Cambridge Journal of Mathematics
Volume 8, Number 2, 407–452, 2020

$(1,1)$ forms with specified Lagrangian phase: a priori estimates and algebraic obstructions

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Let (X,α) be a Kähler manifold of dimension n, and let $[\omega] \in H^{1,1}(X,\mathbb{R})$. We study the problem of specifying the Lagrangian phase of ω with respect to α, which is described by the nonlinear elliptic equation

$$\sum_{i=1}^{n} \arctan(\lambda_i) = h(x)$$

where λ_i are the eigenvalues of ω with respect to α. When $h(x)$ is a topological constant, this equation corresponds to the deformed Hermitian-Yang-Mills (dHYM) equation, and is related by mirror symmetry to the existence of special Lagrangian submanifolds. We introduce a notion of subsolution for this equation, and prove a priori $C^{2,\beta}$ estimates when $|h| > (n-2)\frac{\pi}{2}$ and a subsolution exists. Using the method of continuity we show that the dHYM equation admits a smooth solution in the supercritical phase case, whenever a subsolution exists. Finally, we discover some Bridgeland-stability-type cohomological obstructions to the existence of solutions to the dHYM equation and we conjecture that when these obstructions vanish the dHYM equation admits a solution. We confirm this conjecture for complex surfaces.

AMS 2000 subject classifications: Primary 14J32, 53C07.

1. Introduction

Let (X,α) be a connected, compact Kähler manifold of complex dimension n, and let $\Omega \in H^{1,1}(X,\mathbb{R})$ be a cohomology class. Motivated by Mirror Symmetry, the second and third authors introduced the following problem

*The author is supported in part by NSF grant DMS-1506652, NSF grant DMS-1810924 and an Alfred P. Sloan fellowship.

†The author is supported in part by NSF grant DMS-1204155.

in [26]; does there exist a smooth, closed $(1,1)$ form ω, such that $[\omega] = \Omega$, and

$$\mathrm{Im}(\alpha + \sqrt{-1}\omega)^n = \tan(\hat{\Theta})\mathrm{Re}(\alpha + \sqrt{-1}\omega)^n, \tag{1.1}$$

where $\hat{\Theta}$ is an S^1 valued topological constant determined by $[\alpha], \Omega$? When $\Omega \in H^{1,1}(X, \mathbb{Z})$, this equation is known as the *deformed Hermitian-Yang-Mills* (dHYM) equation, and plays an important role in Mirror Symmetry. Written in terms of the eigenvalues of the relative endomorphism $\Lambda^j_k = \alpha^{j\bar{\ell}}\omega_{k\bar{\ell}}$, equation (1.1) can be written as [26]

$$\Theta_\alpha(\omega) := \sum_{i=1}^{n} \arctan(\lambda_i) = \hat{\Theta} \mod 2\pi. \tag{1.2}$$

Equation (1.2) is the natural generalization to compact Kähler manifolds of the special Lagrangian equation with potential introduced by Harvey-Lawson [24] and since studied extensively; see, for instance, [4, 38, 39, 51, 52, 53, 58, 59] and the references therein. The third author, with Leung and Zaslow [32], showed that when $\Omega = c_1(L)$ for a holomorphic line bundle $L \to X$, and X is a torus fibration over a torus, solutions of equation (1.2) are related via the Fourier-Mukai transform to special Lagrangian sections of the dual torus fibration. In their paper [26], the second and third authors initiated the study of (1.2) over a compact Kähler manifold, and using a parabolic flow they proved the existence of solutions when (X, α) has positive bisectional curvature, and the initial data is sufficiently positive.

A fundamental conjecture in mirror symmetry, dating back to work of Thomas [46] and Thomas-Yau [48] states that, for a Calabi-Yau manifold (X, ω), the class of a Lagrangian $[L]$ in the derived Fukaya category $D^\pi\mathrm{Fuk}(X, \omega)$ contains a special Lagrangian representative if and only if it is *stable* in an algebraic sense. The precise notion of stability has been updated since the original works [46, 48], and is now understood to be *Bridgeland stability*. We therefore state

Conjecture 1.1 (Folklore). *There is a Bridgeland stability condition on the derived Fukaya category such that the class of a Lagrangian contains a special Lagrangian representative if and only if it is stable.*

Joyce has made detailed conjectures in this direction concerning stability and the behavior of the Lagrangian mean curvature flow [27].

In this paper we study the *mirror* of Conjecture 1.1. As remarked above, at least for Lagrangian sections of the SYZ fibration, this is equivalent to understanding conditions for the existence of solutions to the dHYM. Our starting point, at least inspirationally, is the following simple observation; suppose $\Omega = c_1(L)$ is Kähler, and look for hermitian metrics h on L whose curvature form satisfies (1.1). We may also look for metrics h on L, so that the curvature form of h^k on $L^{\otimes k}$ satisfies equation (1.1) with $c_1(L)$ replaced by $kc_1(L)$. These two equations are *different*. It is therefore natural to ask for the limiting equation when $k \to \infty$. Multiplying both sides of (1.1) by k^{-n}, and taking a limit as $k \to \infty$ one easily obtains that the limiting equation is

$$c\omega^n = n\omega^{n-1} \wedge \alpha, \tag{1.3}$$

for $\omega \in c_1(L)$ with c a topological constant. Equation (1.3) is precisely the J-equation, discovered independently by Donaldson [16] and Chen [6, 7]. Let us briefly recall some of the important analytic and algebraic facts about the J-equation to serve as motivation for our work. Analytically, the solvability of the J-equation on general compact Kähler manifolds is well understood thanks to work of Song-Weinkove [42]. Building on previous work of Weinkove [55, 56], Song-Weinkove show that the existence of a solution to the J-equation is equivalent to the existence of a Kähler metric $\chi \in [\omega]$ with

$$c\chi^{n-1} - (n-1)\chi^{n-2} \wedge \alpha > 0 \tag{1.4}$$

in the sense of $(n-1, n-1)$ forms. Very recently, Székelyhidi [44] introduced a notion of subsolutions for a very general class of Hessian type equations on Hermitian manifolds, which encompasses (1.4), and showed that the existence of a subsolution implies a priori estimates to all orders.

The primary goal in this work is to begin building an analytic and algebraic framework for studying the existence problem for solutions of equation (1.2). As a first step, we study the specified Lagrangian phase equation;

$$\Theta_\alpha(\omega) := \sum_i \arctan(\lambda_i) = h(x). \tag{1.5}$$

Our first theorem is that, under the assumption of a subsolution, solutions of the specified Lagrangian phase equation with critical phase admit a priori estimates to all orders.

Theorem 1.2. *Fix $\omega_0 \in \Omega$. Let $u : X \to \mathbb{R}$ be a smooth function such that $\sup_X u = 0$ and $\Theta_\alpha(\omega_0 + \sqrt{-1}\partial\overline{\partial}u) = h(x)$, where $h : X \to [(n-2)\frac{\pi}{2} + \varepsilon_0, n\frac{\pi}{2})$.*

Suppose there exists a $\mathcal{C}$-subsolution $\underline{u} : X \to \mathbb{R}$ in the sense of Definition 3.2 (see also Lemma 3.3). Then for every $\beta \in (0, 1)$ we have an estimate

$$\|u\|_{C^{2,\beta}} \leqslant C(X, \alpha, \beta, h, \varepsilon_0, \omega_0, \underline{u}).$$

Our notion of a subsolution is certainly necessary for the existence of a solution to (1.5), and furthermore agrees with the notion of a $\mathcal{C}$-subsolution recently introduced by Székelyhidi [44]. The Lagrangian phase equation (1.5) fails several of the structural conditions imposed in [44] – most seriously, in general, the operator we study fails to be concave. The main difficulty in the proof of Theorem 1.2 is the C^2 estimate which is rather delicate owing to the lack of concavity. In the real case, a priori second order estimates for graphical solutions of the special Lagrangian equation with constant and critical phase are proved by Wang-Yuan [51]. By contrast, the complex setting studied here introduces several new negative terms into the estimate, which together with the non-constant phase, further complicate the analysis.

We apply these a priori estimates together with the method of continuity to prove an existence theorem for the deformed Hermitian-Yang-Mills equation. Before stating this result, we make two remarks about the topological constant $\hat{\Theta}$. First, by integrating equation (1.1), we see $\hat{\Theta}$ is the argument of the complex number

$$Z_{[\omega]} := \int_X \frac{(\alpha + \sqrt{-1}\omega)^n}{n!}, \tag{1.6}$$

which only depends on the classes $[\alpha], \Omega$. Thus a necessary condition for existence of a solution is that $Z_{[\omega]} \neq 0$. Second, because it is defined as the sum of arc-tangents, the angle $\Theta_\alpha(\omega)$ is real valued, while $\hat{\Theta}$ is valued in S^1. Thus we need to lift $\hat{\Theta}$ to $\mathbb{R}$ to study equation (1.2), which is the formulation of the deformed Hermitian-Yang-Mills we solve in this paper. Fortunately, existence of a subsolution which satisfies (1.7) allows us to specify a natural lift, and guarantees $Z_{[\omega]} \neq 0$, allowing us to prove the following:

Theorem 1.3. *Suppose that there exists a form $\chi := \omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{u} \in \Omega$ defining a subsolution in the sense of Definition 3.2 (see also Lemma 3.3). Furthermore, assume*

$$\Theta_\alpha(\chi) > (n-2)\frac{\pi}{2}. \tag{1.7}$$

Then there exists a unique smooth $(1, 1)$ form ω with $[\omega] = \Omega$ solving the *deformed Hermitian-Yang-Mills equation*

$$\Theta_\alpha(\omega) = \hat{\Theta}.$$

This result removes the hypercritical phase, non-negative sectional curvature, and large initial angle assumptions from [26]. We remark that if there exists a lifted angle such that $\hat{\Theta} \geqslant [(n-2)+\frac{2}{n}]\frac{\pi}{2}$, then any subsolution $\chi \in \Omega$ will automatically satisfy $\Theta_\alpha(\chi) > (n-2)\frac{\pi}{2}$. Thus in this case we do not need an analytic assumption on χ, only an assumption on the topological constant $\hat{\Theta}$. We expect this can be improved to when the average angle lifts and lies in $\hat{\Theta} \in ((n-2)\frac{\pi}{2}, ((n-2)+\frac{2}{n})\frac{\pi}{2})$. This expectation has been verified in dimension 2 [26, Theorem 1.2], and in dimension 3 [35], where it follows from work of Fang-Lai-Ma [21].

In the case of a domain in $\mathbb{C}^n$, we expect the natural extension of the subsolution condition considered here to be equivalent to the solvability of the boundary value problem, in analogy with the work of Guan-Li [23] on the inverse Hessian type equations. In the real setting, the Dirichlet problem posed by Harvey-Lawson [24] was solved by Caffarelli-Niremberg-Spruck [4] under some assumptions on the convexity of the boundary. It is interesting to note the similarities between these convexity conditions and the subsolution condition in Lemma 3.3.

Finally, we show that the existence of a subsolution imposes some cohomological restrictions on X. In particular, we prove the following simple

Proposition 1.4. *For every subvariety $V \subseteq X$, define*

$$Z_V := -\int_V e^{-\sqrt{-1}(\alpha+\sqrt{-1}\omega)}. \tag{1.8}$$

If there exists a solution to the deformed Hermitian-Yang-Mills equation (1.2), then for every proper subvariety $V \subset X$ we have

$$\mathrm{Im}\left(\frac{Z_V}{Z_X}\right) > 0.$$

This condition is a close analog of the stability condition for the J-equation recently discovered by Lejmi-Székelyhidi [31], and we expect the obstruction in Proposition 1.4 to arise from a suitable adaptation of the K-stability framework, a problem we plan to address in future work. In light of [31, Conjecture 1], and recent evidence for this conjecture by the first author and Székelyhidi [10] and Lejmi-Székelyhidi [31], it does not seem irresponsible to pose

Conjecture 1.5. *A solution of the deformed Hermitian-Yang-Mills equation* (1.2) *exists if and only if for every proper subvariety $V \subset X$ we have*

$$\mathrm{Im}\left(\frac{Z_V}{Z_X}\right) > 0.$$

in the notation of Proposition 1.4.

In Proposition 8.5 we show that this conjecture holds in dimension 2. Furthermore, we briefly discuss how the stability condition can be interpreted in terms of a central charge. In future work we hope to understand how Conjecture 1.5 fits into the Mirror Symmetry setting for special Lagrangians and the conjectural picture put forth by Thomas and the third author [48], and Thomas [46, 47]. Finally, we remark that there has recently been considerable interest in the analogy between the problem of finding special Lagrangians in a Calabi-Yau, and that of finding Kähler-Einstein or constant scalar curvature Kähler metrics as outlined by Solomon [40, 41], and studied in recent work of Rubinstein-Solomon [37].

The layout of this paper is as follows; in Section 2 we briefly discuss some background material, mostly taken from earlier work of the second and third authors [26]. In Section 3 we discuss the notion of a $\mathcal{C}$-subsolution, and extract the results from [44] which we will need. In Section 4 we prove an a priori C^2 estimate in terms of the gradient for solutions of the specified Lagrangian phase equation (1.5). This is the most difficult step in the proof of Theorem 1.2. In Section 5 we use a blow-up argument to prove an a priori gradient bound for solutions of (1.5), which implies a uniform C^2 estimates. In Section 6 we discuss the $C^{2,\beta}$ estimates, which follow from the usual Evans-Krylov estimate by a blow-up argument and a reduction to the real case. In Section 7 we take up the method of continuity and prove Theorem 1.3. This actually turns out to be slightly involved, as the natural method of continuity does not obviously preserve the critical phase condition, nor the existence of a subsolution. Instead we adapt a trick of Sun [43], and use a double method of continuity. The first continuity path is used to find a suitable starting point for the second method of continuity, whose ending point is the solution of the deformed Hermitian-Yang-Mills equation. In Section 8 we further discuss the implications of the existence of a subsolution for the deformed Hermitian-Yang-Mills equation, and deduce some algebraic obstructions to the existence of (1, 1) forms with constant Lagrangian phase. We prove Proposition 1.4, and give some evidence for Conjecture 1.5.

Acknowledgements

The first author is grateful to Gábor Székelyhidi for several helpful conversations. The authors would like to thank Valentino Tosatti, Ben Weinkove and the referee for helpful comments.

2. Background and notation

Let us briefly discuss our setup. Fix a compact Kähler manifold X with Kähler form α, and assume the normalization $\int_X \alpha^n = n!$. Throughout this paper, unless otherwise noted, the covariant derivative ∇ and all norms are computed with respect to α.

