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One-sided curvature estimates for H-disks

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In this paper we prove an extrinsic one-sided curvature estimate for disks embedded in $\mathbb{R}^3$ with constant mean curvature, which is independent of the value of the constant mean curvature. We apply this extrinsic one-sided curvature estimate in [26] to prove a weak chord arc result for these disks. In Section 4 we apply this weak chord arc result to obtain an intrinsic version of the one-sided curvature estimate for disks embedded in $\mathbb{R}^3$ with constant mean curvature. In a natural sense, these one-sided curvature estimates generalize respectively, the extrinsic and intrinsic one-sided curvature estimates for minimal disks embedded in $\mathbb{R}^3$ given by Colding and Minicozzi in Theorem 0.2 of [8] and in Corollary 0.8 of [9].

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1. Introduction

In this paper we prove a one-sided curvature estimate for disks embedded in $\mathbb{R}^3$ with constant mean curvature. An important feature of this estimate is its independence on the value of the constant mean curvature.

For clarity of exposition, we will call an oriented surface Σ immersed in $\mathbb{R}^3$ an *H-surface* if it is *embedded*, *connected* and it has *non-negative constant mean curvature H*. We will call an H-surface an *H-disk* if the H-surface is homeomorphic to a closed unit disk in the Euclidean plane. We remark that this definition of H-surface differs from the one given in [27], where we restricted to the case where $H > 0$. In this paper $\mathbb{B}(R)$ denotes the open

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ball in $\mathbb{R}^3$ centered at the origin $\vec{0}$ of radius R, $\overline{\mathbb{B}}(R)$ denotes its closure and for a point p on a surface Σ in $\mathbb{R}^3$, $|A_\Sigma|(p)$ denotes the norm of the second fundamental form of Σ at p.

The main result of this paper, which is Theorem 1.1 below, states that if $\mathcal{D}$ is an H-disk which lies on one side of a plane Π, then the norm of the second fundamental form of $\mathcal{D}$ cannot be arbitrarily large at points sufficiently far from the boundary of $\mathcal{D}$ and sufficiently close to Π. This estimate is inspired by, depends upon and provides a natural generalization of the Colding-Minicozzi one-sided curvature estimate for minimal disks embedded in $\mathbb{R}^3$, which is given in Theorem 0.2 in [8].

Theorem 1.1 (One-sided curvature estimate for H-disks)**.** *There exist $\varepsilon \in (0, \frac{1}{2})$ and $C \geq 2\sqrt{2}$ such that for any $R > 0$, the following holds. Let $\mathcal{D}$ be an H-disk such that*

$$\mathcal{D} \cap \mathbb{B}(R) \cap \{x_3 = 0\} = \varnothing \quad \textit{and} \quad \partial\mathcal{D} \cap \mathbb{B}(R) \cap \{x_3 > 0\} = \varnothing.$$

Then:

$$\sup_{x \in \mathcal{D} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\}} |A_{\mathcal{D}}|(x) \leq \frac{C}{R}. \tag{1}$$

In particular, if $\mathcal{D} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\} \neq \varnothing$, then $H \leq \frac{C}{R}$.

In contrast to the minimal case, the constant C in equation (1) need not improve with smaller choices of ε. To see this, let S be the sphere of radius $\frac{1}{2}$ centered at $(0, 0, \frac{1}{2})$. Each surface in the sequence $E_n = (S + (0, 0, \frac{1}{n})) \cap \overline{\mathbb{B}}(1)$ is a compact disk that satisfies the hypotheses of the theorem for $R = 1$, has $|A_{E_n}| = 2\sqrt{2}$ and, as n tends to infinity, E_n moves arbitrarily close to the origin. In particular these examples show that the constant C in the above theorem must be at least $2\sqrt{2}$ no matter how small ε is.

Theorem 1.1 plays an important role in deriving a weak chord arc property for H-disks in our papers [26, 25], which we describe in Section 4. This weak chord arc property was inspired by and gives a generalization of Proposition 1.1 in [9] by Colding and Minicozzi for 0-disks to the case of H-disks; we apply this weak chord arc property to obtain an intrinsic version of the one-sided curvature estimate in Theorem 1.1, which we describe in Theorem 4.5. In the case $H = 0$, this intrinsic one-sided curvature estimate follows from Corollary 0.8 in [9] by Colding and Minicozzi.

2. Preliminaries

Throughout this paper, we use the following notation. Given $a, b, R > 0$, $p \in \mathbb{R}^3$ and Σ a surface in $\mathbb{R}^3$:

- $\mathbb{B}(p, R)$ is the open ball of radius R centered at p.
- $\mathbb{B}(R) = \mathbb{B}(\vec{0}, R)$, where $\vec{0} = (0, 0, 0)$.
- For $p \in \Sigma$, $B_\Sigma(p, R)$ denotes the open intrinsic ball in Σ of radius R.
- $C(a, b) = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq a^2, \ |x_3| \leq b\}$.
- $A(r_1, r_2) = \{(x_1, x_2, 0) \mid r_2^2 \leq x_1^2 + x_2^2 \leq r_1^2\}$.

We next recall several results from our manuscript [27] that will be used in this paper.

We first introduce the notion of multi-valued graph, see [6] for further discussion and Figure 1. Intuitively, an N-valued graph is a simply-connected

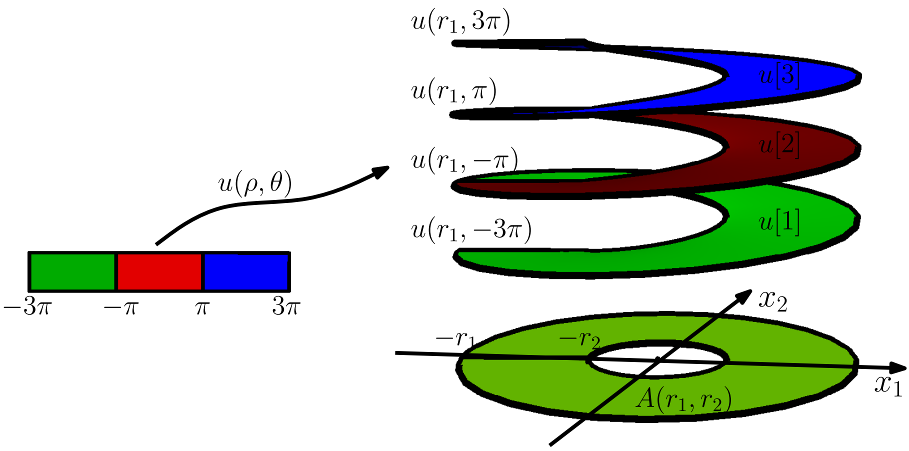

Figure 1: A right-handed 3-valued graph.

embedded surface covering an annulus such that over a neighborhood of each point of the annulus, the surface consists of N graphs. The stereotypical infinite multi-valued graph is half of the helicoid, i.e., half of an infinite double-spiral staircase.

Definition 2.1 (Multi-valued graph)**.** Let $\mathcal{P}$ denote the universal cover of the punctured (x_1, x_2)-plane, $\{(x_1, x_2, 0) \mid (x_1, x_2) \neq (0, 0)\}$, with global coordinates (ρ, θ).

1. An *N-valued graph over the annulus* $A(r_1, r_2)$ is a single valued graph $u(\rho, \theta)$ over $\{(\rho, \theta) \mid r_2 \leq \rho \leq r_1,\ |\theta| \leq N\pi\} \subset \mathcal{P}$, if N is odd, or over $\{(\rho, \theta) \mid r_2 \leq \rho \leq r_1,\ (-N+1)\pi \leq \theta \leq \pi(N+1)\} \subset \mathcal{P}$, if N is even.
2. An N-valued graph $u(\rho, \theta)$ over the annulus $A(r_1, r_2)$ is called *righthanded* [*lefthanded*] if whenever it makes sense, $u(\rho, \theta) < u(\rho, \theta + 2\pi)$ $[u(\rho, \theta) > u(\rho, \theta + 2\pi)]$
3. The set $\{(r_2, \theta, u(r_2, \theta)), \theta \in [-N\pi, N\pi]\}$ when N is odd (or the set $\{(r_2, \theta, u(r_2, \theta)), \theta \in [(-N+1)\pi, (N+1)\pi]\}$ when N is even) is the *inner boundary* of the N-valued graph.

From Theorem 2.23 in [27] one obtains the following, detailed geometric description of an H-disk with large norm of the second fundamental form at the origin. The precise meaning of certain statements below are made clear in [27] and we refer the reader to that paper for further details.

