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Higher $\mathcal{L}$-invariants for $\mathrm{GL}_3(\mathbb{Q}_p)$ and local-global compatibility*

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Let ρ_p be a 3-dimensional semi-stable representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ with Hodge-Tate weights $(0,1,2)$ (up to shift) and such that $N^2 \neq 0$ on $D_{\mathrm{st}}(\rho_p)$. When ρ_p comes from an automorphic representation π of $G(\mathbb{A}_{F^+})$ (for a unitary group G over a totally real field F^+ which is compact at infinite places and GL_3 at p-adic places), we show under mild genericity assumptions that the associated Hecke-isotypic subspaces of the Banach spaces of p-adic automorphic forms on $G(\mathbb{A}_{F^+}^\infty)$ of arbitrary fixed tame level contain (copies of) a unique admissible finite length locally analytic representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ of the form considered in [4] which only depends on and completely determines ρ_p.

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1. Introduction and notation

Let p be a prime number, $n \geq 2$ an integer, F^+ a totally real number field and F a totally imaginary quadratic extension of F^+ such that all places of F^+ dividing p split in F. We fix a unitary algebraic group G over F^+ which becomes GL_n over F and such that $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact and G is split at all places above p. We also fix a place $\wp$ of F^+ above p. Then to each $\mathbb{Q}_p$-algebraic irreducible (finite dimensional) representation $W^{\wp}$ of $\prod_{v|p, v\neq\wp} G(F_v^+)$ over a finite extension E of $\mathbb{Q}_p$ and to each prime-to-$\wp$ level $U^{\wp}$ in $G(\mathbb{A}_{F^+}^{\infty,\wp})$, one can associate the Banach space of p-adic automorphic forms $\widehat{S}(U^{\wp}, W^{\wp})$ (see e.g. § 6.1).

If $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(E)$ is a continuous irreducible representation and $\widetilde{\wp}$ is a place of F above $\wp$, one can consider the associated Hecke isotypic subspace $\widehat{S}(U^{\wp}, W^{\wp})[\mathfrak{m}_\rho]$, which is a continuous admissible representation of $G(F_{\wp}^+) \xrightarrow{\sim} \mathrm{GL}_n(F_{\widetilde{\wp}})$ over E, or its locally $\mathbb{Q}_p$-analytic vectors $\widehat{S}(U^{\wp}, W^{\wp})[\mathfrak{m}_\rho]^{\mathrm{an}}$, which is an admissible locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_n(F_{\widetilde{\wp}})$. When nonzero, these representations of $\mathrm{GL}_n(F_{\widetilde{\wp}})$ are so far only understood when $n=2$ and $F_{\widetilde{\wp}}=\mathbb{Q}_p$ ([24], [42], [53], [20], [25], [59], [35], [18], ...). Indeed, though these representations are expected to be very rich, many results from $\mathrm{GL}_2(\mathbb{Q}_p)$ collapse (see e.g. [65], [79]) and it presently seems an almost impossible task to find a way to completely describe them in general. However, it is (quite reasonably) hoped that they *determine* the local Galois representation $\rho_{\widetilde{\wp}}:=\rho|_{\mathrm{Gal}(\overline{F}_{\widetilde{\wp}}/F_{\widetilde{\wp}})}$ and (may-be less reasonably) hoped that they also *only depend on* $\rho_{\widetilde{\wp}}$. Note that the special case where ρ is automorphic is of particular interest, since then the subspace $\widehat{S}(U^{\wp},W^{\wp})[\mathfrak{m}_\rho]^{\mathrm{lalg}}$ of locally $\mathbb{Q}_p$-algebraic vectors is nonzero, given by the classical local Langlands correspondence for $\mathrm{GL}_n(F_{\widetilde{\wp}})$ tensored by $\mathbb{Q}_p$-algebraic representations of $\mathrm{GL}_n(F_{\widetilde{\wp}})$.

The aim of this work is to consolidate the above hopes in the case of $\mathrm{GL}_3(\mathbb{Q}_p)$. Let St_3^∞ be the usual smooth Steinberg representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ and $v_{\overline{P}_i}^\infty=(\mathrm{Ind}_{\overline{P}_i(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}1)^\infty/1$ for $i=1,2$ the two smooth generalized Steinberg representations where $\overline{P}_1(\mathbb{Q}_p)=\left(\begin{smallmatrix}*&*&0\\ *&*&0\\ *&*&*\end{smallmatrix}\right)$ and $\overline{P}_2(\mathbb{Q}_p):=\left(\begin{smallmatrix}*&0&0\\ *&*&*\\ *&*&*\end{smallmatrix}\right)$. Our main result is the following.

Theorem 1.1 (Corollary 7.54)**.** *Assume* $p\geq 5$*,* $n=3$*,* $F_{\widetilde{\wp}}=\mathbb{Q}_p$ *and* $U^{\wp}=\prod_{v\neq\wp}U_v$ *with* U_v *maximal if* $v|p$*,* $v\neq\wp$*. Assume moreover that:*

- $\overline{\rho}$ *is absolutely irreducible*
- $\widehat{S}(U^{\wp},W^{\wp})[\mathfrak{m}_\rho]^{\mathrm{lalg}}\neq 0$
- $\rho_{\widetilde{\wp}}$ *is semi-stable with consecutive Hodge-Tate weights and* $N^2\neq 0$ *on* $D_{\mathrm{st}}(\rho_{\widetilde{\wp}})$
- *any dimension* 2 *subquotient of* $\overline{\rho}_{\widetilde{\wp}}:=\overline{\rho}|_{\mathrm{Gal}(\overline{F}_{\widetilde{\wp}}/F_{\widetilde{\wp}})}$ *is nonsplit.*

Then $\widehat{S}(U^{\wp},W^{\wp})[\mathfrak{m}_\rho]$ *contains (copies of) a* unique *locally analytic representation* $\Pi\otimes\chi\circ\det$ *of* $\mathrm{GL}_3(\mathbb{Q}_p)$ *with* χ *a locally algebraic character of* $\mathbb{Q}_p^\times$ *and* Π *of the form:*

$$\Pi \cong \begin{tikzcd}
 & & \widetilde{C}_{1,2} \arrow[dr, dash, dashed] & & \widetilde{C}_{1,4} \arrow[dr, dash, dashed] & \\
 & C_{1,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{1,3} \arrow[ur, dash] \arrow[dr, dash] & & C_{1,5} \\
 & & v_{\overline{P}_1}^\infty \arrow[ur, dash] & & v_{\overline{P}_2}^\infty \arrow[ur, dash] & \\
\mathrm{St}_3^\infty \arrow[uur, dash] \arrow[ddr, dash] & & & & & \\
 & & v_{\overline{P}_2}^\infty \arrow[dr, dash] & & v_{\overline{P}_1}^\infty \arrow[dr, dash] & \\
 & C_{2,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{2,3} \arrow[ur, dash] \arrow[dr, dash] & & C_{2,5}. \\
 & & \widetilde{C}_{2,2} \arrow[ur, dash, dashed] & & \widetilde{C}_{2,4} \arrow[ur, dash, dashed] &
\end{tikzcd} \tag{1.1}$$

where the $C_{i,j}$, $\widetilde{C}_{i,j}$ *are certain explicit irreducible subquotients of locally analytic principal series of* $\mathrm{GL}_3(\mathbb{Q}_p)$ *(see § 3.3 or [4, § 4.1]), where* $\mathrm{St}_3^\infty = \mathrm{soc}_{\mathrm{GL}_3(\mathbb{Q}_p)}\Pi$ *and where* $-$ *(resp. the dashed line) means a nonsplit (resp. a possibly split) extension as subquotient. Moreover the representation* $\Pi\otimes\chi\circ\det$ *completely determines and only depends on* $\rho_{\widetilde{\wp}}$*. In particular the locally analytic representation* $\widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]^{\mathrm{an}}$ *of* $\mathrm{GL}_3(\mathbb{Q}_p)$*, hence also the continuous representation* $\widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]$*, determine* $\rho_{\widetilde{\wp}}$*.*

In fact one proves the stronger result that the restriction morphism:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\Pi\otimes\chi\circ\det, \widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]^{\mathrm{an}}\big)&\\ \longrightarrow \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\mathrm{St}_3^\infty\otimes\chi\circ\det, \widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]^{\mathrm{an}}\big)&\end{aligned}\tag{1.2}$$

is bijective. The third assumption in Theorem 1.1 implies that $\rho_{\widetilde{\wp}}$ is up to twist isomorphic to $\left(\begin{smallmatrix}\varepsilon^2 & * & * \\ 0 & \varepsilon & * \\ 0 & 0 & 1\end{smallmatrix}\right)$ where ε is the cyclotomic character. Hence $\overline{\rho}_{\widetilde{\wp}}$ is up to twist isomorphic to $\left(\begin{smallmatrix}\overline{\varepsilon}^2 & * & * \\ 0 & \overline{\varepsilon} & * \\ 0 & 0 & 1\end{smallmatrix}\right)$, and the fourth assumption means that we require the two $*$ above the diagonal in $\overline{\rho}_{\widetilde{\wp}}$ to be nonzero, a kind of assumption which already appears in the $\mathrm{GL}_2(\mathbb{Q}_p)$ case (see e.g. [42, Thm. 1.2.1]).

Without assuming $\overline{\rho}$ absolutely irreducible, consecutive Hodge-Tate weights and the above condition on $\overline{\rho}_{\widetilde{\wp}}$, *but* assuming $F^+ = \mathbb{Q}$, ρ absolutely irreducible and a slightly unpleasant condition on $\widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]^{\mathrm{lalg}}$ (see [4, Rem. 6.2.2(ii)]), it was proven in [4, Thm. 6.2.1] that $\widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]^{\mathrm{an}}$ contains (copies of) a unique locally analytic representation which has the same form as (1.1). However, nothing more was known of its possible link to $\rho_{\widetilde{\wp}}$. So the main novelty in Theorem 1.1 is that the $\mathrm{GL}_3(\mathbb{Q}_p)$-representation $\Pi\otimes\chi\circ\det$ contains *exactly* the same information as the $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-representation $\rho_{\widetilde{\wp}}$. Note however that $\Pi\otimes\chi\circ\det$ is presumably only a small part of the representation $\widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]^{\mathrm{an}}$. For instance one could push a little bit further the methods of this paper to prove that $\widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]^{\mathrm{an}}$ as in Theorem 1.1 in fact contains (copies of) a representation of the form $\widetilde{\Pi}\otimes\chi\circ\det$ with:

$$\begin{tikzcd}
 & v_{\overline{P}_1}^{\mathrm{an}} \arrow[r, dash] & \left(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \varepsilon^{-1}\otimes\varepsilon\otimes 1\right)^{\mathrm{an}} \arrow[r, dash] & v_{\overline{P}_2}^{\mathrm{an}} \\
\widetilde{\Pi} \;\cong\; \mathrm{St}_3^{\mathrm{an}} \arrow[ur, dash] \arrow[dr, dash] & & & \\
 & v_{\overline{P}_2}^{\mathrm{an}} \arrow[r, dash] & \left(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} 1\otimes\varepsilon^{-1}\otimes\varepsilon\right)^{\mathrm{an}} \arrow[r, dash] & v_{\overline{P}_1}^{\mathrm{an}}
\end{tikzcd}\tag{1.3}$$

which still determines and only depends on $\rho_{\widetilde{\wp}}$. In (1.3), we denote by $\mathrm{St}_3^{\mathrm{an}}$, resp. $v_{\overline{P}_i}^{\mathrm{an}}$, the locally analytic Steinberg, resp. generalized Steinberg, and by $(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \cdot)^{\mathrm{an}}$ the locally analytic principal series from lower triangular matrices. In fact the subrepresentation of $\Pi\otimes\chi\circ\det$ without the constituents $\widetilde{C}_{i,4}$, $C_{i,5}$ $(i=1,2)$ in Theorem 1.1 can be seen as the "edge" of the representation $\widetilde{\Pi}\otimes\chi\circ\det$. But even adding those constituents to $\widetilde{\Pi}\otimes\chi\circ\det$ (or more precisely $(\widetilde{\Pi}\otimes\chi\circ\det)^{\oplus d}$ where $d:=\dim_E \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}(\mathrm{St}_3^\infty\otimes\chi\circ\det, \widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]^{\mathrm{an}})$), we are presumably still far from the full representation $\widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]^{\mathrm{an}}$.

Theorem 1.1 (in its stronger form as above) is in fact a special case of a conjecture in arbitrary (distinct) Hodge-Tate weights. In § 3.3, we show that one can associate to $\rho_{\widetilde{\wp}}$, assumed semi-stable with $N^2\neq 0$ on $D_{\mathrm{st}}(\rho_{\widetilde{\wp}})$ and sufficiently generic (we explain this below, any $\rho_{\widetilde{\wp}}$ as in Theorem 1.1 is sufficiently generic), a locally analytic representation $\Pi(\rho_{\widetilde{\wp}})=\Pi\otimes\chi\circ\det$ of $\mathrm{GL}_3(\mathbb{Q}_p)$ containing the same information as $\rho_{\widetilde{\wp}}$ where Π has the same form as (1.1) but replacing St_3^∞, $v_{\overline{P}_i}^\infty$ by $\mathrm{St}_3^\infty(\lambda)=:\mathrm{St}_3^\infty\otimes_E L(\lambda)$, $v_{\overline{P}_i}^\infty(\lambda):=v_{\overline{P}_i}^\infty\otimes_E L(\lambda)$. Here $L(\lambda)$ is the algebraic representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ of highest weight $\lambda=k_1\geq k_2\geq k_3$ where $k_1>k_2-1>k_3-2$ are the Hodge-Tate weights of $\rho_{\widetilde{\wp}}$. We conjecture the following statement.

Conjecture 1.2 (Conjecture 6.2)**.** *Assume $n=3$, $F_{\widetilde{\wp}}=\mathbb{Q}_p$ and:*

- *ρ absolutely irreducible*
- $\widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]^{\mathrm{lalg}}\neq 0$
- *$\rho_{\widetilde{\wp}}$ semi-stable with $N^2\neq 0$ on $D_{\mathrm{st}}(\rho_{\widetilde{\wp}})$ and sufficiently generic.*

Then the following restriction morphism is bijective:

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\Pi(\rho_{\widetilde{\wp}}),\widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]^{\mathrm{an}}\big)\\ &\qquad\xrightarrow{\sim}\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\mathrm{St}_3^\infty\otimes_E L(\lambda)\otimes\chi\circ\det,\widehat{S}(U^\wp,W^\wp)[\mathfrak{m}_\rho]\big).\end{aligned}$$

We now sketch the proof of Theorem 1.1. The preliminary step, which is purely local and holds for arbitrary distinct Hodge-Tate weights, is the definition of $\Pi(\rho_{\widetilde{\wp}})$. Since $N^2\neq 0$, the (φ,Γ)-module $D:=D_{\mathrm{rig}}(\rho_{\widetilde{\wp}})$ over the Robba ring with E-coefficients $\mathcal{R}_E$ can be uniquely written as $\mathcal{R}_E(\delta_1)-\mathcal{R}_E(\delta_2)-\mathcal{R}_E(\delta_3)$ for some locally algebraic characters $\delta_i:\mathbb{Q}_p^\times\to E^\times$ (where, as usual, $\mathcal{R}_E(\delta_1)$ is a submodule, $\mathcal{R}_E(\delta_3)$ a quotient and $-$ means a nonsplit extension). We assume that the triangulation $(\mathcal{R}_E(\delta_1),\mathcal{R}_E(\delta_2),\mathcal{R}_E(\delta_3))$ is noncritical, equivalently the Hodge-Tate weight of δ_i is $k_i-(i-1)$. Twisting $D_{\mathrm{rig}}(\rho_{\widetilde{\wp}})$ if necessary (and twisting $\Pi(\rho_{\widetilde{\wp}})$ accordingly), we can assume $\delta_1=$

x^{k_1}, $\delta_2 = x^{k_2}\varepsilon^{-1}$ and $\delta_3 = x^{k_3}\varepsilon^{-2}$ (note that D is not étale anymore if $k_1 \neq 0$, but this won't be a problem). By the recipe for $\mathrm{GL}_2(\mathbb{Q}_p)$, one can associate to $D_1^2 := \mathcal{R}_E(\delta_1) - \mathcal{R}_E(\delta_2)$ and $D_2^3 := \mathcal{R}_E(\delta_2) - \mathcal{R}_E(\delta_3)$ locally analytic representations $\pi_{1,2}$ and $\pi_{2,3}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$. Then the representations:

$$\begin{tikzcd}
 & & \widetilde{C}_{1,2} \arrow[dr, dash, dashed] & & & & & \widetilde{C}_{2,2} \arrow[dr, dash, dashed] & \\
\mathrm{St}_3^\infty(\lambda) \arrow[r, dash] & C_{1,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{1,3}, & & \mathrm{St}_3^\infty(\lambda) \arrow[r, dash] & C_{2,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{2,3}, \\
 & & v_{\overline{P}_1}^\infty(\lambda) \arrow[ur, dash] & & & & & v_{\overline{P}_1}^\infty(\lambda) \arrow[ur, dash] &
\end{tikzcd}$$

can be defined as subquotients of the locally analytic parabolic inductions $(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \pi_{1,2} \otimes \delta_3\varepsilon^2)^{\mathrm{an}}$ and $(\mathrm{Ind}_{\overline{P}_2(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \delta_1 \otimes (\pi_{2,3} \otimes \varepsilon \circ \det_{\mathrm{GL}_2}))^{\mathrm{an}}$ respectively, see § 3.3.3. Note that these two representations (together) contain what we call the two "simple" $\mathcal{L}$-invariants of $\rho_{\widetilde{\wp}}$ (given by the Hodge filtration on the 2-dimensional filtered (φ, N)-modules associated to D_1^2 and D_2^3). We consider the two following representations (see § 3.3.4 where they are denoted $\Pi^1(\lambda,\psi)^+$ and $\Pi^2(\lambda,\psi)^+$):

$$\begin{tikzcd}
 & & & \widetilde{C}_{1,2} \arrow[dr, dash, dashed] & \\
 & & C_{1,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{1,3} \\
\Pi^1 \;:= & \mathrm{St}_3^\infty(\lambda) \arrow[ur, dash] \arrow[dr, dash] & & v_{\overline{P}_1}^\infty(\lambda) \arrow[ur, dash] & \\
 & & C_{2,1} & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & C_{1,1} & & \\
\Pi^2 \;:= & \mathrm{St}_3^\infty(\lambda) \arrow[ur, dash] \arrow[dr, dash] & & v_{\overline{P}_2}^\infty(\lambda) \arrow[dr, dash] & \\
 & & C_{2,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{2,3}. \\
 & & & \widetilde{C}_{2,2} \arrow[ur, dash, dashed] &
\end{tikzcd}$$

We say that D is sufficiently generic if there are canonical isomorphisms (induced by Colmez's functor [24]):

$$\begin{aligned}
\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{1,2}, \pi_{1,2}) &\xrightarrow{\sim} \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1^2, D_1^2) \\
\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{2,3}, \pi_{2,3}) &\xrightarrow{\sim} \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_2^3, D_2^3)
\end{aligned} \tag{1.4}$$

satisfying the properties of Hypothesis 3.26 in the text. We prove in Lemma 3.29, Proposition 3.30 and Proposition 3.32 that such isomorphisms are true under mild genericity assumptions on the (φ,Γ)-modules D_1^2 and D_2^3. Note that we couldn't find these isomorphisms in the literature (though we suspect they might be known), so we provided our own proofs, see e.g. the proof of Proposition 3.32 in the appendix, where we go through the Galois side and use deformation theory, which forces the aforementioned mild genericity assumptions. Using these isomorphisms, we then prove that there are canonical perfect pairings of 3-dimensional E-vector spaces (see Theorem 3.45):

$$\mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\delta_3),D_1^2\big) \times \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda),\Pi^1\big) \longrightarrow E \tag{1.5}$$

$$\mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D_2^3,\mathcal{R}_E(\delta_1)\big) \times \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_1}(\lambda),\Pi^2\big) \longrightarrow E. \tag{1.6}$$

For instance (1.5) comes from a perfect pairing $\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_3),D_1^2) \times \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1^2,\mathcal{R}_E(\delta_2)) \to E$ and an isomorphism $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^\infty_{\overline{P}_2}(\lambda),\Pi^1) \xrightarrow{\sim} \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1^2,\mathcal{R}_E(\delta_2))$ induced by the isomorphism in (1.4) and locally analytic parabolic induction (see (3.90)). The (φ,Γ)-module D gives an E-line in the left hand side of both (1.5), (1.6), hence its orthogonal space gives a 2-dimensional subspace of $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^\infty_{\overline{P}_2}(\lambda),\Pi^1)$ and a 2-dimensional subspace of $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^\infty_{\overline{P}_1}(\lambda),\Pi^2)$. Choosing a basis of each subspace and amalgamating as much as possible the four corresponding extensions produces a well-defined locally analytic representation of the form (see (3.110)):

$$\begin{tikzcd}
 & & & \widetilde{C}_{1,2} \arrow[dr, dashed, dash] & & \\
 & & C_{1,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{1,3} \arrow[r, dash] \arrow[dl, dash] & v^\infty_{\overline{P}_2}(\lambda) \\
 & & & v^\infty_{\overline{P}_1}(\lambda) & & \\
\Pi(D)^- \arrow[r, phantom, "\cong"] & \mathrm{St}_3^\infty(\lambda) \arrow[uur, dash] \arrow[ddr, dash] & & & & \\
 & & & v^\infty_{\overline{P}_2}(\lambda) \arrow[dl, dash] \arrow[dr, dash] & & \\
 & & C_{2,1} \arrow[dr, dash] & & C_{2,3} \arrow[r, dash] \arrow[dl, dashed, dash] & v^\infty_{\overline{P}_1}(\lambda) \\
 & & & \widetilde{C}_{2,2} & &
\end{tikzcd}$$

which turns out to determine and only depend on D. Then results of [4] show that there is a unique way to add constituents $\widetilde{C}_{1,4}$, $\widetilde{C}_{2,4}$, $C_{1,5}$, $C_{2,5}$ on the right so that the resulting representation $\Pi(D) = \Pi(\rho_{\widetilde{\wp}})$ contains $\Pi(D)^-$ and has the same form as (1.1) (see (3.111)).

We now assume $k_1 = k_2 = k_3$ and recall that $\rho_{\widetilde{\wp}}$ is then upper triangular. The strategy of the proof of Theorem 1.1 is the same as that of [32], [33] when $n = 2$ and $F_{\widetilde{\wp}}$ is arbitrary, and is entirely based on infinitesimal deformations. Very roughly, we replace the diagonal torus $\mathrm{GL}_1 \times \mathrm{GL}_1$ in the arguments of [32] by the two Levi $L_{\overline{P}_1} = \mathrm{GL}_2 \times \mathrm{GL}_1$ and $L_{\overline{P}_2} = \mathrm{GL}_1 \times \mathrm{GL}_2$, and we deal with the GL_2-factors using the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Following Emerton's local-global compatibility work for $\mathrm{GL}_2(\mathbb{Q}_p)$ ([42]), we first study the localized modules $\mathrm{Ord}_{P_i}(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}})$, $i = 1, 2$ where Ord_{P_i} is Emerton's ordinary functor ([40], [41]) with respect to the parabolic subgroup $P_i(\mathbb{Q}_p)$ of $\mathrm{GL}_3(\mathbb{Q}_p)$ opposite to $\overline{P}_i(\mathbb{Q}_p)$. We show that $\mathrm{Ord}_{P_i}(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}})$ is a faithful module over a certain p-adic localized Hecke algebra $\widetilde{\mathbb{T}}(U^\wp)^{P_i-\mathrm{ord}}_{\overline{\rho}}$ (see Lemma 6.7) and using the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ over deformation rings (as in [53] or [67], see also the appendix), we define a continuous admissible representation $\pi^{\otimes}_{P_i}(U^\wp)$ of $L_{P_i}(\mathbb{Q}_p)$ over $\widetilde{\mathbb{T}}(U^\wp)^{P_i-\mathrm{ord}}_{\overline{\rho}}$ (see (7.43)) and a canonical "evaluation" morphism:

$$X_{P_i}(U^\wp) \otimes_{\widetilde{\mathbb{T}}(U^\wp)^{P_i-\mathrm{ord}}_{\overline{\rho}}} \pi^{\otimes}_{P_i}(U^\wp) \longrightarrow \mathrm{Ord}_{P_i}(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}})$$

where $X_{P_i}(U^\wp)$ is the $\widetilde{\mathbb{T}}(U^\wp)^{P_i-\mathrm{ord}}_{\overline{\rho}}$-module

$$\mathrm{Hom}^{\mathrm{cts}}_{\widetilde{\mathbb{T}}(U^\wp)^{P_i-\mathrm{ord}}_{\overline{\rho}}[L_{P_i}(\mathbb{Q}_p)]}(\pi^{\otimes}_{P_i}(U^\wp),\mathrm{Ord}_{P_i}(\widehat{S}(U^\wp,\mathbb{W}^\wp)_{\overline{\rho}})),$$

$\mathbb{W}^\wp$ being an invariant $\mathcal{O}_E$-lattice in the algebraic representation $W^\wp$ (see (7.44)).

Twisting if necessary, we can assume $k_1 = k_2 = k_3 = 0$. We want to prove that the restriction morphism (1.2) (with $\chi = 1$ now) is bijective. Injectivity is not difficult, the hard part is surjectivity. Let w be a nonzero vector in the subspace $D^\perp$ of $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^\infty_{\overline{P}_{3-i}}, \Pi^i)$ orthogonal to D under the pairings (1.5), (1.6) and denote by Π^w the corresponding extension $\Pi^i - v^\infty_{\overline{P}_{3-i}}$. It is enough to prove that the following restriction morphism is surjective for $i = 1, 2$ and any such w:

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\Pi^w, \widehat{S}(U^\wp, W^\wp)[\mathfrak{m}_\rho]^{\mathrm{an}}\big)\\ &\qquad\qquad\longrightarrow \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\mathrm{St}_3^\infty, \widehat{S}(U^\wp, W^\wp)[\mathfrak{m}_\rho]^{\mathrm{an}}\big).\end{aligned}\tag{1.7}$$

We now assume $i=1$, the case $i=2$ being symmetric. Taking ordinary parts induces an isomorphism (see the first isomorphism in (7.78)):

$$(1.8)\quad \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\mathrm{St}_3^\infty, \widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{an}}\big) \\ \xrightarrow{\sim} \mathrm{Hom}_{L_{P_1}(\mathbb{Q}_p)}\big(\mathrm{St}_2^\infty \boxtimes 1, (\mathrm{Ord}_{P_1}(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]))^{\mathrm{an}}\big)$$

where $\boxtimes$ is the exterior tensor product ($\mathrm{GL}_2(\mathbb{Q}_p)$ acting on the left and $\mathbb{Q}_p^\times$ on the right). By a variation/generalization of the arguments in the $\mathrm{GL}_2(\mathbb{Q}_p)$-case, we prove that the restriction induces an isomorphism (see Corollary 7.47):

$$(1.9)\quad \mathrm{Hom}_{L_{P_1}(\mathbb{Q}_p)}\big(\pi_{1,2}\boxtimes 1, (\mathrm{Ord}_{P_1}(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]))^{\mathrm{an}}\big) \\ \xrightarrow{\sim} \mathrm{Hom}_{L_{P_1}(\mathbb{Q}_p)}\big(\mathrm{St}_2^\infty \boxtimes 1, (\mathrm{Ord}_{P_1}(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]))^{\mathrm{an}}\big).$$

Note that the isomorphism (1.9) involves the “simple” $\mathcal{L}$-invariant contained in D_1^2 and is thus already nontrivial.

Denote by V_ρ the tangent space of $\mathrm{Spec}(\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P_1-\mathrm{ord}}[1/p])$ at the closed point associated to the Galois representation ρ. Going through Galois deformation rings, one can prove that there is a canonical morphism of E-vector spaces $d\omega_{1,\rho}^+ : V_\rho \longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1^2, D_1^2)$ such that the image of the composition $d\omega_{1,\rho}^+ : V_\rho \to \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1^2, D_1^2) \twoheadrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1^2, \mathcal{R}_E(\delta_2))$ is exactly $D^\perp$ (see Proposition 7.51). The proof of this statement is based on two main ingredients. The first one (see Theorem 2.7) says that any extension $D_1^2 - D_1^2$ which is contained as a (φ,Γ)-submodule in an extension $D - D$ is sent (after a suitable twist) to an element of $D^\perp$ via $\mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1^2, D_1^2) \twoheadrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1^2, \mathcal{R}_E(\delta_2))$ (the analogue of this statement in dimension 2 was first observed by Greenberg and Stevens [46, Thm. 3.14], see also [23]). This shows that the image is contained in $D^\perp$. The second ingredient is a lower bound on $\dim_E V_\rho$ (see Proposition 7.30) which implies that the image must be all of $D^\perp$.

The vector w is thus the image of a vector $v \in V_\rho$ via the above surjection $d\omega_{1,\rho}^+ : V_\rho \twoheadrightarrow D^\perp$, and by definition of V_ρ, v is an $E[\epsilon]/\epsilon^2$-valued point of $\mathrm{Spec}(\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P_1-\mathrm{ord}})$. Denote by $\mathcal{I}_v$ the corresponding ideal of $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P_1-\mathrm{ord}}$, by a generalization of arguments due to Chenevier ([19]), one can prove that the $E[\epsilon]/\epsilon^2$-module $X_{P_1}(U^\wp)[\mathcal{I}_v][1/p]$ of vectors in $X_{P_1}(U^\wp)[1/p]$ killed by $\mathcal{I}_v$ is free of finite rank (see (7.82)). This implies that any $L_{P_1}(\mathbb{Q}_p)$-equivariant morphism $\pi_{1,2}\boxtimes 1 \to (\mathrm{Ord}_{P_1}(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]))^{\mathrm{an}}$ extends to an $E[\epsilon]/\epsilon^2$-

linear and $L_{P_1}(\mathbb{Q}_p)$-equivariant morphism

$$\widetilde{\pi}_{1,2} \boxtimes_{E[\epsilon]/\epsilon^2} \widetilde{1} \longrightarrow (\mathrm{Ord}_{P_1}(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathcal{I}_v]))^{\mathrm{an}}$$

where $\widetilde{\pi}_{1,2}\boxtimes_{E[\epsilon]/\epsilon^2}\widetilde{1} := \pi_{P_1}^{\otimes}(U^{\wp})\otimes_{\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P_1-\mathrm{ord}}}(\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P_1-\mathrm{ord}}/\mathcal{I}_v)[1/p]$. Note that $\widetilde{\pi}_{1,2}$ (resp. $\widetilde{1}$) is an extension of $\pi_{1,2}$ (resp. 1) by itself. By the adjunction formula for Ord_{P_1}, we obtain a $\mathrm{GL}_3(\mathbb{Q}_p)$-equivariant morphism:

$$\big(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \widetilde{\pi}_{1,2} \boxtimes_{E[\epsilon]/\epsilon^2} \widetilde{1}\big)^{\mathrm{an}} \longrightarrow \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathcal{I}_v]^{\mathrm{an}}. \tag{1.10}$$

The representation Π^w is a multiplicity free subquotient of the representation $(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \widetilde{\pi}_{1,2} \boxtimes_{E[\epsilon]/\epsilon^2} \widetilde{1})^{\mathrm{an}}$, and one can prove that (1.10) induces a $\mathrm{GL}_3(\mathbb{Q}_p)$-equivariant morphism:

$$\Pi^w \longrightarrow \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{an}} \subseteq \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathcal{I}_v]^{\mathrm{an}}$$

which restricts to the unique morphism $\mathrm{St}_3^\infty \to \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{an}}$ corresponding to $\pi_{1,2} \boxtimes 1 \to (\mathrm{Ord}_{P_1}(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]))^{\mathrm{an}}$ via (1.8) and (1.9) (see the proof of Theorem 7.52). This proves the surjectivity of (1.7) (for $i = 1$) and finishes the proof of Theorem 1.1.

The results of this work are used in [9], where a more precise relation is proven between the two "branches" in (1.1) and the filtered (φ, N)-module of $\rho_\wp$ (along the lines of [4, Conj. 6.1.2]). But important questions remain. For instance one can ask for a more explicit (local) construction of the $\mathrm{GL}_3(\mathbb{Q}_p)$-representation $\Pi(\rho_\wp)$. Though there is so far no construction of (analogues of) $\Pi(\rho_\wp)$ for $n \geq 4$, one can also still try to push further the results and methods of this paper in arbitrary dimension. Note that many of the intermediate results used in the proof of Theorem 1.1 are already proven here in a more general setting than just GL_3. For instance we allow an arbitrary parabolic subgroup of GL_n in §§ 4, 5, 6, 7.1.1 and we work in arbitrary dimension n in all sections except §§ 3, 7.2.3 and the appendix.

We finally mention that some of our results in arbitrary dimension have an interest in their own. For instance Corollary 7.34 gives new cases of classicality for certain p-adic automorphic forms with associated Galois representation which is de Rham at $\wp$ and Theorem 7.38 gives a complete description (under certain assumptions) of the P-ordinary part of completed cohomology for a parabolic subgroup P of GL_n with only GL_2 or GL_1 factors in its Levi subgroup, analogous to Emerton's description in the $\mathrm{GL}_2(\mathbb{Q}_p)$-case ([42]).

At the beginning of each section, the reader will find a sentence explaining its contents. We now give the main notation of the paper.

Notation

In the whole text we denote by E a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_E$ its ring of integers, ϖ_E a uniformizer of $\mathcal{O}_E$ and k_E its residue field. Given an E-bilinear map $V \times W \xrightarrow{\cup} E$, for $W' \subseteq W$ we denote:

$$(W')^{\perp} := \{v \in V,\ v \cup w = 0\ \forall\ w \in W'\}.$$

For L a finite extension of $\mathbb{Q}_p$, we let Σ_L be the set of embeddings of L into E (equivalently into $\overline{\mathbb{Q}_p}$ by taking E sufficiently large), $q_L := |k_L|$ with k_L the residue field of L, $\mathrm{Gal}_L := \mathrm{Gal}(\overline{L}/L)$ the Galois group of L, $\mathrm{W}_L \subset \mathrm{Gal}_L$ the Weil group of L, and $\Gamma_L := \mathrm{Gal}(L(\zeta_{p^n}, n \geq 1)/L)$ where $(\zeta_{p^n})_{n\geq 1}$ is a compatible system of primitive p^n-th roots of 1 in $\overline{L}$. When $L = \mathbb{Q}_p$ we write Γ instead of $\Gamma_{\mathbb{Q}_p}$. We denote by $\varepsilon : \mathrm{Gal}_L \twoheadrightarrow \Gamma_L \rightarrow E^{\times}$ the cyclotomic character with the convention $\mathrm{HT}_{\sigma}(\varepsilon) = 1$ for all $\sigma \in \Sigma_L$ where HT_{σ} is the Hodge-Tate weight relative to the embedding $\sigma : L \hookrightarrow E$, and by $\overline{\varepsilon}$ its reduction modulo p. We normalize local class field theory by sending a uniformizer to a (lift of the) geometric Frobenius. In this way, we view characters of Gal_L as characters of $L^{\times}$ without further mention. We let $\mathrm{unr}(a)$ be the unramified character of Gal_L sending a uniformizer of $L^{\times}$ to $a \in E^{\times}$ and $|\cdot| := \mathrm{unr}(q_L^{-1})$. We denote by val_p the valuation normalized by $\mathrm{val}_p(p) = 1$.

If A is a finite dimensional $\mathbb{Q}_p$-algebra, for instance $A = E$ or $E[\epsilon]/\epsilon^2$ (the dual numbers), we write $\mathcal{R}_{A,L}$ for the Robba ring associated to L with A-coefficients (see for example [51, Def. 2.2.2]). When L is fixed, we only write $\mathcal{R}_A$. We denote by $\mathrm{Ext}^i_{(\varphi,\Gamma_L)}(\cdot,\cdot)$ the extensions groups in the category of (φ,Γ_L)-modules over $\mathcal{R}_{E,L}$ and by $H^i_{(\varphi,\Gamma_L)}(\cdot) := \mathrm{Ext}^i_{(\varphi,\Gamma_L)}(\mathcal{R}_{E,L},\cdot)$ ([1, § 2.2.5], [57], [22]). If $\delta : L^{\times} \rightarrow A^{\times}$ is a continuous character, we denote by $\mathcal{R}_{A,L}(\delta)$ the associated rank one (φ,Γ_L)-module (see [51, Cons. 6.2.4]). We have:

$$\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E,\mathcal{R}_E) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Gal}_L}(E,E) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Gal}_L,E) \xrightarrow{\sim} \mathrm{Hom}(L^{\times},E),$$

where the last isomorphism follows from local class field theory, and

$$\mathrm{Hom}(\mathrm{Gal}_L,E)\ (\text{resp. } \mathrm{Hom}(L^{\times},E))$$

is the E-vector space of continuous characters of Gal_L (resp. $L^\times$) to the additive group E. We fix the isomorphism given by the above composition. For any continuous $\delta : L^\times \to E^\times$, the twist by δ^{-1} induces a canonical isomorphism $\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta), \mathcal{R}_E(\delta)) \xrightarrow{\sim} \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E, \mathcal{R}_E)$, and we deduce isomorphisms (uniformly in δ):

$$\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta), \mathcal{R}_E(\delta)) \xrightarrow{\sim} \mathrm{Hom}(L^\times, E). \tag{1.11}$$

By [33, Lem. 1.15], the isomorphism (1.11) induces an isomorphism:

$$\mathrm{Ext}^1_g(\mathcal{R}_E(\delta), \mathcal{R}_E(\delta)) \xrightarrow{\sim} \mathrm{Hom}_\infty(L^\times, E) \tag{1.12}$$

where Ext^1_g denotes the subspace of extensions which are de Rham up to twist by characters, and $\mathrm{Hom}_\infty(L^\times, E)$ denotes the subspace of smooth characters. Finally, if $L = \mathbb{Q}_p$, we denote by $\mathrm{wt}(\delta) := \lim_{z\to 0} \frac{\delta(\exp(z))-1}{z} \in E$ the Sen weight of δ, for instance $\mathrm{wt}(x^k \,\mathrm{unr}(a)) = k$ for $k \in \mathbb{Z}$ and $a \in E^\times$.

Let G be the L-points of a reductive algebraic group over $\mathbb{Q}_p$, we refer without comment to [74], resp. [75], for the background on general, resp. admissible, locally $\mathbb{Q}_p$-analytic representations of G over locally convex E-vector spaces, and to [73] for the background on continuous (admissible) representations of G over E. If V is a continuous representation of G over E, we denote by V^{an} its locally $\mathbb{Q}_p$-analytic vectors ([75, § 7]). If V is a locally $\mathbb{Q}_p$-analytic representation of G over E, we denote by V^{sm}, resp. V^{lalg}, the subrepresentation of its smooth vectors, resp. of its locally $\mathbb{Q}_p$-algebraic vectors ([43, Def. 4.2.6]). If X and Y are topological spaces, we denote by $\mathcal{C}(X, Y)$ the set of continuous functions from X to Y. If P is the L-points of a parabolic subgroup of G and π_P is a continuous representation of P over E, i.e. on a Banach vector space over E, we denote by:

$$(\mathrm{Ind}_P^G \pi_P)^{\mathcal{C}^0} := \{f : G \to \pi_P \text{ continuous},\ f(pg) = p(f(g))\}$$

the continuous parabolic induction endowed with the left action of G by right translation on functions: $(gf)(g') := f(gg')$. It is again a continuous representation of G over E. Likewise, if π_P is a locally analytic representation of P on a locally convex E-vector space of compact type, we denote by:

$$(\mathrm{Ind}_P^G \pi_P)^{\mathrm{an}} := \{f : G \to \pi_P \text{ locally } \mathbb{Q}_p\text{−analytic},\ f(pg) = p(f(g))\}$$

the locally analytic parabolic induction endowed with the same left action of G. It is again a locally analytic representation of G on a locally convex E-vector space of compact type (see e.g. [54, Rem. 5.4]). If π_P is a smooth representation of P over E, we finally denote by $(\mathrm{Ind}_P^G \pi_P)^\infty$ the smooth parabolic induction (taking locally constant functions $f : G \to \pi_P$) endowed with the same G-action. We denote by δ_P the usual (smooth unramified) modulus character of P.

If V, W are two locally $\mathbb{Q}_p$-analytic representations of G over E, we *define* the extension groups $\mathrm{Ext}^i_G(V, W)$ as in [78, Déf. 3.1], that is, as the extension groups $\mathrm{Ext}^i_{D(G,E)}(W^\vee, V^\vee)$ of their strong duals $V^\vee$, $W^\vee$ as algebraic $D(G, E)$-modules where $D(G, E)$ is the algebra of locally analytic E-valued distributions. If the center Z of G (or a subgroup Z of the center of G) acts by the same locally analytic character on V and W, we define the extension groups with that central character $\mathrm{Ext}^i_{G,Z}(V, W)$ as in [78, (3.11)], and there are then functorial morphisms $\mathrm{Ext}^i_{G,Z}(V, W) \to \mathrm{Ext}^i_G(V, W)$. If V, W are smooth representations of G over E, we denote by $\mathrm{Ext}^i_{G,\infty}(V, W)$ the usual extension groups in the category of smooth representations of G over E (see e.g. [28, § 2.1.3] or [62, § 3]). Finally, if $(V_i)_{i=1,\cdots,r}$ are admissible locally analytic representations of G, the notation $V_1 - V_2 - V_3 - \cdots - V_r$ means an admissible locally analytic representation of G such that V_1 is a subobject, V_2 is a subobject of the quotient by V_1 etc. where each subquotient $V_i - V_{i+1}$ is a nonsplit extensions of V_{i+1} by V_i.

If A is a commutative ring, M an A module and I an ideal of A, we denote by $M[I] \subseteq M$ the A-submodule of elements killed by I and by $M\{I\} := \cup_{n\geq 1} M[I^n]$.

2. Higher $\mathcal{L}$-invariants and deformations of (φ, Γ)-modules

In this section we define and study certain subspaces $\mathcal{L}_{\mathrm{FM}}(D : D_1^{n-1})$ and $\ell_{\mathrm{FM}}(D : D_1^{n-1})$ of some Ext^1 groups in the category of (φ, Γ_L)-modules that will be used in the next sections.

We fix a finite extension L of $\mathbb{Q}_p$ (and recall we write $\mathcal{R}_E$ for $\mathcal{R}_{E,L}$). Let D be a trianguline (φ, Γ_L)-module over $\mathcal{R}_E$ of arbitrary rank $n \geq 1$. We denote an arbitrary parameter of D by $(\delta_1, \cdots, \delta_n)$ where the $\delta_i : L^\times \to E^\times$ are continuous characters (see e.g. [11, § 2.1]). Recall that D can have several parameters, see *loc. cit.*

Definition 2.1. *We call a parameter $(\delta_1, \cdots, \delta_n)$ of D special if we have:*

$$\delta_i \delta_{i+1}^{-1} = |\cdot| \prod_{\sigma\in\Sigma_L} \sigma^{k_{\sigma,i}} \quad \forall\ i \in \{1, \cdots, n-1\}$$

for some $k_{\sigma,i} \in \mathbb{Z}$.

We say that $(\delta_1, \cdots, \delta_n)$ as in Definition 2.1 is noncritical if $k_{\sigma,i} \in \mathbb{Z}_{>0}$ for all σ, i. It follows from the proof of [1, Prop. 2.3.4] that a trianguline D with a special noncritical parameter is de Rham up to twist. It then easily follows from Berger's equivalence of categories [2, Thm. A] that such a D has only one special noncritical parameter. In the sequel when we say that $(D, (\delta_1, \cdots, \delta_n))$ is special noncritical, it means that $(\delta_1, \cdots, \delta_n)$ is the unique special noncritical parameter of D.

We now fix a special noncritical $(D, (\delta_1, \cdots, \delta_n))$ and for $1 \leq i < i' \leq n$ we denote by $D_i^{i'}$ the unique (φ, Γ_L)-module subquotient of D of trianguline parameter $(\delta_i, \cdots, \delta_{i'})$. It is then clear that $(D_i^{i'}, (\delta_i, \cdots, \delta_{i'}))$ is also a special noncritical (φ, Γ_L)-module.

We first assume that for $i \in \{1, \cdots, n-2\}$ the extension of $\mathcal{R}_E(\delta_{i+1})$ by $\mathcal{R}_E(\delta_i)$ appearing as a subquotient of D_1^{n-1} is nonsplit. We consider the following cup-product:

$$\operatorname{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^{n-1}) \times \operatorname{Ext}^1_{(\varphi,\Gamma_L)}(D_1^{n-1}, D_1^{n-1}) \xrightarrow{\cup} \operatorname{Ext}^2_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^{n-1}). \tag{2.1}$$

Lemma 2.2. *We have* $\dim_E \operatorname{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^{n-1}) = (n-1)[L:\mathbb{Q}_p]+1$, *and the surjection* $D_1^{n-1} \twoheadrightarrow \mathcal{R}_E(\delta_{n-1})$ *induces an isomorphism:*

$$\operatorname{Ext}^2_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^{n-1}) \xrightarrow{\sim} \operatorname{Ext}^2_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), \mathcal{R}_E(\delta_{n-1})) \cong E.$$

Proof. The lemma follows easily from [61, § 2.2] (see also [57]). □

By functoriality, we have the following commutative diagram of pairings (where we write Ext^i for $\operatorname{Ext}^i_{(\varphi,\Gamma)}$, $\mathcal{R}$ for $\mathcal{R}_E$):

(2.2)

$$\begin{tikzcd}
\operatorname{Ext}^1(\mathcal{R}(\delta_n), D_1^{n-1}) \arrow[d, equal] & \times & \operatorname{Ext}^1(D_1^{n-1}, D_1^{n-2}) \arrow[d, "\iota"'] \arrow[r, "\cup"] & \operatorname{Ext}^2(\mathcal{R}(\delta_n), D_1^{n-2}) \arrow[d] \\
\operatorname{Ext}^1(\mathcal{R}(\delta_n), D_1^{n-1}) \arrow[d, equal] & \times & \operatorname{Ext}^1(D_1^{n-1}, D_1^{n-1}) \arrow[d, "\kappa"'] \arrow[r, "\cup"] & \operatorname{Ext}^2(\mathcal{R}(\delta_n), D_1^{n-1}) \arrow[d, "\sim"'] \\
\operatorname{Ext}^1(\mathcal{R}(\delta_n), D_1^{n-1}) & \times & \operatorname{Ext}^1(D_1^{n-1}, \mathcal{R}(\delta_{n-1})) \arrow[r, "\cup"] & \operatorname{Ext}^2(\mathcal{R}(\delta_n), \mathcal{R}(\delta_{n-1}))
\end{tikzcd}$$

with the bottom right map being an isomorphism by Lemma 2.2.

Proposition 2.3. *Keep the above assumption and notation.*

(1) The map κ is surjective.

(2) The bottom cup-product in (2.2) is a perfect pairing and we have:

$$\mathrm{Ker}(\kappa) = \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^{n-1})^{\perp}$$

with respect to the middle cup-product in (2.2).

Proof. (1) It is enough to show $\mathrm{Ext}^2_{(\varphi,\Gamma_L)}(D_1^{n-1}, D_1^{n-2}) = 0$. By dévissage it is enough to show that $\mathrm{Ext}^2_{(\varphi,\Gamma_L)}(D_1^{n-1}, \mathcal{R}_E(\delta_i)) = 0$ for all $i \in \{1, \cdots, n-2\}$. We have a natural isomorphism:

$$\mathrm{Ext}^2_{(\varphi,\Gamma_L)}(D_1^{n-1}, \mathcal{R}_E(\delta_i)) \cong H^2_{(\varphi,\Gamma_L)}\big((D_1^{n-1})^{\vee} \otimes_{\mathcal{R}_E} \mathcal{R}_E(\delta_i)\big).$$

Together with [57, § 5.2] (see also [33, Prop. 1.7(4)]), we are thus reduced to show $H^0_{(\varphi,\Gamma_L)}\big(D_1^{n-1} \otimes_{\mathcal{R}_E} \mathcal{R}_E(\delta_i^{-1}\varepsilon)\big) = 0$ which follows easily from our assumption on D_1^{n-1}.

(2) Using the natural isomorphisms:

$$\begin{aligned}
\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D_1^{n-1}, \mathcal{R}_E(\delta_{n-1})) &\cong H^1_{(\varphi,\Gamma_L)}((D_1^{n-1})^{\vee} \otimes_{\mathcal{R}_E} \mathcal{R}_E(\delta_{n-1})),\\
\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^{n-1}) &\cong H^1_{(\varphi,\Gamma_L)}(D_1^{n-1} \otimes_{\mathcal{R}_E} \mathcal{R}_E(\delta_n^{-1})),\\
\mathrm{Ext}^2_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), \mathcal{R}_E(\delta_{n-1})) &\cong H^2_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_{n-1}\delta_n^{-1})),
\end{aligned}$$

we are reduced to show for the first statement in (2) that the cup-product:

$$\begin{aligned}
H^1_{(\varphi,\Gamma_L)}((D_1^{n-1})^{\vee} \otimes_{\mathcal{R}_E} \mathcal{R}_E(\delta_{n-1})) \times H^1_{(\varphi,\Gamma_L)}(D_1^{n-1} \otimes_{\mathcal{R}_E} \mathcal{R}_E(\delta_n^{-1}))&\\
\xrightarrow{\cup} H^2_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_{n-1}\delta_n^{-1}))&
\end{aligned}$$

is a perfect pairing. We have a commutative diagram (where we write H^i for $H^i_{(\varphi,\Gamma)}$, $\mathcal{R}$ for $\mathcal{R}_E$):

(2.3)

$$\begin{tikzcd}
H^1((D_1^{n-1})^{\vee} \otimes_{\mathcal{R}} \mathcal{R}(\delta_{n-1})) \arrow[d, equal] & \times & H^1(D_1^{n-1} \otimes_{\mathcal{R}} \mathcal{R}(\delta_n^{-1})) \arrow[d] \arrow[r, "\cup"] & H^2(\mathcal{R}(\delta_{n-1}\delta_n^{-1})) \arrow[d] \\
H^1((D_1^{n-1})^{\vee} \otimes_{\mathcal{R}} \mathcal{R}(\delta_{n-1})) & \times & H^1(D_1^{n-1} \otimes_{\mathcal{R}} \mathcal{R}(\delta_{n-1}^{-1}\varepsilon)) \arrow[r, "\cup"] & H^2(\mathcal{R}(\varepsilon))
\end{tikzcd}$$

where the two vertical maps on the right are induced by the injection $\mathcal{R}_E(\delta_n^{-1}) \hookrightarrow \mathcal{R}_E(\delta_{n-1}^{-1}\varepsilon)$ (see for example [51, Cor. 6.2.9], and recall from Definition 2.1 that we have $\delta_n^{-1} = \delta_{n-1}^{-1}\varepsilon \prod_{\sigma\in\Sigma_L} \sigma^{k_{\sigma,n-1}-1}$ with $k_{\sigma,n-1} - 1 \geq 0$). Moreover, using the same argument as in the proof of [33, Lem. 1.13] (or by [11, Lem. 4.8(i)] together with an easy dévissage argument), the vertical maps are isomorphisms. By Tate duality (see [57, § 5.2] or [33, Prop. 1.7(4)]), the bottom cup-product in (2.3) is a perfect pairing, hence so is the top cup-product. The first part of (2) follows.

By similar (and easier) arguments as in the proof of (1), we have

$$\mathrm{Ext}^2_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^{n-2}) = 0.$$

By (2.2), we deduce:

$$\mathrm{Ker}(\kappa) \subseteq \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^{n-1})^\perp.$$

However, since the bottom cup-product of (2.2) is a perfect pairing and the bottom right map an isomorphism, we easily get $\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^{n-1})^\perp \subseteq \mathrm{Ker}(\kappa)$, hence an equality. ☐

The (φ,Γ_L)-module D gives rise to a nonzero element in

$$\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^{n-1})$$

that we denote by $[D]$. In particular the E-vector subspace $E[D]$ it generates is well defined and we define (with respect to the two bottom pairings in (2.2)):

$$\begin{aligned}\mathcal{L}_{\mathrm{FM}}(D : D_1^{n-1}) &:= (E[D])^\perp \subseteq \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D_1^{n-1}, D_1^{n-1})\\ \ell_{\mathrm{FM}}(D : D_1^{n-1}) &:= (E[D])^\perp \subseteq \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D_1^{n-1}, \mathcal{R}_E(\delta_{n-1})).\end{aligned}$$

By Proposition 2.3 (and the bottom right isomorphism in (2.2)), we deduce a short exact sequence of E-vector spaces:

$$0 \longrightarrow \mathrm{Ker}(\kappa) \longrightarrow \mathcal{L}_{\mathrm{FM}}(D : D_1^{n-1}) \xrightarrow{\kappa} \ell_{\mathrm{FM}}(D : D_1^{n-1}) \longrightarrow 0. \tag{2.4}$$

The following corollary also follows easily from Proposition 2.3 and (2.4).

Corollary 2.4. *(1) The (φ,Γ_L)-module D (seen in $\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n),D_1^{n-1})$) is determined up to isomorphism by D_1^{n-1}, δ_n and $\mathcal{L}_{\mathrm{FM}}(D : D_1^{n-1})$ (resp. and $\ell_{\mathrm{FM}}(D : D_1^{n-1})$).*

(2) If D (seen in $\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^{n-1})$) is nonsplit, then $\mathcal{L}_{\mathrm{FM}}(D : D_1^{n-1})$ (resp. $\ell_{\mathrm{FM}}(D : D_1^{n-1})$) is of codimension 1 in $\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D_1^{n-1}, D_1^{n-1})$ (resp. in $\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D_1^{n-1}, \mathcal{R}_E(\delta_{n-1}))$).

By functoriality we have a commutative diagram for $i < n-1$ (where we write Ext^i for $\mathrm{Ext}^i_{(\varphi,\Gamma)}$, $\mathcal{R}$ for $\mathcal{R}_E$):

(2.5)

$$\begin{tikzcd}
\mathrm{Ext}^1(\mathcal{R}(\delta_n), D_1^{n-1}/D_1^i) & \times & \mathrm{Ext}^1(D_1^{n-1}/D_1^i, \mathcal{R}(\delta_{n-1})) \arrow[r, "\cup"] \arrow[d, "j_i"'] & \mathrm{Ext}^2(\mathcal{R}(\delta_n), \mathcal{R}(\delta_{n-1})) \arrow[d, equal] \\
\mathrm{Ext}^1(\mathcal{R}(\delta_n), D_1^{n-1}) \arrow[u, "u_i"] & \times & \mathrm{Ext}^1(D_1^{n-1}, \mathcal{R}(\delta_{n-1})) \arrow[r, "\cup"] & \mathrm{Ext}^2(\mathcal{R}(\delta_n), \mathcal{R}(\delta_{n-1})).
\end{tikzcd}$$

It is easy to deduce for $i < n-1$ from [57, § 5.2] (see also [33, Prop. 1.7]):

$$\mathrm{Ext}^2_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^i) = \mathrm{Ext}^2_{(\varphi,\Gamma_L)}(D_1^i, \mathcal{R}_E(\delta_{n-1})) = 0$$

and it is clear that $\mathrm{Hom}_{(\varphi,\Gamma_L)}(D_1^i, \mathcal{R}_E(\delta_{n-1})) = \mathrm{Hom}_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^{n-1}/D_1^i) = 0$. By dévissage, we deduce that u_i is surjective, j_i is injective and $\mathrm{Ker}(u_i) \cong \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^i)$. Also the two cup-products in (2.5) are perfect pairings by Proposition 2.3. In particular we obtain the following lemma.

Lemma 2.5. *We have in* $\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D_1^{n-1}, \mathcal{R}_E(\delta_{n-1}))$ *for* $i < n-1$ *(via* j_i*):*

(2.6)

$$\ell_{\mathrm{FM}}(D : D_1^{n-1}) \cap \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D_1^{n-1}/D_1^i, \mathcal{R}_E(\delta_{n-1})) = \ell_{\mathrm{FM}}(D/D_1^i : D_1^{n-1}/D_1^i)$$

and with respect to the bottom pairing in (2.5):

$$\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D_1^{n-1}/D_1^i, \mathcal{R}_E(\delta_{n-1})) \cong \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), D_1^i)^{\perp}.$$

Remark 2.6. In particular, for $i = n-2$, we have a perfect pairing:

(2.7)

$$\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_n), \mathcal{R}_E(\delta_{n-1})) \times \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_{n-1}), \mathcal{R}_E(\delta_{n-1})) \longrightarrow E.$$

Thanks to (1.11) we can thus view:

$$\begin{aligned}
\ell_{\mathrm{FM}}(D : D_1^{n-1}) \cap \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_{n-1}), \mathcal{R}_E(\delta_{n-1}))& \\
= \ell_{\mathrm{FM}}(D/D_1^{n-2} : D_1^{n-1}/D_1^{n-2})& \\
= \mathcal{L}_{\mathrm{FM}}(D/D_1^{n-2} : D_1^{n-1}/D_1^{n-2})&
\end{aligned}$$

as an E-vector subspace of $\mathrm{Hom}(L^{\times}, E)$ of codimension ≤ 1. By [33, Prop. 1.9], the pairing (2.7) induces an equality of subspaces of the extension group $\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_{n-1}), \mathcal{R}_E(\delta_{n-1}))$:

$$\mathrm{Ext}^1_g(\mathcal{R}_E(\delta_{n-1}), \mathcal{R}_E(\delta_{n-1})) \cong \mathrm{Ext}^1_e(\mathcal{R}_E(\delta_n), \mathcal{R}_E(\delta_{n-1}))^{\perp}$$

where Ext^1_e denotes the subspace of extensions which are crystalline up to twist by characters. In particular via (1.12) we have an inclusion in $\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_{n-1}),\mathcal{R}_E(\delta_{n-1}))$:

$$\mathrm{Hom}_\infty(L^\times,E)\subseteq \ell_{\mathrm{FM}}(D/D_1^{n-2}:D_1^{n-1}/D_1^{n-2})$$

if and only if D/D_1^{n-2} is crystalline up to twist by characters.