Fix a cohomology class $\Omega \in H^{1,1}(X,\mathbb{R})$. The deformed Hermitian-Yang-Mills equation seeks a $(1,1)$ form $\omega \in \Omega$ with the property that the map

$$X \ni x \longmapsto \frac{(\alpha + \sqrt{-1}\omega)^n}{\alpha^n} \in \mathbb{C}$$

lies in a fixed ray $\mathbb{R}_{>0}e^{\sqrt{-1}\hat{\Theta}}$. If a solution of this equation exists, then we necessarily have

$$\int_X (\alpha + \sqrt{-1}\omega)^n \in \mathbb{R}_{>0}e^{\sqrt{-1}\hat{\Theta}}$$

and hence $\hat{\Theta} = \mathrm{Arg}_{p.v} \int_X (\alpha + \sqrt{-1}\omega)^n \mod 2\pi$. As shown in [26], this problem is equivalent to both equations (1.1) and (1.2). We will primarily deal with the latter representation. As discussed in the introduction, it is also necessary to consider the specified Lagrangian phase equation for non-constant phase

$$\Theta_\alpha(\omega) := \sum_{i=1}^{n} \arctan(\lambda_i) = h(x),$$

where again λ_i are the eigenvalues of $\alpha^{-1}\omega$.

It is useful to introduce another Hermitian metric on $T^{1,0}(X)$, defined by the formula $\eta_{j\bar{k}} = \alpha_{j\bar{k}} + \omega_{j\bar{\ell}}\alpha^{p\bar{\ell}}\omega_{p\bar{k}}$. Note this metric is never Kähler. With this definition, following [26] one can compute the variation of Θ_α as

$$\delta\Theta_\alpha = \eta^{j\bar{k}}\alpha_{\ell\bar{k}}\delta(\alpha^{\ell\bar{m}}\omega_{j\bar{m}}). \tag{2.1}$$

This computation has two important consequences. First, using the covariant derivative ∇ with respect to α, one sees that $d\Theta = \eta^{j\bar{k}}\nabla\omega_{j\bar{k}}$. Furthermore, since we consider variations of ω which fix α, the linearization of the operator $\Theta_\alpha(\omega)$ is given by

$$\Delta_\eta = \eta^{j\bar{k}}\partial_j\partial_{\bar{k}}. \tag{2.2}$$

It is easy to check that this operator becomes uniformly elliptic as soon as $|\omega|$ is bounded. At a point x_0 in coordinates where $\alpha(x_0)$ is the identity and $\omega(x_0)$ is diagonal with entries λ_i, then the metric $\eta_{j\bar{k}}$ is diagonal with entries

$$\eta_{i\bar{i}}(x_0) = (1+\lambda_i^2)\delta_{i\bar{i}}.$$

We conclude this section by restating the observation that the cohomological data

$$\int_X (\alpha + \sqrt{-1}\omega)^n$$

only provides an S^1 valued target angle $\Theta_\alpha(\omega) \in \mathbb{R}/\mathbb{Z}$. On the other hand, the Lagrangian phase operator, defined as the sum of arc-tangents, is naturally $\mathbb{R}$-valued. Therefore in order to study (1.2), a lift of $\hat{\Theta}$ needs to be chosen. The first observation is that if χ' is any $(1,1)$ form with

$$\mathrm{osc}_X \Theta_\alpha(\chi') < \pi \tag{2.3}$$

then there is a lift of $\hat{\Theta}$ which is in the image of $\Theta_\alpha(\chi') : X \to \mathbb{R}$. Furthermore, an easy application of the maximum principle shows that any $(1,1)$ form satisfying (2.3) must give rise to the same lifted angle. Fortunately, our assumption (1.7) implies (2.3) and hence we obtain *unique* lift of $\hat{\Theta}$ to the branch $((n-2)\frac{\pi}{2}, n\frac{\pi}{2})$. Following [26] and [51], we say that an angle is *supercritical* if it is larger than $(n-2)\frac{\pi}{2}$, and *hypercritical* if it is larger than $(n-1)\frac{\pi}{2}$. For further discussion and background we refer to reader to [26].

3. Subsolutions and the C^0 estimate

In order to introduce the notion of subsolution for the Lagrangian phase equation, we first define the relevant cone in which our solutions takes values. Let $\Gamma_n \subset \mathbb{R}^n$ denote the positive orthant. Recall that (X, α) is a fixed Kähler manifold, and ω_0 is a fixed (1,1) form. In this paper we are interested in finding forms ω such that $[\omega] = [\omega_0]$, and

$$\Theta_\alpha(\omega) := \sum_{\ell=1}^{n} \arctan(\lambda_\ell) = h(x) \tag{3.1}$$

where λ_ℓ are the eigenvalues of the hermitian endomorphism $\Lambda^i_k := \alpha^{i\bar{j}}\omega_{k\bar{j}}$, and $h : X \to ((n-2)\frac{\pi}{2}, n\frac{\pi}{2})$ is a smooth function. We call this the Lagrangian phase equation with *supercritical phase*. To lighten notation, let us define $\Theta : \mathbb{R}^n \to \mathbb{R}$ to be

$$\Theta(x_1, \dots, x_n) = \sum_{i=1}^{n} \arctan(x_i).$$

Let $\Gamma \subset \mathbb{R}^n$ be the cone through the origin over the region

$$\left\{(x_1, \dots, x_n) : \Theta(x) \geqslant (n-2)\frac{\pi}{2}\right\}.$$

Γ is an open, symmetric cone with vertex at the origin containing Γ_n. Additionally, for any $\sigma \in ((n-2)\frac{\pi}{2}, n\frac{\pi}{2})$ we define

$$\Gamma^\sigma := \{\lambda \in \Gamma : \Theta(\lambda) > \sigma\}. \tag{3.2}$$

Note that for any σ such that Γ^σ is not empty, the boundary $\partial\Gamma^\sigma$ is a smooth hypersurface. The geometric and arithmetic properties of the cone Γ, and the sets Γ^σ will play a crucial role in the developments to follow.

Lemma 3.1. *Suppose we have real numbers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ which satisfy $\Theta(\lambda) = \sigma$, for $\sigma \geqslant (n-2)\frac{\pi}{2}$. Then $(\lambda_1, \dots, \lambda_n)$ have the following arithmetic properties;*

(i) $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1} > 0$ *and* $\lambda_{n-1} \geqslant |\lambda_n|$.
(ii) $\lambda_1 + (n-1)\lambda_n \geqslant 0$.
(iii) $\sigma_k(\lambda_1, \dots, \lambda_n) \geqslant 0$ *for all* $1 \leqslant k \leqslant n-1$.

Furthermore,

(iv) If Γ^σ is not empty, the boundary $\partial\Gamma^\sigma$ is a smooth, convex hypersurface.

In addition, if $\sigma \geqslant (n-2)\frac{\pi}{2} + \beta$, then;

(v) if $\lambda_n \leqslant 0$, then $\lambda_{n-1} \geqslant \varepsilon_0(\beta)$.
(vi) $|\lambda_n| \leqslant C(\beta)$.

Proof. Statements *(i)-(iii)* are due to Wang-Yuan [51, Lemma 2.1]. Statement *(iv)* is Yuan [59, Lemma 2.1], while *(v)* and *(vi)* are trivial. □

In particular, it follows from part *(i)* of the above lemma that

$$\Gamma \subset \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \sum_i \lambda_i > 0\}.$$

We recall the definition of a $\mathcal{C}$-subsolution, due to Székelyhidi [44].

Definition 3.2 ([44], Definition 1)**.** *Fix $\omega_0 \in \Omega$. We say that a smooth function $\underline{u} : X \to \mathbb{R}$ is a $\mathcal{C}$-subsolution of (3.1) if the following holds: At each point $x \in X$ define the matrix $\Lambda^i_j := \alpha^{i\bar{k}}(\omega_0 + \sqrt{-1}\partial\overline{\partial}\underline{u})_{j\bar{k}}$. Then we require that the set*

$$\left\{\lambda' \in \Gamma : \sum_{\ell=1}^{n} \arctan(\lambda'_\ell) = h(x), \quad \text{and} \quad \lambda' - \lambda(\Lambda(x)) \in \Gamma_n\right\} \tag{3.3}$$

is bounded, where $\lambda(\Lambda(x))$ denotes the n-tuple of eigenvalues of $\Lambda(x)$.

In the present setting we have the following explicit description of the $\mathcal{C}$-subsolutions.

Lemma 3.3. *A smooth function $\underline{u} : X \to \mathbb{R}$ is a $\mathcal{C}$-subsolution of (3.1) if and only if at each point $x \in X$, if $\mu_1, \dots, \mu_n$ denote the eigenvalues of the Hermitian endomorphism $\Lambda^i_j := \alpha^{i\bar{k}}(\omega_0 + \sqrt{-1}\partial\overline{\partial}\underline{u})_{j\bar{k}}$, then, for all $j = 1, \dots, n$ we have*

$$\sum_{\ell \neq j} \arctan(\mu_\ell) > h(x) - \frac{\pi}{2}. \tag{3.4}$$

Proof. We show that if $\underline{u}$ satisfies (3.4), then it is a $\mathcal{C}$-subsolution. Fix a point $x_0 \in X$, and suppose we have numbers $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n$ such that

$$\sum_i \arctan \lambda_i = h(x_0).$$

It suffices to show that if $\lambda_i \geqslant \mu_i$ for all i, then $\lambda_1 \leqslant C$. Fix $\delta > 0$ such that

$$\sum_{\ell \neq j} \arctan(\mu_\ell) > h(x_0) + \delta - \frac{\pi}{2},$$

and suppose we can find an n-tuple $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n$ as above such that $\arctan \lambda_1 \geqslant \pi/2 - \delta$. Then we clearly have

$$\sum_{i \neq 1} \arctan(\lambda_i) \leqslant h(x_0) + \delta - \frac{\pi}{2} < \sum_{i \neq 1} \arctan(\mu_i).$$

Since $\arctan(\cdot)$ is monotone increasing, we must have that $\mu_j > \lambda_j$ for some j, but this is a contradiction to the assumption that $\lambda_i \geqslant \mu_i$ for all i. The proof of the reverse implication is similar. $\square$

Throughout this paper we will be somewhat abusive in referring to the $(1,1)$ form $\chi := \omega_0 + \sqrt{-1}\partial\overline{\partial}\underline{u}$ as a subsolution. We hope that no confusion will result.

Remark 3.4. *The condition in Lemma 3.3 can be expressed in terms of the positivity of a certain $(n-1, n-1)$ form, which is similar in spirit to the subsolution condition discovered by Song-Weinkove [42] in the setting of the J-flow. We will discuss this fact, as well as some consequences in section 8; see Proposition 8.1 below.*

The following proposition is due to Székelyhidi [44], refining previous work of Guan [22]. This proposition play a fundamental role in proving the C^2 bound for our equation, which we demonstrate in the next section.

Proposition 3.5 ([44], Proposition 6)**.** *Let $[a,b] \subset ((n-2)\frac{\pi}{2}, n\frac{\pi}{2})$ and $\delta, R > 0$. There exists $\kappa > 0$, with the following property: Suppose that $\sigma \in [a,b]$ and B is a hermitian matrix such that*

$$(\lambda(B) - 2\delta Id + \Gamma_n) \cap \partial\Gamma^{\sigma} \subset B_R(0).$$

Then for any hermitian matrix A with $\lambda(A) \in \partial\Gamma^{\sigma}$ and $|\lambda(A)| > R$ we either have

$$\sum_{p,q} \eta^{p\bar{q}}(A)[B_{p\bar{q}} - A_{p\bar{q}}] > \kappa \sum_{p} \eta^{p\bar{p}}(A)$$

or $\eta^{i\bar{i}}(A) > \kappa \sum_p \eta^{p\bar{p}}(A)$ for all i, where $\eta = Id + A^2$.

Proof. Since $\sigma > (n-2)\frac{\pi}{2}$, Lemma 3.1 part *(iv)* implies that $\partial\Gamma^{\sigma}$ is a convex hypersurface. With this observation, the proof in [44] goes through verbatim. $\square$

The following estimate, based on the Alexandroff-Bakelman-Pucci maximum principle, is due to Székelyhidi [44]. Błocki [1] first applied the ABP estimate to the complex Monge-Ampère equation on Kähler manifolds following earlier suggestions by Cheng and the third author. While the operator under consideration here does not have the structural properties imposed in [44], it is straightforward to check that the proof requires only the ellipticity of the operator, and hence applies verbatim here.

Proposition 3.6 ([44], Proposition 10)**.** *Suppose that $\Theta_{\alpha}(\omega_0 + \sqrt{-1}\partial\overline{\partial}u) = h(x)$, where $h : X \to [(n-2)\frac{\pi}{2}, n\frac{\pi}{2})$, and suppose that $\underline{u} = 0$ is a $\mathcal{C}$-subsolution. Then there exists a constant C, depending only on the given data, including ω_0, such that*

$$osc_X u \leqslant C.$$

When a $\mathcal{C}$-subsolution $\underline{u}$ exists, we will denote by $\chi := \omega_0 + \sqrt{-1}\partial\overline{\partial}\underline{u} \in \Omega$ the corresponding $(1,1)$ form.

