Write E'_x for the effective Cartier divisor $E'|_{\bar{Y}'}$ on $\bar{Y}'$. Set $Y'_0 = \bar{Y}' - T'_x = Y' - x \times_B S'$. Consider $H^0(n) := H^0(\bar{Y}', \mathcal{L}(nE'_x)) = \Gamma(\bar{Y}', \mathcal{L}(nE'_x))$. Also, consider the restriction map

$$r_n : H^0(n) \to H^0(T_x, \mathcal{L}(nE'_x)|_{T'_x}).$$

By Serre's vanishing theorem this map is surjective for $n \gg 0$. Denote by $H^0(n)_{\bar{v}(n)}$ the affine subspace $r_n^{-1}(\bar{v}(n))$ in $H^0(n)$. For $n \gg 0$ the dimension $h^0(n)_{\bar{v}(n)}$ of this affine $k(x)$-subspace coincides with the dimension $h^0(nE'_x - T'_x)$ of $H^0(\bar{Y}', \mathcal{L}(nE'_x - T'_x))$ over $k(x)$.

Let $y \in Y'_0$ be a point (not necessarily closed). Set $Y_y = Y \times_x y$, $\bar{Y}'_y = \bar{Y}' \times_x y$, $T'_y = T'_x \times_x y$. Write E'_y for the Cartier divisor $E'_x \times_x y$ on $\bar{Y}'_y$. Let

$\delta_y : y \to \bar{Y}'_y$ be the diagonal embedding. Set $H^0(n,y) := H^0(\bar{Y}'_y, \mathcal{L}(nE'_y - 2\delta_y(y)))$. Also, consider the restriction map

$$r_{n,y} : H^0(n,y) \to H^0(T'_y, \mathcal{L}(nE'_y - 2\delta_y(y))|_{T'_y}).$$

Show that there exists an $n_0 > 0$ such that for all $n \geqslant n_0$ and all $y \in Y$ the map $r_{n,y}$ is an epimorphism. Take locally free sheaves $\mathcal{F}_n = p^*_{\bar{Y}'}(\mathcal{L}(nE'_x - T'_x))$ and $\mathcal{G}_n = \mathcal{F}_n \otimes \mathcal{L}(-2 \cdot \Delta(Y'_0))$ on $Y'_0 \times \bar{Y}'$. It follows from [12, Ch. II, Section 5, Corollary 2] that

$$H^1(Y'_0 \times \bar{Y}', \mathcal{G}_n) \otimes_{k[Y'_0]} k(y) = H^1(\bar{Y}'_y, \mathcal{L}(nE'_y - T'_y - 2 \cdot \delta_y(y))).$$

By Serre's vanishing theorem there exists $n_0 > 0$ such that for any $n \geqslant n_0$ one has $H^1(Y'_0 \times \bar{Y}', \mathcal{G}_n) = 0$. Hence $H^1(\bar{Y}'_y, \mathcal{L}(nE'_y - T'_y - 2 \cdot \delta_y(y))) = 0$ for any $n \geqslant n_0$. Thus $r_{n,y}$ is surjective for any $n \geqslant n_0$. Let $s_y \in H^0(\bar{Y}'_y, \mathcal{L}(\delta_y(y)))$ be the canonical section of the line bundle $\mathcal{L}(\delta_y(y))$ on $\bar{Y}'_y$. Multiplication by $s_y^{\otimes 2}$ identifies $H^0(T_y, \mathcal{L}(nE'_y - 2\delta_y(y))|_{T'_y})$ with $H^0(T_y, \mathcal{L}(nE'_y)|_{T'_y})$, because s_y has no zeros on T'_y. Let $\bar{w}(n,y)$ be a unique element in $H^0(T'_y, \mathcal{L}(nE'_y - 2\delta_y(y))|_{T'_y})$ such that $s_y^{\otimes 2} \otimes \bar{w}(n,y) = \bar{v}(n)$. Denote by $H^0(n,y)_{\bar{w}(n,y)}$ the affine $k(y)$-subspace $r^{-1}_{n,y}(\bar{w}(n,y))$ in the $k(y)$-vector space $H^0(n,y)$. For $n \geqslant n_0$ the dimension $h^0(n,y)_{\bar{w}(n,y)}$ of $H^0(n,y)_{\bar{w}(n,y)}$ over $k(y)$ coincides with the dimension $h^0(nE'_y - T'_y - 2\delta_y(y))$ of $H^0(\bar{Y}'_y, \mathcal{L}(nE'_y - T'_y - 2\delta_y(y)))$ over $k(y)$. Since $H^1(\bar{Y}'_y, \mathcal{L}(nE'_y - T'_y - 2 \cdot \delta_y(y))) = 0$ for any $n \geqslant n_0$, we have $h^0(nE'_y - T'_y - 2\delta_y(y)) = h^0(nE'_y - T'_y) - 2$. Thus for any $n \geqslant n_0$ we have

$$h^0(n,y)_{\bar{w}(n,y)} = h^0(n)_{\bar{v}(n)} - 2. \tag{40}$$

Now regard $H^0(n)_{\bar{v}(n)}$ as a $k(x)$-scheme and for any point $y \in Y'_0$ write $H^0(n)_{\bar{v}(n)} \otimes_{k(x)} k(y)$ for the corresponding $k(y)$-scheme. Consider the scheme $Y'_0 \times H^0(n)_{\bar{v}(n)}$ and its closed subset $Inc_2(n) = \{(y,t) : div_y(t) \geqslant 2\}$. We claim that for any $n \geqslant n_0$ one has

$$\dim(Inc_2(n)) = h^0(n)_{\bar{v}(n)} - 1. \tag{41}$$

In fact, if $n \geqslant n_0$, then for any $y \in Y'_0$ the fibre of $Inc_2(n)$ over y is the affine $k(y)$-subspace $H^0(n,y)_{\bar{w}(n,y)}$ in the affine $k(y)$-space $H^0(n)_{\bar{v}(n)} \otimes_{k(x)} k(y)$. Since $\dim(Y'_0) = 1$, the equality (40) shows that the equality (41) is true. Let $p_2 : Y'_0 \times H^0(n)_{\bar{v}(n)} \to H^0(n)_{\bar{v}(n)}$ be the projection. Then the equality (41) shows that for all $n \geqslant n_0$ the Zariski closure $\overline{p_2(Inc_2(n))}$ is a proper closed subset in $H^0(n)_{\bar{v}(n)}$.

Now consider the scheme $(Y_0' \times_Y Y_0' - \Delta(Y_0)) \times H^0(n)_{\bar{v}(n)}$ and its closed subset $Inc_1(n) = \{(y_1, y_2, t) : t(y_1) = 0 = t(y_2)\}$. We claim that for any integer $n \gg 0$ one has $\dim(Inc_1(n)) = \dim(H^0(n)_{\bar{v}(n)}) - 1$. This equality is proved in the same way as the equality (41). A crucial point in the proof of the latter equality is that $\dim(Y_0' \times_Y Y_0' - \Delta(Y_0)) = 1$. Let $q_2 : Inc_1(n) \to H^0(n)_{\bar{v}(n)}$ be the projection. Then for any $n \geqslant n_1$ the Zariski closure $\overline{q_2(Inc_1(n))}$ is a proper closed subset in $H^0(n)_{\bar{v}(n)}$. Set

$$V(n) = H^0(n)_{\bar{v}(n)} - \overline{q_2(Inc_1(n))} \cup \overline{p_2(Inc_2(n))},$$

where the bar means the Zariski closure. Then $V(n)$ is a non-empty open subset of $H^0(n)_{\bar{v}(n)}$. Let $\bar{s}(n) \in V(n)$ be a $k(x)$-rational point. The Cartier divisor $\bar{Z}(n) := \{\bar{s}(n) = 0\}$ is contained in $Y_0' \subset Y'$ and the scheme $\bar{Z}(n)$ is étale over $Spec(k(x))$, because $\bar{s}(n)$ is not in $\overline{p_2(Inc_2(n))}$. Let $\overline{k(x)}$ be the algebraic closure of $k(x)$. Then for any two different points $y_1, y_2 \in Supp(\bar{Z}(n) \otimes_{k(x)} \overline{k(x)})$ one has

$$(\Pi \otimes_{k(x)} \overline{k(x)})(y_1) \neq (\Pi \otimes_{k(x)} \overline{k(x)})(y_2)$$

in $Y \otimes_{k(x)} \overline{k(x)}$, because $\bar{s}(n)$ is not in $\overline{q_2(Inc_1(n))}$. Hence the morphism $(id \times \Pi)|_{\bar{Z}(n)} : \bar{Z}(n) \to Y$ is a closed embedding. Moreover, $\bar{s}(n)|_{T_x} = \bar{v}(n)$. Since $T_x = x \times_B T$, we have found the desired section $\bar{s}(n)$.