We now assume that for $i\in\{2,\cdots,n-1\}$ the extension of $\mathcal{R}_E(\delta_{i+1})$ by $\mathcal{R}_E(\delta_i)$ appearing as a subquotient of D_2^n is nonsplit. Similarly to the two bottom lines of (2.2) we have a commutative diagram of pairings:

$$\begin{tikzcd}
\mathrm{Ext}^1(D_2^n,D_2^n) \arrow[d, "\kappa'"'] & \times & \mathrm{Ext}^1(D_2^n,\mathcal{R}(\delta_1)) \arrow[d, equal] \arrow[r, "\cup"] & \mathrm{Ext}^2(D_2^n,\mathcal{R}(\delta_1)) \arrow[d, "\wr"'] \\
\mathrm{Ext}^1(\mathcal{R}(\delta_2),D_2^n) & \times & \mathrm{Ext}^1(D_2^n,\mathcal{R}(\delta_1)) \arrow[r, "\cup"] & \mathrm{Ext}^2(\mathcal{R}(\delta_2),\mathcal{R}(\delta_1))
\end{tikzcd} \tag{2.8}$$

where the right vertical map is an isomorphism of 1-dimensional E-vector spaces, the bottom cup-product is a perfect pairing, κ' is surjective, and:

$$\begin{aligned}\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma_L)}&(\mathcal{R}_E(\delta_2),D_2^n)\\ &=\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D_2^n,\mathcal{R}_E(\delta_1))=(n-1)[L:\mathbb{Q}_p]+1.\end{aligned}$$

We define as previously the orthogonal spaces

$$\begin{aligned}\mathcal{L}_{\mathrm{FM}}(D:D_2^n)&\subseteq \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D_2^n,D_2^n)\\ \ell_{\mathrm{FM}}(D:D_2^n)&\subseteq \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_2),D_2^n)\end{aligned}$$

of $E[D]\subseteq \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D_2^n,\mathcal{R}_E(\delta_1))$. We again have a short exact sequence:

$$0\longrightarrow \mathrm{Ker}(\kappa')\longrightarrow \mathcal{L}_{\mathrm{FM}}(D:D_2^n)\xrightarrow{\kappa'}\ell_{\mathrm{FM}}(D:D_2^n)\longrightarrow 0.$$

We have as in (2.5) a commutative diagram for $2<i$:

$$\begin{tikzcd}
\mathrm{Ext}^1(\mathcal{R}(\delta_2),D_2^n) & \times & \mathrm{Ext}^1(D_2^n,\mathcal{R}(\delta_1)) \arrow[d, "u_i"'] \arrow[r, "\cup"] & \mathrm{Ext}^2(\mathcal{R}(\delta_2),\mathcal{R}(\delta_1)) \arrow[d, equal] \\
\mathrm{Ext}^1(\mathcal{R}(\delta_2),D_2^i) \arrow[u, "j_i"] & \times & \mathrm{Ext}^1(D_2^i,\mathcal{R}(\delta_1)) \arrow[r, "\cup"] & \mathrm{Ext}^2(\mathcal{R}(\delta_2),\mathcal{R}(\delta_1))
\end{tikzcd}$$

where the cup-products are perfect pairings, j_i is injective and u_i is surjective. Moreover as in Lemma 2.5, we have in $\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_2), D_2^n)$ for $2 < i$ (via j_i):

$$\ell_{\mathrm{FM}}(D : D_2^n) \cap \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_2), D_2^i) = \ell_{\mathrm{FM}}(D_1^i : D_2^i).$$

Theorem 2.7. *Let $\widetilde{D}_1^{n-1}$ (resp. $\widetilde{D}_2^n$) be a deformation of D_1^{n-1} (resp. D_2^n) over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$ of rank $n-1$ (thus with $\widetilde{D}_1^{n-1} \equiv D_1^{n-1} \pmod{\epsilon}$, resp. $\widetilde{D}_2^n \equiv D_2^n \pmod{\epsilon}$). Then there exist a deformation $\widetilde{D}$ of D over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$ and a deformation $\widetilde{\delta}_n$ (resp. $\widetilde{\delta}_1$) of δ_n (resp. δ_1) over $E[\epsilon]/\epsilon^2$ such that $\widetilde{D}$ sits in an exact sequence of (φ,Γ_L)-modules over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$:*

$$0 \longrightarrow \widetilde{D}_1^{n-1} \longrightarrow \widetilde{D} \longrightarrow \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_n) \longrightarrow 0$$

$$\big(\textit{resp. } 0 \longrightarrow \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1) \longrightarrow \widetilde{D} \longrightarrow \widetilde{D}_2^n \longrightarrow 0\big)$$

if and only if (with notation as for $[D]$):

$$[\widetilde{D}_1^{n-1} \otimes_{\mathcal{R}_{E[\epsilon]/\epsilon^2}} \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_n^{-1}\delta_n)] \in \mathcal{L}_{\mathrm{FM}}(D : D_1^{n-1})$$

$$(\textit{resp. } [\widetilde{D}_2^n \otimes_{\mathcal{R}_{E[\epsilon]/\epsilon^2}} \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1^{-1}\delta_1)] \in \mathcal{L}_{\mathrm{FM}}(D : D_2^n)).$$

Proof. We prove the case D_1^{n-1}, the proof for D_2^n being symmetric. Replacing $\widetilde{D}$ and $\widetilde{D}_1^{n-1}$ by $\widetilde{D} \otimes_{\mathcal{R}_{E[\epsilon]/\epsilon^2}} \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_n^{-1}\delta_n)$ and $\widetilde{D}_1^{n-1} \otimes_{\mathcal{R}_{E[\epsilon]/\epsilon^2}} \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_n^{-1}\delta_n)$ respectively, we can assume $\widetilde{\delta}_n = \delta_n$. By twisting by $\mathcal{R}_E(\delta_n^{-1})$, without loss of generality we can assume $\delta_n = 1$. Now consider the exact sequence $0 \longrightarrow D_1^{n-1} \longrightarrow \widetilde{D}_1^{n-1} \longrightarrow D_1^{n-1} \longrightarrow 0$. Taking cohomology, we get a long exact sequence:

$$0 \to H^1_{(\varphi,\Gamma_L)}(D_1^{n-1}) \to H^1_{(\varphi,\Gamma_L)}(\widetilde{D}_1^{n-1}) \xrightarrow{\mathrm{pr}} H^1_{(\varphi,\Gamma_L)}(D_1^{n-1}) \xrightarrow{c} H^2_{(\varphi,\Gamma_L)}(D_1^{n-1})$$

with the map c equal (up to nonzero scalars) to $\langle [\widetilde{D}_1^{n-1}], \cdot \rangle$ where $\langle\ ,\ \rangle$ is the cup product in (2.1) with $\delta_n = 1$ (and $[\widetilde{D}_1^{n-1}]$ is seen in $\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D_1^{n-1}, D_1^{n-1})$). So we have $\langle [\widetilde{D}_1^{n-1}], [D] \rangle = 0$ if and only if $[D] \in H^1_{(\varphi,\Gamma_L)}(D_1^{n-1})$ lies in the image of pr if and only if a deformation $\widetilde{D}$ of D as in the statement exists. But by definition we also have $\langle [\widetilde{D}_1^{n-1}], [D] \rangle = 0$ if and only if $[\widetilde{D}_1^{n-1}] \in \mathcal{L}_{\mathrm{FM}}(D : D_1^{n-1})$. This concludes the proof. □

Remark 2.8. One can view Theorem 2.7 as a parabolic version of [46, Thm. 3.14] or [23, Thm. 0.5].

When $n = 2$, the two cases in Theorem 2.7 obviously coincide, which in particular implies the following corollary.

Corollary 2.9. *Assume $n = 2$, then we have $\mathcal{L}_{\mathrm{FM}}(D : \mathcal{R}_E(\delta_1)) = \mathcal{L}_{\mathrm{FM}}(D : \mathcal{R}_E(\delta_2))$ when these two vector spaces are viewed as subspaces of* $\mathrm{Hom}(L^\times, E)$ *via (1.11).*

Remark 2.10. For any $\sigma \in \Sigma_L$, denote by $\mathrm{Hom}_\sigma(L^\times, E)$ the subspace of $\mathrm{Hom}(L^\times, E)$ consisting of locally σ-analytic characters on $L^\times$. We have $\mathrm{Hom}_\infty(L^\times, E) \subset \mathrm{Hom}_\sigma(L^\times, E)$ and $\dim_E \mathrm{Hom}_\sigma(L^\times, E) = 2$. Let $\log_p : L^\times \to L$ be the unique character which restricts to the p-adic logarithm on $\mathcal{O}_L^\times$ and such that $\log_p(p) = 0$. We see that $(\mathrm{val}_p, \sigma \circ \log_p)$ form a basis of $\mathrm{Hom}_\sigma(L^\times, E)$. Assume $n = 2$, D special noncritical and noncrystalline (equivalently semi-stable noncrystalline with distinct Hodge-Tate weights) and denote by $\mathcal{L}_{\mathrm{FM}}(D) \subset \mathrm{Hom}(L^\times, E)$ the subspace of Corollary 2.9. Then we have $\mathcal{L}_{\mathrm{FM}}(D) \cap \mathrm{Hom}_\infty(L^\times, E) = 0$ and

$$\mathcal{L}_{\mathrm{FM}}(D)_\sigma := \mathcal{L}_{\mathrm{FM}}(D) \cap \mathrm{Hom}_\sigma(L^\times, E)$$

is 1-dimensional (inside $\mathrm{Hom}(L^\times, E)$). Thus for any $\sigma \in \Sigma_L$ there exists $\mathcal{L}_\sigma \in E$ such that $\mathcal{L}_{\mathrm{FM}}(D)_\sigma$ is generated by the vector $\sigma \circ \log_p - \mathcal{L}_\sigma \mathrm{val}_p$. By comparing Theorem 2.7 with [81, Thm. 1.1] (which generalizes a formula due to Colmez), it follows that this $\mathcal{L}_\sigma$ is equal to Fontaine-Mazur's $\mathcal{L}$-invariant obtained from the Hodge line in the σ-direct summand of the (φ, N)-filtered module associated to D (with the normalization of [23, § 3.1]).

We end this section by a quick speculation. We can call $\mathcal{L}_{\mathrm{FM}}(D : D_1^{n-1})$ (resp. $\mathcal{L}_{\mathrm{FM}}(D : D_2^n)$) the (Fontaine-Mazur) $\mathcal{L}$-invariants of D relative to D_1^{n-1} (resp. to D_2^n). A natural question in the p-adic Langlands program is to understand their counterpart on the automorphic side, e.g. in the setting of locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_n(L)$. The above results suggest that such invariants might be found in deformations of certain representations of (lower rank) Levi subgroups of $\mathrm{GL}_n(L)$. In the following section, we indeed succeed in finding such $\mathcal{L}$-invariants in the locally analytic representations of $\mathrm{GL}_3(\mathbb{Q}_p)$ constructed in [4] by means of the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

3. $\mathcal{L}$-invariants for $\mathrm{GL}_3(\mathbb{Q}_p)$

In this section we use the subspaces $\mathcal{L}_{\mathrm{FM}}(D : D_1^{n-1})$ and $\ell_{\mathrm{FM}}(D : D_1^{n-1})$ defined in § 2 to associate to a given 3-dimensional semi-stable noncrystalline representation of $\mathrm{Gal}_{\mathbb{Q}_p}$ with distinct Hodge-Tate weights one of the finite length locally analytic representations of $\mathrm{GL}_3(\mathbb{Q}_p)$ constructed in [4].

3.1. Preliminaries on locally analytic representations

We recall some useful notation and statements on locally analytic representations. We fix the $\mathbb{Q}_p$-points G of a reductive algebraic group over $\mathbb{Q}_p$ (we will only use its $\mathbb{Q}_p$-points).

Lemma 3.1. *Let V_1, V_2, V be locally $\mathbb{Q}_p$-analytic representations of G over E such that V is a strict extension of V_2 by V_1 in the category of locally analytic representations of G. Suppose $\mathrm{Hom}_G(V_2, V) = 0$, where $\mathrm{Hom}_G(V_2, V)$ is the E-vector space of continuous G-equivariant morphisms, and that V_1, V_2 have the same central character χ. Then V has central character χ.*

Proof. For z in the center of G consider the G-equivariant map $V \to V$, $v \mapsto zv - \chi(z)v$. It is easy to see this map induces a continuous G-equivariant morphism $V_2 \to V$, which has to be zero. The lemma follows. □

Let $V_1 \hookrightarrow V_2 \hookrightarrow V$ be closed embeddings of locally $\mathbb{Q}_p$-analytic representations of G over E with central character χ. Let U be a strict extension of V_1 by V and $W := U/V_2$ (where $V_2 \hookrightarrow V \hookrightarrow U$). We can then view U as a representation of G over $E[\epsilon]/\epsilon^2$ on which ϵ acts via $\epsilon : U \twoheadrightarrow V_1 \hookrightarrow U$. Thus the closed subrepresentation V of U is exactly the subspace annihilated by ϵ. We also see W as a representation over $E[\epsilon]/\epsilon^2$ by making ϵ act trivially, so that $U \twoheadrightarrow W$ is a surjection of $E[\epsilon]/\epsilon^2$-modules. Let $\psi : \mathbb{Q}_p^\times \to E$ be a continuous additive character and define the character $1 + \psi\epsilon : \mathbb{Q}_p^\times \to 1 + E\epsilon \subset (E[\epsilon]/\epsilon^2)^\times$. Set $U' := U \otimes_{E[\epsilon]/\epsilon^2} (1 + \psi\epsilon) \circ \det$ and $W' := U'/V_2$ (where we still denote by V_2 the image of $V_2 \otimes_{E[\epsilon]/\epsilon^2} (1 + \psi\epsilon) \circ \det$).

Lemma 3.2. *We have $W \cong W'$ as G-representations.*

Proof. Let e be a basis of the underlying $E[\epsilon]/\epsilon^2$-module of the representation $(1 + \psi\epsilon) \circ \det$, we have a natural E-linear bijection $f : U \xrightarrow{\sim} U'$, $v \mapsto v \otimes e$. For $v \in V$, we have:

$$g(f(v)) = g(v \otimes e) = g(v) \otimes ((1 + \psi\epsilon) \circ \det(g))e = g(v) \otimes e = f(g(v))$$

where the last equality follows from the fact that $g(v) \in V \hookrightarrow U$ is annihilated by ϵ. Thus $f|_V$ induces a G-equivariant automorphism of V if we still denote by V the image of $V \otimes_{E[\epsilon]/\epsilon^2} (1 + \psi\epsilon) \circ \det$ in U'. We now consider the induced map (still denoted by f):

$$f : U/V_2 \xrightarrow{\sim} U'/V_2.$$

The same argument using the fact that W is killed by ϵ shows that f is G-equivariant. $\square$

The following lemma will often be tacitly used in the sequel.

Lemma 3.3. *Let $Z := (\mathbb{Q}_p^\times)^r$ for some integer r and χ, χ' be locally analytic characters of Z over E. Assume $\chi \neq \chi'$, then we have $\mathrm{Ext}^i_Z(\chi', \chi) = 0$ for $i \geq 0$.*

Proof. This follows from [54, Cor. 8.8] together with [54, Thm. 4.8] and [54, Thm. 6.5]. $\square$

Notation 3.4. *Let V_1, V_2 be admissible locally $\mathbb{Q}_p$-analytic representations of G over E, $W \subseteq \mathrm{Ext}^1_G(V_2, V_1)$ be a finite dimensional E-vector subspace and $d := \dim_E W$. Then we denote by $\mathscr{E}(V_1, V_2^{\oplus d}, W)$ the extension of $V_2^{\oplus d}$ by V_1 naturally associated to W.*

Explicitly, let $e_1, \cdots, e_d$ be a basis of W over E and denote by

$$\mathscr{E}(V_1, V_2, e_i) \in \mathrm{Ext}^1_G(V_2, V_1)$$

the extension corresponding to e_i, then we have:

$$\mathscr{E}(V_1, V_2^{\oplus d}, W) := \bigoplus_{V_1}^{i=1,\cdots,d} \mathscr{E}(V_1, V_2, e_i)$$

where the subscript V_1 means the amalgamate sum over V_1. This is an admissible locally $\mathbb{Q}_p$-analytic representation of G over E which only depends on W.

3.2. p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and deformations

We study Ext^1 groups of rank 2 special (φ, Γ)-modules over $\mathcal{R}_E$ and relate them to Ext^1 groups of their associated locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. We prove several results on these Ext^1 groups that are used in the next sections. Some statements in this section might already be known or hidden in the literature, but we provide complete proofs.

3.2.1. Deformations of rank 2 special (φ, Γ)-modules. We define and study certain subspaces of Ext^1 groups of (φ, Γ)-module over $\mathcal{R}_E$ and relate them to infinitesimal deformations of rank 2 special (φ, Γ)-modules. We now assume $L = \mathbb{Q}_p$ and let $(D, (\delta_1, \delta_2))$ be a special, noncritical and nonsplit (φ, Γ)-module over $\mathcal{R}_E$ (see the beginning of § 2).

Lemma 3.5. *We have* $\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, D) = 5$ *and a short exact sequence:*

$$0 \to \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_1)) \xrightarrow{\iota} \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, D) \xrightarrow{\kappa} \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_2)) \to 0 \tag{3.1}$$

where $\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_1)) = 2$ *and* $\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_2)) = 3$.

Proof. By the hypothesis on D we have a long exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_1)) &\longrightarrow \mathrm{Hom}_{(\varphi,\Gamma)}(D, D) \\ \longrightarrow \mathrm{Hom}_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_2)) &\longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_1)) \\ \xrightarrow{\iota} \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, D) &\xrightarrow{\kappa} \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_2)) \longrightarrow \cdots \end{aligned} \tag{3.2}$$

By Proposition 2.3(1), κ is surjective, and ι is injective since the third arrow is obviously an isomorphism (using the fact that D is non-split and noting that both source and target are 1-dimensional E-vector spaces). By (1.11) we have $\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_2)) = 2$. Using [57, Thm. 5.3, Thm. 5.7], we get:

$$\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_1), \mathcal{R}_E(\delta_2)) = 1,\ \mathrm{Ext}^2_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_2)) = 0$$

which implies $\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_2)) = 3$ by an obvious dévissage. By [57, Thm. 5.3] together with [57, § 5.2] we obtain (where $D^\vee$ is the dual of D):

$$\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_1)) = \dim_E H^1_{(\varphi,\Gamma)}(D^\vee \otimes_{\mathcal{R}_E} \mathcal{R}_E(\delta_1)) = 2.$$

The lemma follows. □

We have (see Lemma 2.2 for the latter)

$$\dim_E \mathrm{Hom}_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_1), \mathcal{R}_E(\delta_1)) = \dim_E \mathrm{Ext}^2_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_1)) = 1.$$

From [57, § 5.2] and the proof of Lemma 3.5 we have:

$$\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_1)) = \dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_1))$$

$$= \dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_1), \mathcal{R}_E(\delta_1)) = 2.$$

Moreover we also have $\mathrm{Ext}^2_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_1)) = H^0_{(\varphi,\Gamma)}(D^\vee \otimes_{\mathcal{R}_E} \mathcal{R}_E(\delta_1)) = 0$. We deduce a long exact sequence:

$$\begin{aligned} (3.3)\quad 0 \longrightarrow \mathrm{Hom}_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_1), \mathcal{R}_E(\delta_1)) &\longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_1)) \\ \xrightarrow{\iota_1} \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_1)) &\xrightarrow{\kappa_1} \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_1), \mathcal{R}_E(\delta_1)) \\ &\longrightarrow \mathrm{Ext}^2_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_1)) \longrightarrow 0 \end{aligned}$$

where $\dim_E \mathrm{Im}(\iota_1) = \dim_E \mathrm{Im}(\kappa_1) = 1$. Since

$$\mathrm{Ext}^2_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_2)) = H^0_{(\varphi,\Gamma)}(\mathcal{R}_E(\varepsilon)) = 0,$$

we also have a short exact sequence:

$$\begin{aligned} (3.4)\quad 0 \longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_2)) &\xrightarrow{\iota_2} \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_2)) \\ &\xrightarrow{\kappa_2} \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_1), \mathcal{R}_E(\delta_2)) \longrightarrow 0 \end{aligned}$$

with $\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_2)) = 2$ and $\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_1), \mathcal{R}_E(\delta_2)) = 1$ (see the proof of Lemma 3.5). We denote by κ_0 the following composition:
(3.5)

$$\kappa_0 : \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, D) \xrightarrow{\kappa} \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_2)) \xrightarrow{\kappa_2} \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_1), \mathcal{R}_E(\delta_2)).$$

In the sequel we loosely identify $\mathrm{Ext}^1_{(\varphi,\Gamma)}(D, D)$ with deformations $\widetilde{D}$ of D over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$, dropping the $[\cdot]$ (this won't cause any ambiguity). We define:

$$\mathrm{Ext}^1_{\mathrm{tri}}(D, D) := \mathrm{Ker}(\kappa_0) \subseteq \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, D).$$

It is then easy to check that those $\widetilde{D}$ in $\mathrm{Ext}^1_{\mathrm{tri}}(D, D)$ can be written as a (nonsplit) extension of $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_2)$ by $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1)$ as a (φ, Γ)-module over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$ where $\widetilde{\delta}_i$ for $i \in \{1, 2\}$ is a deformation of the character δ_i over $E[\epsilon]/\epsilon^2$.

Lemma 3.6. *We have* $\dim_E \mathrm{Ext}^1_{\mathrm{tri}}(D, D) = 4$.

Proof. It follows from the surjectivity of κ (Lemma 3.5) that $\mathrm{Ker}(\kappa_0)$ is the inverse image (under the map κ) of $\mathrm{Ker}(\kappa_2)$ in $\mathrm{Ext}^1_{(\varphi,\Gamma)}(D, D)$. The lemma follows then from (3.4) and a dimension count using the first equality in Lemma 3.5. ☐

By (3.4), (3.1) and the proof of Lemma 3.6, we get a short exact sequence:
(3.6)

$$0 \to \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_1)) \xrightarrow{\iota} \mathrm{Ext}^1_{\mathrm{tri}}(D, D) \xrightarrow{\kappa} \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_2)) \to 0.$$

The map κ in (3.6) is given by sending $(\widetilde{D}, (\widetilde{\delta}_1, \widetilde{\delta}_2)) \in \mathrm{Ext}^1_{\mathrm{tri}}(D, D)$ (with the above notation) to $\widetilde{\delta}_2 \in \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_2))$. In particular we deduce from (3.6) the following lemma.

Lemma 3.7. *Let $\widetilde{D} \in \mathrm{Ext}^1_{\mathrm{tri}}(D, D)$ and $(\widetilde{\delta}_1, \widetilde{\delta}_2)$ be the above trianguline parameter of $\widetilde{D}$ over $E[\epsilon]/\epsilon^2$. Then $\widetilde{D} \in \mathrm{Ker}(\kappa)$ if and only if $\widetilde{\delta}_2 = \delta_2$.*

Let $D_1 := D_1^1 = \mathcal{R}_E(\delta_1) \subset D$ (notation of § 2), identifying $\mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1, D_1)$ with $\mathrm{Hom}(\mathbb{Q}_p^\times, E)$ by (1.11) we view $\mathcal{L}_{\mathrm{FM}}(D : D_1) \subseteq \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1, D_1)$ (see § 2) as an E-vector subspace of $\mathrm{Hom}(\mathbb{Q}_p^\times, E)$. Since D is assumed to be nonsplit, $\mathcal{L}_{\mathrm{FM}}(D : D_1)$ is one dimensional by Corollary 2.4(2). The following formula (sometimes called a Colmez-Greenberg-Stevens formula) is a special case of Theorem 2.7 (via the identification (1.11)).

Corollary 3.8. *Let $\widetilde{D} \in \mathrm{Ext}^1_{\mathrm{tri}}(D, D)$ and $(\widetilde{\delta}_1, \widetilde{\delta}_2)$ its above trianguline parameter. Let $\psi \in \mathrm{Hom}(\mathbb{Q}_p^\times, E)$ such that $\widetilde{\delta}_2\widetilde{\delta}_1^{-1} = \delta_2\delta_1^{-1}(1 + \psi\epsilon)$, then $\psi \in \mathcal{L}_{\mathrm{FM}}(D : D_1)$.*

Likewise one checks that the composition:

$$\mathrm{Ker}(\kappa) \xrightarrow{\iota^{-1}} \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_1)) \xrightarrow{\kappa_1} \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_1), \mathcal{R}_E(\delta_1)) \tag{3.7}$$

(see (3.1) for ι and (3.3) for κ_1) is given by sending $(\widetilde{D}, (\widetilde{\delta}_1, \delta_2)) \in \mathrm{Ker}(\kappa)$ (cf. Lemma 3.7) to $\widetilde{\delta}_1$. Hence, by Corollary 3.8 and $\dim_E \mathrm{Im}(\kappa_1) = 1$, (3.7) has image equal to $\mathcal{L}_{\mathrm{FM}}(D : D_1)$. Denote by ι_0 the following composition:

$$\iota_0 : \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_1)) \xrightarrow{\iota_1} \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_1)) \xrightarrow{\iota} \mathrm{Ext}^1_{\mathrm{tri}}(D, D)$$

(see (3.3) for ι_1). By Lemma 3.5 and $\dim_E \mathrm{Im}(\iota_1) = 1$ we see that $\mathrm{Im}(\iota_0)$ is a one dimensional subspace of $\mathrm{Ker}(\kappa)$. From (3.3) and (the discussion after) (3.7), we deduce a short exact sequence:

$$0 \longrightarrow \mathrm{Im}(\iota_0) \longrightarrow \mathrm{Ker}(\kappa) \xrightarrow{\kappa_1} \mathcal{L}_{\mathrm{FM}}(D : D_1) \longrightarrow 0. \tag{3.8}$$

In particular, $\mathrm{Im}(\iota_0)$ is generated by $(\widetilde{D}, \widetilde{\delta}_1, \widetilde{\delta}_2) \in \mathrm{Ext}^1_{\mathrm{tri}}(D, D)$ with $\widetilde{\delta}_1 = \delta_1$ and $\widetilde{\delta}_2 = \delta_2$.

We denote by $\mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D,D)$ the E-vector subspace of $\mathrm{Ext}^1_{(\varphi,\Gamma)}(D,D)$ consisting of (φ,Γ)-modules $\widetilde{D}$ over $\mathcal{R}_{E[\epsilon]/\epsilon^2}$ such that $\wedge^2_{\mathcal{R}_{E[\epsilon]/\epsilon^2}}\widetilde{D}\cong\mathcal{R}_{E[\epsilon]/\epsilon^2}(\delta_1\delta_2)$ (i.e. with "constant" determinant), and by $\mathrm{Ext}^1_g(D,D)$ the E-vector subspace of $\mathrm{Ext}^1_{(\varphi,\Gamma)}(D,D)$ consisting of $\widetilde{D}$ such that $\widetilde{D}\otimes_{\mathcal{R}_E}\mathcal{R}_E(\delta_1^{-1})$ is de Rham.

Lemma 3.9. *We have* $\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D,D)=3$.

Proof. We have a natural exact sequence:

$$0\longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D,D)\longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}(D,D)\longrightarrow \mathrm{Hom}(\mathbb{Q}_p^\times,E) \tag{3.9}$$

where the last map sends $\widetilde{D}\in\mathrm{Ext}^1_{(\varphi,\Gamma)}(D,D)$ to ψ' with ψ' satisfying:

$$\wedge^2_{\mathcal{R}_{E[\epsilon]/\epsilon^2}}\widetilde{D}\cong\mathcal{R}_{E[\epsilon]/\epsilon^2}(\delta_1\delta_2(1+\psi'\epsilon)).$$

On the other hand, we have an injection $j:\mathrm{Hom}(\mathbb{Q}_p^\times,E)\hookrightarrow\mathrm{Ext}^1_{(\varphi,\Gamma)}(D,D)$, $\psi\mapsto D\otimes_E\mathcal{R}_{E[\epsilon]/\epsilon^2}(1+(\psi/2)\epsilon)$ and it is clear that $\mathrm{Im}(j)$ gives a section of the last map of (3.9). Hence the latter is surjective and the lemma follows from the first equality in Lemma 3.5. ☐

Lemma 3.10. *We have* $\dim_E\left(\mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D,D)\cap\mathrm{Ext}^1_{\mathrm{tri}}(D,D)\right)=2$.

Proof. From Lemma 3.5, Lemma 3.6 and Lemma 3.9 it is sufficient to show that $\mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D,D)$ is not contained in $\mathrm{Ext}^1_{\mathrm{tri}}(D,D)$. However with the notation of the proof of Lemma 3.9, we have $\mathrm{Im}(j)\cap\mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D,D)=0$ inside $\mathrm{Ext}^1_{(\varphi,\Gamma)}(D,D)$ and it is clear that $\mathrm{Im}(j)\subseteq\mathrm{Ext}^1_{\mathrm{tri}}(D,D)$. If $\mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D,D)\subseteq\mathrm{Ext}^1_{\mathrm{tri}}(D,D)$, this would imply

$$\dim_E\mathrm{Ext}^1_{\mathrm{tri}}(D,D)\geq\dim_E\mathrm{Im}(j)+\dim_E\mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D,D)=2+3=5,$$

contradicting Lemma 3.6. ☐

Lemma 3.11. *(1) We have* $\mathrm{Ext}^1_g(D,D)\subset\mathrm{Ext}^1_{\mathrm{tri}}(D,D)$.

(2) For $\widetilde{D}\in\mathrm{Ext}^1_{\mathrm{tri}}(D,D)$ *of trianguline parameter* $\widetilde{\delta}_1=\delta_1(1+\psi_1\epsilon)$, $\widetilde{\delta}_2=\delta_2(1+\psi_2\epsilon)$, *we have* $\widetilde{D}\in\mathrm{Ext}^1_g(D,D)$ *if and only if* $\psi_i\in\mathrm{Hom}_\infty(\mathbb{Q}_p^\times,E)$ *for* $i=1,2$.

(3) We have:

$$\dim_E\mathrm{Ext}^1_g(D,D)=\begin{cases}2 & \text{if } \mathcal{L}_{\mathrm{FM}}(D:D_1)\neq\mathrm{Hom}_\infty(\mathbb{Q}_p^\times,E)\\ 3 & \text{if } \mathcal{L}_{\mathrm{FM}}(D:D_1)=\mathrm{Hom}_\infty(\mathbb{Q}_p^\times,E).\end{cases} \tag{3.10}$$

Proof. (1) Twisting by a character, we can (and do) assume that the δ_i for $i = 1, 2$ are locally algebraic (see Definition 2.1). Since $\mathrm{wt}(\delta_2\delta_1^{-1}) \in \mathbb{Z}_{<0}$ we have $\mathrm{Ext}^1_g(\mathcal{R}_E(\delta_1), \mathcal{R}_E(\delta_2)) = H^1_g(\mathcal{R}_E(\delta_2\delta_1^{-1})) = 0$ and we deduce from (3.5) (since being de Rham is preserved by taking subquotients) $\mathrm{Ext}^1_g(D, D) \subseteq \mathrm{Ker}(\kappa_0) = \mathrm{Ext}^1_{\mathrm{tri}}(D, D)$.

(2) We know that $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\delta_i(1 + \psi_i\epsilon))$ is de Rham if and only if $\psi_i \in \mathrm{Hom}_\infty(\mathbb{Q}_p^\times, E)$ (see e.g. [33, Rem. 2.2(2)]). The "only if" part follows. For $i \in \{1, 2\}$ let $\psi_i \in \mathrm{Hom}_\infty(\mathbb{Q}_p^\times, E)$, $\widetilde{\delta}_i := \delta_i(1 + \psi_i\epsilon)$ and

$$\widetilde{D} \in \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_2), \mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1)) \subset \mathrm{Ext}^1_{\mathrm{tri}}(D, D).$$

Since $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_2)$ is de Rham, we are reduced to show that:

$$D \otimes_{\mathcal{R}_{E[\epsilon]/\epsilon^2}} \mathcal{R}_{E[\epsilon]/\epsilon^2}\big(\widetilde{\delta}_2^{-1}\big) \in H^1_{(\varphi,\Gamma)}\big(\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1\widetilde{\delta}_2^{-1})\big)$$

is de Rham. However, since $\mathrm{wt}(\delta_1\delta_2^{-1}) \in \mathbb{Z}_{>0}$ and $\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1\widetilde{\delta}_2^{-1})$ is de Rham, we know (e.g. by [33, Lem. 1.11]) that any element in $H^1_{(\varphi,\Gamma)}\big(\mathcal{R}_{E[\epsilon]/\epsilon^2}(\widetilde{\delta}_1\widetilde{\delta}_2^{-1})\big)$ is de Rham. The "if" part follows.

(3) The exact sequence (3.9) induces a short exact sequence:

$$0 \to \mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D, D) \cap \mathrm{Ext}^1_g(D, D) \to \mathrm{Ext}^1_g(D, D) \to \mathrm{Hom}_\infty(\mathbb{Q}_p^\times, E) \to 0 \tag{3.11}$$

where the last map is surjective since the map j in the proof of Lemma 3.9 induces an injection $\mathrm{Hom}_\infty(\mathbb{Q}_p^\times, E) \hookrightarrow \mathrm{Ext}^1_g(D, D)$. We have

$$\widetilde{D} \in \mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D, D) \cap \mathrm{Ext}^1_g(D, D)$$

if and only if $\psi_1, \psi_2 \in \mathrm{Hom}_\infty(\mathbb{Q}_p^\times, E)$ and $\psi_1 + \psi_2 = 0$ (for ψ_i as in (2)). Moreover, for any $\widetilde{D} \in \mathrm{Ext}^1_{\mathrm{tri}}(D, D)$ we have $\psi_1 - \psi_2 \in \mathcal{L}_{\mathrm{FM}}(D : D_1)$ by Corollary 3.8. If $\mathcal{L}_{\mathrm{FM}}(D : D_1) \neq \mathrm{Hom}_\infty(\mathbb{Q}_p^\times, E)$, we see this implies $\psi_i = 0$, hence $\mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D, D) \cap \mathrm{Ext}^1_g(D, D) = \mathrm{Im}(\iota_0)$ is one dimensional by the sentences before and after (3.8). If $\mathcal{L}_{\mathrm{FM}}(D : D_1) = \mathrm{Hom}_\infty(\mathbb{Q}_p^\times, E)$, we have $\mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D, D) \cap \mathrm{Ext}^1_g(D, D) = \mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D, D) \cap \mathrm{Ext}^1_{\mathrm{tri}}(D, D)$ by (2) since $\psi_1 + \psi_2 = 0$ and $\psi_1 - \psi_2 \in \mathrm{Hom}_\infty(\mathbb{Q}_p^\times, E)$ is equivalent to $\psi_1, \psi_2 \in \mathrm{Hom}_\infty(\mathbb{Q}_p^\times, E)$, hence $\mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D, D) \cap \mathrm{Ext}^1_g(D, D)$ is 2-dimensional by Lemma 3.10. The result then follows from (3.11). □

Now fix $k \in \mathbb{Z}_{\geq 1}$, set $\delta_3 := \delta_2 x^{-k}|\cdot|^{-1}$ and consider the special case of the pairing (2.1):

(3.12)

$$\mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_3), D) \times \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(D, D) \xrightarrow{\cup} \mathrm{Ext}^2_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_3), D) \simeq E.$$

Recall the map $\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_3), \mathcal{R}_E(\delta_1)) \to \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\delta_3), D)$ is injective from our assumptions on the δ_i, $i \in \{1,2,3\}$.

Lemma 3.12. *We have* $\mathrm{Ext}^1_{\mathrm{tri}}(D,D) = \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_3), \mathcal{R}_E(\delta_1))^{\perp}$ *in (3.12) and a commutative diagram:*

(3.13)

$$\begin{tikzcd}
\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_3), \mathcal{R}_E(\delta_2)) \arrow[d, equal] & \times & \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_2)) \arrow[r, "\cup"] & E \arrow[d, equal] \\
\mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\delta_3), \mathcal{R}_E(\delta_2)\big) & \times & \mathrm{Ext}^1_{\mathrm{tri}}\big(D,D\big) \arrow[u, "\kappa"] \arrow[d] \arrow[r, "\cup"] & E \arrow[d, equal] \\
\mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\delta_3), D\big) \arrow[u] & \times & \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D,D\big) \arrow[r, "\cup"] & E.
\end{tikzcd}$$

Proof. The top squares of (3.13) are induced from the bottom squares of (2.2). Recall (see the proof of Lemma 3.6)

$$\mathrm{Ext}^1_{\mathrm{tri}}\big(D,D\big) = \kappa^{-1}\big(\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_2))\big) \subseteq \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D,D\big).$$

Replacing the middle objects in (2.5) (for $\delta_n = \delta_3$, $\delta_{n-1} = \delta_2$, $D_1^{n-1}/D_1^i = D/D_1 = \mathcal{R}_E(\delta_2)$ and $D_1^{n-1} = D$) by their preimage under the map $\kappa : \mathrm{Ext}^1_{(\varphi,\Gamma)}(D,D) \to \mathrm{Ext}^1_{(\varphi,\Gamma)}(D, \mathcal{R}_E(\delta_2))$, we obtain the bottom squares of (3.13). This gives the commutativity. Together with the second part of Lemma 2.5, the first statement also follows. □

3.2.2. Deformations of $\mathbf{GL_2(\mathbb{Q}_p)}$-representations in special cases.

By the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, we can associate a locally analytic representation $\pi(D)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ to the (φ,Γ)-module D of § 3.2.1. Moreover, under mild hypothesis, we may identify $\mathrm{Ext}^1_{(\varphi,\Gamma)}(D,D)$ to $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(D), \pi(D))$ (see Hypothesis 3.26(1) and Proposition 3.30). In this section, we recall the explicit structure of $\pi(D)$ (twisting by characters, $\pi(D)$ will be isomophic to $\pi(\lambda,\psi)$ in (3.26) below). We then study Ext^1 groups of certain subquotients of $\pi(D)$ and we construct analogues of the groups Ext^1_g, $\mathrm{Ext}^1_{\mathrm{tri}}$ for $\pi(D)$ (such that Hypothesis 3.26 (2) and Lemma 3.28

hold, see Proposition 3.22, Proposition 3.25). Many of the results in this section may not be really new, but we include the proofs for completeness.

For an integral weight μ of $\mathrm{GL}_2(\mathbb{Q}_p)$, we denote by δ_μ the algebraic character of the diagonal torus $T(\mathbb{Q}_p)$ of weight μ. We fix $\lambda = (k_1, k_2) \in \mathbb{Z}^2$ a dominant weight of $\mathrm{GL}_2(\mathbb{Q}_p)$ with respect to the Borel subgroup $B(\mathbb{Q}_p)$ of upper triangular matrices (i.e. $k_1 \geq k_2$), and denote by $L(\lambda)$ the associated algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E. If s is the nontrivial element of the Weyl group of GL_2, we have $s\cdot\lambda = (k_2-1, k_1+1)$ (dot action with respect to $B(\mathbb{Q}_p)$). We denote by St_2^∞ be the usual smooth Steinberg representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E and set:

$$I(\lambda) := \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_\lambda\big)^{\mathrm{an}}, \quad I(s\cdot\lambda) := \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_{s\cdot\lambda}\big)^{\mathrm{an}}$$

where $\overline{B}(\mathbb{Q}_p)$ is the subgroup of lower triangular matrices. Then $I(\lambda)$ has the form $I(\lambda) \cong L(\lambda) - \mathrm{St}_2^\infty(\lambda) - I(s\cdot\lambda)$ (recall $-$ denotes a nonsplit extension), $\mathrm{St}_2^\infty(\lambda) := \mathrm{St}_2^\infty \otimes_E L(\lambda)$ and where the subrepresentation $L(\lambda) - \mathrm{St}_2^\infty(\lambda)$ is isomorphic to $i(\lambda) := (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1)^\infty \otimes_E L(\lambda)$. We denote by $\mathrm{St}_2^{\mathrm{an}}(\lambda) := I(\lambda)/L(\lambda) = \mathrm{St}_2^\infty(\lambda) - I(s\cdot\lambda)$ and set:

$$\begin{aligned} \widetilde{I}(\lambda) &:= \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_\lambda(|\cdot|^{-1} \otimes |\cdot|)\big)^{\mathrm{an}} \\ \widetilde{I}(s\cdot\lambda) &:= \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_{s\cdot\lambda}(|\cdot|^{-1} \otimes |\cdot|)\big)^{\mathrm{an}}. \end{aligned}$$

Then $\widetilde{I}(\lambda)$ has the form $\mathrm{St}_2^\infty(\lambda) - L(\lambda) - \widetilde{I}(s\cdot\lambda)$ where the subrepresentation $\mathrm{St}_2^\infty(\lambda) - L(\lambda)$ is isomorphic to $\widetilde{i}(\lambda) := (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\cdot|^{-1} \otimes |\cdot|)^\infty \otimes_E L(\lambda)$. If V is a locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, we define the locally analytic homology groups $H_i(\overline{N}(\mathbb{Q}_p), V)$ as in [54, Def. 2.7] where $\overline{N}(\mathbb{Q}_p)$ is the unipotent radical of $\overline{B}(\mathbb{Q}_p)$. The homology groups in the following lemma (combined with Schraen's spectral sequence [78, (4.37), (4.38)]) will be frequently used in our study of the extension groups of locally analytic representations.

Lemma 3.13. *We have the following isomorphisms:*

$$H_i(\overline{N}(\mathbb{Q}_p), L(\lambda)) \cong \begin{cases} \delta_\lambda & i=0 \\ \delta_{s\cdot\lambda} & i=1 \\ 0 & i\geq 2 \end{cases} \tag{3.14}$$

$$(3.15)\qquad H_i(\overline{N}(\mathbb{Q}_p), \mathrm{St}_2^\infty(\lambda)) \cong \begin{cases} \delta_\lambda(|\cdot|^{-1}\otimes|\cdot|) & i=0\\ \delta_{s\cdot\lambda}(|\cdot|^{-1}\otimes|\cdot|) & i=1\\ 0 & i\geq 2\end{cases}$$

$$(3.16)\qquad H_i(\overline{N}(\mathbb{Q}_p), I(s\cdot\lambda)) \cong \begin{cases} \delta_{s\cdot\lambda} & i=0\\ \delta_\lambda(|\cdot|^{-1}\otimes|\cdot|) & i=1\\ 0 & i\geq 2\end{cases}$$

$$(3.17)\qquad H_i(\overline{N}(\mathbb{Q}_p), \widetilde{I}(s\cdot\lambda)) \cong \begin{cases} \delta_{s\cdot\lambda}(|\cdot|^{-1}\otimes|\cdot|) & i=0\\ \delta_\lambda & i=1\\ 0 & i\geq 2.\end{cases}$$

Proof. The isomorphisms (3.14) and (3.15) follow from results on classical Jacquet module together with [78, (4.41) & Thm. 4.10]. The isomorphisms (3.16) and (3.17) follow from [54, Thm. 8.13] and [78, Thm. 3.15]. ☐

The following statement is not new, we include a proof for the reader's convenience.

Theorem 3.14. *We have natural isomorphisms:*

$$(3.18)\qquad \mathrm{Hom}(\mathbb{Q}_p^\times, E) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda)) \xleftarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\widetilde{I}(\lambda)/\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda)).$$

Proof. The first isomorphism follows from [5, § 2.1], but we include a proof. By [78, (4.38)], we have a spectral sequence (where $Z(\mathbb{Q}_p)\cong\mathbb{Q}_p^\times$ is the center of $\mathrm{GL}_2(\mathbb{Q}_p)$, that we often shorten into Z):

$$\mathrm{Ext}^r_{T(\mathbb{Q}_p),Z}(H_s(\overline{N}(\mathbb{Q}_p), L(\lambda)), I(\lambda)) \Rightarrow \mathrm{Ext}^{r+s}_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(L(\lambda), I(\lambda)).$$

Together with (3.14), we have isomorphisms:

$$\mathrm{Hom}(T(\mathbb{Q}_p)/Z(\mathbb{Q}_p), E) \xrightarrow{\sim} \mathrm{Ext}^1_{T(\mathbb{Q}_p),Z}(\delta_\lambda,\delta_\lambda) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(L(\lambda), I(\lambda)).$$

By [78, Cor. 4.8], we have

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(L(\lambda),L(\lambda)) = \mathrm{Ext}^2_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(L(\lambda),L(\lambda)) = 0.$$

By dévissage, Lemma 3.1 and [4, Lem. 2.1.1], we have then:

$$(3.19)\qquad \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(L(\lambda), I(\lambda)) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(L(\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda)) \\ \cong \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda)).$$

The first isomorphism follows. By [4, Prop. 3.1.6] we have

$$\mathrm{Ext}^i_{\mathrm{GL}_2(\mathbb{Q}_p)}(\widetilde{I}(s\cdot\lambda), \mathrm{St}_2^{\infty}(\lambda)) = 0 \text{ for } i = 1, 2.$$

By [78, (4.37)] and (3.17), we have $\mathrm{Ext}^i_{\mathrm{GL}_2(\mathbb{Q}_p)}(\widetilde{I}(s\cdot\lambda), I(s\cdot\lambda)) = 0$ for $i \in \mathbb{Z}_{\geq 0}$. By dévissage, we deduce $\mathrm{Ext}^i_{\mathrm{GL}_2(\mathbb{Q}_p)}(\widetilde{I}(s\cdot\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda)) = 0$ for $i = 1, 2$. The second isomorphism then follows by dévissage again. □

By [78, (4.37) & (4.38)], we have a commutative diagram (with the notation of the last proof):

(3.20)
$$\begin{tikzcd}
\mathrm{Hom}(T(\mathbb{Q}_p)/Z(\mathbb{Q}_p), E) \arrow[r, "\sim"] \arrow[d] & \mathrm{Ext}^1_{T(\mathbb{Q}_p),Z}(\delta_\lambda, \delta_\lambda) \arrow[r, "\sim"] \arrow[d] & \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(L(\lambda), I(\lambda)) \arrow[d] \\
\mathrm{Hom}(T(\mathbb{Q}_p), E) \arrow[r, "\sim"] & \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_\lambda, \delta_\lambda) \arrow[r, "\sim"] & \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), I(\lambda)).
\end{tikzcd}$$

Consider the short exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), L(\lambda)) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), I(\lambda)) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda)) \longrightarrow 0$$

where the last map is surjective by (3.19). Contrary to $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(L(\lambda), L(\lambda)) = 0$, we have $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), L(\lambda)) \neq 0$. More precisely, we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(Z(\mathbb{Q}_p), E) \arrow[r, "\sim"] \arrow[d] & \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), L(\lambda)) \arrow[d] \\
\mathrm{Hom}(T(\mathbb{Q}_p), E) \arrow[r, "\sim"] & \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), I(\lambda))
\end{tikzcd}$$

where the bottom horizontal map is the composition of the bottom line in (3.20) and where the left vertical map is given by $\psi \mapsto \psi \circ \det$. In particular, the natural surjective map:

$$\mathrm{Hom}(T(\mathbb{Q}_p), E) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda)) \tag{3.21}$$

is zero on $\mathrm{Hom}(Z(\mathbb{Q}_p), E)$.

We can make this map more explicit (e.g. by unwinding the spectral sequence [78, (4.37)]). Let $\psi \in \mathrm{Hom}(\mathbb{Q}_p^\times, E)$ and choose $\psi_i \in \mathrm{Hom}(\mathbb{Q}_p^\times, E)$

for $i = 1, 2$ such that $\psi_1 - \psi_2 = \psi$. Let $\sigma(\psi_1, \psi_2)$ be the following two dimensional representation of $T(\mathbb{Q}_p)$:

$$\sigma(\psi_1,\psi_2)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} := \begin{pmatrix} 1 & \psi_1(a)+\psi_2(d) \\ 0 & 1 \end{pmatrix}$$

and consider the natural exact sequence:

$$0 \longrightarrow I(\lambda) \longrightarrow \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_\lambda \otimes_E \sigma(\psi_1,\psi_2)\big)^{\mathrm{an}} \xrightarrow{\mathrm{pr}} I(\lambda) \longrightarrow 0.$$

Then the locally analytic representation:

$$\pi(\lambda,\psi)^- := \mathrm{pr}^{-1}(L(\lambda))/L(\lambda) \;\cong\; \mathrm{St}_2^{\mathrm{an}}(\lambda) - L(\lambda) \tag{3.22}$$

only depends on ψ and not on the choice of ψ_1 and ψ_2, and the map (3.21) is given by sending ψ to $\pi(\lambda,\psi)^-$ (that is seen as an element of $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda))$). Moreover:

$$\pi(\lambda,\psi_1,\psi_2)^- := \mathrm{pr}^{-1}(i(\lambda))/L(\lambda) \tag{3.23}$$

actually depends on (and is determined by) both ψ_1 and ψ_2. By [78, (4.37)] and (3.14), (3.15), we have

$$\mathrm{Hom}(T(\mathbb{Q}_p), E) \xrightarrow{\sim} \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_\lambda, \delta_\lambda) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(i(\lambda), I(\lambda))$$

and the composition is given by mapping (ψ_1, ψ_2) to $\mathrm{pr}^{-1}(i(\lambda))$. By [62, Prop. 15] and the same argument as in the proof of [62, Cor. 2] (see also [28]), we have:

$$\mathrm{Ext}^i_{\mathrm{GL}_2(\mathbb{Q}_p),\infty}(i(0), 1) = 0, \quad i \in \mathbb{Z}_{\geq 0}. \tag{3.24}$$

By a version without central character of the spectral sequence [78, (4.27)] (which follows exactly by the same argument), we deduce from (3.24)

$$\mathrm{Ext}^i_{\mathrm{GL}_2(\mathbb{Q}_p)}(i(\lambda), L(\lambda)) = 0 \text{ for } i \in \mathbb{Z}_{\geq 0}.$$

Hence the natural push-forward map:

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(i(\lambda), I(\lambda)) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(i(\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda))$$

is a bijection. Putting the above maps together, we obtain:

$$\mathrm{Hom}(T(\mathbb{Q}_p), E) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(i(\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda)), \;\; (\psi_1,\psi_2) \longmapsto \pi(\lambda,\psi_1,\psi_2)^-.$$

Let $0 \neq \psi \in \mathrm{Hom}(\mathbb{Q}_p^\times, E)$ and define:

$$\pi(\lambda, \psi) \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widetilde{I}(\lambda)/\,\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda)\big) \tag{3.25}$$

to be the preimage of $\pi(\lambda, \psi)^-$ in (3.22) via the second isomorphism of (3.18) (we should write $[\pi(\lambda, \psi)]$ to denote some element of the above Ext^1 associated to the representation $\pi(\lambda, \psi)$, but as in § 3.2.1 we drop the $[\cdot]$, which won't cause any ambiguity). So one has:

$$\pi(\lambda, \psi) \simeq \mathrm{St}_2^{\mathrm{an}}(\lambda) - (L(\lambda) - \widetilde{I}(s \cdot \lambda)) \tag{3.26}$$

and we recall the irreducible constituents of $\mathrm{St}_2^{\mathrm{an}}(\lambda)$ are $\mathrm{St}_2^\infty(\lambda)$ and $I(s\cdot\lambda)$.

We now study the extension groups:

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi)) \text{ and } \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(\pi(\lambda,\psi),\pi(\lambda,\psi)).$$

Note first that by [4, Lem. 2.1.1] one can identify $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi))$ with deformations $\widetilde{\pi}$ of $\pi(\lambda,\psi)$ over $E[\epsilon]/\epsilon^2$. Let $\chi_\lambda := \delta_\lambda|_{Z(\mathbb{Q}_p)}$, which is the central character of $\pi(\lambda,\psi)$.

Lemma 3.15. *For any $\widetilde{\pi} \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi))$, there exists a unique lifting $\widetilde{\chi}_\lambda : Z(\mathbb{Q}_p) \to (E[\epsilon]/\epsilon^2)^\times$ of χ_λ such that $Z(\mathbb{Q}_p)$ acts on $\widetilde{\pi}$ via $\widetilde{\chi}_\lambda$.*

Proof. For $v \in \widetilde{\pi}$, we have $(z-\chi_\lambda(z))v \in \pi(\lambda,\psi)$ and thus $(z-\chi_\lambda(z))^2 v = 0$ for all $z \in Z(\mathbb{Q}_p)$. The map $v \mapsto (z-\chi_\lambda(z))v$ induces a morphism from $\pi(\lambda,\psi)$ (as quotient of $\widetilde{\pi}$) to $\pi(\lambda,\psi)$ (as subobject of $\widetilde{\pi}$) which is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant since z is in the center. But any such endomorphism of $\pi(\lambda,\psi)$ is a scalar by [36, § 3.4] since all (absolutely) irreducible constituents of $\pi(\lambda,\psi)$ are distinct. So for any $v \in \widetilde{\pi}$ and $z \in Z(\mathbb{Q}_p)$, we have $(z-\chi_\lambda(z))v \in E(\epsilon v)$, and hence there exists $\widetilde{\chi}_\lambda : Z(\mathbb{Q}_p) \to (E[\epsilon]/\epsilon^2)^\times$ (which *a priori* depends on v) such that $zv = \widetilde{\chi}_\lambda(z)v$ for all $z \in Z(\mathbb{Q}_p)$. We fix a v which is not in $\pi(\lambda,\psi) = \epsilon\widetilde{\pi}$ and define:

$$\pi(\widetilde{\chi}_\lambda) := \big\{w \in \widetilde{\pi},\ (z-\widetilde{\chi}_\lambda(z))w = 0\ \forall\ z \in Z(\mathbb{Q}_p)\big\}$$

which is a $\mathrm{GL}_2(\mathbb{Q}_p)$-subrepresentation of $\widetilde{\pi}$ strictly containing $\pi(\lambda,\psi) = \epsilon\widetilde{\pi}$. Thus we have $\mathrm{St}_2^\infty(\lambda) \subseteq \pi(\widetilde{\chi}_\lambda)/\pi(\lambda,\psi)$ since $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Q}_p)}\pi(\lambda,\psi) \cong \mathrm{St}_2^\infty(\lambda)$ and $\pi(\widetilde{\chi}_\lambda)/\pi(\lambda,\psi)$ is a nonzero subrepresentation of $\widetilde{\pi}/\epsilon\widetilde{\pi} \simeq \pi(\lambda,\psi)$. We need to prove $\pi(\widetilde{\chi}_\lambda) = \widetilde{\pi}$. If $\pi(\widetilde{\chi}_\lambda) \neq \widetilde{\pi}$, then there exists another lifting

$\widetilde{\chi}'_\lambda \neq \widetilde{\chi}_\lambda$ such that $\pi(\widetilde{\chi}'_\lambda)$ strictly contains $\pi(\lambda, \psi)$, and hence $\mathrm{St}_2^\infty(\lambda) \subseteq \pi(\widetilde{\chi}'_\lambda)/\pi(\lambda,\psi)$. This implies $Z(\mathbb{Q}_p)$ acts on the subextension $V_1 \subset \widetilde{\pi}$ of $\mathrm{St}_2^\infty(\lambda)$ by $\pi(\lambda,\psi) = \epsilon\widetilde{\pi}$ via χ_λ. However, by Lemma 3.1, this then implies $Z(\mathbb{Q}_p)$ acts on the whole $\widetilde{\pi}$ by the character χ_λ, a contradiction. Hence $\pi(\widetilde{\chi}_\lambda) = \widetilde{\pi}$. ☐

Lemma 3.16. *We have a short exact sequence:*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(\pi(\lambda,\psi),\pi(\lambda,\psi)) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi)) \xrightarrow{\mathrm{pr}} \mathrm{Hom}(\mathbb{Q}_p^\times, E) \longrightarrow 0$$

where pr *sends* $\widetilde{\pi}$ *to* $(\widetilde{\chi}_\lambda\chi_\lambda^{-1}-1)/\epsilon$ *where* $\widetilde{\chi}_\lambda$ *is the central character of* $\widetilde{\pi}$ *given by Lemma 3.15.*

Proof. It is sufficient to prove pr is surjective. However, it is easy to check that $\psi' \mapsto \pi(\lambda,\psi) \otimes_E (1+\frac{\psi'}{2}\epsilon) \circ \det$ gives a section of the map pr. The lemma follows. ☐

The following lemma consists of some Ext calculations (using Schraen's spectral sequences [78, (4.37), (4.38)], Lemma 3.13 and dévissage). We suggest to skip the proof on first reading.