4. The C^2 estimate

The main result of this section is

Theorem 4.1. *Suppose $u : X \to \mathbb{R}$ is a smooth function solving the equation*

$$\Theta_\alpha(\chi + \sqrt{-1}\partial\overline{\partial}u) = h(x) \tag{4.1}$$

where $h : X \to [(n-2)\frac{\pi}{2} + \beta, n\frac{\pi}{2})$ for some $\beta > 0$. Then there exists a constant C depending only on the subsolution χ, as well as $|h|_{C^2(X,\alpha)}$, $\mathrm{osc}_X u, \alpha, \beta$, such that

$$|\partial\overline{\partial}u| \leqslant C\left(1 + \sup_X |\nabla u|^2\right).$$

Proof. The proof is via the maximum principle. Let $\omega := \chi + \sqrt{-1}\partial\overline{\partial}u$. We begin by defining functions $\varphi(t), \psi(t)$ as follows. Let $K = 1 + \sup_X |\nabla u|^2$, and set

$$\varphi(t) = -\frac{1}{2}\log(1 - \frac{t}{2K}), \qquad t \in [0, K-1].$$

Note that $\varphi(t)$ satisfies

$$(4K)^{-1} < \varphi' < (2K)^{-1}, \qquad \varphi'' = 2(\varphi')^2, \qquad 0 \leqslant \varphi(t) \leqslant \frac{1}{2}\log 2$$

Normalize u so that $\inf_X u = 0$. By Proposition 3.6 we have a bound on $\sup_X u$. Define $\psi : [0, \sup_X u] \to \mathbb{R}$ by

$$\psi(t) = -2At + \frac{A\tau}{2}t^2$$

where $A \gg 0$ and $\tau > 0$ are constants to be determined. We choose τ sufficiently small so that

$$A \leqslant -\psi' \leqslant 2A, \quad \psi'' = A\tau$$

Define the Hermitian endomorphisms

$$\Lambda := \alpha^{i\bar{j}}(\chi + \sqrt{-1}\partial\overline{\partial}u)_{k\bar{j}}, \qquad \Lambda_0 := \alpha^{i\bar{j}}\chi_{k\bar{j}}$$

and recall that we are assuming χ is a $\mathcal{C}$-subsolution. Let λ_{max} denote the largest eigenvalue of Λ, which is a continuous function from X to $\mathbb{R}$. We want to apply the maximum principle to the quantity

$$G_0(x) := \frac{1}{2}\log(1+\lambda_{max}^2) + \varphi(|\nabla u|^2) + \psi(u).$$

This quantity is inspired by the one considered by Hou-Ma-Wu [25] for the complex Hessian equations and subsequently used by Székelyhidi [44] for a large class of concave equations. The gradient term used appearing in G_0 was first used by Chou-Wang [8] in their study of the real Hessian equations. The function G_0 differs from the one considered in [25, 44] in its highest order term, where we have used a function of the eigenvalues which first appeared in the study of the real special Lagrangian equation in work of Wang-Yuan [51]. This modification is not merely cosmetic – the added convexity of this higher order term appears essential to the estimate. Finally, we note that, unlike the estimates in the real case, we require extra lower order terms in order to counter additional negative terms which appear when differentiating the eigenvalues of a Hermitian (rather than symmetric) matrix.

The function $G_0(x)$ is clearly continuous, and hence achieves its maximum at some point $x_0 \in X$. Fix local coordinates $(z_1, \ldots, z_n)$ centered at x_0 which are normal for the background Kähler metric α, and such that $\omega(x_0)$ is diagonal with entries $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. By Lemma 3.1 we may assume that λ_1 is sufficiently large so that $\lambda_1 > \max\{2|\lambda_n|, |\lambda_n|+1\}$. Since χ is a subsolution, we can find $\delta, R > 0$ depending only on h, ω_0 such that

$$[\lambda(\Lambda_0(x_0)) - 2\delta Id + \Gamma_n] \cap \partial\Gamma^{h(x_0)} \subset B_R(0).$$

We may assume that $|\lambda(\Lambda(x_0))| > R$, so that Proposition 3.5 applies. In particular, there exists $\kappa > 0$ depending only on δ, R and h such that either

$$\sum_p \frac{\chi_{p\bar{p}} - \lambda_p}{1+\lambda_p^2} > \kappa \sum_p \frac{1}{1+\lambda_p^2}. \tag{4.2}$$

or $(1+\lambda_i^2)^{-1} > \kappa \sum_p (1+\lambda_p^2)^{-1}$ for all i. Since λ_n is uniformly bounded by Lemma 3.1 part *(vi)*, we may assume that λ_1 is sufficiently large so that

$$\frac{1}{1+\lambda_1^2} \leqslant \kappa \frac{1}{1+\lambda_n^2}.$$

In particular, (4.2) must hold.

In order to apply the maximum principle, we must differentiate the function G_0 twice. Since the eigenvalues of Λ need not be distinct at x_0, the function G_0 may only be continuous. To circumvent this difficulty we use a perturbation argument similar to the one used in [44]. We choose a constant matrix B, defined in our fixed local coordinates to be a constant diagonal matrix B_{pq} with real entries satisfying $B_{11} = B_{nn} = 0$ and $0 < B_{22} < \cdots < B_{n-1\,n-1}$, and such that

$$\sum_j B_{jj} \leqslant (n-1)\frac{\varepsilon_0}{2}$$

where ε_0 is the constant from Lemma 3.1 part *(v)*. We work with the matrix $\tilde{\Lambda} = \Lambda - B$, and apply the maximum principle to the smooth function

$$G(x) = \frac{1}{2}\log(1+\tilde{\lambda}_{max}^2) + \varphi(|\nabla u|^2) + \psi(u)$$

where $\tilde{\lambda}_{max}$ denotes the largest eigenvalue of $\tilde{\Lambda}$. Note that $G(x) \leqslant G_0(x)$ and that $G(x)$ achieves its maximum at x_0, where we have $G(x_0) = G_0(x_0)$. If we denote by $\tilde{\lambda}_i$ are the eigenvalues of $\tilde{\Lambda}$, then $\tilde{\lambda}_1 = \lambda_1$, and all the remaining eigenvalues are distinct from $\tilde{\lambda}_1$. In particular, $\tilde{\lambda}_1$ is a smooth function near x_0 and we may differentiate it freely. Computing derivatives of $\tilde{\lambda}_1$ yields

$$\begin{aligned}
\nabla_s\tilde{\lambda}_1 &= \nabla_s\omega_{1\bar{1}} - \nabla_s B_{11}\\
\nabla_s\nabla_{\bar{s}}\tilde{\lambda}_1 &= \nabla_s\nabla_{\bar{s}}\omega_{1\bar{1}} + \sum_{q>1}\frac{|\nabla_s\omega_{q\bar{1}}|^2 + |\nabla_s\omega_{1\bar{q}}|^2}{(\lambda_1 - \tilde{\lambda}_q)}\\
&\quad + \nabla_s\nabla_{\bar{s}}B_{11} - 2\mathrm{Re}\sum_{q>1}\frac{\nabla_s\omega_{q\bar{1}}\nabla_{\bar{s}}B_{1\bar{q}} + \nabla_s\omega_{1\bar{q}}\nabla_{\bar{s}}B_{p\bar{1}}}{\lambda_1 - \tilde{\lambda}_p}\\
&\quad + \tilde{\lambda}_1^{pq,r\ell}\nabla_s(B_{pq})\nabla_{\bar{s}}(B_{r\ell}),
\end{aligned}\tag{4.3}$$

where

$$\tilde{\lambda}_1^{pq,r\ell} = (1-\delta_{1p})\frac{\delta_{1q}\delta_{1r}\delta_{p\ell}}{\tilde{\lambda}_1 - \tilde{\lambda}_p} + (1-\delta_{1r})\frac{\delta_{1\ell}\delta_{1p}\delta_{rq}}{\tilde{\lambda}_1 - \tilde{\lambda}_r}$$

see, for example, [44, Equation (70)] or [45]. Evaluating this expression at $x_0 \in X$, and using that B is constant, $B_{11} = 0$, and that we are working in normal coordinates for α, we have

$$\begin{aligned}
\nabla_s\tilde{\lambda}_1 &= \nabla_s\omega_{1\bar{1}}\\
\nabla_s\nabla_{\bar{s}}\tilde{\lambda}_1 &= \nabla_s\nabla_{\bar{s}}\omega_{1\bar{1}} + \sum_{q>1}\frac{|\nabla_s\omega_{q\bar{1}}|^2 + |\nabla_s\omega_{1\bar{q}}|^2}{(\lambda_1 - \tilde{\lambda}_q)}
\end{aligned}\tag{4.4}$$

We are thus reduced to differentiating equation (4.1). Using (2.1), and computing at x_0, we have

$$\begin{aligned}\nabla_{\bar b}\nabla_a h &= \nabla_{\bar b}(\eta^{s\bar q}\nabla_a\omega_{s\bar q}) \\ &= \eta^{s\bar q}\nabla_{\bar b}\nabla_a\omega_{s\bar q} + (\nabla_{\bar b}\eta^{s\bar q})\nabla_a\omega_{s\bar q} \\ &= \eta^{s\bar q}\nabla_s\nabla_{\bar q}\omega_{a\bar b} + \eta^{s\bar q}[\nabla_{\bar b},\nabla_s]\omega_{a\bar q} + (\nabla_{\bar b}\eta^{s\bar q})\nabla_a\omega_{s\bar q}.\end{aligned} \tag{4.5}$$

Expanding the third term, we have

$$\begin{aligned}(\nabla_{\bar b}\eta^{s\bar q}) &= -\eta^{s\bar k}\eta^{j\bar q}\nabla_{\bar b}\eta_{j\bar k} \\ &= -\eta^{s\bar k}\eta^{j\bar q}\left(\alpha^{p\bar m}\omega_{p\bar k}\nabla_{\bar b}\omega_{j\bar m} + \alpha^{p\bar m}\omega_{j\bar m}\nabla_{\bar b}\omega_{p\bar k}\right),\end{aligned} \tag{4.6}$$

Using that α,ω are diagonal at x_0, we can now solve for $\Delta_\eta\omega_{1\bar 1}$,

$$\begin{aligned}\eta^{s\bar q}\nabla_s\nabla_{\bar q}\omega_{1\bar 1} &= \nabla_1\nabla_{\bar 1}h - \sum_s \frac{\lambda_s R_1{}^s{}_{s\bar 1}}{1+\lambda_s^2} \\ &\quad + \sum_s \frac{\lambda_1 R^{\bar 1}{}_{\bar s s\bar 1}}{1+\lambda_s^2} + \sum_{s,q}\frac{\lambda_s+\lambda_q}{(1+\lambda_s^2)(1+\lambda_q^2)}|\nabla_1\omega_{q\bar s}|^2.\end{aligned} \tag{4.7}$$

Combining this expression with (4.4) allows us to solve for $\Delta_\eta\tilde\lambda_1$. This allows us to compute the Laplacian of the highest order term from $G(x)$ at the point x_0

$$\begin{aligned}\Delta_\eta\frac{1}{2}\log(1+\tilde\lambda_1^2) &= \frac{\lambda_1\Delta_\eta\tilde\lambda_1}{(1+\lambda_1^2)} + \frac{1-\lambda_1^2}{(1+\lambda_1^2)^3}|\nabla_1\omega_{1\bar 1}|^2 \\ &\quad + \sum_{s>1}\frac{1-\lambda_1^2}{(1+\lambda_1^2)^2(1+\lambda_s^2)}|\nabla_s\omega_{1\bar 1}|^2.\end{aligned}$$

The Laplacian term can be computed as

$$\begin{aligned}\Delta_\eta\tilde\lambda_1 &= \nabla_1\nabla_{\bar 1}h - \sum_s\frac{\lambda_s R_1{}^s{}_{s\bar 1}}{1+\lambda_s^2} + \sum_s\frac{\lambda_1 R^{\bar 1}{}_{\bar s s\bar 1}}{1+\lambda_s^2} \\ &\quad + \sum_{s,q}\frac{\lambda_s+\lambda_q}{(1+\lambda_s^2)(1+\lambda_q^2)}|\nabla_1\omega_{q\bar s}|^2 + \sum_s\sum_{q>1}\frac{|\nabla_s\omega_{q\bar 1}|^2+|\nabla_s\omega_{1\bar q}|^2}{(1+\lambda_s^2)(\lambda_1-\tilde\lambda_q)}.\end{aligned}$$

After some algebra we arrive at the formula

(4.8)
$$\begin{aligned}\Delta_\eta \frac{1}{2}\log(1+\tilde{\lambda}_1^2) &= \frac{\lambda_1}{1+\lambda_1^2}\nabla_1\nabla_{\bar{1}}h + \sum_s \frac{-\lambda_1\lambda_s R_1{}^s{}_{s\bar{1}} + \lambda_1^2 R^{\bar{1}}{}_{\bar{s}s\bar{1}}}{(1+\lambda_1^2)(1+\lambda_s^2)} \\ &+ \sum_s\sum_{q>1} \frac{\lambda_1[1+\lambda_1(\lambda_s+\lambda_q) - \lambda_q\lambda_s + (\lambda_s+\lambda_q)(\lambda_q-\tilde{\lambda}_q)]}{(1+\lambda_1^2)(1+\lambda_s^2)(1+\lambda_q^2)(\lambda_1-\tilde{\lambda}_q)}|\nabla_1\omega_{s\bar{q}}|^2 \\ &+ \frac{1}{(1+\lambda_1^2)^2}|\nabla_1\omega_{1\bar{1}}|^2 + \sum_{s>1}\frac{\lambda_1^2\lambda_s + 2\lambda_1 - \tilde{\lambda}_s + \lambda_1\lambda_s(\lambda_s-\tilde{\lambda}_s)}{(1+\lambda_1^2)^2(1+\lambda_s^2)(\lambda_1-\tilde{\lambda}_s)}|\nabla_s\omega_{1\bar{1}}|^2 \\ &+ \sum_{s,q>1}\frac{\lambda_1}{(1+\lambda_1^2)(1+\lambda_s^2)(\lambda_1-\tilde{\lambda}_q)}|\nabla_q\omega_{s\bar{1}}|^2.\end{aligned}$$

The main difficulty is finding a useful estimate for this quantity. For the remainder of this section we let C denote a constant depending only on the stated data, but which may change from line to line. The first two terms contribute only a negative constant. For the third term, we require the following simple lemma

Lemma 4.2. *If $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, and these numbers satisfy $\Theta(\lambda) \geqslant (n-2)\frac{\pi}{2}$, then*

$$1+\lambda_1(\lambda_j+\lambda_\ell) - \lambda_j\lambda_\ell \geqslant 0$$

unless $j=\ell=n$ and $\lambda_n<0$.