Next, patching $\bar{s}(n)$ and $v(n)$ we get a section $s_{ext}(n)$ of $\mathcal{L}(nE')|_{T' \cup x \times_B \bar{X}'}$ such that $s_{ext}(n)|_{T'} = v(n)$ and $s_{ext}(n)|_{x \times_B \bar{X}'} = \bar{s}(n)$. If $n \gg 0$, then by Serre's vanishing theorem we can lift the section $s_{ext}(n)$ to a section $s_1'(n) \in \Gamma(U \times_B \bar{X}', \mathcal{L}(nE'))$. The latter section $s_1'(n)$ is the desired one as one can easily check. □

Lemma 15.11. *Let $n \gg 0$ and $Z_1(n)$ be as in Construction 15.10. Properties* (1*a*), (1*a*′) *and* (1*b*) *yield the following property:*
(1*c*) *one has a scheme equality* $(id \times \bar{\Pi})^{-1}(Z_1(n)) = Z_1'(n) \sqcup \bar{Z}_2'(n)$.

Proof. The morphism $id \times \Pi : U \times_B X' \to U \times_B X$ is étale. It follows that the morphism $(id \times \Pi)|_{(id \times \Pi)^{-1}(Z_1(n))} : (id \times \Pi)^{-1}(Z_1(n)) \to Z_1(n)$ is étale. Since $i : Z_1'(n) \to Z_1(n)$ is an isomorphism, hence the étale morphism $(id \times \Pi)|_{(id \times \Pi)^{-1}(Z_1(n))}$ has a section whose image is $Z_1'(n)$. Thus $(id \times \Pi)^{-1}(Z_1(n)) = Z_1'(n) \sqcup Z_2'(n)$. The property (1*a*′) shows now that

$$(id \times \bar{\Pi})^{-1}(Z_1(n)) = Z_1'(n) \cup \bar{Z}_2'(n),$$

where $\bar{Z}_2'(n)$ is the closure of $Z_2'(n)$ in $U \times \bar{X}'$. One has $Z_1'(n) \cap \bar{Z}_2'(n) \subset (U \times_B X') \cap \bar{Z}_2'(n) = Z_2'(n)$. Hence $Z_1'(n) \cap \bar{Z}_2'(n) \subset Z_1'(n) \cap Z_2'(n) = \emptyset$. Thus $(id \times \bar{\Pi})^{-1}(Z_1) = Z_1'(n) \sqcup \bar{Z}_2'(n)$. □

Lemma 15.12. *Let $n \gg 0$ and $\bar{Z}_2(n)$ be as Lemma 15.11. Then one has $\bar{Z}'_2(n) \cap U \times_B S' = \emptyset$.*

Proof. One has a chain of inclusions

$$(id \times \bar{\Pi})((U \times_B S') \cap \bar{Z}'_2(n)) \subset (id \times \bar{\Pi})((U \times_B S') \cap (id \times \bar{\Pi})^{-1}(Z_1(n))) = \\ = (U \times_B S) \cap Z_1(n).$$

This inclusion and the fact that the morphism $(id \times \Pi)|_{U \times_B S'} : U \times_B S' \to U \times_B S$ is an isomorphism yield the following inclusions $(U \times_B S') \cap \bar{Z}'_2(n) \subset (U \times_B S') \cap Z'_1(n) \subset Z'_1(n)$. Since $U \times_B S' \cap \bar{Z}'_2(n) \subset \bar{Z}'_2(n)$, we see that $U \times_B S' \cap \bar{Z}'_2(n) \subset Z'_1(n) \cap \bar{Z}'_2(n) = \emptyset$ by Lemma 15.11. □

Note that the Cartier divisor $Z_1(n)$ in $U \times_B \bar{X}$ is equivalent to the Cartier divisor dnE, where $d = [k(X') : k(X)]$. Let $s_1(n) \in \Gamma(U \times_B \bar{X}, \mathcal{L}(Z_1(n)))$ be the canonical section (its vanishing locus is $Z_1(n)$). By property (1c) from Lemma 15.11 one has an equality

$$(id \times \bar{\Pi})^*(s_1(n)) = (s'_1(n) \otimes s'_2(n)) \cdot \mu(n), \tag{42}$$

where $\mu(n) \in k[U]^\times$ and $s'_2(n) \in \Gamma(U \times_B \bar{X}', \mathcal{L}(\bar{Z}'_2(n)))$ is the canonical section of the line bundle $\mathcal{L}(\bar{Z}'_2(n))$.

Definition 15.13. *For $n \gg 0$ set $t_1(n) = s_1(n) \in \Gamma(U \times_B \bar{X}, \mathcal{L}(Z_1(n))) = \Gamma(U \times_B \bar{X}, \mathcal{L}(dnE))$.*

Similar to Construction 15.7 we are able to do the following

Construction 15.14. *For $n \gg 0$ construct a section $t_0(n) \in \Gamma(U \times_B \bar{X}, \mathcal{L}(dnE))$ of the form $t_0(n) = s_{\Delta(U)} \otimes t'_0(n)$, where $t'_0(n) \in \Gamma(U \times_B \bar{X}, \mathcal{L}(dnE - \Delta(U)))$ and $s_{\Delta(U)} \in \Gamma(U \times_B \bar{X}, \mathcal{L}(\Delta(U)))$ is the canonical section (its vanishing locus is $\Delta(U)$) and $t'_0(n)$ has the following properties:*
(1′) $t'_0(n)|_E = (t_1(n)|_E) \otimes (s_{\Delta(U)}|_E)^{-1}$;
(2′) $((id \times \bar{\Pi})|_{U \times_B S'})^*(t'_0(n)|_{U \times_B S}) = ((\mathrm{id} \times \Pi)|_{U \times_B S'})^*(r_1(n)) \otimes (s'_2(n)|_{U \times_B S'}) \cdot (\mu(n)|_{U \times_B S'})$, where $r_1(n)$ is defined just above Construction 15.10, $s'_2(n)$ and $\mu(n) \in k[U]^\times$ are defined just above the present construction (since $U \times_B S' \cong U \times_B S$, then condition (2′) on $t'_0(n)$ is a condition on $t'_0(n)|_{U \times_B S}$).

Proof. Set $T = U \times_B S \sqcup E$. If $n \gg 0$, then by Serre's vanishing theorem the restriction map

$$\Gamma(U \times_B \bar{X}, \mathcal{L}(dnE - \Delta(U))) \to \Gamma(T, \mathcal{L}(dnE - \Delta(U))|_T)$$

is surjective. This completes the proof. □

Lemma 15.15. *For $n \gg 0$ the following statements are true:*
(1″) $t_0(n)|_E = t_1(n)|_E$ and both sections have no zeros on E;
(2″) $t_0(n)|_{U\times_B S} = t_1(n)|_{U\times_B S}$ and both sections have no zeros on $(U-S)\times_B S$;
(3″) $t'_0(n)|_{U\times_B S}$ has no zeros.

Proof. Indeed, the first equality is obvious. To prove the second one, it suffices to prove the equality

$$((id\times\bar\Pi)|_{U\times_B S'})^*(t_0(n)|_{U\times_B S}) = ((id\times\bar\Pi)|_{U\times_B S'})^*(t_1(n)|_{U\times_B S}).$$

This equality is a consequence of the following chain of equalities:

$$((id\times\bar\Pi)|_{U\times_B S'})^*(t_0(n)|_{U\times_B S}) =$$

$$(id\times\bar\Pi)^*(s_{\Delta(U)})|_{U\times_B S'}\otimes((\mathrm{id}\times\Pi)|_{U\times_B S'})^*(r_1(n))\otimes(s'_2(n)|_{U\times_B S'})\cdot(\mu(n)|_{U\times_B S'}) =$$
$$= s'_1(n)|_{U\times_B S'}\otimes(s'_2(n)|_{U\times_B S'})\cdot(\mu(n)|_{U\times_B S'}) = ((id\times\bar\Pi)|_{U\times_B S'})^*(t_1(n)|_{U\times_B S}).$$

The first equality holds by property (2′) from Construction 15.14, the second equality holds by property (2) from Construction 15.10. The third equality follows from the relation (42). The assertion (3") follows from Lemma 15.12 and Construction 15.14(2'). □

Definition 15.16. Choose $n \gg 0$ and set

$$h_\theta = \frac{((1-\theta)\cdot t_0(n)+\theta\cdot t_1(n))|_{\mathbb{A}^1\times U\times X}}{(s_E^{\otimes dn})|_{\mathbb{A}^1\times U\times X}} \in k[\mathbb{A}^1\times U\times_B X],$$

where $s_E\in\Gamma(U\times_B\bar X,\mathcal{L}(E))$ is the canonical section.