Theorem 2.2. *Given $\varepsilon, \tau > 0$ and $\overline{\varepsilon} \in (0, \varepsilon/4)$ there exist constants $\Omega_\tau := \Omega(\tau)$, $\omega_\tau := \omega(\tau)$ and $G_\tau := G(\varepsilon, \tau, \overline{\varepsilon})$ such that if M is an H-disk, $H \in (0, \frac{1}{2\varepsilon})$, $\partial M \subset \partial \mathbb{B}(\varepsilon)$, $\vec{0} \in M$ and $|A_M|(\vec{0}) > \frac{1}{\eta} G_\tau$, for $\eta \in (0, 1]$, then for any $p \in \overline{\mathbb{B}}(\vec{0}, \eta\overline{\varepsilon})$ that is a maximum of the function $|A_M|(\cdot)(\eta\bar{\varepsilon} - |\cdot|)$, after translating M by $-p$, the following geometric description of M holds:*

On the scale of the norm of the second fundamental form M looks like one or two helicoids nearby the origin and, after a rotation that turns these helicoids into vertical helicoids, M contains a 3-valued graph u over $A(\varepsilon/\Omega_\tau, \frac{\omega_\tau}{|A_M|(\vec{0})})$ with the norm of its gradient less than τ and with its inner boundary in $\mathbb{B}(10\frac{\omega_\tau}{|A_M|(\vec{0})})$.

Theorem 2.2 was inspired by the pioneering work of Colding and Minicozzi in the minimal case [5, 6, 7, 8]; however in the constant positive mean curvature setting this description has led to a different conclusion, that is the existence of extrinsic radius and curvature estimates stated below, which do not depend on the results in this paper.

Theorem 2.3 (Extrinsic radius estimates [27])**.** *There exists an $\mathcal{R}_0 \geq \pi$ such that for any H-disk $\mathcal{D}$ with $H > 0$,*

$$\sup_{p\in\mathcal{D}}\{d_{\mathbb{R}^3}(p, \partial\mathcal{D})\} \leq \frac{\mathcal{R}_0}{H},$$

where $d_{\mathbb{R}^3}$ refers to extrinsic distance.

Theorem 2.4 (Extrinsic curvature estimates [27])**.** *Given $\delta, \mathcal{H} > 0$, there exists a constant $K_0(\delta, \mathcal{H})$ such that for any H-disk $\mathcal{D}$ with $H \geq \mathcal{H}$,*

$$\sup_{\{p\in\mathcal{D}\,|\,d_{\mathbb{R}^3}(p,\partial\mathcal{D})\geq\delta\}}|A_{\mathcal{D}}| \leq K_0(\delta, \mathcal{H}).$$

Indeed, since the plane and the helicoid are complete simply-connected minimal surfaces properly embedded in $\mathbb{R}^3$, a radius estimate does not hold in the minimal case. Moreover rescalings of a helicoid give a sequence of embedded minimal disks with arbitrarily large norm of the second fundamental form at points arbitrarily far from their boundary curves; therefore in the minimal setting, the extrinsic curvature estimates do not hold.

Next, we recall the notion of the flux of a 1-cycle in an H-surface; see for instance the references [13, 14, 30] for further discussions of this invariant.

Definition 2.5. Let γ be a 1-cycle in an H-surface M. The *flux* of γ is $\int_\gamma (H\gamma + \xi) \times \dot{\gamma}$, where ξ is the unit normal to M along γ and γ is parameterized by arc length.

The flux of a 1-cycle in an H-surface M is a homological invariant and we say that M has *zero flux* if the flux of any 1-cycle in M is zero; in particular, since the first homology group of a disk is zero, an H-disk has zero flux.

Finally, we recall the following definition.

Definition 2.6. Let U be an open set in $\mathbb{R}^3$. We say that a sequence of surfaces $\Sigma(n)$ in $\mathbb{R}^3$, has *locally bounded norm of the second fundamental form in U* if for every compact subset B in U, the norms of the second fundamental forms of the surfaces $\Sigma(n)$ are uniformly bounded in B.

3. The proof of Theorem 1.1

Proof of Theorem 1.1. After homothetically scaling, it suffices to prove Theorem 1.1 for H-disks E, where the radius R of the related ambient balls is fixed. Henceforth, we will assume that $R = 1$.

Arguing by contradiction, suppose that Theorem 1.1 fails. Then there exists a sequence of H_n-disks $E(n)$ satisfying the hypotheses of Theorem 1.1 and numbers ε_n going to zero, such that $E(n) \cap \mathbb{B}(\varepsilon_n) \cap \{x_3 > 0\}$ contains points $\widetilde{p}_n$ with $\lim_{n\to\infty} |A_{E(n)}|(\widetilde{p}_n) = \infty$. Since we may assume that $\partial\mathbb{B}(1)$ is transverse to $E(n)$ then, after replacing $E(n)$ by a subdisk containing $\widetilde{p}_n$, we may also assume that $\partial E(n) \subset \partial\mathbb{B}(1) \cap \{x_3 > 0\}$. Note that when $H > 0$, after such a replacement of $E(n)$ by a subdisk, it might be the case that $E(n)$ is not contained in $\mathbb{B}(1)$ or in $\{x_3 > 0\}$ because the convex hull property need not hold.

By the extrinsic curvature estimates given in Theorem 2.4 for H-disks with positive mean curvature, the mean curvatures H_n of the disks $E(n)$ must be tending to zero as n goes to infinity. Also, note that for n large, there exist points $p(n) \in E(n)$ with vertical tangent planes and with $d_{E(n)}(p(n),\widetilde{p}_n)$

converging to zero, where $d_{E(n)}$ refers to intrinsic distance. Otherwise, after replacing by a subsequence, small but fixed sized intrinsic balls centered at the points $\widetilde{p}_n$ would be stable (the inner product of the unit normal to M with the vector $(0,0,1)$ is a nowhere vanishing Jacobi function), thereby having uniform curvature estimates (see for instance Rosenberg, Souam and Toubiana [29] for these estimates) and contradicting our choices of the points $\widetilde{p}_n$ with their norms of the second fundamental form becoming arbitrarily large.

To obtain a contradiction, we are going to analyze the behavior of the set of points $\widehat{\alpha}_n$ in $E(n)$ where the tangent planes are vertical; in particular, $\widehat{\alpha}_n$ contains $p(n)$. We will prove for n large that $\widehat{\alpha}_n$ contains a smooth arc α_n beginning at $p(n)$ that moves downward at a much faster rate than it moves sideways and so α_n must cross the (x_1,x_2)-plane near the origin. The existence of such a curve $\alpha_n \subset E(n)$ will then contradict the fact $E(n)$ is disjoint from the intersection of $\mathbb{B}(1)$ with the (x_1,x_2)-plane.

The next proposition describes the geometry of $E(n)$ around some of its points which are above and close to the (x_1,x_2)-plane and where the tangent planes are vertical. The proposition states that intrinsically close to such points $E(n)$ must look like a homothetically scaled *vertical* helicoid. The proof of this result relies heavily on Theorem 2.2 and on the uniqueness of the helicoid by Meeks and Rosenberg [22]; see also Bernstein and Breiner [1] for a proof of this uniqueness result.

Proposition 3.1. *Consider a sequence of points $q_n \in E(n)\cap C(\frac{1}{2},\frac{1}{2})\cap\{x_3 > 0\}$ with $x_3(q_n)$ converging to zero and where the tangent planes $T_{q_n}E(n)$ to $E(n)$ are vertical. Then the numbers $\lambda_n := |A_{E(n)}|(q_n)$ diverge to infinity and a subsequence of the surfaces $M(n) = \lambda_n(E(n) - q_n)$ converges on compact subsets of $\mathbb{R}^3$ to a vertical helicoid $\mathcal{H}$ containing the x_3-axis and with maximal absolute Gaussian curvature $\frac{1}{2}$ at the origin. Furthermore, after replacing by a further subsequence, the multiplicity of the convergence of the surfaces $M(n)$ to $\mathcal{H}$ is one or two.*

Proof. Crucial to the proof of the proposition is understanding the appropriate scale to study the geometry of the disks $E(n)$ near q_n, which in principle might not be related to the norms of the second fundamental forms of $E(n)$ at the points q_n; later we will relate this new scale to the numbers λ_n appearing in its statement.

Intrinsically near q_n, the surface $E(n)$ is graphical over its tangent plane $T_{q_n}E(n)$. Recall that each tangent plane at q_n is vertical, the sequence of positive numbers $x_3(q_n)$ is converging to zero and $E(n)$ lies above the (x_1,x_2)-plane near q_n. It follows that $B_{E(n)}(q_n, 2x_3(q_n))$ cannot be a graph with the norm of its gradient less than or equal to 1 over its (orthogonal) projection to $T_{q_n}E(n)$.