Lemma 3.17. *(1) We have* $\mathrm{Ext}^i_{\mathrm{GL}_2(\mathbb{Q}_p)}(I(s\cdot\lambda),\pi(\lambda,\psi)) = \mathrm{Ext}^i_{\mathrm{GL}_2(\mathbb{Q}_p)}(\widetilde{I}(s\cdot\lambda),\pi(\lambda,\psi)) = 0$ *for all* $i \in \mathbb{Z}_{\geq 0}$.

(2) We have:

$$\begin{aligned}
&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)) = 2,\\
&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)) = 4,\\
&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(L(\lambda),\pi(\lambda,\psi)) = \dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda),\pi(\lambda,\psi)) = 1,\\
&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(\pi(\lambda,\psi)/\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi))\\
&= \dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi)/\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)) = 1.
\end{aligned}$$

(3) The following natural sequences are exact:

$$\begin{aligned}
(3.27)\quad 0 &\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(\pi(\lambda,\psi)/\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi))\\
&\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(\pi(\lambda,\psi),\pi(\lambda,\psi))\\
&\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi))
\end{aligned}$$

$$(3.28)\quad 0 \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi)/\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi))$$

$$\xrightarrow{\iota_0} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi)) \xrightarrow{\kappa_1} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi))$$

with $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(\pi(\lambda,\psi),\pi(\lambda,\psi)) \leq 3$ *and*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi)) \leq 5.$$

(4) The following natural sequences are exact (see (3.22) for $\pi(\lambda,\psi)^-$*):*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)^-) \xrightarrow{\iota_1} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\widetilde{I}(s\cdot\lambda)) \longrightarrow 0 \tag{3.29}$$

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\mathrm{St}_2^\infty(\lambda)) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)^-) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)^-/\,\mathrm{St}_2^\infty(\lambda)) \longrightarrow 0 \tag{3.30}$$

with $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\mathrm{St}_2^\infty(\lambda)) = 2$, $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\widetilde{I}(s\cdot\lambda)) = 1$ *and*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),L(\lambda)) = \dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)^-/\,\mathrm{St}_2^\infty(\lambda)) = 1.$$

Proof. In this proof, we write Ext^i (resp. Ext^i_Z) for $\mathrm{Ext}^i_{\mathrm{GL}_2(\mathbb{Q}_p)}$ (resp. $\mathrm{Ext}^i_{\mathrm{GL}_2(\mathbb{Q}_p),Z}$).

(1) We prove the case of $I(s\cdot\lambda)$, the proof for $\widetilde{I}(s\cdot\lambda)$ being parallel. By [78, (4.37)], (3.16) and Lemma 3.3, we have $\mathrm{Ext}^i(I(s\cdot\lambda),I(\lambda))=0$ for all $i\in\mathbb{Z}_{\geq 0}$. By [78, Cor. 4.3], we have:

$$\mathrm{Ext}^i(I(s\cdot\lambda),L(\lambda)) \cong \mathrm{Ext}^{i-1}\left(L(\lambda)^\vee, (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\delta_\mu\delta_B)^{\mathrm{an}}\right)$$

where μ is the weight such that $L(\lambda)^\vee \cong L(\mu)$. However, by [78, (4.37)], (3.14) (with λ replaced by μ) and Lemma 3.3, we have

$$\mathrm{Ext}^i\left(L(\lambda)^\vee, (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\delta_\mu\delta_B)^{\mathrm{an}}\right) = 0 \text{ for } i\in\mathbb{Z}_{\geq 0},$$

hence $\mathrm{Ext}^i(I(s\cdot\lambda), L(\lambda))=0$ for all $i\in\mathbb{Z}_{\geq 0}$. By dévissage, we deduce:

$$\mathrm{Ext}^i(I(s\cdot\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda))=0,\ \forall\ i\geq 0. \tag{3.31}$$

By [78, (4.37)] and (3.16) (+ Lemma 3.3), we also have:

$$\mathrm{Ext}^i(I(s\cdot\lambda), \widetilde{I}(s\cdot\lambda))=0,\ \forall\ i\geq 0 \tag{3.32}$$

and by dévissage we deduce $\mathrm{Ext}^i(I(s\cdot\lambda), L(\lambda)-\widetilde{I}(s\cdot\lambda))=0$ for $i\geq 0$. Together with (3.31) this implies (again by dévissage) $\mathrm{Ext}^i(I(s\cdot\lambda), \pi(\lambda,\psi))=0$ for all $i\geq 0$. This concludes the proof of (1).

(2) By [78, Cor. 4.8], we have $\mathrm{Ext}^i_Z(\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^\infty(\lambda))=0$ for $i=1,2$. Consider the following map:

$$\mathrm{Hom}(\mathbb{Q}_p^\times, E)\to \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^\infty(\lambda)),\ \psi'\mapsto \mathrm{St}_2^\infty(\lambda)\otimes_E (1+\psi'\epsilon)\circ\det. \tag{3.33}$$

It is straightforward to see this map is injective. We claim it is also surjective. For any nonsplit extension $\widetilde{\pi}\in \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^\infty(\lambda))$ (which we view as a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $E[\epsilon]/\epsilon^2$), let $\psi'\in\mathrm{Hom}(\mathbb{Q}_p^\times, E)$ be such that the central character of $\widetilde{\pi}$ is given by $\chi_\lambda(1+\psi'\epsilon)$ (argue as in Lemma 3.15 for the latter, though this is simpler here). Then the representation $\widetilde{\pi}\otimes_{E[\epsilon]/\epsilon^2}(1-(\psi'/2)\epsilon)\circ\det$ has central character χ_λ and hence is isomorphic to $\mathrm{St}_2^\infty(\lambda)^{\oplus 2}\cong \mathrm{St}_2^\infty(\lambda)\otimes_E E[\epsilon]/\epsilon^2$ since $\mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^\infty(\lambda))=0$ ([78, Cor. 4.8]). So we have:

$$\begin{aligned}\widetilde{\pi} &\cong \big(\widetilde{\pi}\otimes_{E[\epsilon]/\epsilon^2}(1-(\psi'/2)\epsilon)\circ\det\big)\otimes_{E[\epsilon]/\epsilon^2}(1+(\psi'/2)\epsilon)\circ\det\\ &\cong \big(\mathrm{St}_2^\infty(\lambda)\otimes_E E[\epsilon]/\epsilon^2\big)\otimes_{E[\epsilon]/\epsilon^2}(1+(\psi'/2)\epsilon)\circ\det\\ &\cong \mathrm{St}_2^\infty(\lambda)\otimes_E(1+(\psi'/2)\epsilon)\circ\det.\end{aligned}$$

Thus $\dim_E \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^\infty(\lambda))=2$. By [78, (4.38)] and (3.15), we have:

$$\begin{aligned}\mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \widetilde{I}(\lambda)) &\cong \mathrm{Ext}^1_{T(\mathbb{Q}_p),Z}(\delta_\lambda(|\cdot|^{-1}\otimes|\cdot|), \delta_\lambda(|\cdot|^{-1}\otimes|\cdot|))\\ \mathrm{Ext}^i_Z(\mathrm{St}_2^\infty(\lambda), I(s\cdot\lambda)) &= 0\ \ \forall\ i\in\mathbb{Z}_{\geq 0}.\end{aligned}$$

Putting these together we deduce by dévissage:

$$\begin{aligned}&\mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)/\mathrm{St}_2^\infty(\lambda))\cong \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)/\mathrm{St}_2^\infty(\lambda))\\ &\cong \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), L(\lambda)-\widetilde{I}(s\cdot\lambda))\cong \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \widetilde{I}(\lambda))\cong \mathrm{Hom}(\mathbb{Q}_p^\times, E)\end{aligned} \tag{3.34}$$

where, for the second last isomorphism, we use the exact sequence:

$$0 \longrightarrow \mathrm{St}_2^\infty(\lambda) \longrightarrow \widetilde{I}(\lambda) \longrightarrow (L(\lambda) - \widetilde{I}(s\cdot\lambda)) \longrightarrow 0$$

together with $\mathrm{Ext}^i_Z(\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^\infty(\lambda)) = 0$, $i = 1, 2$. Likewise we have:

$$\begin{aligned} (3.35)\quad 0 = \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^\infty(\lambda)) &\longrightarrow \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)) \\ \longrightarrow \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)/\,\mathrm{St}_2^\infty(\lambda)) &\longrightarrow \mathrm{Ext}^2_Z(\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^\infty(\lambda)) = 0 \end{aligned}$$

from which together with (3.34) we deduce $\dim_E \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)) = 2$. Similarly:

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^\infty(\lambda)) &\longrightarrow \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)) \\ &\longrightarrow \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)/\,\mathrm{St}_2^\infty(\lambda)) \longrightarrow 0 \end{aligned}$$

where the last map is surjective by (3.35) and the first isomorphism of (3.34). By the above dimension computations we deduce

$$\dim_E \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)) = 4.$$

To prove the remaining equalities in (2), we only need to prove $\dim_E \mathrm{Ext}^1_Z(L(\lambda), \pi(\lambda,\psi)) = 1$ since the other equalities follow easily from (1) and Lemma 3.1. By [78, (4.38)] and (3.14), we have $\mathrm{Ext}^i_Z(L(\lambda), \widetilde{I}(s\cdot\lambda)) = 0$ for $i \geq 0$ and by [78, Cor. 4.8], we have $\mathrm{Ext}^1_Z(L(\lambda), L(\lambda)) = 0$. By dévissage, we deduce then $\mathrm{Ext}^1_Z(L(\lambda), \widetilde{I}(\lambda)/\,\mathrm{St}_2^\infty(\lambda)) = 0$. By the exact sequence:

$$\begin{aligned} (3.36)\quad 0 \longrightarrow \mathrm{Hom}(L(\lambda), \widetilde{I}(\lambda)/\,\mathrm{St}_2^\infty(\lambda)) &\longrightarrow \mathrm{Ext}^1_Z(L(\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda)) \\ \longrightarrow \mathrm{Ext}^1_Z(L(\lambda), \pi(\lambda,\psi)) &\longrightarrow \mathrm{Ext}^1_Z(L(\lambda), \widetilde{I}(\lambda)/\,\mathrm{St}_2^\infty(\lambda)) = 0, \end{aligned}$$

Theorem 3.14 and an easy dimension count, we get $\dim_E \mathrm{Ext}^1_Z(L(\lambda), \pi(\lambda,\psi)) = 1$.

(3) This follows easily from (1), (2) and Lemma 3.16.

(4) To get the exact sequences, it is sufficient to prove that the maps:

$$\begin{array}{rcl} \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)) & \longrightarrow & \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \widetilde{I}(s\cdot\lambda)) \\ \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)^-) & \longrightarrow & \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)^-/\,\mathrm{St}_2^\infty(\lambda)) \end{array}$$

are surjective and it is sufficient to prove they are surjective with Ext^1 replaced by Ext^1_Z (since the vector spaces on the right hand side do not change). The second one follows easily from $\mathrm{Ext}^2_Z(\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^\infty(\lambda)) = 0$ (see the proof of (2) above). By [78, (4.38)] and (3.15), we have

$$\dim_E \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \widetilde{I}(s \cdot \lambda)) = 1 \text{ and } \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), I(s \cdot \lambda)) = 0.$$

And by [78, Cor. 4.8] we have $\dim_E \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), L(\lambda)) = 1$. The last two equalities imply by dévissage

$$\dim_E \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)^-/\,\mathrm{St}_2^\infty(\lambda)) \leq 1.$$

The first, together with $\dim_E \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)) = 2$ in (2), imply the surjectivity of $\mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)) \to \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \widetilde{I}(s \cdot \lambda))$, and then $\dim_E \mathrm{Ext}^1_Z(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)^-/\,\mathrm{St}_2^\infty(\lambda)) = 1$. We have seen

$$\dim_E \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \mathrm{St}_2^\infty(\lambda)) = 2$$

in the proof of (2), and the rest of (4) follows from lemma 3.1. □

Remark 3.18. It follows from (3.36) and (2) that, if $\psi' \notin E\psi \subset \mathrm{Hom}(\mathbb{Q}_p^\times, E)$, the image of $\pi(\lambda,\psi')^-$, seen as an element of $\mathrm{Ext}^1(L(\lambda), \mathrm{St}_2^{\mathrm{an}}(\lambda))$, in $\mathrm{Ext}^1(L(\lambda), \pi(\lambda,\psi))$ is the unique nonsplit extension V of $L(\lambda)$ by $\pi(\lambda,\psi)$. Moreover V contains the unique extension V_0 of $L(\lambda)^{\oplus 2}$ by $\mathrm{St}_2^{\mathrm{an}}(\lambda) = \mathrm{St}_2^\infty(\lambda) - I(s\cdot\lambda)$ with socle $\mathrm{St}_2^\infty(\lambda)$ and we have $(V_0)^{\mathrm{lalg}} \cong V^{\mathrm{lalg}} \cong \mathrm{St}_2^\infty(\lambda) - L(\lambda) \cong \widetilde{i}(\lambda)$. By Lemma 3.17(1) and (3.28), we have:

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), \pi(\lambda,\psi)) &\cong \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi)/\,\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi))\\ &\lhook\joinrel\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi), \pi(\lambda,\psi))\end{aligned}$$

and we let $\widetilde{\pi}$ be the image of $V \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(L(\lambda), \pi(\lambda,\psi))$ via the above injection. It is not difficult then to deduce:

$$\widetilde{\pi}^{\mathrm{lalg}} \cong \begin{cases} \mathrm{St}_2^\infty(\lambda)^{\oplus 2} & \psi \text{ not smooth} \\ \mathrm{St}_2^\infty(\lambda) \oplus \widetilde{i}(\lambda) & \psi \text{ smooth.} \end{cases} \tag{3.37}$$

Thus if ψ is smooth, the map $\widetilde{\pi}^{\mathrm{lalg}} \to \pi(\lambda,\psi)^{\mathrm{lalg}} \cong \widetilde{i}(\lambda)$ induced by $\widetilde{\pi} \twoheadrightarrow \pi(\lambda,\psi)$ is nonzero but not surjective.

We next construct $\mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda,\psi), \pi(\lambda,\psi)) \subseteq \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi), \pi(\lambda,\psi))$ using the Jacquet-Emerton functor. We first make the following hypotheses, which will be proved (under some mild technical assumption) in Proposition 3.30 below.

Hypothesis 3.19. *(1) Any representation in* $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi))$ *is very strongly admissible in the sense of [37, Def. 0.12] (which implies* $\pi(\lambda,\psi)$ *itself is very strongly admissible).*

(2) We have

$$\begin{aligned}&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(\pi(\lambda,\psi),\pi(\lambda,\psi))=3\\ &dim_E\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi))=5.\end{aligned}$$

In particular (by Lemma 3.17(2)), the last maps in (3.27) and (3.28) are surjective.

Denote by $\mathrm{Hom}(T(\mathbb{Q}_p),E)_\psi$ the subspace of $\mathrm{Hom}(T(\mathbb{Q}_p),E)$ generated by those $(\psi_1,\psi_2)\in\mathrm{Hom}(\mathbb{Q}_p^\times,E)^2$ such that $\psi_1-\psi_2\in E\psi$. For a locally analytic character $\delta:T(\mathbb{Q}_p)\to E^\times$, denote by $\mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta,\delta)_\psi\subseteq\mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta,\delta)$ the E-vector subspace corresponding to $\mathrm{Hom}(T(\mathbb{Q}_p),E)_\psi$ via the natural bijection $\mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta,\delta)\cong\mathrm{Hom}(T(\mathbb{Q}_p),E)$. Denoting by J_B the Jacquet-Emerton functor relative to the Borel subgroup B (where the $T(\mathbb{Q}_p)$-action is normalized as in [38]), we have since $\psi\neq 0$:

$$J_B(\pi(\lambda,\psi))=\begin{cases}J_B(\mathrm{St}_2^\infty(\lambda))\cong\delta_\lambda(|\cdot|\otimes|\cdot|^{-1}) & \psi\text{ not smooth}\\ J_B(\mathrm{St}_2^\infty(\lambda))\oplus J_B(L(\lambda))\cong\delta_\lambda(|\cdot|\otimes|\cdot|^{-1})\oplus\delta_\lambda & \psi\text{ smooth.}\end{cases}\tag{3.38}$$

It is clear that the right hand side is contained in the left hand side. Since $\pi(\lambda,\psi)$ is very strongly admissible, it is not difficult to prove they are equal using [7, Thm. 4.3] together with the left exactness of J_B and [37, Ex. 5.1.9].

Lemma 3.20. *(1) Let* $V\in\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi))$*, then* $J_B(V)\neq J_B(\pi(\lambda,\psi))$ *if and only if* V *lies in the image of* $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)^-)$*.*

(2) The functor J_B *induces a bijection:*

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)^-)\xrightarrow{\sim}\mathrm{Ext}^1_{T(\mathbb{Q}_p)}\big(\delta_\lambda(|\cdot|\otimes|\cdot|^{-1}),\delta_\lambda(|\cdot|\otimes|\cdot|^{-1})\big)_\psi.\tag{3.39}$$

Proof. In this proof we write $\chi:=\delta_\lambda(|\cdot|\otimes|\cdot|^{-1})$ for simplicity.

(1) We first prove the "only if" part, and for that we can assume that V is nonsplit. If $J_B(V)\neq J_B(\pi(\lambda,\psi))$, then by (3.38) we see that $J_B(V)$ is isomorphic to an extension of χ by $J_B(\pi(\lambda,\psi))$ and that there exists an extension $\widetilde{\chi}$ of χ by χ such that $j_1:\widetilde{\chi}\hookrightarrow J_B(V)$ (recall $\mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\chi,\delta_\lambda)=0$). Denote by $\widetilde{\delta}_\lambda:=\widetilde{\chi}\otimes_E(|\cdot|^{-1}\otimes|\cdot|)$, which is thus isomorphic to an extension of δ_λ by δ_λ. One can check (e.g. by the proof of [31, Lem. 4.11]) that the morphism j_1 is balanced in the sense of [38, Def. 0.8]. From Hypothesis 3.19

(both (1) and (2) are needed), we deduce that V is very strongly admissible. Let $I_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\widetilde{\delta}_\lambda$ denote the closed subrepresentation of $\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\widetilde{\delta}_\lambda\big)^{\mathrm{an}}$ generated by $\widetilde{\chi}$ via the natural embedding (see [37, Lem. 0.3] for details):

$$\widetilde{\chi} \hookrightarrow J_B\Big(\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\widetilde{\delta}_\lambda\big)^{\mathrm{an}}\Big) \hookrightarrow \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\widetilde{\delta}_\lambda\big)^{\mathrm{an}}.$$

By [37, Thm. 0.13], the map j_1 then induces a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map:

$$j_2 : I_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\widetilde{\delta}_\lambda \longrightarrow V$$

such that the morphism j_1 can be recovered from j_2 by applying the functor $J_B(\cdot)$. We have $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{Im}(j_2) \xrightarrow{\sim} \mathrm{soc}_{\mathrm{GL}_2(\mathbb{Q}_p)}V \cong \mathrm{St}_2^\infty(\lambda)$ (as V is nonsplit). This implies $\mathrm{St}_2^\infty(\lambda)$ has multiplicity 2 in the irreducible constituents of $\mathrm{Im}(j_2)$, since otherwise we would have $\mathrm{Im}(j_2) \subseteq \pi(\lambda,\psi)$ and thus $\widetilde{\chi} \hookrightarrow J_B(\mathrm{Im}(j_2)) \subseteq J_B(\pi(\lambda,\psi))$ which is a contradiction. By the exact sequence (3.29) together with the fact that $\widetilde{I}(s\cdot\lambda)$ is not an irreducible constituent of $I_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\widetilde{\delta}_\lambda$ (since it is not an irreducible constituent of $(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\widetilde{\delta}_\lambda)^{\mathrm{an}}$), we obtain that V comes from an element in $\mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)^-)$.

We prove the "if" part. For $\psi' \in \mathrm{Hom}(\mathbb{Q}_p^\times, E)$, let $U(\psi') := \mathrm{St}_2^\infty(\lambda) \otimes_E (1+\psi'\epsilon)\circ\det$, hence $J_B(U(\psi')) \cong \chi \otimes_E (1+\psi'\epsilon)\circ\det$. In particular, taking J_B induces a bijection by (3.33):

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\mathrm{St}_2^\infty(\lambda)) \xrightarrow{\sim} \mathrm{Hom}(Z(\mathbb{Q}_p),E)\big(\hookrightarrow \mathrm{Hom}(T(\mathbb{Q}_p),E)\big).$$

Denote by $W(\psi') \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)^-)$ the image of $U(\psi')$ via the injection in (3.30) (so $U(\psi') \subset W(\psi')$). By left exactness of J_B we have:

$$\chi \otimes_E (1+\psi'\epsilon)\circ\det \hookrightarrow J_B(W(\psi')). \tag{3.40}$$

Now let $\Psi = (\psi_1,\psi_2) \in \mathrm{Hom}(T(\mathbb{Q}_p),E)_\psi \setminus \mathrm{Hom}(Z(\mathbb{Q}_p),E)$ (i.e. $\psi_1 \neq \psi_2$ and $\psi_1 - \psi_2 \in E\psi$) and consider the representation $\pi(\lambda,\psi_1,\psi_2)^-$ in (3.23). We know $\pi(\lambda,\psi)^- \subseteq \pi(\lambda,\psi_1,\psi_2)^-$ and thus $\pi(\lambda,\psi_1,\psi_2)^-$ gives a *nonsplit* extension of $\mathrm{St}_2^\infty(\lambda)$ by $\pi(\lambda,\psi)^-$ (since the quotient $i(\lambda)$ is nonsplit). Note that by construction $\pi(\lambda,\psi_1,\psi_2)^-$ is a subquotient of $W(\Psi) := (\mathrm{Ind}_{\overline{B}}^{\mathrm{GL}_2}\delta_\lambda(1+\Psi\epsilon))^{\mathrm{an}}$ and that we have a natural injection $\chi \otimes_E (1+\Psi\epsilon) \hookrightarrow J_B(W(\Psi))$ (cf. [37, Lem. 0.3]). Moreover $\pi(\lambda,\psi_1,\psi_2)^- \hookrightarrow W(\Psi)/L(\lambda)$ and neither $J_B(L(\lambda))$ nor $J_B((W(\Psi)/L(\lambda))/\pi(\lambda,\psi_1,\psi_2)^-) = J_B(I(s\cdot\lambda))$ contains χ as a subquotient (the latter by [37, Ex. 5.1.9]). By left exactness of J_B we deduce:

$$\chi \otimes_E (1+\Psi\epsilon) \hookrightarrow J_B(\pi(\lambda,\psi_1,\psi_2)^-). \tag{3.41}$$

From Lemma 3.17(2)&(4) we deduce $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)^-) = 3$ and we let Π be the unique extension of $\mathrm{St}_2^\infty(\lambda)^{\oplus 3}$ by $\pi(\lambda,\psi)^-$ with $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Q}_p)} \Pi \cong \mathrm{St}_2^\infty(\lambda)$. The above discussion implies $J_B(\Pi)$ contains the unique extension of $\chi^{\oplus 3}$ by χ with socle χ attached to the 3-dimensional space $\mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\chi,\chi)_\psi$. Indeed, let $\{\psi_1'\circ\det, \psi_2'\circ\det, \Psi_3 := \Psi\}$ be a basis of the 3-dimensional space $\mathrm{Hom}(T(\mathbb{Q}_p),E)_\psi \cong \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\chi,\chi)_\psi$ where $\{\psi_1',\psi_2'\}$ is a basis of $\mathrm{Hom}(\mathbb{Q}_p^\times, E)$ and $\Psi = (\psi_1,\psi_2)$ is as after (3.40), then we have by (3.30) again:

$$\Pi \cong W(\psi_1') \oplus_{\pi(\lambda,\psi)^-} W(\psi_2') \oplus_{\pi(\lambda,\psi)^-} \pi(\lambda,\psi_1,\psi_2)^-.$$

By (3.40), (3.41), left exactness of J_B and (3.38), we deduce that applying J_B to $\Pi \twoheadrightarrow \mathrm{St}_2^\infty(\lambda)^{\oplus 3}$ induces a surjective map:

$$J_B(\Pi) \twoheadrightarrow J_B(\mathrm{St}_2^\infty(\lambda)^{\oplus 3}) \cong \chi^{\oplus 3}. \tag{3.42}$$

This, together with (3.38) and left exactness of J_B, imply that any $U \in \mathrm{Ext}^1(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)^-)$ satisfies $J_B(U) \neq J_B(\pi(\lambda,\psi))$.

(2) By the proof of (1) (see (3.40), (3.41), (3.42)) together with (3.38) and $\mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\chi,\delta_\lambda) = 0$, we see that taking J_B induces a map:

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)^-) \to \mathrm{Ext}^1_{T(\mathbb{Q}_p)}\big(\chi, J_B(\pi(\lambda,\psi)^-)\big) \cong \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\chi,\chi)$$

which induces an isomorphism

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda), \pi(\lambda,\psi)^-) \xrightarrow{\sim} \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\chi,\chi)_\psi.$$

This finishes the proof. ☐

Remark 3.21. From the proof of Lemma 3.20, we can explicitly describe the inverse of (3.39) as follows. Let $\Psi \in \mathrm{Hom}(T(\mathbb{Q}_p),E)_\psi$, define:

$$\widetilde{\chi} = \delta_\lambda(|\cdot|\otimes|\cdot|^{-1})(1+\Psi\epsilon) \in \mathrm{Ext}^1_{T(\mathbb{Q}_p)}\big(\delta_\lambda(|\cdot|\otimes|\cdot|^{-1}), \delta_\lambda(|\cdot|\otimes|\cdot|^{-1})\big)_\psi$$

and consider the short exact sequence:

$$0 \longrightarrow I(\lambda) \longrightarrow \big(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_\lambda(1+\Psi\epsilon)\big)^{\mathrm{an}} \xrightarrow{\mathrm{pr}} I(\lambda) \longrightarrow 0.$$

If $\Psi = (\psi',\psi')$, i.e. $\Psi \in \mathrm{Hom}(Z(\mathbb{Q}_p),E)$, then $\mathrm{pr}^{-1}(i(\lambda))/L(\lambda)$ has a subrepresentation isomorphic to $\mathrm{St}_2^\infty(\lambda)\otimes_E(1+\psi'\epsilon)\circ\det$. The inverse image of $\widetilde{\chi}$ in (3.39) is then given by the push-forward of this representation via

$\mathrm{St}_2^\infty(\lambda) \hookrightarrow \pi(\lambda,\psi)^-$. If $\Psi \notin \mathrm{Hom}(Z(\mathbb{Q}_p), E)$, the inverse image of Ψ is then isomorphic to $\mathrm{pr}^{-1}(i(\lambda))/L(\lambda)$.

We now denote by $\mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda,\psi),\pi(\lambda,\psi))$ the kernel of the composition:

$$\begin{aligned}(3.43)\quad \kappa_0 : \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi)) &\xrightarrow{\kappa_1} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)) \\ &\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\widetilde{I}(s\cdot\lambda))\end{aligned}$$

with κ_1 as in (3.28). In particular, by (3.28) we have

$$\mathrm{Im}(\iota_0) \subset \mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda,\psi),\pi(\lambda,\psi)).$$

Proposition 3.22. *(1) We have* $\dim_E \mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda,\psi),\pi(\lambda,\psi)) = 4$.

(2) For $\widetilde{\pi} \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi))$, *we have*

$$\widetilde{\pi} \in \mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda,\psi),\pi(\lambda,\psi))$$

if and only if $\delta_\lambda(|\cdot|\otimes|\cdot|^{-1})$ *appears with multiplicity* 2 *in* $J_B(\widetilde{\pi})$.

(3) We have a natural short exact sequence:

$$\begin{aligned}(3.44)\quad 0 \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi)/\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)) &\xrightarrow{\iota_0} \mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda,\psi),\pi(\lambda,\psi)) \\ &\longrightarrow \mathrm{Ext}^1_{T(\mathbb{Q}_p)}\big(\delta_\lambda(|\cdot|\otimes|\cdot|^{-1}),\delta_\lambda(|\cdot|\otimes|\cdot|^{-1})\big)_\psi \longrightarrow 0.\end{aligned}$$

Proof. By Hypothesis 3.19(2) and Lemma 3.17(4) (see in particular (3.29)), there is a natural exact sequence:

$$(3.45)\quad 0\to\mathrm{Ext}^1(\pi(\lambda,\psi)/\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi))\xrightarrow{\iota_0}\mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda,\psi),\pi(\lambda,\psi))\to\mathrm{Im}(\iota_1)\to 0.$$

(1) follows by Lemma 3.17(2), (3.29) and a dimension count. Together with (the proof of) Lemma 3.20(1), left exactness of J_B and (3.38), we easily deduce (2) and (3), where the third map of (3.44) is given by:

$$\begin{aligned}(3.46)\quad \mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda,\psi),\pi(\lambda,\psi)) \twoheadrightarrow \mathrm{Im}(\iota_1) &\xrightarrow[\sim]{\iota_1^{-1}} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)^-) \\ &\xrightarrow[\sim]{(3.39)} \mathrm{Ext}^1_{T(\mathbb{Q}_p)}\big(\delta_\lambda(|\cdot|\otimes|\cdot|^{-1}),\delta_\lambda(|\cdot|\otimes|\cdot|^{-1})\big)_\psi.\end{aligned}$$

□

Remark 3.23. By Lemma 3.20(2) and its proof, for any

$$\widetilde{\pi}\in \mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda,\psi),\pi(\lambda,\psi)),$$

the composition (3.46) sends $\widetilde{\pi}$ to the (unique) deformation $\widetilde{\chi}\in \mathrm{Ext}^1_{T(\mathbb{Q}_p)}\big(\delta_\lambda(|\cdot|\otimes|\cdot|^{-1}),\delta_\lambda(|\cdot|\otimes|\cdot|^{-1})\big)_\psi$ such that $\widetilde{\chi}\hookrightarrow J_B(\widetilde{\pi})$.

We denote by κ the following composition:

$$\begin{aligned}(3.47)\quad \kappa:\ &\mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda,\psi),\pi(\lambda,\psi))\\ &\xrightarrow{(3.46)} \mathrm{Ext}^1_{T(\mathbb{Q}_p)}\big(\delta_\lambda(|\cdot|\otimes|\cdot|^{-1}),\delta_\lambda(|\cdot|\otimes|\cdot|^{-1})\big)_\psi\\ &\qquad\cong \mathrm{Hom}(T(\mathbb{Q}_p),E)_\psi \xrightarrow{\mathrm{pr}_2} \mathrm{Hom}(\mathbb{Q}_p^\times,E)\end{aligned}$$

where the last map sends $\Psi=(\psi_1,\psi_2)$ to ψ_2 (and hence is surjective). From the exact sequence:

$$0\longrightarrow E\psi\longrightarrow \mathrm{Hom}(T(\mathbb{Q}_p),E)_\psi\xrightarrow{\mathrm{pr}_2}\mathrm{Hom}(\mathbb{Q}_p^\times,E)\longrightarrow 0$$

(where the injection is $\psi\mapsto\Psi=(\psi,0)$), we obtain with (3.44) an exact sequence (compare with (3.8)):

(3.48)

$$0\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi)/\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi))\xrightarrow{\iota_0}\mathrm{Ker}(\kappa)\longrightarrow E\psi\longrightarrow 0.$$

Lemma 3.24. *(1) We have* $\dim_E \mathrm{Ker}(\kappa)=2$, *and* $\widetilde{\pi}\in\mathrm{Ker}(\kappa)$ *if and only if* $\kappa_1(\widetilde{\pi})\in E\iota_1(\pi(\lambda,\psi,0)^-)$ *where* κ_1 *is as in (3.28) and* ι_1 *as in (3.29).*

(2) We have $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(\pi(\lambda,\psi),\pi(\lambda,\psi))\cap\mathrm{Ker}(\kappa)=\mathrm{Im}(\iota_0)$ *(where the intersection is in* $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi))$*), and it is a* 1*-dimensional* E*-vector space.*

Proof. (1) The first statement follows from (3.48) and Lemma 3.17(2). By (3.41) applied to $\Psi=(\psi,0)$ and Remark 3.23, the "if" part follows. However, it is straightforward from (3.28) and Lemma 3.17(2) that the E-vector space $\kappa_1^{-1}(E\iota_1([\pi(\lambda,\psi,0)^-]))$ is also 2-dimensional. The "only if" part follows.

(2) The direction $\supseteq$ is clear from the definitions and Lemma 3.17(3). By (1), it is sufficient to show that if $\kappa_1([\widetilde{\pi}])\neq 0$, i.e. $\kappa_1([\widetilde{\pi}])\in E^\times\iota_1([\pi(\lambda,\psi,0)^-])$, then $\widetilde{\pi}$ does not have central character χ_λ (which is the central character of $\pi(\lambda,\psi)$). It is then enough to show that $\pi(\lambda,\psi,0)^-$ does not have central character χ_λ. By the construction following Theorem 3.14 and by Lemma 3.1 (applied first to the extension W of $V_1=\pi(\lambda,\psi,0)^-$ by $V_2=L(\lambda)$ inside $V:=(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\delta_\lambda\otimes_E\sigma(\psi,0))^{\mathrm{an}}$, then to $V_1=W$, $V_2=\mathrm{St}_2^{\mathrm{an}}(\lambda)$), if

$\pi(\lambda,\psi,0)^-$ has central character χ_λ, so does $(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\delta_\lambda\otimes_E\sigma(\psi,0))^{\mathrm{an}}$, a contradiction. □

We denote by $\mathrm{Ext}^1_g(\pi(\lambda,\psi),\pi(\lambda,\psi))\subset\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi))$ the E-vector subspace generated by those $\widetilde{\pi}$ such that $\widetilde{\pi}^{\mathrm{lalg}}\neq\pi(\lambda,\psi)^{\mathrm{lalg}}$.

Proposition 3.25. *(1) We have (ι_0 as in (3.28)):*

$$\mathrm{Im}(\iota_0)\subseteq\mathrm{Ext}^1_g(\pi(\lambda,\psi),\pi(\lambda,\psi))\subseteq\mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda,\psi),\pi(\lambda,\psi)).$$

(2) The exact sequence (3.44) induces an exact sequence:

$$\begin{aligned}(3.49)\quad 0\to\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi)/\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi))\xrightarrow{\iota_0}\mathrm{Ext}^1_g(\pi(\lambda,\psi),\pi(\lambda,\psi))\\ \to\mathrm{Hom}_\infty(T(\mathbb{Q}_p),E)_\psi\to 0\end{aligned}$$

where we have identified $\mathrm{Ext}^1_{T(\mathbb{Q}_p)}\big(\delta_\lambda(|\cdot|\otimes|\cdot|^{-1}),\delta_\lambda(|\cdot|\otimes|\cdot|^{-1})\big)_\psi$ *with* $\mathrm{Hom}(T(\mathbb{Q}_p),E)_\psi$ *and* $\mathrm{Hom}_\infty(T(\mathbb{Q}_p),E)_\psi$ *is the subspace of smooth characters in* $\mathrm{Hom}(T(\mathbb{Q}_p),E)_\psi$. *In particular:*

$$\dim_E\mathrm{Ext}^1_g(\pi(\lambda,\psi),\pi(\lambda,\psi))=\begin{cases}2 & \psi\textit{ non smooth}\\ 3 & \psi\textit{ smooth.}\end{cases}$$

Proof. (1) It is easy to see $\mathrm{Im}(\iota_0)\subseteq\mathrm{Ext}^1_g(\pi(\lambda,\psi),\pi(\lambda,\psi))$. Since we have $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Q}_p)}\pi(\lambda,\psi)\cong\mathrm{St}_2^\infty(\lambda)$, for any $\widetilde{\pi}\in\mathrm{Ext}^1_g(\pi(\lambda,\psi),\pi(\lambda,\psi))$, its image $\kappa_1(\widetilde{\pi})$ in $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi))$ (see (3.28)) in fact lies in the image of:

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)^{\mathrm{lalg}})\longrightarrow\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda),\pi(\lambda,\psi)).$$

This easily implies $\kappa_0(\widetilde{\pi})=0$ (κ_0 as in (3.43)), and (1) follows.

(2) Let $\widetilde{\pi}\in\mathrm{Ext}^1_g(\pi(\lambda,\psi),\pi(\lambda,\psi))$. By (1) and Remark 3.23, we know that there exists $\Psi\in\mathrm{Hom}(T(\mathbb{Q}_p),E)_\psi$ such that:

$$\delta_\lambda(|\cdot|\otimes|\cdot|^{-1})\otimes_E(1+\Psi\epsilon)\lhook\joinrel\longrightarrow J_B(\widetilde{\pi}).\tag{3.50}$$

Moreover, the natural surjection $\widetilde{\pi}\twoheadrightarrow\pi(\lambda,\psi)$ induces a nonzero map

$$\widetilde{\pi}^{\mathrm{lalg}}/\pi(\lambda,\psi)^{\mathrm{lalg}}\to\pi(\lambda,\psi)^{\mathrm{lalg}},$$

and hence we have $\mathrm{St}_2^\infty(\lambda)\hookrightarrow\widetilde{\pi}^{\mathrm{lalg}}/\pi(\lambda,\psi)^{\mathrm{lalg}}$ (since $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Q}_p)}\pi(\lambda,\psi)^{\mathrm{lalg}}\cong\mathrm{St}_2^\infty(\lambda)$). Thus $\mathrm{St}_2^\infty(\lambda)$ is not an irreducible constituent of $\widetilde{\pi}/\widetilde{\pi}^{\mathrm{lalg}}$, from which we see (together with the left exactness of J_B and [37, Ex. 5.1.9]) that the map (3.50) must have image in the subspace $J_B(\widetilde{\pi}^{\mathrm{lalg}})$. However, $J_B(\widetilde{\pi}^{\mathrm{lalg}})$ is locally algebraic since so is $\widetilde{\pi}^{\mathrm{lalg}}$, which implies $\Psi \in \mathrm{Hom}_\infty(T(\mathbb{Q}_p), E) \cap \mathrm{Hom}(T(\mathbb{Q}_p), E)_\psi = \mathrm{Hom}_\infty(T(\mathbb{Q}_p), E)_\psi$.

By (1), the sequence (3.44) hence induces (3.49), except for the surjectivity on the right. By Lemma 3.17(2) and an easy dimension count, it is enough to prove this surjectivity. However, by the construction in Remark 3.21, if Ψ in Remark 3.21 is smooth, then we see that the inverse image $\widetilde{\pi}_1$ of Ψ in (3.39) has extra locally algebraic vectors than $(\pi(\lambda,\psi)^-)^{\mathrm{lalg}} = \pi(\lambda,\psi)^{\mathrm{lalg}}$. Let $\widetilde{\pi} \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi), \pi(\lambda,\psi))$ such that $\kappa_1(\widetilde{\pi}) = \iota_1(\widetilde{\pi}_1)$, it is easy to see that we have an injection $\widetilde{\pi}_1 \subset \widetilde{\pi}$. Hence $\widetilde{\pi} \in \mathrm{Ext}^1_g(\pi(\lambda,\psi), \pi(\lambda,\psi))$, and $\widetilde{\pi}$ is sent (up to nonzero scalars) to Ψ via (3.46) (use Remark 3.23 and that $\widetilde{\pi}_1$ is sent to Ψ via (3.39)). This concludes the proof. □

Finally, for any locally algebraic character $\delta : \mathbb{Q}_p^\times \to E^\times$, it is obvious that all the above results hold if we twist all the representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ by $\delta \circ \det$.

3.2.3. *p*-adic correspondence for $\mathbf{GL_2}(\mathbb{Q}_p)$ and deformations.

We relate the Ext^1 groups of § 3.2.1 to those of § 3.2.2 via the local p-adic correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. Part of the argument (the proof of Proposition 3.32), which is essentially independent from the rest of the paper, is given in the appendix.

We keep all the previous notation. For $k \in \mathbb{Z}_{>0}$ and $0 \neq \psi \in \mathrm{Hom}(\mathbb{Q}_p^\times, E)$, we denote by $D(k,\psi) \in \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E, \mathcal{R}_E(|\cdot|x^k))$ the unique (nonsplit) extension up to isomorphism such that:

$$(ED(k,\psi))^\perp = E\psi \in \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(|\cdot|x^k), \mathcal{R}_E(|\cdot|x^k)) \overset{(1.11)}{\cong} \mathrm{Hom}(\mathbb{Q}_p^\times, E)$$

for the perfect pairing given by the cup-product:

$$\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E, \mathcal{R}_E(|\cdot|x^k)) \times \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(|\cdot|x^k), \mathcal{R}_E(|\cdot|x^k)) \longrightarrow E.$$

For $\lambda = (k_1, k_2) \in \mathbb{Z}^2$ with $k_1 > k_2$, we denote by $D(\lambda,\psi) := D(k_1 - k_2, \psi) \otimes_{\mathcal{R}_E} \mathcal{R}_E(x^{k_2})$ and $\lambda^\flat := (k_1, k_2+1)$. For $\alpha \in E^\times$, we set:

$$D(\alpha,\lambda,\psi) := D(\lambda,\psi) \otimes_{\mathcal{R}_E} \mathcal{R}_E(\mathrm{unr}(\alpha)). \tag{3.51}$$

We also make the following hypotheses.

Hypothesis 3.26. *(1) There exists an isomorphism of E-vector spaces:*

$$\mathrm{pLL}:\mathrm{Ext}^1_{(\varphi,\Gamma)}(D(p,\lambda,\psi),D(p,\lambda,\psi))\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda^\flat,\psi),\pi(\lambda^\flat,\psi)) \tag{3.52}$$

and any representation in $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda,\psi),\pi(\lambda,\psi))$ is very strongly admissible.

(2) The isomorphism (3.52) induces an isomorphism:

$$\mathrm{Ext}^1_{\mathrm{tri}}\big(D(p,\lambda,\psi),D(p,\lambda,\psi)\big)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{tri}}\big(\pi(\lambda^\flat,\psi),\pi(\lambda^\flat,\psi)\big). \tag{3.53}$$

(3) Let $\widetilde{D}\in\mathrm{Ext}^1_{\mathrm{tri}}(D(p,\lambda,\psi),D(p,\lambda,\psi))$ and $(\psi_1,\psi_2)\in\mathrm{Hom}(\mathbb{Q}_p^\times,E)^2$ such that:

$$\big(x^{k_1}(1+\psi_1\epsilon),|\cdot|^{-1}x^{k_2}(1+\psi_2\epsilon)\big)$$

is a trianguline parameter of $\widetilde{D}$ (see § 3.2.1). If $\widetilde{\pi}\in\mathrm{Ext}^1_{\mathrm{tri}}\big(\pi(\lambda^\flat,\psi),\pi(\lambda^\flat,\psi)\big)$ is the image of $\widetilde{D}$ via the isomorphism (3.53), we have an embedding:

$$\delta_{\lambda^\flat}(|\cdot|\otimes|\cdot|^{-1})(1+\Psi\epsilon)\longleftrightarrow J_B(\widetilde{\pi})$$

where $\Psi:=(\psi_1,\psi_2)\in\mathrm{Hom}(T(\mathbb{Q}_p),E)$.

Remark 3.27. (1) By Lemma 3.5 and Lemma 3.17(3), Hypothesis 3.26(1) implies Hypothesis 3.19.

(2) In Proposition 3.30 and Proposition 3.32 below, under mild hypothesis and using some deformation theory, we will show that Colmez's functor induces an isomorphism (3.52) such that Hypothesis 3.26 holds. The resulting isomorphism (3.52) should also induce a bijection:

$$\mathrm{Ext}^1_{(\varphi,\Gamma),Z}(D(p,\lambda,\psi),D(p,\lambda,\psi))\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),Z}(\pi(\lambda^\flat,\psi),\pi(\lambda^\flat,\psi)), \tag{3.54}$$

but we won't need this property in the paper.

Lemma 3.28. *Assuming Hypothesis 3.26, then (3.52) induces isomorphisms:*

$$\begin{aligned}
\mathrm{Ext}^1_g\big(D(p,\lambda,\psi),D(p,\lambda,\psi)\big) &\xrightarrow{\sim} \mathrm{Ext}^1_g\big(\pi(\lambda^\flat,\psi),\pi(\lambda^\flat,\psi)\big) &\quad (3.55)\\
\mathrm{Ker}(\kappa^{\mathrm{gal}}) &\xrightarrow{\sim} \mathrm{Ker}(\kappa^{\mathrm{aut}}) &\quad (3.56)
\end{aligned}$$

where we denote by κ^{gal} the morphism κ in (3.2) and by κ^{aut} the morphism κ in (3.47).

Proof. For $\widetilde{D} \in \mathrm{Ext}^1_{(\varphi,\Gamma)}(D(\lambda,\psi), D(\lambda,\psi))$ it follows from Lemma 3.11 that we have $\widetilde{D} \in \mathrm{Ext}^1_g(D(\lambda,\psi), D(\lambda,\psi))$ if and only if $\widetilde{D}$ is trianguline and the trianguline parameter of $\widetilde{D}$ is locally algebraic. Together with Remark 3.23, Proposition 3.25(2) and Hypothesis 3.26(2)&(3), the first isomorphism follows. The second follows from Lemma 3.7 together with Remark 3.23, (3.47) and Hypothesis 3.26(2)&(3). ☐

The following lemma is a trivial consequence of the Colmez-Fontaine theorem ([27, Thm. A]) and of the main result of [2].

Lemma 3.29. *Let $\alpha \in E^\times$ such that $\mathrm{val}_p(\alpha) = \frac{1-(k_1+k_2)}{2}$, then $D(\alpha,\lambda,\psi)$ is étale, i.e. $D(\alpha,\lambda,\psi) \cong D_{\mathrm{rig}}(\rho)$ for a 2-dimensional continuous representation ρ of $\mathrm{Gal}_{\mathbb{Q}_p}$ over E.*

If $\alpha' \in E^\times$ is such that $D(\alpha',\lambda,\psi) \cong D_{\mathrm{rig}}(\rho')$ is also étale, then $\alpha^{-1}\alpha' \in \mathcal{O}_E^\times$ and $\rho' \cong \rho \otimes_E \mathrm{unr}(\alpha'\alpha^{-1})$, hence ρ as in Lemma 3.29 is unique up to twist by characters. Let ρ be as in Lemma 3.29 (for a choice of α) and denote by $\widehat{\pi}(\rho)$ the continuous Banach representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E attached to ρ via the local p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ ([24]). Then we have using Remark 2.10 together with [59]:

$$\widehat{\pi}(\rho)^{\mathrm{an}} \cong \pi(p^{-1}\alpha, \lambda^\flat, \psi) := \pi(\lambda^\flat, \psi) \otimes_E \mathrm{unr}(p^{-1}\alpha) \circ \det .$$

Proposition 3.30. *Assume ρ admits an invariant lattice such that its mod ϖ_E reduction $\overline{\rho}$ satisfies (A.2) (in the appendix), then Hypothesis 3.26(1) (hence Hypothesis 3.19 by Remark 3.27(1)) is true.*

Proof. Let $\alpha \in E^\times$ such that $D(\alpha,\lambda,\psi) \cong D_{\mathrm{rig}}(\rho)$. By Corollary A.2, Colmez's functor $\mathbf{V}_{\varepsilon^{-1}}$ (see § A.1) induces a surjection:

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widehat{\pi}(\rho), \widehat{\pi}(\rho)\big) \twoheadrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D(\alpha,\lambda,\psi), D(\alpha,\lambda,\psi)\big) \tag{3.57}$$

where the Ext^1 on the left is in the category of admissible *unitary* Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ (recall unitary means that there exists a unit ball preserved by $\mathrm{GL}_2(\mathbb{Q}_p)$). By the exactness of locally ($\mathbb{Q}_p$-)analytic vectors ([75, Thm. 7.1]), we have a morphism:

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widehat{\pi}(\rho), \widehat{\pi}(\rho)\big) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widehat{\pi}(\rho)^{\mathrm{an}}, \widehat{\pi}(\rho)^{\mathrm{an}}\big) \tag{3.58}$$

which we claim is injective. Indeed, assume there is a continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant section $\widehat{\pi}(\rho)^{\mathrm{an}} \hookrightarrow \widetilde{\pi}^{\mathrm{an}} \subset \widetilde{\pi}$ for $\widetilde{\pi} \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\widehat{\pi}(\rho), \widehat{\pi}(\rho))$. By

[26], the universal unitary completion of $\widehat{\pi}(\rho)^{\mathrm{an}} \cong \pi(p^{-1}\alpha, \lambda^\flat, \psi)$ is isomorphic to $\widehat{\pi}(\rho)$. By the universal property of this universal completion and the exactness in [75, Thm. 7.1], we easily deduce that the above continuous injection $\widehat{\pi}(\rho)^{\mathrm{an}} \hookrightarrow \widetilde{\pi}$ canonically factors through a continuous injection $\widehat{\pi}(\rho) \hookrightarrow \widetilde{\pi}$ which provides a section to $\widetilde{\pi} \twoheadrightarrow \widehat{\pi}(\rho)$. However, by Lemma 3.5 we have $\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(D(\alpha,\lambda,\psi), D(\alpha,\lambda,\psi)) = 5$, and by Lemma 3.17(3) (and twisting by $\mathrm{unr}(p^{-1}\alpha) \circ \det$) we have $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\widehat{\pi}(\rho)^{\mathrm{an}}, \widehat{\pi}(\rho)^{\mathrm{an}}) \leq 5$. Thus both (3.58) and (3.57) are bijective. The composition of (3.57) with the inverse of (3.58) gives an isomorphism:

$$\mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D(\alpha,\lambda,\psi), D(\alpha,\lambda,\psi)\big) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\widehat{\pi}(\rho)^{\mathrm{an}}, \widehat{\pi}(\rho)^{\mathrm{an}}\big). \tag{3.59}$$

Twisting by $\mathcal{R}_E(\mathrm{unr}(p\alpha^{-1}))$ on the left hand side and by $\mathrm{unr}(p\alpha^{-1}) \circ \det$ on the right hand side, we deduce an isomorphism:

(3.60)

$$\mathrm{pLL} : \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D(p,\lambda,\psi), D(p,\lambda,\psi)\big) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda^\flat,\psi), \pi(\lambda^\flat,\psi)).$$

The first part of Hypothesis 3.26(1) follows.

From the bijectivity of (3.58), we see any element in

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(p^{-1}\alpha,\lambda^\flat,\psi), \pi(p^{-1}\alpha,\lambda^\flat,\psi))$$

is isomorphic to the locally analytic vectors of an extension of $\widehat{\pi}(\rho)$ by $\widehat{\pi}(\rho)$ (in the category of admissible unitary Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$) and in particular is very strongly admissible. Twisting by $\mathrm{unr}(p\alpha^{-1}) \circ \det$, we deduce any element in $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\lambda^\flat,\psi), \pi(\lambda^\flat,\psi))$ is also very strongly admissible, which is the second part of Hypothesis 3.26(1). ☐

Remark 3.31. (1) Keeping the assumptions of Proposition 3.30, by the same argument together with a version with fixed central character of (A.3) (see (A.9)), we can show that (3.60) induces an isomorphism as in (3.54).

(2) Assume $\mathrm{End}_{\mathrm{Gal}_{\mathbb{Q}_p}}(\overline{\rho}) = k_E$, any element t in the left hand side set of (3.59) gives rise to an ideal $\mathcal{I}_t \subseteq R_{\overline{\rho}}$ with $R_{\overline{\rho}}/\mathcal{I}_t \cong \mathcal{O}_E[\epsilon]/\epsilon^2$ ($R_{\overline{\rho}}$ is the universal deformation ring of $\overline{\rho}$, see § 5.1). With the notation of § A.2, the map (3.59) then sends t to $((\pi^{\mathrm{univ}}(\overline{\rho}) \otimes_{R_{\overline{\rho}}} R_{\overline{\rho}}/\mathcal{I}_t) \otimes_{\mathcal{O}_E} E)^{\mathrm{an}}$.

The following proposition is presumably not new, but we couldn't find the precise statement in the existing literature. We provide a complete proof in § A.4.

Proposition 3.32. *Keep the assumptions of Proposition 3.30 and assume moreover* $\mathrm{End}_{\mathrm{Gal}_{\mathbb{Q}_p}}(\overline{\rho}) = k_E$*, and* $p \geq 5$ *if* $\overline{\rho}$ *is nongeneric (see just* *before Proposition A.4 for this terminology). Then Hypothesis 3.26 is true. Consequently, the statements in Lemma 3.28 also hold.*

Remark 3.33. Assume ψ smooth, let $\widetilde{\pi}\in \mathrm{Ext}^1_g(\pi(\lambda^\flat,\psi),\pi(\lambda^\flat,\psi))$ as in Remark 3.18 (with λ replaced by $\lambda^\flat$), and let $\widetilde{D}\in \mathrm{Ext}^1_g(D(p,\lambda,\psi),D(p,\lambda,\psi))$ the inverse image of $\widetilde{\pi}$ via the isomorphism (3.55). By Remark 3.18 (see in particular (3.37)), the existence of $\widetilde{D}$ confirms the discussion in [34, Rem. 1.6(a)].

3.3. $\mathcal{L}$-invariants for $\mathrm{GL}_3(\mathbb{Q}_p)$

We use the previous results for $\mathrm{GL}_2(\mathbb{Q}_p)$ and the results of § 2 to associate to a 3-dimensional semi-stable representation of $\mathrm{Gal}_{\mathbb{Q}_p}$ with $N^2\neq 0$ and distinct Hodge-Tate weights one of the finite length locally analytic representations of $\mathrm{GL}_3(\mathbb{Q}_p)$ constructed in [4].

3.3.1. Notation and preliminaries. We introduce some notation and define some locally analytic representations of $\mathrm{GL}_3(\mathbb{Q}_p)$ that will be used to describe $\mathcal{L}$-invariants for $\mathrm{GL}_3(\mathbb{Q}_p)$.

We now switch to $\mathrm{GL}_3(\mathbb{Q}_p)$ and we let $B(\mathbb{Q}_p)$ (resp. $\overline{B}(\mathbb{Q}_p)$) be the Borel subgroup of upper (resp. lower) triangular matrices, $T(\mathbb{Q}_p)$ the diagonal torus and $N(\mathbb{Q}_p)$ (resp. $\overline{N}(\mathbb{Q}_p)$) the unipotent radical of $B(\mathbb{Q}_p)$ (resp. $\overline{B}(\mathbb{Q}_p)$). We set:

$$P_1(\mathbb{Q}_p):=\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix},\quad P_2(\mathbb{Q}_p):=\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

For $i\in\{1,2\}$ we denote by $L_i(\mathbb{Q}_p)$ the Levi subgroup of $P_i(\mathbb{Q}_p)$ containing $T(\mathbb{Q}_p)$, $N_i(\mathbb{Q}_p)$ the unipotent radical of $P_i(\mathbb{Q}_p)$, $\overline{P}_i(\mathbb{Q}_p)$ the parabolic subgroup opposite to $P_i(\mathbb{Q}_p)$ and $\overline{N}_i(\mathbb{Q}_p)$ the unipotent radical of $\overline{P}_i$. Finally we let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{n}$, $\mathfrak{p}_i$, $\mathfrak{l}_i$, $\mathfrak{n}_i$, $\overline{\mathfrak{n}}_i$ the respective $\mathbb{Q}_p$-Lie algebras.