Proof. The lemma is obvious if $\lambda_j, \lambda_\ell \geqslant 0$, since $\lambda_1 \geqslant \max\{\lambda_j,\lambda_\ell\}$. By symmetry we can consider the case when $j=n, \lambda_n<0$, and $\ell<n$. In this case Lemma 3.1 part *(i)* guarantees that $\lambda_\ell+\lambda_n \geqslant 0$, and so again we are done, since the final term above is positive. ☐

The fourth term in (4.8) is positive, as is the fifth term, unless $s=n$ and $\lambda_n<0$. The sixth term is also clearly positive. Thus, if $\lambda_n \geqslant 0$, then

$$\Delta_\eta \frac{1}{2}\log(1+\tilde{\lambda}_1^2) \geqslant -C.$$

If $\lambda_n<0$ then the estimate is much worse, due to the presence of several negative terms. Throwing away some but not all of the positive terms, we rewrite (4.8) as

$$
\begin{aligned}
\Delta_\eta \frac{1}{2}\log(1+\tilde{\lambda}_1^2) \geqslant -C &+ \frac{1}{(1+\lambda_1^2)^2}|\nabla_1 \omega_{1\bar{1}}|^2 \\
&+ \sum_{q>1} \frac{\lambda_1[1+2\lambda_1\lambda_q - \lambda_q^2 + 2\lambda_q(\lambda_q - \tilde{\lambda}_q)]}{(1+\lambda_1^2)(1+\lambda_q^2)^2(\lambda_1 - \tilde{\lambda}_q)}|\nabla_1 \omega_{q\bar{q}}|^2 \\
&+ \sum_{q>1} \frac{\lambda_1^2\lambda_q + 2\lambda_1 - \tilde{\lambda}_q + \lambda_1\lambda_q(\lambda_q - \tilde{\lambda}_q)}{(1+\lambda_1^2)^2(1+\lambda_q^2)(\lambda_1 - \tilde{\lambda}_q)}|\nabla_q \omega_{1\bar{1}}|^2.
\end{aligned}
\tag{4.9}
$$

Let us analyze this more difficult case. We first estimate the second line above. Note that we can write

$$
\begin{aligned}
&\frac{\lambda_1[1+2\lambda_1\lambda_q - \lambda_q^2 + 2\lambda_q(\lambda_q - \tilde{\lambda}_q)]}{(1+\lambda_1^2)(1+\lambda_q^2)^2(\lambda_1 - \tilde{\lambda}_q)} \\
&\quad = \frac{\lambda_q}{(1+\lambda_q^2)^2(\lambda_1 - \tilde{\lambda}_q)} \\
&\qquad + \frac{(\lambda_1 - \lambda_q)(1+\lambda_1\lambda_q) + 2\lambda_q\lambda_1(\lambda_q - \tilde{\lambda}_q)}{(1+\lambda_1^2)(1+\lambda_q^2)^2(\lambda_1 - \tilde{\lambda}_q)} \\
&\quad = \frac{\lambda_q}{(1+\lambda_q^2)^2(\lambda_1 - \tilde{\lambda}_q)} + \frac{1+\lambda_1\lambda_q}{(1+\lambda_1^2)(1+\lambda_q^2)^2} \\
&\qquad + \frac{(\lambda_q\lambda_1 - 1)(\lambda_q - \tilde{\lambda}_q)}{(1+\lambda_1^2)(1+\lambda_q^2)^2(\lambda_1 - \tilde{\lambda}_q)},
\end{aligned}
$$

and so we can rewrite the first and second lines in (4.9) (excluding the constant) as three separate sums

$$
\begin{aligned}
\mathbf{(I)} &= \sum_{q>1} \frac{\lambda_q}{(1+\lambda_q^2)^2(\lambda_1 - \tilde{\lambda}_q)}|\nabla_1 \omega_{q\bar{q}}|^2 + \frac{1}{(1+\lambda_1^2)^2}|\nabla_1 \omega_{1\bar{1}}|^2 \\
\mathbf{(II)} &= \sum_{q} \frac{1+\lambda_1\lambda_q}{(1+\lambda_1^2)(1+\lambda_q^2)^2}|\nabla_1 \omega_{q\bar{q}}|^2 - \frac{1}{(1+\lambda_1^2)^2}|\nabla_1 \omega_{1\bar{1}}|^2 \\
\mathbf{(III)} &= \sum_{1<q<n} \frac{(\lambda_q\lambda_1 - 1)(\lambda_q - \tilde{\lambda}_q)}{(1+\lambda_1^2)(1+\lambda_q^2)^2(\lambda_1 - \tilde{\lambda}_q)}|\nabla_1 \omega_{q\bar{q}}|^2
\end{aligned}
\tag{4.10}
$$

where we have used that $B_{nn} = 0$. We may assume that λ_1 is sufficiently large so that for $q < n$ we have $\lambda_1\lambda_q \geqslant 1$, since $\lambda_{n-1} \geqslant \varepsilon_0$ by Lemma 3.1

part *(v)*. In particular, the third sum is positive. We next consider terms (**I**) and (**II**) individually, beginning with term (**I**). The only negative contribution to the sum occurs when $q = n$. Differentiating our main equation (4.1), we have, for any $\delta, \alpha_j > 0$, $j = 1, \dots, n-1$

$$\begin{aligned}
\frac{|\nabla_1 \omega_{n\bar{n}}|^2}{(1+\lambda_n^2)^2} &= \left|\nabla_1 h - \sum_{q<n} \frac{\nabla_1 \omega_{q\bar{q}}}{1+\lambda_q^2}\right|^2 \\
&\leqslant (1+\frac{\lambda_1}{\delta})|\nabla_1 h|^2 + (1+\frac{\delta}{\lambda_1})\left|\sum_{q<n} \frac{\nabla_1 \omega_{q\bar{q}}}{1+\lambda_q^2}\right|^2 \\
&\leqslant (1+\frac{\lambda_1}{\delta})|\nabla_1 h|^2 + (1+\frac{\delta}{\lambda_1})\left(\sum_{q<n} \frac{|\nabla_1 \omega_{q\bar{q}}|^2 \alpha_q}{(1+\lambda_q^2)^2}\right) \cdot \left(\sum_{j<n} \frac{1}{\alpha_j}\right).
\end{aligned} \tag{4.11}$$

In the above we have used Young's inequality for the first line, and the Cauchy-Schwartz inequality in the third line. Now, set $\alpha_q = \frac{\lambda_q}{\lambda_1 - \tilde{\lambda}_q}$ for $1 < q < n$, and $\alpha_1 = 1$, and choose $\delta = \varepsilon_0/2$, where ε_0 is as in Lemma 3.1 part *(v)*. Let us denote

$$\begin{aligned}
\Upsilon &:= \left(\sum_{q<n} \frac{|\nabla_1 \omega_{q\bar{q}}|^2 \alpha_q}{(1+\lambda_q^2)^2}\right) \\
&= \sum_{1<q<n} \frac{\lambda_q}{(1+\lambda_q^2)^2(\lambda_1 - \tilde{\lambda}_q)} |\nabla_1 \omega_{q\bar{q}}|^2 + \frac{|\nabla_1 \omega_{1\bar{1}}|^2}{(1+\lambda_1^2)^2}.
\end{aligned}$$

Multiplying (4.11) by $\frac{\lambda_n}{(\lambda_1 - \lambda_n)}$ and observing that $\frac{\lambda_n(\delta+\lambda_1)|\nabla_1 h|^2}{\delta(\lambda_1 - \lambda_n)} \geqslant -C$, by our choice of α_j we have,

$$\frac{\lambda_n |\nabla_1 \omega_{n\bar{n}}|^2}{(1+\lambda_n^2)^2(\lambda_1 - \lambda_n)} \geqslant -C + \frac{\lambda_n}{(\lambda_1 - \lambda_n)}(1+\frac{\delta}{\lambda_1})\left(1 + \sum_{1<j<n} \frac{\lambda_1 - \tilde{\lambda}_j}{\lambda_j}\right)\Upsilon. \tag{4.12}$$

Note that the left hand side above is the $q = n$ term from (**I**), while the remaining terms from (**I**) are equal to Υ. Using that $\tilde{\lambda}_n = \lambda_n < 0$, we estimate

(**I**) as follows

(4.13)

$$(\mathbf{I}) \geqslant -C+\Upsilon\frac{\lambda_n}{\lambda_1-\lambda_n}\left\{\frac{\lambda_1-\lambda_n}{\lambda_n}+\sum_{j<n}\frac{\lambda_1}{\lambda_j}+\delta\sum_{j<n}\frac{1}{\lambda_j}-(1+\frac{\delta}{\lambda_1})\sum_{1<j<n}\frac{\tilde{\lambda}_j}{\lambda_j}\right\}$$
$$\geqslant -C+\Upsilon\frac{\lambda_n}{\lambda_1-\lambda_n}\left\{\lambda_1\sum_{j}\frac{1}{\lambda_j}-\sum_{j>1}\frac{\tilde{\lambda}_j}{\lambda_j}+\delta\sum_{j<n}\frac{1}{\lambda_j}\right\}$$
$$\geqslant -C+\Upsilon\frac{\lambda_n}{\lambda_1-\lambda_n}\left\{\lambda_1\frac{\sigma_{n-1}(\lambda)}{\sigma_n(\lambda)}-\sum_{j>1}\frac{\tilde{\lambda}_j}{\lambda_j}+\delta\frac{n-1}{\lambda_{n-1}}\right\}.$$

Since $\sigma_{n-1}(\lambda(\Lambda)) \geqslant 0$, and $\sigma_n(\lambda(\Lambda)) < 0$ by Lemma 3.1 part *(iii)*, the first term in the brackets is negative. Furthermore, by our choice of B we know that

$$\sum_{j>1}\frac{\tilde{\lambda}_j}{\lambda_j} = (n-1)-\sum_{1<j<n}\frac{B_{jj}}{\lambda_j}$$
$$\geqslant (n-1)-\frac{1}{\varepsilon_0}\sum_{j}B_{jj}$$
$$\geqslant \frac{n-1}{2},$$

and hence our choice of δ implies that the final two terms combine to be negative as well. Thus, we obtain that the term (**I**) in equation (4.10) is bounded below by a negative constant depending only on the stated data.

Next we consider the sign of the sum (**II**). Again, the only negative contribution to the sum occurs when $q = n$.

We use an estimate similar to that in (4.11) to get that, for any $\delta, \alpha_j, \alpha_j' > 0$, $1 \leqslant j < n$

(4.14)

$$\frac{\lambda_1\lambda_n|\nabla_1\omega_{n\bar{n}}|^2}{(1+\lambda_1^2)(1+\lambda_n^2)^2} \geqslant -\frac{C}{\delta}+\frac{\lambda_1\lambda_n}{(1+\lambda_1^2)}(1+\frac{\delta}{\lambda_1})\left|\sum_{q<n}\frac{\nabla_1\omega_{q\bar{q}}}{1+\lambda_q^2}\right|^2$$
$$\geqslant -\frac{C}{\delta}+\lambda_n\left(\sum_{q<n}\frac{|\nabla_1\omega_{q\bar{q}}|^2\lambda_1\alpha_q}{(1+\lambda_1^2)(1+\lambda_q^2)^2}\right)\cdot\left(\sum_{j<n}\frac{1}{\alpha_j}\right)$$
$$+\,\delta\lambda_n\left(\sum_{q<n}\frac{|\nabla_1\omega_{q\bar{q}}|^2\alpha_q'}{(1+\lambda_1^2)(1+\lambda_q^2)^2}\right)\cdot\left(\sum_{j<n}\frac{1}{\alpha_j'}\right)$$

where in the last line we have used the Cauchy-Schwartz inequality twice. We take $\alpha_q = \lambda_q$, and $\alpha'_q = 1$ for $1 \leqslant q < n$. To simplify notation, let us define

$$\tilde{\Upsilon} = \sum_{q<n} \frac{|\nabla_1 \omega_{q\bar{q}}|^2 \lambda_1 \lambda_q}{(1+\lambda_1^2)(1+\lambda_q^2)^2}.$$

Substituting the estimate in (4.14) into the expression for term (**II**) and simplifying we obtain

$$\begin{aligned}(\mathbf{II}) \geqslant& -\frac{C}{\delta} - \frac{1}{(1+\lambda_1^2)^2}|\nabla_1 \omega_{1\bar{1}}|^2 + \frac{|\nabla_1 \omega_{n\bar{n}}|^2}{(1+\lambda_1^2)(1+\lambda_n^2)^2} \\ &+ \left\{1 + \delta(n-1)\lambda_n\right\} \sum_{q<n} \frac{|\nabla_1 \omega_{q\bar{q}}|^2}{(1+\lambda_1^2)(1+\lambda_q^2)^2} \\ &+ \tilde{\Upsilon}\left\{1 + \lambda_n \left(\sum_{j<n} \frac{1}{\lambda_j}\right)\right\}.\end{aligned} \tag{4.15}$$

If we choose δ sufficiently small depending only on the uniform lower bound for λ_n provided by Lemma 3.1 part *(vi)* then the first term on the second line is positive, while the final term is always positive by Lemma 3.1 part *(iii)*. Thus

$$(\mathbf{II}) \geqslant -C - \frac{1}{(1+\lambda_1^2)^2}|\nabla_1 \omega_{1\bar{1}}|^2.$$

Summarizing, we have proven the estimate

$$\begin{aligned}\Delta_\eta \frac{1}{2}\log(1+\tilde{\lambda}_1^2) \geqslant& -C - \frac{1}{(1+\lambda_1^2)^2}|\nabla_1 \omega_{1\bar{1}}|^2 \\ &+ \sum_{q>1} \frac{\lambda_1^2 \lambda_q + 2\lambda_1 - \tilde{\lambda}_q + \lambda_1 \lambda_q(\lambda_q - \tilde{\lambda}_q)}{(1+\lambda_1^2)^2(1+\lambda_q^2)(\lambda_1 - \tilde{\lambda}_q)}|\nabla_q \omega_{1\bar{1}}|^2 \\ \geqslant& -C - \frac{1}{(1+\lambda_1^2)^2}|\nabla_1 \omega_{1\bar{1}}|^2 + \frac{\lambda_1^2 \lambda_n |\nabla_n \omega_{1\bar{1}}|^2}{(1+\lambda_1^2)^2(1+\lambda_n^2)(\lambda_1 - \lambda_n)}.\end{aligned} \tag{4.16}$$

where in the last line we have used the obvious fact that

$$\lambda_1^2 \lambda_q + 2\lambda_1 - \tilde{\lambda}_q + \lambda_1 \lambda_q(\lambda_q - \tilde{\lambda}_q) \geqslant 0, \qquad 1 < q < n.$$

We now compute the action of the linearized operator on the lower order terms in the definition of G.

$$\Delta_\eta \psi(u) = \psi''(u)\sum_q \frac{|u_q|^2}{1+\lambda_q^2} + \psi'(u)\sum_q \frac{\lambda_q - \chi_{q\bar q}}{1+\lambda_q^2}$$

$$\begin{aligned}\Delta_\eta \varphi(|\nabla u|^2) = {} & \frac{\varphi''(|\nabla u|^2)}{1+\lambda_q^2}\sum_q \left|\sum_j u_{q\bar j}u_j + u_{qj}u_{\bar j}\right|^2 \\ & + 2\varphi'(|\nabla u|^2)\sum_j \mathrm{Re}\left(u_j h_{\bar j} - \sum_q \frac{u_j \nabla_{\bar j}\chi_{q\bar q}}{1+\lambda_q^2}\right) \\ & + \varphi'(|\nabla u|^2)\sum_q\left(\sum_j \frac{|u_{j\bar q}|^2 + |u_{qj}|^2}{1+\lambda_q^2} + \frac{R_{q\bar q}{}^{\bar k\ell}u_\ell u_{\bar k}}{1+\lambda_q^2}\right).\end{aligned}$$