Proof of Proposition 10.9. Let $pr_{23}:\mathbb{A}^1\times U\times_B\bar X\to U\times_B\bar X$ be the projection. Consider two sections $(1-\theta)\cdot t_0(n)+\theta\cdot t_1(n)$ and $s_E^{\otimes dn}$ of the line bundle $pr_{23}^*(\mathcal{L}(dnE))$ on $\mathbb{A}^1\times U\times_B\bar X$. By Lemma 15.15 these two sections have no common zeros. Thus one has a morphism

$$[pr_{12},(1-\theta)\cdot t_0(n)+\theta\cdot t_1(n):s_E^{\otimes dn}]:\mathbb{A}^1\times U\times_B\bar X\to\mathbb{A}^1\times U\times\mathbb{P}^1,$$

where $pr_{12}:\mathbb{A}^1\times U'\times_B\bar X'\to\mathbb{A}^1\times U'$ is the projection. This morphism is quasi-finite and projective. Hence it is finite and surjective. It follows that any of its base changes is finite and surjective. Particularly, the morphism $(pr_{12},h_\theta):\mathbb{A}^1\times U\times X\to\mathbb{A}^1\times U\times\mathbb{A}^1$ is finite and surjective, because the closed subset $\{s_E^{\otimes dn}=0\}$ in $\mathbb{A}^1\times U\times_B\bar X$ coincides with the one $\mathbb{A}^1\times E$. This proves the assertion (a) of Proposition 10.9.

Lemma 15.15 yields the assertion (d) of Proposition 10.9. The property (1b) from Construction 15.10 and Lemma 15.11 yields the assertion

(c) of Proposition 10.9. Lemma 15.12 and the property (2') from Construction 15.14 yield the assertion (b) of Proposition 10.9. Proposition 10.9 follows. □

16. Nisnevich cohomology with coeffitients in a $\mathbb{Z}F_*$-sheaf

Lemma 16.1. *The category of Nisnevich sheaves of Abelian groups with $\mathbb{Z}F_*$-transfers is a Grothendieck category.*

Proof. The category of presheaves of Abelian groups with $\mathbb{Z}F_*$-transfers is plainly a Grothendieck category. Since for every radditive framed presheaf of Abelian groups F the associated sheaf in the Nisnevich topology has a unique structure of a framed presheaf such that the map $F \to F_{\text{nis}}$ is a map of framed presheaves by Corollary 2.17, our lemma is now proved similar to [7, 6.4]. □

The main purpose of this section is to prove the following

Proposition 16.2. *For any Nisnevich sheaf $\mathcal{F}$ with $\mathbb{Z}F_*$-transfers, any integer n and any k-smooth scheme X, there is a natural isomorphism*

$$H^n_{Nis}(X, \mathcal{F}) = \operatorname{Ext}^n(\mathbb{Z}F_*(X)_{Nis}, \mathcal{F}),$$

where the Ext*-groups are taken in the Grothendieck category of Nisnevich sheaves with $\mathbb{Z}F_*$-transfers.*

Recall that for a morphism $f : Y \to X$ we denote by $\check{C}(f)$ or $\check{C}(Y)$ the Cech simplicial object defined by f.

Lemma 16.3 ([20], Theorem 4.4)**.** *Let $f : Y \to X$ be an etale (respectively Nisnevich) covering of a k-smooth scheme X. Then for any n the map of simplicial presheaves*

$$Fr_n(-, \check{C}(Y)) \to Fr_n(-, X)$$

is a local equivalence in the etale (respectively Nisnevich) topology.

Definition 16.4. For any $U \in Sm'/k$ and $X \in Sm/k$ define $F_m(U, X) \subset Fr_m(U, X)$ as a subset consisting of $(Z, W, \varphi; g) \in Fr_m(U, X)$ such that Z is connected.

Clearly, the set $F_m(U, X) - \emptyset_m$ is a free basis of the abelian group $\mathbb{Z}F_m(U, X)$. However, the assignment $U \mapsto F_m(U, X)$ is not a presheaf even on the category Sm/k.

Applying the proof of [20, Theorem 4.4] one can conclude that the following lemma holds:

Lemma 16.5. *Let $f : Y \to X$ be an etale (respectively Nisnevich) covering of a k-smooth scheme X. Then for any local essentially k-smooth henselian scheme U and for any integer $n \geqslant 0$ the map of pointed simplicial sets*

$$F_n(U, \check{C}(Y)) \to F_n(U, X)$$

is a weak equivalence.

Corollary 16.6. *Let $f : Y \to X$ be an etale (respectively Nisnevich) covering of a k-smooth scheme X. Then for any n the maps of simplicial presheaves*

$$\mathbb{Z}F_n(-, \check{C}(Y)) \to \mathbb{Z}F_n(-, X), \quad \mathbb{Z}F_*(-, \check{C}(Y)) \to \mathbb{Z}F_*(-, X)$$

are local equivalences in the Nisnevich topology.

Corollary 16.7. *Let I be an injective object in the category of Nisnevich sheaves with $\mathbb{Z}F_*$-transfers. Then for any k-smooth scheme X and for all $i > 0$, one has $H^i_{Nis}(X, I) = 0$.*

Proof. Using the preceding corollary, our proof is similar to that of [18, 1.7]. □

Proof of Proposition 16.2. Corollary 16.7 implies the proposition. □

Proposition 16.2 implies the following useful

Corollary 16.8. *For any Nisnevich sheaf $\mathcal{F}$ with $\mathbb{Z}F_*$-transfers and any integer n, the presheaf $X \mapsto H^n_{Nis}(X, \mathcal{F})$ has a canonical structure of a $\mathbb{Z}F_*$-presheaf.*

In fact, this holds for the presheaf $X \mapsto \operatorname{Ext}^n(\mathbb{Z}F_*(X)_{Nis}, \mathcal{F})$.

17. Homotopy invariance of cohomology presheaves

In this section we prove Theorems 17.15 and 17.16. They complete the proof of Theorem 1.1, which is the main result of the paper. Each statement in this section except Lemma 17.3 is split in two parts depending on whether the characteristic of the base field does not equal 2 or equals 2. We will only prove the case when the characteristic is not 2, because the case when $\operatorname{char} k = 2$ is proved similarly and is left to the reader. Throughout this section the base field k is supposed to be infinite and perfect.

We refer the reader to [6] or [5] for the definition and basic properties of the henselization Y^h_Z of an affine scheme $Y \in Sm'/k$ along a closed subscheme Z. If $y \in Y$ is a point we sometimes write Y_y for $Spec(\mathcal{O}_{Y,y})$ and Y^h_y for $(Y_y)^h_y$. All k-schemes of the form Y^h_Z we work with in this section belong to the category Sm'/k.

Definition 17.1. Let G be a homotopy invariant presheaf of abelian groups with $\mathbb{Z}F_*$-transfers. Then the presheaf $X \mapsto G_{-1}(X) := G(X \times (\mathbb{A}^1 - 0))/G(X)$ is also a homotopy invariant presheaf of abelian groups with $\mathbb{Z}F_*$-transfers. If the presheaf G is a Nisnevich sheaf, then the presheaf G_{-1} is also a Nisnevich sheaf. If the presheaf G is quasi-stable, then so is the presheaf G_{-1}.

If G is a presheaf on Sm/k and Y is a k-smooth scheme, then denote by $G|_Y$ the restriction of G to the small Nisnevich site of Y.

Consider the inclusion of categories $inc : Sm/k \to Sm'/k$ where Sm'/k is the category of essentially smooth schemes over k. Then for any presheaf G of Abelian groups on Sm/k the restriction of the presheaf $inc_*(G)$ on Sm'/k to Sm/k equals G (that is $inc^*(inc_*(G)) = G$ on Sm/k). For any essentially smooth scheme Y over k we will use notation $G(Y)$ instead of $inc_*(G)(Y)$.

Let $Y \in Sm'/k$. Since Y is Noetherian it makes sense to consider the small Nisnevich site Y_{Nis}. We will write $G|_Y$ for the presheaf $W \mapsto G(W)$ on Y_{Nis}. Particularly, this notation will be used for $Y \in Sm/k$. One can show that for any $Y \in Sm'/k$ the presheaf $G|_Y$ is a Nisnevich sheaf on Y_{Nis} whenever G is a Nisnevich sheaf on Sm/k.

For any closed subset Z in Y, any integer n and any Nisnevich sheaf G of abelian groups on the small Nisnevich site Y_{Nis} of Y write $H^n_Z(Y, G)$ for the Nisnevich cohomology with support on Z.

The following useful result will be used in this section several times (cf. [19, Lemma E.6]).

Proposition 17.2. *Let Y be in Sm'/k. Write Y as a filtered limit $\lim Y_i$ over a small filtered category I, where Y_i are in Sm/k and the transition morphisms $\varphi_{ij} : Y_i \to Y_j$ are affine étale morphisms. Let $\mathcal{F}$ be a Nisnevich sheaf on Sm/k. Then for any integer $n \geqslant 0$ the canonical map*

$$\operatorname{colim}_{i\in I} H^n_{Nis}(Y_i, \mathcal{F}|_{Y_i}) \to H^n_{Nis}(Y, \mathcal{F}|_Y)$$

is an isomorphism. More generally, for any $i \in I$ let $\varphi_i : Y_i \to Y$ be the canonical morphism. Let $Z \subset Y$ be a closed subset. Then there is an $i \in I$ and a closed subset Z_i in Y_i such that $Z = \varphi^{-1}(Z_i)$. Moreover, if for any

$j \in I$ with $i \leqslant j$ we write Z_j for $\varphi_{ji}^{-1}(Z_i)$, then for any integer $n \geqslant 0$ the canonical map

$$\mathrm{colim}_{i \leqslant j \in I} H^n_{Z_j}(Y_j, \mathcal{F}|_{Y_j}) \to H^n_Z(Y, \mathcal{F}|_Y)$$

is an isomorphism.