We fix $\lambda=(k_1,k_2,k_3)$ a dominant integral weight of $\mathfrak{t}$ with respect to the Borel subgroup B, i.e. $k_1\geq k_2\geq k_3$. We let $L(\lambda)$ (resp. $L_i(\lambda)$ for $i\in\{1,2\}$) be the algebraic representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ (resp. of $L_i(\mathbb{Q}_p)$) of highest weight λ and δ_λ be the algebraic character of $T(\mathbb{Q}_p)$ of weight λ. To lighten notation we set:

$$\begin{aligned} I_{\overline{B}}^{\mathrm{GL}_3}(\lambda) &:= \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\delta_\lambda\big)^{\mathrm{an}}\\ I_{\overline{P}_i}^{\mathrm{GL}_3}(\lambda) &:= \big(\mathrm{Ind}_{\overline{P}_i(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}L_i(\lambda)\big)^{\mathrm{an}}\end{aligned}$$

$$\begin{aligned} i_{\overline{B}}^{\mathrm{GL}_3}(\lambda) &:= \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} 1\big)^{\infty} \otimes_E L(\lambda) \\ i_{\overline{P}_i}^{\mathrm{GL}_3}(\lambda) &:= \big(\mathrm{Ind}_{\overline{P}_i(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} 1\big)^{\infty} \otimes_E L(\lambda). \end{aligned}$$

We also set:

$$\begin{aligned} \mathrm{St}_3^{\mathrm{an}}(\lambda) &:= I_{\overline{B}}^{\mathrm{GL}_3}(\lambda)/\sum_{i=1,2} I_{\overline{P}_i}^{\mathrm{GL}_3}(\lambda) \\ \mathrm{St}_3^{\infty}(\lambda) &:= i_{\overline{B}}^{\mathrm{GL}_3}(\lambda)/\sum_{i=1,2} i_{\overline{P}_i}^{\mathrm{GL}_3}(\lambda) \\ v_{\overline{P}_i}^{\mathrm{an}}(\lambda) &:= I_{\overline{P}_i}^{\mathrm{GL}_3}(\lambda)/L(\lambda) \\ v_{\overline{P}_i}^{\infty}(\lambda) &:= i_{\overline{P}_i}^{\mathrm{GL}_3}(\lambda)/L(\lambda). \end{aligned}$$

We have $\mathrm{St}_3^{\infty}(\lambda) \xrightarrow{\sim} \mathrm{St}_3^{\mathrm{an}}(\lambda)^{\mathrm{lalg}}$, $v_{\overline{P}_i}^{\infty}(\lambda) \xrightarrow{\sim} v_{\overline{P}_i}^{\mathrm{an}}(\lambda)^{\mathrm{lalg}}$ and long exact sequences (cf. [78, Prop. 5.4]):

$$\begin{aligned} (3.61)\ & 0 \to L(\lambda) \to I_{\overline{P}_1}^{\mathrm{GL}_3}(\lambda) \oplus I_{\overline{P}_2}^{\mathrm{GL}_3}(\lambda) \to I_{\overline{B}}^{\mathrm{GL}_3}(\lambda) \to \mathrm{St}_3^{\mathrm{an}}(\lambda) \to 0 \\ & 0 \to L(\lambda) \to i_{\overline{P}_1}^{\mathrm{GL}_3}(\lambda) \oplus i_{\overline{P}_2}^{\mathrm{GL}_3}(\lambda) \to i_{\overline{B}}^{\mathrm{GL}_3}(\lambda) \to \mathrm{St}_3^{\infty}(\lambda) \to 0. \end{aligned}$$

For an integral weight μ, we denote by $\overline{L}(\mu)$ the unique simple quotient of the Verma module $\overline{M}(\mu) := \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\overline{\mathfrak{b}})} \mu$. Note that $\overline{L}(-\lambda)$ is isomorphic to the dual $L(\lambda)'$ of $L(\lambda)$. We use without comment the theory of [63], see e.g. [8, § 2] for a summary. We often write GL_3, $\overline{P}_i$, Z (= the center of GL_3) instead of $\mathrm{GL}_3(\mathbb{Q}_p)$, $\overline{P}_i(\mathbb{Q}_p)$, $Z(\mathbb{Q}_p)$ etc.

We now give several useful short exact sequences of admissible locally analytic representations of $\mathrm{GL}_3(\mathbb{Q}_p)$ over E. For $i = 1, 2$, we have a *nonsplit* exact sequence:

$$(3.62) \qquad 0 \longrightarrow v_{\overline{P}_i}^{\infty}(\lambda) \longrightarrow v_{\overline{P}_i}^{\mathrm{an}}(\lambda) \longrightarrow \mathcal{F}_{\overline{P}_i}^{\mathrm{GL}_3}(\overline{L}(-s_j \cdot \lambda), 1) \longrightarrow 0$$

where $j \neq i$ and s_i denotes the simple reflection corresponding to the simple root of $L_i(\mathbb{Q}_p)$. Indeed, by [4, Lem. 5.3.1], the theory of [63] and [8, Cor. 2.5], we have a nonsplit exact sequence:

$$0 \longrightarrow i_{\overline{P}_i}^{\mathrm{GL}_3}(\lambda) \longrightarrow I_{\overline{P}_i}^{\mathrm{GL}_3}(\lambda) \longrightarrow \mathcal{F}_{\overline{P}_i}^{\mathrm{GL}_3}(\overline{L}(-s_j \cdot \lambda), 1) \longrightarrow 0,$$

which together with the fact $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(\mathcal{F}_{\overline{P}_i}^{\mathrm{GL}_3}(\overline{L}(-s_j \cdot \lambda), 1), L(\lambda)) = 0$ (cf. [78, Cor. 4.3]) implies that (3.62) is nonsplit by a straightforward dévissage.

We let $\lambda_{1,2} := (k_1, k_2)$ (which is thus a dominant weight for $\mathrm{GL}_2(\mathbb{Q}_p)$ as in § 3.2.2), it is easy to see that we have a commutative diagram (where we write GL_3 for $\mathrm{GL}_3(\mathbb{Q}_p)$ etc.):

$$\begin{tikzcd}
\big(\mathrm{Ind}_{\overline{P}_1}^{\mathrm{GL}_3}\big((\mathrm{Ind}_{\overline{B}\cap L_1}^{L_1} 1)^\infty \otimes_E L_1(\lambda)\big)\big)^{\mathrm{an}} \arrow[r] \arrow[d] & \big(\mathrm{Ind}_{\overline{P}_1}^{\mathrm{GL}_3} \mathrm{St}_2^\infty(\lambda_{1,2}) \otimes x^{k_3}\big)^{\mathrm{an}} \arrow[d] \\
\big(\mathrm{Ind}_{\overline{P}_1}^{\mathrm{GL}_3}\big((\mathrm{Ind}_{\overline{B}\cap L_1}^{L_1} \delta_\lambda)^{\mathrm{an}}\big)\big)^{\mathrm{an}} \arrow[r] & \big(\mathrm{Ind}_{\overline{P}_1}^{\mathrm{GL}_3} \mathrm{St}_2^{\mathrm{an}}(\lambda_{1,2}) \otimes x^{k_3}\big)^{\mathrm{an}}
\end{tikzcd}$$

where all the vertical maps are injective and all the horizontal maps are surjective. Using the exactness and transitivity of parabolic induction, the bottom surjection induces an isomorphism

$$I_{\overline{B}}^{\mathrm{GL}_3}(\lambda)/I_{\overline{P}_1}^{\mathrm{GL}_3}(\lambda) \xrightarrow{\sim} (\mathrm{Ind}_{\overline{P}_1}^{\mathrm{GL}_3} \mathrm{St}_2^{\mathrm{an}}(\lambda_{1,2}) \otimes x^{k_3})^{\mathrm{an}}.$$

Together with (3.61), we deduce an exact sequence:

$$0 \longrightarrow v_{\overline{P}_2}^{\mathrm{an}}(\lambda) \longrightarrow \big(\mathrm{Ind}_{\overline{P}_1}^{\mathrm{GL}_3} \mathrm{St}_2^{\mathrm{an}}(\lambda_{1,2}) \otimes x^{k_3}\big)^{\mathrm{an}} \longrightarrow \mathrm{St}_3^{\mathrm{an}}(\lambda) \longrightarrow 0. \tag{3.63}$$

By the theory of [63] and [8, Cor. 2.5], we have a nonsplit exact sequence:

$$\begin{aligned} 0 \longrightarrow \big(\mathrm{Ind}_{\overline{P}_1}^{\mathrm{GL}_3} \mathrm{St}_2^\infty \otimes 1\big)^\infty \otimes_E L(\lambda) \longrightarrow \big(\mathrm{Ind}_{\overline{P}_1}^{\mathrm{GL}_3} \mathrm{St}_2^\infty(\lambda_{1,2}) \otimes x^{k_3}\big)^{\mathrm{an}} \\ \longrightarrow \mathcal{F}_{\overline{P}_1}^{\mathrm{GL}_3}(\overline{L}(-s_2 \cdot \lambda), \mathrm{St}_2^\infty \otimes 1) \longrightarrow 0. \end{aligned} \tag{3.64}$$

We also have another exact sequence (see e.g. [4, (53)]):

$$0 \longrightarrow v_{\overline{P}_2}^{\infty}(\lambda) \longrightarrow \big(\mathrm{Ind}_{\overline{P}_1}^{\mathrm{GL}_3} \mathrm{St}_2^\infty \otimes 1\big)^\infty \otimes_E L(\lambda) \longrightarrow \mathrm{St}_3^\infty(\lambda) \longrightarrow 0. \tag{3.65}$$

From (3.63), (3.64), (3.62) and (3.65) and by comparing irreducible constituents, we easily deduce that in $(\mathrm{Ind}_{\overline{P}_1}^{\mathrm{GL}_3} \mathrm{St}_2^{\mathrm{an}}(\lambda_{1,2}) \otimes x^{k_3})^{\mathrm{an}}$ we have:

$$\big(\mathrm{Ind}_{\overline{P}_1}^{\mathrm{GL}_3} \mathrm{St}_2^\infty(\lambda_{1,2}) \otimes x^{k_3}\big)^{\mathrm{an}} \cap v_{\overline{P}_2}^{\mathrm{an}}(\lambda) = v_{\overline{P}_2}^{\infty}(\lambda). \tag{3.66}$$

Let $C_{2,1} := \mathcal{F}_{\overline{P}_1}^{\mathrm{GL}_3}(\overline{L}(-s_2 \cdot \lambda), \mathrm{St}_2^\infty \otimes 1)$ and:

$$S_{1,0} := \big(\mathrm{Ind}_{\overline{P}_1}^{\mathrm{GL}_3} \mathrm{St}_2^\infty(\lambda_{1,2}) \otimes x^{k_3}\big)^{\mathrm{an}} / v_{\overline{P}_2}^{\infty}(\lambda)$$

which is a subrepresentation of $\mathrm{St}_3^{\mathrm{an}}(\lambda)$ by (3.63) and (3.66) and sits in an exact sequence by (3.64) and (3.65):

$$0 \longrightarrow \mathrm{St}_3^\infty(\lambda) \longrightarrow S_{1,0} \longrightarrow C_{2,1} \longrightarrow 0. \tag{3.67}$$

We claim the latter is nonsplit. Indeed, as in the proof of [4, Prop. 4.6.1], we have $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(C_{2,1}, v^{\infty}_{\overline{P}_2}(\lambda)) = 0$. Together with the fact that (3.64) is nonsplit, the claim follows by a straightforward dévissage. By replacing P_1 by P_2 and s_2 by s_1, we define in the same way $C_{1,1}$ as $C_{2,1}$ and $S_{2,0}$ as $S_{1,0}$, and we have similar results for $C_{1,1}$ and $S_{2,0}$. In particular, we have $\mathrm{soc}_{\mathrm{GL}_3(\mathbb{Q}_p)} S_{i,0} \cong \mathrm{St}^{\infty}_3(\lambda)$.

In the sequel, we define several locally analytic representations $C_{i,j}$ and $S_{i,j}$ of $\mathrm{GL}_3(\mathbb{Q}_p)$ for $i \in \{1, 2\}$ and $j \in \{0, 1, 2, 3\}$, these representations being such that $C_{i,0} = \mathrm{St}^{\infty}_3(\lambda)$ for $i \in \{1, 2\}$ and $C_{i,j} \hookrightarrow \mathrm{soc}_{\mathrm{GL}_3(\mathbb{Q}_p)} S_{i,j}$ for all i, j.

3.3.2. Simple $\mathcal{L}$-invariants.

We recall some facts on simple $\mathcal{L}$-invariants. We keep all the previous notation.

Lemma 3.34. *Let $i, j \in \{1, 2\}$, $i \neq j$.*

(1) We have $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_i}(\lambda), \mathrm{St}^{\mathrm{an}}_3(\lambda)/S_{j,0}) = 0$ and an isomorphism:

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_i}(\lambda), S_{j,0}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_i}(\lambda), \mathrm{St}^{\mathrm{an}}_3(\lambda)).$$

(2) We have $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(L(\lambda), \mathrm{St}^{\mathrm{an}}_3(\lambda)) = 0$ and an isomorphism:

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_i}(\lambda), \mathrm{St}^{\mathrm{an}}_3(\lambda)) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(i^{\mathrm{GL}_3}_{\overline{P}_i}(\lambda), \mathrm{St}^{\mathrm{an}}_3(\lambda)).$$

Proof. In each case, the isomorphism follows from the first equality by an obvious dévissage.

(1) It is enough to prove $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_i}(\lambda), C) = 0$ for all the irreducible constituents C of $\mathrm{St}^{\mathrm{an}}_3(\lambda)/S_{j,0}$. By the theory [63], we know that C is of the form $\mathcal{F}^{\mathrm{GL}_3}_{\overline{P}_w}(\overline{L}(-w \cdot \lambda), \pi^{\infty})$ were w is a nontrivial element of the Weyl group distinct from s_i (since we mod out by $S_{j,0}$), $P_w \subset \mathrm{GL}_3$ is the maximal parabolic subgroup containing B such that $w \cdot \lambda$ is dominant for L_{P_w} (with respect to $B \cap L_{P_w}$) and π^{∞} is a smooth irreducible representation of $L_{P_w}(\mathbb{Q}_p)$ over E. If $w \neq s_j$, i.e. w has length > 1, by [31, Lem. 2.6 (2)], we have $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_i}(\lambda), C) = 0$. If $w = s_j$, then we have $\pi^{\infty} = \mathrm{St}^{\infty}_2 \otimes 1$ if $i = 1$ or $\pi^{\infty} = 1 \otimes \mathrm{St}^{\infty}_2$ if $i = 2$, and $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_i}(\lambda), C) = 0$ (via Lemma 3.1) is then one of the cases of [78, (4.45)] (or its symmetric).

(2) The first equality follows directly from [78, Prop. 5.6] and Lemma 3.1, and the isomorphism follows by an obvious dévissage. □

Let $\Psi = (\psi_1, \psi_2, \psi_3) \in \mathrm{Hom}(T(\mathbb{Q}_p), E)$ (with obvious notation) and consider the exact sequence:

$$0 \longrightarrow I^{\mathrm{GL}_3}_{\overline{B}}(\lambda) \longrightarrow \left(\mathrm{Ind}^{\mathrm{GL}_3(\mathbb{Q}_p)}_{\overline{B}(\mathbb{Q}_p)} \delta_{\lambda}(1 + \Psi\epsilon)\right)^{\mathrm{an}} \xrightarrow{\mathrm{pr}} I^{\mathrm{GL}_3}_{\overline{B}}(\lambda) \longrightarrow 0. \tag{3.68}$$

For $i = 1, 2$, we see that $\mathrm{pr}^{-1}(i_{\overline{P}_i}^{\mathrm{GL}_3}(\lambda))/\sum_{j=1,2} I_{\overline{P}_j}^{\mathrm{GL}_3}(\lambda)$ is by construction an extension of $i_{\overline{P}_i}^{\mathrm{GL}_3}(\lambda)$ by $\mathrm{St}_3^{\mathrm{an}}(\lambda)$. By Lemma 3.34, it comes from a unique extension $\Pi^i(\lambda, \Psi)_0$ of $v_{\overline{P}_i}^{\infty}(\lambda)$ by $S_{i,0}$. If Ψ is smooth (i.e. all ψ_j are smooth, $j \in \{1, 2, 3\}$), by considering the following exact sequence (which is then "contained" in (3.68)):

$$0 \longrightarrow i_{\overline{B}}^{\mathrm{GL}_3}(\lambda) \longrightarrow \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}(1 + \Psi\epsilon)\big)^{\infty} \otimes_E L(\lambda) \xrightarrow{\mathrm{pr}'} i_{\overline{B}}^{\mathrm{GL}_3}(\lambda) \longrightarrow 0,$$

we see that $\Pi^i(\lambda, \Psi)_0$ then comes via the embedding $\mathrm{St}_3^{\infty}(\lambda) \hookrightarrow S_{i,0}$ from a (unique) locally algebraic extension of $v_{\overline{P}_i}^{\infty}(\lambda)$ by $\mathrm{St}_3^{\infty}(\lambda)$.

Proposition 3.35. *For $i \in \{1, 2\}$ the extension*

$$\Pi^i(\lambda, \Psi)_0 \in \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_i}^{\infty}(\lambda), S_{i,0}\big)$$

is split if and only if $\psi_i = \psi_{i+1}$, i.e. $\Psi \in \mathrm{Hom}(Z_{L_i}(\mathbb{Q}_p), E)$ where $Z_{L_i}(\mathbb{Q}_p)$ is the center of $L_i(\mathbb{Q}_p)$. Moreover, we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\infty}(\mathbb{Q}_p^{\times}, E) \arrow[r, "\sim"] \arrow[d] & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v_{\overline{P}_i}^{\infty}(\lambda), \mathrm{St}_3^{\infty}(\lambda)) \arrow[d] \\
\mathrm{Hom}(\mathbb{Q}_p^{\times}, E) \arrow[r, "\sim"] & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v_{\overline{P}_i}^{\infty}(\lambda), S_{j,0})
\end{tikzcd} \tag{3.69}$$

where the vertical maps are the natural injections, the bottom horizontal map is given by the composition of

$$\mathrm{Hom}(\mathbb{Q}_p^{\times}, E) \xrightarrow{\sim} \mathrm{Hom}(T(\mathbb{Q}_p), E)/\,\mathrm{Hom}(Z_{L_i}(\mathbb{Q}_p), E)$$

with $\Psi \mapsto \Pi^i(\lambda, \Psi)_0$, and the top horizontal map is induced by the bottom map.

Proof. See [31, Thm. 2.17 & Rem. 2.18(ii)]. ☐

We now let $\delta_1 := x^{k_1}$, $\delta_2 := |\cdot|^{-1}x^{k_2-1}$, $\delta_3 := |\cdot|^{-2}x^{k_3-2}$ and identify $\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_i), \mathcal{R}_E(\delta_i))$ with $\mathrm{Hom}(\mathbb{Q}_p^{\times}, E)$ by (1.11).

Corollary 3.36. *For $i, j \in \{1, 2\}$, $i \neq j$, we have a natural perfect pairing:*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v_{\overline{P}_i}^{\infty}(\lambda), S_{j,0}) \times \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_{i+1}), \mathcal{R}_E(\delta_i)) \xrightarrow{\cup_1} E,$$

and the same holds with $S_{j,0}$ *replaced by* $\mathrm{St}_3^{\mathrm{an}}(\lambda)$ *(for* $i \in \{1,2\}$*). Moreover, the one dimensional subspace* $\mathrm{Ext}^1_e(\mathcal{R}_E(\delta_{i+1}), \mathcal{R}_E(\delta_i))$ *of crystalline extensions is exactly annihilated by the subspace* $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_i}(\lambda), \mathrm{St}_3^{\infty}(\lambda))$*.*

Proof. By Proposition 3.35 (together with the above identification) and Proposition 2.3(2) (applied to $\mathcal{R}_E(\delta_n) = \mathcal{R}_E(\delta_{i+1})$, $D_1^{n-1} = \mathcal{R}_E(\delta_i)$ for $i = 1, 2$), we obtain the perfect pairing of the statement. By Lemma 3.34(1), we have a similar perfect pairing with $S_{j,0}$ replaced by $\mathrm{St}_3^{\mathrm{an}}(\lambda)$. The last part follows then from (3.69) and the discussion in Remark 2.6. ☐

3.3.3. Parabolic inductions.

We study the locally analytic representation $(\mathrm{Ind}^{\mathrm{GL}_3(\mathbb{Q}_p)}_{\overline{P}_1(\mathbb{Q}_p)} \pi(\lambda_{1,2}, \psi) \otimes x^{k_3})^{\mathrm{an}}$ (cf. § 3.2.2) and some of its subquotients, that we will use to describe $\mathcal{L}$-invariants for $\mathrm{GL}_3(\mathbb{Q}_p)$.

We keep the previous notation and fix $0 \neq \psi \in \mathrm{Hom}(\mathbb{Q}_p^{\times}, E)$. For a locally analytic representation V of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E we use the notation:

$$I^{\mathrm{GL}_3}_{\overline{P}_1}(V, k_3) := \big(\mathrm{Ind}^{\mathrm{GL}_3(\mathbb{Q}_p)}_{\overline{P}_1(\mathbb{Q}_p)} V \otimes x^{k_3}\big)^{\mathrm{an}}.$$

We have studied the subrepresentation $I^{\mathrm{GL}_3}_{\overline{P}_1}(\mathrm{St}_2^{\infty}(\lambda_{1,2}), k_3)$ in § 3.3.1. Exactness of parabolic induction gives the isomorphism (recalling that s is the unique nontrivial element in the Weyl group of GL_2):

$$I^{\mathrm{GL}_3}_{\overline{P}_1}(I(s\cdot\lambda_{1,2}), k_3) \cong I^{\mathrm{GL}_3}_{\overline{P}_1}(\mathrm{St}_2^{\mathrm{an}}(\lambda_{1,2}), k_3)/I^{\mathrm{GL}_3}_{\overline{P}_1}(\mathrm{St}_2^{\infty}(\lambda_{1,2}), k_3).$$

From (3.66) and (3.62) (for $i = 2$) we deduce an injection $\mathcal{F}^{\mathrm{GL}_3}_{\overline{P}_2}(\overline{L}(-s_1 \cdot \lambda), 1) \hookrightarrow I^{\mathrm{GL}_3}_{\overline{P}_1}(I(s\cdot\lambda_{1,2}), k_3)$ and together with (3.63) an isomorphism:

$$S_{1,1} := I^{\mathrm{GL}_3}_{\overline{P}_1}(I(s\cdot\lambda_{1,2}), k_3)/\mathcal{F}^{\mathrm{GL}_3}_{\overline{P}_2}(\overline{L}(-s_1\cdot\lambda), 1) \xrightarrow{\sim} \mathrm{St}_3^{\mathrm{an}}(\lambda)/S_{1,0}.$$

Since $C_{1,1} = \mathcal{F}^{\mathrm{GL}_3}_{\overline{P}_2}(\overline{L}(-s_1\cdot\lambda), \mathrm{St}_2^{\infty} \otimes 1) \hookrightarrow \mathrm{St}_3^{\mathrm{an}}(\lambda)/\mathrm{St}_3^{\infty}(\lambda)$ and $C_{1,1}$ is not an irreducible constituent of $S_{1,0}$ by (3.67), we have a commutative diagram:

$$\begin{tikzcd} S_{2,0} \arrow[r] \arrow[d] & \mathrm{St}_3^{\mathrm{an}}(\lambda) \arrow[d] \\ C_{1,1} \arrow[r] & S_{1,1} \end{tikzcd}$$

where the vertical maps are the natural surjections and the horizontal maps are injections. From the theory of [63], one moreover easily deduces that the irreducible constituents of $S_{1,1}/C_{1,1}$ are:

(3.70)
$$\Big\{\mathcal{F}_{\overline{P}_2}^{\mathrm{GL}_3}(\overline{L}(-s_1s_2\cdot\lambda),1),\ \mathcal{F}_{\overline{P}_2}^{\mathrm{GL}_3}(\overline{L}(-s_1s_2\cdot\lambda),1\otimes\mathrm{St}_2^{\infty}),\ \mathcal{F}_{\overline{P}_1}^{\mathrm{GL}_3}(\overline{L}(-s_2s_1\cdot\lambda),1),$$
$$\mathcal{F}_{\overline{P}_1}^{\mathrm{GL}_3}(\overline{L}(-s_2s_1\cdot\lambda),\mathrm{St}_2^{\infty}\otimes 1),\ \mathcal{F}_{\overline{B}}^{\mathrm{GL}_3}(\overline{L}(-s_1s_2s_1\cdot\lambda),1)\Big\},$$

all of them occurring with multiplicity one. Since $\pi(\lambda_{1,2},\psi)^- \cong \mathrm{St}_2^{\mathrm{an}}(\lambda_{1,2}) - L(\lambda_{1,2})$ (see (3.22)), we have an exact sequence:

(3.71)
$$0\rightarrow I_{\overline{P}_1}^{\mathrm{GL}_3}(\mathrm{St}_2^{\mathrm{an}}(\lambda_{1,2}),k_3)\rightarrow I_{\overline{P}_1}^{\mathrm{GL}_3}(\pi(\lambda_{1,2},\psi)^-,k_3)\xrightarrow{\mathrm{pr}} I_{\overline{P}_1}^{\mathrm{GL}_3}(L(\lambda_{1,2}),k_3)\rightarrow 0$$

where $I_{\overline{P}_1}^{\mathrm{GL}_3}(L(\lambda_{1,2}),k_3)\cong I_{\overline{P}_1}^{\mathrm{GL}_3}(\lambda)$. Denote by:

$$S_{1,2}:=v_{\overline{P}_1}^{\mathrm{an}}(\lambda),\quad C_{1,2}:=v_{\overline{P}_1}^{\infty}(\lambda)\cong \mathrm{soc}_{\mathrm{GL}_3(\mathbb{Q}_p)}\, S_{1,2}.$$

Since $I_{\overline{P}_1}^{\mathrm{GL}_3}(\mathrm{St}_2^{\mathrm{an}}(\lambda_{1,2}),k_3)/v_{\overline{P}_2}^{\mathrm{an}}(\lambda)\cong \mathrm{St}_3^{\mathrm{an}}(\lambda)$ (see (3.63)) and $L(\lambda)\hookrightarrow I_{\overline{P}_1}^{\mathrm{GL}_3}(\lambda)$, it follows from (3.71) together with Lemma 3.34(2) that we have an injection:

$$L(\lambda)\longhookrightarrow I_{\overline{P}_1}^{\mathrm{GL}_3}(\pi(\lambda_{1,2},\psi)^-,k_3)/v_{\overline{P}_2}^{\mathrm{an}}(\lambda).$$

We let $\widetilde{\Pi}^1(\lambda,\psi)^-$ be the cokernel, which is thus isomorphic to an extension of $I_{\overline{P}_1}^{\mathrm{GL}_3}(\lambda)/L(\lambda)\cong v_{\overline{P}_1}^{\mathrm{an}}(\lambda)$ by $\mathrm{St}_3^{\mathrm{an}}(\lambda)$. Finally we denote by T_2, resp. $\overline{B}_2$, the diagonal torus, resp. the lower triangular matrices, of GL_2.

Lemma 3.37. *We have a commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & S_{2,0} \arrow[r] \arrow[d] & \Pi^1(\lambda,\psi)_0 \arrow[r] \arrow[d] & v_{\overline{P}_1}^{\infty}(\lambda) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{St}_3^{\mathrm{an}}(\lambda) \arrow[r] & \widetilde{\Pi}^1(\lambda,\psi)^- \arrow[r] & v_{\overline{P}_1}^{\mathrm{an}}(\lambda) \arrow[r] & 0
\end{tikzcd}$$

where $\Pi^1(\lambda,\psi)_0$ denotes the image of ψ via the bottom isomorphism of (3.69).

Proof. Let $\Psi_1:=(\psi_1,\psi_2)\in \mathrm{Hom}(T_2(\mathbb{Q}_p),E)$ and

$$\Psi:=(\psi_1,\psi_2,0)\in \mathrm{Hom}(T(\mathbb{Q}_p),E)$$

with $0\neq\psi_1-\psi_2\in E\psi$. We have (by the transitivity of parabolic inductions):

$$\begin{aligned} I_{\overline{P}_1}^{\mathrm{GL}_3}(\pi(\lambda_{1,2},\psi)^-, k_3) \longhookrightarrow\ & I_{\overline{P}_1}^{\mathrm{GL}_3}\big((\mathrm{Ind}_{\overline{B}_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_{\lambda_{1,2}}(1+\Psi_1\epsilon))^{\mathrm{an}}/L(\lambda_{1,2}), k_3\big) \\ & \cong \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \delta_\lambda(1+\Psi\epsilon)\big)^{\mathrm{an}}/I_{\overline{P}_1}^{\mathrm{GL}_3}(\lambda) \end{aligned}$$

which induces an injection by (3.61) together with Lemma 3.34(2):

$$\widetilde{\Pi}^1(\lambda,\psi)^- \longhookrightarrow W := \Big(\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \delta_\lambda(1+\Psi\epsilon)\big)^{\mathrm{an}}\Big/ \sum_{i=1,2} I_{\overline{P}_i}^{\mathrm{GL}_3}(\lambda)\Big)\Big/L(\lambda).$$

By Proposition 3.35 and the discussion above it, W contains $\Pi^1(\lambda,\psi)_0$ as subrepresentation, and it is easy to see that the injection $\Pi^1(\lambda,\psi)_0 \hookrightarrow W$ factors through $\widetilde{\Pi}^1(\lambda,\psi)^-$ (e.g. by comparing the irreducible constituents). The lemma follows. $\square$

We set $\widetilde{C}_{1,2} := \mathcal{F}_{\overline{P}_1}^{\mathrm{GL}_3}(\overline{L}(-s_2s_1\cdot\lambda), 1)$. By [4, Prop. 4.2.1 (ii)] and the proof of [4, Lem. 4.4.1], we know that there exits a unique (up to isomorphism) non-split extension $C_{1,1} - \widetilde{C}_{1,2}$, and it is a subrepresentation of $S_{1,1}$. Using the formula in [4, § 5.2] and [78, (4.37)], it is not difficult to show:

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\widetilde{C}_{1,2}, \mathcal{F}_{\overline{P}_1}^{\mathrm{GL}_3}(\overline{M}(-s_2\cdot\lambda), \mathrm{St}_2^\infty \otimes 1)\big) = 0,$$

and hence (by dévissage) $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\widetilde{C}_{1,2}, C_{2,1}\big) = 0$. We deduce that $\mathrm{St}_3^{\mathrm{an}}(\lambda)$ (which is of the form $S_{1,0} - S_{1,1}$) has a unique subrepresentation of the form $\mathrm{St}_3^\infty(\lambda) - C_{1,1} - \widetilde{C}_{1,2}$, containing $S_{2,0}$. Denote by $\Pi^1(\lambda,\psi)^-$ the push-forward of $\Pi^1(\lambda,\psi)_0$ via $S_{2,0} \hookrightarrow \mathrm{St}_3^\infty(\lambda) - C_{1,1} - \widetilde{C}_{1,2}$, which, by Lemma 3.37, is a subrepresentation of $\widetilde{\Pi}^1(\lambda,\psi)^-$.

Remark 3.38. If ψ is not smooth then $\Pi^1(\lambda,\psi)^-$ has the form:

$$\begin{tikzcd} & & \widetilde{C}_{1,2} \\ \mathrm{St}_3^\infty(\lambda) \arrow[r, dash] & C_{1,1} \arrow[ur, dash] \arrow[dr, dash] & \\ & & C_{1,2} \end{tikzcd}$$

whereas if ψ is smooth it has the form:

$$\begin{tikzcd} & C_{1,1} \arrow[r, dash] & \widetilde{C}_{1,2} \\ \mathrm{St}_3^\infty(\lambda) \arrow[ur, dash] \arrow[r, dash] & C_{1,2}. & \end{tikzcd}$$

In all cases $\widetilde{\Pi}^1(\lambda,\psi)^-$ has the form $S_{1,0} - S_{1,1} - S_{1,2} \cong \mathrm{St}_3^{\mathrm{an}}(\lambda) - S_{1,2}$.

We now set:

$$\begin{aligned}S_{1,3}:=I_{\overline{P}_1}^{\mathrm{GL}_3}(\widetilde{I}(s\cdot\lambda_{1,2}),k_3)\cong\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\delta_{s_1\cdot\lambda}(|\cdot|^{-1}\otimes|\cdot|\otimes 1)\big)^{\mathrm{an}}\\ \cong\mathcal{F}_{\overline{B}}^{\mathrm{GL}_3}\big(\overline{M}(-s_1\cdot\lambda),|\cdot|^{-1}\otimes|\cdot|\otimes 1\big)\end{aligned}$$

$$\begin{aligned}C_{1,3}&:=\mathcal{F}_{\overline{B}}^{\mathrm{GL}_3}\big(\overline{L}(-s_1\cdot\lambda),|\cdot|^{-1}\otimes|\cdot|\otimes 1\big)\\ &\cong\mathcal{F}_{\overline{P}_2}^{\mathrm{GL}_3}\big(\overline{L}(-s_1\cdot\lambda),|\cdot|^{-1}\otimes(\mathrm{Ind}_{\overline{B}_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}|\cdot|\otimes 1)^{\infty}\big)\cong\mathrm{soc}_{\mathrm{GL}_3(\mathbb{Q}_p)}S_{1,3},\end{aligned}$$

where the last isomorphism follows from [8, Cor. 2.5]. The irreducible constituents of $S_{1,3}/C_{1,3}$ are (from [63]):

(3.72)
$$\begin{aligned}\Big\{\mathcal{F}_{\overline{P}_2}^{\mathrm{GL}_3}\big(\overline{L}(-s_1s_2\cdot\lambda),|\cdot|^{-1}\otimes(\mathrm{Ind}_{\overline{B}_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}|\cdot|\otimes 1)^{\infty}\big),\ \mathcal{F}_{\overline{P}_1}^{\mathrm{GL}_3}\big(\overline{L}(-s_2s_1\cdot\lambda),\mathrm{St}_2^{\infty}\otimes 1\big),\\ \mathcal{F}_{\overline{P}_1}^{\mathrm{GL}_3}\big(\overline{L}(-s_2s_1\cdot\lambda),1\big),\ \mathcal{F}_{\overline{B}}^{\mathrm{GL}_3}\big(\overline{L}(-s_2s_1s_2\cdot\lambda),|\cdot|^{-1}\otimes|\cdot|\otimes 1\big)\Big\},\end{aligned}$$

all of them occurring with multiplicity one.

Lemma 3.39. *The natural map:*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(C_{1,3},\Pi^1(\lambda,\psi)^-)\longrightarrow\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(C_{1,3},\widetilde{\Pi}^1(\lambda,\psi)^-)\tag{3.73}$$

is an isomorphism of 1*-dimensional vector spaces.*

Proof. (a) By [4, Prop. 4.6.1], we have:

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(C_{1,3},\Pi^1(\lambda,\psi)^-)\xrightarrow{\sim}\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(C_{1,3},\Pi^1(\lambda,\psi)^-/\,\mathrm{St}_3^{\infty}(\lambda)).$$

By [4, Prop. 4.4.2 & Prop. 4.2.1(i)] (resp. by [4, Lem. 4.4.1 & Prop. 4.2.1(i)]), we deduce:

$$\dim_E\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(C_{1,3},\Pi^1(\lambda,\psi)^-/\,\mathrm{St}_3^{\infty}(\lambda))=1$$

in the case where ψ is not smooth (resp. in the case where ψ is smooth).

(b) Since $\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}(C_{1,3},\widetilde{\Pi}^1(\lambda,\psi)^-/\Pi^1(\lambda,\psi)^-)=0$, we see (3.73) is injective, and it is sufficient to prove $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(C_{1,3},C)=0$ for any irreducible constituent of $\widetilde{\Pi}^1(\lambda,\psi)^-/\Pi^1(\lambda,\psi)^-$. By Step 3 of [4, Prop. 4.4.2], it is left to show $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(C_{1,3}, C_{2,1}) = 0$. However, using [4, Cor. 5.3.2(ii) & Lem. 5.3.3] and ([78, (4.37)]), one can show:

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(C_{1,3}, \mathcal{F}^{\mathrm{GL}_3}_{\overline{P}_1}(\overline{M}(-s_2 \cdot \lambda), \mathrm{St}_2^\infty \otimes 1)) = 0$$

and hence $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(C_{1,3}, C_{2,1}) = 0$. The lemma follows. $\square$

Now consider the exact sequence (see (3.25)):

$$0 \longrightarrow I^{\mathrm{GL}_3}_{\overline{P}_1}(\pi(\lambda_{1,2}, \psi)^-, x^{k_3}) \longrightarrow I^{\mathrm{GL}_3}_{\overline{P}_1}(\pi(\lambda_{1,2}, \psi), x^{k_3}) \xrightarrow{\mathrm{pr}} S_{1,3} \longrightarrow 0. \tag{3.74}$$

The push-forward of $\mathrm{pr}^{-1}(C_{1,3})$ via $I^{\mathrm{GL}_3}_{\overline{P}_1}(\pi(\lambda_{1,2}, \psi)^-, x^{k_3}) \twoheadrightarrow \widetilde{\Pi}^1(\lambda, \psi)^-$ gives an extension of $C_{1,3}$ by $\widetilde{\Pi}^1(\lambda, \psi)^-$, which by Lemma 3.39 comes from an extension of $C_{1,3}$ by $\Pi^1(\lambda, \psi)^-$ denoted by $\Pi^1(\lambda, \psi)$.

Lemma 3.40. *The extension* $\Pi^1(\lambda, \psi) \in \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(C_{1,3}, \Pi^1(\lambda, \psi)^-)$ *is nonsplit.*

Proof. The lemma follows from Step 2 of the proof of [4, Prop. 4.4.2]. $\square$

Remark 3.41. (1) If ψ is not smooth then $\Pi^1(\lambda, \psi)$ has the form:

$$\begin{tikzcd}
 & & \widetilde{C}_{1,2} \arrow[dr, dash, dashed] & \\
\mathrm{St}_3^\infty(\lambda) \arrow[r, dash] & C_{1,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{1,3}, \\
 & & C_{1,2} \arrow[ur, dash] &
\end{tikzcd} \tag{3.75}$$

whereas if ψ is smooth it has the form:

$$\begin{tikzcd}
 & C_{1,1} \arrow[r, dash] & \widetilde{C}_{1,2} \\
\mathrm{St}_3^\infty(\lambda) \arrow[ur, dash] \arrow[r, dash] & C_{1,2} \arrow[r, dash] & C_{1,3}.
\end{tikzcd}$$

(2) One can actually show that the subquotient $\widetilde{C}_{1,2} - C_{1,3}$ in (3.75) is also non-split (see [4, Rk. 4.4.3(ii)]). But we don't need this fact in the paper.

Denote by $\widetilde{\Pi}^1(\lambda, \psi)$ the push-forward of (3.74) along

$$I^{\mathrm{GL}_3}_{\overline{P}_1}(\pi(\lambda_{1,2}, \psi)^-, k_3) \longrightarrow\!\!\!\!\!\rightarrow \widetilde{\Pi}^1(\lambda, \psi)^-,$$

which thus has the following form by Lemma 3.37:

$$\widetilde{\Pi}^1(\lambda,\psi)\cong S_{1,0}-S_{1,1}-S_{1,2}-S_{1,3}\cong \mathrm{St}_3^{\mathrm{an}}(\lambda)-S_{1,2}-S_{1,3} \tag{3.76}$$

and contains $\Pi^1(\lambda,\psi)$ by Lemma 3.39.

We define $C_{2,i}$, $S_{2,i}$ for $i\in\{1,2,3\}$, $\widetilde{C}_{2,2}$, $\Pi^2(\lambda,\psi)_0$, $\Pi^2(\lambda,\psi)^-$, $\widetilde{\Pi}^2(\lambda,\psi)^-$, $\Pi^2(\lambda,\psi)$ and $\widetilde{\Pi}^2(\lambda,\psi)$ in a similar way be replacing $\overline{P}_1$ by $\overline{P}_2$ (and modifying everything accordingly, e.g. $I(s\cdot\lambda_{1,2})\otimes x^{k_3}$ is replaced by $x^{k_1}\otimes I(s\cdot\lambda_{2,3})$ with $\lambda_{2,3}:=(k_2,k_3)$ etc.). In particular all these representations are subquotients of $\big(\mathrm{Ind}_{\overline{P}_2(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}x^{k_1}\otimes\pi(\lambda_{2,3},\psi)\big)^{\mathrm{an}}$ and all the above results have their symmetric version with $\overline{P}_1$ replaced by $\overline{P}_2$.

3.3.4. $\mathcal{L}$-invariants. We associate a finite length locally analytic representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ to a 3-dimensional semi-stable representation of $\mathrm{Gal}_{\mathbb{Q}_p}$ with $N^2\neq 0$ and distinct Hodge-Tate weights. Roughly speaking, the results in § 2 and § 3.2.3 allow us to associate to such a $\mathrm{Gal}_{\mathbb{Q}_p}$-representation certain deformations (i.e. extensions) of locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. We then use Schraen's spectral sequences [78, (4.37), (4.38)] combined with parabolic induction to go from extensions of locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to extensions of locally analytic representations of $\mathrm{GL}_3(\mathbb{Q}_p)$.

We keep the notation of the previous sections (in particular we have fixed $\lambda=(k_1,k_2,k_3)$ and $0\neq\psi\in\mathrm{Hom}(\mathbb{Q}_p^\times,E)$). From the constructions of $\Pi^1(\lambda,\psi)$ and $\widetilde{\Pi}^1(\lambda,\psi)$ (and from Lemma 3.39), it is not difficult to see that one has an injection:

$$\Pi^1(\lambda,\psi)^+:=\Pi^1(\lambda,\psi)\oplus_{\mathrm{St}_3^\infty(\lambda)}S_{1,0}\ \hookrightarrow\ \widetilde{\Pi}^1(\lambda,\psi).$$

From Remark 3.41, we see that $\Pi^1(\lambda,\psi)^+$ has the following form (ψ not smooth on the left, ψ smooth on the right):

$$\begin{tikzcd}
 & & \widetilde{C}_{1,2} \arrow[dr, dash, dashed] & \\
 & C_{1,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{1,3} \\
\mathrm{St}_3^\infty(\lambda) \arrow[ur, dash] \arrow[dr, dash] & & C_{1,2} \arrow[ur, dash] & \\
 & C_{2,1} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C_{1,1} \arrow[r, dash] & \widetilde{C}_{1,2} \\
\mathrm{St}_3^\infty(\lambda) \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & C_{1,2} \arrow[r, dash] & C_{1,3} \\
 & C_{2,1}. &
\end{tikzcd}$$

We will show that the extension group $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_2}^\infty(\lambda),\Pi^1(\lambda,\psi)^+\big)$ can encode the information on (higher) $\mathcal{L}$-invariants. We start with some lemmas.

Lemma 3.42. *(1) The natural map:*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)^+\big) \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), \widetilde{\Pi}^1(\lambda,\psi)\big) \tag{3.77}$$

is an isomorphism.

(2) We have an exact sequence:

$$\begin{aligned}0 \to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), \mathrm{St}_3^{\infty}(\lambda)\big) \to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)^+\big)\\ \to \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)/\mathrm{St}_3^{\infty}(\lambda)\big) \oplus \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), C_{2,1}\big) \to 0\end{aligned} \tag{3.78}$$

where:

$$\begin{aligned}&\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), \mathrm{St}_3^{\infty}(\lambda)\big) = 1\\ &\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)^+\big) = 3\\ &\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)/\mathrm{St}_3^{\infty}(\lambda)\big) = 1\\ &\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), C_{2,1}\big) = 1.\end{aligned}$$

Proof. (1) It is easy to see $\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_2}(\lambda), \widetilde{\Pi}^1(\lambda,\psi)/\Pi^1(\lambda,\psi)^+) = 0$, and thus (3.77) is injective. It is sufficient to show

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_2}(\lambda), \widetilde{\Pi}^1(\lambda,\psi)/\Pi^1(\lambda,\psi)^+) = 0.$$

From [78, (4.37) & (4.41)] and [78, Prop. 4.10], we easily deduce that for any irreducible representation W in the union (3.70) $\cup$ (3.72) we have $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_2}(\lambda), W) = 0$. As in Step 4 of the proof of [4, Prop. 4.3.1], we also have $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_2}(\lambda), \mathcal{F}^{\mathrm{GL}_3}_{\overline{P}_1}(\overline{L}(-s_2\cdot\lambda), 1)) = 0$. Since the irreducible constituents of $\widetilde{\Pi}^1(\lambda,\psi)/\Pi^1(\lambda,\psi)^+$ are exactly given by the representations in the set (3.70) $\cup$ (3.72) $\cup$ $\{\mathcal{F}^{\mathrm{GL}_3}_{\overline{P}_1}(\overline{L}(-s_2\cdot\lambda), 1)\}$, the result follows by dévissage.

(2) First note that by Lemma 3.1 the extension groups in (3.78) do not change if $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}$ is replaced by $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),Z}$. By [78, Cor. 4.8], we have $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbb{Q}_p),Z}(v^{\infty}_{\overline{P}_2}(\lambda), \mathrm{St}_3^{\infty}(\lambda)) = 0$, from which we easily deduce (3.78). By *loc. cit.* we also have:

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),Z}(v^{\infty}_{\overline{P}_2}(\lambda), \mathrm{St}_3^{\infty}(\lambda)) = \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_2}(\lambda), \mathrm{St}_3^{\infty}(\lambda)) = 1.$$

It follows from (3.69) and $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbb{Q}_p),Z}(v^{\infty}_{\overline{P}_2}(\lambda), \mathrm{St}_3^{\infty}(\lambda)) = 0$ that we have:

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), C_{2,1}\big) = \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), S_{1,0}/\mathrm{St}_3^{\infty}(\lambda)\big) = 1.$$

From Remark 3.41 and [4, Prop. 4.2.2 (ii) & Prop. 4.2.3 (ii)] we easily deduce:

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)/\operatorname{St}^{\infty}_3(\lambda)\big) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), C_{1,2} - C_{1,3}\big).$$

By [4, Prop. 4.3.1 & Prop. 4.2.1 (i)] the latter is one dimensional. This concludes the proof. □

By [78, (4.38)], we have a spectral sequence:[1]

$$\begin{aligned}(3.79)\quad \mathrm{Ext}^i_{L_1(\mathbb{Q}_p),Z}&\big(H_j(\overline{N}_1(\mathbb{Q}_p), v^{\infty}_{\overline{P}_2}(\lambda)), \pi(\lambda_{1,2},\psi)\otimes x^{k_3}\big)\\ &\Rightarrow \mathrm{Ext}^{i+j}_{\mathrm{GL}_3(\mathbb{Q}_p),Z}\big(v^{\infty}_{\overline{P}_2}(\lambda), I^{\mathrm{GL}_3}_{\overline{P}_1}(\pi(\lambda_{1,2},\psi),k_3)\big).\end{aligned}$$

From [78, (4.41) & (4.42)] and the discussion after [4, (52)] we have (with obvious notation):

$$H_i(\overline{N}_1(\mathbb{Q}_p), v^{\infty}_{\overline{P}_2}(\lambda)) \cong \Big(\bigoplus_{\substack{\lg w = i\\ w\cdot\lambda \text{ is}\\ B\cap L_1\text{-dominant}}} L_1(w\cdot\lambda)\Big)\otimes\big((\operatorname{St}^{\infty}_2\otimes 1)\oplus(|\cdot|^{-1}\circ\det\otimes|\cdot|^2)\big).$$

For all w with $w\cdot\lambda$ dominant with respect to $B(\mathbb{Q}_p)\cap L_1(\mathbb{Q}_p)$ we have by considering the action of the center of $L_1(\mathbb{Q}_p)$:

$$\begin{aligned}\mathrm{Hom}_{L_1(\mathbb{Q}_p)}\big(L_1(w\cdot\lambda)\otimes_E(|\cdot|^{-1}\circ\det\otimes|\cdot|^2), \pi(\lambda_{1,2},\psi)\otimes x^{k_3}\big) &= 0\\ \mathrm{Ext}^1_{L_1(\mathbb{Q}_p)}\big(L_1(w\cdot\lambda)\otimes_E(|\cdot|^{-1}\circ\det\otimes|\cdot|^2), \pi(\lambda_{1,2},\psi)\otimes x^{k_3}\big) &= 0.\end{aligned}$$

It is then easy to see from the above formula:

$$\mathrm{Hom}_{L_1(\mathbb{Q}_p)}\big(H_1(\overline{N}_1(\mathbb{Q}_p), v^{\infty}_{\overline{P}_2}(\lambda)), \pi(\lambda_{1,2},\psi)\otimes x^{k_3}\big) = 0.$$

Thus we deduce from (3.79) an isomorphism:

$$\begin{aligned}(3.80)\quad \mathrm{Ext}^1_{L_1(\mathbb{Q}_p),Z}&\big(\operatorname{St}^{\infty}_2(\lambda_{1,2})\otimes x^{k_3}, \pi(\lambda_{1,2},\psi)\otimes x^{k_3}\big)\\ &\xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),Z}\big(v^{\infty}_{\overline{P}_2}(\lambda), I^{\mathrm{GL}_3}_{\overline{P}_1}(\pi(\lambda_{1,2},\psi),k_3)\big).\end{aligned}$$

---

[1] Actually, to apply [78, (4.38)], one needs to show that the (dual of the) $\overline{P}_1$-representation $\pi(\lambda_{1,2},\psi)\otimes x^{k_3}$ satisfies the condition (FIN) of [76, § 6]. However, any irreducible constituent of $\pi(\lambda_{1,2},\psi)\otimes x^{k_3}$ is either locally algebraic or isomorphic to a locally analytic principal series, and hence satisfies the condition (FIN) (see the discussion in the beginning of [78, § 4.4] for the locally algebraic case, and the discussion before Step 1 in the proof of [4, Prop. 4.3.1] for the case of principal series). One deduces then that the dual of $\pi(\lambda_{1,2},\psi)\otimes x^{k_3}$ also satisfies (FIN).

Denote by W be the kernel of $I_{P_1}^{\mathrm{GL}_3}(\pi(\lambda_{1,2},\psi),k_3) \twoheadrightarrow \widetilde{\Pi}^1(\lambda,\psi)$, which (by the definition of $\widetilde{\Pi}^1(\lambda,\psi)$ and $\widetilde{\Pi}^1(\lambda,\psi)^-$) is an extension of $L(\lambda)$ by $v_{P_2}^{\mathrm{an}}(\lambda)$. By [31, Cor. 2.13], we have $\mathrm{Ext}^i_{\mathrm{GL}_3(\mathbb{Q}_p),Z}(v_{P_2}^\infty(\lambda), I_{P_2}^{\mathrm{GL}_3}(\lambda))=0$ for all $i\geq 0$ and:

$$\mathrm{Ext}^i_{\mathrm{GL}_3(\mathbb{Q}_p),Z}\big(v_{P_2}^\infty(\lambda), L(\lambda)\big) = \begin{cases} E & \text{if } i=1 \\ 0 & \text{otherwise.}\end{cases}$$

By dévissage (recall $I_{P_2}^{\mathrm{GL}_3}(\lambda)\cong L(\lambda) - v_{P_2}^{\mathrm{an}}(\lambda)$, see § 3.3.1), we get:

$$\mathrm{Ext}^i_{\mathrm{GL}_3(\mathbb{Q}_p),Z}\big(v_{P_2}^\infty(\lambda), v_{P_2}^{\mathrm{an}}(\lambda)\big) = \begin{cases} E & \text{if } i=0 \\ 0 & \text{otherwise.}\end{cases} \tag{3.81}$$

Again by dévissage, we deduce $\mathrm{Ext}^2_{\mathrm{GL}_3(\mathbb{Q}_p),Z}(v_{P_2}^\infty(\lambda), W)=0$ and an isomorphism:

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),Z}(v_{P_2}^\infty(\lambda), W) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),Z}\big(v_{P_2}^\infty(\lambda), L(\lambda)\big)$$

of 1-dimensional E-vector spaces (with [78, Cor. 4.8] for the dimension). From the former equality we obtain an exact sequence:

$$\begin{aligned} 0 \longrightarrow &\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),Z}\big(v_{P_2}^\infty(\lambda), W\big) \\ &\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),Z}\big(v_{P_2}^\infty(\lambda), I_{P_1}^{\mathrm{GL}_3}(\pi(\lambda_{1,2},\psi),k_3)\big) \\ &\qquad\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),Z}\big(v_{P_2}^\infty(\lambda), \widetilde{\Pi}^1(\lambda,\psi)\big) \longrightarrow 0. \end{aligned} \tag{3.82}$$

Together with Lemma 3.42, (3.80) and a dimension count we obtain:

$$\begin{aligned} &\dim_E \mathrm{Ext}^1_{L_1(\mathbb{Q}_p),Z}\big(\mathrm{St}_2^\infty(\lambda_{1,2})\otimes x^{k_3}, \pi(\lambda_{1,2},\psi)\otimes x^{k_3}\big) \\ &\qquad = \dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),Z}\big(v_{P_2}^\infty(\lambda), I_{P_1}^{\mathrm{GL}_3}(\pi(\lambda_{1,2},\psi),k_3)\big)=4. \end{aligned} \tag{3.83}$$

By (3.80) and (3.82), we have a natural surjection:

$$\begin{aligned} &\mathrm{Ext}^1_{L_1(\mathbb{Q}_p),Z}\big(\mathrm{St}_2^\infty(\lambda_{1,2})\otimes x^{k_3}, \pi(\lambda_{1,2},\psi)\otimes x^{k_3}\big) \\ &\qquad\qquad \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),Z}\big(v_{P_2}^\infty(\lambda), \widetilde{\Pi}^1(\lambda,\psi)\big). \end{aligned} \tag{3.84}$$

Similarly, we have natural maps (without fixing the central character of $\mathrm{GL}_3(\mathbb{Q}_p)$ and using (3.64) and (3.65)):

$$\begin{aligned}(3.85)\quad &\mathrm{Ext}^1_{L_1(\mathbb{Q}_p)}\big(\mathrm{St}_2^\infty(\lambda_{1,2})\otimes x^{k_3}, \pi(\lambda_{1,2},\psi)\otimes x^{k_3}\big)\\ &\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_2}^\infty(\lambda), I_{\overline{P}_1}^{\mathrm{GL}_3}(\pi(\lambda_{1,2},\psi),k_3)\big)\\ &\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_2}^\infty(\lambda), \widetilde{\Pi}^1(\lambda,\psi)\big)\end{aligned}$$

whose composition is surjective by (3.84) and the isomorphism (Lemma 3.1):

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p),Z}\big(v_{\overline{P}_2}^\infty(\lambda), \widetilde{\Pi}^1(\lambda,\psi)\big) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_2}^\infty(\lambda), \widetilde{\Pi}^1(\lambda,\psi)\big).$$

Remark 3.43. We can describe (3.85) (and similarly for (3.80) and (3.84)) in the following explicit way. For any

$$\widetilde{\pi}\in \mathrm{Ext}^1_{L_1(\mathbb{Q}_p)}(\mathrm{St}_2^\infty(\lambda_{1,2})\otimes x^{k_3}, \pi(\lambda_{1,2},\psi)\otimes x^{k_3}),$$

the parabolic induction $(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\widetilde{\pi})^{\mathrm{an}}$ lies in an exact sequence:

$$\begin{aligned}(3.86)\quad 0\longrightarrow I_{\overline{P}_1}^{\mathrm{GL}_3}(\pi(\lambda_{1,2},\psi),k_3)\longrightarrow \big(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\widetilde{\pi}\big)^{\mathrm{an}}&\\ \xrightarrow{\mathrm{pr}} \big(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\mathrm{St}_2^\infty(\lambda_{1,2})\otimes x^{k_3}\big)^{\mathrm{an}}\longrightarrow 0.&\end{aligned}$$

Then the first map of (3.85) is given by sending $\widetilde{\pi}$ to $\mathrm{pr}^{-1}(v_{\overline{P}_2}^\infty(\lambda))$ and the second map is given by quotienting by the subspace W. In particular the composition sends $\widetilde{\pi}$ to $\mathrm{pr}^{-1}(v_{\overline{P}_2}^\infty(\lambda))/W$.

Consider the following composition:

$$\begin{aligned}(3.87)\quad &\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathrm{St}_2^\infty(\lambda_{1,2}), \pi(\lambda_{1,2},\psi)\big)\\ &\longrightarrow \mathrm{Ext}^1_{L_1(\mathbb{Q}_p)}\big(\mathrm{St}_2^\infty(\lambda_{1,2})\otimes x^{k_3}, \pi(\lambda_{1,2},\psi)\otimes x^{k_3}\big)\\ &\xrightarrow{(3.85)} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_2}^\infty(\lambda), \widetilde{\Pi}^1(\lambda,\psi)\big)\end{aligned}$$

where the first map sends $\widetilde{\pi}$ to $\widetilde{\pi}\otimes x^{k_3}$.

Lemma 3.44. *(1) The composition (3.87) is surjective.*

(2) The kernel of the composition (3.87) is 1-dimensional and is generated by $\iota_1(\pi(\lambda,\psi,0)^-)$ (see (3.29) and (3.23)).

Proof. (1) For any $\widetilde{\pi}\in \mathrm{Ext}^1_{L_1(\mathbb{Q}_p),Z}(\mathrm{St}_2^\infty(\lambda_{1,2})\otimes x^{k_3}, \pi(\lambda_{1,2},\psi)\otimes x^{k_3})$, we can view $\widetilde{\pi}$ as a representation of $L_1(\mathbb{Q}_p)$ over $E[\epsilon]/\epsilon^2$ by making ϵ act as the composition (unique up to nonzero scalars):

$$\widetilde{\pi}\twoheadrightarrow \mathrm{St}_2^\infty(\lambda_{1,2})\otimes x^{k_3}\hookrightarrow \pi(\lambda_{1,2},\psi)\otimes x^{k_3}\hookrightarrow \widetilde{\pi}.$$

Let $Z_2 := \mathbb{Q}_p^\times \hookrightarrow L_1(\mathbb{Q}_p) \cong \mathrm{GL}_2(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$, $a \mapsto (1, a)$, which acts on $\widetilde{\pi}$ by a character $\widetilde{\chi}$ of $\mathbb{Q}_p^\times$ over $E[\epsilon]/\epsilon^2$ (by the same argument as in the proof of Lemma 3.15). Consider $\widetilde{\pi}' := \widetilde{\pi} \otimes_{E[\epsilon]/\epsilon^2} (\widetilde{\chi}^{-1} \circ \det)$, on which Z_2 acts thus by x^{k_3}. So there exists $\widetilde{\pi}'_0 \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathrm{St}_2^\infty(\lambda_{1,2}), \pi(\lambda_{1,2}, \psi)\big)$ such that $\widetilde{\pi}' \cong \widetilde{\pi}'_0 \otimes x^{k_3}$ ("external" tensor product). However, by Lemma 3.2, Remark 3.43 and the fact that:

$$\big(\mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \widetilde{\pi}'\big)^{\mathrm{an}} \cong \big(\mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \widetilde{\pi}\big)^{\mathrm{an}} \otimes_{E[\epsilon]/\epsilon^2} \widetilde{\chi}^{-1} \circ \det,$$

we see that the image of $\widetilde{\pi}$ via (3.84) is isomorphic to the image of $\widetilde{\pi}'_0$ via (3.87). Since (3.84) is surjective, so is (3.87).