Now, it is easy to see that

$$\frac{R_{q\bar q}{}^{\bar k\ell}u_\ell u_{\bar k}}{1+\lambda_q^2} + 2\mathrm{Re}\left(u_j h_{\bar j} - \frac{u_j\nabla_{\bar j}\chi_{p\bar p}}{1+\lambda_p^2}\right) \geqslant -C_0K,$$

so at x_0, where G achieves its maximum, we have

$$\begin{aligned}0 \geqslant \Delta_\eta G \geqslant {} & -C_1 - \frac{|\nabla_1\omega_{1\bar 1}|^2}{(1+\lambda_1^2)^2} + \frac{\lambda_1^2\lambda_n|\nabla_n\omega_{1\bar 1}|^2}{(1+\lambda_1^2)^2(1+\lambda_n^2)(\lambda_1-\lambda_n)} \\ & + \varphi''\sum_q \frac{1}{1+\lambda_q^2}\left|\sum_j u_{q\bar j}u_j + u_{qj}u_{\bar j}\right|^2 - \varphi' C_0 K \\ & + \sum_q\left(\varphi'\sum_j \frac{|u_{j\bar q}|^2+|u_{qj}|^2}{1+\lambda_q^2} + \psi''\frac{|u_q|^2}{1+\lambda_q^2} + \psi'\frac{\lambda_q-\chi_{q\bar q}}{1+\lambda_q^2}\right).\end{aligned}$$

Furthermore, we have $\nabla_p G(x_0) = 0$, and so

$$\frac{\lambda_1\nabla_p\omega_{1\bar 1}}{1+\lambda_1^2} = -\varphi'\sum_j\left(u_{pj}u_{\bar j} + u_j u_{p\bar j}\right) - \psi' u_p$$

In particular,

$$\begin{aligned}\frac{\lambda_1^2\lambda_n|\nabla_n\omega_{1\bar 1}|^2}{(1+\lambda_1^2)^2(1+\lambda_n^2)(\lambda_1-\lambda_n)} \geqslant {} & \frac{\lambda_n(1+\delta)(\varphi')^2}{(1+\lambda_n^2)(\lambda_1-\lambda_n)}\left|\sum_j u_{nj}u_{\bar j} + u_j u_{n\bar j}\right|^2 \\ & + \frac{\lambda_n(1+\delta^{-1})(\psi')^2}{(1+\lambda_n^2)(\lambda_1-\lambda_n)}|u_n|^2.\end{aligned}$$

In a similar fashion we have

$$\frac{|\nabla_1 \omega_{1\bar{1}}|^2}{(1+\lambda_1^2)^2} \leqslant \frac{(1+\delta)(\varphi')^2}{\lambda_1^2}\left|\sum_j u_{1j}u_{\bar{j}} + u_j u_{1\bar{j}}\right|^2 + \frac{(1+\delta^{-1})(\psi')^2}{\lambda_1^2}|u_1|^2.$$

We now use that $\varphi'' = 2(\varphi')^2$. If we take $\delta = 1/2$, then we have at x_0

$$\begin{aligned}
0 \geqslant & -C_1 + (\varphi')^2\left|\sum_j u_{1\bar{j}}u_j + u_{1j}u_{\bar{j}}\right|^2\left(\frac{2}{1+\lambda_1^2} - \frac{1+\delta}{\lambda_1^2}\right) \\
& + \frac{(\varphi')^2}{1+\lambda_n^2}\left|\sum_j u_{n\bar{j}}u_j + u_{nj}u_{\bar{j}}\right|^2\left(2 + \frac{\lambda_n(1+\delta)}{\lambda_1 - \lambda_n}\right) \\
& - \frac{(1+\delta^{-1})(\psi')^2}{\lambda_1^2}|u_1|^2 + \frac{\lambda_n(1+\delta^{-1})(\psi')^2}{(1+\lambda_n^2)(\lambda_1-\lambda_n)}|u_n|^2 \\
& + \varphi''\sum_{1<q<n}\frac{1}{1+\lambda_q^2}\left|\sum_j u_{q\bar{j}}u_j + u_{qj}u_{\bar{j}}\right|^2 - \varphi' C_0 K \\
& + \sum_q\left(\varphi'\sum_j \frac{|u_{j\bar{q}}|^2 + |u_{qj}|^2}{1+\lambda_q^2} + \psi''\frac{|u_q|^2}{1+\lambda_q^2} + \psi'\frac{\lambda_q - \chi_{q\bar{q}}}{1+\lambda_q^2}\right).
\end{aligned}$$

If λ_1 is sufficiently large, depending only on the lower bound for λ_n, then

$$\left(\frac{2}{1+\lambda_1^2} - \frac{1+\delta}{\lambda_1^2}\right) \geqslant 0, \qquad \left(2 + \frac{\lambda_n(1+\delta)}{\lambda_1-\lambda_n}\right) \geqslant 0.$$

In particular, since $\varphi'' \geqslant 0$ we have

$$\begin{aligned}
0 \geqslant & -C_1 - \varphi' C_0 K - \frac{(1+\delta^{-1})(\psi')^2}{\lambda_1^2}|u_1|^2 \\
& + \left(\psi'' + \frac{\lambda_n(1+\delta^{-1})(\psi')^2}{\lambda_1 - \lambda_n}\right)\frac{|u_n|^2}{1+\lambda_n^2} \\
& + \varphi'\sum_{q,j}\frac{|u_{j\bar{q}}|^2 + |u_{qj}|^2}{1+\lambda_q^2} + \psi''\sum_{q<n}\frac{|u_q|^2}{1+\lambda_q^2} + \psi'\sum_q \frac{\lambda_q - \chi_{q\bar{q}}}{1+\lambda_q^2}.
\end{aligned}$$

As long as λ_1 is sufficiently large, depending only on τ, A, the bracketed term on the second line is positive. For the last term on the first line we clearly have the estimate

$$\frac{(1+\delta^{-1})(\psi')^2}{\lambda_1^2}|u_1|^2 \leqslant \frac{C_0A^2K}{\lambda_1^2},$$

and so

$$\begin{aligned} 0 \geqslant& -C_1 - \varphi' C_0 K - \frac{C_0 A^2 K}{\lambda_1^2} \\ &+ \varphi' \sum_{q,j} \frac{|u_{j\bar{q}}|^2 + |u_{qj}|^2}{1+\lambda_p^2} + \psi' \sum_q \frac{\lambda_q - \chi_{q\bar{q}}}{1+\lambda_q^2}. \end{aligned}$$

Now, if λ_1 is sufficiently large relative to $\chi_{1\bar{1}}$, then we have

$$|u_{1\bar{1}}|^2 \geqslant \frac{1}{2}\lambda_1^2$$

and so, since $(4K)^{-1} < \varphi' < (2K)^{-1}$ we have

$$\begin{aligned} 0 \geqslant& -C_1 - C_0 - \frac{C_0 A^2 K}{\lambda_1^2} \\ &+ \frac{1}{2}\varphi' \frac{\lambda_1^2}{1+\lambda_1^2} + \psi' \sum_q \frac{\lambda_q - \chi_{q\bar{q}}}{1+\lambda_q^2}. \end{aligned} \tag{4.17}$$

Recall from equation (4.2) that we have

$$\sum_q \frac{\chi_{q\bar{q}} - \lambda_q}{1+\lambda_q^2} \geqslant \kappa \sum_q \frac{1}{1+\lambda_q^2}.$$

Since $|\lambda_n| \leqslant C_3$ by Lemma 3.1, we can choose A sufficiently large so that

$$A\frac{\kappa}{1+\lambda_n^2} \geqslant -C_1 - C_0,$$

then, since $A < -\psi' < 2A$, we have

$$0 \geqslant \frac{\lambda_1^2}{8K(1+\lambda_1^2)} - \frac{C_0A^2K}{\lambda_1^2}.$$

In other words,

$$\frac{\lambda_1^2}{K^2} \leqslant \frac{8C_0A^2(1+\lambda_1^2)}{\lambda_1^2} \leqslant C_5.$$

Thus, at the maximum of G we have $\lambda_1 \leqslant C_5 K$. At this point we have

$$\frac{1}{2}\log(1+\lambda_1^2) - \frac{1}{2}\log(1-\frac{|\nabla u|^2}{2K}) + \psi(u) \leqslant \frac{1}{2}\log(1+C_5K^2) + C$$

which after simplification yields the desired estimate;

$$\sqrt{1+\lambda_1^2} \leqslant C_6 K. \qquad \square$$

5. The blow-up argument and the gradient estimate

We now apply a blow-up argument to the estimate in Theorem 4.1 to obtain a gradient bound. By contrast with the general setting considered by Szḱelyhidi [44], or the complex Hessian equation studied by Dinew-Kołodziej [14], the argument here is rather simple. By the lower bound for ω from Lemma 3.1, part *(vi)* it suffices to prove

Proposition 5.1. *Suppose $u : X \to \mathbb{R}$ satisfies*

(i) $\omega_0 + \sqrt{-1}\partial\overline{\partial}u \geqslant -K\alpha$,
(ii) $\sup_X |u| \leqslant K$,
(iii) $|\partial\overline{\partial}u| \leqslant K(1+\sup_X |\nabla u|^2)$,

for a uniform constant $K < +\infty$. Then there exists a constant C, depending only on $(X,\alpha), \omega_0$, and K such that

$$\sup_X |\nabla u| \leqslant C.$$

Proof. We argue by contradiction. Suppose we have a Kähler manifold (X,α) where the estimate fails. Then we have smooth functions $u_k : X \to \mathbb{R}$, and a $(1,1)$ form ω_0 such that the assumptions $(i) - (iii)$ hold uniformly in $n \in \mathbb{N}$, but

$$\sup_X |\nabla u_k| = C_k \geqslant k.$$

Let $x_k \in X$ be a point where $\sup_X |\nabla u_k|$ is attained. Up to passing to a subsequence we may assume that $\{x_k\}$ converges to some point $x \in X$. In particular, we may assume that about each x_k there is a coordinate chart $U_k \subset X$ with coordinates $(z_1, \ldots, z_n)$ defined on a ball of radius 1, centered at x_k, such that

$$\alpha(z) = Id + O(|z|^2)$$

on U_k. In particular, estimates *(i)-(iii)* hold uniformly on $B_1(0)$ with α replaced by the Euclidean metric, after possibly increasing K slightly. Define $\hat{u}_k(z) := u_k(\frac{z}{C_k})$, defined in the ball of radius C_k. From properties *(i)-(iii)* and the above remark we have

- $\partial\overline{\partial}\hat{u}_k \geqslant \frac{-KId-\omega_0}{C_k^2}$ for all $z \in B_{C_k}(0)$,
- $\mathrm{osc}_{B_{C_k}(0)}\hat{u}_k \leqslant K$,
- $|\partial\overline{\partial}\hat{u}_k| \leqslant 2K$ for all $z \in B_{C_k}(0)$
- $|\nabla\hat{u}_k(z)| \leqslant 1 = |\nabla\hat{u}_k(0)|$ for all $z \in B_{C_k}(0)$.

Since $C_k \to \infty$, a standard diagonal argument yields, for a fixed $\beta \in (0,1)$, the existence of a $C^{1,\beta}$ function $u_\infty : \mathbb{C}^n \to \mathbb{R}$ so that $\hat{u}_j \to u_\infty$ in $C^{1,\beta}$ topology on compact subsets. Furthermore, by the above estimates u is continuous, uniformly bounded, has $|\nabla u(0)| = 1$, and satisfies $\sqrt{-1}\partial\overline{\partial}u \geqslant 0$ in the sense of distributions. Hence, u is bounded, non-constant plurisubharmonic function defined on all of $\mathbb{C}^n$. By a standard result in several complex variables, no such functions exist [36]. $\square$

6. Higher order estimates

The higher order estimates follow from the Evans-Krylov theory. The equation (1.5) is only concave when $h : X \to [(n-1)\frac{\pi}{2}, n\frac{\pi}{2})$, the so called *hypercritical phase* case. However, as long as $h \geqslant (n-2)\frac{\pi}{2}$, we can exploit the convexity of the level sets $\partial\Gamma^\sigma$ (see Lemma 3.1 part *(iv)*) to obtain the $C^{2,\beta}$ estimates by a blow-up argument. The first step in this direction is to prove a Liouville theorem. The following proposition implies the complex analog of [59, Theorem 1.1] except that we also assume a second derivative bound. Let $\mathrm{Herm}(n)$ denote the space of $n \times n$ Hermitian matrices.

Lemma 6.1. *Suppose $u : \mathbb{C}^n \to \mathbb{R}$ is a C^3 function satisfying*

$$F(\partial\overline{\partial}u) = \sigma.$$

where $F : \mathrm{Herm}(n) \to \mathbb{R}$ is smooth and elliptic. Assume that the set

$$\Gamma^\sigma = \{M \in \mathrm{Herm}(n) : F(M) > \sigma\}$$

is convex. If $|\partial\overline{\partial}u|_{L^\infty(\mathbb{C}^n)} \leqslant K < +\infty$, then u is a quadratic polynomial.

The proof follows by combining the convexity of the level sets of the equation $F(\partial\overline{\partial}u) = \sigma$ with an extension trick in order to apply the standard Evans-Krylov estimate. The extension trick occurs in two steps. First we find a concave elliptic operator $F_0(\cdot)$, such that $F_0(\partial\overline{\partial}u) = 0$ if and only if $F(\partial\overline{\partial}u) = \sigma$. Secondly, we use a trick due to Wang [54], which was used also by Tosatti-Wang-Weinkove-Yang [49], to extend F_0 to a real uniformly elliptic concave operator, to which we apply the Evans-Krylov theory. While we expect this is well-known to experts, we give the details for the readers' convenience.