Proof. The proof is like that for the Zariski topology case and is quite standard. Therefore we will only sketch the proof. For any $Y \in Sm'/k$ and a Nisnevich sheaf G of abelian groups on Y_{Nis} we say that G is *flasque* if for any $V \in Y_{Nis}$ and any Zariski open subset W in V the restriction map $G(V) \to G(W)$ is an epimorphism. It is easy to check the following: (1) for any flasque sheaf G on Y_{Nis} and any $n > 0$ the group $H^n_Y(Y, G)$ vanishes; (2) for any injective Nisnevich sheaf of abelian groups $\mathcal{I}$ on Sm/k the Nisnevich sheaf $\mathcal{I}|_Y$ on Y_{Nis} is flasque.

The property (1) shows that the groups $H^n_{Nis}(Y, G)$ can be computed using flasque resolutions. Take a Nisnevich sheaf $\mathcal{F}$ of abelian groups on Sm/k. Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be its injective resolution on the big Nisnevich site $(Sm/k)_{Nis}$. Then for any integer $n \geqslant 0$ the Nisnevich sheaf $\mathcal{I}|_Y$ is flasque and for any $i \in I$ the Nisnevich sheaf $\mathcal{I}|_{Y_i}$ is flasque. Thus

$$\begin{aligned} H^n_{Nis}(Y, \mathcal{F}|_Y) = H^n(\mathcal{I}(Y^\bullet)) = H^n(\mathrm{colim}_{i \in I} \mathcal{I}^\bullet(Y_i)) = \\ = \mathrm{colim}_{i \in I} H^n(\mathcal{I}^\bullet(Y_i)) = \mathrm{colim}_{i \in I} H^n_{Nis}(Y_i, \mathcal{F}|_{Y_i}). \end{aligned}$$

This proves the first assertion.

To prove the second one consider a long exact sequence

$$\begin{aligned} H^n_{Nis}(Y, \mathcal{F}|_Y) \to H^n_{Nis}(Y - Z, \mathcal{F}|_{Y-Z}) \to H^{n+1}_Z(Y, \mathcal{F}|_Y) \to \\ \to H^{n+1}_{Nis}(Y, \mathcal{F}|_Y) \to H^{n+1}_{Nis}(Y - Z, \mathcal{F}|_{Y-Z}), \end{aligned}$$

and for any $j \in I$ with $i \leqslant j$ consider similar sequences corresponding to the pair (Y_j, Z_j). The first assertion of the proposition now implies the second one. □

For any $Y \in Sm'/k$, any closed subset Z in Y, any integer n and any Nisnevich sheaf G of abelian groups on the Nisnevich site Y_{Nis} of Y consider the presheaf $Y' \mapsto H^n_{Z'}(Y', G)$, where $Z' = Y' \times_Y Z$. We will write $\mathcal{H}^n_Z(Y, G)$ for the associated Nisnevich sheaf on Y_{Nis}.

Lemma 17.3. *For any $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-sheaf of abelian groups $\mathcal{F}$, any k-smooth scheme Y and any k-smooth divisor D in Y the canonical morphism $H^1_D(Y, \mathcal{F}) \to H^0_{Nis}(Y, \mathcal{H}^1_D(Y, \mathcal{F}))$ is an isomorphism.*

Proof. The local-global spectral sequence yields an exact sequence

$$H^1_{Nis}(Y, \mathcal{H}^0_D(Y, \mathcal{F})) \to H^1_D(Y, \mathcal{F}) \to \\ \to H^0_{Nis}(Y, \mathcal{H}^1_D(Y, \mathcal{F})) \to H^2_{Nis}(Y, \mathcal{H}^0_D(Y, \mathcal{F})).$$

By Theorem 3.15(3') all Nisnevich stalks of the sheaf $\mathcal{H}^0_D(Y, \mathcal{F})$ vanishes. Thus the sheaf $\mathcal{H}^0_D(Y, \mathcal{F})$ vanish. This proves the lemma. □

Proposition 17.4. *Let $S \in Sm/k$, $s \in S$ be a point, $V := Spec(\mathcal{O}_{S,s})$, $W = V^h_s$. Let $\mathcal{V} := Spec(\mathcal{O}_{W \times \mathbb{A}^1,(s,0)})$ and $can : \mathcal{V} \hookrightarrow W \times \mathbb{A}^1$ be the canonical embedding. Let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-presheaf of abelian groups. Then the pull-back map*

$$[[can]]^* : \mathcal{F}(W \times (\mathbb{A}^1 - \{0\}))/\mathcal{F}(W \times \mathbb{A}^1) \to \mathcal{F}(\mathcal{V} - W \times \{0\})/\mathcal{F}(\mathcal{V})$$

is an isomorphism.

Proof. For any étale neighborhood V_i of the point s in V let s_i be the unique point in V_i lying over s. Set $W_i = Spec(\mathcal{O}_{V_i,s_i})$ and $\mathcal{V}_i = Spec(\mathcal{O}_{W_i \times \mathbb{A}^1,(s_i,0)})$. Then $W = \lim W_i$ and $\mathcal{V} = \lim \mathcal{V}_i$. Consider the quotients $\mathcal{F}(W_i \times (\mathbb{A}^1 - \{0\}))/\mathcal{F}(W_i \times \mathbb{A}^1)$ and $\mathcal{F}(\mathcal{V}_i - W_i \times \{0\})/\mathcal{F}(\mathcal{V}_i)$. Both quotients make sense: the first quotient makes sense due to $\mathbb{A}^1$-invariance of $\mathcal{F}$, the second one makes sense due to Theorem 3.15(3). Thus, the quotients $\mathcal{F}(W \times (\mathbb{A}^1 - \{0\}))/\mathcal{F}(W \times \mathbb{A}^1)$ and $\mathcal{F}(\mathcal{V} - W \times \{0\})/\mathcal{F}(\mathcal{V})$ make sense as well. For any i the map

$$[[can_i]]^* : \mathcal{F}(W_i \times (\mathbb{A}^1 - \{0\}))/\mathcal{F}(W_i \times \mathbb{A}^1) \to \mathcal{F}(\mathcal{V}_i - W_i \times \{0\})/\mathcal{F}(\mathcal{V}_i)$$

is an isomorphism by Corollary 3.19. Since $[[can]]^* = \operatorname{colim}_{i \in I}[[can_i]]^*$ we see that the map $[[can]]^*$ is an isomorphism. □

Consider smooth k-schemes X_l, X_r and D. Let $i_l : D \to X_l$ and $i_r : D \to X_r$ be closed embeddings. Let $(X_m, \pi_l : X_m \to X_l, s_l : D \to X_m)$ and $(X_m, \pi_r : X_m \to X_r, s_r : D \to X_m)$ be étale neighborhoods of $i_l(D)$ in X_l and $i_r(D)$ in X_r respectively. These data are called *geometric data* if $s_l = s_r$. In this case $\pi_r \circ s_l = i_r$. We also write a zigzag

$$D \xrightarrow{i_l} X_l \xleftarrow{\pi_l} X_m \xrightarrow{\pi_r} X_r \xleftarrow{i_r} D$$

to denote the geometric data.

Definition 17.5. We will say that geometric data X_l, X_m, X_r, D, i_l, i_r, (X_m, π_l, s_l), (X_m, π_r, s_l) are *Voevodsky's data* if $X_r = D \times \mathbb{A}^1$ and $i_r : D \to X_r = D \times \mathbb{A}^1$ is the zero section. Clearly, in this case $i_l(D)$, $s_l(D) = s_r(D)$ are smooth divisors in X_l and X_m respectively. We also denote Voevodsky's data by writing a tuple $(D, X_l, X_m, X_r, ...)$.

If $(D, X_l, X_m, X_r, ...)$ are Voevodsky's data, then for any $Y \in Sm/k$ the data $(Y \times D, Y \times X_l, Y \times X_m, Y \times X_r, ...)$ are Voevodsky's data as well.

If $(D, X_l, X_m, X_r, ...)$ are Voevodsky's data and $X' \to X$ is an étale morphism, then the data $(X' \times_X D, X' \times_X X_l, X' \times_X X_r, ...)$ are Voevodsky's data. Voevodsky's data $(D, X_l, X_m, X_r, ...)$ can be written as the following zigzag (with i_r is the zero section)

$$D \xrightarrow{i_l} X_l \xleftarrow{\pi_l} X_m \xrightarrow{\pi_r} D \times \mathbb{A}^1 \xleftarrow{i_r} D.$$

Suppose $\mathcal{F}$ is a Nisnevich sheaf of abelian groups on Sm/k. Consider three Nisnevich sheaves $\mathcal{H}_l := \mathcal{H}^1_{i_l(D)}(X_l, \mathcal{F})$, $\mathcal{H}_m := \mathcal{H}^1_{s_l(D)}(X_m, \mathcal{F}) = \mathcal{H}^1_{s_r(D)}(X_m, \mathcal{F})$ and $\mathcal{H}_r := \mathcal{H}^1_{i_r(D)}(X_r, \mathcal{F})$ on the small Nisnevich sites of X_l, X_m and X_r respectively.