(2) Since (3.87) is surjective, by counting dimensions using Lemma 3.42 and (3.83) we see that the kernel of (3.87) is one dimensional. It is thus sufficient to prove $\iota_1(\pi(\lambda, \psi, 0)^-)$ is sent to zero. Let $\Psi_{1,2} := (\psi, 0)$ and $\Psi := (\psi, 0, 0)$. By construction (cf. (3.23)), $\pi(\lambda, \psi, 0)^-$ is a subquotient of $(\mathrm{Ind}_{B_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_{\lambda_{1,2}}(1 + \Psi_{1,2}\epsilon))^{\mathrm{an}}$, and thus $(\mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \pi(\lambda, \psi, 0)^- \otimes x^{k_3})^{\mathrm{an}}$ is a subquotient of $(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \delta_\lambda(1 + \Psi\epsilon))^{\mathrm{an}}$. However, from the first part of Proposition 3.35 and Lemma 3.34(1), we deduce (see Proposition 3.35 for $\Pi^2(\lambda, \Psi)_0$):

$$v_{\overline{P}_2}^\infty(\lambda) \longhookrightarrow \Pi^2(\lambda, \Psi)_0 \longhookrightarrow \big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \delta_\lambda(1 + \Psi\epsilon)\big) / \sum_{i=1,2} I_{\overline{P}_i}^{\mathrm{GL}_3}(\lambda).$$

In particular the image of $\iota_1(\pi(\lambda, \psi, 0)^-)$ via (3.87) contains $v_{\overline{P}_2}^\infty(\lambda)$ as a subrepresentation, hence the associated extension is split. This concludes the proof. □

We now can prove the main result of the section. We let $\lambda^\sharp := (k_1, k_2 - 1, k_3 - 2)$, $\lambda^\sharp_{1,2} := (k_1, k_2 - 1)$, $\lambda^\sharp_{2,3} := (k_2 - 1, k_3 - 2)$ and $D_1^2 := D(p, \lambda^\sharp_{1,2}, \psi)$ (see (3.51)), the notation D_1^2 is for (future) compatibility with the notation at the beginning of § 2).

Theorem 3.45. *Assume Hypothesis 3.26 for D_1^2. The cup product (2.1) together with the isomorphisms:*

$$\mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D_1^2, D_1^2\big) \cong \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D(p, \lambda^\sharp_{1,2}, \psi), D(p, \lambda^\sharp_{1,2}, \psi)\big)$$
$$\xrightarrow[\sim]{(3.52)} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi(\lambda_{1,2}, \psi), \pi(\lambda_{1,2}, \psi)\big)$$

induce a perfect pairing of 3*-dimensional* E*-vector spaces:*

$$\mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(x^{k_3-2}|\cdot|^{-2}), D_1^2\big) \times \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda), \Pi\big) \xrightarrow{\cup} E \tag{3.88}$$

with $\Pi = \widetilde{\Pi}^1(\lambda,\psi)$ *or* $\Pi^1(\lambda,\psi)^+$*.*

Proof. The dimension 3 comes from Lemma 3.42(2). We have morphisms (see (3.28) for κ_1):

(3.89)
$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)\big) \xrightarrow{\kappa_1} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathrm{St}_2^{\infty}(\lambda_{1,2}),\pi(\lambda_{1,2},\psi)\big)\\ &\xrightarrow{(3.87)} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\widetilde{\Pi}^1(\lambda,\psi)\big) \cong \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\Pi^1(\lambda,\psi)^+\big)\end{aligned}$$

where the first morphism is the surjection in (3.28) (it is surjective by Remark 3.27(1)) and the last isomorphism is Lemma 3.42(1). By Lemma 3.44(2) and Lemma 3.24(1), we obtain that the kernel of the composition in (3.89) is equal to $\mathrm{Ker}(\kappa^{\mathrm{aut}})$ where we use the notation of Lemma 3.28. Note that this composition is surjective by Lemma 3.44(1) (and the surjectivity of κ_1). Now consider:

$$\begin{aligned}&\mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D_1^2, D_1^2\big) \xrightarrow{\sim} \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D(p,\lambda^{\sharp}_{1,2},\psi), D(p,\lambda^{\sharp}_{1,2},\psi)\big) \xrightarrow[\sim]{(3.52)}\\ &\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)\big) \overset{(3.89)}{\twoheadrightarrow} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\Pi^1(\lambda,\psi)^+\big).\end{aligned} \tag{3.90}$$

By (3.56), the kernel of the composition in (3.90) is thus isomorphic to $\mathrm{Ker}(\kappa^{\mathrm{gal}})$. Since this composition is moreover surjective, the theorem then follows from Proposition 2.3 (where κ there is denoted κ^{gal} here). □

We let $\delta_1 := x^{k_1}$, $\delta_2 := x^{k_2-1}|\cdot|^{-1}$, and $\delta_3 := x^{k_3-2}|\cdot|^{-2}$. The following proposition shows that the pairing (3.88) is compatible with the one in Corollary 3.36 for simple $\mathcal{L}$-invariants.

Proposition 3.46. *Assume Hypothesis 3.26 for* D_1^2*. We have a commutative diagram:*

(3.91)
$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\mathrm{St}_3^{\mathrm{an}}(\lambda)\big) \arrow[d] & \times & \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\delta_3),\mathcal{R}_E(\delta_2)\big) \arrow[r, "\cup_1"] & E \arrow[d, equal] \\
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\widetilde{\Pi}^1(\lambda,\psi)\big) & \times & \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\delta_3),D_1^2\big) \arrow[u, "u_1"'] \arrow[r, "\cup"] & E
\end{tikzcd}$$

where the left vertical map is the natural injection, the middle vertical map is the natural surjection, the bottom (perfect) pairing is the one in Theorem 3.45 and the top (perfect) pairing is the one in Corollary 3.36. The same holds with $(\mathrm{St}_3^{\mathrm{an}}(\lambda), \widetilde{\Pi}^1(\lambda,\psi))$ *replaced by* $(S_{1,0}, \Pi^1(\lambda,\psi)^+)$*.*

Proof. (a) We first show that the composition:

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{tri}}\big(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)\big) &\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)\big)\\ &\twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathrm{St}_2^{\infty}(\lambda_{1,2}),\pi(\lambda_{1,2},\psi)\big) \overset{(3.87)}{\twoheadrightarrow} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\widetilde{\Pi}^1(\lambda,\psi)\big)\end{aligned} \tag{3.92}$$

factors through:

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{tri}}\big(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)\big) &\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\mathrm{St}_3^{\mathrm{an}}(\lambda)\big)\\ &\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\widetilde{\Pi}^1(\lambda,\psi)\big).\end{aligned} \tag{3.93}$$

By (3.45), the composition of the first two maps in (3.92) has image equal to $\mathrm{Im}(\iota_1)$ (cf. (3.29)). It is thus sufficient to show that the composition:

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathrm{St}_2^{\infty}(\lambda_{1,2}),\pi(\lambda_{1,2},\psi)^-\big) &\xrightarrow{\iota_1} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathrm{St}_2^{\infty}(\lambda_{1,2}),\pi(\lambda_{1,2},\psi)\big)\\ &\xrightarrow{(3.87)} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\widetilde{\Pi}^1(\lambda,\psi)\big)\end{aligned} \tag{3.94}$$

factors through:

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathrm{St}_2^{\infty}(\lambda_{1,2}),\pi(\lambda_{1,2},\psi)^-\big) &\longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\mathrm{St}_3^{\mathrm{an}}(\lambda)\big)\\ &\hookrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\widetilde{\Pi}^1(\lambda,\psi)\big).\end{aligned} \tag{3.95}$$

By the construction in Remark 3.43, it is easy to see that any element in the image of (3.94) comes by push-forward from a certain extension of $v^{\infty}_{\overline{P}_2}(\lambda)$ by $\widetilde{\Pi}^1(\lambda,\psi)^-$. By the proof of Lemma 3.42(1), one has:

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\mathrm{St}_3^{\mathrm{an}}(\lambda)\big) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\widetilde{\Pi}^1(\lambda,\psi)^-\big).$$

We deduce that the map (3.94) factors through $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_2}(\lambda),\mathrm{St}_3^{\mathrm{an}}(\lambda))$.

(b) We prove the following map is surjective:

$$\mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^{\infty}_{\overline{P}_2}(\lambda),\mathrm{St}_3^{\mathrm{an}}(\lambda)).$$

The composition of the last two maps in (3.92) is equal to (3.89) and has kernel equal to $\mathrm{Ker}(\kappa^{\mathrm{aut}})$ by the proof of Theorem 3.45. From (3.47) we have $\mathrm{Ker}(\kappa^{\mathrm{aut}}) \subseteq \mathrm{Ext}^1_{\mathrm{tri}}(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi))$, so the kernel of the composition in (3.92) is $\mathrm{Ker}(\kappa^{\mathrm{aut}})$. From Lemma 3.24(1), we get that the kernel of the composition in (3.93) is (also) $\mathrm{Ker}(\kappa^{\mathrm{aut}})$ and is 2-dimensional. From Lemma 3.34(1) and Proposition 3.35 we deduce $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^\infty_{\overline{P}_2}(\lambda), \mathrm{St}^{\mathrm{an}}_3(\lambda)) = 2$. Together with Proposition 3.22(1) and a dimension count, we obtain that the first map in (3.93) is surjective. From the proof of (a), it follows that the first map in (3.95) is also surjective. In summary, we have a natural commutative diagram:

(3.96)

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathrm{tri}}\big(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)\big) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \mathrm{St}^{\mathrm{an}}_3(\lambda)\big) \arrow[d] \\
\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)\big) \arrow[r] & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \widetilde{\Pi}^1(\lambda,\psi)\big)
\end{tikzcd}$$

where the horizontal maps are surjective and the vertical maps are injective.

(c) By the discussion in (a), the morphism (3.95) can be constructed in a similar way as in Remark 3.43. In particular, for

$$\widetilde{\pi} \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{St}^\infty_2(\lambda_{1,2}), \pi(\lambda_{1,2},\psi)^-),$$

its image in $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^\infty_{\overline{P}_2}(\lambda), \mathrm{St}^{\mathrm{an}}_3(\lambda))$ is a subquotient of $(\mathrm{Ind}^{\mathrm{GL}_3(\mathbb{Q}_p)}_{\overline{P}_1(\mathbb{Q}_p)} \widetilde{\pi} \otimes x^{k_3})^{\mathrm{an}}$. By the transitivity of parabolic inductions, one can check the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(T_2(\mathbb{Q}_p), E)_\psi \arrow[r, "\mathrm{pr}_2"] \arrow[d] & \mathrm{Hom}(\mathbb{Q}_p^\times, E) \arrow[d] \\
\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathrm{St}^\infty_2(\lambda_{1,2}), \pi(\lambda_{1,2},\psi)^-\big) \arrow[r, "(3.95)"] & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \mathrm{St}^{\mathrm{an}}_3(\lambda)\big)
\end{tikzcd}$$

where the left vertical map is given by the inverse of (3.39) (see Remark 3.21 for its construction), and the right vertical map is given as in Proposition 3.35 (see the discussion above Proposition 3.35 for its construction). We deduce that the following diagram commutes (see (3.47) for $\kappa^{\mathrm{aut}} = \kappa$ and recall (3.93) comes from (3.95) by the proof of (a)):

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathrm{tri}}\big(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)\big) \arrow[r, "(3.93)"] \arrow[d, "\kappa^{\mathrm{aut}}"'] & \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\mathrm{St}_3^{\mathrm{an}}(\lambda)\big) \arrow[d, "\wr"'] \\
\mathrm{Hom}(\mathbb{Q}_p^{\times},E) \arrow[r, "\sim"] & \mathrm{Hom}(\mathbb{Q}_p^{\times},E)
\end{tikzcd} \tag{3.97}$$

where the right vertical map is the inverse of the bottom horizontal map in (3.69) (via Lemma 3.34(1)).

(d) By Hypothesis 3.26(2)&(3), the bottom squares of (3.13) induce a commutative diagram:

(3.98)

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathrm{tri}}\big(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)\big) \arrow[d] & \times & \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\delta_3),\mathcal{R}_E(\delta_2)\big) \arrow[r, "\cup"] & E \arrow[d, equal] \\
\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)\big) & \times & \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\delta_3),D_1^2\big) \arrow[u] \arrow[r, "\cup"] & E.
\end{tikzcd}$$

And the top squares of (3.13) induce another commutative diagram:

(3.99)

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathrm{tri}}\big(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)\big) \arrow[d, "\kappa^{\mathrm{aut}}"'] & \times & \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\delta_3),\mathcal{R}_E(\delta_2)\big) \arrow[r, "\cup"] \arrow[d, equal] & E \arrow[d, equal] \\
\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2),\mathcal{R}_E(\delta_2)) & \times & \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\delta_3),\mathcal{R}_E(\delta_2)\big) \arrow[r, "\cup_1"] & E
\end{tikzcd}$$

where we identify $\mathrm{Hom}(\mathbb{Q}_p^{\times},E)$ with $\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2),\mathcal{R}_E(\delta_2))$ (see (1.11)).

(e) We finally prove the proposition. By (3.96), (3.98) and Theorem 3.45, we deduce a commutative diagram as in (3.91) but with the top pairing $\cup_1$ replaced by the pairing induced by the top pairing of (3.98) via the surjection (see (b)):

$$\mathrm{Ext}^1_{\mathrm{tri}}\big(\pi(\lambda_{1,2},\psi),\pi(\lambda_{1,2},\psi)\big) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\mathrm{St}_3^{\mathrm{an}}(\lambda)\big).$$

However, by (3.99) and (3.97), we see these two pairings actually coincide. This concludes the proof. □

We fix a *nonsplit* extension $D\in\mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\delta_3),D_1^2\big)$ and we let (assuming Hypothesis 3.26 for D_1^2):

$$\mathcal{L}_{\mathrm{aut}}(D:D_1^2)\subseteq\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^{\infty}_{\overline{P}_2}(\lambda),\widetilde{\Pi}^1(\lambda,\psi)\big) \tag{3.100}$$

$$\cong \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)^+\big)$$

be the 2-dimensional E-vector subspace annihilated by D via (3.88).

Remark 3.47. By Theorem 3.45 and its proof, the composition (3.90) actually induces an isomorphism $\mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1^2, \mathcal{R}_E(\delta_2)) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^\infty_{\overline{P}_2}(\lambda), \Pi)$ with $\Pi = \widetilde{\Pi}^1(\lambda,\psi)$ or $\Pi^1(\lambda,\psi)^+$. Moreover, for a nonsplit

$$D \in \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_3), D_1^2)$$

as above and from the definitions of $\ell_{\mathrm{FM}}(D : D_1^2)$ and $\mathcal{L}_{\mathrm{aut}}(D : D_1^2)$, this isomorphism induces an isomorphism:

$$\ell_{\mathrm{FM}}(D : D_1^2) \xrightarrow{\sim} \mathcal{L}_{\mathrm{aut}}(D : D_1^2) \tag{3.101}$$

since both are annihilated by D via the corresponding pairing.

We define (cf. Notation 3.4):

$$\widetilde{\Pi}^1(D)^- := \mathscr{E}\big(\widetilde{\Pi}^1(\lambda,\psi), v^\infty_{\overline{P}_2}(\lambda)^{\oplus 2}, \mathcal{L}_{\mathrm{aut}}(D : D_1^2)\big) \tag{3.102}$$

$$\Pi^1(D)^- := \mathscr{E}\big(\Pi^1(\lambda,\psi)^+, v^\infty_{\overline{P}_2}(\lambda)^{\oplus 2}, \mathcal{L}_{\mathrm{aut}}(D : D_1^2)\big). \tag{3.103}$$

It follows from the perfect pairing (3.88) and Lemma 3.42(1) that D is *determined* by the subspace $\mathcal{L}_{\mathrm{aut}}(D : D_1^2)$, hence by $\widetilde{\Pi}^1(D)^-$ and $\Pi^1(D)^-$. Let:

$$\mathcal{L}_{\mathrm{aut}}(D : D_1^2)_0 := \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \mathrm{St}_3^{\mathrm{an}}(\lambda)\big) \cap \mathcal{L}_{\mathrm{aut}}(D : D_1^2)$$

which we also view as a subspace of $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), S_{2,0}\big)$ by Lemma 3.34(1). By Proposition 3.46, we have via the pairing $\cup_1$ (u_1 as in Proposition 3.46 and identifying D with its corresponding extension):

$$\mathcal{L}_{\mathrm{aut}}(D : D_1^2)_0 = (Eu_1(D))^\perp. \tag{3.104}$$

We assume now that the extension $u_1(D)$ of $\mathcal{R}_E(\delta_3)$ by $\mathcal{R}_E(\delta_2)$ inside D is nonsplit. As $\cup_1$ is perfect (cf. (3.91)) this implies $\dim_E \mathcal{L}_{\mathrm{aut}}(D : D_1^2)_0 = 1$. We define (cf. Notation 3.4):

$$\widetilde{\Pi}^1(D)_2^- := \mathscr{E}\big(\mathrm{St}_3^{\mathrm{an}}(\lambda), v^\infty_{\overline{P}_2}(\lambda), \mathcal{L}_{\mathrm{aut}}(D : D_1^2)_0\big)$$

$$\Pi^1(D)_2^- := \mathscr{E}\big(S_{2,0}, v^\infty_{\overline{P}_2}(\lambda), \mathcal{L}_{\mathrm{aut}}(D : D_1^2)_0\big).$$

We have injections $\widetilde{\Pi}^1(D)_2^- \hookrightarrow \widetilde{\Pi}^1(D)^-$ and $\Pi^1(D)_2^- \hookrightarrow \Pi^1(D)^-$.

Remark 3.48. Identifying $\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_2))$ with $\mathrm{Hom}(\mathbb{Q}_p^\times, E)$, the vector space $(Eu_1(D))^\perp$ via the top perfect pairing $\cup_1$ of (3.13) is thus a one dimensional subspace of $\mathrm{Hom}(\mathbb{Q}_p^\times, E)$. We let ψ' be a basis. By Corollary 3.36, we have $\Pi^1(D)_2^- \cong \Pi^2(\lambda, \psi')_0$ where we denote by $\Pi^2(\lambda, \psi')_0$ the image of ψ' via the bottom bijection of (3.69).

Proposition 3.49. *Assume Hypothesis 3.26 for D_1^2 and $N^2 \neq 0$ on the filtered (φ, N)-module associated to D ([2, Thm. A], and note that the latter implies that $u_1(D)$ is nonsplit and ψ is non smooth). Then there exists a unique subrepresentation:*

$$\Pi^1(D)_1^- \in \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)\big) \setminus \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \mathrm{St}_3^\infty(\lambda)\big)$$

of $\Pi^1(D)^-$ such that $\Pi^1(D)^- \cong \Pi^1(D)_1^- \oplus_{\mathrm{St}_3^\infty(\lambda)} \Pi^1(D)_2^-$. In particular, $\Pi^1(D)^-$ has the following form:

$$\begin{tikzcd}
& & \widetilde{C}_{1,2} \arrow[dl, dash] \arrow[dr, dash, dashed] & & \\
& C_{1,1} \arrow[dl, dash] \arrow[dr, dash] & & C_{1,3} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{St}_3^\infty(\lambda) \arrow[dr, dash] & & C_{1,2} & & v^\infty_{\overline{P}_2}(\lambda) \\
& C_{2,1} \arrow[r, dash] & v^\infty_{\overline{P}_2}(\lambda). & &
\end{tikzcd}$$

Proof. Considering the surjection in (3.78):

$$\begin{aligned}
&\mathrm{pr} : \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)^+\big) \\
&\xrightarrow{(\mathrm{pr}_1,\mathrm{pr}_2)} \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)/\mathrm{St}_3^\infty(\lambda)\big) \oplus \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), C_{2,1}\big).
\end{aligned}$$

We see with Lemma 3.42(2), Remark 3.41 and the form of $\Pi^1(\lambda,\psi)^+$ at the beginning of § 3.3.4 that we have:

$$\begin{aligned}
\mathrm{Ker}(\mathrm{pr}) &\cong \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \mathrm{St}_3^\infty(\lambda)\big) \\
\mathrm{Ker}(\mathrm{pr}_1) &\cong \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), S_{1,0}\big) \\
\mathrm{Ker}(\mathrm{pr}_2) &\cong \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)\big).
\end{aligned}$$

And it follows from Lemma 3.42(2) that the first kernel has dimension 1 and the two others dimension 2. We first show:

$$\mathrm{Ker}(\mathrm{pr}) \cap \mathcal{L}_{\mathrm{aut}}(D : D_1^2) = \mathrm{Ker}(\mathrm{pr}) \cap \mathcal{L}_{\mathrm{aut}}(D : D_1^2)_0 = 0. \tag{3.105}$$

The first equality is clear since by definition and Lemma 3.34(1) we have $\mathcal{L}_{\mathrm{aut}}(D : D_1^2)_0 = \mathrm{Ker}(\mathrm{pr}_1) \cap \mathcal{L}_{\mathrm{aut}}(D : D_1^2)$. As $N^2 \neq 0$, the quotient $u_1(D)$ (as an extension of $\mathcal{R}_E(\delta_3)$ by $\mathcal{R}_E(\delta_2)$) is *not* crystalline, hence by the second part of Corollary 3.36, $u_1(D)$ is *not* annihilated by $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^\infty_{\overline{P}_2}(\lambda), \mathrm{St}_3^\infty(\lambda))$. Since the latter vector space has dimension 1, one deduces from (3.104):

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \mathrm{St}_3^\infty(\lambda)\big) \cap \mathcal{L}_{\mathrm{aut}}(D : D_1^2)_0 = 0 \tag{3.106}$$

and the second equality in (3.105) follows. As $\mathcal{L}_{\mathrm{aut}}(D : D_1^2)$ has dimension 2 and $\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v^\infty_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)^+)$ dimension 3, we easily deduce from (3.105) and $\dim_E \mathrm{Ker}(\mathrm{pr}_i) = 2$ that $\dim_E \mathrm{Ker}(\mathrm{pr}_i) \cap \dim_E \mathcal{L}_{\mathrm{aut}}(D : D_1^2) = 1$ for $i = 1, 2$. Let $\mathcal{L}_{\mathrm{aut}}(D : D_1^2)_1 := \mathrm{Ker}(\mathrm{pr}_2) \cap \mathcal{L}_{\mathrm{aut}}(D : D_1^2) \subseteq \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_2}(\lambda), \Pi^1(\lambda,\psi)\big)$ and set (with Notation 3.4):

$$\Pi^1(D)_1^- := \mathscr{E}\big(\Pi^1(\lambda,\psi), v^\infty_{\overline{P}_2}(\lambda), \mathcal{L}_{\mathrm{aut}}(D : D_1^2)_1\big).$$

Since we have $\mathcal{L}_{\mathrm{aut}}(D : D_1^2)_1 \oplus \mathcal{L}_{\mathrm{aut}}(D : D_1^2)_0 = \mathcal{L}_{\mathrm{aut}}(D : D_1^2)$ (as follows from (3.105)) and $\mathrm{Ker}(\mathrm{pr}) \cap \mathcal{L}_{\mathrm{aut}}(D : D_1^2)_1 = 0$ (*ibid.*), one easily checks the statements in the proposition. □

Replacing $\overline{P}_1$ by $\overline{P}_2$, we define $\Pi^2(\lambda,\psi)^+ := \Pi^2(\lambda,\psi) \oplus_{\mathrm{St}_3^\infty(\lambda)} S_{2,0} \hookrightarrow \widetilde{\Pi}^2(\lambda,\psi)$ as for $\Pi^1(\lambda,\psi)^+$ at the beginning of § 3.3.4. All the above results have their analogue (or symmetric) version. Let $D_2^3 := D(p^2, \lambda^\sharp_{2,3}, \psi)$ (see the beginning of § 3.2.3). The following theorem is the analogue of Theorem 3.45 and Proposition 3.46.

Theorem 3.50. *Assume Hypothesis 3.26 for D_2^3. The isomorphism (3.52) and (2.8) induce a perfect pairing:*

$$\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_1}(\lambda), \Pi\big) \times \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D_2^3, \mathcal{R}_E(\delta_1)\big) \xrightarrow{\cup} E$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_1}(\lambda), \Pi^-\big) \arrow[d] & \times & \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\delta_2), \mathcal{R}_E(\delta_1)\big) \arrow[r, "\cup_1"] & E \arrow[d, equal] \\
\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v^\infty_{\overline{P}_1}(\lambda), \Pi\big) & \times & \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D_2^3, \mathcal{R}_E(\delta_1)\big) \arrow[u, "u_1"] \arrow[r, "\cup"] & E
\end{tikzcd}$$

with $(\Pi^-, \Pi) = (S_{2,0}, \Pi^2(\lambda,\psi)^+)$ or $(\mathrm{St}_3^{\mathrm{an}}(\lambda), \widetilde{\Pi}^2(\lambda,\psi))$ and where the top perfect pairing is given as in Corollary 3.36 (via Lemma 3.34(1)).

For $D \in \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(D_2^3, \mathcal{R}_E(\delta_1)\big)$, we define (using the symmetric version of Lemma 3.42(1)):

$$(3.107)\quad \mathcal{L}_{\mathrm{aut}}(D : D_2^3) := (ED)^{\perp} \subseteq \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_1}^{\infty}(\lambda), \Pi^2(\lambda,\psi)^+\big) \\ \cong \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_1}^{\infty}(\lambda), \widetilde{\Pi}^2(\lambda,\psi)\big)$$

and likewise using Lemma 3.34(1):

$$\mathcal{L}_{\mathrm{aut}}(D : D_2^3)_0 := \mathcal{L}_{\mathrm{aut}}(D : D_2^3) \cap \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_1}^{\infty}(\lambda), S_{2,0}\big) \\ \subseteq \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_1}^{\infty}(\lambda), S_{2,0}\big) \cong \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_1}^{\infty}(\lambda), \mathrm{St}_3^{\mathrm{an}}(\lambda)\big).$$

We have $\mathcal{L}_{\mathrm{aut}}(D : D_2^3)_0 = (Eu_1(D))^{\perp}$ via the pairing $\cup_1$ in Theorem 3.50. We also define:

$$\begin{aligned}
(3.108)\qquad \Pi^2(D)^- &:= \mathscr{E}\big(\Pi^2(\lambda,\psi)^+, v_{\overline{P}_1}^{\infty}(\lambda)^{\oplus 2}, \mathcal{L}_{\mathrm{aut}}(D : D_2^3)\big)\\
(3.109)\qquad \widetilde{\Pi}^2(D)^- &:= \mathscr{E}\big(\widetilde{\Pi}^2(\lambda,\psi), v_{\overline{P}_1}^{\infty}(\lambda)^{\oplus 2}, \mathcal{L}_{\mathrm{aut}}(D : D_2^3)\big)\\
\Pi^2(D)_1^- &:= \mathscr{E}\big(S_{2,0}, v_{\overline{P}_1}^{\infty}(\lambda), \mathcal{L}_{\mathrm{aut}}(D : D_2^3)_0\big)\\
\widetilde{\Pi}^2(D)_1^- &:= \mathscr{E}\big(\mathrm{St}_3^{\mathrm{an}}(\lambda), v_{\overline{P}_1}^{\infty}(\lambda), \mathcal{L}_{\mathrm{aut}}(D : D_2^3)_0\big)
\end{aligned}$$

and we have $\Pi^2(D)_1^- \hookrightarrow \Pi^2(D)^-$ and $\widetilde{\Pi}^2(D)_1^- \hookrightarrow \widetilde{\Pi}^2(D)^-$. Similarly as in Proposition 3.49, assuming Hypothesis 3.26 there exists a unique representation if $N^2 \neq 0$:

$$\Pi^2(D)_2^- \in \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_1}^{\infty}(\lambda), \Pi^2(\lambda,\psi)\big) \setminus \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(v_{\overline{P}_1}^{\infty}(\lambda), \mathrm{St}_3^{\infty}(\lambda)\big)$$

such that $\Pi^2(D)^- \cong \Pi^2(D)_1^- \oplus_{\mathrm{St}_3^{\infty}(\lambda)} \Pi^2(D)_2^-$.

Now we fix $(D, (\delta_1, \delta_2, \delta_3))$ a special noncritical (φ,Γ)-module of rank 3 over $\mathcal{R}_E$ (see the beginning of § 2) with $\delta_1 = x^{k_1}$, $\delta_2 = x^{k_2-1}|\cdot|^{-1}$, and $\delta_3 = x^{k_3-2}|\cdot|^{-2}$. We assume the extension of $\mathcal{R}_E(\delta_2)$ (resp. of $\mathcal{R}_E(\delta_3)$) by $\mathcal{R}_E(\delta_1)$ (resp. by $\mathcal{R}_E(\delta_2)$) is nonsplit and we let ψ_1 be a basis of $\mathcal{L}_{\mathrm{FM}}(D_1^2 : \mathcal{R}_E(\delta_1)) \subseteq \mathrm{Hom}(\mathbb{Q}_p^{\times}, E)$ and ψ_2 a basis of $\mathcal{L}_{\mathrm{FM}}(D_2^3 : \mathcal{R}_E(\delta_2)) \subseteq \mathrm{Hom}(\mathbb{Q}_p^{\times}, E)$ (see § 2), i.e. we have $D_1^2 \cong D(p, \lambda_{1,2}^{\sharp}, \psi_1)$ and $D_2^3 \cong D(p^2, \lambda_{2,3}^{\sharp}, \psi_2)$ (see the beginning of § 3.2.3 and (3.51)). We assume $N^2 \neq 0$, which is equivalent to ψ_i not smooth for $i = 1, 2$ and we also assume that Hypothesis 3.26 holds for D_1^2 and D_2^3 (recall that under quite mild genericity assumptions this is automatic by Lemma 3.29, Proposition 3.30 and Proposition 3.32). We can then associate to D the above representations:

$$\begin{aligned}\Pi^1(D)^- &\cong \Pi^1(D)_1^- \oplus_{\mathrm{St}_3^\infty(\lambda)} \Pi^1(D)_2^- \hookrightarrow \widetilde{\Pi}^1(D)^-\\ \Pi^2(D)^- &\cong \Pi^2(D)_1^- \oplus_{\mathrm{St}_3^\infty(\lambda)} \Pi^2(D)_2^- \hookrightarrow \widetilde{\Pi}^2(D)^-.\end{aligned}$$

By the symmetric version of Lemma 3.37, the subrepresentation $S_{1,0}-v_{\overline{P}_1}^\infty(\lambda)$ of $\Pi^2(\lambda,\psi_2)\subseteq \Pi^2(D)_2^- \subseteq \Pi^2(D)^-$ is isomorphic to the image $\Pi^2(\lambda,\psi_2)_0$ of ψ_2 via the bottom map of (3.69). By Lemma 2.5, we deduce:

$$\begin{aligned}\mathcal{L}_{\mathrm{FM}}(D_2^3:\mathcal{R}_E(\delta_2)) &= \ell_{\mathrm{FM}}(D_2^3:\mathcal{R}_E(\delta_2))\\ &= \ell_{\mathrm{FM}}(D:D_1^2)\cap \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\delta_2),\mathcal{R}_E(\delta_2))\\ &= E\psi_2 \subseteq \mathrm{Hom}(\mathbb{Q}_p^\times, E^\times).\end{aligned}$$

Thus by Remark 3.48, $\Pi^1(D)_2^-$ is also isomorphic to $\Pi^2(\lambda,\psi_2)_0$. In particular, we have an injection $\Pi^1(D)_2^- \hookrightarrow \Pi^2(D)_2^-$. Similarly, we have an injection $\Pi^2(D)_1^- \hookrightarrow \Pi^1(D)_1^-$. Denote by $\Pi^0(D)^-$ the following subrepresentation of $\Pi^1(D)^-$ and $\Pi^2(D)^-$:

$$\begin{tikzcd}
 & & C_{1,1} \arrow[r, dash] & C_{1,2} \\
\Pi^0(D)^- \cong & \mathrm{St}_3^\infty(\lambda) \arrow[ur, dash] \arrow[dr, dash] & & \\
 & & C_{2,1} \arrow[r, dash] & C_{2,2}
\end{tikzcd}$$

and put $\Pi(D)^- := \Pi^1(D)^- \oplus_{\Pi^0(D)^-} \Pi^2(D)^-$, which is thus of the following form (where $C_{1,4}\cong v_{\overline{P}_2}^\infty(\lambda)\cong C_{2,2}$ and $C_{2,4}\cong v_{\overline{P}_1}^\infty(\lambda)\cong C_{1,2}$):

$$\begin{tikzcd}
 & & & \widetilde{C}_{1,2} \arrow[dl, dash] \arrow[dr, dash, dashed] & & \\
 & & C_{1,1} \arrow[dr, dash] & & C_{1,3} \arrow[r, dash] & C_{1,4} \\
 & & & C_{1,2} \arrow[ur, dash] & & \\
(3.110)\quad \Pi(D)^- \cong & \mathrm{St}_3^\infty(\lambda) \arrow[uur, dash] \arrow[ddr, dash] & & & & \\
 & & & C_{2,2} \arrow[dr, dash] & & \\
 & & C_{2,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{2,3} \arrow[r, dash] & C_{2,4}. \\
 & & & \widetilde{C}_{2,2} \arrow[ur, dash, dashed] & &
\end{tikzcd}$$

It follows from the previous results that the (φ,Γ)-module D and the $\mathrm{GL}_3(\mathbb{Q}_p)$-representation $\Pi(D)^-$ determine each other. From the results of

[4, § 4] (see in particular [4, Rem. 4.6.3]), there is a unique locally analytic representation $\Pi(D)$ containing $\Pi(D)^-$ of the form:

$$\begin{tikzcd}[column sep=small]
 & & \widetilde{C}_{1,2} \arrow[dr, dashed, dash] & & \widetilde{C}_{1,4} \arrow[dr, dashed, dash] & \\
 & C_{1,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{1,3} \arrow[ur, dash] \arrow[dr, dash] & & C_{1,5} \\
 & & C_{1,2} \arrow[ur, dash] & & C_{1,4} \arrow[ur, dash] & \\
\Pi(D) \cong \mathrm{St}_3^{\infty}(\lambda) \arrow[uur, dash] \arrow[ddr, dash] & & & & & \\
 & & C_{2,2} \arrow[dr, dash] & & C_{2,4} \arrow[dr, dash] & \\
 & C_{2,1} \arrow[ur, dash] \arrow[dr, dash] & & C_{2,3} \arrow[ur, dash] \arrow[dr, dash] & & C_{2,5}. \\
 & & \widetilde{C}_{2,2} \arrow[ur, dashed, dash] & & \widetilde{C}_{2,4} \arrow[ur, dashed, dash] &
\end{tikzcd} \tag{3.111}$$

where the irreducible constituents $C_{1,5}$, $C_{2,5}$, $\widetilde{C}_{1,4}$, $\widetilde{C}_{2,4}$ are defined in [4, § 4.1].

For $\chi : \mathbb{Q}_p^\times \to E^\times$ and $D' := D \otimes_{\mathcal{R}_E} \mathcal{R}_E(\chi)$, we finally set $\Pi(D')^- := \Pi(D)^- \otimes \chi \circ \det$, $\Pi(D') := \Pi(D) \otimes \chi \circ \det$, and if $D' \cong D_{\mathrm{rig}}(\rho)$ for a certain $\rho\colon \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_3(E)$, we set $\Pi(\rho) := \Pi(D')$. In particular, we have thus associated to any sufficiently generic semi-stable $\rho\colon \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_3(E)$ with distinct Hodge-Tate weights and with $N^2 \neq 0$ on $D_{\mathrm{st}}(\rho) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \rho)^{\mathrm{Gal}_{\mathbb{Q}_p}}$ a locally analytic representation $\Pi(\rho)$ of $\mathrm{GL}_3(\mathbb{Q}_p)$ over E which has the form (3.111) and which only depends on and completely determines ρ.

4. Ordinary part functor

In this section we give several properties of the ordinary part functor of [40] and review the ordinary part of a locally algebraic representation that has an invariant lattice ([42, § 5.6]).

4.1. Notation and preliminaries

We start with some preliminary notation. We fix finite extensions L and E of $\mathbb{Q}_p$ as in § 1 and denote by ϖ_L a uniformizer of L. We let G be a connected reductive algebraic group over L (G will be split from § 4.3 on), B a Borel subgroup of G, P a parabolic subgroup of G containing B with N_P the unipotent radical of P and L_P a Levi subgroup of P. We let $\overline{P}$ be the parabolic subgroup of G opposite to P, $N_{\overline{P}}$ its unipotent radical and Z_{L_P} the center of $L_P = L_{\overline{P}}$.

Let K be a compact open subgroup of $G(L)$, as in [40, § 3.3] we say K admits an *Iwahori decomposition* (with respect to P and $\overline{P}$) if the following natural map:

$$(K \cap N_{\overline{P}}(L)) \times (K \cap Z_{L_P}(L)) \times (K \cap N_P(L)) \longrightarrow K$$

is an isomorphism. We let $I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_i \supset I_{i+1} \supset \cdots$ be a cofinal family of compact open subgroups of $G(L)$ such that:

- I_i is normal in I_0
- I_i admits an Iwahori decomposition.

For $i \in \mathbb{Z}_{\geq 0}$, we put $N_i := N_P(L) \cap I_i$, $L_i := L_P(L) \cap I_i$ and $\overline{N}_i := N_{\overline{P}}(L) \cap I_i$. For $i \geq j \geq 0$, we put $I_{i,j} := \overline{N}_i L_j N_0$, which can be checked to be a compact open subgroup of I_0 such that:

$$\overline{N}_i \times L_j \times N_0 \xrightarrow{\sim} I_{i,j}.$$

Remark 4.1. For any $i \in \mathbb{Z}_{\geq 0}$, the subgroups $\overline{N}_i$, L_i, and N_i of I_0 are normalized by L_0, and hence $I_{i,j}$ is normalized by L_0 for any $i \geq j \geq 0$. We show this for $\overline{N}_i$ (the other cases are similar). Let $z \in L_0$, we have $zN_{\overline{P}}(L)z^{-1} = N_{\overline{P}}(L)$, which together with the fact $zI_iz^{-1} = I_i$ implies $z\overline{N}_iz^{-1} = z(N_{\overline{P}}(L) \cap I_i)z^{-1} = N_{\overline{P}}(L) \cap I_i = \overline{N}_i$.

Now we set:

$$L_P^+ := \{z \in L_P(L),\ zN_0z^{-1} \subseteq N_0\}$$

and $Z_{L_P}^+ := L_P^+ \cap Z_{L_P}(L)$. We will assume moreover the following hypothesis.

Hypothesis 4.2. *For any* $z \in Z_{L_P}^+$ *and* $i \in \mathbb{Z}_{\geq 0}$, *we have* $\overline{N}_i \subseteq z\overline{N}_iz^{-1}$.

Example 4.3. (1) Let $G = \mathrm{GL}_n$, P a parabolic subgroup containing the Borel subgroup B of upper triangular matrices, and let $L_P \cong \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$ be the Levi subgroup of P containing the diagonal subgroup T. Let $I_i := \{g \in \mathrm{GL}_n(\mathcal{O}_L),\ g \equiv 1 \pmod{\varpi_L^{i+1}}\}$, we have:

$$Z_{L_P}^+ = \{(a_1, \cdots, a_k) \in Z_{L_P}(L),\ \mathrm{val}_p(a_1) \geq \cdots \geq \mathrm{val}_p(a_k)\}$$

where $a_j \in L^\times$ is seen in (the center of) $\mathrm{GL}_{n_j}(L)$ by the diagonal map. It is straightforward to check that Hypothesis 4.2 is satisfied for $\{I_i\}_{i \in \mathbb{Z}_{\geq 0}}$.

(2) Let $G = \mathrm{GSp}_4$, P the Siegel (resp. Klingen) parabolic subgroup and $I_i := \{g \in \mathrm{GSp}_4(\mathcal{O}_L),\ g \equiv 1 \pmod{\varpi_L^{i+1}}\}$. The Iwahori decomposition of I_i in both cases follows from [70, (2.6) & (2.7)], one has in the Siegel case:

$$Z_{L_P}^+ = \{\mathrm{diag}(a_1, a_1, a_2, a_2) \in Z_{L_P}(L),\ \mathrm{val}_p(a_1) \geq \mathrm{val}_p(a_2)\},$$

and in the Klingen case:

$$Z_{L_P}^+ = \{\mathrm{diag}(a_1, a_2, a_2, a_2^2/a_1) \in Z_{L_P}(L),\ \mathrm{val}_p(a_1) \geq \mathrm{val}_p(a_2)\}.$$

Hypothesis 4.2 is again satisfied for $\{I_i\}_{i \in \mathbb{Z}_{\geq 0}}$.

4.2. The functor Ord_P

We review and/or prove useful results on the functor Ord_P of [40], [41].

Let A be a complete noetherian local $\mathcal{O}_E$-algebra with finite residue field, and $\mathfrak{m}_A$ be the maximal ideal of A. Let V be a smooth representation of $G(L)$ over A in the sense of [40, Def. 2.2.5]. Recall we have in particular $V \cong \varinjlim_n V[\mathfrak{m}_A^n]$. The A-submodule V^{N_0} of elements fixed by N_0 is equipped with a natural Hecke action of L_P^+ given by (cf. [40, Def. 3.1.3]):

$$z \cdot v := \sum_{x \in N_0 / z N_0 z^{-1}} \widetilde{x}(zv) \tag{4.1}$$

where $z \in L_P^+$, $v \in V^{N_0}$, and $\widetilde{x}$ is an arbitrary lift of x in N_0. Note that the A-module V^{N_0} is a smooth representation of L_0 over A. Following [40, Def. 3.1.9], we define:

$$\mathrm{Ord}_P(V) := \mathrm{Hom}_{A[Z_{L_P}^+]}\left(A[Z_{L_P}(L)], V^{N_0}\right)_{Z_{L_P}(L)-\mathrm{finite}}, \tag{4.2}$$

which is called the P-ordinary part of V. Here the A-module

$$\mathrm{Hom}_{A[Z_{L_P}^+]}(A[Z_{L_P}(L)], V^{N_0})$$

is naturally equipped with an A-linear action of $Z_{L_P}(L)$ given by $(z \cdot f)(x) := f(zx)$, and $(\cdot)_{Z_{L_P}(L)-\mathrm{finite}}$ denotes the A-submodule of locally $Z_{L_P}(L)$-finite elements (cf. [40, Def. 2.3.1 (2)]). By [40, Lem. 3.1.7],

$$\mathrm{Hom}_{A[Z_{L_P}^+]}(A[Z_{L_P}(L)], V^{N_0})$$

and $\mathrm{Ord}_P(V)$ are smooth representations of $L_P(L)$ over A. By [40, Thm. 3.3.3], if V is moreover admissible (cf. [40, Def. 2.2.9]), then $\mathrm{Ord}_P(V)$ is a smooth admissible representation of $L_P(L)$ over A. As in [40, Def. 3.1.10], we have the canonical lifting map:

$$\iota_{\mathrm{can}} : \mathrm{Ord}_P(V) \longrightarrow V^{N_0}, \ f \mapsto f(1) \tag{4.3}$$

which is L_P^+-linear, and injective if V is admissible (cf. [40, Thm. 3.3.3]). We put:

$$\mathrm{NOrd}_P(V) := \big\{v \in V^{N_0} \text{ such that there exists } z \in Z_{L_P}^+ \text{ with } z \cdot v = 0\big\}$$

which is an A-submodule of V^{N_0} stable by L_P^+. The following theorem is a consequence of the results in [40, § 3], but we include a proof.

Theorem 4.4. *Assume V is an admissible representation of $G(L)$, then we have:*

$$\mathrm{Ord}_P(V) \oplus \mathrm{NOrd}_P(V) \xrightarrow{\sim} V^{N_0}$$

as smooth representations of L_0, where $\mathrm{Ord}_P(V)$ is sent to V^{N_0} by ι_{can}.

Proof. We easily reduce to the case where V is annihilated by $\mathfrak{m}_A^n$ for a certain $n \in \mathbb{Z}_{>0}$.

(a) Set $V_i := V^{I_{i,i}}$, since V is smooth we have $V^{N_0} = \varinjlim_i V_i$. By Hypothesis 4.2 and [40, Lem. 3.3.2] (applied to $I_0 = I_1 = I_{i,i}$), we see that V_i is stable by the action of $Z_{L_P}^+$. Since V is admissible, V_i is a finitely generated A-module. Let B_i be the A-subalgebra of $\mathrm{End}_A(V_i)$ generated by $Z_{L_P}^+$, then B_i is a finite commutative A-algebra. Note that B_i is actually a finite $A/\mathfrak{m}_A^n$-algebra since V_i is annihilated by $\mathfrak{m}_A^n$, so in particular it is Artinian. For a maximal ideal $\mathfrak{m}$ of B_i, we call $\mathfrak{m}$ ordinary (resp. nonordinary) if $\mathrm{Image}(Z_{L_P}^+) \cap \mathfrak{m} = \emptyset$ (resp. $\mathrm{Image}(Z_{L_P}^+) \cap \mathfrak{m} \neq \emptyset$) where $\mathrm{Image}(Z_{L_P}^+)$ is the image of $Z_{L_P}^+$ in $\mathrm{End}_A(V_i)$ (or in B_i). Since B_i is artinian we have a natural decomposition:

$$B_i \cong \prod_{\mathfrak{m} \text{ ordinary}} (B_i)_{\mathfrak{m}} \times \prod_{\mathfrak{m} \text{ non ordinary}} (B_i)_{\mathfrak{m}} =: B_{i,\mathrm{ord}} \times B_{i,\mathrm{nord}}$$

and another decomposition:

$$V_i \cong (V_i)_{\mathrm{ord}} \oplus (V_i)_{\mathrm{nord}} := \prod_{\mathfrak{m} \text{ ordinary}} (V_i)_{\mathfrak{m}} \times \prod_{\mathfrak{m} \text{ non ordinary}} (V_i)_{\mathfrak{m}}. \tag{4.4}$$

Note that, for $v \in V_i$, we have $v \in (V_i)_{\mathrm{nord}}$ if and only if there exists $z \in Z_{L_P}^+$ such that $z \cdot v = 0$. In particular $(V_i)_{\mathrm{nord}} = \mathrm{NOrd}_P(V) \cap V_i$. Note also that V_i is stable by L_0 since $I_{i,i}$ is normalized by L_0. Since the action of L_0 and $Z_{L_P}^+$ commute, (4.4) is equivariant under the action of L_0.

(b) For $j > i$, the natural injection $V_i \hookrightarrow V_j$ is equivariant under the action of $Z_{L_P}^+$ and L_0. Therefore the restriction to the subspace V_i induces a surjection $\kappa_{j,i} : B_j \twoheadrightarrow B_i$ of finite $A/\mathfrak{m}_A^n$-algebras (it is surjective because both A-algebras are generated by the image of $Z_{L_P}^+$). For a maximal ideal $\mathfrak{n}$ of B_i, it is clear that $\mathfrak{n}$ is ordinary (resp. nonordinary) if and only if $\kappa_{j,i}^{-1}(\mathfrak{n})$ is ordinary (resp. nonordinary). Thus the inclusion $V_i \hookrightarrow V_j$ induces injections $(V_i)_{\mathrm{ord}} \hookrightarrow (V_j)_{\mathrm{ord}}$ and $(V_i)_{\mathrm{nord}} \hookrightarrow (V_j)_{\mathrm{nord}}$ which are equivariant under the action of L_0 and $Z_{L_P}^+$. From $(V_i)_{\mathrm{nord}} = \mathrm{NOrd}_P(V) \cap V_i$ in (a), we also see $\mathrm{NOrd}_P(V) \cong \varinjlim_i (V_i)_{\mathrm{nord}}$.

(c) By [40, Thm. 3.3.3], we have $\mathrm{Ord}_P(V) = \varinjlim_i \mathrm{Ord}_P(V)^{L_i}$ and ι_{can} is injective. Moreover, we have:

$$\begin{aligned}\mathrm{Ord}_P(V)^{L_i} &= \mathrm{Hom}_{A[Z_{L_P}^+]}\big(A[Z_{L_P}(L)], V^{L_i N_0}\big)_{Z_{L_P}(L)-\mathrm{finite}} \\ &= \mathrm{Hom}_{A[Z_{L_P}^+]}\big(A[Z_{L_P}(L)], V^{I_{i,i}}\big)\end{aligned} \tag{4.5}$$

where the first equality follows by definition (recall L_0, and hence L_i, normalize N_0 and commute with $Z_{L_P}^+$), and the second follows by the proof of *loc.cit* as we now explain. Since $V^{I_{i,i}}$ is a finitely generated A-module, any element in $\mathrm{Hom}_{A[Z_{L_P}^+]}(A[Z_{L_P}(L)], V^{I_{i,i}})$ is locally $Z_{L_P}(L)$-finite, hence we have an inclusion:

$$\mathrm{Hom}_{A[Z_{L_P}^+]}\big(A[Z_{L_P}(L)], V^{I_{i,i}}\big) \subseteq \mathrm{Hom}_{A[Z_{L_P}^+]}\big(A[Z_{L_P}(L)], V^{L_i N_0}\big)_{Z_{L_P}(L)-\mathrm{finite}}. \tag{4.6}$$

However, by the proof of [40, Thm. 3.3.3], we have $\iota_{\mathrm{can}}(\mathrm{Ord}_P(V)^{L_i}) \subseteq V^{I_{i,i}}$, in other words, any element in the right hand side set of (4.6) has image in $V^{I_{i,i}}$ and thus is contained in the left hand side (note that by Hypothesis 4.2, the A-module U in the proof of [40, Thm. 3.3.3] is actually equal to $V^{I_{j,j}}$ with the notation of *loc. cit.*).

(d) Combining (4.5) with the isomorphism at the end of the proof of [40, Lem. 3.1.5] (applied to $U = V^{I_{i,i}} = V_i$), the map ι_{can} induces an isomorphism $\mathrm{Ord}_P(V)^{L_i} \xrightarrow{\sim} (V_i)_{\mathrm{ord}}$ which is equivariant under the action of L_0 and $Z_{L_P}^+$. Thus we deduce $\mathrm{Ord}_P(V) \cong \varinjlim_i (V_i)_{\mathrm{ord}}$ and together with (4.4) and (b):

$$V^{N_0} \cong \varinjlim_i V_i \cong \varinjlim_i \big((V_i)_{\mathrm{ord}} \oplus (V_i)_{\mathrm{nord}}\big) \cong \mathrm{Ord}_P(V) \oplus \mathrm{NOrd}_P(V)$$

which concludes the proof. ☐

Corollary 4.5. *Assume* $A := \mathcal{O}_E/\varpi_E^n$ *for some* $n > 0$*,* V *is an admissible representation of* $G(L)$ *over* A *and* V *is an injective object in the category of smooth representations of* I_0 *over* A*. Then* $\mathrm{Ord}_P(V)$ *is an injective object in the category of smooth representations of* L_0 *over* A*.*

Proof. By the same argument as in the proof of [42, Cor. 5.3.19], there exists $r > 0$ such that V is a direct factor of $\mathcal{C}(I_0, A)^{\oplus r}$ as a representation of I_0 where $\mathcal{C}(I_0, A)$ (= the A-module of continuous, hence locally constant, functions from I_0 to A with the discrete topology on the latter) is endowed with the left action of I_0 by right translation. Since I_0 admits an Iwahori decomposition, we deduce from this that V^{N_0} is a direct factor of:

$$W := (\mathcal{C}(I_0, A)^{N_0})^{\oplus r} \cong \big(\mathcal{C}(\overline{N}_0, A) \otimes_{\mathcal{O}_E/\varpi_E^n} \mathcal{C}(L_0, A)\big)^{\oplus r} \tag{4.7}$$

where L_0 acts on the latter by $l(f \otimes h) := f \otimes l(h)$. By [41, Prop. 2.1.3], W is an injective object in the category of smooth representations of L_0 over A. It follows from Theorem 4.4 that $\mathrm{Ord}_P(V)$ is a direct factor of W, and hence also an injective object. ☐

Let now V be a ϖ_E-adically continuous representation of G over A in the sense of [40, Def. 2.4.1]. Then V/ϖ_E^n is a smooth representation of G over A/ϖ_E^n for all $n \in \mathbb{Z}_{>0}$. Following [40, Def. 3.4.1], we define:

$$\mathrm{Ord}_P(V) := \varprojlim_n \mathrm{Ord}_P(V/\varpi_E^n V) \tag{4.8}$$

which is a ϖ_E-adically continuous representation of $L_P(L)$ over A (cf. [40, Prop. 3.4.6]). We have the canonical lifting map (cf. [40, (3.4.7)]):

$$\iota_{\mathrm{can}} : \mathrm{Ord}_P(V) \longrightarrow V^{N_0} \tag{4.9}$$

which is L_P^+-equivariant. By [40, Thm. 3.4.8], if V is moreover admissible ([40, Def. 2.4.7]), $\mathrm{Ord}_P(V)$ is also admissible and ι_{can} is a closed embedding (where the target and the source are equipped with the ϖ_E-adic topology).

Let V be a unitary Banach space representation of $G(L)$ over E and V^0 an open bounded $G(L)$-invariant lattice of V (i.e. a unit ball preserved by $G(L)$, which exists by definition as the representation is unitary). Then V^0 is a ϖ_E-adically continuous representation of G over $\mathcal{O}_E$ and we put $\mathrm{Ord}_P(V) := \mathrm{Ord}_P(V^0)[1/p]$, which is easily checked to be independent of the choice of V^0. For any compact group K we endow $\mathcal{C}(K, \mathcal{O}_E)$ and $\mathcal{C}(K, E)$ with the left action of K by right translation on functions.

Corollary 4.6. *Assume moreover that* $V^0|_{I_0}$ *is isomorphic to a direct factor of* $\mathcal{C}(I_0, \mathcal{O}_E)^{\oplus r}$ *for some integer* $r > 0$. *Then* $\mathrm{Ord}_P(V^0)|_{L_0}$ *(resp.* $\mathrm{Ord}_P(V)|_{L_0}$*) is isomorphic to a direct factor of* $\mathcal{C}(L_0, \mathcal{O}_E)^{\oplus r}$ *(resp.* $\mathcal{C}(L_0, E)^{\oplus r}$*) for some integer* $s \geq 0$.

Proof. Let $n_1, n_2 \in \mathbb{Z}_{>0}$ with $n_2 > n_1$ and consider the exact sequence:

$$0 \to V^0/\varpi_E^{n_2-n_1} \xrightarrow{\varpi_E^{n_1}} V^0/\varpi_E^{n_2} \to V^0/\varpi_E^{n_1} \to 0.$$

Since $V^0|_{I_0}$ is a direct factor of $\mathcal{C}(I_0, \mathcal{O}_E)^{\oplus r}$, arguing as in (4.7) we deduce an exact sequence (which is equivariant for the action of $Z_{L_P}^+$ and L_0):

$$0 \to (V^0/\varpi_E^{n_2-n_1})^{N_0} \xrightarrow{\varpi_E^{n_1}} (V^0/\varpi_E^{n_2})^{N_0} \to (V^0/\varpi_E^{n_1})^{N_0} \to 0.$$

Together with Theorem 4.4, it follows that

$$\mathrm{Ord}_P(V^0/\varpi_E^{n_2})/\varpi_E^{n_1} \cong \mathrm{Ord}_P(V^0/\varpi_E^{n_1}).$$

Moreover, from Corollary 4.5 we deduce that the dual

$$\mathrm{Hom}_{\mathcal{O}_E}(\mathrm{Ord}_P(V^0/\varpi_E^{n}), \mathcal{O}_E/\varpi_E^{n})$$

is a finitely generated projective $\mathcal{O}_E/\varpi_E^n[[L_0]]$-module. By a projective limit argument, it is then not difficult to deduce that $\mathrm{Hom}_{\mathcal{O}_E}(\mathrm{Ord}_P(V^0), \mathcal{O}_E)$ is also a finitely generated projective $\mathcal{O}_E[[L_0]]$-module. Dualizing back using [73, Lem. 2.1] the corollary follows. □

4.3. Ordinary parts of locally algebraic representations

We review and generalize the ordinary part of a locally algebraic representation of $G(L)$ that admits an invariant lattice (see [42, § 5.6]).