Proof. Let $\mathrm{Sym}(2n)$ denote the space of real symmetric $2n \times 2n$ matrices. Note that we have a canonical inclusion $\iota : \mathrm{Herm}(n) \hookrightarrow \mathrm{Sym}(2n)$, and so we will always regard $\mathrm{Herm}(n) \subset \mathrm{Sym}(2n)$. Let $\mathcal{H}_{\lambda,\Lambda} \subset \mathrm{Sym}(2n)$ denote the set of symmetric matrices with eigenvalues lying in $[\lambda, \Lambda]$.

As in [44], we define $F_0 : \mathrm{Herm}(n) \to \mathbb{R}$ by

$$F_0(A) := \inf\left\{t : \lambda(A) - t \cdot Id \in \overline{\Gamma}^{\sigma}\right\}, \tag{6.1}$$

where $\lambda(A)$ denotes the eigenvalues of A. The reader can check that F_0 is a smooth, elliptic, non-linear operator on $\mathrm{Herm}(n)$. The convexity of Γ^{σ} implies that $F_0(\cdot)$ is a concave operator. Furthermore, $F_0(\partial\overline{\partial}u) = 0$ if and only if $F(\partial\overline{\partial}u) = \sigma$. Consider the compact, convex set

$$B_{2K} := \{M \in \mathrm{Herm}(n) : \|M\| \leqslant 2K\}.$$

Since $F_0(\cdot)$ is smooth, and elliptic, and B_{2K} is compact, $F_0(\cdot)$ is uniformly elliptic on B_{2K}.

The next step is to extend F_0 to a uniformly elliptic, concave operator outside of B_{2K}. We use an envelope trick due to Wang [54] (see also [49]). The complex structure J on $\mathbb{C}^n$ gives a canonical projection $p : \mathrm{Sym}(2n) \to \mathrm{Herm}(n)$, by setting

$$p(M) = \frac{M + J^T M J}{2}.$$

Define

$$\mathcal{B}_{2K} := \{N \in \mathrm{Sym}(2n) : p(N) \in B_{2K}\},$$

and extend F_0 to a smooth, concave operator $\hat{F}_0 : \mathcal{B}_{2K} \to \mathbb{R}$ by setting

$$\hat{F}_0(N) := F_0(p(N)).$$

We claim that $\hat{F}_0$ is uniformly elliptic on $\mathcal{B}_{2K}$. This is just a matter of linear algebra. First, observe that if $M \geqslant 0$ is positive semi-definite, then so is $p(M)$, since, for any vector $v \in \mathbb{R}^{2n}$,

$$\langle v, p(M)v\rangle = \frac{\langle v, Mv\rangle + \langle Jv, MJv\rangle}{2}.$$

Furthermore, we clearly have $\mathrm{Tr}(p(M)) = \mathrm{Tr}(M)$. From these two facts the uniform ellipticity of $\hat{F}_0$ on $\mathcal{B}_{2K}$ easily follows from the uniform ellipticity of F_0 on B_{2K}. Hence, there are constants $0 < \lambda < \Lambda < +\infty$ such that, for all $A \in \mathcal{B}_{2K}$ the differential of F_0, denoted DF_0, at A lies in $\mathcal{H}_{\lambda,\Lambda}$. We define

(6.2)

$$F_1(N) := \inf \Big\{ L(N) : \quad L : \mathrm{Sym}(2n) \to \mathbb{R} \text{ affine linear },$$
$$DL \in \mathcal{H}_{\lambda,\Lambda}, \text{ and } L(A) \geqslant \hat{F}_0(A), \ \forall A \in \mathcal{B}_{2K} \Big\}$$

where DL denotes the differential of L. In words, F_1 is the concave envelope of the graph of $\hat{F}_0$ over $\mathcal{B}_{2K}$. As in [49, Lemma 4.1] it is straightforward to check that $F_1 : \mathrm{Sym}(2n) \to \mathbb{R}$ is uniformly elliptic, concave and agrees with $\hat{F}_0$ over $\mathcal{B}_{2K}$. Since $\partial\overline{\partial}u : \mathbb{C}^n \to \mathcal{B}_{2K}$ we have

$$F_1(D^2u) = 0.$$

By the Evans-Krylov theorem [20, 29], [3, Theorem 6.1] and a standard scaling argument we have; for some $\beta = \beta(n, \lambda, \Lambda) \in (0, 1)$ and for every $R > 0$ there holds

$$|D^2u|_{C^\beta(B_R(0))} \leqslant C(n, \lambda, \Lambda) R^{-\beta} \|D^2u\|_{L^\infty(B_{2R}(0))} \leqslant C(n, \lambda, \Lambda) R^{-\beta} K.$$

Letting $R \to +\infty$ we get the result. □

We use this Liouville type result to conclude $C^{2,\beta}$ estimates by a blow-up argument.

Lemma 6.2. *Suppose $u : B_2 \subset \mathbb{C}^n \to \mathbb{R}$ is a smooth function satisfying*

$$F(x, \partial\overline{\partial}u) = h(x),$$

for some smooth map $F : B_2 \times \mathrm{Herm}(n) \to \mathbb{R}$. Suppose that $F(x, \cdot)$ is uniformly elliptic on $B_2 \times \partial\overline{\partial}u(B_2)$ with ellipticity constant $0 < \lambda < \Lambda < +\infty$. Assume $h : B_2 \to [a, b]$ is C^2 and, for every $\sigma \in [a, b]$ and $x \in B_2$ the set $\Gamma^\sigma := \{M \in \mathrm{Herm}(n) : F(x, M) > \sigma\}$ is convex. Then, for every $\beta \in (0, 1)$ we have the estimate

$$|\partial\overline{\partial}u|_{C^\beta(B_{1/2})} \leqslant C(n, \beta, \lambda, \Lambda, |\partial\overline{\partial}u|_{L^\infty(B_2)}, \|h\|_{C^2(B_2)}).$$

Proof. The proof is by a standard blow-up argument; see, for instance [9]. We give the details for the convenience of the reader. For each $x \in B_1$ consider the quantity

$$N_u := \sup_{B_1} d_x |\partial\partial\overline{\partial} u|(x)$$

where $d_x := \operatorname{dist}(x, \partial B_1)$. Suppose the supremum is achieved at $x_0 \in B_1$. Consider the function $\tilde{u} : B_{N_u}(0) \to R$ defined by

$$\tilde{u}(z) := \frac{N_u^2}{d_{x_0}^2} u\left(x_0 + \frac{d_{x_0}}{N_u} z\right) - A - A_i z_i$$

where A, A_i are chosen so that $\tilde{u}(0) = \partial\tilde{u}(0) = 0$. Note that

$$\partial\overline{\partial}\tilde{u} = \partial\overline{\partial} u, \qquad \|\partial\partial\overline{\partial} u\|_{L^\infty(B_{N_u}(0))} = |\partial\partial\overline{\partial} u(0)| = 1.$$

In particular, we have $|\partial\overline{\partial} u|_{C^\beta(B_1)} \leqslant 1$ for every $\beta \in (0,1)$ and $\tilde{u}$ solves

$$F(x_0 + \frac{d_{x_0}}{N_u} z, \partial\overline{\partial}\tilde{u}(z)) = h\left(x_0 + \frac{d_{x_0}}{N_u} z\right), \qquad z \in B_{N_u}(0).$$

Differentiating the equation in the ∂_ℓ direction yields

$$F^{i\bar{j}}(x_0 + \frac{d_{x_0}}{N_u} z, \partial\overline{\partial}\tilde{u})\partial_i\partial_{\bar{j}}\partial_\ell\tilde{u} = \frac{d_{x_0}}{N_u} h'\left(x_0 + \frac{d_{x_0}}{N_u} z\right).$$

Since $F(x, \cdot)$ is uniformly elliptic and h is smooth, the Schauder theory implies $\partial\tilde{u}$ is bounded in $C^{2,\beta}(B_{N_u/2}(0))$, and so $\tilde{u}$ is controlled in $C^{3,\beta}(B_{N_u/2}(0))$.

Now, for the sake of finding a contradiction, suppose we have:

- a sequence $u_n : B_2 \to \mathbb{R}$ such that $\|\partial\overline{\partial} u_n\|_{L^\infty(B_2)} \leqslant K$, but so that $N_{u_n} \geqslant n$
- functions $h_n : B_2 \to [a, b]$ such that $\|h_n\|_{C^2(B_2)} \leqslant K$

For each n let $x_n \in B_1$ be a point where N_{u_n} is achieved. By compactness, after passing to a subsequence (not relabelled) we may assume that:

- $x_n \to x_\infty \in \overline{B_1}$.
- h_n converges to some function h uniformly in $C^{1,\beta'}$ topology on $B_{3/2}$ for some fixed $\beta' \in (0,1)$.

By the above rescaling we find functions $\tilde{u}_n : B_{N_{u_n}}(0) \to \mathbb{R}$ such that

- $\|\tilde{u}_n\|_{C^{3,\beta}(B_{N_{u_n}}(0))} \leqslant C$ and
- $F(x_n + \frac{d_{x_n}}{N_{u_n}} z, \partial\overline{\partial}\tilde{u}_n) = h_n\left(x_n + \frac{d_{x_n}}{N_{u_n}} z\right) \qquad z \in B_{N_{u_n}}(0).$

Since $N_{u_n} \geqslant n$, a diagonal argument yields the existence of a function $u : \mathbb{C}^n \to \mathbb{R}$ and a subsequence (again, not relabelled) such that $\{u_n\}_{n\geqslant k}$ converges uniformly to u in $C^{3,\alpha/2}(B_k(0))$. In particular, we have

$$F(x_0, \partial\overline{\partial}u) = h(x_0), \qquad |\partial\partial\overline{\partial}u|(0) = 1.$$

Clearly $h(x_0) \in [a, b]$, and so we may apply Lemma 6.1 to conclude that u is a quadratic polynomial, which is a contradiction. □

By arguing locally, Lemma 6.2 immediately implies the following corollary, whose proof we leave to the reader, and finishes the proof of Theorem 1.2.

Corollary 6.3. *Suppose $u : X \to \mathbb{R}$ is a solution of*

$$\Theta_\alpha(\omega + \sqrt{-1}\partial\overline{\partial}u) = h(x)$$

where $h(x) \geqslant (n-2)\frac{\pi}{2} + \varepsilon$ for some $\varepsilon > 0$. Then for every $\beta \in (0, 1)$ we have the estimate

$$|\partial\overline{\partial}u|_{C^\beta(X)} \leqslant C(n, X, \alpha, \beta, \|h\|_{C^2}, \|\partial\overline{\partial}u\|_{L^\infty(X)})$$

7. The method of continuity and the proof of Theorem 1.3

In this section we prove Theorem 1.3, using the method of continuity. Unfortunately, the naive method of continuity does not work due essentially to the fact that the subsolution condition is non-trivial; for related discussion see [44]. Instead, adapting an idea of Sun [43] in the setting of the J-equation, the proof of Theorem 1.3 requires two applications of the method of continuity. Let us first prove openness along a general method of continuity.

Lemma 7.1. *Fix $k \geqslant 2, \beta \in (0, 1)$ and suppose we have $C^{k-2,\beta}$ functions $H_0, H_1 : X \to \mathbb{R}$, and a $C^{k,\beta}$ function $u : X \to \mathbb{R}$ such that*

$$\Theta_\alpha(\omega + \sqrt{-1}\partial\overline{\partial}u) = H_0.$$

Consider the family of equations

$$\Theta_\alpha(\omega + \sqrt{-1}\partial\overline{\partial}u_t) = (1-t)H_0 + tH_1 + c_t \tag{7.1}$$

for c_t a constant. There exists $\varepsilon > 0$ such that, for every $|t| < \varepsilon$ a unique pair $(u_t, c_t) \in C^{k,\beta} \times \mathbb{R}$ solving (7.1). Furthermore, if H_0, H_1 are smooth, then so is u_t.

Proof. The proof is by the implicit function theorem. Fix $\beta > 0$, $k \geqslant 2$ and consider the map $F : [0,1] \times C^{k,\beta} \times \mathbb{R} \to C^{k-2,\beta}$ given by

$$(t, c, u) \longmapsto \Theta_\alpha(\omega + \sqrt{-1}\partial\overline{\partial}u) - (1-t)H_0 - tH_1 - c.$$

Let Δ_η denote the linearization of Θ_α around $(u_0, c_0) := (u, 0)$. The operator Δ_η is homotopic to the Laplacian with respect to α, and so has index 0. By the maximum principle, the kernel of Δ_η consists of the constants, and hence the cokernel of Δ_η has dimension 1. Another application of the maximum principle shows that the constants are not in the image of Δ_η. It follows that the linearization of F at time 0, given by

$$(v, c) \longmapsto \Delta_\eta v + c,$$

is a surjective map from $C^{k,\beta} \times \mathbb{R}$ to $C^{k-2,\beta}$. In particular, by the implicit function theorem we conclude that there exists $\varepsilon > 0$ such that, for all $|t| < \varepsilon$ we can find a unique pair $(u_t, c_t) \in C^{k,\beta} \times \mathbb{R}$ solving (7.1). By a standard boot strapping argument, we find that u_t is in fact smooth provided H_0, H_1 are smooth. $\square$

Suppose now that we have a subsolution $\chi \in [\Omega]$ to the deformed Hermitian-Yang-Mills equation satisfying the assumptions of Theorem 1.3. Let us denote by

$$\Theta_0 := \Theta_\alpha(\chi).$$

As stated above, by assumption (1.7), the average angle $\hat{\Theta}$ lifts naturally to $\mathbb{R}$. Without loss of generality, we will assume that $\Theta_0 \neq \hat{\Theta}$, for otherwise we are finished. Now, and for the remainder of this section, we let $\mu_1, \cdots \mu_n$ be the eigenvalues of the relative endomorphism $\alpha^{-1}\chi$ at an arbitrary point of X. We clearly have

$$\sum_{i \neq j} \arctan(\mu_i) > \Theta_0 - \frac{\pi}{2} \qquad \forall j.$$

In particular, we can find $\delta_0 > 0$ such that

$$\sum_{i \neq j} \arctan(\mu_i) > \max\{\Theta_0, \hat{\Theta}\} + 100\delta_0 - \frac{\pi}{2} \qquad \forall j.$$

Furthermore, since

$$\mathrm{Arg}\int_X (\alpha+\sqrt{-1}\chi)^n = \hat{\Theta}$$

we must have that $\inf_X \Theta_0 < \hat{\Theta}$. Choose $\delta_1 > 0$ such that

$$\inf_X \Theta_0 + 100\delta_1 = \hat{\Theta}$$

Set $\delta = \min\{\delta_0, \delta_1\}$, and define

$$\Theta_1 = \widetilde{\max}_\delta\{\hat{\Theta}, \Theta_0\}$$

where $\widetilde{\max}_\delta$ denotes the regularized maximum [12]. We have

Lemma 7.2. *Fix a point $p \in X$ where Θ_0 achieves its infimum. The function Θ_1 has the following properties:*

(i) Θ_1 *is smooth.*
(ii) $\max\{\Theta_0, \hat{\Theta}\} \leqslant \Theta_1 \leqslant \max\{\Theta_0, \hat{\Theta}\} + \delta$.
(iii) $\Theta_1(x) = \hat{\Theta}$ *on the set* $\{x \in X : \Theta_0 + \delta \leqslant \hat{\Theta} - \delta\}$. *In particular,* $\Theta_1(x) = \hat{\Theta}$ *in a neighbourhood of* $p \in X$.
(iv) $\Theta_1(x) = \Theta_0(x)$ *on the set* $\{x \in X : \hat{\Theta} + \delta \leqslant \Theta_0 - \delta\}$.
(v) *For every* $t \in [0, 1]$

$$\inf_X[(1-t)\Theta_0 + t\Theta_1] = (1-t)\inf_X \Theta_0 + t\hat{\Theta} = (1-t)\Theta_0(p) + t\hat{\Theta}.$$

(vi) $\sup_X[\Theta_1 - \Theta_0] = \Theta_1(p) - \Theta_0(p) = \hat{\Theta} - \inf_X \Theta_0(p)$.