Lemma 17.6. *For any geometric data X_l, X_m, X_r, D, i_l, i_r (X_m, π_l, s_l), (X_m, π_r, s_l) above Definition 17.5, there is a natural sheaf isomorphism $i_l^*(\mathcal{H}_l) \cong i_r^*(\mathcal{H}_r)$ on the small Nisnevich site of D. Particularly, this holds if these geometric data are Voevodsky's data.*

Proof. Clearly, $\pi_l^*(\mathcal{H}_l) \cong \mathcal{H}_m \cong \pi_r^*(\mathcal{H}_r)$. Thus we now have a chain of canonical isomorphisms $i_l^*(\mathcal{H}_l) = s_l^*(\pi_l^*(\mathcal{H}_l)) = s_l^*(\pi_r^*(\mathcal{H}_r)) = i_r^*(\mathcal{H}_r)$. □

The following lemma follows from the proof of [21, Theorem 4.14].

Lemma 17.7. *For any $X \in Sm/k$, any smooth divisor D in X and any point $x \in X$ there is a Zariski neiborhood X_l of the point x that can be fit in Voevodsky's data $(D \cap X_l, X_l, X_m, (D \cap X_l) \times \mathbb{A}^1, ...)$.*

Remark 17.8. Let $T \in Sm/k$ and $i : S \hookrightarrow T$ be a k-smooth closed subscheme, $j : T - S \hookrightarrow T$ be the open subscheme. Let G be a Nisnevich sheaf of abelian groups on the small Nisnevih site of T. Then the sequence of Nisnevich sheaves $G \xrightarrow{adj} j_*j^*(G) \to \mathcal{H}^1_S(T, G) \to 0$ on T is exact and induces a sheaf isomorphism $\partial : \mathrm{coker}_{T,S} := \mathrm{coker}(adj)_{Nis} \cong \mathcal{H}^1_S(T, G)$.

Proposition 17.9. *Let $D \in Sm/k$, $i_D : D \hookrightarrow D \times \mathbb{A}^1$ be the zero section, $j_D : D \times (\mathbb{A}^1 - 0) \hookrightarrow D \times \mathbb{A}^1$ be the open embedding. Let G be an $\mathbb{A}^1$-invariant* *quasi-stable $\mathbb{Z}F_*$-Nisnevich sheaf. Then there is a natural Nisnevich sheaf isomorphism on the small Nisnevich site D_{Nis} of D*

$$\varphi_D : G_{-1}|_D \cong i_D^*(coker_{D\times\mathbb{A}^1,D}).$$

If char $k = 2$, *then the same statement is true if we assume that the $\mathbb{Z}F_*$-sheaf $\mathcal{F}$ is a sheaf of $\mathbb{Z}[1/2]$-modules.*

Proof. Consider G, $j_D^*(G)$ and $j_{D,*}j_D^*(G)$ as presheaves on the small site D_{Nis}. Take the adjuction morphism $a_D : G \to j_{D,*}j_D^*(G)$ and let $\operatorname{coker}(a_D)$ be its cokernel in the category of presheaves on D_{Nis}. Clearly, $\operatorname{coker}_{D\times\mathbb{A}^1,D}$ is the Nisnevich sheaf associated with the presheaf $\operatorname{coker}(a_D)$.

Recall that $G_{-1}(U) = G(U\times(\mathbb{A}^1-0))/G(U\times\mathbb{A}^1)$ for any $U \in D_{Nis}$. For any $a \in G_{-1}(U)$ let $\tilde{a} \in G(U \times (\mathbb{A}^1 - 0))$ be a lift of a. Translating literally the proof of [21, Proposition 4.11] to the context of the Nisnevich site D_{Nis}, we get a presheaf morphism $\varphi_D : G_{-1}|_D \to i_D^*(\operatorname{coker}(a_D))$ on D_{Nis}. For any $a \in G_{-1}(U)$ the element $\varphi_D(a) \in i_D^*(\operatorname{coker}(a_D))(U)$ is the class of the pair $(U \times \mathbb{A}^1, \tilde{a})$, where $\tilde{a} \in G(U \times (\mathbb{A}^1 - 0))$.

To prove that the corresponding morphism φ_D^{nis} of the associated sheaves in the Nisnevich topology on D is an isomorphism, it sufficient to show that φ_D is an isomorphism on Nisnevich stalks. Take any point $x \in D$ and note that if $V = (D \times \mathbb{A}^1)^h_{(x,0)}$, then V contains D^h_x as a closed subscheme (namely as the zero section). Furthermore $V = (D^h_x \times \mathbb{A}^1)^h_{(x,0)} = \mathcal{V}^h_{(x,0)}$, where $\mathcal{V} = Spec(\mathcal{O}_{D^h_x\times\mathbb{A}^1,(x,0)})$. We need to check that the homomorphism

$$G(D^h_x \times (\mathbb{A}^1 - 0))/G(D^h_x \times \mathbb{A}^1) \to G(\mathcal{V}^h_x - D^h_x \times 0)/G(\mathcal{V}^h_x) \tag{43}$$

is a group isomorphism. This homomorphism is a composition of the homomorphism $G(D^h_x \times (\mathbb{A}^1 - 0))/G(D^h_s \times \mathbb{A}^1) \to G(\mathcal{V} - D^h_x \times 0)/G(\mathcal{V})$ and the homomorphism $G(\mathcal{V} - D^h_x \times 0)/G(\mathcal{V}) \to G(\mathcal{V}^h_{(x,0)} - D^h_x \times 0)/G(\mathcal{V}^h_{(x,0)})$. The first one is an isomorphism by Proposition 17.4. Applying Theorem 3.15(5) and standard colimit arguments, one can show that the second homomorphism is an isomorphism as well. □

The proof of the latter proposition has the following

Corollary 17.10. *Let G be an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-presheaf. Suppose the associated Nisnevich sheaf G_{nis} vanishes. Then for any $X, S \in Sm/k$, any smooth divisor $D \subset X$, any points $x \in D$ and $z \in S \times D$ one has $G(X^h_x - D^h_x) = 0$ and $G((S \times X)^h_z - (S \times D)^h_z) = 0$. If* char $k = 2$, *then the same statement is true if the $\mathbb{Z}F_*$-sheaf $\mathcal{F}$ is a sheaf of $\mathbb{Z}[1/2]$-modules.*

Proof. The second assertion follows from the first one. Indeed, apply the first assertion to the pair $(S \times X, S \times D)$. We now prove the first assertion. The assertion is local for the Zariski topology on X. Using Lemma 17.7 and shrinking X, we may assume that there are Voevodsky's data $D, X = X_l, X_m, X_r = D \times \mathbb{A}^1$. Let $y = s_l(x)$, then $y = s_r(x, 0)$. Let $D_m = s_l(D)$. Set $V = (D \times \mathbb{A}^1)^h_{(x,0)}$. Then V contains D^h_x as a closed subscheme (as the zero section). Clearly,

$$G(X^h_x - D^h_x) \cong G((X_m)^h_y - (D_m)^h_y) \cong G(V - D^h_x).$$

Since $G(V) = 0$ we have $G(V - D^h_x) = G(V - D^h_x)/G(V)$. The homomorphism (43) is an isomorphism and $\mathcal{V}^h_{(x,0)} = V$. Thus $G(V - D^h_x)/G(V) \cong G(D^h_x \times (\mathbb{A}^1 - 0))/G(D^h_x \times \mathbb{A}^1)$. Since G is $\mathbb{A}^1$-invariant and $G_{nis} = 0$ we see that $G(D^h_s \times \mathbb{A}^1) = 0$. It remains to check that $G(D^h_x \times (\mathbb{A}^1 - 0)) = 0$. The presheaf $X \mapsto G'(X) = G(X \times (\mathbb{A}^1 - 0))$ is an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-presheaf. By Theorem 3.15(3') the map $G'(D^h_x) \to G'(Spec(K))$ is injective, where K is the field of fractions for the henselian ring $\mathcal{O}^h_{D,x}$. One has $G'(Spec(K)) = G((\mathbb{A}^1 - 0)_K)$. The latter group embeds into $G(Spec(K(t)))$ by Theorem 3.15(1). The latter group vanishes because $G_{Nis} = 0$. □

Proposition 17.11. *Suppose* char $k \neq 2$. *Let $X \in Sm/k$ and $i : D \hookrightarrow X$ be a smooth divisor in X. Let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-sheaf of abelian groups. Then the pull-back map*

$$pr^*_X : H^0_{Nis}(X, \mathcal{H}^1_D(X, \mathcal{F})) \to H^0_{Nis}(\mathbb{A}^1 \times X, \mathcal{H}^1_{\mathbb{A}^1 \times D}(\mathbb{A}^1 \times X, \mathcal{F}))$$

is an isomorphism. If char $k = 2$, *then the same statement is true if the $\mathbb{Z}F_*$-sheaf $\mathcal{F}$ is a sheaf of $\mathbb{Z}[1/2]$-modules.*