We keep the notation of §§ 4.1 & 4.2 and now assume that G is split. We fix a split torus T over L and a Borel subgroup containing T such that $B \subseteq P$ (where P is the parabolic subgroup of *loc. cit.*). We let V_∞ be a smooth *admissible* representation of $G(L)$ over E, $L(\lambda)$ the irreducible $\mathbb{Q}_p$-algebraic representation of $G(L)$ over E of highest weight $\lambda \in \mathrm{Hom}(\mathrm{Res}_{L/\mathbb{Q}_p} T, \mathbb{G}_{\mathrm{m}/\mathbb{Q}_p})$ where λ is dominant with respect to $\mathrm{Res}_{L/\mathbb{Q}_p} B$ and we set:

$$V := V_\infty \otimes_E L(\lambda).$$

We denote by $L_P(\lambda)$ the irreducible $\mathbb{Q}_p$-algebraic representation of L_P over E of highest weight λ and by $\delta_{L_P,\lambda}$ the central character of $L_P(\lambda)$. Note that we have $L_P(\lambda) \cong L(\lambda)^{N_0} \cong L(\lambda)^{N_P(L)}$ and by [38, Prop. 4.3.6]:

$$J_P(V) \cong J_P(V_\infty) \otimes_E L_P(\lambda) \tag{4.10}$$

where $J_P(V)$ on the left is the Jacquet-Emerton functor of the locally algebraic representation V relative to the parabolic subgroup $P(L)$ and $J_P(V_\infty)$ is the usual Jacquet functor of the smooth representation V_∞.

For $i \geq 0$ consider:

$$V_i := V_\infty^{I_{i,i}} \otimes_E L_P(\lambda) \subseteq V^{N_0} \cong V_\infty^{N_0} \otimes_E L_P(\lambda)$$

which is finite dimensional over E since V_∞ is admissible. We equip $V_\infty^{N_0}$ and V^{N_0} with the Hecke action of L_P^+ given by (4.1). Note that we have $z \cdot (v \otimes u) = (z \cdot v) \otimes (zu)$ for $z \in L_P^+$, $v \in V_\infty^{N_0}$ and $u \in L_P(\lambda)$. In particular by Hypothesis 4.2, $V_i \subseteq V^{N_0}$ is invariant under this $Z_{L_P}^+$-action. Denote by B_i the E-subalgebra of $\mathrm{End}_E(V_i)$ generated by the operators in $Z_{L_P}^+$, then B_i is an Artinian E-algebra. Similarly to what we did in the proof of Theorem 4.4, a maximal ideal $\mathfrak{m}$ of B_i is called *of finite slope* if $\mathrm{Image}(Z_{L_P}^+) \cap \mathfrak{m} = \emptyset$ (inside $\mathrm{End}_E(V_i)$). Let $\mathfrak{m}$ be such a maximal ideal of finite slope and consider:

$$Z_{L_P}^+ \longrightarrow B_i \twoheadrightarrow B_i/\mathfrak{m} \hookrightarrow \overline{\mathbb{Q}_p}.$$

Note that the image of $Z_{L_P}^+$ lies in $\overline{\mathbb{Q}_p}^\times$. We call $\mathfrak{m}$ *of slope zero* if the above composition has image contained in the units $\overline{\mathbb{Z}_p}^\times$ (this is independent of the choice of the last embedding). Denote by $(V_i)_*$ with $* \in \{\mathrm{fs},\ \mathrm{null},\ 0,\ >0\}$ the direct sum of the localizations of V_i at maximal ideals which are respectively: of finite slope, not of finite slope, of slope zero, not of slope zero. We have thus:

$$V_i \cong (V_i)_{\mathrm{fs}} \oplus (V_i)_{\mathrm{null}} \cong (V_i)_0 \oplus (V_i)_{>0} \tag{4.11}$$

and note that $v \in (V_i)_{\mathrm{null}}$ if and only if there exists $z \in Z_{L_P}^+$ such that $z \cdot v = 0$. Moreover, as in the proof of Theorem 4.4, for $j \geq i$ the natural injection $V_i \hookrightarrow V_j$ induces a $Z_{L_P}^+$-equivariant map for $* \in \{\mathrm{fs}, \mathrm{null}, 0, >0\}$:

$$(V_i)_* \longhookrightarrow (V_j)_*.$$

For $* \in \{\mathrm{fs}, 0\}$, this action (uniquely) extends to $Z_{L_P}(L)$ since the action of $Z^+_{L_P}$ on $(V_i)_*$ is invertible. For $* \in \{\mathrm{fs}, 0, \mathrm{null}, > 0\}$, we set:

$$(V^{N_0})_* := \varinjlim_i (V_i)_* \tag{4.12}$$

which is an E-vector subspace of V^{N_0} stable by L^+_P (indeed, each $(V_i)_*$ is a generalized eigenspace of some sort for the action of $Z^+_{L_P}$ on V_i, and the action of L^+_P on $V^{N_0} = \varinjlim_i V_i$ commutes with that of $Z^+_{L_P}$, so preserves generalized eigenspaces of $Z^+_{L_P}$ even though it may send a vector of V_i to V_j for some $j \gg i$). Moreover, for $* \in \{\mathrm{fs}, 0\}$ this action of L^+_P on $(V^{N_0})_*$ uniquely extends to $L_P(L)$ by [38, Prop. 3.3.6]. The decomposition (4.11) induces L^+_P-equivariant decompositions:

$$V^{N_0} \cong (V^{N_0})_{\mathrm{fs}} \oplus (V^{N_0})_{\mathrm{null}} \cong (V^{N_0})_0 \oplus (V^{N_0})_{>0}. \tag{4.13}$$

It follows from (4.10), $V^{N_0} \cong V^{N_0}_\infty \otimes_E L_P(\lambda)$ and the proof of [38, Prop. 4.3.2] (we leave here the details to the reader) that we have an isomorphism of locally algebraic representations of $L_P(L)$ (called the canonical lifting):

$$J_P(V) \xrightarrow{\sim} (V^{N_0})_{\mathrm{fs}}(\delta_P) \tag{4.14}$$

where (δ_P) means the twist by the modulus character δ_P.

If W is an E-vector space, recall an $\mathcal{O}_E$-lattice of W is by definition an $\mathcal{O}_E$-submodule which generates W over E and doesn't contain any nonzero E-line. If W is a $E[Z_{L_P}(L)]$-module such that the $Z_{L_P}(L)$-orbit of any element of W is of finite dimension, by the very same construction as above we have a decomposition $W = W_0 \oplus W_{>0}$ analogous to (4.13).

Lemma 4.7. *Let W be an E-vector space equipped with a $Z^+_{L_P}$-action and let $f : W \to V^{N_0}$ be an E-linear $Z^+_{L_P}$-equivariant map.*

(1) If W is moreover an $E[Z_{L_P}(L)]$-module, then f factors through a $Z_{L_P}(L)$-equivariant map $f : W \longrightarrow (V^{N_0})_{\mathrm{fs}}$.

(2) If W is an $E[Z_{L_P}(L)]$-module such that the $Z_{L_P}(L)$-orbit of any element of W is of finite dimension, then f restricts to a $Z_{L_P}(L)$-equivariant map $W_0 \longrightarrow (V^{N_0})_0$. In particular, if W admits a $Z_{L_P}(L)$-invariant $\mathcal{O}_E$-lattice, then f factors through $W \longrightarrow (V^{N_0})_0$.

Proof. (1) For $v \in V^{N_0}$, we have $v \in (V^{N_0})_{\mathrm{null}}$ if and only if there exists $z \in Z^+_{L_P}$ such that $z \cdot v = 0$, which easily implies (1) using the first isomorphism in (4.13).

(2) From the assumption on W we can write $W = \varinjlim_\alpha (W_\alpha)$ where the $W_\alpha \subseteq W$ are finite dimensional and preserved by $Z_{L_P}(L)$. By (1) it is enough to prove $f((W_\alpha)_0) \subseteq (V^{N_0})_0$, but this is clear from the definition. If W^0 is a $Z_{L_P}(L)$-invariant $\mathcal{O}_E$-lattice of W, then $W^0 \cap W_\alpha$ is a $Z_{L_P}(L)$-invariant $\mathcal{O}_E$-lattice in W_α which easily implies $(W_\alpha)_0 = W_\alpha$ and (2) follows. □

Remark 4.8. It easily follows from the first statement in Lemma 4.7(2) and the fact the $L_P(L)$-representations $(V^{N_0})_{\mathrm{fs}}$ doesn't depend on the choice of N_0 up to isomorphism (see [38, Prop. 3.4.11]) that the $L_P(L)$-representation $(V^{N_0})_0$ also doesn't depend on the choice of N_0 up to isomorphism.

Assume from now on that V is a *unitary* $G(L)$-representation, i.e. admits an $\mathcal{O}_E$-lattice V^0 which is stable by $G(L)$, and set $V_i^0 := V_i \cap V^0$, which is thus an $\mathcal{O}_E$-lattice of V_i stable by $Z_{L_P}^+$ (note that $(V^0)^{N_0} = \varinjlim_i V_i^0$). Denote by A_i the $\mathcal{O}_E$-subalgebra of $\mathrm{End}_{\mathcal{O}_E}(V_i^0)$ generated by $Z_{L_P}^+$. Then A_i is an $\mathcal{O}_E$-algebra which is a free $\mathcal{O}_E$-module of finite type. We have $B_i \cong A_i \otimes_{\mathcal{O}_E} E$ and $A_i = \prod_{\mathfrak{n}} (A_i)_{\mathfrak{n}}$ where the product runs over the maximal ideals $\mathfrak{n}$ of A_i. As in the proof of Theorem 4.4, a maximal ideal $\mathfrak{n}$ of A_i is called ordinary if $\mathrm{Image}(Z_{L_P}^+) \cap \mathfrak{n} = \emptyset$. And we put:

$$(V_i^0)_{\mathrm{ord}} := \oplus_{\mathfrak{n}\ \mathrm{ordinary}} (V_i^0)_{\mathfrak{n}} \qquad (V_i^0)_{\mathrm{nord}} := \oplus_{\mathfrak{n}\ \mathrm{nonordinary}} (V_i^0)_{\mathfrak{n}}.$$

We have $V_i^0 \cong (V_i^0)_{\mathrm{ord}} \oplus (V_i^0)_{\mathrm{nord}}$ and we set $(V_i)_{\mathrm{ord}} := (V_i^0)_{\mathrm{ord}} \otimes_{\mathcal{O}_E} E$.

Lemma 4.9. *We have $(V_i^0)_{\mathrm{ord}} \cong V_i^0 \cap (V_i)_0$, and hence $(V_i)_{\mathrm{ord}} \cong (V_i)_0$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of B_i and $\mathfrak{n}$ the unique maximal ideal of A_i containing $\mathfrak{m} \cap A_i$ and $j : B_i/\mathfrak{m} \hookrightarrow \overline{\mathbb{Q}_p}$ an embedding as above. Then the restriction of j to $A_i/(\mathfrak{m} \cap A_i)$ induces $j : A_i/(\mathfrak{m} \cap A_i) \hookrightarrow \overline{\mathbb{Z}_p}$ and we have $\mathfrak{n}/(\mathfrak{m} \cap A_i) = j^{-1}(\mathfrak{m}_{\overline{\mathbb{Z}_p}})$ (where $\mathfrak{m}_{\overline{\mathbb{Z}_p}}$ is the maximal ideal of $\overline{\mathbb{Z}_p}$). It is then easy to see that $\mathfrak{n}$ is ordinary if and only if $\mathfrak{m}$ is of slope zero. The inclusions $(V_i^0)_{\mathfrak{n}} \subseteq (V_i)_{\mathfrak{n}} \subseteq (V_i)_{\mathfrak{m}}$ thus imply $(V_i^0)_{\mathrm{ord}} \subseteq V_i^0 \cap (V_i)_0$. On the other hand, we have $V_i^0 \cap (V_i)_{\mathfrak{m}} \subseteq (V_i^0)_{\mathfrak{n}}$ and thus $V_i^0 \cap (V_i)_0 \subseteq (V_i^0)_{\mathrm{ord}}$. The lemma follows. □

The action of $Z_{L_P}^+$ on $(V_i^0)_{\mathrm{ord}}$ being invertible, it (uniquely) extends to an action of $Z_{L_P}(L)$ and the isomorphism $(V_i^0)_{\mathrm{ord}} \otimes_{\mathcal{O}_E} E \cong (V_i)_0$ of Lemma 4.9 is equivariant under the action of $Z_{L_P}(L)$. We set (using Lemma 4.9 for the second equality):

$$\mathrm{Ord}_P(V^0) := \varinjlim_i (V_i^0)_{\mathrm{ord}} = V^0 \cap (V^{N_0})_0 \longrightarrow (V^0)^{N_0} = \varinjlim_i (V_i^0) \tag{4.15}$$

and $\mathrm{Ord}_P(V) := \mathrm{Ord}_P(V^0)\otimes_{\mathcal{O}_E} E \hookrightarrow V^{N_0}$. The combined actions of $Z_{L_P}(L)$ and of L_P^+ on $\mathrm{Ord}_P(V^0)$ (the action of L_P^+ being induced by that on $(V^0)^{N_0}$) imply with [38, Prop. 3.3.6] that this L_P^+-action uniquely extends to $L_P(L)$. We deduce that $\mathrm{Ord}_P(V)$ is a unitary representation of $L_P(L)$ over E and we call it the *P-ordinary part* of V.

Lemma 4.10. *We have an isomorphism* $\mathrm{Ord}_P(V) \cong (V^{N_0})_0$*, in particular the* $L_P(L)$*-representation* $\mathrm{Ord}_P(V)$ *is independent of the choice of* V^0 *and* N_0*, and* $(V^{N_0})_0$ *is a unitary representation of* $L_P(L)$ *over* E*.*

Proof. The isomorphism follows from the second equality in (4.15). The lemma follows since $(V^{N_0})_0$ doesn't depend on any lattice. ☐

Remark 4.11. If we drop the assumption that V admits an invariant $\mathcal{O}_E$-lattice, then the $L_P(L)$-representation $(V^{N_0})_0$ might not be a unitary representation of $L_P(L)$ over E.

Lemma 4.12. *Let* $P' \supseteq P$ *be another parabolic subgroup of* G *and* $L_{P'}$ *the Levi subgroup of* P' *(containing* L_P*). Then we have:*

$$\mathrm{Ord}_P(V) \cong \mathrm{Ord}_{P\cap L_{P'}}\big(\mathrm{Ord}_{P'}(V)\big).$$

Proof. Let $N_0' := N_0 \cap N_{P'}(L)$ and $N_0'' := N_0 \cap N_{P\cap L_{P'}}(L)$. We have $N_0 \cong N_0' \rtimes N_0''$ and thus an isomorphism $V^{N_0} \cong (V^{N_0'})^{N_0''}$. By Lemma 4.10 and (the first statement in) Lemma 4.7(2), we see that the embedding $(((V^{N_0'})_0)^{N_0''})_0 \hookrightarrow V^{N_0}$ factors through $(V^{N_0})_0$. On the other hand, we have an embedding $(V^{N_0})_0 \hookrightarrow (V^{N_0'})_0$ (using $Z_{L_{P'}}(L) \subseteq Z_{L_P}(L)$) which factors through an embedding $(V^{N_0})_0 \hookrightarrow (((V^{N_0'})_0)^{N_0''})_0$ using $L_{P\cap L_{P'}} \cong L_P$ and (again the first statement in) Lemma 4.7(2). We deduce an isomorphism $(V^{N_0})_0 \cong (((V^{N_0'})_0)^{N_0''})_0$ whence the result by Lemma 4.10. ☐

Remark 4.13. If we drop the assumption that V admits an invariant $\mathcal{O}_E$-lattice, the proof of Lemma 4.12 still gives $(V^{N_0})_0 \cong (((V^{N_0'})_0)^{N_0''})_0$ (with the notation in the proof of *loc. cit.*). And if we use Lemma 4.7(1) instead of Lemma 4.7(2), the same proof gives $(V^{N_0})_{\mathrm{fs}} \cong (((V^{N_0'})_{\mathrm{fs}})^{N_0''})_{\mathrm{fs}}$ (which can also be deduced from (4.14)).

Fix $n \in \mathbb{Z}_{>0}$ and consider V^0/ϖ_E^n which is a smooth representation of $G(L)$ over $\mathcal{O}_E/\varpi_E^n$. We have $(V^0)^{N_0}/\varpi_E^n = \varinjlim_i (V_i^0/\varpi_E^n)$. For $i \in \mathbb{Z}_{\geq 0}$ the quotient A_i/ϖ_E^n of A_i is isomorphic to the $\mathcal{O}_E/\varpi_E^n$-subalgebra of $\mathrm{End}_{\mathcal{O}_E/\varpi_E^n}(V_i^0/\varpi_E^n)$ generated by $Z_{L_P}^+$. We have a natural bijection between the maximal ideals $\mathfrak{m}$ of A_i and the maximal ideals $\overline{\mathfrak{m}}$ of A_i/ϖ_E^n (since any maximal ideal of A_i contains ϖ_E). And it is easy to see that $\mathfrak{m} \subset A_i$ is ordinary if and only if $\overline{\mathfrak{m}}$ is ordinary (see the proof of Theorem 4.4). We deduce an isomorphism of A_i/ϖ_E^n-modules (see (4.4)):

$$(V_i^0)_{\mathrm{ord}}/\varpi_E^n \xrightarrow{\sim} (V_i^0/\varpi_E^n)_{\mathrm{ord}}. \tag{4.16}$$

Lemma 4.14. *We have an $L_P(L)$-equivariant injection where $\mathrm{Ord}_P(V^0/\varpi_E^n)$ is defined as in (4.2):*

$$\varinjlim_i (V_i^0/\varpi_E^n)_{\mathrm{ord}} \lhook\joinrel\longrightarrow \mathrm{Ord}_P(V^0/\varpi_E^n). \tag{4.17}$$

Moreover, the composition of (4.17) with the canonical lifting (4.3) gives the natural injection $\varinjlim_i (V_i^0/\varpi_E^n)_{\mathrm{ord}} \lhook\joinrel\longrightarrow (V^0/\varpi_E^n)^{N_0}$.

Proof. For any $i \geq 0$, by the last isomorphism in the proof of [40, Lem. 3.1.5] we have:

$$\begin{aligned}(V_i^0/\varpi_E^n)_{\mathrm{ord}} &\cong \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^n[Z_{L_P}^+]}\big(\mathcal{O}_E/\varpi_E^n[Z_{L_P}(L)], V_i^0/\varpi_E^n\big)\\ &\cong \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^n[Z_{L_P}^+]}\big(\mathcal{O}_E/\varpi_E^n[Z_{L_P}(L)], V_i^0/\varpi_E^n\big)_{Z_{L_P}(L)-\mathrm{finite}}\\ &\hookrightarrow \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^n[Z_{L_P}^+]}\big(\mathcal{O}_E/\varpi_E^n[Z_{L_P}(L)], (V^0/\varpi_E^n)^{N_0}\big)_{Z_{L_P}(L)-\mathrm{finite}}\end{aligned}$$

where the second isomorphism follows from the fact V_i^0 is of finite rank over $\mathcal{O}_E$. The first part of the lemma follows. By unwinding the maps, the second part also easily follows. ☐

Remark 4.15. (1) The embedding

$$\varinjlim_i V_i^0/\varpi_E^n \cong (V^0)^{N_0}/\varpi_E^n \lhook\joinrel\longrightarrow (V^0/\varpi_E^n)^{N_0}$$

is not surjective in general. Consequently (e.g. by the proof of Lemma 4.14), (4.17) might not be surjective in general.

(2) If the inclusion $V_i^0/\varpi_E^n \hookrightarrow (V^0/\varpi_E^n)^{I_{i,i}}$ is an isomorphism for all i (which in particular implies $(V^0)^{N_0}/\varpi_E^n \xrightarrow{\sim} (V^0/\varpi_E^n)^{N_0}$ and that the $G(L)$-representation V^0/ϖ_E^n is admissible), it follows from (4.5) and the proof of Lemma 4.14 that (4.17) is an isomorphism.

Lemma 4.16. *We have a natural $L_P(L)$-equivariant injection:*

$$\mathrm{Ord}_P(V^0) \lhook\joinrel\longrightarrow \varprojlim_n \mathrm{Ord}_P(V^0/\varpi_E^n) = \mathrm{Ord}_P(\widehat{V}^0) \quad (\textit{see } (4.8)) \tag{4.18}$$

where $\widehat{V}^0 := \varprojlim_n V^0/\varpi_E^n$. *Moreover, the composition of (4.18) with the (projective limit over* n *of the) canonical lifting (4.9) coincides with the composition of the natural injections:*

$$\mathrm{Ord}_P(V^0) \hookrightarrow (V^0)^{N_0} \hookrightarrow (\widehat{V}^0)^{N_0}.$$

Proof. For any $n \in \mathbb{Z}_{>0}$, by (4.15) and (4.16) we have:

$$\mathrm{Ord}_P(V^0) \cong \varinjlim_i (V_i^0)_{\mathrm{ord}} \twoheadrightarrow \varinjlim_i (V_i^0/\varpi_E^n)_{\mathrm{ord}}.$$

It is easy to see $(\varpi_E^n V_{i+1}^0) \cap V_i^0 = (\varpi_E^n V^0) \cap V_i^0 = \varpi_E^n V_i^0$. Hence the above surjection induces an isomorphism $\mathrm{Ord}_P(V^0)/\varpi_E^n \xrightarrow{\sim} \varinjlim_i (V_i^0/\varpi_E^n)_{\mathrm{ord}}$. We also have $\cap_n \varpi_E^n \mathrm{Ord}_P(V^0) = 0$ since the same holds for V^0. Thus we obtain an injection:

$$\mathrm{Ord}_P(V^0) \hookrightarrow \varprojlim_n \big(\mathrm{Ord}_P(V^0)/\varpi_E^n\big) \cong \varprojlim_n \big(\varinjlim_i (V_i^0/\varpi_E^n)_{\mathrm{ord}}\big).$$

By (4.17) and taking the projective limit over n, (4.18) follows. The second part of the lemma follows from the second part of Lemma 4.14. ☐

Remark 4.17. By (4.3) and Remark 4.15(2), if $V_i^0/\varpi_E^n \xrightarrow{\sim} (V^0/\varpi_E^n)^{I_{i,i}}$ for all i, then we see that (4.18) has dense image where $\mathrm{Ord}_P(\widehat{V}^0)$ is endowed with the ϖ_E-adic topology.

The proof of the following lemma is straightforward, we omit it.

Lemma 4.18. *Let* W *be a unitary Banach representation of* $G(L)$ *over* E, $W^0 \subset W$ *an open bounded* $G(L)$*-invariant lattice and* $f : V^0 \to W^0$ *an* $\mathcal{O}_E$*-linear* $G(L)$*-equivariant morphism, which induces a* $G(L)$*-equivariant morphism* $f : V \to W$. *Then* f *induces an* $L_P(L)$*-equivariant morphism:*

$$\mathrm{Ord}_P(V^0) \longrightarrow \mathrm{Ord}_P(W^0) \quad (\textit{resp. } \mathrm{Ord}_P(V) \longrightarrow \mathrm{Ord}_P(W)) \tag{4.19}$$

such that the following diagram commutes (resp. with V^0, W^0 *replaced by* V, W*):*

$$\begin{tikzcd} \mathrm{Ord}_P(V^0) \arrow[r] \arrow[d] & \mathrm{Ord}_P(W^0) \arrow[d] \\ (V^0)^{N_0} \arrow[r] & (W^0)^{N_0}. \end{tikzcd} \tag{4.20}$$

Moreover, if f *is injective and* $V^0 = W^0 \cap V$, *then the morphisms in (4.19) are injective.*

4.4. An adjunction property

We study some adjunction property of the functor $\mathrm{Ord}_P(\cdot)$ of § 4.3 on locally algebraic representations.

We keep the notation of §§ 4.1, 4.2 & 4.3. If U is any E-vector space, denote by $\mathcal{C}_c^\infty(N_P(L), U)$ the E-vector space of U-valued locally constant functions with compact support in $N_P(L)$ endowed with the left action of $N_P(L)$ by right translation on function. If U_∞ is a smooth representation of $L_P(L)$ over E, recall that there is a natural $N_P(L)$-equivariant injection:

$$\mathcal{C}_c^\infty(N_P(L), U_\infty) \lhook\joinrel\longrightarrow \big(\mathrm{Ind}_{\overline{P}(L)}^{G(L)} U_\infty\big)^\infty \tag{4.21}$$

sending $f \in \mathcal{C}_c^\infty(N_P(L), U_\infty)$ to $F \in (\mathrm{Ind}_{\overline{P}(L)}^{G(L)} U_\infty)^\infty$ such that:

$$F(g) = \begin{cases} \overline{p}(f(n)) & \text{for } g = \overline{p}n \in \overline{P}(L)N_P(L) \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.19. *Let U_∞ be a smooth admissible representation of $L_P(L)$ over E and assume that $U := U_\infty \otimes_E L_P(\lambda)$ is unitary as representation of $L_P(L)$ ($L_P(\lambda)$ as in the beginning of § 4.3).*

(1) The locally algebraic representation $(\mathrm{Ind}_{\overline{P}(L)}^{G(L)} U_\infty)^\infty \otimes_E L(\lambda)$ is unitary as representation of $G(L)$.

(2) There is a natural $L_P(L)$-equivariant injection:

$$U = U_\infty \otimes_E L_P(\lambda) \lhook\joinrel\longrightarrow \mathrm{Ord}_P\big((\mathrm{Ind}_{\overline{P}(L)}^{G(L)} U_\infty)^\infty \otimes_E L(\lambda)\big) \tag{4.22}$$

such that the composition of (4.22) with the natural injection (see just after (4.15)):

$$\mathrm{Ord}_P\big((\mathrm{Ind}_{\overline{P}(L)}^{G(L)} U_\infty)^\infty \otimes_E L(\lambda)\big) \lhook\joinrel\longrightarrow \big((\mathrm{Ind}_{\overline{P}(L)}^{G(L)} U_\infty)^\infty \otimes_E L(\lambda)\big)^{N_0}$$

has image in

$$\mathcal{C}_c^\infty(N_P(L), U)^{N_0} \cong (\mathcal{C}_c^\infty(N_P(L), U^\infty) \otimes_E L(\lambda))^{N_0}$$

via (4.21) (tensored with $L(\lambda)$) and maps $u \in U$ to the unique function $f_u \in \mathcal{C}_c^\infty(N_P(L), U)^{N_0}$ with $f_u(n) = u$ for all $n \in N_0$ and $f_u(n) = 0$ otherwise.

Proof. For simplicity, we write $V := (\mathrm{Ind}_{\overline{P}(L)}^{G(L)} U_\infty)^\infty \otimes_E L(\lambda)$. Let U^0 be an $L_P(L)$-invariant $\mathcal{O}_E$-lattice of U and $\widehat{U^0} := \varprojlim_n U^0/\varpi_E^n$. We have $G(L)$-equivariant embeddings:

$$V \longrightarrow \big(\mathrm{Ind}_{\overline{P}(L)}^{G(L)} U\big)^{\mathrm{an}} \hookrightarrow \big(\mathrm{Ind}_{\overline{P}(L)}^{G(L)} \widehat{U^0} \otimes_{\mathcal{O}_E} E\big)^{\mathcal{C}^0}. \tag{4.23}$$

Since the right hand side of (4.23) has an obvious invariant lattice given by $(\mathrm{Ind}_{\overline{P}(L)}^{G(L)} \widehat{U^0})^{\mathcal{C}^0}$, its intersection with the left hand side also gives an invariant lattice on V, hence V is unitary. We have:

$$U \xrightarrow{\sim} J_P\big(\mathcal{C}_c^\infty(N_P(L), U)\big)(\delta_P^{-1}) \longrightarrow J_P(V)(\delta_P^{-1})$$

where the first isomorphism follows from [38, Lem. 3.5.2] (the above action of $N_P(L)$ on $\mathcal{C}_c^\infty(N_P(L), U)$ being extended to $P(L)$ as in [38, § 3.5]) and the second injection follows from the left exactness of $J_P(\cdot)$. Since U is unitary, by Lemma 4.7(2) and Lemma 4.10 we deduce an injection:

$$U \longrightarrow \mathrm{Ord}_P(V)\big(\longrightarrow J_P(V)(\delta_P^{-1}) \longrightarrow V^{N_0}\big)$$

(recall the second embedding follows from (4.14) and the third from (4.13)). Moreover the composition is equal to the composition:

$$U \xrightarrow{\sim} J_P\big(\mathcal{C}_c^\infty(N_P(L), U)\big)(\delta_P^{-1}) \longrightarrow \mathcal{C}_c^\infty(N_P(L), U)^{N_0} \longrightarrow V^{N_0}$$

sending $u \in U$ to $f_u \in \mathcal{C}_c^\infty(N_P(L), U)^{N_0}$ as in the statement of the lemma (see [38, § 3.5], in particular the proof of [38, Lem. 3.5.2], see also the beginning of [37, § 2.8]). ☐

Lemma 4.20. *Keep the notation and assumptions of Lemma 4.19 and let U^0 be an $L_P(L)$-invariant $\mathcal{O}_E$-lattice of U and $\widehat{U} := (\varprojlim_n U^0/\varpi_E^n) \otimes_{\mathcal{O}_E} E$. Assume that $\widehat{U}$ is an admissible Banach representation of $G(L)$ over E ([73, § 3]). We have a natural commutative diagram:*

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "(4.22)"'] & \widehat{U} \arrow[d, "\wr"'] \\
\mathrm{Ord}_P\big((\mathrm{Ind}_{\overline{P}(L)}^{G(L)} U_\infty)^\infty \otimes_E L(\lambda)\big) \arrow[r] & \mathrm{Ord}_P\big((\mathrm{Ind}_{\overline{P}(L)}^{G(L)} \widehat{U})^{\mathcal{C}^0}\big)
\end{tikzcd}$$

where the bottom map is induced by (4.23) and Lemma 4.18, and where the isomorphism on the right is [40, Cor. 4.3.5].

Proof. By (4.20) and the fact (4.9) (with $V = (\mathrm{Ind}_{\overline{P}(L)}^{G(L)} \widehat{U})^{\mathcal{C}^0}$) is an embedding (note that V is admissible by assumption), it is sufficient to prove that the following diagram:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \widehat{U} \arrow[d] \\ \big((\mathrm{Ind}_{\overline{P}(L)}^{G(L)} U_\infty)^\infty \otimes_E L(\lambda)\big)^{N_0} \arrow[r] & \big((\mathrm{Ind}_{\overline{P}(L)}^{G(L)} \widehat{U})^{\mathcal{C}^0}\big)^{N_0} \end{tikzcd} \tag{4.24}$$

is commutative. By Lemma 4.19 the left map sends $u \in U^0$ to

$$f_u \in \mathcal{C}_c^\infty(N_P(L), U^0)^{N_0}.$$

By [40, § 4] the right map is induced by the maps (with obvious notation):

$$U^0/\varpi_E^n \longrightarrow \mathcal{C}_c^\infty(N_P(L), U^0/\varpi_E^n)^{N_0}, \quad \overline{u} \longmapsto \overline{f_u}$$

then taking the inverse limit over n and inverting p. We see (4.24) commutes. □

Proposition 4.21. *Let U_∞ be a smooth admissible representation of $L_P(L)$ over E, $U := U_\infty \otimes_E L_P(\lambda)$ and V a unitary admissible Banach representation of $G(L)$ over E. Let $f : U \hookrightarrow \mathrm{Ord}_P(V)$ be an $L_P(L)$-equivariant injection and denote by $\widehat{U}$ the closure of U in the Banach space $\mathrm{Ord}_P(V)$. Then f induces $G(L)$-equivariant morphisms:*

$$(\mathrm{Ind}_{\overline{P}(L)}^{G(L)} U_\infty)^\infty \otimes_E L(\lambda) \hookrightarrow \big(\mathrm{Ind}_{\overline{P}(L)}^{G(L)} \widehat{U}\big)^{\mathcal{C}^0} \longrightarrow V \tag{4.25}$$

from which f can be recovered as the following composition:

$$U \xrightarrow{(4.22)} \mathrm{Ord}_P\big((\mathrm{Ind}_{\overline{P}(L)}^{G(L)} U_\infty)^\infty \otimes_E L(\lambda)\big) \longrightarrow \mathrm{Ord}_P(V) \tag{4.26}$$

where the last map is induced from the composition (4.25) and Lemma 4.18.

Proof. Note that U is a unitary representation of $L_P(L)$ and that $\widehat{U}$ is a unitary admissible Banach representation of $L_P(L)$ over E by [40, Thm. 3.4.8]. The second map in (4.25) is then obtained by applying [40, Thm. 4.4.6], and the first map is obtained as in (4.23) (with $U^0 := \mathrm{Ord}_P(V^0) \cap U$ where V^0 is an open bounded $G(L)$-invariant lattice in V). The second part of the proposition follows from [40, Thm. 4.4.6] together with Lemma 4.20. □

5. *P*-ordinary Galois representations and local Langlands correspondence

In this section, for P a parabolic subgroup of GL_n we define P-ordinary Galois representations and prove some standard compatibility with classical local Langlands correspondence which will be used later. We denote by L a finite extension of $\mathbb{Q}_p$.

5.1. *P*-ordinary Galois deformations

We define P-ordinary Galois deformations and recall some useful standard statements.

We fix P a parabolic subgroup of GL_n containing the Borel subgroup of upper triangular matrices and with a Levi subgroup L_P given by (where $\sum_{i=1}^k n_i = n$):

$$\begin{pmatrix} \mathrm{GL}_{n_1} & 0 & \cdots & 0 \\ 0 & \mathrm{GL}_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \mathrm{GL}_{n_k} \end{pmatrix}. \tag{5.1}$$

Definition 5.1. *Let A be a topological commutative ring and (ρ_A, T_A) a continuous A-linear representation of* Gal_L *on a free A-module T_A of rank n (we often just write ρ_A for simplicity). The representation ρ_A is P-ordinary (over A) if there exists an increasing filtration of T_A by invariant free A-submodules which are direct summands as A-modules such that the graded pieces are of rank n_1, n_2, $\cdots$, n_k over A.*

Choosing a basis of T_A over A, we see that a P-ordinary representation gives rise to a continuous group homomorphism $\mathrm{Gal}_L \to P(A)$. Fix a P-ordinary representation $\overline{\rho} = (\overline{\rho}, T_{k_E})$ of Gal_L over k_E together with an invariant increasing filtration $0 = T_{k_E,0} \subsetneq T_{k_E,1} \subsetneq \cdots \subsetneq T_{k_E,k} = T_{k_E}$ as in Definition 5.1. Denote by $(\overline{\rho}_i, \mathrm{gr}_i\, T_{k_E,\bullet} := T_{k_E,i}/T_{k_E,i-1})$, $i \in \{1, \cdots, k\}$, or simply $\overline{\rho}_i$, for the representations of Gal_L over k_E given by the graded pieces (thus $\overline{\rho}_i$ is of dimension n_i). We assume the following hypothesis on $\overline{\rho}$ and the $\overline{\rho}_i$.

Hypothesis 5.2. *We have* $\mathrm{End}_{\mathrm{Gal}_L}(\overline{\rho}) \cong k_E$, $\mathrm{End}_{\mathrm{Gal}_L}(\overline{\rho}_i) \cong k_E$ *for* $i = 1, \cdots, k$ *and* $\mathrm{Hom}_{\mathrm{Gal}_L}(\overline{\rho}_i, \overline{\rho}_j) = 0$ *for all* $i \neq j$.

Let $\mathrm{Art}(\mathcal{O}_E)$ be the category of local artinian $\mathcal{O}_E$-algebras with residue field k_E. Let $\mathrm{Def}_{\overline{\rho}}$ (resp. $\mathrm{Def}_{\overline{\rho}_i}$) be the usual functor of deformations of $\overline{\rho}$ (resp. of $\overline{\rho}_i$), i.e. the functor from $\mathrm{Art}(\mathcal{O}_E)$ to sets which sends $A \in \mathrm{Art}(\mathcal{O}_E)$ to the set $\{((\rho_A, T_A), i_A)\}/\sim$ where (ρ_A, T_A) is a representation of Gal_L over A as above, i_A is a Gal_L-equivariant isomorphism $T_A \otimes_A k_E \xrightarrow{\sim} T_{k_E}$ (T_{k_E} being the underlying vector space of $\overline{\rho}$) and $\sim$ means modulo the Gal_L-equivariant isomorphisms $T_A \xrightarrow{\sim} T'_A$ such that the following induced diagram commutes:

$$\begin{tikzcd}
T_A \otimes_A k_E \arrow[r, "\sim"] \arrow[d, "\iota_A"', "\wr"] & T'_A \otimes_A k_E \arrow[d, "\iota'_A"', "\wr"] \\
T_{k_E} \arrow[r, equal] & T_{k_E}
\end{tikzcd} \tag{5.2}$$

(resp. with $\overline{\rho}_i$ instead of $\overline{\rho}$). If $A \to B$ in $\mathrm{Art}(\mathcal{O}_E)$ then T_A is sent to $T_A \otimes_A B$ (and i_A to itself via $T_A \otimes_A B \otimes_B k_E \cong T_A \otimes_A k_E$). By choosing basis, the functor $\mathrm{Def}_{\overline{\rho}}(A)$ can also be described as the set:

$$\{\rho_A : \mathrm{Gal}_L \to \mathrm{GL}_n(A) \text{ such that the composition with } \\ \mathrm{GL}_n(A) \twoheadrightarrow \mathrm{GL}_n(k_E) \text{ gives} \overline{\rho} \colon \mathrm{Gal}_L \to \mathrm{GL}_n(k_E)\}/\sim$$

where $\sim$ means modulo conjugation by matrices in $\mathrm{GL}_n(A)$ which are congruent to 1 modulo the maximal ideal $\mathfrak{m}_A$ of A. Since $\mathrm{End}_{\mathrm{Gal}_L}(\overline{\rho}) \cong k_E$ (resp. $\mathrm{End}_{\mathrm{Gal}_L}(\overline{\rho}_i) \cong k_E$), it is a standard result (first due to Mazur) that this functor is pro-representable. We denote by $R_{\overline{\rho}}$ (resp. $R_{\overline{\rho}_i}$) the universal deformation ring of $\overline{\rho}$ (resp. of $\overline{\rho}_i$), which is a complete local noetherian $\mathcal{O}_E$-algebra of residue field k_E.

We now switch to P-ordinary deformations. We define the functor

$$\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}} : \mathrm{Art}(\mathcal{O}_E) \to \{\mathrm{Sets}\}$$

by sending $A \in \mathrm{Art}(\mathcal{O}_E)$ to the set:

$$\big\{\big((\rho_A, T_A), T_{A,\bullet}, i_A\big)\big\}/\sim$$

where $((\rho_A, T_A), i_A)$ is as above, $T_{A,\bullet} = (0 = T_{A,0} \subsetneq T_{A,1} \subsetneq \cdots \subsetneq T_{A,k} = T_A)$ is an increasing filtration of T_A by invariant free A-submodules which are direct summands as A-modules such that i_A induces a Gal_L-equivariant isomorphism $T_{A,i} \otimes_A k_E \xrightarrow{\sim} T_{k_E,i}$ for $i \in \{1, \cdots, k\}$, and where $\sim$ means modulo the Gal_L-equivariant isomorphisms $T_A \xrightarrow{\sim} T'_A$ satisfying (5.2) and which moreover respect the increasing filtration on both sides. Alternatively, by choosing adapted basis one can describe $\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}(A)$ as the set:

$$\begin{aligned}(5.3)\quad \{\rho\colon \mathrm{Gal}_L \to P(A) \text{ such that the composition with}\\ P(A) \twoheadrightarrow P(k_E) \text{ gives } \overline{\rho}\colon \mathrm{Gal}_L \to P(k_E)\}/\sim\end{aligned}$$

where $\sim$ means modulo conjugation by matrices in $P(A)$ which are congruent to 1 modulo the maximal ideal $\mathfrak{m}_A$ of A. The following two propositions are standard, we provide short proofs for the convenience of the reader.

Lemma 5.3. *The functor* $\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}$ *is a subfunctor of* $\mathrm{Def}_{\overline{\rho}}$.

Proof. Let $A \in \mathrm{Art}(\mathcal{O}_E)$, starting from $((\rho_A, T_A), i_A) \in \mathrm{Def}_{\overline{\rho}}(A)$, it is enough to prove that there is at most one filtration $T_{A,\bullet}$ on T_A such that i_A induces isomorphisms $T_{A,i} \otimes_A k_E \xrightarrow{\sim} T_{k_E,i}$ and that any isomorphism $T_A \xrightarrow{\sim} T'_A$ satisfying (5.2) is automatically compatible with the filtrations (when they exist). For the first statement, by dévissage it is enough to prove $T^{(1)}_{A,1} = T^{(2)}_{A,1}$ (where $T^{(1)}_{A,\bullet}, T^{(2)}_{A,\bullet}$ are two filtrations). But the equivariant map $T^{(1)}_{A,1} \to T_A/T^{(2)}_{A,1}$ must be zero (and hence $T^{(1)}_{A,1} = T^{(2)}_{A,1}$) since the Gal_L-representation $T_A/T^{(2)}_{A,1}$ is by definition a successive extension of $\overline{\rho}_i$, $i \neq 1$ and we have $\mathrm{Hom}_{\mathrm{Gal}_L}(T^{(1)}_{A,1}, \overline{\rho}_i) = 0$ for $i \neq 1$ by Hypothesis 5.2 (and an obvious dévissage). The same argument replacing $T_A/T^{(2)}_{A,1}$ by $T'_A/T'_{A,1}$ shows that any equivariant isomorphism $T_A \xrightarrow{\sim} T'_A$ must send $T_{A,i}$ to $T'_{A,i}$. ☐

Proposition 5.4. *The functor* $\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}$ *is pro-representable by a complete local noetherian* $\mathcal{O}_E$*-algebra* $R^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}$ *of residue field* k_E.

Proof. By Schlessinger's criterion ([72]), Lemma 5.3 and the fact that $\mathrm{Def}_{\overline{\rho}}$ is pro-representable, it is enough to check that, given morphisms $f_1 : A \to C$, $f_2 : B \to C$ in $\mathrm{Art}(\mathcal{O}_E)$ with f_2 surjective and small, the induced map:

$$\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}(A \times_C B) \longrightarrow \mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}(A) \times_{\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}(C)} \mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}(B)$$

is surjective. But this is immediate from the description (5.3). ☐

By Proposition 5.4, Lemma 5.3 and the fact that $R_{\overline{\rho}}$ is a complete local noetherian $\mathcal{O}_E$-algebra, we see (e.g. by [45, Lem. 2.1]) that the natural morphism $R_{\overline{\rho}} \to R^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}$ is surjective. Moreover, for $i \in \{1, \cdots, k\}$, we have a natural transformation of functors $\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}} \to \mathrm{Def}_{\overline{\rho}_i}$ sending

$((\rho_A,T_A),T_{A,\bullet},i_A)$ to $\mathrm{gr}_i\,T_{A,\bullet}$ with the induced i_A. It corresponds to a canonical morphism of $\mathcal{O}_E$-algebras $R_{\overline{\rho}_i}\to R^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}$, and we deduce a morphism of local complete $\mathcal{O}_E$-algebras (with obvious notation):

$$\widehat{\bigotimes}_{i=1,\cdots,k}R_{\overline{\rho}_i}\longrightarrow R^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}. \tag{5.4}$$

Let us now consider equal characteristic 0 deformations. Fix a P-ordinary representation ρ of Gal_L over E together with an invariant increasing filtration $0=T_{E,0}\subsetneq T_{E,1}\subsetneq\cdots\subsetneq T_{E,k}=T_E$ as in Definition 5.1 and denote by $(\rho_i,\mathrm{gr}_i\,T_{E,\bullet}:=T_{E,i}/T_{E,i-1})$, $i\in\{1,\cdots,k\}$ the graded pieces. As previously we assume the following hypothesis on ρ and the ρ_i.

Hypothesis 5.5. *We have* $\mathrm{End}_{\mathrm{Gal}_L}(\rho)\cong E$, $\mathrm{End}_{\mathrm{Gal}_L}(\rho_i)\cong E$ *for* $i=1,\cdots,k$ *and* $\mathrm{Hom}_{\mathrm{Gal}_L}(\rho_i,\rho_j)=0$ *for* $i\neq j$.

Let $\mathrm{Art}(E)$ be the category of local artinian E-algebras with residue field E and define Def_{ρ} (resp. Def_{ρ_i}) as $\mathrm{Def}_{\overline{\rho}}$ (resp. $\mathrm{Def}_{\overline{\rho}_i}$) but replacing $\mathrm{Art}(E)$ by $\mathrm{Art}(\mathcal{O}_E)$ and $\overline{\rho}$ (resp. $\overline{\rho}_i$) by ρ (resp. ρ_i). Then from Hypothesis 5.5 the functor Def_{ρ} (resp. Def_{ρ_i}) is pro-representable by a complete local noetherian E-algebra of residue field E denoted by R_ρ (resp. R_{ρ_i}). Likewise we define the functor $\mathrm{Def}^{P-\mathrm{ord}}_{\rho,\{\rho_i\}}$ of P-ordinary deformations of ρ on $\mathrm{Art}(E)$ in a similar way as (5.3) and before by replacing $\overline{\rho}$, $T_{k_E,i}$ and $\overline{\rho}_i$ by ρ, $T_{E,i}$ and ρ_i. By the same proof as for Lemma 5.3 and Proposition 5.4, we obtain the following proposition.

Proposition 5.6. *The functor* $\mathrm{Def}^{P-\mathrm{ord}}_{\rho,\{\rho_i\}}$ *is a subfunctor of* Def_{ρ} *and is pro-representable by a complete local noetherian E-algebra* $R^{P-\mathrm{ord}}_{\rho,\{\rho_i\}}$ *of residue field E.*

Let $(\overline{\rho},\{\overline{\rho}_i\})$ as before satisfying Hypothesis 5.2. Let $\xi:R^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}\twoheadrightarrow\mathcal{O}_E$ be a homomorphism of local $\mathcal{O}_E$-algebras and denote by ρ^0_ξ (resp. $\rho^0_{\xi,i}$) the deformation of $\overline{\rho}$ (resp. of $\overline{\rho}_i$) over $\mathcal{O}_E$ associated to ξ via $\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}\to\mathrm{Def}_{\overline{\rho}}$ (resp. $\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}\to\mathrm{Def}_{\overline{\rho}_i}$). In particular, ρ^0_ξ is a representation of Gal_L over a free $\mathcal{O}_E$-module $T_{\mathcal{O}_E}$ endowed with an invariant filtration by direct summands $T_{\mathcal{O}_E,i}$ as $\mathcal{O}_E$-modules such that the graded pieces give the representations $\rho^0_{\xi,i}$, $i=1,\cdots,k$. Let $\rho_\xi:=\rho^0_\xi\otimes_{\mathcal{O}_E}E$ and $\rho_{\xi,i}:=\rho^0_{\xi,i}\otimes_{\mathcal{O}_E}E$.

Proposition 5.7. *(1) We have that* $(\rho_\xi,\{\rho_{\xi,i}\})$ *satisfies Hypothesis 5.5.*

(2) The E-algebra $R^{P-\mathrm{ord}}_{\rho_\xi,\{\rho_{\xi,i}\}}$ *is isomorphic to the* $(\mathrm{Ker}(\xi)\otimes_{\mathcal{O}_E}E)$*-adic completion of* $R^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}\otimes_{\mathcal{O}_E}E$.

Proof. (1) is straightforward from Hypothesis 5.2 and a dévissage.

(2) Denote by $\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\},(\xi)}$ (resp. $\mathrm{Def}_{\overline{\rho},(\xi)}$) the generic fiber of $\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}$ (resp. $\mathrm{Def}_{\overline{\rho}}$) at ξ in the sense of [52, § 2.3]. By [52, Lem. 2.3.3], it is sufficient to prove $\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\},(\xi)} \cong \mathrm{Def}^{P-\mathrm{ord}}_{\rho_\xi,\{\rho_{\xi,i}\}}$. By [52, Prop. 2.3.5], the generic fiber $\mathrm{Def}_{\overline{\rho},(\xi)}$ is isomorphic to Def_{ρ_ξ}. Moreover, by the argument in the proof of *loc. cit.* (together with Lemma 5.3 and Proposition 5.7), the isomorphism $\mathrm{Def}_{\overline{\rho},(\xi)} \xrightarrow{\sim} \mathrm{Def}_{\rho_\xi}$ induces an injection of functors:

$$\mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\},(\xi)} \hookrightarrow \mathrm{Def}^{P-\mathrm{ord}}_{\rho_\xi,\{\rho_{\xi,i}\}}. \tag{5.5}$$

For $A \in \mathrm{Art}(E)$, let A_0 be an $\mathcal{O}_E$-subalgebra of A such that A_0 is finitely generated as $\mathcal{O}_E$-module and $A_0[1/p] \cong A$. The canonical surjection of E-algebras $A \twoheadrightarrow E$ induces a surjection of $\mathcal{O}_E$-algebras $A_0 \twoheadrightarrow \mathcal{O}_E$. Let $((\rho_A, T_A), T_{A,\bullet}, i_A) \in \mathrm{Def}^{P-\mathrm{ord}}_{\rho_\xi,\{\rho_{\xi,i}\}}(A)$. As in the proof of [52, Prop. 2.3.5], the free A-module T_A admits a Gal_L-invariant A_0-lattice T_{A_0} such that $T_{A_0} \otimes_{A_0} \mathcal{O}_E \cong \rho^0_\xi$. We define an invariant filtration on T_{A_0} by $T_{A_0,i} := T_{A,i} \cap T_{A_0}$ (inside T_A). It is not difficult to check that $T_{A_0,i}$ is a direct summand of T_{A_0} as A_0-module and that $T_{A_0,i} \otimes_{A_0} \mathcal{O}_E \cong T_{\mathcal{O}_E,i}$. Hence $((\rho_A, T_A), T_{A,\bullet}, i_A) \in \mathrm{Def}^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\},(\xi)}(A)$ (see [52, § 2.3]) which implies (5.5) is also surjective, and thus an isomorphism. □

Definition 5.8. *Let $\overline{\rho}$ (resp. ρ) be a P-ordinary representation of* Gal_L *over k_E (resp. E) and fix an invariant increasing filtration of the underlying space T_{k_E} (resp. T_E) as in Definition 5.1 leading to representations $\overline{\rho}_i$ (resp. ρ_i) for $i \in \{1, \cdots, k\}$ on the graded pieces. The representation $\overline{\rho}$ (resp. ρ) is strictly P-ordinary if the following conditions are satisfied:*

- *$(\overline{\rho}, \{\overline{\rho}_i\})$ satisfies Hypothesis 5.2 (resp. $(\rho, \{\rho_i\})$ satisfies Hypothesis 5.5)*
- *if $\overline{\rho}$ (resp. ρ) is isomorphic to a successive extension of n_i-dimensional representations $\overline{\rho}'_i$ (resp. ρ'_i) for $i = 1, \cdots, k$, then $\overline{\rho}'_i \cong \overline{\rho}_i$ (resp. $\rho'_i \cong \rho_i$) for all $i = 1, \cdots, k$.*

In particular, if $\overline{\rho}$ (resp. ρ) is strictly P-ordinary, there is a *unique* invariant increasing filtration on its underlying space as in Definition 5.1.

Lemma 5.9. *Let $\overline{\rho}$ be a strictly P-ordinary representation of* Gal_L *over k_E, $\xi : R_{\overline{\rho}} \twoheadrightarrow \mathcal{O}_E$ a surjection of local $\mathcal{O}_E$-algebras and ρ^0_ξ the deformation of $\overline{\rho}$ over $\mathcal{O}_E$ associated to ξ. Assume that ρ^0_ξ, and thus $\rho_\xi := \rho^0_\xi \otimes_{\mathcal{O}_E} E$, are P-ordinary.*

(1) The morphism ξ factors through the quotient $R^{P-\mathrm{ord}}_{\overline{\rho},\{\overline{\rho}_i\}}$ of $R_{\overline{\rho}}$.

(2) The representation ρ_ξ is strictly P-ordinary.

Proof. Any choice of filtration as in Definition 5.1 on the underlying space of ρ_ξ^0 satisfies that its reduction modulo ϖ_E gives the above unique filtration on the underlying space of $\overline{\rho}$, from which (1) follows easily. The proof of (2) is by the same argument as for Lemma 5.3. $\square$

When $\overline{\rho}$ (resp. ρ) is strictly P-ordinary, by Definition 5.8 the representations $\overline{\rho}_i$ (resp. ρ_i) are defined without ambiguity and we then write $R_{\overline{\rho}}^{P-\mathrm{ord}} := R_{\overline{\rho},\{\overline{\rho}_i\}}^{P-\mathrm{ord}}$ (resp. $R_{\rho}^{P-\mathrm{ord}} := R_{\rho,\{\rho_i\}}^{P-\mathrm{ord}}$).

5.2. Classical local Langlands correspondence

We give a sufficient condition in terms of the (usual) local Langlands correspondence for a p-adic Galois representation to be P-ordinary. The results of this section will be used in §§ 6.3 & 7.1.

Let $\rho\colon \mathrm{Gal}_L \to \mathrm{GL}_n(E)$ be a potentially semi-stable representation of Gal_L over E and L' a finite Galois extension of L such that $\rho|_{\mathrm{Gal}_{L'}}$ is semi-stable. Following Fontaine we can associate to ρ a Deligne-Fontaine module:

$$\mathrm{DF}(\rho) := \big((B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \rho)^{\mathrm{Gal}_{L'}}, \varphi, N, \mathrm{Gal}(L'/L)\big),$$

where $D_{L'} := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \rho)^{\mathrm{Gal}_{L'}}$ is a finite free $L_0' \otimes_{\mathbb{Q}_p} E$-module of rank n, L_0' being the maximal unramified subextension of L' (over $\mathbb{Q}_p$), where the (φ, N)-action on $D_{L'}$ is induced from the (φ, N)-action on B_{st}, and where the $\mathrm{Gal}(L'/L)$-action on $D_{L'}$ is the residual action of Gal_L. As in [16, § 4], we associate to $\mathrm{DF}(\rho)$ an n-dimensional Weil-Deligne representation $\mathrm{WD}(\rho)$ in the following way. By enlarging E, we assume E contains all the embeddings of L' (and hence L_0') in $\overline{\mathbb{Q}_p}$. We have thus $L_0' \otimes_{\mathbb{Q}_p} E \cong \prod_{\sigma: L_0' \hookrightarrow E} E$ and therefore an isomorphism $D_{L'} \xrightarrow{\sim} \prod_{\sigma: L_0' \hookrightarrow E} D_{L',\sigma}$ where $D_{L',\sigma} := D_{L'} \otimes_{L_0' \otimes_{\mathbb{Q}_p} E, \sigma \otimes 1} E$. Each $D_{L',\sigma}$ is stable by the N-action. Moreover, for $w \in \mathrm{W}_L$ (the Weil group of L), we have that $r(w) := \varphi^{-\alpha(w)} \circ \overline{w}$ acts $L_0' \otimes_{\mathbb{Q}_p} E$-linearly on $D_{L'}$ where $\alpha(w) \in [L_0 : \mathbb{Q}_p]\mathbb{Z}$ is such that the image of w in $\mathrm{Gal}_{\mathbb{F}_p}$ is equal to $\mathrm{Frob}^{\alpha(w)}$, Frob being the absolute arithmetic Frobenius, and where $\overline{w}$ denotes the image of w in $\mathrm{Gal}(L'/L)$. We still denote by $r(w)$ the induced map $D_{L',\sigma} \to D_{L',\sigma}$ for $\sigma : L_0' \hookrightarrow E$, then we denote by $\mathrm{W}(\rho)$ the representation $(D_{L',\sigma}, r)$ of W_L and by $\mathrm{WD}(\rho) := (\mathrm{W}(\rho), N)$ the Weil-Deligne representation obtained when taking N into account. Both $\mathrm{W}(\rho)$ and $\mathrm{WD}(\rho)$ are independent of the choice of σ: if we replace σ by $\sigma \circ \mathrm{Frob}^{-j}$ for $j \in \mathbb{Z}$ (Frob being the absolute arithmetic Frobenius on L_0'), then $\varphi^j : D_{L'} \to D_{L'}$ induces an isomorphism of Weil-Deligne representations $D_{L',\sigma} \xrightarrow{\sim} D_{L',\sigma \circ \mathrm{Frob}^{-j}}$ (cf. [15, Lem. 2.2.1.2]). In fact, we only make use of $\mathrm{W}(\rho)$ in the sequel. We let $W(\rho)^{\mathrm{ss}}$ be the semi-simplification of $W(\rho)$.