Proof. Statements $(i) - (iv)$ are just the properties of the regularized maximum, [12, Chapter 1, Lemma 5.18]. We prove (v). From our choice of δ, and the definition of Θ_1 we have $\Theta_1(p) = \hat{\Theta}$. Thus

$$\begin{aligned}
(1-t)\Theta_0(p) + t\hat{\Theta} &= (1-t)\Theta_0(p) + t\Theta_1(p) \\
&\geqslant \inf_X[(1-t)\Theta_0 + t\Theta_1] \\
&\geqslant (1-t)\inf_X \Theta_0 + t\inf_X \Theta_1 \\
&= (1-t)\Theta_0(p) + t\hat{\Theta},
\end{aligned}$$

establishing the fifth point. For (vi), we first consider the set $U_1 := \{x \in X : \Theta_0 + \delta \leqslant \hat{\Theta} - \delta\}$. On this set we have $\Theta_1 - \Theta_0 = \hat{\Theta} - \Theta_0$ by property (iii). This difference is maximized at the point $p \in U_1$, where we have

$$\Theta_1(p) - \Theta_0(p) = \hat{\Theta} - \Theta_0(p) = 100\delta_1 \geqslant 100\delta.$$

Now consider the set $U_2 := \{x \in X : \hat{\Theta} + \delta \leqslant \Theta_0 - \delta\}$. On this set we have $\Theta_1 - \Theta_0 \equiv 0$ by (iv). Finally, we consider the set $U_3 = \{x \in X : |\Theta_0 - \hat{\Theta}| < 2\delta\}$. On U_3 we have

$$\Theta_1 - \Theta_0 \leqslant \max\{\Theta_0, \hat{\Theta}\} + \delta - \Theta_0 \leqslant 3\delta < 100\delta,$$

and the lemma follows. $\square$

We use the function Θ_1 as the first target for the method of continuity.

Proposition 7.3. *There exists a smooth function $u_1 : X \to \mathbb{R}$, and a constant $b_1 < 0$ such that*

$$\Theta_\alpha(\omega + \sqrt{-1}\partial\overline{\partial}u_1) = \Theta_1 + b_1, \qquad \textit{and} \qquad \Theta_1 + b_1 > (n-2)\frac{\pi}{2}.$$

Proof. We use the method of continuity. Consider the family of equations

$$\Theta_\alpha(\chi + \sqrt{-1}\partial\overline{\partial}u_t) = (1-t)\Theta_0 + t\Theta_1 + b_t. \tag{7.2}$$

Define

$$I = \left\{t \in [0,1] : \exists\ (u_t, b_t) \in C^\infty(X) \times \mathbb{R} \text{ solving } (7.2)\right\}.$$

Since $(0,0)$ is a solution at time $t = 0$, we have that I is non-empty. By Lemma 7.1 the set I is open. It suffices to prove that I is closed. This will follow from the a priori estimates in Theorem 1.2 together with a standard bootstrapping argument provided we can show

- χ is a subsolution of equation (7.2) for all $t \in [0,1]$
- $(1-t)\Theta_0 + t\Theta_1 + b_t > (n-2)\frac{\pi}{2}$ uniformly for $t \in [0,1]$.

In order to do each of these things, we must control the constant b_t. First, we claim that $b_t \leqslant t\sup_X(\Theta_0 - \Theta_1) \leqslant 0$. To see this, choose $q \in X$ where u_t achieves its maximum. Then at q, ellipticity implies

$$\Theta_0(q) \geqslant \Theta_\alpha(\chi + \sqrt{-1}\partial\overline{\partial}u_t)(q) = (1-t)\Theta_0(q) + t\Theta_1(q) + b_t.$$

Rearranging this equation yields

$$b_t \leqslant t\sup_X(\Theta_0 - \Theta_1) \leqslant 0$$

where the final inequality follows from the fact that $\Theta_1 \geqslant \Theta_0$ by construction. It follows that for every $1 \leqslant j \leqslant n$ there holds,

$$\begin{aligned}\sum_{i \neq j} \arctan(\mu_i) &> \max\{\Theta_0, \hat{\Theta}\} + 100\delta - \frac{\pi}{2} \\ &> \Theta_1 - \frac{\pi}{2} \\ &\geqslant (1-t)\Theta_0 + t\Theta_1 + b_t - \frac{\pi}{2}\end{aligned}$$

and so χ is a subsolution of equation (7.2) for all $t \in [0,1]$, taking care of the first point. To take care of the second point we look at a point $q \in X$ where u_t achieves its minimum to find

$$b_t \geqslant -t \sup_X (\Theta_1 - \Theta_0).$$

Combining this estimate with the results of Lemma 7.2, we have

$$\begin{aligned}\inf_X [(1-t)\Theta_0 + t\Theta_1 + b_t] &= (1-t)\Theta_0(p) + t\Theta_1(p) + b_t \\ &= \Theta_0(p) + t(\Theta_1(p) - \Theta_0(p)) + b_t \\ &= \Theta_0(p) + t\sup_X(\Theta_1 - \Theta_0) + b_t \\ &\geqslant \Theta_0(p) \\ &> (n-2)\frac{\pi}{2}.\end{aligned} \tag{7.3}$$

By Theorem 1.2, together with the usual Schauder estimates and bootstrapping argument we conclude that I is closed. Proposition 7.3 follows. ☐

We now turn to the proof of the main theorem. Let $\omega_1 = \chi + \sqrt{-1}\partial\overline{\partial}u_1$, where u_1 is the function from Proposition 7.3. We consider the method of continuity

$$\Theta_\alpha(\omega_1 + \sqrt{-1}\partial\overline{\partial}v_t) = (1-t)\Theta_1 + t\hat{\Theta} + c_t. \tag{7.4}$$

Define

$$I = \{t \in [0,1] : \exists\ (v_t, c_t) \in C^\infty(X) \times \mathbb{R} \text{ solving } (7.4)\}.$$

By Proposition 7.3 we have a solution at time $t = 0$, with constant $c_0 = b_1$. Thanks to Lemma 7.1, the set I is open, and so it suffices to prove I is closed. Again this will follow from the a priori estimates in Theorem 1.2, if we can show that

- χ is a subsolution along the whole method of continuity (7.4).
- $(1-t)\Theta_1 + t\hat{\Theta} + c_t > (n-2)\frac{\pi}{2}$ for all $t \in [0,1]$.

as in the proof of Proposition 7.3, it suffices to control the constant c_t. To control c_t from above we observe that since $\Theta_1 \geqslant \hat{\Theta}$, the cohomological condition

$$\mathrm{Arg}\int_X \sqrt{\frac{\det \eta_t}{\det \alpha}} e^{i\left((1-t)\Theta_1 + t\hat{\Theta} + c_t\right)}\alpha^n = \hat{\Theta}$$

implies that $c_t \leqslant 0$ for all $t \in [0,1]$. Arguing as in the proof of Proposition 7.3, we conclude that χ is again a subsolution along the whole method of continuity. Furthermore, if $p \in X$ is a point where Θ_0 achieves its infimum, then Lemma 7.2 part (iii), combined with Proposition 7.3 implies

$$\hat{\Theta} + b_1 = \Theta_1(p) + b_1 > (n-2)\frac{\pi}{2},$$

and so in particular, we have

$$(1-t)[\Theta_1 + b_1] + t[\hat{\Theta} + b_1] > (n-2)\frac{\pi}{2}.$$

In order to show that $(1-t)\Theta_1 + t\hat{\Theta} + c_t > (n-2)\frac{\pi}{2}$ it suffices to show that $c_t \geqslant b_1$ for all t. This is easy. If the maximum of v_t is achieved at the point $q \in X$, then we have

$$\Theta_1(q) + b_1 \leqslant (1-t)\Theta_1(q) + t\hat{\Theta} + c_t,$$

or in other words,

$$c_t \geqslant b_1 + t[\Theta_1(q) - \hat{\Theta}] \geqslant b_1$$

since $\Theta_1 \geqslant \hat{\Theta}$. As a result we can apply the a priori estimates in 1.2 uniformly in t to conclude that I is closed. The higher regularity follows in the usual way from the Schauder estimates and bootstrapping. Since we clearly have $c_1 = 0$ by the cohomological condition, Theorem 1.3 follows.

Remark 7.4. *It is easy to establish the following weaker existence theorem using the parabolic flow introduced in [26]: If $\hat{\Theta} > (n-1)\frac{\pi}{2}$, and $\chi \in \Omega$ is a subsolution with $\Theta_\alpha(\chi) > (n-1)\frac{\pi}{2}$, then the flow in [26] starting at χ converges smoothly to a solution of the deformed Hermitian-Yang-Mills equation.*

8. Subsolutions, class conditions and stability

In this section we briefly elaborate on the subsolution condition as well as pose some natural conjectures related to the existence of solutions to the deformed Hermitian-Yang-Mills equation. The first step is to observe that the subsolution condition in Lemma 3.3 is equivalent to a class condition, as we alluded to in Remark 3.4. Recall that $\Omega \in H^{1,1}(X, \mathbb{R})$ is a fixed cohomology class. We then have the following proposition.

Proposition 8.1. *Let $\hat{\Theta} \in ((n-2)\frac{\pi}{2}, n\frac{\pi}{2})$ be the fixed constant defined in Section 2. Then a $(1,1)$ form $\chi \in \Omega$, satisfying $\Theta_\alpha(\chi) > (n-2)\frac{\pi}{2}$ is a subsolution to equation (1.2) if and only if*

1. $\dim_{\mathbb{C}} X = n$ *is even and*

$$-(i^n)\left(\mathrm{Im}(\alpha+\sqrt{-1}\chi)^{n-1} + \cot(\hat{\Theta})\mathrm{Re}(\alpha+\sqrt{-1}\chi)^{n-1}\right) > 0 \tag{8.1}$$

2. $\dim_{\mathbb{C}} X = n$ *is odd and*

$$i^{n-1}\left(\tan(\hat{\Theta})\mathrm{Im}(\alpha+\sqrt{-1}\chi)^{n-1} + \mathrm{Re}(\alpha+\sqrt{-1}\chi)^{n-1}\right) > 0.$$

In each line, positivity is to be understood in the sense of $(n-1, n-1)$ forms.

Proof. We will prove the statement in the case $n \equiv 0 \pmod 4$, as all other cases are similar. Suppose that χ is a subsolution in the sense of Lemma 3.3. Since the statement is pointwise, it suffices to fix a point $x_0 \in X$, and coordinates so that α is the identity at x_0 and $\chi(x_0)$ is diagonal with entries $\mu_1, \ldots, \mu_n$. By assumption, for every $1 \leqslant j \leqslant n$ we have

$$(n-1)\frac{\pi}{2} > \sum_{i\neq j} \arctan(\mu_i) > \hat{\Theta} - \frac{\pi}{2}.$$

In other words

$$(n-1)\frac{\pi}{2} > \mathrm{Arg}\left(\prod_{j\neq i}(1+\sqrt{-1}\mu_j)\right) > \hat{\Theta} - \frac{\pi}{2}, \tag{8.2}$$

where again we have fixed the branch cut of Arg by setting it to be zero when $\mu_1 = \cdots = \mu_n = 0$. If $\hat{\Theta} = (n-1)\frac{\pi}{2}$, then the fact that $n \equiv 0 \pmod 4$

along with (8.2) implies

$$\mathrm{Im}\left(\prod_{j\neq i}(1+\sqrt{-1}\mu_j)\right) < 0.$$

Since $\cot(\hat{\Theta}) = 0$ in this case, we obtain (8.1). Otherwise, (8.2) implies

$$\arctan\left(\frac{\mathrm{Im}\prod_{i\neq j}(1+\sqrt{-1}\mu_i)}{\mathrm{Re}\prod_{i\neq j}(1+\sqrt{-1}\mu_i)}\right) > \hat{\Theta} - \frac{\pi}{2} - k\pi$$

where on the right hand side, we choose $k \in \mathbb{Z}$ so that $\hat{\Theta} - \frac{\pi}{2} - k\pi \in (-\frac{\pi}{2}, 0)\cup(0, \frac{\pi}{2})$. Since $\tan(\cdot)$ is increasing and non-zero on $(-\frac{\pi}{2}, \frac{\pi}{2})$, we obtain

$$\frac{\mathrm{Im}\prod_{i\neq j}(1+\sqrt{-1}\mu_i)}{\mathrm{Re}\prod_{i\neq j}(1+\sqrt{-1}\mu_i)} > -\cot(\hat{\Theta}).$$

Above we have used the elementary fact that $\tan(x-\pi/2) = -\cot(x)$ for $x \neq 0 \pmod{\pi}$. By (8.2), the complex number $\prod_{j\neq i}(1+\sqrt{-1}\mu_j)$ has argument lying in the interval $((n-3)\frac{\pi}{2}, (n-1)\frac{\pi}{2})$. Since $n \equiv 0 \pmod 4$, this implies that it has negative real part. As a result, we have

$$\mathrm{Im}\prod_{i\neq j}(1+\sqrt{-1}\mu_i) < -\cot(\hat{\Theta})\mathrm{Re}\prod_{i\neq j}(1+\sqrt{-1}\mu_i).$$

Since this holds for all j, we obtain that (8.1) holds in the sense of $(n-1, n-1)$ forms.