Proof. This statement is local for the Zariski topology on X. Shrinking X we may assume that there are Voevodsky's data $D, X = X_l, X_m, X_r = D \times \mathbb{A}^1$. This time we write $i_D : D \to D \times \mathbb{A}^1$ for the 0-section. By the comment from Definition 17.5 the data $Y \times D, Y \times X = Y \times X_l, Y \times X_m, Y \times X_r = (Y \times D) \times \mathbb{A}^1$ are also Voevodsky's data. Using these data, Proposition 17.9, Remark 17.8 and Lemma 17.6, we get a chain of sheaf isomorphisms on the small site $(Y \times D)_{Nis}$

$$\begin{aligned}\mathcal{F}_{-1}|_{Y \times D} &\cong (id_Y \times i_D)^*(\mathrm{coker}_{Y \times D \times \mathbb{A}^1, Y \times D}) \cong \\ &\cong (id_Y \times i_D)^*(\mathcal{H}^1_{Y \times D \times 0}(Y \times D \times \mathbb{A}^1, \mathcal{F})) \cong \\ &\cong (id_Y \times i)^*(\mathcal{H}^1_{Y \times D}(Y \times X, \mathcal{F})).\end{aligned}$$

Since the adjunction map $\mathcal{H}^1_{Y\times D}(Y\times X,\mathcal{F})\to(id_Y\times i)_*(id_Y\times i)^*(\mathcal{H}^1_{Y\times D}(Y\times X,\mathcal{F}))$ is a sheaf isomorphism, we get a natural sheaf isomorphism on the small Nisnevich site of $Y\times X$

$$(id_Y\times i)_*(\mathcal{F}_{-1}|_{Y\times D})\cong\mathcal{H}^1_{Y\times D}(Y\times X,\mathcal{F}). \tag{44}$$

Taking global sections we get a group isomorphism, natural in Y,

$$\alpha_{Y\times D}:\mathcal{F}_{-1}(Y\times D)\to H^0(Y\times X,\mathcal{H}^1_{Y\times D}(Y\times X,\mathcal{F})).$$

The functoriality of this isomorphism with respect to Y means that for the projections $pr_D:Y\times D\to D$ and $pr_X:Y\times X\to X$ one has $\alpha_{Y\times D}\circ pr_D^*=pr_X^*\circ\alpha_D$. To complete the proof, take $Y=\mathbb{A}^1$ and use the $\mathbb{A}^1$-invariance of the sheaf $\mathcal{F}_{-1}$. □

Corollary 17.12. *Suppose* $\operatorname{char}k\neq 2$. *Let* $X\in Sm/k$ *and* D *be a smooth divisor in* X. *Let* $\mathcal{F}$ *be an* $\mathbb{A}^1$*-invariant quasi-stable* $\mathbb{Z}F_*$*-sheaf of abelian groups. Let* $x\in D$ *be a point. Then the map*

$$H^1_{Nis}(X^h_x\times\mathbb{A}^1,\mathcal{F})\to H^1_{Nis}((X^h_x-D^h_x)\times\mathbb{A}^1,\mathcal{F})$$

is injective. If $\operatorname{char}k=2$, *then the same statement is true if the* $\mathbb{Z}F_*$*-sheaf* $\mathcal{F}$ *is a sheaf of* $\mathbb{Z}[1/2]$*-modules.*

Proof. By Lemma 17.3 and Proposition 17.11 for any étale morphism $X'\to X$ and $D'=X'\times_X D$ the pullback map $H^1_{D'}(X',\mathcal{F})\to H^1_{\mathbb{A}^1\times D'}(\mathbb{A}^1\times X',\mathcal{F})$ is an isomorphism. Thus the map $H^1_{D^h_x}(X^h_x,\mathcal{F})\to H^1_{\mathbb{A}^1\times D^h_x}(\mathbb{A}^1\times X^h_x,\mathcal{F})$ is an isomorphism. The map $\partial:\mathcal{F}(X^h_x-D^h_x)\to H^1_{D^h_x}(X^h_x,\mathcal{F})$ is an epimorphism, because $H^1_{Nis}(X^h_x,\mathcal{F})=0$. Therefore the map $\mathcal{F}(\mathbb{A}^1\times(X^h_x-D^h_x))\to H^1_{\mathbb{A}^1\times D^h_x}(\mathbb{A}^1\times X^h_x,\mathcal{F})$ is an epimorphism. This proves the corollary. □

Proposition 17.13. *Let* $\operatorname{char}k\neq 2$ *and* K *be a field such that* $Spec(K)\in Sm'/k$. *Let* $\mathcal{F}$ *be an* $\mathbb{A}^1$*-invariant quasi-stable* $\mathbb{Z}F_*$*-sheaf of abelian groups. Then* $H^1_{Nis}(\mathbb{A}^1_K,\mathcal{F})=0$. *If* $\operatorname{char}k=2$, *then the same statement is true if the* $\mathbb{Z}F_*$*-sheaf* $\mathcal{F}$ *is a sheaf of* $\mathbb{Z}[1/2]$*-modules.*

Proof. Let $a\in H^1_{Nis}(\mathbb{A}^1_K,\mathcal{F})$. We want to prove that $a=0$. The Nisnevich topology is trivial at the generic point of the affine line $\mathbb{A}^1_K$. Therefore there is a Zariski open subset U in $\mathbb{A}^1_K$ such that the restriction of a to U vanishes. Let Z be the complement of U in $\mathbb{A}^1_K$ regarded as a closed subscheme with the reduced structure (it consists of finitely many closed points). Let $V:=$

$\sqcup_{z\in Z}(\mathbb{A}^1)^h_z$, where each summand is the henselization of the affine line at $z \in Z$. Then the cartesian square

$$\begin{array}{ccc} V - Z & \longrightarrow & V \\ \downarrow & & \downarrow \Pi \\ U & \longrightarrow & \mathbb{A}^1_K \end{array}$$

gives rise to a long exact sequence

$$\mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(V - Z) \xrightarrow{\partial} H^1_{Nis}(\mathbb{A}^1_K, \mathcal{F}) \to H^1_{Nis}(U, \mathcal{F}) \oplus H^1_{Nis}(V, \mathcal{F}).$$

The left arrow is surjective by Theorem 3.15 (items (2), (5)). The group $H^1_{Nis}(V, \mathcal{F})$ vanishes by the choice of V. Thus the map $H^1_{Nis}(\mathbb{A}^1_K, \mathcal{F}) \to H^1_{Nis}(U, \mathcal{F})$ is injective, and hence $a = 0$. □

Proposition 17.14. *Suppose the base field k is infinite and perfect with* $\operatorname{char} k \neq 2$. *Let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-sheaf of Abelian groups. Let X be a k-smooth scheme and let $a \in H^1_{Nis}(X \times \mathbb{A}^1, \mathcal{F})$ be an element such that its restriction to $X \times \{0\}$ vanishes. Then $a = 0$. If* $\operatorname{char} k = 2$, *then the same statement is true if the $\mathbb{Z}F_*$-sheaf $\mathcal{F}$ is a sheaf of* $\mathbb{Z}[1/2]$*-modules.*

Proof. Let $p : X \times \mathbb{A}^1 \to X$ be the projection. Since the sheaf $\mathcal{F}$ is homotopy invariant we have $p_*(\mathcal{F}|_{X\times\mathbb{A}^1}) = \mathcal{F}|_X$. Consider the exact sequence

$$0 \to H^1_{Nis}(X, p_*(\mathcal{F}|_{X\times\mathbb{A}^1})) \xrightarrow{\alpha} H^1_{Nis}(X \times \mathbb{A}^1, \mathcal{F}|_{X\times\mathbb{A}^1}) \xrightarrow{\beta} \\ \to H^0_{Nis}(X, R^1 p_*(\mathcal{F}|_{X\times\mathbb{A}^1})).$$

Let $i_0 : X \to X \times \mathbb{A}^1$ be the zero section. The identification $p_*(\mathcal{F}|_{X\times\mathbb{A}^1}) = \mathcal{F}|_X$ implies the map α is just the pullback map p^*. Thus $i_0^* \circ \alpha = i_0^* \circ p^* = id : H^1_{Nis}(X, \mathcal{F}_X) \to H^1_{Nis}(X, \mathcal{F}_X)$. Set,

$$A := Ker[i_0^* : H^1_{Nis}(X \times \mathbb{A}^1, \mathcal{F}|_{X\times\mathbb{A}^1}) \to H^1_{Nis}(X, \mathcal{F}|_X)].$$

We now show that $\ker(\beta) \cap \ker(i_0^*) = \{0\}$. Indeed, if $a \in \ker(\beta)$, then $a = \alpha(a')$ for some $a' \in H^1_{Nis}(X, p_*(\mathcal{F}|_{X\times\mathbb{A}^1}))$. If a is also in $\ker(i_0^*)$, then $0 = i_0^*(a) = i_0^*(\alpha(a')) = a'$. Thus $a' = 0$ and $a = 0$.