By compactness of $\overline{B}_{E(n)}(q_n, 2x_3(q_n))$, there is a largest number $r(n) \in (0, 2x_3(q_n))$ such that $B_{E(n)}(q_n, r(n))$ is a graph with the norm of its gradient at most 1 over its projection to $T_{q_n}E(n)$. Since $r(n) < 2x_3(q_n)$, then $\lim_{n\to\infty} r(n) = 0$.

Consider the sequence of translated and scaled surfaces

$$\Sigma(n) = \frac{1}{r(n)}(E(n) - q_n).$$

We claim that it suffices to prove that a subsequence of the $\Sigma(n)$ converges with multiplicity one or two to a vertical helicoid containing the x_3-axis. To see this claim holds, suppose that a subsequence $\Sigma(n_i)$ of these surfaces converges with multiplicity one or two to a vertical helicoid $\mathcal{H}'$ containing the x_3-axis, then $\lambda := |A_{\mathcal{H}'}|(\vec{0}) \in (0, \infty)$ and

$$\lambda = \lim_{i\to\infty} |A_{\Sigma(n_i)}|(\vec{0}) = \lim_{i\to\infty} r(n_i)|A_{E(n_i)}|(q_{n_i}) = \lim_{i\to\infty} r(n_i)\lambda_{n_i}. \tag{2}$$

Since the numbers $r(n_i)$ are converging to zero, equation (2) implies that the numbers λ_{n_i} must diverge to infinity, the sequence of surfaces

$$M(n_i) = \lambda_{n_i}(E(n_i) - q_{n_i}) = \lambda_{n_i} r(n_i)\Sigma(n_i)$$

converges with multiplicity one or two to $\mathcal{H} = \lambda\mathcal{H}'$ and the proposition follows. Thus, it suffices to prove that a subsequence of the surfaces $\Sigma(n)$ converges with multiplicity one or two to a vertical helicoid containing the x_3-axis.

There are two cases to consider.

Case A: The sequence of surfaces $\Sigma(n)$ has locally bounded norm of the second fundamental form in $\mathbb{R}^3$.

Case B: The sequence of surfaces $\Sigma(n)$ does *not* have locally bounded norm of the second fundamental form in $\mathbb{R}^3$.

Suppose that Case A holds. A standard compactness argument gives that a subsequence of the $\Sigma(n)$ converges C^α to a minimal lamination of $\mathbb{R}^3$ for any $\alpha \in (0, 1)$: see for example any of the references [2, 8, 22, 31] for these arguments when the surfaces $\Sigma(n)$ are minimal surfaces. After possibly replacing by a subsequence, we will assume that the original sequence $\Sigma(n)$ converges to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$.

Let L be the leaf of $\mathcal{L}$ that passes through the origin and recall that L has a vertical tangent plane at the origin. Since L is not a plane, because it is not a graph over this vertical tangent plane with the norm of its gradient less than 1, then L is a non-flat leaf of $\mathcal{L}$. By construction L has bounded norm of the second fundamental form in compact subsets of $\mathbb{R}^3$. By Theorem 1.6 in [22], L is a non-limit leaf of $\mathcal{L}$ and one of the following must hold:

- L is properly embedded in $\mathbb{R}^3$.
- L is properly embedded in an open half-space of $\mathbb{R}^3$.
- L is properly embedded in an open slab of $\mathbb{R}^3$.

Since L is properly embedded in a simply-connected open set $\mathcal{O}$ of $\mathbb{R}^3$, then it separates $\mathcal{O}$ and so it is orientable. Since L is non-flat and it is complete, then L is not stable by the classical results of do Carmo, Peng, Fisher-Colbrie and Schoen [10, 11]. We claim that the instability of L implies that the multiplicity of convergence of domains on $\Sigma(n)$ can be at most one or two. Otherwise, suppose that the multiplicity of convergence of portions of the surfaces $\Sigma(n)$ to L is greater than two and let $\Omega \subset L$ be a smooth compact unstable domain. A standard argument that we now sketch produces a contradiction to the existence of Ω. By separation properties, the uniform boundedness of the norms of the second fundamental forms of the compact oriented surfaces $\Sigma(n)$ in a small ε-neighborhood of Ω in $\mathcal{O}$ and the properness of L, together with the fact that L is not a limit leaf, we have that for n large there exist three pairwise disjoint compact domains $\Omega_1(n), \Omega_2(n), \Omega_3(n)$ in $\Sigma(n)$ that are converging to Ω, each domain is a normal graph over Ω and the unit normal vectors of $\Omega_1(n)$ and of $\Omega_3(n)$ at corresponding points over points of Ω have positive inner products converging to 1 as n goes to infinity. Moreover, if we let $f_1(n), f_3(n)\colon \Omega \to \mathbb{R}$ denote the related graphing functions, we can assume that $(f_1(n) - f_3(n)) > 0$. After renormalizing this difference as

$$F(n) = \frac{f_1(n) - f_3(n)}{(f_1(n) - f_3(n))(q)}$$

for some $q \in \mathrm{Int}(\Omega)$, elliptic PDE theory implies that a subsequence of the $F(n)$ converges to a positive Jacobi function on Ω, which implies Ω is stable. This contradiction implies that the multiplicity of convergence is one or two.

Since the multiplicity of convergence of portions of the $\Sigma(n)$ to L is one or two, then, for n large, we can lift any simple closed curve γ on the orientable surface L to one or two pairwise disjoint normal graphs over γ and contained in $\Sigma(n)$, where the number of such lifts depends on the multiplicity of the convergence. This construction gives either one or two simple closed lifted curves in $\Sigma(n)$. Hence, since the domains $\Sigma(n)$ have genus zero, it follows that any pair of transversely intersecting simple closed curves on L cannot intersect in exactly one point; therefore, L also has genus zero. By the properness of finite genus leaves of a minimal lamination of $\mathbb{R}^3$ given by Meeks, Perez and Ros in Theorem 7 in [15], L must be properly embedded in $\mathbb{R}^3$. In fact, this discussion demonstrates that all the leaves of $\mathcal{L}$ are proper. Since the leaf L is not flat, then the strong halfspace theorem by Hoffman and Meeks in [12] implies that L is the only leaf in $\mathcal{L}$. If L has more than one end then, by the main result of Choi, Meeks and White in [3], L has non-zero flux and so, by the nature of the convergence of the $\Sigma(n)$ to L, the domains $\Sigma(n)$ must have non-zero flux as well. However, this leads to a contradiction since flux is a homological invariant and thus $\Sigma(n)$, being topologically a disk, has zero flux. Therefore L must have genus zero and one end, which implies that it is simply-connected. By [22], L is a helicoid and it remains to show that it is a vertical helicoid.

Claim 3.2. *The leaf L is a vertical helicoid.*

Proof. Recall that L is the limit of

$$\Sigma(n) = \frac{1}{r(n)}(E(n) - q_n).$$

Therefore the helicoid L contains the origin, the tangent plane at the origin is vertical and, by the definition of $r(n)$, the geodesic ball centered at the origin is not a graph with norm of its gradient bounded by one over its tangent plane at the origin.

Let p be the point on the axis of the helicoid L that is closest to the origin. By the geometric properties of a helicoid and the discussion in the previous paragraph, it follows that there exist $k_1, k_2 > 0$ such that $|p| < k_1$ and that $|A_L|(p) > k_2 \in (0,1)$. Let $p'_n \in \Sigma(n)$ be a sequence of points such that $\lim_{n\to\infty} p'_n = p$ and let $p_n \in E(n)$ be the sequence of points such that $p'_n = \frac{p_n - q_n}{r(n)}$. Recall that $\lim_{n\to\infty} x_3(q_n) = 0$ and that $|q_n|^2 - x_3(q_n)^2 \leq \frac{1}{4}$. Thus, for n sufficiently large the following holds:

- $|p_n - q_n| \leq r(n)|p'_n| \leq 2k_1 r(n)$;
- $|p_n| \leq |p_n - q_n| + |q_n| \leq 2k_1 r(n) + \sqrt{\frac{1}{4} + x_3^2(q_n)}$;
- $|x_3(p_n)| \leq |x_3(p_n - q_n)| + |x_3(q_n)| \leq |p_n - q_n| + x_3(q_n) \leq 2k_1 r(n) + x_3(q_n)$.

In particular, on the original scale, the points p_n and q_n have the same limit.