For a representation W of W_L and an integer s, we set $W(s) := W \otimes |\cdot|^s = W \otimes_E \mathrm{unr}(q_L^{-s})$. Let π^∞ be a smooth irreducible (hence admissible) representation of $\mathrm{GL}_n(L)$ over E such that $\mathrm{rec}(\pi^\infty)\big(\frac{1-n}{2}\big) \cong \mathrm{W}(\rho)^{\mathrm{ss}}$, where $\mathrm{rec}(\pi^\infty)$ denotes the semi-simple representation of W_L associated to π^∞ normalized as in [47, Thm. A]. As in § 4.3, let $\lambda = (\lambda_{\sigma,1}, \cdots, \lambda_{\sigma,n})_{\sigma \in \Sigma_L} \in \mathrm{Hom}(\mathrm{Res}_{L/\mathbb{Q}_p} T, \mathbb{G}_m)$ be a dominant weight with respect to $\mathrm{Res}_{L/\mathbb{Q}_p} B$ (so $\lambda_{\sigma,i} \geq \lambda_{\sigma,i+1}$ for all σ). Put $\pi := \pi^\infty \otimes_E L(\lambda)$. Assume that, for all $\sigma \in \Sigma_L$, the σ-Hodge-Tate weights $\mathrm{HT}_\sigma(\rho)$ of ρ are given by $\mathrm{HT}_\sigma(\rho) := \{\lambda_{\sigma,n} + 1 - n, \cdots, \lambda_{\sigma,i} + 1 - i, \cdots, \lambda_{\sigma,1}\}$. Let $P \subseteq \mathrm{GL}_n$ as in (5.1) and choose N_0 a compact open subgroup of the unipotent radical $N_P(L)$ as in § 4.1. Recall that we defined a canonical representation $(\pi^{N_0})_0$ of $L_P(L) = \prod_{i=1}^k \mathrm{GL}_{n_i}(L)$ in (4.12) (see Remark 4.8). For $i \in \{1, \cdots, k\}$, we denote by $s_i := \sum_{j=0}^{i-1} n_j$ where we set $n_0 := 0$ (hence $s_1 = 0$).

Proposition 5.10. *For $i = 1, \cdots, k$ let π_i^∞ be a smooth irreducible representation of $\mathrm{GL}_{n_i}(L)$ over E. If there is an embedding $(\otimes_{i=1}^k \pi_i^\infty) \otimes_E L_P(\lambda) \hookrightarrow (\pi^{N_0})_0$ of locally algebraic representations of $L_P(L) = \prod_{i=1}^k \mathrm{GL}_{n_i}(L)$, then there exist $\rho_i \colon \mathrm{Gal}_L \to \mathrm{GL}_{n_i}(E)$ for $i = 1, \cdots, k$ such that:*

- *ρ is isomorphic to a successive extension of the ρ_i (thus ρ_i is potentially semi-stable for all i),*
- $\mathrm{rec}(\pi_i^\infty)(\frac{1-n_i}{2} - s_i) \cong \mathrm{W}(\rho_i)^{\mathrm{ss}}$,
- $\mathrm{HT}_\sigma(\rho_i) = \{\lambda_{\sigma,j} + 1 - j\}_{j = s_i+1, \cdots, s_{i+1}}$ *for $\sigma \in \Sigma_L$.*

In particular, if $(\pi^{N_0})_0 \neq 0$, then ρ is P-ordinary over E in the sense of Definition 5.1.

Proof. The very last assertion easily follows from the others and the finite length of the $L_P(L)$-representation $(\pi^{N_0})_0$ (which follows from $(\pi^{N_0})_0 \subseteq J_P(\pi)(\delta_P^{-1})$, see (4.14), and the finite length of $J_P(\pi)$). The general idea of the proof below is the following: by classical local Langlands correspondence, we deduce first a "P-filtration" of the Weil representation $\mathrm{W}(\rho)^{\mathrm{ss}}$, then we show that this filtration actually comes from a filtration of Galois representations.

(a) First we reduce to the case $k = 2$ (i.e. P maximal). Take $P' \supseteq P$ such that the Levi subgroup $L_{P'}$ of P' satisfies $L_{P'} \cong \begin{pmatrix} \mathrm{GL}_{n-n_k} & 0 \\ 0 & \mathrm{GL}_{n_k} \end{pmatrix}$. By the proof of Lemma 4.12 (see Remark 4.13), we have with the notation as *loc. cit.* $(\pi^{N_0})_0 \cong (((\pi^{N_0'})_0)^{N_0''})_0$. Thus if $(\otimes_{i=1}^k \pi_i^\infty) \otimes_E L_P(\lambda) \hookrightarrow (\pi^{N_0})_0$, there exists a smooth irreducible representation $\pi_{P'}^\infty = (\pi')^\infty \otimes \pi_k^\infty$ of $L_{P'}(L)$ over E such that $\pi_{P'}^\infty \otimes_E L_{P'}(\lambda) \hookrightarrow (\pi^{N_0'})_0$ and $(\otimes_{i=1}^k \pi_i^\infty) \otimes_E L_P(\lambda) \hookrightarrow \big((\pi_{P'}^\infty \otimes_E L_{P'}(\lambda))^{N_0''}\big)_0$. Assume the statement holds for $k = 2$, we then obtain ρ', ρ_k corresponding to $(\pi')^\infty$, π_k^∞ respectively as in the proposition. Applying the same argument with ρ', $(\pi')^\infty$, $\lambda'=(\lambda_{\sigma,i})_{\substack{\sigma\in\Sigma_L\\ i=1,\cdots,n-n_k}}$ instead of ρ, π^∞, λ and using an easy induction, we deduce the statement for arbitrary k.

(b) Assume now $L_P\cong\begin{pmatrix}\mathrm{GL}_{n_1} & 0\\ 0 & \mathrm{GL}_{n_2}\end{pmatrix}$. The composition

$$(\pi_1^\infty\otimes\pi_2^\infty)\otimes_E L_P(\lambda)\hookrightarrow(\pi^{N_0})_0\hookrightarrow J_P(\pi)(\delta_P^{-1})\cong J_P(\pi^\infty)(\delta_P^{-1})\otimes_E L_P(\lambda) \tag{5.6}$$

corresponds to an injection $\pi_1^\infty\otimes\pi_2^\infty\hookrightarrow J_P(\pi^\infty)(\delta_P^{-1})$. The latter injection induces (for example see [38, (0.2)]) a nonzero, hence surjective, morphism (recall $\overline{P}$ is the opposite parabolic):

$$\big(\mathrm{Ind}_{\overline{P}(L)}^{\mathrm{GL}_n(L)}\pi_1^\infty\otimes\pi_2^\infty\big)^\infty\twoheadrightarrow\pi^\infty. \tag{5.7}$$

Let $\mathrm{W}_i:=\mathrm{rec}(\pi_i^\infty)(\frac{1-n_i}{2})$, $i\in\{1,2\}$ be the semi-simple representation of W_L associated to π_i^∞, we have (see for example [77, Thm. 1.2(b)], noting that our $\mathrm{rec}(-)$ is $\sigma(-)$ of *loc. cit.*):

$$\mathrm{W}(\rho)^{\mathrm{ss}}\cong\mathrm{W}_1\oplus\mathrm{W}_2(-n_1)$$

with $\mathrm{W}_i|_{\mathrm{W}_{L'}}$ being unramified. For $i\in\{1,2\}$ let $\mathrm{DF}_i:=(\mathscr{D}_i,\varphi,N=0,\mathrm{Gal}(L'/L))$ be the Deligne-Fontaine module associated to $(\mathrm{W}_i,N=0)$ ([16, Prop. 4.1]). Enlarging E if needed, there exists a φ-submodule D_1 of $D_{L'}=(B_{\mathrm{st}}\otimes_{\mathbb{Q}_p}\rho)^{\mathrm{Gal}_{L'}}$ such that the $\varphi^{[L_0':\mathbb{Q}_p]}$-semi-simplification of D_1 is isomorphic to $\mathscr{D}_1$ as φ-modules over $L_0'\otimes_{\mathbb{Q}_p}E$. Indeed, for $\sigma:L_0'\hookrightarrow E$, by choosing appropriate generalized $\varphi^{[L_0':\mathbb{Q}_p]}$-eigenvectors, we see there exists a $\varphi^{[L_0':\mathbb{Q}_p]}$-submodule $D_{1,\sigma}$ of $D_{L',\sigma}$ such that $D_{1,\sigma}^{\mathrm{ss}}\cong\mathscr{D}_{1,\sigma}$ (since $\mathscr{D}_{1,\sigma}$ is a $\varphi^{[L_0':\mathbb{Q}_p]}$-submodule of $D_{L',\sigma}^{\mathrm{ss}}$ and E is sufficiently large). We can then take D_1 to be the φ-submodule of $D_{L'}$ generated by $D_{1,\sigma}$. We will show that D_1 is stable by N and by $\mathrm{Gal}(L'/L)$ (hence is a Deligne-Fontaine submodule of D) and that the induced filtration on D_1 is admissible.

(c) We first show that we have (where $t_N(\cdot):=\frac{1}{[L_0':\mathbb{Q}_p]}\mathrm{val}_p(\det_{L_0'}(\varphi^{[L_0':\mathbb{Q}_p]}))$):

$$t_N(D_1)=\Big(\sum_{\sigma\in\Sigma_L}\sum_{j=1}^{n_1}(j-1-\lambda_{\sigma,j})\Big)[E:L]. \tag{5.8}$$

From the discussion above [47, Thm. A], the central character $\omega_{\pi_1^\infty}$ of π_1^∞ coincides with $\wedge_E^{n_1}\mathrm{rec}(\pi_1^\infty)\cong\wedge_E^{n_1}(W_1(\frac{n_1-1}{2}))$. On the other hand, since

$(\pi_1^\infty \otimes \pi_2^\infty) \otimes_E L_P(\lambda) \hookrightarrow (\pi^{N_0})_0$, we deduce

$$\mathrm{val}_p\left(\omega_{\pi_1^\infty}(\varpi_L)\right) + \mathrm{val}_p(\varpi_L)\Big(\sum_{\sigma\in\Sigma_L}\sum_{j=1}^{n_1}\lambda_{\sigma,j}\Big) = 0,$$

and hence (5.8). We equip $D_1\otimes_{L_0'}L'$ with the Hodge filtration $\mathrm{Fil}^\bullet(D_1\otimes_{L_0'}L')$ induced by $D_{L'}\otimes_{L_0'}L'$. Since $\mathrm{HT}_\sigma(\rho) := \{\lambda_{\sigma,n}+1-n, \cdots, \lambda_{\sigma,i}+1-i, \cdots, \lambda_{\sigma,1}\}$ for $\sigma \in \Sigma_L$, it is easy to deduce (where $t_H(\cdot\otimes_{L_0'}L') := \sum_{i\in\mathbb{Z}} \dim_{L'} i\,\mathrm{Fil}^i(\cdot\otimes_{L_0'} L')/\,\mathrm{Fil}^{i+1}(\cdot \otimes_{L_0'} L')$):

$$t_H(D_1 \otimes_{L_0'} L') \geq \Big(\sum_{\sigma\in\Sigma_L}\sum_{j=1}^{n_1}(j-1-\lambda_{\sigma,j})\Big)[E:L].$$

(d) We show that D_1 is stable by the monodromy operator N of $D_{L'}$. Let $\sigma : L_0' \hookrightarrow E$, by the relation $N\varphi = p\varphi N$ and the fact that φ^j induces an isomorphism $D_{1,\sigma} \to D_{1,\sigma\circ\mathrm{Frob}^{-j}}$ for $j \in \mathbb{Z}_{\geq 0}$, it is sufficient to prove that $D_{1,\sigma}$ is stable by N. Let $f' := [L_0' : \mathbb{Q}_p]$ and denote by D_σ' the $(\varphi^{f'}, N)$-submodule of $D_{L',\sigma}$ generated by $D_{1,\sigma}$. Let D' be the (φ, N)-submodule of D generated by D_σ', i.e. $D'_{\sigma\circ\mathrm{Frob}^{-j}} = \varphi^j(D_\sigma')$ for $j \in \mathbb{Z}_{\geq 0}$.

Claim. If $D_\sigma' \neq D_{1,\sigma}$ then there exists a $(\varphi^{f'}, N)$-submodule D_σ'' of D_σ' such that:

$$\dim_E D_\sigma'' = \dim_E D_{1,\sigma} = n_1 \quad\text{and}\quad t_N(D'') < t_N(D_1)$$

where D'' is the (φ, N)-submodule of D' generated by D_σ''.

We first prove the claim in the case where there is $\alpha \in E^\times$ and $m \in \mathbb{Z}_{>0}$ such that the $\varphi^{f'}$-eigenvalues on D_σ' lie in $\{\alpha, p^{-f'}\alpha, \cdots, p^{-f'm}\alpha\}$ (enlarging E if necessary) and α is an eigenvalue of $\varphi^{f'}$. Since D_σ' is generated by $D_{1,\sigma}$, we see from $N\varphi = p\varphi N$ that α is also a $\varphi^{f'}$-eigenvalue on $D_{1,\sigma}$. Since N is nilpotent on D_σ', there exists $s \in \mathbb{Z}_{\geq 0}$ such that $\dim_E \mathrm{Ker}(N^s) \geq n_1$ and $\dim_E \mathrm{Ker}(N^{s-1}) < n_1$ as $(\varphi^{f'}, N)$-submodule of D_σ'. Consider the short exact sequence:

$$0 \longrightarrow \mathrm{Ker}(N^{s-1}) \longrightarrow \mathrm{Ker}(N^s) \xrightarrow{N^{s-1}} N^{s-1}(\mathrm{Ker}(N^s)) \to 0.$$

Let M be a $\varphi^{f'}$-submodule of $N^{s-1}(\mathrm{Ker}(N^s))$ of dimension $n_1 - \dim_E \mathrm{Ker}(N^{s-1})$ and let D_σ'' be the preimage of M in $\mathrm{Ker}(N^s)$, which is thus a $(\varphi^{f'}, N)$-submodule of D_σ' of dimension n_1. Since $D_\sigma' \neq D_{1,\sigma}$, we have $\mathrm{Ker}(N^{s-1}) \nsubseteq D_{1,\sigma}$ or $D_{1,\sigma} \nsubseteq \mathrm{Ker}(N^s)$ (indeed, otherwise we have $\mathrm{Ker}(N^{s-1}) \subseteq D_{1,\sigma} \subseteq$

$\mathrm{Ker}(N^s)$ which implies $N(D_{1,\sigma})\subseteq \mathrm{Ker}(N^{s-1})\subseteq D_{1,\sigma}$ hence $D_{1,\sigma}$ stable by N and $D'_\sigma=D_{1,\sigma}$). In both cases, by comparing the $\varphi^{f'}$-eigenvalues, it is not difficult to see $t_N(D_1)>t_N(D'')$. The claim in this case follows. In general, we have a decomposition $D_{1,\sigma}\cong\oplus_{j\in J}D_{1,\sigma,j}$ where the $\varphi^{f'}$-eigenvalues on the $D_{1,\sigma,j}$ lie in disjoint finite sets of elements of $E^\times$ of the form $\{\alpha_j,p^{-f'}\alpha_j,\cdots,p^{-f'm'_j}\alpha_j\}$ with α_j an eigenvalue of $\varphi^{f'}$ on $D_{1,\sigma,j}$. Since D'_σ is generated by $D_{1,\sigma}$, from $N\varphi=p\varphi N$ we have $D'_\sigma\cong\oplus_{j\in J}D'_{\sigma,j}$ where $D'_{\sigma,j}$ is generated by $D_{1,\sigma,j}$ and the $\varphi^{f'}$-eigenvalues on $D'_{\sigma,j}$ lie in $\{\alpha_j,p^{-f'}\alpha_j,\cdots,p^{-f'm_j}\alpha_j\}$ for $m_j\geq m'_j$. We put $D''_{\sigma,j}:=D_{1,\sigma,j}$ if $D_{1,\sigma,j}=D'_{\sigma,j}$ and define $D''_{\sigma,j}\subseteq D'_{\sigma,j}$ as above when $D_{1,\sigma,j}\neq D'_{\sigma,j}$. The claim then follows with $D''_\sigma:=\oplus_{j\in J}D''_{\sigma,j}$.

Assume now $D'_\sigma\neq D_{1,\sigma}$ and let D'' be as in the claim. The same argument as in (c) with the induced Hodge filtration gives then $t_H(D''\otimes_{L'_0}L')\geq\big(\sum_{\sigma\in\Sigma_L}\sum_{j=1}^{n_1}(j-1-\lambda_{\sigma,j})\big)[E:L]>t_N(D'')$, which contradicts the fact that $D_{L'}$ is admissible. So we have $D'_\sigma=D_{1,\sigma}$, $D_1=D'$ and these spaces are stable by N. By (c) and the fact that $D_{L'}$ is admissible, we deduce:

$$t_H(D_1\otimes_{L'_0}L')=\Big(\sum_{\sigma\in\Sigma_L}\sum_{j=1}^{n_1}(j-1-\lambda_{\sigma,j})\Big)[E:L]$$

and hence together with (5.8) that D_1 is a weakly admissible (φ,N)-submodule of $D_{L'}$.

(e) For a $\varphi^{f'}$-module W over E and $a\in\mathbb{R}$, denote by $W_{<a}$ (resp. $W_{\leq a}$) the E-vector subspace of W generated by the generalized $\varphi^{f'}$-eigenvectors of eigenvalues β satisfying $\mathrm{val}_p(\beta)<a$ (resp. $\mathrm{val}_p(\beta)\leq a$). If W is moreover a (φ,N)-module over $L'_0\otimes_{\mathbb{Q}_p}E$, it is easy to see that $W_{<a}$ and $W_{\leq a}$ are still (φ,N)-submodules (over $L'_0\otimes_{\mathbb{Q}_p}E$) of W. Let $\mu_j:=\frac{[E:L]}{[E:L'_0]}\sum_{\sigma\in\Sigma_L}(j-1-\lambda_{\sigma,j})$. We now show $(D_{L'})_{<\mu_1}=0$ and $D_1=(D_{L'})_{\leq\mu_{n_1}}$. Since $t_H(W)\geq\big(\sum_{\sigma\in\Sigma_L}(-\lambda_{\sigma,1})\big)[E:L]$ for any nonzero E-vector subspace W of $D_{L'}$ (with the induced filtration) and since $D_{L'}$ is admissible, it follows that $(D_{L'})_{<\mu_1}=0$. We show $(D_1)_{\leq\mu_{n_1}}=D_1$ (and hence $D_1=(D_{L'})_{\leq\mu_{n_1}}$, since otherwise one easily deduces $t_H((D_{L'})_{\leq\mu_{n_1}})>t_N((D_{L'})_{\leq\mu_{n_1}})$). Assume not and let $n'_1<n_1$ such that $\dim_E(D_1)_{\leq\mu_{n_1}}=n'_1f'$ (note that $\dim_E D_1=n_1f'$ and that $(D_1)_{\leq\mu_{n_1}}$ is free over $L'_0\otimes_{\mathbb{Q}_p}E$ (as is easily checked)). Then we deduce:

$$t_N\big((D_1)_{\leq\mu_{n_1}}\big)<t_N(D_1)-\Big(\sum_{\sigma\in\Sigma_L}(n_1-1-\lambda_{\sigma,n_1})\Big)(n_1-n'_1)[E:L]$$

$$=\Big(\sum_{\sigma\in\Sigma_L}\sum_{j=1}^{n_1}(j-1-\lambda_{\sigma,j})\Big)[E:L]-\Big(\sum_{\sigma\in\Sigma_L}(n_1-1-\lambda_{\sigma,n_1})\Big)(n_1-n'_1)[E:L].$$

But we also have (with the induced Hodge filtration and by the same argument as in (c)):

$$\begin{aligned}
t_H\big((D_1)_{\leq \mu_{n_1}}\big) &\geq \Big(\sum_{\sigma\in\Sigma_L}\sum_{j=1}^{n_1'}(j-1-\lambda_{\sigma,j})\Big)[E:L]\\
&= \Big(\sum_{\sigma\in\Sigma_L}\sum_{j=1}^{n_1}(j-1-\lambda_{\sigma,j})\Big)[E:L]-\Big(\sum_{\sigma\in\Sigma_L}\sum_{j=n_1'+1}^{n_1}(j-1-\lambda_{\sigma,j})\Big)[E:L].
\end{aligned}$$

Since $j-1-\lambda_{\sigma,j}\leq n_1-1-\lambda_{\sigma,n_1}$ for all $j\leq n_1$ and $\sigma\in\Sigma_L$, we deduce $t_H((D_1)_{\leq\mu_{n_1}})>t_N((D_1)_{\leq\mu_{n_1}})$, contradicting the fact D_1 is weakly admissible (see the end of (d)).

(f) Since $\mathrm{Gal}(L'/L)$ commutes with φ, we see that $D_1=(D_{L'})_{\leq\mu_{n-1}}$ is stable by $\mathrm{Gal}(L'/L)$. Let ρ_1 be the continuous representation of Gal_L over E associated to D_1 by the Colmez-Fontaine theorem ([27, Thm. A]) and $\rho_2:=\rho/\rho_1$. Thus $\mathrm{W}(\rho)^{\mathrm{ss}}\cong\mathrm{W}(\rho_1)^{\mathrm{ss}}\oplus\mathrm{W}(\rho_2)^{\mathrm{ss}}$ and the first and third properties in the statement are then clear. To finish the proof, we only need to show that the W_L-representations $\mathrm{W}(\rho_1)^{\mathrm{ss}}$ and W_1 (see (b)) are isomorphic. Let $\mathrm{DF}_1':=(D_1^{\mathrm{ss}},\varphi,N=0,\mathrm{Gal}(L'/L))$ be the Deligne-Fontaine module associated to $(\mathrm{W}(\rho_1)^{\mathrm{ss}},N=0)$ ([16, Prop. 4.1]) where D_1^{ss} denotes the semi-simplification of D_1 for the $\varphi^{f'}$-action, we are reduced to show that DF_1' and $\mathrm{DF}_1=(\mathscr{D}_1,\varphi,N=0,\mathrm{Gal}(L'/L))$ (see (b)) are isomorphic (that is, one has to take care of the $\mathrm{Gal}(L'/L)$-action). The natural inclusion $\mathrm{W}_1\hookrightarrow\mathrm{W}(\rho)^{\mathrm{ss}}$ induces an embedding of Deligne-Fontaine modules:

$$\mathrm{DF}_1\longhookrightarrow\mathrm{DF}:=(D_{L'}^{\mathrm{ss}},\varphi,N=0,\mathrm{Gal}(L'/L))$$

where the latter is isomorphic to the Deligne-Fontaine module associated to $(\mathrm{W}(\rho)^{\mathrm{ss}},N=0)$ and where $D_{L'}^{\mathrm{ss}}$ denotes the semi-simplification of $D_{L'}$ for the $\varphi^{f'}$-action. Similarly, the inclusion $\mathrm{W}(\rho_1)^{\mathrm{ss}}\hookrightarrow\mathrm{W}(\rho)^{\mathrm{ss}}$ induces an injection $\mathrm{DF}_1'\longhookrightarrow\mathrm{DF}$. By construction, we also know $\mathscr{D}_1\cong D_1^{\mathrm{ss}}$ as φ-module. However, by (e) we have $D_1^{\mathrm{ss}}=(D_{L'}^{\mathrm{ss}})_{\leq\mu_{n_1}}$, thus we also have $\mathscr{D}_1=(D_{L'}^{\mathrm{ss}})_{\leq\mu_{n_1}}$ since $(D_{L'}^{\mathrm{ss}})_{\leq\mu_{n_1}}$ is only defined in terms of the φ-action. So both DF_1 and DF_1' are isomorphic to the Deligne-Fontaine submodule $((D_{L'}^{\mathrm{ss}})_{\leq\mu_{n_1}},\varphi,N=0,\mathrm{Gal}(L'/L))$ of DF. This concludes the proof. ☐

6. Automorphic and *P*-ordinary automorphic representations

In this section we start the global theory: we give the global setup, state our local-global compatibility conjecture for $\mathrm{GL}_3(\mathbb{Q}_p)$, and prove several useful results on the P-ordinary part of (localized) Banach spaces of p-adic automorphic forms on definite unitary groups.

6.1. Global setup and main conjecture

We introduce the global setup and state our main local-global compatibility conjecture for $\mathrm{GL}_3(\mathbb{Q}_p)$.

We fix field embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. We also fix F^+ a totally real number field, F a quadratic totally imaginary extension of F^+ and G/F^+ a unitary group attached to the quadratic extension F/F^+ as in [1, § 6.2.2] such that $G\times_{F^+}F \cong \mathrm{GL}_n$ $(n \geq 2)$ and $G(F^+\otimes_{\mathbb{Q}}\mathbb{R})$ is compact. For a finite place v of F^+ which is totally split in F and $\tilde{v}$ a place of F dividing v, we have thus isomorphisms $i_{G,\tilde{v}} : G(F_v^+) \xrightarrow{\sim} G(F_{\tilde{v}}) \xrightarrow{\sim} \mathrm{GL}_n(F_{\tilde{v}})$. We let Σ_p denote the set of places of F^+ dividing p and we assume that each place in Σ_p is split in F.

We fix a place $\wp$ of F^+ above p, a place $\widetilde{\wp}$ of F dividing $\wp$ and we set $L := F_\wp^+ \cong F_{\widetilde{\wp}}$. We have thus an isomorphism $i_{G,\widetilde{\wp}} : G(F_\wp^+) \xrightarrow{\sim} \mathrm{GL}_n(L)$. We also fix an irreducible $\mathbb{Q}_p$-algebraic representation $W^\wp$ of $\prod_{v|p,v\neq\wp} G(F_v^+)$ over E and a compact open subgroup $U_p^\wp = \prod_{v|p,v\neq\wp} U_v$ of $\prod_{v|p,v\neq\wp} G(F_v^+)$. We fix an open compact subgroup $U^p = \prod_{v\nmid p} U_v$ of $G(\mathbb{A}_{F^+}^{p,\infty})$ and we put $U^\wp := U^pU_p^\wp$. Set:

$$\widehat{S}(U^\wp, W^\wp) = \Big\{ f : G(F^+) \setminus G(\mathbb{A}_{F^+}^\infty)/U^p \longrightarrow W^\wp, \quad f \text{ is continuous and } f(gg_p^\wp) = (g_p^\wp)^{-1}(f(g)) \text{ for all } g \in G(\mathbb{A}_{F^+}^\infty) \text{ and all } g_p^\wp \in U_p^\wp \Big\}. \tag{6.1}$$

Let $\mathbb{W}^\wp$ be an $\mathcal{O}_E$-lattice of $W^\wp$ stable by $U_p^\wp$, we define $\widehat{S}(U^\wp, *)$ by replacing $W^\wp$ by $*$ in (6.1) for $* \in \{\mathbb{W}^\wp, \mathbb{W}^\wp/\varpi_E^s\}$ (where $s \geq 1$). Since $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact, $G(F^+) \setminus G(\mathbb{A}_{F^+}^\infty)/U^p$ is a profinite set. We see that $\widehat{S}(U^\wp, W^\wp)$ is a Banach space over E with the norm defined by the (complete) $\mathcal{O}_E$-lattice $\widehat{S}(U^\wp, \mathbb{W}^\wp)$. Moreover, $\widehat{S}(U^\wp, W^\wp)$ is equipped with a continuous action of $G(F_\wp^+) \cong \mathrm{GL}_n(L)$ given by $(g'f)(g) = f(gg')$ for $f \in \widehat{S}(U^\wp, W^\wp)$, $g' \in G(F_\wp^+)$, $g \in G(\mathbb{A}_{F^+}^\infty)$. The lattice $\widehat{S}(U^\wp, \mathbb{W}^\wp)$ is obviously stable by this action, so the Banach representation $\widehat{S}(U^\wp, W^\wp)$ of $\mathrm{GL}_n(L)$ is unitary. We also know (see e.g. the proof of Lemma 6.1) that $\widehat{S}(U^\wp, W^\wp)$ is admissible. Let $\Sigma(U^p)$ be the set of primes v of F^+ satisfying:

- $v \nmid p$ and v is totally split in F

- U_v is a maximal compact open subgroup of $G(F_v^+)$.

Let $\mathbb{T}(U^p) := \mathcal{O}_E[T_{\tilde{v}}^{(j)}]$ be the commutative polynomial $\mathcal{O}_E$-algebra generated by the formal variables $T_{\tilde{v}}^{(j)}$ where $j \in \{1, \cdots, n\}$ and $\tilde{v}$ is a finite place of F above a finite place v in $\Sigma(U^p)$. The $\mathcal{O}_E$-algebra $\mathbb{T}(U^p)$ acts on $\widehat{S}(U^\wp, W^\wp)$ and $\widehat{S}(U^\wp, \mathbb{W}^\wp)$ by making $T_{\tilde{v}}^{(j)}$ act by the double coset operator:

$$T_{\tilde{v}}^{(j)} := \left[U_v g_v i_{G,\tilde{v}}^{-1} \begin{pmatrix} \mathbf{1}_{n-j} & 0 \\ 0 & \varpi_{\tilde{v}}\, \mathbf{1}_j \end{pmatrix} g_v^{-1} U_v\right] \tag{6.2}$$

where $\varpi_{\tilde{v}}$ is a uniformizer of $F_{\tilde{v}}$, and where $g_v \in G(F_v^+)$ is such that $i_{G,\tilde{v}}(g_v^{-1} U_v g_v) = \mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$. This action commutes with that of $\mathrm{GL}_n(L)$.

Recall that the automorphic representations of $G(\mathbb{A}_{F^+})$ are the irreducible constituents of the $\mathbb{C}$-vector space of functions $f : G(F^+)\backslash G(\mathbb{A}_{F^+}) \to \mathbb{C}$ which are:

- $\mathcal{C}^\infty$ when restricted to $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$
- locally constant when restricted to $G(\mathbb{A}_{F^+}^\infty)$
- $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$-finite,

where $G(\mathbb{A}_{F^+})$ acts on this space via right translation. An automorphic representation π is isomorphic to $\pi_\infty \otimes_{\mathbb{C}} \pi^\infty$ where $\pi_\infty = W_\infty$ is an irreducible algebraic representation of $(\mathrm{Res}_{F^+/\mathbb{Q}}\, G)(\mathbb{R}) = G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ over $\mathbb{C}$ and $\pi^\infty \cong \mathrm{Hom}_{G(F^+\otimes_{\mathbb{Q}}\mathbb{R})}(W_\infty, \pi) \cong \otimes'_v \pi_v$ is an irreducible smooth representation of $G(\mathbb{A}_{F^+}^\infty)$. The algebraic representation $W_\infty|_{(\mathrm{Res}_{F^+/\mathbb{Q}}\, G)(\mathbb{Q})}$ is defined over $\overline{\mathbb{Q}}$ via ι_∞ and we denote by W_p its base change to $\overline{\mathbb{Q}_p}$ via ι_p, which is thus an irreducible algebraic representation of $(\mathrm{Res}_{F^+/\mathbb{Q}}\, G)(\mathbb{Q}_p) = G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ over $\overline{\mathbb{Q}_p}$. Via the decomposition $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p) \xrightarrow{\sim} \prod_{v \in \Sigma_p} G(F_v^+)$, one has $W_p \cong \otimes_{v \in \Sigma_p} W_v$ where W_v is an irreducible $\mathbb{Q}_p$-algebraic representation of $G(F_v^+)$ over $\overline{\mathbb{Q}_p}$. One can also prove π^∞ is defined over a number field via ι_∞ (e.g. see [1, § 6.2.3]). Denote by $\pi^{\infty,p} := \otimes'_{v \nmid p} \pi_v$, so that we have $\pi^\infty \cong \pi^{\infty,p} \otimes_{\overline{\mathbb{Q}}} \pi_p$ (seen over $\overline{\mathbb{Q}}$ via ι_∞), and by $m(\pi) \in \mathbb{Z}_{\geq 1}$ the multiplicity of π in the above space of functions $f : G(F^+)\backslash G(\mathbb{A}_{F^+}) \to \mathbb{C}$. Denote by $\widehat{S}(U^\wp, W^\wp)^{\mathrm{lalg}}$ the subspace of $\widehat{S}(U^\wp, W^\wp)$ of locally algebraic vectors for the $\mathrm{GL}_n(L)$-action, which is stable by $\mathbb{T}(U^p)$. We have an isomorphism which is equivariant under the action of $\mathrm{GL}_n(L) \times \mathbb{T}(U^p)$:

$$\widehat{S}(U^\wp, W^\wp)^{\mathrm{lalg}} \otimes_E \overline{\mathbb{Q}_p} \cong \bigoplus_\pi \left((\pi^{\infty,p})^{U^p} \otimes_{\overline{\mathbb{Q}}} (\otimes_{v|p, v \neq \wp} \pi_v^{U_v}) \otimes_{\overline{\mathbb{Q}}} (\pi_\wp \otimes_{\overline{\mathbb{Q}}} W_\wp)\right)^{m(\pi)} \tag{6.3}$$

where $\pi \cong \pi_\infty \otimes_{\mathbb{C}} \pi^\infty$ runs through the automorphic representations of $G(\mathbb{A}_{F_+})$ such that the algebraic representation W_p associated to π_∞ as above is of the form $W_p \cong W_\wp \otimes_E (W^\wp)^\vee$, where $(W^\wp)^\vee$ is the dual of $W^\wp$ and $W_\wp$ is a $\mathbb{Q}_p$-algebraic representation of $\mathrm{GL}_n(L)$ over $\overline{\mathbb{Q}_p}$, and where $T_{\tilde{v}}^{(j)} \in \mathbb{T}(U^p)$ acts on $(\pi^{\infty,p})^{U^p}$ by the double coset operator (6.2). Indeed, let $\widehat{S}(U^p, E)$ be as in [7, § 5], then we have $\widehat{S}(U^\wp, W^\wp) \cong \big(\widehat{S}(U^p, E) \otimes_E W^\wp\big)^{U_p^\wp}$. The isomorphism in (6.3) follows easily from [7, Prop. 5.1]. We also define for $* \in \{W^\wp, \mathbb{W}^\wp, \mathbb{W}^\wp/\varpi_E^s\}$:

$$\begin{aligned} S(U^\wp, *) := \Big\{ & f : G(F^+) \setminus G(\mathbb{A}_{F^+}^\infty)/U^p \longrightarrow *, \quad f \text{ is locally constant} \\ & \text{and } f(g g_p^\wp) = (g_p^\wp)^{-1}(f(g)) \text{ for all } g \in G(\mathbb{A}_{F^+}^\infty) \text{ and all } g_p^\wp \in U_p^\wp \Big\}. \end{aligned}$$

All these spaces are equipped with the action of $\mathrm{GL}_n(L) \times \mathbb{T}(U^p)$ by right translation on functions for $\mathrm{GL}_n(L)$ and by the double coset operators (6.2) for $\mathbb{T}(U^p)$. We have moreover $\mathrm{GL}_n(L) \times \mathbb{T}(U^p)$-equivariant isomorphisms:

$$\begin{aligned} &(6.4) \qquad S(U^\wp, \mathbb{W}^\wp/\varpi_E^s) \cong \widehat{S}(U^\wp, \mathbb{W}^\wp/\varpi_E^s) \cong S(U^\wp, \mathbb{W}^\wp)/\varpi_E^s \\ &(6.5) \qquad \widehat{S}(U^\wp, \mathbb{W}^\wp) \cong \varprojlim_s S(U^\wp, \mathbb{W}^\wp/\varpi_E^s) \\ &\qquad\qquad \widehat{S}(U^\wp, W^\wp) \cong \widehat{S}(U^\wp, \mathbb{W}^\wp) \otimes_{\mathcal{O}_E} E \\ &(6.6) \qquad S(U^\wp, W^\wp) \cong S(U^\wp, \mathbb{W}^\wp) \otimes_{\mathcal{O}_E} E \cong \widehat{S}(U^\wp, W^\wp)^{\mathrm{sm}}. \end{aligned}$$

Finally, for a compact open subgroup $U_\wp$ of $\mathrm{GL}_n(L)$, we define for $* \in \{W^\wp, \mathbb{W}^\wp, \mathbb{W}^\wp/\varpi_E^s\}$:

$$\begin{aligned} S(U^\wp U_\wp, *) := \Big\{ & f : G(F^+) \setminus G(\mathbb{A}_{F^+}^\infty)/U^p U_\wp \longrightarrow *, \\ & f(g g_p^\wp) = (g_p^\wp)^{-1}(f(g)) \text{ for all } g \in G(\mathbb{A}_{F^+}^\infty) \text{ and all } g_p^\wp \in U_p^\wp \Big\}. \end{aligned}$$

We thus have:

$$\varinjlim_{U_\wp} S(U^\wp U_\wp, *) = S(U^\wp, *). \tag{6.7}$$

Following [21, § 3.3], we say that U^p is *sufficiently small* if there is a place $v \nmid p$ such that 1 is the only element of finite order in U_v. The following (standard) lemma will be useful.

Lemma 6.1. *Assume U^p sufficiently small, then for any $U_p^\wp$, $\mathbb{W}^\wp$ as above and any compact open subgroup $U_\wp$ of $G(F^+_\wp)$ there is an integer $r \geq 1$ such that $\widehat{S}(U^\wp, \mathbb{W}^\wp)|_{U_\wp}$ is isomorphic to $\mathcal{C}(U_\wp, \mathcal{O}_E)^{\oplus r}$.*

Proof. Let $\widehat{S}(U^p, \mathcal{O}_E) := \{f : G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p \to \mathcal{O}_E,\ f \text{ is continuous}\}$. The $\mathcal{O}_E$-module $\widehat{S}(U^p, \mathcal{O}_E)$ is equipped with a natural action of $G(F^+ \otimes_\mathbb{Q} \mathbb{Q}_p) \times \mathbb{T}(U^p)$, and we have an isomorphism $\widehat{S}(U^\wp, \mathbb{W}^\wp) \cong (\widehat{S}(U^p, \mathcal{O}_E) \otimes_{\mathcal{O}_E} \mathbb{W}^\wp)^{U_p^\wp}$.

(a) We first show that for any compact open subgroup U_p of $G(F^+ \otimes_\mathbb{Q} \mathbb{Q}_p)$ there exist an integer r' such that $\widehat{S}(U^p, \mathcal{O}_E)|_{U_p} \cong C(U_p, \mathcal{O}_E)^{r'}$. Since U^p is sufficiently small, we have $U^p U_p \cap gG(F^+)g^{-1} = \{1\}$ for all $g \in G(\mathbb{A}^\infty_{F^+})$ (the left hand side is a finite group as $G(F^+)$ is discrete in $G(\mathbb{A}^\infty_{F^+})$, then U^p being sufficiently small implies it has to be $\{1\}$). From which we deduce a U_p-invariant isomorphism:

$$\amalg_s\, sU_p \xrightarrow{\sim} G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p, \quad sh \longmapsto sh \tag{6.8}$$

where $h \in U_p$ and s runs through a representative set of $G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p U_p$. Indeed, first (6.8) is clearly surjective. If $s_1 h_1 = s_2 h_2$ in $G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^p$ (for $h_1, h_2 \in U_p$), we have $s_1 = s_2 = s$, and there exist $g \in G(F^+)$, $u \in U^p$ such that $sh_1 = gsh_2u$ in $G(\mathbb{A}^\infty_{F^+})$. This implies g lies in $s^{-1}(U^p U_p)s \cap G(F^+) = \{1\}$, and the injectivity follows. From (6.8), we deduce (a).

(b) From (a) we deduce using $U_p = U_\wp U_p^\wp$:

$$\widehat{S}(U^\wp, \mathbb{W}^\wp)|_{U_\wp} \cong \mathcal{C}(U_\wp, \mathcal{O}_E) \widehat{\otimes}_{\mathcal{O}_E} [\mathcal{C}(U_p^\wp, \mathcal{O}_E)^{r'} \otimes_{\mathcal{O}_E} \mathbb{W}^\wp]^{U_p^\wp}.$$

Since $[\mathcal{C}(U_p^\wp, \mathcal{O}_E)^{r'} \otimes_{\mathcal{O}_E} \mathbb{W}^\wp]^{U_p^\wp}$ is easily checked to be a finite free $\mathcal{O}_E$-module, the lemma follows with r the rank of this $\mathcal{O}_E$-module. ☐

Let $\mathrm{Gal}_F := \mathrm{Gal}(\overline{F}/F)$, $\rho\colon \mathrm{Gal}_F \to \mathrm{GL}_n(E)$ a continuous representation and assume ρ is unramified for $v \in \Sigma(U^p)$. We associate to ρ the unique maximal ideal $\mathfrak{m}_\rho$ of residue field E of $\mathbb{T}(U^p)[1/p]$ such that for any $v \in \Sigma(U^p)$ and $\tilde{v}$ a place of F above v, the characteristic polynomial of $\rho(\mathrm{Frob}_{\tilde{v}})$, where $\mathrm{Frob}_{\tilde{v}}$ is a *geometric* Frobenius at $\tilde{v}$, is given by (compare [14, § 4.2]):

$$X^n + \cdots + (-1)^j (N\tilde{v})^{\frac{j(j-1)}{2}} \theta_\rho(T_{\tilde{v}}^{(j)}) X^{n-j} + \cdots + (-1)^n (N\tilde{v})^{\frac{n(n-1)}{2}} \theta_\rho(T_{\tilde{v}}^{(n)}) \tag{6.9}$$

where $N\tilde{v}$ is the cardinality of the residue field at $\tilde{v}$ and $\theta_\rho : \mathbb{T}(U^p)[1/p]/\mathfrak{m}_\rho \xrightarrow{\sim} E$. Recall that (see for example [21, Prop. 3.3.4]) if $\widehat{S}(U^\wp, W^\wp)[\mathfrak{m}_\rho]^{\mathrm{lalg}} \neq 0$

then $\rho_{\widetilde{\wp}}$ is in particular de Rham with distinct Hodge-Tate weights. We end this section by our main local-global compatibility conjecture when $n = 3$ and $L = \mathbb{Q}_p$. If $\rho_p : \mathrm{Gal}_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_3(E)$ is a semi-stable representation such that $N^2 \neq 0$ on $D_{\mathrm{st}}(\rho_p)$, there exists a unique triangulation $\mathcal{R}_E(\delta_1) - \mathcal{R}_E(\delta_2) - \mathcal{R}_E(\delta_3)$ on the (φ, Γ)-module $D_{\mathrm{rig}}(\rho_p)$ (with $\mathcal{R}_E(\delta_1)$ as unique subobject and $\mathcal{R}_E(\delta_3)$ as unique quotient). If $(D_{\mathrm{rig}}(\rho_p), (\delta_1, \delta_2, \delta_3))$ is (special) noncritical and if the (φ, Γ)-modules $\mathcal{R}_E(\delta_1) - \mathcal{R}_E(\delta_2)$ and $\mathcal{R}_E(\delta_2) - \mathcal{R}_E(\delta_3)$ satisfy Hypothesis 3.26, we say that $D_{\mathrm{rig}}(\rho_p)$ is *sufficiently generic.* We have then associated to such a ρ_p a finite length locally analytic representation $\Pi(\rho_p)$ at the end of § 3.3.4 which determines and only depends on ρ_p.

Conjecture 6.2. *Assume $n = 3$ and $F^+_{\wp} \cong F_{\widetilde{\wp}} = \mathbb{Q}_p$. Let $\rho : \mathrm{Gal}_F \to \mathrm{GL}_3(E)$ be a continuous absolutely irreducible representation which is unramified at the places of $\Sigma(U^p)$ and such that:*

- $\widehat{S}(U^{\wp}, W^{\wp})[\mathfrak{m}_\rho]^{\mathrm{lalg}} \neq 0$
- $\rho_{\widetilde{\wp}} := \rho|_{\mathrm{Gal}_{F_{\widetilde{\wp}}}}$ *is semi-stable with $N^2 \neq 0$ on $D_{\mathrm{st}}(\rho_{\widetilde{\wp}})$*
- $D_{\mathrm{rig}}(\rho_{\widetilde{\wp}})$ *is sufficiently generic.*

Let $\Pi(\rho_{\widetilde{\wp}})$ be the locally analytic representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ at the very end of § 3.3.4, then the following restriction morphism is bijective (recall we have $\Pi(\rho_{\widetilde{\wp}})^{\mathrm{lalg}} = \mathrm{soc}_{\mathrm{GL}_3(\mathbb{Q}_p)} \Pi(\rho_{\widetilde{\wp}})$):

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\Pi(\rho_{\widetilde{\wp}}), \widehat{S}(U^{\wp}, W^{\wp})[\mathfrak{m}_\rho]\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\Pi(\rho_{\widetilde{\wp}})^{\mathrm{lalg}}, \widehat{S}(U^{\wp}, W^{\wp})[\mathfrak{m}_\rho]\big).$$

6.2. Hecke operators

We give (or recall) the definition of some useful pro-p-Hecke algebras and of their localisations.

We keep the notation of § 6.1. For $s \in \mathbb{Z}_{>0}$ and a compact open subgroup $U_\wp$ of $\mathrm{GL}_n(L)$, we let $\mathbb{T}(U^{\wp}U_\wp, \mathbb{W}^{\wp}/\varpi_E^s)$ (resp. $\mathbb{T}(U^{\wp}U_\wp, \mathbb{W}^{\wp})$) be the $\mathcal{O}_E/\varpi_E^s$-subalgebra (resp. $\mathcal{O}_E$-subalgebra) of the endomorphism ring of $S(U^{\wp}U_\wp, \mathbb{W}^{\wp}/\varpi_E^s)$ (resp. $S(U^{\wp}U_\wp, \mathbb{W}^{\wp})$) generated by the operators in $\mathbb{T}(U^p)$. Since $S(U^{\wp}U_\wp, \mathbb{W}^{\wp}/\varpi_E^s)$ (resp. $S(U^{\wp}U_\wp, \mathbb{W}^{\wp})$) is a finite free $\mathcal{O}_E/\varpi_E^s$-module (resp. $\mathcal{O}_E$-module), $\mathbb{T}(U^{\wp}U_\wp, \mathbb{W}^{\wp}/\varpi_E^s)$ is a finite $\mathcal{O}_E/\varpi_E^s$-algebra (resp. $\mathbb{T}(U^{\wp}U_\wp, \mathbb{W}^{\wp})$ an $\mathcal{O}_E$-algebra which is finitely free as $\mathcal{O}_E$-module). For $s' \leq s$, since we have:

$$S(U^{\wp}U_\wp, \mathbb{W}^{\wp}/\varpi_E^s) \otimes_{\mathcal{O}_E/\varpi_E^s} \mathcal{O}_E/\varpi_E^{s'} \cong S(U^{\wp}U_\wp, \mathbb{W}^{\wp}/\varpi_E^{s'}), \tag{6.10}$$

it is easy to see:

$$\mathbb{T}(U^{\wp}U_{\wp},\mathbb{W}^{\wp}) \xrightarrow{\sim} \varprojlim_{s} \mathbb{T}(U^{\wp}U_{\wp},\mathbb{W}^{\wp}/\varpi_E^s). \tag{6.11}$$

For $U_{\wp,2} \subseteq U_{\wp,1}$ an inclusion of compact open subgroups of $\mathrm{GL}_n(L)$, the natural injections:

$$\begin{aligned} S(U^{\wp}U_{\wp,1},\mathbb{W}^{\wp}/\varpi_E^s) &\hookrightarrow S(U^{\wp}U_{\wp,2},\mathbb{W}^{\wp}/\varpi_E^s) \\ S(U^{\wp}U_{\wp,1},\mathbb{W}^{\wp}) &\hookrightarrow S(U^{\wp}U_{\wp,2},\mathbb{W}^{\wp}) \end{aligned}$$

induce natural surjections:

$$\begin{aligned} \mathbb{T}(U^{\wp}U_{\wp,2},\mathbb{W}^{\wp}/\varpi_E^s) &\twoheadrightarrow \mathbb{T}(U^{\wp}U_{\wp,1},\mathbb{W}^{\wp}/\varpi_E^s) \\ \mathbb{T}(U^{\wp}U_{\wp,2},\mathbb{W}^{\wp}) &\twoheadrightarrow \mathbb{T}(U^{\wp}U_{\wp,1},\mathbb{W}^{\wp}) \end{aligned}$$

giving rise to projective systems when $U_{\wp}$ gets smaller. From (6.11) we deduce isomorphisms:

$$\begin{aligned} \widetilde{\mathbb{T}}(U^{\wp}) &:= \varprojlim_{s} \varprojlim_{U_{\wp}} \mathbb{T}(U^{\wp}U_{\wp},\mathbb{W}^{\wp}/\varpi_E^s) \\ &\cong \varprojlim_{U_{\wp}} \varprojlim_{s} \mathbb{T}(U^{\wp}U_{\wp},\mathbb{W}^{\wp}/\varpi_E^s) \cong \varprojlim_{U_{\wp}} \mathbb{T}(U^{\wp}U_{\wp},\mathbb{W}^{\wp}). \end{aligned} \tag{6.12}$$

Lemma 6.3. *The $\mathcal{O}_E$-algebra $\widetilde{\mathbb{T}}(U^{\wp})$ is reduced and acts faithfully on $\widehat{S}(U^{\wp},W^{\wp})$.*

Proof. By construction, the algebra $\widetilde{\mathbb{T}}(U^{\wp})$ acts $\mathcal{O}_E$-linearly and faithfully on $S(U^{\wp},\mathbb{W}^{\wp}) \cong \varinjlim_{U_{\wp}} S(U^{\wp}U_{\wp},\mathbb{W}^{\wp})$. By (6.5) and (6.4), this action extends naturally to an $\mathcal{O}_E$-linear faithful action of $\widetilde{\mathbb{T}}(U^{\wp})$ on $\widehat{S}(U^{\wp},\mathbb{W}^{\wp})$ and hence to an E-linear faithful action on $\widehat{S}(U^{\wp},W^{\wp})$. Since the operators in $\mathbb{T}(U^p)$ acting on $S(U^{\wp},W^{\wp})$ are semi-simple (which easily follows from (6.6) and (6.3)), we deduce $\widetilde{\mathbb{T}}(U^{\wp})$ is reduced. $\square$

To a continuous representation $\overline{\rho}\colon \mathrm{Gal}_F \to \mathrm{GL}_n(k_E)$ which is unramified for $v \in \Sigma(U^p)$, we associate a maximal ideal $\mathfrak{m}_{\overline{\rho}}$ of residue field k_E of $\mathbb{T}(U^p)$ by the same formula as (6.9) replacing θ_ρ by $\theta_{\overline{\rho}} : \mathbb{T}(U^p)/\mathfrak{m}_{\overline{\rho}} \xrightarrow{\sim} k_E$.

Definition 6.4. *A maximal ideal $\mathfrak{m}$ of $\mathbb{T}(U^p)$ is called $(U^{\wp},\mathbb{W}^{\wp})$-automorphic if there exist an integer s and a compact open subgroup $U_{\wp}$ as above such that the image of $\mathfrak{m}$ in $\mathbb{T}(U^{\wp}U_{\wp},\mathbb{W}^{\wp}/\varpi_E^s)$ is still a maximal ideal, or* *equivalently such that the localisation* $S(U^{\wp}U_{\wp}, \mathbb{W}^{\wp}/\varpi_E^s)_{\mathfrak{m}}$ *is nonzero. A continuous representation* $\overline{\rho}\colon \mathrm{Gal}_F \to \mathrm{GL}_n(k_E)$ *is called* $(U^{\wp}, \mathbb{W}^{\wp})$*-automorphic if* $\mathfrak{m}_{\overline{\rho}}$ *is* $(U^{\wp}, \mathbb{W}^{\wp})$*-automorphic.*

Lemma 6.5. *There are finitely many* $(U^{\wp}, \mathbb{W}^{\wp})$*-automorphic maximal ideals of* $\mathbb{T}(U^p)$*.*

Proof. By (6.10) and (6.7), the maximal ideal $\mathfrak{m}$ is $(U^{\wp}, \mathbb{W}^{\wp})$-automorphic if and only if the $\mathrm{GL}_n(L)$-representation $S(U^{\wp}, \mathbb{W}^{\wp}/\varpi_E)_{\mathfrak{m}}$ is nonzero. Let $U_{\wp}$ be a pro-p compact open subgroup of $\mathrm{GL}_n(L)$. Suppose $S(U^{\wp}, \mathbb{W}^{\wp}/\varpi_E)_{\mathfrak{m}} \neq 0$, then we have (using exactness of localization):

$$S(U^{\wp}U_{\wp}, \mathbb{W}^{\wp}/\varpi_E)_{\mathfrak{m}} = \big(S(U^{\wp}, \mathbb{W}^{\wp}/\varpi_E)_{\mathfrak{m}}\big)^{U_{\wp}} \neq 0.$$

Hence the image of $\mathfrak{m} \subset \mathbb{T}(U^p)$ in $\mathbb{T}(U^{\wp}U_{\wp}, k_E)$ is still a maximal ideal in $\mathbb{T}(U^{\wp}U_{\wp}, k_E)$. Since $\mathbb{T}(U^{\wp}U_{\wp}, k_E)$ is Artinian, it only has a finite number of maximal ideals, and the lemma follows. $\square$

If $\mathfrak{m}$ is $(U^{\wp}, \mathbb{W}^{\wp})$-automorphic, by (6.12), Lemma 6.5 and the proof, we can associate to $\mathfrak{m}$ a maximal ideal (still denoted) $\mathfrak{m}$ of $\widetilde{\mathbb{T}}(U^{\wp})$ of residue field k_E. We have $\mathbb{T}(U^{\wp}U_{\wp}, \mathbb{W}^{\wp}/\varpi_E^s) \cong \prod_{\mathfrak{m}} \mathbb{T}(U^{\wp}U_{\wp}, \mathbb{W}^{\wp}/\varpi_E^s)_{\mathfrak{m}}$ for any pro-p compact open subgroup of $\mathrm{GL}_n(L)$, where the product is over the $(U^{\wp}, \mathbb{W}^{\wp})$-automorphic maximal ideals $\mathfrak{m}$ of $\mathbb{T}(U^p)$. We then deduce by (6.12) $\widetilde{\mathbb{T}}(U^{\wp}) \xrightarrow{\sim} \prod_{\mathfrak{m}} \widetilde{\mathbb{T}}(U^{\wp})_{\mathfrak{m}}$ and isomorphisms:

$$\widetilde{\mathbb{T}}(U^{\wp})_{\mathfrak{m}} \cong \varprojlim_{s} \varprojlim_{U_{\wp}} \mathbb{T}(U^{\wp}U_{\wp}, \mathbb{W}^{\wp}/\varpi_E^s)_{\mathfrak{m}} \cong \varprojlim_{U_{\wp}} \mathbb{T}(U^{\wp}U_{\wp}, \mathbb{W}^{\wp})_{\mathfrak{m}}. \tag{6.13}$$

Note that $\mathbb{T}(U^{\wp}U_{\wp}, \mathbb{W}^{\wp}/\varpi_E^s)_{\mathfrak{m}}$ (resp. $\mathbb{T}(U^{\wp}U_{\wp}, \mathbb{W}^{\wp})_{\mathfrak{m}}$) is isomorphic to the $\mathcal{O}_E$-subalgebra of the endomorphism ring of $S(U^{\wp}U_{\wp}, \mathbb{W}^{\wp}/\varpi_E^s)_{\mathfrak{m}}$ (resp. of the endomorphism ring of $S(U^{\wp}U_{\wp}, \mathbb{W}^{\wp})_{\mathfrak{m}} \cong \varprojlim_s S(U^{\wp}U_{\wp}, \mathbb{W}^{\wp}/\varpi_E^s)_{\mathfrak{m}}$) generated by the operators in $\mathbb{T}(U^p)$. It is also easy to see that:

$$\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\mathfrak{m}} \cong \varprojlim_{s} \varinjlim_{U_{\wp}} S(U^{\wp}U_{\wp}, \mathbb{W}^{\wp}/\varpi_E^s)_{\mathfrak{m}} \tag{6.14}$$

is a direct summand of $\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})$ (where the localisation $\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\mathfrak{m}}$ is with respect to the $\widetilde{\mathbb{T}}(U^{\wp})$-module structure on $\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})$, which might be different from the localisation at $\mathfrak{m}$ with respect to the $\mathbb{T}(U^p)$-module structure). When $\mathfrak{m} = \mathfrak{m}_{\overline{\rho}}$ comes from a continuous $\overline{\rho}\colon \mathrm{Gal}_F \to \mathrm{GL}_n(k_E)$ as at the beginning of § 6.2, we simply denote by $M_{\overline{\rho}}$ the localisation of a $\widetilde{\mathbb{T}}(U^{\wp})$-module (resp. of a $\mathbb{T}(U^p)$-module) M at $\mathfrak{m}_{\overline{\rho}}$. We easily check $\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}} \cong$

$\widehat{S}(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}} \otimes_{\mathcal{O}_E} E$. The following result is then a consequence of Lemma 6.3 and its proof.

Lemma 6.6. *Let $\overline{\rho}$ be $(U^\wp, \mathbb{W}^\wp)$-automorphic, then the local $\mathcal{O}_E$-algebra $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}$ is reduced and acts faithfully on $\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}$.*

6.3. *P*-ordinary automorphic representations

We relate the space $\mathrm{Ord}_P(S(U^\wp, W^\wp)_{\overline{\rho}})$ to P-ordinary Galois representations (§ 5.1).