Since $\ker(\beta|_A) = \ker(\beta) \cap \ker(i_0^*) = \{0\}$, it follows that the map $\beta|_A : A \to H^0_{Nis}(X, R^1 p_*(\mathcal{F}|_{X\times\mathbb{A}^1}))$ is injective. The stalk of the sheaf $R^1 p_*(\mathcal{F})$ at a point $x \in X$ is $H^1_{Nis}(X^h_x \times \mathbb{A}^1, \mathcal{F})$, where $X^h_x = Spec(\mathcal{O}^h_{X,x})$ is the henselization of the local scheme $Spec(\mathcal{O}_{X,x})$ at x. By Proposition 17.13 there is a closed subset Z in X such that $\beta(a)|_{X-Z} = 0$. Since the field k is perfect, there is a proper closed subset $Z_1 \subset Z$ such that $Z - Z_1$ is k-smooth. Then $Z - Z_1$ is a k-smooth closed subscheme in $X - Z_1$.

We claim that $a_1 := a|_{(X-Z_1)\times\mathbb{A}^1} = 0$. By assumption, $a_1|_{(X-Z_1)\times 0} = 0$. Thus it suffices to check that all Nisnevich stalks of the element $\beta(a_1)$ vanish. Let $x \in X - Z_1$ be a point. If $x \in X - Z$ then $\beta(a_1)_x = 0$, because $\beta(a_1)|_{X-Z} = 0$. If $x \in Z - Z_1$ then shrinking $X - Z_1$ around x we may assume that there is a k-smooth divisor D in $X - Z_1$ containing $Z - Z_1$. We now have $\beta(a_1)|_{X-D} = 0$, because $\beta(a_1)|_{X-Z} = 0$. Now Corollary 17.12 shows that $\beta(a_1)_x = 0$. We have proved that $a_1 = 0$.

Now there is a proper closed subset $Z_2 \subset Z_1$ such that $Z_1 - Z_2$ is k-smooth (we use here that k is a perfect field). Then $Z_1 - Z_2$ is a k-smooth closed subscheme in $X - Z_2$. Arguing just as above, we conclude that $a_2 := a|_{(X-Z_2)\times\mathbb{A}^1} = 0$.

Continuing this process finitely many times, we find a strictly decreasing chain of closed subsets $X \supset Z_1 \supset Z_2 \supset ... \supset Z_n = \emptyset$ in X such that for any integer $i = \{1, 2, ..., n\}$ one has $a|_{(X-Z_i)\times\mathbb{A}^1} = 0$. Taking $i = n$ we get $a = a|_{(X-Z_n)\times\mathbb{A}^1} = 0$. □

Theorem 17.15. *Suppose the base field k is infinite and perfect with* $\operatorname{char} k \neq 2$. *If $\mathcal{F}$ is an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-sheaf of abelian groups, then the $\mathbb{Z}F_*$-presheaf of abelian groups $X \mapsto H^1_{Nis}(X, \mathcal{F})$ is $\mathbb{A}^1$-invariant and quasi-stable. If* $\operatorname{char} k = 2$, *then the same statement is true if the $\mathbb{Z}F_*$-sheaf $\mathcal{F}$ is a sheaf of $\mathbb{Z}[1/2]$-modules.*

Proof. By Corollary 16.8 the presheaf $X \mapsto H^1_{Nis}(X, \mathcal{F})$ has a canonical structure of a $\mathbb{Z}F_*$-presheaf. Let X be a k-smooth scheme. Let $\sigma_X \in Fr_1(X, X)$ be the distinguished morphism of level one. The assignement $X \mapsto (\sigma_X^* : \mathcal{F}(X) \to \mathcal{F}(X))$ is an endomorphism of the Nisnevich sheaf $\mathcal{F}|_{Sm/k}$. Thus for each n it induces an endomorphism of the cohomology presheaf $\sigma^* : H^n(-, \mathcal{F}) \to H^n(-, \mathcal{F})$. Since σ^* acts on $\mathcal{F}$ as an isomorphism, it acts as an isomorphism on the presheaf $H^n(-, \mathcal{F})$. We see that the $\mathbb{Z}F_*$-presheaf $H^n(-, \mathcal{F})$ is quasi-stable.

To show that the presheaf $X \mapsto H^1_{Nis}(X, \mathcal{F})$ is $\mathbb{A}^1$-invariant, note that the pullback map $i_0^* : H^1_{Nis}(X \times \mathbb{A}^1, \mathcal{F}) \to H^1_{Nis}(X, \mathcal{F})$ is surjective. It is also injective by Proposition 17.14. Our theorem now follows. □

Theorem 17.16. *Suppose the base field k is infinite and perfect with* $\operatorname{char} k \neq 2$. *Let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-sheaf of abelian* *groups. Then for any integer* $n \geqslant 2$*, the presheaf* $X \mapsto H^n_{Nis}(X, \mathcal{F})$ *is an* $\mathbb{A}^1$*-invariant and quasi-stable* $\mathbb{Z}F_*$*-presheaf of abelian groups. If* $\operatorname{char} k = 2$*, then the same statement is true if the* $\mathbb{Z}F_*$*-sheaf* $\mathcal{F}$ *is a sheaf of* $\mathbb{Z}[1/2]$*-modules.*

Proof. We can apply the same arguments as in the proof of Theorem 17.15 to show that the presheaf $X \mapsto H^n_{Nis}(X, \mathcal{F})$ is a $\mathbb{Z}F_*$-presheaf of abelian groups, which is, moreover, quasi-stable.

It remains to check that the presheaf is homotopy invariant. We may assume till the end of the proof that each presheaf $X \mapsto H^j_{Nis}(X, \mathcal{F})$ with $j < n$ is an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-presheaf.

In order to complete the proof of the theorem, we shall need the following lemma.

Lemma 17.17. *Suppose the base field* k *is infinite and perfect with* $\operatorname{char} k \neq 2$*. Let* X *be in* Sm/k*,* $i : D \hookrightarrow X$ *be a* k*-smooth divisor and* $x \in D$ *be a point. Then for any* $\mathbb{A}^1$*-invariant quasi-stable* $\mathbb{Z}F_*$*-sheaf of abelian groups* $\mathcal{F}$ *and any* $n \geqslant 2$ *one has*

$$H^n_{D^h_x \times \mathbb{A}^1}(X^h_x \times \mathbb{A}^1, \mathcal{F}) = 0.$$

If $\operatorname{char} k = 2$*, then the same statement is true if the* $\mathbb{Z}F_*$*-sheaf* $\mathcal{F}$ *is a sheaf of* $\mathbb{Z}[1/2]$*-modules.*

Proof. The main part of the proof is dedicated to verifying the following claim: the pullback map $p^*_{X^h_x} : H^n_{D^h_x}(X^h_x, \mathcal{F}) \to H^n_{D^h_x \times \mathbb{A}^1}(X^h_x \times \mathbb{A}^1, \mathcal{F})$ is an isomorphism. Afterwards we prove that $H^n_{D^h_x}(X^h_x, \mathcal{F}) = 0$.

Using Proposition 17.2, the above claim reduces to showing the following assertion: for any $X \in Sm/k$, any k-smooth divisor D in X and any point $x \in D$ the pullback map $p^*_{X_x} : H^n_{D_x}(X_x, \mathcal{F}) \to H^n_{D_x \times \mathbb{A}^1}(X_x \times \mathbb{A}^1, \mathcal{F})$ is an isomorphism.

Firstly, we consider any $Y \in Sm/k$ and any k-smooth divisor $E \subset Y$. We now use the notation introduced above Lemma 17.3. Analyzing the local global spectral sequence of the form

$$H^i_{Nis}(Y, \mathcal{H}^j_E(Y, \mathcal{F})) \Longrightarrow H^{i+j}_E(Y, \mathcal{F}),$$

we will show that $H^n_E(Y, \mathcal{F})$ is naturally isomorphic to $H^{n-1}_{Nis}(Y, \mathcal{H}^1_E(Y, \mathcal{F}))$. The nearest aim is to show the following computational claim: for any integers $0 \leqslant j \leqslant n$ with $j \neq 1$ the Nisnevich sheaves $\mathcal{H}^j_E(Y, \mathcal{F})$ on the small site Y_{Nis} vanish.

Consider the case $j = 0$. Obviously, the stalk of the sheaf $\mathcal{H}^j_E(Y, \mathcal{F})$ vanishes at every point $z \in Y - E$. Let $z \in E$ then the stalk at z equals

$H^0_{E^h_z}(Y^h_z, \mathcal{F})$. Since $H^0_{E^h_z}(Y^h_z, \mathcal{F}) = \ker[\mathcal{F}(Y^h_z) \to \mathcal{F}(Y^h_z - E^h_z)]$, Theorem 3.15(3') implies that the stalk at z vanishes. This proves the computational claim for $j = 0$.