The new sequence of surfaces

$$\Sigma'(n) = \frac{1}{r(n)}(E(n) - p_n)$$

converges to the translated helicoid $\widehat{L} = L - p$ whose axis contains the origin. Clearly, in order to prove the claim, it suffices to show that $\widehat{L}$ is a vertical helicoid. Suppose that the axis of $\widehat{L}$ makes an angle $\theta > 0$ with the x_3-axis. Let $\tau > 0$ and let $G_\tau := G(\frac{1}{4}, \tau, \frac{1}{20})$ as given by Theorem 2.2 with $\varepsilon = \frac{1}{4}$ and $\overline{\varepsilon} = \frac{1}{20}$. Let $z'_n \in \Sigma'(n) \cap \mathbb{B}(\frac{G_\tau}{20k_2})$ be a sequence of points where the maximum of the functions

$$|A_{\Sigma'(n)\cap\mathbb{B}(\frac{G_\tau}{20k_2})}|(\cdot)(\frac{G_\tau}{20k_2} - |\cdot|)$$

is obtained. Clearly, $\lim_{n\to\infty} |z'_n| = 0$. Let $z_n \in E(n)$ be the sequence of points such that $z'_n = \frac{1}{r(n)}(z_n - p_n)$. By the previous discussion, $\frac{1}{r(n)}(E(n) - z_n)$ converges to the helicoid $\widehat{L}$.

Consider the sequence of surfaces

$$M_\tau(n) = E(n) - p_n = r(n)\Sigma'(n).$$

Recall that $\partial E(n) \subset \partial\mathbb{B}(1) \cap \{x_3 > 0\}$, $E(n) \cap \{x_3 = 0\} = \varnothing$, $|p_n| \leq 2k_1 r(n) + \sqrt{\frac{1}{4} + x_3^2(q_n)}$ and $|x_3(p_n)| \leq 2k_1 r(n) + x_3(q_n)$. Therefore, when n is sufficiently large and abusing the notation, we can let $M_\tau(n)$ denote the subdisk of $M_\tau(n)$ containing the component of $M_\tau(n) \cap \mathbb{B}(\frac{1}{4})$ that contains the origin and with boundary in $\partial\mathbb{B}(\frac{1}{4})$. Recall that the constant mean curvatures H_n of $E(n)$, and thus of $M_\tau(n)$, are going to zero as n goes to infinity. Therefore, when n is sufficiently large, we have that $H_n \in (0, \frac{1}{8})$. Note that

$$M_\tau(n) \cap \{x_3 = -2k_1 r_n - x_3(q_n)\} = \varnothing.$$

Let $\eta_n = \frac{G_\tau r(n)}{k_2}$. Then, $\lim_{n\to\infty} \eta_n = 0$ and for n sufficiently large, the sequence $M_\tau(n)$ satisfies the following properties.

- $|A_{M_\tau(n)}|(\vec{0}) > \frac{k_2}{r(n)} = \frac{G_\tau}{\eta_n}$;
- the maximum of the function $|A_{M_\tau(n)\cap\mathbb{B}(\frac{\eta_n}{20})}|(\cdot)(\frac{\eta_n}{20} - |\cdot|)$ is obtained at the point $z_n - p_n$;
- on the scale of the norm of the second fundamental form $M_\tau(n) - z_n$ looks like the helicoid $\widehat{L}$.

Let Π denote the plane containing $z_n - p_n$ and perpendicular to the axis of $\widehat{L}$. Then, under these hypotheses, Theorem 2.2 implies that there exist constants $\Omega(\tau)$ and $\omega(\tau)$ such that $M_\tau(n)$ contains a 3-valued graph over the annulus in Π centered at z_n of outer radius $\frac{1}{4\Omega(\tau)}$ and inner radius $\frac{\omega(\tau)}{|A_{M_\tau(n)}|(\vec{0})}$. Note that $\lim_{n\to\infty} z_n = \vec{0}$ and that $\lim_{n\to\infty} \frac{\omega(\tau)}{|A_{M_\tau(n)}|(\vec{0})} = 0$. Therefore, by taking τ sufficiently small, depending on θ, and n sufficiently large, this geometric description contradicts the fact that $M_\tau(n) \cap \{x_3 = -2k_1 r(n) + -x_3(q_n)\} = \emptyset$. This contradiction finishes the proof of the claim. $\square$

This finishes the proof of the proposition when Case A holds. To complete the proof of Proposition 3.1, it suffices to demonstrate the following assertion, which is the difficult point in the proof of Proposition 3.1.

Assertion 3.3. *Case B does not occur.*

Proof. Some of the techniques used to eliminate Case B are motivated by techniques and results developed in [4, 19, 21] and their corresponding proofs. Arguing by contradiction, assume that the sequence of surfaces $\Sigma(n)$ does not have locally bounded norm of the second fundamental form in $\mathbb{R}^3$.

For clarity of exposition, we will replace the sequence of surfaces $\Sigma(n)$ by a specific subsequence such that for some non-empty closed set χ in $\mathbb{R}^3$ with χ different from $\mathbb{R}^3$, the new sequence of surfaces has locally bounded norm of the second fundamental form in $\mathbb{R}^3 - \chi$, converges C^α, for any $\alpha \in (0,1)$, to a non-empty minimal lamination $\mathcal{L}_\chi$ of $\mathbb{R}^3 - \chi$ and no further subsequence has locally bounded norm of the second fundamental form in $\mathbb{R}^3 - \chi'$, where χ' is a proper closed subset of χ. This reduction is explained in the next claim.

Lemma 3.4. *After replacing by a subsequence, the sequence of surfaces $\Sigma(n)$ satisfies the following properties:*

1. *There exists a closed non-empty set $\chi \subset \mathbb{R}^3$ such that for every point $s \in \chi$ and for each $k \in \mathbb{N}$, there exists an $N(s,k) \in \mathbb{N}$ such that for each $j \geq N(s,k)$, there exists a point $p(j) \in \Sigma(j) \cap \mathbb{B}(s, \frac{1}{k})$ with $|A_{\Sigma(j)}|(p(j)) \geq k$.*
2. *The sequence $\Sigma(n)$ has locally bounded norm of the second fundamental form in $\mathbb{R}^3 - \chi$.*
3. *The sequence $\Sigma(n)$ converges C^α, for any $\alpha \in (0,1)$, on compact domains of $\mathbb{R}^3 - \chi$ to a non-empty minimal lamination $\mathcal{L}_\chi$ of $\mathbb{R}^3 - \chi$.*
4. *There exists a maximal open horizontal slab or open half-space W in $\mathbb{R}^3 - \chi$, $\vec{0} \in W$ and $\mathcal{L} = \mathcal{L}_\chi \cap W$ is a non-empty minimal lamination of W.*

5. *The leaf L of $\mathcal{L}$ that passes through the origin is non-flat and contains an intrinsic open disk Ω passing through the origin which is the limit of the surfaces $B_{\Sigma(n)}(\vec{0},1)$ and Ω is a graph over its projection to $T_{\vec{0}}\,\Omega$ which is a vertical plane, and with norm of the gradient of the graphing function at most one.*

Proof of Lemma 3.4. We begin by constructing the set χ and the related subsequence of surfaces $\Sigma(n)$ described in the claim. The assumption that the original sequence of surfaces does not have locally bounded norm of the second fundamental form in $\mathbb{R}^3$ implies there exists a point $q(1) \in \mathbb{R}^3$ such that, after replacing this sequence of surfaces by a subsequence $\Gamma_1 := \{\Sigma(1,n)\}_{n\in\mathbb{N}}$, the surfaces in Γ_1 satisfy the following property: For each $k \in \mathbb{N}$, there is a point $p(1,k) \in \Sigma(1,k) \cap \mathbb{B}(q(1),\frac{1}{k})$ with $|A_{\Sigma(1,k)}|(p(1,k)) \geq k$.

Let $\mathbb{Q}$, $\mathbb{Q}^+$ denote the set of rational numbers and the subset of positive rational numbers, respectively. Consider the countable collection of balls

$$\mathcal{B} = \{\mathbb{B}(x,q) \mid x \in \mathbb{Q}^3,\, q \in \mathbb{Q}^+\},$$

and let $\mathcal{B} = \{B_1, B_2, \ldots, B_n, \ldots\}$ be an ordered listing of the elements in $\mathcal{B}$ where $q(1) \in B_1$. If Γ_1 has locally bounded norm of the second fundamental form in $\mathbb{R}^3 - \{q(1)\}$, then $\chi = \{q(1)\}$ and we stop our construction of the set χ. Assume now that Γ_1 does not have locally bounded norm of the second fundamental form in $\mathbb{R}^3 - \{q(1)\}$. Let $B_{n(2)}$ be the first indexed ball in the ordered listing of $\mathcal{B} - \{B_1\}$, such that the following happens: there is a point $q(2) \in B_{n(2)} - \{q(1)\}$, a subsequence $\Gamma_2 := \{\Sigma(2,1), \Sigma(2,2), \ldots, \Sigma(2,k), \ldots\}$ of Γ_1 together with points $p(2,k) \in \Sigma(2,k) \cap \mathbb{B}(q(2),\frac{1}{k})$ where $|A_{\Sigma(2,k)}|(p(2,k)) \geq k$ for all $k \in \mathbb{N}$. Note that $B_{n(2)}$ is just the first ball in the list $\mathcal{B} - \{B_1\}$ that contains a point q different from $q(1)$ such that the norms of the second fundamental forms of the surfaces in the sequence Γ_1 are not bounded in any neighborhood of q, and after choosing such a point, we label it as $q(2)$.