The reverse implication holds by essentially the same argument. Suppose that χ satisfies (8.1). Since $\chi \in \Omega$ we get

$$\mathrm{Arg}\int_X (\alpha+\sqrt{-1}\chi)^n = \hat{\Theta} \mod 2\pi.$$

By assumption $\Theta_\alpha(\chi) > (n-2)\frac{\pi}{2}$, and so it follows that there exists a point $x_0 \in X$ such that $\Theta_\alpha(\chi) = \hat{\Theta} \in ((n-2)\frac{\pi}{2}, n\frac{\pi}{2})$. In particular, in a neighbourhood of x_0, χ defines a subsolution in the sense of Lemma 3.3. The set of points $U \subset X$ where χ defines a subsolution is thus open and non-empty. On the other hand, it is also closed. To see this assume we can find points $p_j \in U$ converging to p, and at p there exists a j such that

$$\sum_{i\neq j}\arctan(\mu_j) = \hat{\Theta} - \frac{\pi}{2}.$$

The above computation implies that, at p the $(n-1, n-1)$ form

$$-\mathrm{Im}(\alpha + \sqrt{-1}\chi)^{n-1} - \cot(\hat{\Theta})\mathrm{Re}(\alpha + \sqrt{-1}\chi)^{n-1}$$

is positive, but *not* strictly positive, which is a contradiction. Since X is connected, it follows that χ is a subsolution everywhere. $\square$

Notice that if χ is a subsolution to the deformed Hermitian-Yang-Mills equation (1.2) in the sense of Lemma 3.3, then in fact we obtain a large set of inequalities that the eigenvalues of χ with respect to α must satisfy. Namely, at a point $x_0 \in X$, and in coordinates so that α is the identity at x_0 and $\chi(x_0)$ is diagonal with entries $\mu_1, \ldots, \mu_n$, then for every choice of ℓ distinct indices $j_1, \ldots, j_\ell$, and every $1 \leqslant \ell \leqslant n-1$, we must have

$$\sum_{i \notin \{j_1, \ldots, j_n\}} \arctan(\mu_i) \geqslant \hat{\Theta} - \ell\frac{\pi}{2}.$$

Of course, for any $\ell > 1$, these inequalities are all implied by the definition of a subsolution, so we have not really gained anything new. On the other hand, this observation suggests a cohomological obstruction to the existence of solutions for the deformed Hermitian-Yang-Mills equation. In order to explain this, we first prove

Lemma 8.2. *A* $(1,1)$ *form* $\chi \in [\omega]$ *with* $\Theta_\alpha(\chi) \in ((n-2)\frac{\pi}{2}, n\frac{\pi}{2})$ *is a subsolution to the deformed Hermitian-Yang-Mills equation if and only if, for any* $1 \leqslant p \leqslant n-1$, *and any non-zero, simple, positive* $(n-p, n-p)$ *form* β, *we have*

$$\mathrm{Im}\left[\frac{-(-\sqrt{-1})^p(\alpha + \sqrt{-1}\chi)^p \wedge \beta}{-(-\sqrt{-1})^n e^{\sqrt{-1}\hat{\Theta}}\alpha^n}\right] > 0. \tag{8.3}$$

Proof. The proof is a matter of linear algebra. Recall that a smooth (k,k) form β defined on an open set is said to be a simple positive form if it can be written as

$$\beta = (\sqrt{-1})^k \beta_1 \wedge \overline{\beta_1} \wedge \beta_2 \wedge \overline{\beta_2} \wedge \cdots \wedge \beta_k \wedge \overline{\beta_k}$$

for smooth $(1,0)$ forms β_j [28]. Since the statement is pointwise, we again fix a point $x_0 \in X$, and coordinates so that α is the identity at x_0 and $\chi(x_0)$ is diagonal with entries $\mu_1, \ldots, \mu_n$. For any p, we can have

$$(\alpha + \sqrt{-1}\chi)^p = (\sqrt{-1})^p p! \sum_J \prod_{j \in J}(1 + \sqrt{-1}\mu_j) dz^J \wedge \overline{dz^J}$$

where the sum is over ordered sets $J \subset \{1, \dots, n\}$ of cardinality p. Suppose that χ is a subsolution. Then by the above remarks we know that, for any J we have

$$\prod_{j\in J}(1+\sqrt{-1}\mu_j) \in \mathbb{R}_{>0}e^{\sqrt{-1}\Theta_J},$$

where $\Theta_J > \hat{\Theta} - (n-p)\frac{\pi}{2}$. In particular, for each J

$$\mathrm{Im}\left[\frac{-(-\sqrt{-1})^p\prod_{j\in J}(1+\sqrt{-1}\mu_j)}{-(-\sqrt{-1})^n e^{\sqrt{-1}\hat{\Theta}}}\right] > 0.$$

Let β be any non-zero simple positive $(n-p, n-p)$ form. Then we can write

$$\beta = (\sqrt{-1})^{n-p}\sum_J c_J dz^{J^c} \wedge \overline{dz^{J^c}} + \tilde{\beta}$$

for a smooth $(n-p, n-p)$ form $\tilde{\beta}$ satisfying $\tilde{\beta} \wedge dz^J \wedge \overline{dz^J} = 0$ for all J. Here again the sum is over ordered sets $J \subset \{1, \dots, n\}$ of cardinality p, and J^c denotes the ordered complement of J. The coefficients c_J are necessarily real, non-negative, and at least one c_J must be strictly positive since

$$p!(\sqrt{-1})^n\sum_J c_J dz_1 \wedge \overline{dz_1} \wedge \cdots \wedge dz_n \wedge \overline{dz_n} = \beta \wedge \alpha^p.$$

The right hand side is positive and not identically zero, since β is non-zero. Thus we have

$$\frac{(\alpha+\sqrt{-1}\chi)^p\wedge\beta}{\alpha^n} = \sum_J c_J\left(\prod_{j\in J}(1+\sqrt{-1}\mu_j)\right).$$

The right hand side is a positive linear combination of complex numbers lying in the upper half-plane and so lies in the upper-half plane, proving one implication. The reverse implication is trivial, by taking $\beta = (\sqrt{-1})^{n-p}dz^{J^c} \wedge \overline{dz^{J^c}}$ for every ordered set J of cardinality p.

□

The upshot of this linear algebra is the following proposition, which is essentially a corollary of Lemma 8.2

Proposition 8.3. *For every subvariety $V \subset X$, define*

$$Z_V := -\int_V e^{-\sqrt{-1}(\alpha+\sqrt{-1}\omega)}. \tag{8.4}$$

If there exists a solution to the deformed Hermitian-Yang-Mills equation, then for every proper subvariety $V \subset X$ we have

$$\mathrm{Im}\left(\frac{Z_V}{Z_X}\right) > 0.$$

Note that by convention we only integrate the term in the formal expansion of $e^{-\sqrt{-1}(\alpha+\sqrt{-1}\omega)}$ which has order $\dim V$. When $[\omega] = c_1(L)$, this formula is equivalent to

$$Z_V(L) = -\int_V e^{-\sqrt{-1}\alpha}\mathrm{ch}(L). \tag{8.5}$$

It is easy to check that if L admits a solution of the dHYM equation with $\hat{\Theta} \in ((n-2)\frac{\pi}{2}, n\frac{\pi}{2})$ then $Z_X(L)$ lies in the upper half plane. Let us denote by $\mathrm{Arg}_{p.v.}$ the principal value of Arg, valued in $(-\pi, \pi]$. Then, in the notation of Proposition 8.3 we have $\mathrm{Im}\left(\frac{Z_V}{Z_X}\right) > 0$ implies

$$\mathrm{Arg}_{p.v.}\, Z_V(L) > \mathrm{Arg}_{p.v.}\, Z_X(L).$$

The numbers Z_V appearing in (8.5) bear a resemblance to the various notions central charge appearing in stability conditions in several physical and mathematical theories. For example, we refer the reader to the works of Douglas [17, 18, 19], Bridgeland [2], and Thomas [47] to name just a few. We hope to further elucidate this observation in future work.

Additionally, the condition appearing in Proposition 8.3 is, at least heuristically, similar to the algebro-geometric stability notions appearing in other problems in complex geometry. Perhaps most notably, the notion of Mumford-Takemoto stability pertaining to the existence of Hermitian-Einstein metrics on holomorphic vector bundles [15, 50], and the recent stability condition posed by Lejmi-Székelyhidi for the convergence of the J-equation, and more generally existence of solutions to the inverse σ_k-equations [31]. Let us briefly recount this conjecture in the setting of the J-equation.

Conjecture 8.4 ([31])**.** *Let (X, α) be a Kähler manifold, and $[\omega]$ another Kähler class. For every subvariety $V \subset X$ with $\dim V = p$ define*

$$c_V := \frac{p\int_V \omega^{\dim p-1} \wedge \alpha}{\int_V \omega^p}.$$

Then there exists a solution to the J-equation if and only if $c_X > c_V$ for all proper subvarieties $V \subset X$.

This conjecture is known to hold when $\dim X = 2$, thanks to the third authors solution of the Calabi conjecture [57] and work of Demailly-Păun [13]. The conjecture also holds when X is a complex torus, due to Lejmi-Székelyhidi. Recently, the first author and Székelyhidi [10] have proven the conjecture in the case that X is toric. It is interesting to note that the stability condition in Conjecture 8.4 arises from a modification of K-stability by considering certain special test configurations arising from deformation to the normal cone. We expect that the stability type condition in Proposition 8.3 can be realized in a similar manner, a point which we will address in future work. Finally, we note;

Proposition 8.5. *If $\dim X = 2$, then a solution to the deformed Hermitian-Yang-Mills equation exists if and only if, for every curve $C \subset X$ we have*

$$\mathrm{Im}\left(\frac{Z_C}{Z_X}\right) > 0. \tag{8.6}$$

Proof. First, an application of the Hodge index theorem shows that $Z_X \neq 0$. Let us assume that $\mathrm{Im}(Z_X) > 0$. If $\mathrm{Im}(Z_X) < 0$, then we can replace $[\omega]$ with $[-\omega]$, and if $\mathrm{Im}(Z_X) = 0$, then the condition in (8.6) is vacuous, and a solution always exists, as observed in [26]. We can therefore assume that $\hat{\Theta} \in (0, \pi)$, and so

$$1 - \int_X \omega^2 = 2\cot(\hat{\Theta}) \int_X \alpha \wedge \omega. \tag{8.7}$$

It was observed in [26] that a solution to the deformed Hermitian-Yang-Mills equation exists if and only if the class $[\cot(\hat{\Theta})\alpha + \omega]$ is Kähler, thanks to the third authors solution of the Calabi conjecture [57]. Since $[\alpha]$ is Kähler, the class $[\Omega_T] := [(T + \cot(\hat{\Theta}))\alpha + \omega]$ is a Kähler class for $T \gg 0$. Suppose there exists a time $T \geqslant 0$ where $[\Omega_T]$ lies on the boundary of the Kähler cone – that is, $[\Omega_T]$ is nef, but not Kähler. First, we claim that $[\Omega_T]$ is big. By [13, Theorem 2.12] it suffices to check that $\int_X \Omega_T^2 > 0$. We compute

$$\begin{aligned}\int_X \Omega_T^2 &= (T + \cot(\hat{\Theta}))^2 + 2(T + \cot(\hat{\Theta_X})) \int_X \alpha \wedge \omega + \int_X \omega^2 \\ &= (T + \cot(\hat{\Theta}))^2 + 1 + 2T \int_X \alpha \wedge \omega,\end{aligned}$$

where we have used (8.7). Note that since $\mathrm{Im} Z_X > 0$ we have

$$2\int_X \alpha \wedge \omega = \mathrm{Im}(\alpha + \sqrt{-1}\omega)^2 > 0,$$

and so the above computation implies

$$\int_X \Omega_T^2 \geqslant 1.$$

Finally, by the main theorem of [13] (see also [11]), we can conclude that $[\Omega_T]$ is Kähler provided $\int_C \Omega_T > 0$ for any curve $C \subset X$. Fix $C \subset X$. Define

$$\hat{\Theta}_C = \mathrm{Arg}_{p.v} \int_C (\alpha + \sqrt{-1}\omega)$$

Since $\mathrm{Im}\left(\frac{Z_C}{Z_X}\right) > 0$, and $\mathrm{Im} Z_X > 0$, one easily see that $\hat{\Theta}_C \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\hat{\Theta}_C > \hat{\Theta} - \frac{\pi}{2}$, and so the following equality makes sense;

$$\tan(\hat{\Theta}_C) \int_C \alpha = \int_C \omega.$$

Because $\tan(\cdot)$ is defined an increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have

$$\tan(\hat{\Theta}_C) > \tan(\hat{\Theta} - \frac{\pi}{2}) = -\cot(\hat{\Theta}).$$

We obtain

$$\int_C \omega = \tan(\hat{\Theta}_C) \int_C \alpha \geqslant -\cot(\hat{\Theta}) \int_C \alpha.$$

Since $T \geqslant 0$,

$$\int_C \Omega_T = T \int_C \alpha + \int_C \cot(\hat{\Theta})\alpha + \omega > T \int_C \alpha > 0,$$

and so $[\Omega_T]$ is Kähler as long as $T \geqslant 0$, and the proposition follows. □

We end by remarking that one could hope for a similar framework for the lower branches of the deformed Hermitian-Yang-Mills equation – that is, when $\Theta_X \leqslant (n-2)\frac{\pi}{2}$. However, due to the lack of convexity in the lower branches we expect that the deformed Hermitian-Yang-Mills equation with subcritical phase may be extremely poorly behaved from an analytic and algebraic stand point. For example, in the real case Nadirashvili-Vlăduţ [34] and Wang-Yuan [52] have demonstrated the existence of $C^{1,\beta}$ viscosity solutions to the special Lagrangian equation with subcritical phase on a ball in $\mathbb{R}^3$ for $n \geqslant 3$ which are not C^2 in the interior. Furthermore, Wang-Yuan [52] have shown the existence of smooth solutions $\{u^\varepsilon\}$ to the special Lagrangian equation with fixed, subcritical phase on a ball in $\mathbb{R}^3$ such that $\|Du^\varepsilon\|_{L^\infty} < C$, but so that $|D^2u^\varepsilon|(0)$ blows up as $\varepsilon \to 0$.

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Received August 5, 2019