Let $2 \leqslant j \leqslant n$. Obviously, the stalks of the sheaf $\mathcal{H}^j_E(Y, \mathcal{F})$ vanish at every point $z \in Y - E$. If $z \in E$ then the stalk of this sheaf at the point z is equal to the group $H^j_{E^h_z}(Y^h_z, \mathcal{F})$. Since $H^j_{Nis}(Y^h_z, \mathcal{F}) = 0$ one has the equality $H^j_{E^h_z}(Y^h_z, \mathcal{F}) = H^{j-1}_{Nis}(Y^h_z - E^h_z, \mathcal{F})/Im[H^{j-1}_{Nis}(Y^h_z, \mathcal{F})]$. Since $j-1 < n$ the presheaf $Y \mapsto G(U) := H^{j-1}_{Nis}(U, \mathcal{F})$ is an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-presheaf. Since $j - 1 > 0$ the associated Nisnevich sheaf G_{nis} vanishes. Thus, $G(Y^h_z - E^h_z) = 0$ by Lemma 17.10. This completes the proof of the computational claim.

The computational claim shows that the only nonzero term of the second page of the above spectal sequence lying on the diagonal $i + j = n$ is the group $H^{n-1}_{Nis}(Y, \mathcal{H}^1_E(Y, \mathcal{F}))$. Moreover, there are no incoming differentials to this term and no outcoming differentials from this term. We see that the canonical map $H^{n-1}_{Nis}(Y, \mathcal{H}^1_E(Y, \mathcal{F})) \to H^n_E(Y, \mathcal{F})$ is an isomorphism.

The assertion that $p^*_{X_x} : H^n_{D_x}(X_x, \mathcal{F}) \to H^n_{D_x \times \mathbb{A}^1}(X_x \times \mathbb{A}^1, \mathcal{F})$ is an isomorphism is local for the Zariski topology on X. Thus shrinking X, we may assume that there are Voevodsky's data $D, X = X_l, X_m, X_r = D \times \mathbb{A}^1$. Let $in : D \hookrightarrow X$ be the closed embedding and $I = in \times id_{\mathbb{A}^1}$. We have isomorphisms $I_*(\mathcal{F}_{-1}) \cong \mathcal{H}^1_{D \times \mathbb{A}^1}(X \times \mathbb{A}^1, \mathcal{F})$ and $in_*(\mathcal{F}_{-1}) \cong \mathcal{H}^1_D(X, \mathcal{F})$ of the form (44) on the small Nisnevich sites of $X \times \mathbb{A}^1$ and X respectively. They give rise to a commutative diagram

$$\begin{array}{ccccc} H^{n-1}_{Nis}(X, \mathcal{H}^1_D(X, \mathcal{F})) & \xrightarrow{\cong} & H^{n-1}_{Nis}(X, in_*(\mathcal{F}_{-1})) & \xrightarrow{\cong} & H^{n-1}_{Nis}(D, \mathcal{F}_{-1}) \\ \downarrow{\scriptstyle p^*_X} & & \downarrow{\scriptstyle p^*_X} & & \downarrow{\scriptstyle p^*_X} \\ H^{n-1}_{Nis}(X \times \mathbb{A}^1, \mathcal{H}^1_{D \times \mathbb{A}^1}(X \times \mathbb{A}^1, \mathcal{F})) & \xrightarrow{\cong} & H^{n-1}_{Nis}(X \times \mathbb{A}^1, I_*(\mathcal{F}_{-1})) & \xrightarrow{\cong} & H^{n-1}_{Nis}(D \times \mathbb{A}^1, \mathcal{F}_{-1}) \end{array}$$

The right vertical map is an isomorphism by the inductive assumption, and hence so is the map $p^*_X : H^{n-1}_{Nis}(X, \mathcal{H}^1_D(X, \mathcal{F})) \to H^{n-1}_{Nis}(X \times \mathbb{A}^1, \mathcal{H}^1_{D \times \mathbb{A}^1}(X \times \mathbb{A}^1, \mathcal{F}))$. As we have shown above the first of these groups is naturally isomorphic to $H^n_D(X, \mathcal{F})$. The second group is naturally isomorphic to $H^n_{D \times \mathbb{A}^1}(X \times \mathbb{A}^1, \mathcal{F})$. Thus the map $p^*_X : H^n_D(X, \mathcal{F}) \to H^n_{D \times \mathbb{A}^1}(X \times \mathbb{A}^1, \mathcal{F})$ is an isomorphism.

If $D, X = X_l, X_m, X_r = D \times \mathbb{A}^1$ are Voevodsky's data, then for any non-empty Zariski open X^0 in X and $D^0 := D \cap X^0$ the data $D^0, X^0, \pi_l^{-1}(X^0) \cap \pi_r^{-1}(D^0 \times \mathbb{A}^1), D^0 \times \mathbb{A}^1$ are Voevodsky's data as well. This observation and Proposition 17.2 imply the pullback map

$$p^*_{X_x} : H^n_{D_x}(X_x, \mathcal{F}) \to H^n_{D_x \times \mathbb{A}^1}(X_x \times \mathbb{A}^1, \mathcal{F})$$

is an isomorphism. Since $p^*_{X_x}$ is an isomorphism for any $X \in Sm/k$, any k-smooth divisor D in X and any point $x \in D$, we conclude that the pullback map $p^*_{X^h_x} : H^n_{D^h_x}(X^h_x, \mathcal{F}) \to H^n_{D^h_x \times \mathbb{A}^1}(X^h_x \times \mathbb{A}^1, \mathcal{F})$ is an isomorphism.

It remains to check that $H^n_{D^h_x}(X^h_x, \mathcal{F}) = 0$. Since $H^n_{Nis}(X^h_x, \mathcal{F}) = 0 = H^{n-1}_{Nis}(X^h_x, \mathcal{F})$, we see that $H^{n-1}_{Nis}(X^h_x - D^h_x, \mathcal{F}) = H^n_{D^h_x}(X^h_x, \mathcal{F})$. By induction, the presheaf $U \mapsto G(U) := H^{n-1}_{Nis}(U, \mathcal{F})$ is an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-presheaf. Since $n-1 > 0$ the associated Nisnevich sheaf G_{nis} vanishes. Thus, $G(X^h_x - D^h_x) = 0$ by Lemma 17.10. Hence $0 = G(X^h_x - D^h_x) = H^n_{D^h_x}(X^h_x, \mathcal{F})$. This completes the proof of the lemma. □

Returning to the proof of Theorem 17.16, let X be in Sm/k, $i : D \hookrightarrow X$ be a k-smooth divisor and $x \in D$ be a point. By Lemma 17.17 the map

$$H^n_{Nis}(X^h_x \times \mathbb{A}^1, \mathcal{F}) \to H^n_{Nis}((X^h_x - D^h_x) \times \mathbb{A}^1, \mathcal{F}) \tag{45}$$

is injective.

Next, we claim that for a k-smooth scheme X and the projection $p : X \times \mathbb{A}^1 \to X$ the Nisnevich sheaves $R^j p_*(\mathcal{F})$ vanish for $j = 1, ..., n-1$. In fact, such a sheaf is associated with the presheaf $U \mapsto H^j_{Nis}(U \times \mathbb{A}^1, \mathcal{F})$. The presheaf $U \mapsto H^j_{Nis}(U, \mathcal{F})$ is $\mathbb{A}^1$-invariant. Thus $H^j_{Nis}(U \times \mathbb{A}^1, \mathcal{F}) = H^j_{Nis}(U, \mathcal{F})$. Since $j \geqslant 1$ the associated Nisnevich sheaf vanishes. This proves the claim.

Since the Nisnevich sheaves $R^j p_*(\mathcal{F})$ vanish for $j = 1, ..., n-1$, one has an exact sequence

$$0 \to H^n_{Nis}(X, p_*(\mathcal{F})) \xrightarrow{\alpha} H^n_{Nis}(X \times \mathbb{A}^1, \mathcal{F}) \xrightarrow{\beta} H^0_{Nis}(X, R^n p_*(\mathcal{F})).$$

Set $A := Ker[i^*_0 : H^n_{Nis}(X \times \mathbb{A}^1, \mathcal{F}) \to H^n_{Nis}(X, \mathcal{F})]$. Arguing as in the proof of Proposition 17.14, we conclude that the map $\beta|_A : A \to H^0_{Nis}(X, R^n p_*(\mathcal{F}))$ is injective. Arguing again as in the proof of Proposition 17.14 and using the fact that the map (45) is injective, we get the following

Lemma 17.18. *Suppose the base field k is infinite and perfect with* $\operatorname{char} k \neq 2$*. Let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_*$-sheaf of Abelian groups. Let X be a k-smooth scheme and let $a \in H^n_{Nis}(X \times \mathbb{A}^1, \mathcal{F})$ be an element such that its restriction to $X \times \{0\}$ vanishes. Then $a = 0$. If* $\operatorname{char} k = 2$*, then the same statement is true if the $\mathbb{Z}F_*$-sheaf $\mathcal{F}$ is a sheaf of $\mathbb{Z}[1/2]$-modules.*

Finally, the pullback map $i^*_0 : H^n_{Nis}(X \times \mathbb{A}^1, \mathcal{F}) \to H^n_{Nis}(X, \mathcal{F})$ is surjective by functoriality. By Lemma 17.18 it is also injective. This completes the proof of Theorem 17.16. □

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