If Γ_2 does not have locally bounded norm of the second fundamental form in $\mathbb{R}^3 - \{q(1), q(2)\}$, then there exists a first ball $B_{n(3)}$ in the ordered list $\mathcal{B} - \{B_1, B_{n(2)}\}$ such that there is a point $q(3) \in [B_{n(3)} - \{q(1), q(2)\}]$ and such that after replacing Γ_2 by a subsequence $\Gamma_3 := \{\Sigma(3,1), \ldots, \Sigma(3,k), \ldots\}$, there are points $p(3,k) \in \Sigma(3,k) \cap \mathbb{B}(q(3),\frac{1}{k})$ with $|A_{\Sigma(3,k)}|(p(3,k)) \geq k$ for all $k \in \mathbb{N}$.

Define $n(1) = 1$. Then, continuing the above construction inductively, we obtain at the m-th stage a subsequence $\Gamma_m = \{\Sigma(m,1), \ldots, \Sigma(m,k), \ldots\}$, $\Gamma_m \subset \Gamma_{m-1}$, a set of distinct points $\{q(1), q(2), \ldots, q(m)\}$ and related balls $B_{n(1)}$, $B_{n(2)}, \ldots, B_{n(m)}$ in $\mathcal{B}$ such that for each $i \in \{1, \ldots, m\}$, each $\Sigma(m,k)$

contains points $p(m,k,i)$ in the balls $\mathbb{B}(q(i),\frac{1}{k})$, where the norm of the second fundamental form of $\Sigma(m,k)$ is at least k. Note that $n(i+1)>n(i)$ for all i.

Let $\Gamma_\infty=\{\Sigma(1,1),\Sigma(2,2),\dots,\Sigma(n,n),\dots\}$ be the related diagonal sequence and let χ be the closure of the set $\{q(n)\}_{n\in\mathbb{N}}$. By the definitions of χ and of Γ_∞, for each point $s\in\chi$ and for each $k\in\mathbb{N}$, there exists $N(s,k)\in\mathbb{N}$ such that for $j\geq N(s,k)$, there are points $p(j)\in\Sigma(j,j)\cap\mathbb{B}(s,\frac{1}{k})$ with $|A_{\Sigma(j,j)}|(p(j))\geq k$.

We claim that the sequence $\{\Sigma(n,n)\}_{n\in\mathbb{N}}=\Gamma_\infty$ has locally bounded norm of the second fundamental form in $\mathbb{R}^3-\chi$. To prove this, it suffices to check that given a point $y\in\mathbb{R}^3-\chi$, there exists a closed ball $\overline{B}$ such that $y\in\mathrm{Int}(\overline{B})$ and Γ_∞ has uniformly bounded norm of the second fundamental form in $\overline{B}$. Choose an $r\in(0,\frac{1}{4}d_{\mathbb{R}^3}(y,\chi))\cap\mathbb{Q}^+$ and let $x\in\mathbb{Q}^3$ be a point of distance less than r from y. By the triangle inequality, one has $\mathbb{B}(x,2r)\subset\mathbb{R}^3-\chi$ and suppose $\mathbb{B}(x,2r)=B_J\in\mathcal{B}$, for some $J\in\mathbb{N}$. We claim that Γ_∞ has uniformly bounded norm of the second fundamental form on $\overline{\mathbb{B}(y,r)}\subset B_J$. Otherwise, by compactness of $\overline{\mathbb{B}(y,r)}$, there exists a point $q\in\overline{\mathbb{B}(y,r)}\subset B_J$ such that the norms of the second fundamental forms of the surfaces in the sequence Γ_∞ are not bounded in any neighborhood of q. By the inductive construction then $\{q(1),\dots,q(J)\}\cap B_J\neq\emptyset$; however, $B_J\subset[\mathbb{R}^3-\chi]$ and $\{q(1),\dots,q(J)\}\subset\chi$. This contradiction implies that Γ_∞ has locally bounded norm of the second fundamental form in $\mathbb{R}^3-\chi$.

Now replace Γ_∞ by a subsequence, which after relabeling we denote by $\Gamma=\Sigma(n)$, such that the surfaces $\Sigma(n)$ converge C^α to a minimal lamination $\mathcal{L}_\chi$ of $\mathbb{R}^3-\chi$ for any $\alpha\in(0,1)$. This completes the proofs of the first three items of Lemma 3.4.

Recall that the intrinsic open balls $B_{\Sigma(n)}(\vec{0},1)\subset\Sigma(n)$ are graphs of functions with the norms of their gradients at most one over their vertical tangent planes. Hence, after refining the subsequence further, assume that the graphical surfaces $B_{\Sigma(n)}(\vec{0},1)$ converge smoothly to a graph Ω over a vertical plane. In particular for some $\delta\in(0,\frac{1}{4})$ small, depending on curvature estimates for H-graphs, we can find arcs

$$\gamma(n)\subset B_{\Sigma(n)}(\vec{0},1)\cap\{|x_3|\leq\delta\}$$

with one end point in the plane $\{x_3=\delta\}$ and the other end point in $\{x_3=-\delta\}$ and the δ-neighborhood of $\gamma(n)$ in $\Sigma(n)$ is contained in $B_{\Sigma(n)}(\vec{0},\frac{1}{2})$. In particular, the curves $\gamma(n)$ stay an intrinsic distance at least δ from any points of $\Sigma(n)$ with very large curvature.

Claim 3.5. *Let $p\in\chi$. Then, after choosing a subsequence, there is a horizontal plane $Q(p)$ passing through p such that $Q(p)-\{p\}$ is the limit of a* *sequence of 3-valued graphs* $G(n) \subset \Sigma(n)$ *over the annulus* $p + A(n, \frac{1}{n})$ *with the norms of their gradients less than* $\frac{1}{n}$.

Proof. Recall that

$$\Sigma(n) = \frac{1}{r(n)}(E(n) - q_n),$$

and as n goes to infinity, the constant mean curvatures $H(n) = r(n)H_n$ of the surfaces $\Sigma(n)$ are converging to zero, while the distances from $\partial\Sigma(n)$ to the origin are diverging to infinity.

By item 1 of Lemma 3.4, for each $k \in \mathbb{N}$, there exists an $N(k) \in \mathbb{N}$ such that for each $n \geq N(k)$, there exists a point $p_k(n) \in \Sigma(n) \cap \mathbb{B}(p, \frac{1}{k})$ with $|A_{\Sigma(n)}|(p_k(n)) \geq k$. Let $\widetilde{\Sigma}(n) = \Sigma(n) - p_k(n)$ and note that $|A_{\widetilde{\Sigma}(n)}|(\vec{0}) \geq k$. Without loss of generality, we may assume that as n goes to infinity, $\partial\widetilde{\Sigma}(n) \subset \partial\mathbb{B}(R(n))$ with $\lim_{n\to\infty} R(n) = \infty$ and that its mean curvature $H(n) \in (0, \frac{1}{2R(n)})$.

Given $i \in \mathbb{N}$, let τ_i and $\overline{\varepsilon}_i$ be two sequences of positive numbers going to zero as i goes to infinity and let ε_i be a sequence of positive numbers going to infinity such that $\varepsilon_i/\Omega_{\tau_i}$ is also going to infinity, where Ω_{τ_i} is given by Theorem 2.2. By the discussion at the beginning of the proof and after possibly considering $H(n)$-subdisks of $\widetilde{\Sigma}(n)$ that, abusing the notation, we still call $\widetilde{\Sigma}(n)$, the following holds: for each $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that $|A_{\widetilde{\Sigma}(n_i)}|(\vec{0}) > \max(i\omega_{\tau_i}, G_{\tau_i})$, $\partial\widetilde{\Sigma}(n_i) \subset \partial\mathbb{B}(\varepsilon_i)$ and $H(n_i) \in (0, \frac{1}{2\varepsilon_i})$, where ω_{τ_i} and G_{τ_i} are again obtained by applying Theorem 2.2.

By Theorem 2.2 with $\eta = 1$ there exist points $p(n_i) \in \overline{\mathbb{B}}(\overline{\varepsilon}_i)$ such that after translating $\widetilde{\Sigma}(n_i)$ by $-p(n_i)$, the following geometric description of $\widetilde{\Sigma}(n_i)$ holds:

On the scale of the norm of the second fundamental form $\widetilde{\Sigma}(n_i)$ looks like one or two helicoids nearby the origin and, after a rotation that turns these helicoids into vertical helicoids, $\widetilde{\Sigma}(n_i)$ contains a 3-valued graph $u(n_i)$ over $A(\varepsilon_i/\Omega_{\tau_i}, \frac{1}{i})$ with the norm of its gradient less than τ_i and with its inner boundary in $\mathbb{B}(\frac{10}{i})$. Since, as i goes to infinity, $\varepsilon_i/\Omega_{\tau_i}$ goes to infinity, and τ_i and $\overline{\varepsilon}_i$ are going to zero, the sequence of 3-valued graphs $u(n_i)$ is converging to the (x_1, x_2)-plane minus the origin.

Thus, after possibly reindexing the elements of the sequence of surfaces $\Sigma(n)$, choosing a subsequence and applying a rotation $\mathcal{R}$, the previous discussion shows that there is a horizontal plane $Q(p)$ passing through p such that $Q(p) - \{p\}$ is the limit of a sequence of 3-valued graphs $G(n) \subset \Sigma(n)$ over the annulus $p + A(n, \frac{1}{n})$ with the norms of their gradients less than $\frac{1}{n}$. In order to finish the proof of the claim, it suffices to show that the rotation

$\mathcal{R}$ is in fact the identity, namely that the helicoids forming on the scale of the norm of the second fundamental form are vertical. This follows using arguments that are similar to the ones used in the proof of Claim 3.2. This observation finishes the proof of Claim 3.5. □

We now continue with the proofs of items 3 and 4 in Lemma 3.4. The previous claim gives that in the slab $S = \{-\frac{\delta}{2} < x_3 < \frac{\delta}{2}\}$, the sequence of surfaces $\Sigma(n) \cap S$ has locally bounded norm of the second fundamental form. Otherwise there would be a point $p \in \chi \cap S$ and a horizontal plane $Q(p)$ passing through p such that $Q(p) - \{p\}$ is the limit of 3-valued graphs $G(n) \subset \Sigma(n)$. This is impossible since these 3-valued graphs would intersect the arcs $\gamma(n)$ contradicting embeddedness of the surfaces $\Sigma(n)$.

Let $\mathcal{T}$ be the set of open horizontal slabs containing the origin and disjoint from χ. The set $\mathcal{T}$ is non-empty as it contains S. Since $\chi \neq \emptyset$, the union $W = \cup_{\widehat{S} \in \mathcal{T}} \widehat{S}$ is the largest open slab in $\mathcal{T}$ or the largest open half-space containing S for which $W \cap \chi = \emptyset$ and containing the origin. The fact that $\mathcal{L} = \mathcal{L}_\chi \cap W$ is a non-empty lamination of W is clear by definition of lamination and this proves item 4. The validity of item 5 is also clear. This completes the proof of Lemma 3.4. □

Lemma 3.6. *The leaf L given in item 5 of Lemma 3.4 has at most one point of incompleteness in each component of ∂W and any such point lies in χ. Here we refer to a point $p \in \mathbb{R}^3$ as being a point of* incompleteness *of L, if p is the limiting end point in $\mathbb{R}^3$ of some proper arc $\alpha\colon [0,1) \to L$ of finite length.*

Proof. Since L is a leaf of a lamination of W, a point of incompleteness of L must lie in ∂W. Let P be one of the horizontal planes in ∂W and let $y_1 \in P$. Suppose that there exists some smooth, proper, finite length arc $\alpha_{y_1}\colon [0,1) \to L$ with limiting end point y_1 and with beginning point $\alpha_{y_1}(0)$ in L; then such a point y_1 is a point of incompleteness of the leaf L. Note that the half-open arc α_{y_1} can be taken to be a C^1-limit of smooth embedded compact arcs $\alpha^n_{y_1} \subset [\Sigma(n) \cap W]$ of lengths converging to the length of α_{y_1}. Let $y_1(n) \in \alpha^n_{y_1}$ be the sequence of end points of $\alpha^n_{y_1}$, which are converging to y_1.

We claim that there must be positive numbers τ_n with $\lim_{n\to\infty} \tau_n = 0$, such that $B_{\Sigma(n)}(y_1(n), \tau_n)$ contains points of arbitrarily large curvature as n goes to ∞. In particular, it would then follow that $y_1 \in \chi$ and that we could modify the choice of the curves $\alpha^n_{y_1}$ so that the endpoints $y_1(n)$ are points with arbitrarily large norm of the second fundamental form as n goes to infinity. To see that this claim holds, suppose to the contrary that after choosing a subsequence, for some small fixed $\varepsilon > 0$, the norms of the second fundamental forms of the intrinsic balls $B_{\Sigma(n)}(y_1(n), \varepsilon)$ are bounded, and by choosing ε smaller, we may assume that these intrinsic balls are disks that are graphs over their projections to their tangent planes at $y_1(n)$ with the norms of their gradients at most 1. Then a subsequence of these intrinsic balls would converge to an open minimal disk D with $y_1 \in D$ and $D \cap L$ contains a subarc of α_{y_1}. By the maximum principle, D contains points on both sides of P. Since the distance from P to χ is zero, then Claim 3.5 implies that P is also the limit of a sequence of 3-valued graphs $G(n) \subset \Sigma(n)$. But then for n large, this sequence $G(n)$ must intersect D transversally at some point $q_n \in D$. Since the disks $B_{\Sigma(n)}(y_1(n), \varepsilon)$ converge smoothly to D, then for n sufficiently large, $G(n)$ also intersects $B_{\Sigma(n)}(y_1(n), \varepsilon)$ transversely at some point. This contradiction then proves the claim that $y_1 \in \chi$ and that we could modify the choice of the curves $\alpha^n_{y_1}$ so that the endpoints $y_1(n)$ are points with arbitrarily large norm of the second fundamental form as n goes to infinity.

Without loss of generality, assume that W lies below P. Suppose that there is a point $y_2 \in P$, $y_2 \neq y_1$, of incompleteness for L with related proper arc α_{y_2} beginning at $\alpha_{y_2}(0)$ with limiting end point y_2, and related approximating curves $\alpha^n_{y_2}$ in $\Sigma(n)$ with end points $y_2(n)$ converging to y_2, where the norm of the second fundamental form is arbitrarily large. Also assume that $\alpha_{y_1}, \alpha_{y_2}$ are sufficiently close to their different limiting end points so that, after possibly replacing them by subarcs, they are contained in closed balls in $\mathbb{R}^3$ that are a positive distance from each other. By Claim 3.5 there exist sequences of 3-valued graphs $G_1(n)$, $G_2(n)$ in $\Sigma(n)$ with inner boundary curves converging to the points $y_1(n)$, $y_2(n)$, respectively, and these 3-valued graphs can be chosen so that each of them collapses to the punctured plane P punctured at y_1, y_2, respectively. For n large, $G_1(n)$ must lie "above" the point $\alpha^n_{y_2}(0)$ which implies that near $y_2(n)$, $G_1(n)$ must lie above $G_2(n)$ as well, otherwise, $G_1(n)$ would intersect the arc $\alpha^n_{y_2}$, which would contradict the embeddedness of $\Sigma(n)$. Reversing the roles of $G_1(n)$ and $G_2(n)$, we find that $G_2(n)$ must lie above $G_1(n)$ near $y_1(n)$; hence, for n sufficiently large, the multigraph $G_1(n)$ must intersect $G_2(n)$. This contradicts that $\Sigma(n)$ is embedded and thereby proves that there is at most one point $y \in P \cap \chi$ that is a point of incompleteness of L. $\square$

So far we have shown that L can have at most one point of incompleteness on each of the components of ∂W and all the points of incompleteness are contained in ∂W. Define $\mathbf{S}$ to be the set of points of incompleteness of L. By the previous claim, the possibly empty set $\mathbf{S}$ has at most two points. The next claim describes some consequences of this finiteness of $\mathbf{S}$.

Lemma 3.7. *The leaf L has genus zero and the closure $\overline{L}$ of L in $\mathbb{R}^3 - \mathbf{S}$ is a minimal lamination of $\mathbb{R}^3 - \mathbf{S}$ that is contained in $\overline{W} - \mathbf{S}$.*

Proof. We first prove that L is not a limit leaf of $\mathcal{L}$. If L were a limit leaf of $\mathcal{L}$, then its oriented 1 or 2-sheeted cover would be stable by Theorem 4.3 in [16] or Theorem 1 in [17] by Meeks, Perez and Ros. Stability gives curvature estimates on L away from $\mathbf{S}$ and thus the closure $\overline{L}$ of L in $\mathbb{R}^3 - \mathbf{S}$, is a minimal lamination of $\mathbb{R}^3 - \mathbf{S}$. By Corollary 7.2 in [19] by Meeks, Perez and Ros, the closure of $\overline{L}$ in $\mathbb{R}^3$ is a minimal lamination of $\mathbb{R}^3$ and each leaf is stable; in particular, by stability the closure of $\overline{L}$ in $\mathbb{R}^3$ must be a plane. This is a contradiction because L lies in a horizontal slab and its tangent plane at the origin is vertical.

Let $\widetilde{W} \subset W$ be the smallest open slab or half-space containing L. We claim that L is properly embedded in $\widetilde{W}$. If not, then the closure $\overline{L}$ of L in $\widetilde{W}$ is a sublamination of $\mathcal{L}$ with a limit leaf T. Note that since $\widetilde{W}$ is the smallest open slab or half-space in W containing L, T cannot be a plane. The proof of Lemma 3.6 applies to T to show that it has at most one point of incompleteness on each component of ∂W and any such point lies in χ. By the arguments in the previous paragraph, T must be a plane. This is a contradiction and therefore L is properly embedded in $\widetilde{W}$.

Since L is properly embedded in a simply-connected open set of $\mathbb{R}^3$, then it separates the open set and so it is orientable. As L is not stable, the convergence of portions of $\Sigma(n)$ to L has multiplicity one or two which implies that L has genus zero and zero flux; see the discussion when Case A holds for further details on this multiplicity of convergence bound and the genus-zero and zero-flux properties.

It remains to prove that the closure $\overline{L}$ of L in $\mathbb{R}^3 - \mathbf{S}$ is a minimal lamination of $\mathbb{R}^3 - \mathbf{S}$. By the minimal lamination closure theorem by Meeks and Rosenberg in [23] (specifically see Remark 2 at the end of this paper), this desired result is equivalent to proving that the injectivity radius function of L is bounded away from zero on compact subsets of $\mathbb{R}^3 - \mathbf{S}$. Otherwise, the local picture theorem on the scale of the topology by Meeks, Perez and Ros in [20] implies that there exists a sequence of compact domains $\Delta_n \subset L$ such that related homothetically scaled and translated domains $\widetilde{\Delta}_n$ converge smoothly with multiplicity one to a properly embedded genus-zero minimal surface L_∞ in $\mathbb{R}^3$ with injectivity radius one and there exist closed geodesics $\gamma_n \subset \widetilde{\Delta}_n$ with nontrivial flux. So, the latter cannot happen as L has zero flux. On the other hand, if the former happens then, by the classification of non-simply-connected, properly embedded minimal planar domains in $\mathbb{R}^3$ in [18] by Meeks, Perez and Ros, the surface L_∞ is a catenoid or a Riemann minimal example and so has non-zero flux; therefore L_∞ cannot be the limit with multiplicity one of translated and scaled domains in L. In either case, we have obtained a contradiction, which proves that the injectivity radius function of L is bounded away from zero on compact subsets of $\mathbb{R}^3 - \mathbf{S}$. This completes the proof of the claim. $\square$

Finally we prove that Case B cannot occur. By Lemma 3.7, the closure $\overline{L}$ of L in $\mathbb{R}^3 - \mathbf{S}$ is a minimal lamination; in the technical language developed in [21] by Meeks, Perez and Ros, $\overline{L}$ is a singular minimal lamination of $\mathbb{R}^3$ with a countable (in fact at most two) set of singular points and its regular part in $\mathbb{R}^3 - \mathbf{S}$ contains the genus-zero leaf L. By item 6 of Theorem 1.8 in [21], the closure of L in $\mathbb{R}^3$ must be a properly embedded minimal surface in $\mathbb{R}^3$. Since L is non-flat and is contained in the open slab or half-space W, the half-space theorem in [12] applied to $\overline{L}$ gives a contradiction, which proves that Case B cannot occur. This completes the proof of Assertion 3.3. $\square$

Since Case B does not occur, then Case A must occur. We have already proven Proposition 3.1 when Case A holds, thus, the proof of Proposition 3.1 is finished. $\square$

The next corollary follows immediately from Proposition 3.1.

Corollary 3.8. *Given $\varepsilon_1 \in (0, 1/2)$, there exist $\varepsilon_2 > 0$ such that the following holds. Let E be an H-disk satisfying*

$$E \cap \mathbb{B}(1) \cap \{x_3 = 0\} = \emptyset \quad \text{and} \quad \partial E \cap \mathbb{B}(1) \cap \{x_3 > 0\} = \emptyset.$$

Suppose $p \in E \cap \mathbb{B}(\varepsilon_2) \cap \{x_3 > 0\}$ where the tangent plane to E is vertical. Then there exists a vertical helicoid $\mathcal{H}$ with maximal absolute Gaussian curvature $\frac{1}{2}$ at the origin such that the connected component of $[|A_E|(p)(E - p)] \cap \mathbb{B}(1)$ containing the origin is a normal graph u over its projection $\Omega \subset [\mathbb{B}(1 + 2\varepsilon_1) \cap \mathcal{H}]$, where $\Omega \supset [\mathbb{B}(1 - 2\varepsilon_1) \cap \mathcal{H}]$ and $\|u\|_{C^2} \leq \varepsilon_1$. Furthermore, $[|A_E|(p)(E - p)] \cap \mathbb{B}(1)$ consists of 1 or 2 components and if there are two components, then the component not passing through the origin is a normal graph u' over its projection $\Omega' \subset [\mathbb{B}(1 + 2\varepsilon_1) \cap \mathcal{H}]$, where $\Omega' \supset [\mathbb{B}(1 - 2\varepsilon_1) \cap \mathcal{H}]$ and $\|u'\|_{C^2} \leq \varepsilon_1$.

As a consequence of Proposition 3.1 and Corollary 3.8, we obtain the next claim.

Claim 3.9. *There exists an $\varepsilon > 0$ such that the following holds. If E is an H-disk satisfying the hypotheses of Theorem 1.1 for $R = 1$, then $E \cap \mathbb{B}(\varepsilon) \cap \{x_3 > 0\}$ contains no vertical tangent planes.*

Proof. Arguing by contradiction, let $E(n)$ be a sequence of H_n-disks satisfying the hypothesis of Theorem 1.1 for $R = 1$, together with a sequence of points $p(n)$ in $E(n) \cap \mathbb{B}(\frac{1}{n}) \cap \{x_3 > 0\}$ with vertical tangent planes. Given $\delta > 0$, let $\Gamma_n(\delta)$ be the connected component of

$$E(n) \cap \mathbb{B}(\delta) \cap \{x_3 > 0\} \cap N_n^{-1}(\{x_1^2 + x_2^2 = 1\})$$

containing $p(n)$, where N_n is the Gauss map of $E(n)$. It follows from Corollary 3.8 that given $\rho \in (0, 1/2)$, there exists $\delta > 0$ such that for n large, $\Gamma_n(\delta)$ is an analytic curve that can be parameterized to have unit speed and so that $|\langle \dot{\Gamma}_n(\delta), (0, 0, -1)\rangle| > 1 - \rho$. By taking ρ sufficiently small, and for n sufficiently large, this curve crosses the (x_1, x_2)-plane nearby the origin. This is impossible because the disks $E(n)$ are disjoint from $\mathbb{B}(1) \cap \{x_3 = 0\}$. This contradiction completes the proof of the claim. $\square$

To complete the proof of Theorem 1.1, recall that at the beginning of the proof of Theorem 1.1, we showed that if the theorem fails, then there exists a sequence of H_n-disks $E(n)$ with $\partial E(n) \subset \partial \mathbb{B}(1)$ and $H_n \leq 1$ satisfying the hypotheses of Theorem 1.1 and points $p_k(n) \in E(n)$ with vertical tangent plane and converging to the origin. This contradicts Claim 3.9 and the proof of Theorem 1.1 is completed. $\square$

The next corollary follows from Theorem 1.1 by a rescaling argument. It roughly states that we can replace the (x_1, x_2)-plane by any surface that has a fixed uniform estimate on the norm of its second fundamental form.

Corollary 3.10. *Given $a \geq 0$, there exist $\varepsilon \in (0, \frac{1}{2})$ and $C_a > 0$ such that for any $R > 0$, the following holds. Let Δ be a compact immersed surface in $\mathbb{B}(R)$ with $\partial\Delta \subset \partial\mathbb{B}(R)$, $\vec{0} \in \Delta$ and satisfying $|A_\Delta| \leq a/R$. Let $\mathcal{D}$ be an H-disk such that*

$$\mathcal{D} \cap \mathbb{B}(R) \cap \Delta = \emptyset \quad \textit{and} \quad \partial\mathcal{D} \cap \mathbb{B}(R) = \emptyset.$$

Then:

$$\sup_{x \in \mathcal{D} \cap \mathbb{B}(\varepsilon R)} |A_{\mathcal{D}}|(x) \leq \frac{C_a}{R}. \tag{3}$$

In particular, if $\mathcal{D} \cap \mathbb{B}(\varepsilon R) \neq \emptyset$, then $H \leq \frac{C_a}{R}$.

Proof. Let $f\colon (\Delta, \partial\Delta) \looparrowright (\mathbb{B}(R), \partial\mathbb{B}(R))$ be the immersion associated to the immersed surface Δ and choose a point $p_f \in f^{-1}(\vec{0})$. Let $P \subset \mathbb{R}^3$ be the plane passing through the origin and tangent to the immersion f at $f(p_f)$.

After a rotation of Δ, we may assume that P is the (x_1, x_2)-plane. After scaling by $1/R$, we may also assume that $R = 1$.

If $a = 0$, then $\Delta = P \cap \mathbb{B}(1)$. In this case the corollary follows from Theorem 1.1 by setting $C_a = C$, where C is given in Theorem 1.1 and $\varepsilon > 0$ is also given by that theorem.

Assume now that $a > 0$ is fixed. Arguing by contradiction, suppose that the corollary fails to hold. Then, as in the proof of Theorem 1.1, there exists a sequence of compact domains Δ_n with associated immersions $f(n)\colon (\Delta_n, \partial\Delta_n) \looparrowright (\mathbb{B}(1), \partial\mathbb{B}(1))$ that satisfy the conditions of Δ with corresponding H_n-disks $E(n)$ satisfying the hypotheses of $\mathcal{D}$,

$$E(n) \cap \mathbb{B}(1) \cap \Delta_n = \emptyset \quad \text{and} \quad \partial E(n) \cap \mathbb{B}(1) = \emptyset,$$

together with points $\widetilde{p}_n$ in $E(n)$ and satisfying:

$$|A_{E(n)}|(\widetilde{p}_n) \geq n^2, \; d_{\mathbb{R}^3}(\widetilde{p}_n, \vec{0}) \leq \frac{1}{n^2}. \tag{4}$$

For each of the immersions $f(n)$, make a choice $p_{f(n)} \in (f(n))^{-1}(\vec{0})$.

Since we are assuming that $|A_{\Delta_n}| \leq a$, there exists an $\varepsilon_1 = \varepsilon_1(a) \in (0, \frac{1}{2})$ such that the component Δ_n' of $(f(n))^{-1}(\mathbb{B}(\varepsilon_1))$ containing $p_{f(n)}$ satisfies that $f(\Delta_n')$ is a graph of gradient less than 1 over its projection to P. Let $E'(n)$ be the component of $E(n) \cap \mathbb{B}(\varepsilon_1)$ containing $\widetilde{p}_n$, and, without loss of generality, assume that $E'(n)$ lies in the component of $\mathbb{B}(\varepsilon_1) \setminus f(\Delta_n')$ that contains the positive x_3-axis.

For $n > 1/\varepsilon_1$, replace the immersed surfaces $f(\Delta_n)$ by the connected embedded scaled surfaces $\widetilde{\Delta}_n = nf(\Delta_n') \cap \mathbb{B}(1)$, replace the H_n-disks $E(n)$ by the scaled $(\frac{1}{n}H_n)$-disks $nE(n)$ and let $\widetilde{E}(n)$ be the connected component of $nE(n) \cap \mathbb{B}(1)$ that contains $n\widetilde{p}_n$.

Notice that as $n \to \infty$, the graphs $\widetilde{\Delta}_n$ converge C^1 to the disk $P \cap \mathbb{B}(1)$. It follows that there exists a strictly increasing sequence $\{n_k \in \mathbb{N}\}_{k \in \mathbb{N}}$ such that each of the vertically translated planar domains $\widehat{E}(k) = \widetilde{E}(n_k) + (0, 0, \frac{1}{k})$ lies "above" the vertically translated graph $\widetilde{\Delta}_{n_k} + (0, 0, \frac{1}{k})$. Moreover, from (4), the points $\widehat{p}_k = n_k \widetilde{p}_{n_k} + (0, 0, \frac{1}{k})$ in $\widehat{E}(k)$ converge to the origin as $k \to \infty$ and $|A_{\widehat{E}(k)}|(\widehat{p}_k) \geq n_k$.

By Theorem 1.1, the translated disks $n_k E(n_k) + (0, 0, \frac{1}{k})$ must intersect $P \cap \mathbb{B}(1)$ at some point below the positive graph $\widetilde{\Delta}_{n_k} + (0, 0, \frac{1}{k})$. But Theorem 1.1 gives a contradiction by only considering the component of $\widehat{E}(k) \cap \mathbb{B}(1) \subset \{x_3 > 0\}$ that contains the point $\widehat{p}_k$ instead of the entire set

$(n_kE(n_k) + (0, 0, \frac{1}{k})) \cap \mathbb{B}(1)$. This contradiction completes the proof of the corollary. $\square$

4. Consequences of the one-sided curvature estimate

In this section we state several theorems that depend on the one-sided curvature estimate in Theorem 1.1. These theorems appear in [26, 24, 25, 28]. We begin by making the following definition.

Definition 4.1. *Given a point p on a surface $\Sigma \subset \mathbb{R}^3$, $\Sigma(p, R)$ denotes the closure of the component of $\Sigma \cap \mathbb{B}(p, R)$ passing through p.*

In [26], we apply the one-sided curvature estimate in Theorem 1.1 to prove a relation between intrinsic and extrinsic distances in an H-disk, which can be viewed as a *weak chord arc* property. This result was motivated by and generalizes Proposition 1.1 in [9] by Colding and Minicozzi. More precisely, the statement is the following.

Theorem 4.2 (Weak chord arc property [26])**.** *There exists a $\delta_1 \in (0, \frac{1}{2})$ such that the following holds.*

Let Σ be an H-disk in $\mathbb{R}^3$. Then for all intrinsic closed balls $\overline{B}_\Sigma(x, R)$ in $\Sigma - \partial\Sigma$:

1. *$\Sigma(x, \delta_1 R)$ is a disk with piecewise smooth boundary $\partial\Sigma(\vec{0}, \delta_1 R) \subset \partial\mathbb{B}(\delta_1 R)$.*
2. *$\Sigma(x, \delta_1 R) \subset B_\Sigma(x, \frac{R}{2})$.*

By the relation between extrinsic and intrinsic distances described in Theorem 4.2, the extrinsic radius and curvature estimates given in Theorems 2.3 and 2.4, that were used to prove Theorem 4.2, become radius and curvature estimates that depend only on intrinsic distances; see [27] for details on the proofs of the two theorems below in the extrinsic distance case.

In the next theorem the *radius* of a compact Riemannian surface Σ with boundary is the maximum intrinsic distance of points in the surface to its boundary; in the second theorem, d_Σ denotes the intrinsic distance function of Σ.

Theorem 4.3 (Radius estimates [24])**.** *There exists an $\mathcal{R} \geq \pi$ such that any compact disk embedded in $\mathbb{R}^3$ of constant mean curvature $H > 0$ has radius less than $\mathcal{R}/H$.*

Theorem 4.4 (Curvature Estimates [24])**.** *Given δ, $\mathcal{H} > 0$, there exists a $K(\delta, \mathcal{H}) \geq \sqrt{2}\mathcal{H}$ such that any compact disk M embedded in $\mathbb{R}^3$ with constant mean curvature $H \geq \mathcal{H}$ satisfies*

$$\sup_{\{p \in M \,|\, d_M(p, \partial M) \geq \delta\}} |A_M| \leq K(\delta, \mathcal{H}).$$

An immediate consequence of the triangle inequality and Theorem 4.2 is the following intrinsic version of the one-sided curvature estimate given in Theorem 1.1. In the case that $H = 0$, the next theorem follows from Corollary 0.8 in [9].

Theorem 4.5 (Intrinsic one-sided curvature estimate for H-disks)**.** *There exist $\varepsilon_I \in (0, \frac{1}{2})$ and $C_I \geq 2\sqrt{2}$ such that for any $R > 0$, the following holds. Let $\mathcal{D}$ be an H-disk such that*

$$\mathcal{D} \cap \mathbb{B}(R) \cap \{x_3 = 0\} = \varnothing$$

and $x \in \mathcal{D} \cap \mathbb{B}(\varepsilon_I R)$ where $d_{\mathcal{D}}(x, \partial\mathcal{D}) \geq R$. Then:

$$|A_{\mathcal{D}}|(x) \leq \frac{C_I}{R}. \tag{5}$$

In particular, $H \leq \frac{C_I}{R}$.

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