We keep the previous notation. We let $\overline{\rho}\colon \mathrm{Gal}_F \to \mathrm{GL}_n(k_E)$ be $(U^\wp, \mathbb{W}^\wp)$-automorphic and *absolutely irreducible*. We fix P a parabolic subgroup of GL_n as in § 5.1. Recall we have from (4.15):

$$\mathrm{Ord}_P(S(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}}) \cong \varinjlim_i \left(S(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}}^{I_{i,i}}\right)_{\mathrm{ord}}$$

where $(I_{i,i})_i$ is as in § 4.1 with $(I_i)_i$ as in Example 4.3. For any $i \geq 0$, $(S(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}}^{I_{i,i}})_{\mathrm{ord}} = (S(U^\wp I_{i,i}, \mathbb{W}^\wp)_{\overline{\rho}})_{\mathrm{ord}}$ is stable by $\mathbb{T}(U^\wp)$ (since the action of $\mathbb{T}(U^\wp)$ on $S(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}}^{I_{i,i}}$ commutes with that of L_P^+), and we denote by $\mathbb{T}(U^\wp I_{i,i}, \mathbb{W}^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}$ the $\mathcal{O}_E$-subalgebra of the endomorphism ring of $(S(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}}^{I_{i,i}})_{\mathrm{ord}}$ generated by the operators in $\mathbb{T}(U^p)$. Since:

$$\left(S(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}}^{I_{i,i}}\right)_{\mathrm{ord}} \hookrightarrow S(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}}^{I_{i,i}} \cong S(U^\wp I_{i,i}, \mathbb{W}^\wp)_{\overline{\rho}},$$

we have a natural surjection of local $\mathcal{O}_E$-algebras (finite free over $\mathcal{O}_E$):

$$\mathbb{T}(U^\wp I_{i,i}, \mathbb{W}^\wp)_{\overline{\rho}} \twoheadrightarrow \mathbb{T}(U^\wp I_{i,i}, \mathbb{W}^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}.$$

We set:

$$\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}} := \varprojlim_i \mathbb{T}(U^\wp I_{i,i}, \mathbb{W}^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}$$

which is thus easily checked to be a quotient of $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}$ and is also a complete local $\mathcal{O}_E$-algebra of residue field k_E. Moreover, as in the proof of Lemma 6.3, the operators in $\mathbb{T}(U^p)$ acting on $(S(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}}^{I_{i,i}})_{\mathrm{ord}} \otimes_{\mathcal{O}_E} E$ are semi-simple (since they are so on $S(U^\wp, W^\wp)$). We have as in *loc. cit.* the following consequence.

Lemma 6.7. *The $\mathcal{O}_E$-algebra $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}$ is reduced and the natural action of $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}$ on $\mathrm{Ord}_p(S(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}})$ and $\mathrm{Ord}_P(S(U^\wp, W^\wp)_{\overline{\rho}})$ is faithful.*

From now on we assume that the compact open subgroup U^p is sufficiently small (see the end of § 6.1).

Lemma 6.8. *(1) The $\mathcal{O}_E$-module $\mathrm{Ord}_P(S(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}})$ is dense for the p-adic topology in $\mathrm{Ord}_P(\widehat{S}(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}})$ (see (4.8) for the latter). Consequently, the action of $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$ on $\mathrm{Ord}_P(S(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}})$ extends to a faithful action on $\mathrm{Ord}_P(\widehat{S}(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}})$.*

(2) The representation $\mathrm{Ord}_P(\widehat{S}(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}})|_{L_P(\mathcal{O}_L)}$ is isomorphic to a direct summand of $\mathcal{C}(L_P(\mathcal{O}_L),\mathcal{O}_E)^{\oplus r}$ for some $r\geq 1$.

Proof. (1) From Lemma 6.1 we deduce that there exist $r\geq 1$ and a $\mathrm{GL}_n(\mathcal{O}_L)$-representation Q such that:

$$\widehat{S}(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}}|_{\mathrm{GL}_n(\mathcal{O}_L)}\oplus Q\cong \mathcal{C}(\mathrm{GL}_n(\mathcal{O}_L),\mathcal{O}_E)^{\oplus r} \tag{6.15}$$

which implies using (6.6) that $S(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}}|_{\mathrm{GL}_n(\mathcal{O}_L)}$ is a direct summand of $\mathcal{C}^{\infty}(\mathrm{GL}_n(\mathcal{O}_L),\mathcal{O}_E)^{\oplus r}$. It is easy to see that the condition in Remark 4.15(2) is satisfied with $V^0=\mathcal{C}^{\infty}(\mathrm{GL}_n(\mathcal{O}_L),\mathcal{O}_E)$, which then implies it is also satisfied with $V^0=S(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}}$. Thus the natural injection from (4.17):

$$\mathrm{Ord}_P(S(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}})/\varpi_E^s\cong\Big(\varinjlim_i\big(S(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}}^{I_{i,i}}\big)_{\mathrm{ord}}\Big)/\varpi_E^s\lhook\joinrel\longrightarrow \mathrm{Ord}_P\big(\widehat{S}(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}}/\varpi_E^s\big) \tag{6.16}$$

is actually an isomorphism for all $s\geq 1$ by the proof of Lemma 4.16. Then (1) follows (see also Remark 4.17).

(2) The statement follows from (6.15) and Corollary 4.6. □

We now make the following further hypothesis on G and F *till the end of the paper.*

Hypothesis 6.9. *We have either ($p>2$, $n\leq 3$) or ($p>2$, F/F^+ is unramified and G is quasi-split at all finite places of F^+).*

When $n\leq 3$, Rogawski's well-known results ([71]) imply that strong base change holds from G/F^+ to GL_n/F. When F/F^+ is moreover unramified, it also holds by well-known results of Labesse ([55]).

Remark 6.10. It is possible that for $n>3$ the recent results ([60], [50]) now allow to relax (for this paper) some of the assumptions in Hypothesis 6.9. Note that the main result of the paper will be anyway for $n=3$.

We now also assume that U_v is maximal in $\mathrm{GL}_n(L)=\mathrm{GL}_n(F_{\tilde{v}})$ for all $v|p$, $v\neq\wp$. Let $S(U^p)$ be the union of Σ_p and of the places $v\notin\Sigma_p$ such that

U_v is not hyperspecial. Since $\overline{\rho}$ is $(U^\wp, \mathbb{W}^\wp)$-automorphic, recall we have in particular that $\overline{\rho}$ is unramified outside $S(U^p)$ and $\overline{\rho}^\vee \circ c \cong \overline{\rho} \otimes \overline{\varepsilon}^{n-1}$ where $\overline{\rho}^\vee$ is the dual of $\overline{\rho}$ and c is the nontrivial element in $\mathrm{Gal}(F/F^+)$. The functor $A \mapsto \rho_A$ of (isomorphism classes of) deformations of $\overline{\rho}$ on the category of local artinian $\mathcal{O}_E$-algebras A of residue field k_E satisfying that ρ_A is unramified outside $S(U^p)$ and that $\rho_A^\vee \circ c \cong \rho_A \otimes \varepsilon^{n-1}$ is pro-representable by a complete local noetherian algebra of residue field k_E denoted by $R_{\overline{\rho}, S(U^p)}$. By [80, Prop. 6.7] (which holds under Hypothesis 6.9, this is the place where $p > 2$ is required), for any compact open subgroup $U_\wp$ of $\mathrm{GL}_n(L)$, we have a natural surjection of local $\mathcal{O}_E$-algebras $R_{\overline{\rho}, S(U^p)} \twoheadrightarrow \mathbb{T}(U^\wp U_\wp, \mathbb{W}^\wp)_{\overline{\rho}}$, from which we easily deduce using (6.13) a surjection of local complete $\mathcal{O}_E$-algebras:

$$R_{\overline{\rho}, S(U^p)} \longrightarrow\!\!\!\!\!\rightarrow \widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}. \tag{6.17}$$

In particular, $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}$ and $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}$ are noetherian (local complete) $\mathcal{O}_E$-algebras.

Lemma 6.11. *The representation* $\mathrm{Ord}_P(\widehat{S}(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}})$ *is a* ϖ_E*-adically admissible representation of* $L_P(L)$ *over* $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}$ *in the sense of [42, Def. 3.1.1].*

Proof. The lemma follows by the same argument as in the proof of [42, Lem. 5.3.5] with (5.3.3) of *loc. cit.* replaced by the isomorphism (6.16). ☐

Assume now that $\overline{\rho}_{\widetilde{\wp}} := \overline{\rho}|_{\mathrm{Gal}_{F_{\widetilde{\wp}}}}$ is strictly P-ordinary (cf. Definition 5.8) and is isomorphic to a successive extension of $\overline{\rho}_i$ for $i = 1, \cdots, k$ with $\overline{\rho}_i : \mathrm{Gal}_L \to \mathrm{GL}_{n_i}(k_E)$ (recall $L \cong F_{\widetilde{\wp}}$). The restriction to $\mathrm{Gal}_{F_{\widetilde{\wp}}}$ gives a natural morphism:

$$R_{\overline{\rho}_{\widetilde{\wp}}} \longrightarrow R_{\overline{\rho}, S(U^p)}. \tag{6.18}$$

We fix $\rho\colon \mathrm{Gal}_F \to \mathrm{GL}_n(E)$ a continuous representation such that ρ is unramified outside $S(U^p)$ and $\rho^\vee \circ c \cong \rho \otimes \varepsilon^{1-n}$. We set $\mathfrak{p}_\rho := \mathfrak{m}_\rho \cap \mathbb{T}(U^p)$, which is a prime ideal of $\mathbb{T}(U^p)$ (see (6.9) for $\mathfrak{m}_\rho$), and $\rho_{\widetilde{\wp}} := \rho|_{\mathrm{Gal}_{F_{\widetilde{\wp}}}}$. We assume $\widehat{S}(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}}[\mathfrak{p}_\rho] \neq 0$, then $\mathfrak{p}_\rho$ can also be seen as a prime ideal of $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}$ (using (6.12)). Note that this implies that the mod p semi-simplification of ρ is isomorphic to $\overline{\rho}$ (and is thus irreducible).

Theorem 6.12. *(1) The action of* $R_{\overline{\rho}_{\widetilde{\wp}}}$ *on* $\mathrm{Ord}_P(\widehat{S}(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}})$ *via (6.18) and (6.17) factors through* $R_{\overline{\rho}_{\widetilde{\wp}}}^{P-\mathrm{ord}}$ *(see the very end of § 5.1).*

(2) If $\mathrm{Ord}_P(\widehat{S}(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}}[\mathfrak{p}_\rho]) \neq 0$ *then* $\rho_{\widetilde{\wp}}$ *is* P*-ordinary.*

Proof. (1) Assume first $S(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}}[\mathfrak{p}_{\rho}] \neq 0$. By (6.6) and (6.3), there is an automorphic representation π of $G(\mathbb{A}_{F_+})$ (with $W_{\wp}$ trivial in (6.3)) which contributes to:

$$S(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_{\rho}] \cong S(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}}[\mathfrak{p}_{\rho}] \otimes_{\mathcal{O}_E} E.$$

By the local-global compatibility for classical local Langlands correspondence (see e.g. [80, Thm. 6.5(v)] and [17] taking into account our various normalisations and note that this uses Hypothesis 6.9 via strong base change), $\rho_{\widetilde{\wp}}$ is potentially semi-stable with $\mathrm{HT}_{\sigma}(\rho_{\widetilde{\wp}}) = \{1-n, \cdots, 0\}$ for all $\sigma : L \hookrightarrow E$ and $\mathrm{rec}(\pi_{\wp})(\frac{1-n}{2}) \cong \mathrm{W}(\rho_{\widetilde{\wp}})^{\mathrm{ss}}$ where $\pi_{\wp}$ is the $\wp$-th component of π and is viewed as a representation of $\mathrm{GL}_n(L)$ via $i_{G,\widetilde{\wp}}$ (see § 5.2 for the notation). If $\mathrm{Ord}_P(S(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_{\rho}]) \neq 0$, then there exists π as above such that moreover $\mathrm{Ord}_P(\pi_{\wp}) \neq 0$ (since we actually have $S(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_{\rho}] \cong \pi_{\wp}^{\oplus r}$ as $\mathrm{GL}_n(L)$-representations for some $r \geq 1$). It follows from Lemma 4.10 and Proposition 5.10 that $\rho_{\widetilde{\wp}}$ is P-ordinary. Denote by $I^{P-\mathrm{ord}}$ the kernel of the natural surjection $R_{\overline{\rho}_{\widetilde{\wp}}} \twoheadrightarrow R_{\overline{\rho}_{\widetilde{\wp}}}^{P-\mathrm{ord}}$, which we also view as an ideal of $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}$ via:

$$R_{\overline{\rho}_{\widetilde{\wp}}} \xrightarrow{(6.18)} R_{\overline{\rho}, S(U^p)} \xrightarrow{(6.17)} \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}. \tag{6.19}$$

Then Lemma 5.9 easily implies $I^{P-\mathrm{ord}} \subseteq \mathfrak{p}_{\rho}$, in particular $S(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}}[\mathfrak{p}_{\rho}]$ is killed by $I^{P-\mathrm{ord}}$. With (6.6) and (6.3) we deduce that $\mathrm{Ord}_P(S(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}})$ is also killed by $I^{P-\mathrm{ord}}$. By Lemma 6.8(1), $\mathrm{Ord}_P(S(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}})$ is dense in $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}})$ for the ϖ_E-adic topology. We deduce then:

$$I^{P-\mathrm{ord}} \,\mathrm{Ord}_P\left(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}}\right) \subset \cap_{i \in \mathbb{Z}_{\geq 0}} \varpi_E^i \,\mathrm{Ord}_P\left(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}}\right) = 0$$

and (1) follows.

(2) Let $\mathfrak{p}_{\rho_{\widetilde{\wp}}}$ be the prime ideal of $R_{\overline{\rho}_{\widetilde{\wp}}}$ attached to $\rho_{\widetilde{\wp}}$, which is just the preimage of $\mathfrak{p}_{\rho}$ via (6.19), and $\mathfrak{m}_{\rho_{\widetilde{\wp}}} := \mathfrak{p}_{\rho_{\widetilde{\wp}}}[1/p]$, which is a maximal ideal of $R_{\overline{\rho}_{\widetilde{\wp}}}[1/p]$. If $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}}[\mathfrak{p}_{\rho}]) \neq 0$ then we have $I^{P-\mathrm{ord}}[1/p] \subseteq \mathfrak{m}_{\rho_{\widetilde{\wp}}}$, since otherwise $1 \in \mathfrak{m}_{\rho_{\widetilde{\wp}}} + I^{P-\mathrm{ord}}[1/p]$ annihilates $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_{\rho}]) = \mathrm{Ord}_P(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}}[\mathfrak{p}_{\rho}]) \otimes_{\mathcal{O}_E} E$ by the first part. From the discussion above Proposition 5.7, we obtain that $\rho_{\widetilde{\wp}}$ is P-ordinary. $\square$

By Theorem 6.12(1) and the last part in Lemma 6.8(1), the surjection $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}} \twoheadrightarrow \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$ factors through:

$$\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}} \otimes_{R_{\overline{\rho}_{\widetilde{\wp}}}} R_{\overline{\rho}_{\widetilde{\wp}}}^{P-\mathrm{ord}} \twoheadrightarrow \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}.$$

In particular, we have natural morphisms of local complete noetherian $\mathcal{O}_E$-algebras of residue field k_E:

$$\omega : \widehat{\bigotimes}_{i=1,\cdots,k} R_{\overline{\rho}_i} \xrightarrow{(5.4)} R_{\overline{\rho}_{\widetilde{\wp}}}^{P-\mathrm{ord}} \longrightarrow \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}. \tag{6.20}$$

We end this section by the following proposition.

Proposition 6.13. *Let π_∞ be a smooth admissible representation of $L_P(L)$, λ be a dominant weight as in the beginning of § 4.3 (for $G = \mathrm{GL}_n$ and P as above), x be a closed point of $\mathrm{Spec}(\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p])$, and $\mathfrak{m}_x$ be the associated maximal ideal. Then any $L_P(L)$-equivariant morphism:*

$$\pi_\infty \otimes_E L_P(\lambda) \longrightarrow \mathrm{Ord}_P\left(\widehat{S}(U^{\wp}, W^{\wp})\right)\{\mathfrak{m}_x\}$$

has image in $\mathrm{Ord}_P\left(\widehat{S}(U^{\wp}, W^{\wp})^{\mathrm{lalg}}\right)[\mathfrak{m}_x]$.

Proof. Replacing $\pi_\infty \otimes_E L_P(\lambda)$ by its image, we can assume the morphism is injective. From Proposition 4.21 we deduce that the image is in $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})^{\mathrm{lalg}})$, hence also in $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})^{\mathrm{lalg}})\{\mathfrak{m}_x\}$. From (6.3) it is easy to check that $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})^{\mathrm{lalg}})[\mathfrak{m}_x] = \mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})^{\mathrm{lalg}})\{\mathfrak{m}_x\}$, whence the result. □

7. $\mathcal{L}$-invariants, $\mathrm{GL}_2(\mathbb{Q}_p)$-ordinary families and local-global compatibility

We now assume that the field $L = F_{\wp}^{+} \cong F_{\widetilde{\wp}}$ in § 6.1 is $\mathbb{Q}_p$ and study $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}})$ when the factors in the Levi L_P of the parabolic subgroup P are either GL_1 or GL_2. We derive several local-global compatibility results in this case. In particular we prove Conjecture 6.2 when $\mathrm{HT}(\rho_{\widetilde{\wp}}) = \{k_1, k_1 - 1, k_1 - 2\}$ for some integer k_1 (under mild genericity assumptions).

7.1. $\mathrm{GL}_2(\mathbb{Q}_p)$-ordinary families and local-global compatibility

When the factors of the L_P are either GL_1 or GL_2 we prove local-global compatibility results for the $L_P(\mathbb{Q}_p)$-representation $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}})$ by generalizing Emerton's method ([42]).

7.1.1. Dominant algebraic vectors. In this section, which is purely local, we prove density results of subspaces of algebraic functions.

We fix H a connected reductive algebraic group over $\mathbb{Z}_p$ and denote by A the finitely generated $\mathbb{Z}_p$-algebra which represents H. For any $f \in A$, the natural map $\mathrm{Hom}_{\mathbb{Z}_p-\mathrm{alg}}(A, \mathbb{Z}_p) = H(\mathbb{Z}_p) \to \mathbb{Z}_p,\ z \mapsto z(f)$ lies in $\mathcal{C}(H(\mathbb{Z}_p), \mathbb{Z}_p)$ and induces an E-linear morphism $A \otimes_{\mathbb{Z}_p} E \to \mathcal{C}(H(\mathbb{Z}_p), E)$. We denote by $\mathcal{C}^{\mathrm{alg}}(H(\mathbb{Z}_p), E)$ its image, which is called the vector space of algebraic functions on the compact group $H(\mathbb{Z}_p)$. By [66, Lem. 6.A.15], $\mathcal{C}^{\mathrm{alg}}(H(\mathbb{Z}_p), E)$ is dense in the Banach space $\mathcal{C}(H(\mathbb{Z}_p), E)$. For $f \in \mathcal{C}(H(\mathbb{Z}_p), E)$, we set $v(f) := \inf_{z \in H(\mathbb{Z}_p)} \mathrm{val}_p(f(z))$ and note that the associated norm gives the Banach topology on $\mathcal{C}(H(\mathbb{Z}_p), E)$. Now we let $H = \mathrm{GL}_r$, $r \geq 1$. By [66, Prop. 6.A.17] we have a $\mathrm{GL}_r(\mathbb{Z}_p)$-equivariant isomorphism:

$$\mathcal{C}^{\mathrm{alg}}(\mathrm{GL}_r(\mathbb{Z}_p), E) \cong \bigoplus_{\sigma} \mathrm{Hom}_{\mathrm{GL}_r(\mathbb{Z}_p)}(\sigma, \mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p), E)) \otimes_E \sigma \tag{7.1}$$

where σ runs through the irreducible algebraic representations of GL_r over E and where $\mathrm{Hom}_{\mathrm{GL}_r(\mathbb{Z}_p)}(\sigma, \mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p), E))$ denotes the E-linear $\mathrm{GL}_r(\mathbb{Z}_p)$-equivariant morphisms with $\mathrm{GL}_r(\mathbb{Z}_p)$ acting on $\mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p), E)$ by the usual right translation on functions. Recall there exists a one-to-one correspondence between the integral dominant weights $\lambda = (\lambda_1, \cdots, \lambda_r)$ for GL_r with respect to the Borel subgroup of upper triangular matrices, i.e. such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, and the irreducible algebraic representations $L(\lambda)$ of GL_r. For $a \in \mathbb{Z}$, we put:

$$\mathcal{C}^{\mathrm{alg}}_{\leq a}(\mathrm{GL}_r(\mathbb{Z}_p), E) := \bigoplus_{\substack{\lambda = (\lambda_1, \cdots, \lambda_n) \\ \lambda_1 \leq a}} \mathrm{Hom}_{\mathrm{GL}_r(\mathbb{Z}_p)}\big(L(\lambda), \mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p), E)\big) \otimes_E L(\lambda). \tag{7.2}$$

Lemma 7.1. *For any $a \in \mathbb{Z}$, the vector space $\mathcal{C}^{\mathrm{alg}}_{\leq a}(\mathrm{GL}_r(\mathbb{Z}_p), E)$ is dense in $\mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p), E)$.*

Proof. We first prove the lemma for $r = 1$, in which case we have by (7.1) (with obvious notation) $\mathcal{C}^{\mathrm{alg}}(\mathbb{Z}_p^\times, E) \cong \oplus_{j \in \mathbb{Z}} E x^j$. Let W be the closure of $\oplus_{j \leq a} E x^j$, we have to prove $x^j \in W$ for any $j \in \mathbb{Z}$. It is enough to prove that, for any $j \in \mathbb{Z}$ and $M > 0$, there exists $j' \leq a$ such that $v(x^{j'} - x^j) \geq M$. If we consider $j' := j - (p-1)p^{M'}$ with $M' > M$ sufficiently large so that $j' \leq a$, then we indeed have $\mathrm{val}_p(x^{j'} - x^j) = \mathrm{val}_p(x^{(p-1)p^{M'}} - 1) > M$ for any $z \in \mathbb{Z}_p^\times$. The case $r = 1$ follows.

For general r, denote by $\iota_{11} : \mathbb{Z}_p^\times \hookrightarrow \mathrm{GL}_r(\mathbb{Z}_p)$, $u \mapsto \mathrm{diag}(u, 1, \cdots, 1)$ and consider the induced map $\mathrm{SL}_r(\mathbb{Z}_p) \times \mathbb{Z}_p^\times \to \mathrm{GL}_r(\mathbb{Z}_p)$, $(u, v) \mapsto u\iota_{11}(v)$. This map is a homeomorphism and thus induces an isomorphism:

$$h:\mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p),E)\xrightarrow{\sim}\mathcal{C}(\mathrm{SL}_r(\mathbb{Z}_p)\times\mathbb{Z}_p^\times,E)\cong\mathcal{C}(\mathrm{SL}_r(\mathbb{Z}_p),E)\widehat{\otimes}_E\mathcal{C}(\mathbb{Z}_p^\times,E).$$

For a dominant weight $\lambda=(\lambda_1,\cdots,\lambda_r)$ as above, let $L(\lambda)_0:=L(\lambda)|_{\mathrm{SL}_r(\mathbb{Z}_p)}$. We claim that $h|_{\mathcal{C}^{\mathrm{alg}}(\mathrm{GL}_r(\mathbb{Z}_p),E)}$ induces an isomorphism via (7.1):

$$\begin{aligned}(7.3)\quad &\mathrm{Hom}_{\mathrm{GL}_r(\mathbb{Z}_p)}\big(L(\lambda),\mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p),E)\big)\otimes_E L(\lambda)\\ &\qquad\xrightarrow{\sim}\big(\mathrm{Hom}_{\mathrm{SL}_r(\mathbb{Z}_p)}\big(L(\lambda)_0,\mathcal{C}(\mathrm{SL}_r(\mathbb{Z}_p),E)\big)\otimes_E L(\lambda)_0\big)\otimes_E Ex^{\lambda_1}.\end{aligned}$$

Indeed, we have a natural commutative diagram (induced by the restriction map):

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{GL}_r(\mathbb{Z}_p)}\big(L(\lambda),\mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p),E)\big)\otimes_E L(\lambda) \arrow[r] \arrow[d] & \mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p),E) \arrow[d] \\
\mathrm{Hom}_{\mathrm{SL}_r(\mathbb{Z}_p)}\big(L(\lambda)_0,\mathcal{C}(\mathrm{SL}_r(\mathbb{Z}_p),E)\big)\otimes_E L(\lambda)_0 \arrow[r] & \mathcal{C}(\mathrm{SL}_r(\mathbb{Z}_p),E)
\end{tikzcd}$$

where the horizontal maps are the evaluation maps and are injective by (7.1). The morphism $\mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p),E)\to\mathcal{C}(\mathbb{Z}_p^\times,E)$ induced by ι_{11} is easily checked to send (via (7.1)) $\mathrm{Hom}_{\mathrm{GL}_r(\mathbb{Z}_p)}(L(\lambda),\mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p),E))\otimes_E L(\lambda)$ (on)to Ex^{λ_1}. We thus obtain the morphism in (7.3), which is moreover injective since h is. Since we have from the proof of [66, Prop. 6.A.17]:

$$\begin{aligned}(7.4)\quad &\dim_E\mathrm{Hom}_{\mathrm{GL}_r(\mathbb{Z}_p)}\big(L(\lambda),\mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p),E)\big)\\ &\qquad=\dim_E\mathrm{Hom}_{\mathrm{SL}_r(\mathbb{Z}_p)}\big(L(\lambda)_0,\mathcal{C}(\mathrm{SL}_r(\mathbb{Z}_p),E)\big)=\dim_E L(\lambda),\end{aligned}$$

we deduce that (7.3) is an isomorphism. The isomorphism h then induces a bijection:

$$\mathcal{C}^{\mathrm{alg}}_{\leq a}(\mathrm{GL}_r(\mathbb{Z}_p),E)\xrightarrow{\sim}\mathcal{C}^{\mathrm{alg}}(\mathrm{SL}_r(\mathbb{Z}_p),E)\otimes_E\mathcal{C}^{\mathrm{alg}}_{\leq a}(\mathbb{Z}_p^\times,E).$$

Since $\mathcal{C}^{\mathrm{alg}}_{\leq a}(\mathbb{Z}_p^\times,E)$ is dense in $\mathcal{C}(\mathbb{Z}_p^\times,E)$ and $\mathcal{C}^{\mathrm{alg}}(\mathrm{SL}_r(\mathbb{Z}_p),E)$ is dense in $\mathcal{C}(\mathrm{SL}_r(\mathbb{Z}_p),E)$, we deduce that $\mathcal{C}^{\mathrm{alg}}(\mathrm{SL}_r(\mathbb{Z}_p),E)\otimes_E\mathcal{C}^{\mathrm{alg}}_{\leq a}(\mathbb{Z}_p^\times,E)$ is dense in $\mathcal{C}(\mathrm{SL}_r(\mathbb{Z}_p),E)\widehat{\otimes}_E\mathcal{C}(\mathbb{Z}_p^\times,E)$, that is $\mathcal{C}^{\mathrm{alg}}_{\leq a}(\mathrm{GL}_r(\mathbb{Z}_p),E)$ is dense in $\mathcal{C}(\mathrm{GL}_r(\mathbb{Z}_p),E)$. □

We fix P a parabolic subgroup of GL_n as in § 5.1 (or § 6.3) with L_P as in (5.1). We have in particular:

$$\mathcal{C}(L_P(\mathbb{Z}_p),E)\ \cong\ \widehat{\bigotimes}_{i=1,\cdots,k}\mathcal{C}(\mathrm{GL}_{n_i}(\mathbb{Z}_p),E)$$

$$\mathcal{C}^{\mathrm{alg}}(L_P(\mathbb{Z}_p),E) \;\cong\; \bigotimes_{i=1,\cdots,k} \mathcal{C}^{\mathrm{alg}}(\mathrm{GL}_{n_i}(\mathbb{Z}_p),E).$$

For $i \in \{1,\cdots,k\}$ we define $s_i := \sum_{j=0}^{i-1} n_j$ (with $n_0 := 0$) as in § 5.2 and set:

$$\mathcal{C}^{\mathrm{alg}}_{+}(L_P(\mathbb{Z}_p),E) := \bigoplus_{\substack{(\lambda_1,\cdots,\lambda_n)\in\mathbb{Z}^n\\ \lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n}} \Big(\bigotimes_{i=1,\cdots,k} \mathcal{C}^{\underline{\lambda}_i-\mathrm{alg}}(\mathrm{GL}_{n_i}(\mathbb{Z}_p),E)\Big) \tag{7.5}$$

where $\underline{\lambda}_i := (\lambda_{s_i+1},\cdots,\lambda_{s_{i+1}})$ and:

$$\mathcal{C}^{\underline{\lambda}_i-\mathrm{alg}}(\mathrm{GL}_{n_i}(\mathbb{Z}_p),E) := \mathrm{Hom}_{\mathrm{GL}_{n_i}(\mathbb{Z}_p)}\big(L(\underline{\lambda}_i),\mathcal{C}(\mathrm{GL}_{n_i}(\mathbb{Z}_p),E)\big)\otimes_E L(\underline{\lambda}_i).$$

We define the subspace $\mathcal{C}^{\mathrm{alg}}_{++}(L_P(\mathbb{Z}_p),E)$ of $\mathcal{C}^{\mathrm{alg}}_{+}(L_P(\mathbb{Z}_p),E)$ in the same way but taking in (7.5) the direct sum only over those (dominant) λ such that $\lambda_{s_i} > \lambda_{s_i+1}$ for $i = 2,\cdots,k$. We call vectors in $\mathcal{C}^{\mathrm{alg}}_{+}(L_P(\mathbb{Z}_p),E)$ dominant $L_P(\mathbb{Z}_p)$-algebraic vectors.

Proposition 7.2. *The vector spaces $\mathcal{C}^{\mathrm{alg}}_{++}(L_P(\mathbb{Z}_p),E)$ and $\mathcal{C}^{\mathrm{alg}}_{+}(L_P(\mathbb{Z}_p),E)$ are dense in $\mathcal{C}(L_P(\mathbb{Z}_p),E)$.*

Proof. It is enough to prove the result for the first one. Using an easy induction argument, we can reduce to the case where $k = 2$. In this case, we have (see (7.2)):

$$\begin{aligned}&\mathcal{C}^{\mathrm{alg}}_{++}(L_P(\mathbb{Z}_p),E)\\ &\quad\cong \bigoplus_{\substack{\underline{\lambda}_1=(\lambda_1,\cdots,\lambda_{n_1})\\ \lambda_1\geq\cdots\geq\lambda_{n_1}}} \Big(\mathcal{C}^{\underline{\lambda}_1-\mathrm{alg}}(\mathrm{GL}_{n_1}(\mathbb{Z}_p),E)\otimes_E \mathcal{C}^{\mathrm{alg}}_{\leq\lambda_{n_1}-1}(\mathrm{GL}_{n_2}(\mathbb{Z}_p),E)\Big).\end{aligned}$$

From (7.4) we have that $\dim_E \mathcal{C}^{\underline{\lambda}_1-\mathrm{alg}}(\mathrm{GL}_{n_1}(\mathbb{Z}_p),E) < +\infty$, which implies that $F_{\underline{\lambda}_1} := \mathcal{C}^{\underline{\lambda}_1-\mathrm{alg}}(\mathrm{GL}_{n_1}(\mathbb{Z}_p),E)\otimes_E \mathcal{C}(\mathrm{GL}_{n_2}(\mathbb{Z}_p),E)$ is a Banach space. From Lemma 7.1 we have that

$$\mathcal{C}^{\underline{\lambda}_1-\mathrm{alg}}(\mathrm{GL}_{n_1}(\mathbb{Z}_p),E)\otimes_E \mathcal{C}^{\mathrm{alg}}_{\leq\lambda_{n_1}-1}(\mathrm{GL}_{n_2}(\mathbb{Z}_p),E)$$

is dense in $F_{\underline{\lambda}_1}$. We deduce that the closure of $\mathcal{C}^{\mathrm{alg}}_{+}(L_P(\mathbb{Z}_p),E)$ in $\mathcal{C}(L_P(\mathbb{Z}_p),E)$ contains $\oplus_{\underline{\lambda}_1} F_{\underline{\lambda}_1} \cong \mathcal{C}^{\mathrm{alg}}(\mathrm{GL}_{n_1}(\mathbb{Z}_p),E)\otimes_E\mathcal{C}(\mathrm{GL}_{n_2}(\mathbb{Z}_p),E)$. But $\mathcal{C}^{\mathrm{alg}}(\mathrm{GL}_{n_1}(\mathbb{Z}_p),E)$ is dense in $\mathcal{C}(\mathrm{GL}_{n_1}(\mathbb{Z}_p),E)$, hence $\sum_{\underline{\lambda}_1} F_{\underline{\lambda}_1}$ is dense in $\mathcal{C}(L_P(\mathbb{Z}_p),E)$ and the lemma follows. □

Let V be an admissible continuous Banach representation of $L_P(\mathbb{Q}_p)$ over E and put:

$$(7.6)\quad V^{L_P(\mathbb{Z}_p)-\mathrm{alg}} := \bigoplus_{\sigma} \mathrm{Hom}_{L_P(\mathbb{Z}_p)}(\sigma, V)\otimes_E \sigma \cong \bigoplus_{\sigma} (V\otimes_E \sigma^\vee)^{L_P(\mathbb{Z}_p)}\otimes_E \sigma$$

where σ runs through the irreducible algebraic representations of L_P and $\sigma^\vee$ is the dual of σ. By [43, Prop. 4.2.4], the evaluation map induces a natural injection $V^{L_P(\mathbb{Z}_p)-\mathrm{alg}} \hookrightarrow V$. We denote by $V_+^{L_P(\mathbb{Z}_p)-\mathrm{alg}}$ (resp. $V_{++}^{L_P(\mathbb{Z}_p)-\mathrm{alg}}$) the subspace of $V^{L_P(\mathbb{Z}_p)-\mathrm{alg}}$ defined as in (7.6) but taking the direct sum over those irreducible algebraic representations of L_P of highest weight $(\lambda_1,\cdots,\lambda_n)$ such that $\lambda_1\geq\cdots\geq\lambda_n$ (resp. such that $\lambda_1\geq\cdots\geq\lambda_n$ and $\lambda_{s_i}>\lambda_{s_i+1}$ for $i=2,\cdots,k$). If W is a closed subrepresentation of V, one easily checks that $W_*^{L_P(\mathbb{Z}_p)-\mathrm{alg}}\cong W\cap V_*^{L_P(\mathbb{Z}_p)-\mathrm{alg}}$ with $*\in\{\emptyset,+,++\}$.

Corollary 7.3. *Assume that $V|_{L_P(\mathbb{Z}_p)}$ is isomorphic to a direct summand of $\mathcal{C}(L_P(\mathbb{Z}_p),E)^{\oplus r}$ for some $r\geq 1$. Then $V_*^{L_P(\mathbb{Z}_p)-\mathrm{alg}}$ is dense in V for $*\in\{\emptyset,+,++\}$.*

Proof. If V_1, V_2 are two locally convex E-vector spaces and $X_i\subseteq V_i$, $i=1,2$ two E-vector subspaces, then $X_1\oplus X_2$ is dense in $V_1\oplus V_2$ (with the direct sum topology) if and only if X_i is dense in V_i for $i=1,2$. The result follows then from Proposition 7.2 together with $(V_1\oplus V_2)_*^{L_P(\mathbb{Z}_p)-\mathrm{alg}} = (V_1)_*^{L_P(\mathbb{Z}_p)-\mathrm{alg}}\oplus(V_2)_*^{L_P(\mathbb{Z}_p)-\mathrm{alg}}$ for $*\in\{\emptyset,+,++\}$. ☐

7.1.2. Benign points. We define benign points of $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$ and prove several results on them.

We keep the previous notation. We also keep all the notation and assumption of § 6.3 with $L=\mathbb{Q}_p$ (in particular U^p is sufficiently small, U_v is maximal for $v|p$, $v\neq\wp$, and we assume Hypothesis 6.9). We denote by B the subgroup of upper triangular matrices in GL_n and by T the torus of diagonal matrices. We assume moreover $n_i\leq 2$ for all $i=1,\cdots,k$ (though many results in this section hold more generally). For x a closed point of $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$, we denote by $\mathfrak{m}_x$ the associated maximal ideal, $k(x)$ the residue field (a finite extension of E) and by $\mathfrak{p}_x := \mathfrak{m}_x\cap\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}$ (a prime ideal). We also denote by $\mathfrak{m}_x$ (resp. $\mathfrak{p}_x$) the corresponding maximal ideal of $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}[1/p]$ (resp. the corresponding prime ideal of $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}$). We easily deduce from the left exactness of Ord_P ([40, Prop. 3.2.4]) an $L_P(\mathbb{Q}_p)$-equivariant isomorphism:

$$\mathrm{Ord}_P\left(\widehat{S}(U^\wp,\mathbb{W}^\wp)_{\overline{\rho}}[\mathfrak{p}_x]\right)\cong\mathrm{Ord}_P\left(\widehat{S}(U^\wp,\mathbb{W}^\wp)_{\overline{\rho}}\right)[\mathfrak{p}_x].$$

and we recall that $\mathrm{Ord}_P(\widehat{S}(U^\wp,\mathbb{W}^\wp)_{\overline{\rho}}[\mathfrak{p}_x])$ is an invariant lattice in the admissible unitary $L_p(\mathbb{Q}_p)$-representation $\mathrm{Ord}_P(\widehat{S}(U^\wp,W^\wp)_{\overline{\rho}}[\mathfrak{m}_x])$. We denote by:

$$\rho_x : \mathrm{Gal}_F \longrightarrow \mathrm{GL}_n(R_{\overline{\rho},S(U^p)}) \longrightarrow \mathrm{GL}_n(\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}) \longrightarrow \mathrm{GL}_n(k(x))$$

the continuous representation attached to x and set $\rho_{x,\widetilde{\wp}} := \rho_x|_{\mathrm{Gal}_{F_{\widetilde{\wp}}}}$. We also denote by x_i for $i \in \{1,\cdots,k\}$ the associated point of $\mathrm{Spec}\, R_{\overline{\rho}_i}[1/p]$ via (6.20) and $\rho_{x_i} : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_{n_i}(k(x))$ the attached representation. Thus $\rho_{x,\widetilde{\wp}}$ is a successive extension of the ρ_{x_i} for $i=1,\cdots,k$ and is strictly P-ordinary by Lemma 5.9 (applied with $E=k(x)$). In particular each ρ_{x_i} is indecomposable by Hypothesis 5.5.

Definition 7.4. *A closed point* $x \in \mathrm{Spec}\, \widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$ *is benign if:*

$$\mathrm{Ord}_P\left(\widehat{S}(U^\wp,W^\wp)_{\overline{\rho}}[\mathfrak{m}_x]\right)_+^{L_P(\mathbb{Z}_p)-\mathrm{alg}} \neq 0.$$

We recall that a closed point $x \in \mathrm{Spec}\, \widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}[1/p]$ is *classical* if

$$\widehat{S}(U^\wp,W^\wp)_{\overline{\rho}}[\mathfrak{m}_x]^{\mathrm{lalg}} \neq 0.$$

If x classical, then it follows [43, Prop. 4.2.4] that there is an integral dominant $\lambda=(\lambda_1,\cdots,\lambda_n)$ as in § 7.1.1 such that:

$$\begin{aligned}\left(\widehat{S}(U^\wp,W^\wp)_{\overline{\rho}})[\mathfrak{m}_x] \otimes_E L(\lambda)^\vee\right)^{\mathrm{sm}} \otimes_E L(\lambda)&\\ \hookrightarrow \widehat{S}(U^\wp,W^\wp)_{\overline{\rho}}[\mathfrak{m}_x]^{\mathrm{lalg}} &\hookrightarrow (\widehat{S}(U^\wp,W^\wp)_{\overline{\rho}})[\mathfrak{m}_x].\end{aligned}$$

One then easily deduces from (6.3) and, e.g. [80, Thm. 6.5(v)] (taking into account the normalisations) and [14, Rem. 4.2.4], that $\mathrm{HT}(\rho_{x,\widetilde{\wp}})=\{\lambda_1,\lambda_2-1,\cdots,\lambda_n-(n-1)\}$. In particular, λ is uniquely determined by x.

Proposition 7.5. *(1) A benign point is classical.*

(2) The set of benign points is Zariski-dense in $\mathrm{Spec}\, \widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$.

Proof. (1) Let x be a benign point. The admissibility of the $L_P(\mathbb{Q}_p)$-continuous representation $\mathrm{Ord}_P(\widehat{S}(U^\wp,W^\wp)_{\overline{\rho}}[\mathfrak{m}_x]$ together with [75, Thm. 7.1] and [43, Prop. 6.3.6] imply that there exist a smooth admissible representation π_x^∞ of $L_P(\mathbb{Q}_p)$ over $k(x)$ with $(\pi_x^\infty)^{L_P(\mathbb{Z}_p)} \neq 0$ and λ integral dominant such that:

$$\pi_x := \pi_x^\infty \otimes_E L_P(\lambda) \longrightarrow \mathrm{Ord}_P\left(\widehat{S}(U^\wp,W^\wp)_{\overline{\rho}}[\mathfrak{m}_x]\right). \tag{7.7}$$

Denote by $\widehat{\pi}_x$ the closure of π_x in $\mathrm{Ord}_P\left(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\right)$, by Proposition 4.21 we have continuous $L_P(\mathbb{Q}_p)$-equivariant morphisms:

$$\begin{aligned}(\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \pi_x^{\infty})^{\infty} \otimes_E L(\lambda) \lhook\joinrel\longrightarrow (\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \pi_x)^{\mathrm{an}} \lhook\joinrel\longrightarrow (\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \widehat{\pi}_x)^{\mathcal{C}^0} \\ \longrightarrow \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x],\end{aligned}$$

the composition of which is nonzero. (1) follows (and λ is the unique dominant weight as discussed just before Proposition 7.5).

(2) Let $\mathcal{I} := \bigcap_{x\in Z_0} \mathfrak{m}_x$ where Z_0 is the set of benign points of the scheme $\mathrm{Spec}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$, we have to prove $\mathcal{I} = 0$. By Lemma 6.8(2) and Corollary 7.3, $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}})_{+}^{L_P(\mathbb{Z}_p)-\mathrm{alg}}$ is dense in $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}})$. Since by Lemma 6.8(1) the $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$-action on $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}})$ is faithful, it is thus sufficient to prove that $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}})_{+}^{L_P(\mathbb{Z}_p)-\mathrm{alg}}$ is annihilated by $\mathcal{I}$. Let λ be an integral dominant weight, by (7.6) we are reduced to prove that any

$$v \in (\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}) \otimes_E L_P(\lambda)^{\vee})^{L_P(\mathbb{Z}_p)} \otimes_E L_P(\lambda)$$

is annihilated by $\mathcal{I}$. It is enough to consider the case $v = v_{\infty} \otimes u$ with $v_{\infty} \in (\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}) \otimes_E L_P(\lambda)^{\vee})^{L_P(\mathbb{Z}_p)}$ and $u \in L_P(\lambda)$. Let V_{∞} be the smooth $L_P(\mathbb{Q}_p)$-subrepresentation of $(\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}) \otimes_E L_P(\lambda)^{\vee})^{\mathrm{sm}}$ generated by v_{∞}. Consider the $L_P(\mathbb{Q}_p)$-equivariant injection (see [43, Prop. 4.2.4]):

$$V_{\infty} \otimes_E L_P(\lambda) \lhook\joinrel\longrightarrow \mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}). \tag{7.8}$$

By Proposition 4.21 again, this injection induces:

$$\begin{aligned}(\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} V_{\infty})^{\infty} \otimes_E L(\lambda) \longrightarrow \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}^{\mathrm{lalg}} \\ \cong \bigoplus_{x \text{ classical}} \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]^{\mathrm{lalg}} \lhook\joinrel\longrightarrow \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}\end{aligned} \tag{7.9}$$

where the middle isomorphism follows from (6.3). Since we can recover the injection (7.8) from (7.9) by applying the functor $\mathrm{Ord}_P(\cdot)$ (cf. Proposition 4.21), we see (7.8) factors through:

$$\mathrm{Ord}_P\Big(\bigoplus_{x \text{ classical}} \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\Big) \cong \bigoplus_{x \text{ classical}} \mathrm{Ord}_P\left(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\right)$$

$$\hookrightarrow \mathrm{Ord}_P\big(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\big).$$

Since V_∞ is generated by v_∞ and each $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x])$ is preserved by $L_P(\mathbb{Q}_p)$, there is a finite set C of classical points such that (7.8) has image in $\oplus_{x\in C}\mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x])$. In particular $v \in \mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}})_+^{L_P(\mathbb{Z}_p)-\mathrm{alg}}$ is contained in:

$$\bigoplus_{x\in C}\mathrm{Ord}_P\big(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\big)_+^{L_P(\mathbb{Z}_p)-\mathrm{alg}} = \bigoplus_{x\in C\cap Z_0}\mathrm{Ord}_P\big(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\big)_+^{L_P(\mathbb{Z}_p)-\mathrm{alg}}$$

and hence is annihilated by $\mathcal{I}$. (2) follows. □

Let x be a closed point of $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$. For $i = 1, \cdots, k$ we denote by $\widehat{\pi}(\rho_{x_i})$ the continuous finite length representation of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$ over $k(x)$ associated to ρ_{x_i} via the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ ([24]) normalized as in [6, § 3.1] when $n_i = 2$, via local class field theory for $\mathrm{GL}_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times$ normalized as in § 1 when $n_i = 1$. Recall that $\overline{B}_2$ denotes the lower triangular matrices of GL_2.

Proposition 7.6. *(1) If x is a benign point then $\rho_{x,\widetilde{\wp}}$ is semi-stable.*

(2) If x is benign (hence classical by Proposition 7.5(1)) and if $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is the unique integral dominant weight associated to x before Proposition 7.5, then for $i = 1, \cdots, k$, ρ_{x_i} is semi-stable with $\mathrm{HT}(\rho_{x_i}) = \{\lambda_{s_i+1} - s_i, \lambda_{s_i+n_i} - (s_i + n_i - 1)\}$ *(note these two integers are the same when $n_i = 1$ and recall $s_i = \sum_{j=0}^{i-1} n_j$).*

Proof. We fix a benign point x and use the notation of the proof of Proposition 7.5(1).

(1) Let $0 \neq v \in (\pi_x^\infty)^{L_P(\mathbb{Z}_p)}$ be an eigenvector for the spherical Hecke algebra of $L_P(\mathbb{Q}_p)$ with respect to $L_P(\mathbb{Z}_p)$ and let π^∞ be the $L_P(\mathbb{Q}_p)$-subrepresentation of π_x^∞ generated by v. Then it is easy to check that we have:

$$\pi^\infty \cong \bigotimes_{i=1,\cdots,k}\pi_i^\infty$$

where if $n_i = 1$, $\psi_{s_i+1} := \pi_i^\infty$ is an unramified character of $\mathbb{Q}_p^\times$ and if $n_i = 2$, either there exist unramified characters $\psi_{s_i+1}, \psi_{s_i+2}$ of $\mathbb{Q}_p^\times$ such that $\pi_i^\infty \cong (\mathrm{Ind}_{\overline{B}_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi_{s_i+1} \otimes \psi_{s_i+2})^\infty$ with $\psi_{s_i+1} \neq \psi_{s_i+2}$ or π_i^∞ is isomorphic to the composition of an unramified character of $\mathbb{Q}_p^\times$ with the determinant character (note that we can assume $\psi_{s_i+1} \neq \psi_{s_i+2}$ in the first case since otherwise we would in fact be in the second). As in (7.7), we have an $L_P(\mathbb{Q}_p)$-equivariant embedding:

$$\Big(\bigotimes_{i=1,\cdots,k}\pi_i^\infty\Big)\otimes_E L_P(\lambda)\hookrightarrow \mathrm{Ord}_P\big(\widehat{S}(U^\wp,W^\wp)_{\overline{\rho}}[\mathfrak{m}_x]\big) \tag{7.10}$$

which, by Proposition 4.21, induces a nonzero morphism:

$$\big(\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}\otimes_{i=1,\cdots,k}\pi_i^\infty\big)^\infty\otimes_E L(\lambda)\longrightarrow \widehat{S}(U^\wp,W^\wp)_{\overline{\rho}}[\mathfrak{m}_x]. \tag{7.11}$$

By (6.3) and the local-global compatibility at $\ell=p$ in the classical local Langlands correspondence (cf. [17]), there exists an automorphic representation π of G associated to ρ_x such that the factor of π at the place $\wp$ is of the form $\pi_\wp\otimes_{k(x)}\overline{\mathbb{Q}_p}$ where $\pi_\wp$ is an irreducible constituent of $(\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}\otimes_{i=1,\cdots,k}\pi_i^\infty)^\infty$ (note that the action of $\mathrm{GL}_n(\mathbb{Q}_p)$ on $\pi_\wp$ actually also depends on $\widetilde{\wp}$). Since the representation π_i^∞ is unramified for all i, it easily follows from [17] and properties of the local Langlands correspondence that the potentially semi-stable $\rho_{x,\widetilde{\wp}}$ must be semi-stable. This proves (1). Moreover, since $\overline{\rho}$ is irreducible so is ρ_x. Thus $\pi_\wp$ is a generic representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ by genericity of local components of cuspidal automorphic representations of GL_n using base change to GL_n ([71], [55]). This implies that π_i^∞ is infinite dimensional when $n_i=2$ since otherwise it is easy to check that $(\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}\otimes_{i=1,\cdots,k}\pi_i^\infty)^\infty$ has no generic irreducible constituent.

(2) The fact that ρ_{x_i} is semi-stable follows from (1). By [17], there exists $m(x)\in\mathbb{Z}_{\geq 1}$ such that:

$$\widehat{S}(U^\wp,W^\wp)_{\overline{\rho}}[\mathfrak{m}_x]^{\mathrm{lalg}}\cong(\pi_\wp\otimes_E L(\lambda))^{\oplus m(x)}. \tag{7.12}$$

In fact, we have (the second equality following from the fact that U_v is maximal for $v|p$, $v\neq\wp$):

$$m(x)=\sum_\pi m(\pi)\dim_{\overline{\mathbb{Q}_p}}(\pi^{\infty,\wp})^{U^\wp}=\sum_\pi m(\pi)\dim_{\overline{\mathbb{Q}_p}}(\pi^{\infty,p})^{U^p} \tag{7.13}$$

where π runs through the automorphic representations of $G(\mathbb{A}_{F_+})$ such that $\mathbb{T}(U^p)[1/p]$ acts on $(\pi^{\infty,p})^{U^p}$ via $\mathbb{T}(U^p)[1/p]/\mathfrak{m}_x\cong k(x)$ (hence the factor of π at the place of $\wp$ is of the form $\pi_\wp\otimes_{k(x)}\overline{E}$). Since each π_i^∞ for $i=1,\cdots,k$, and thus $\otimes_{i=1,\cdots,k}\pi_i^\infty$, has an irreducible socle, the injection (7.10) factors through an $L_P(\mathbb{Q}_p)$-equivariant injection:

$$(7.14)\qquad \Big(\bigotimes_{i=1,\cdots,k}\pi_i^{\infty}\Big)\otimes_E L_P(\lambda)\hookrightarrow \mathrm{Ord}_P\big(\pi_{\wp}\otimes_E L(\lambda)\big).$$

Applying Proposition 5.10, we see $\rho_{x,\wp}$ is isomophic to a successive extension of ρ'_{x_i} with $\mathrm{HT}(\rho'_{x_i})=\{\lambda_{s_i+1}-s_i,\lambda_{s_i+n_i}-(s_i+n_i-1)\}$. Since $\rho_{x,\wp}$ is strictly P-ordinary, we have $\rho'_{x_i}\cong\rho_{x_i}$ for all i, which finishes the proof of (2). □

Remark 7.7. If $n_i=1$, set $\alpha_{s_i+1}:=p^{s_i}\psi_{s_i+1}(p)$; if $n_i=2$, set $\alpha_{s_i+1}:=\psi_{s_i+1}(p)p^{s_i}$, $\alpha_{s_i+2}:=\psi_{s_i+2}(p)p^{s_i+1}$ (so $\alpha_{s_i+1}\alpha_{s_i+2}^{-1}\neq p^{-1}$, by the proof of Proposition 7.6 (1)). It follows from [17] and [77, Thm. 1.2(b)] that we have:

$$(7.15)\qquad \mathrm{rec}(\pi_{\widetilde{\wp}})\Big(\frac{1-n}{2}\Big)\cong W(\rho_{x,\widetilde{\wp}})^{\mathrm{ss}}\cong\oplus_{j=1}^n\mathrm{unr}(\alpha_j)$$

(rec := semi-simplified local Langlands correspondence, see § 5.2). Since $\big(\otimes_{i=1,\cdots,k}\pi_i^{\infty}\big)\otimes_E L_P(\lambda)$ is unitary by (7.10), we have if $n_i=1$:

$$(7.16)\qquad \mathrm{val}_p(\alpha_{s_i+1})=-\lambda_{s_i+1}+s_i$$

and if $n_i=2$:

$$(7.17)\qquad \mathrm{val}_p(\alpha_{s_i+1})+\mathrm{val}_p(\alpha_{s_i+2})=-\lambda_{s_i+1}-\lambda_{s_i+2}+s_i+(s_i+1).$$

If $n_i=2$, we have $W(\rho_{x_i})^{\mathrm{ss}}\cong\mathrm{unr}(\alpha_{s_i+1})\oplus\mathrm{unr}(\alpha_{s_i+2})$ (by Proposition 5.10). Hence by weak admissibility, we see:

$$(7.18)\qquad -\lambda_{s_i+1}+s_i\leq\mathrm{val}_p(\alpha_{s_i+l})\leq-\lambda_{s_i+2}+s_i+1,\quad\forall\, l=1,2.$$

Together with (7.16), we see $\alpha_j\neq\alpha_{j'}$ if j, j' do not lie in $\{s_i+1,s_i+n_i\}$ for any $i\in\{1,\cdots,k\}$. If λ is moreover strictly dominant, i.e. $\lambda_j>\lambda_{j+1}$ for all j, we deduce $\alpha_j\alpha_{j'}^{-1}\notin\{1,p,p^{-1}\}$ if j, j' do not lie in $\{s_i+1,s_i+n_i\}$ for any $i\in\{1,\cdots,k\}$.

Lemma 7.8. *The injection (7.14) is bijective.*

Proof. Denote by $I_P:=\{i=1,\cdots,k,\ n_i=2\}$. Let S_{n_i} be the Weyl group of GL_{n_i} identified with the permutations on the set $\{1,n_i\}$. For $w\in S_{n_i}$ we set $\beta_{w,s_i+1}:=p^{-s_i}\alpha_{s_i+1}$ if $n_i=1$; and $\beta_{w,s_i+1}:=p^{-s_i-1}\alpha_{s_i+w_i(1)}$, $\beta_{w,s_i+2}:=p^{-s_i}\alpha_{s_i+w_i(2)}$ if $n_i=2$. We have:

$$(7.19)\qquad J_{B\cap L_P}\big((\otimes_{i=1,\cdots,k}\pi_i^{\infty})\otimes_E L_P(\lambda)\big)^{\mathrm{ss}}\cong\delta_\lambda\otimes\big(\oplus_{w=(w_i)\in S_2^{|I_P|}}(\otimes_{j=1}^n\mathrm{unr}(\beta_{w,j}))\big)$$

where δ_λ is the algebraic character of $T(\mathbb{Q}_p)$ of weight λ, ss denotes the semi-simplification as $T(\mathbb{Q}_p)$-representations. On the other hand, we deduce from Remark 4.13:

$$J_{B\cap L_P}\big(\mathrm{Ord}_P(\pi_\wp \otimes_E L(\lambda))\big) \lhook\joinrel\longrightarrow J_B(\pi_\wp \otimes_E L(\lambda))(\delta_P^{-1}).$$

Comparing [69, Thm. 5.4] with (7.19) (recall that $\pi_\wp$ is a constituent of $(\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \otimes_{i=1,\cdots,k}\pi_i^\infty)^\infty$) and using Remark 7.7, one can check that any character:

$$\chi' \lhook\joinrel\longrightarrow J_B(\pi_\wp \otimes_E L(\lambda))[\delta_P^{-1}]^{\mathrm{ss}}/J_{B\cap L_P}\big((\otimes_{i=1,\cdots,k}\pi_i^\infty)\otimes_E L_P(\lambda)\big)^{\mathrm{ss}} \tag{7.20}$$

does not appear on the right hand-side of (7.19). Let π_P^∞ be the smooth admissible representation of $L_P(\mathbb{Q}_p)$ over $k(x)$ such that $\mathrm{Ord}_P(\pi_\wp \otimes_E L(\lambda)) \cong \pi_P^\infty \otimes_E L_P(\lambda)$. Let χ' be as in (7.20). If χ' injects into $J_{B\cap L_P}\big(\pi_P^\infty \otimes_E L_P(\lambda)\big)$ (which is equivalent to $\chi'\delta_\lambda^{-1} \hookrightarrow J_{B\cap L_P}(\pi_P^\infty)$), by [37, (0.1)] we deduce a nonzero morphism:

$$\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)\cap L_P(\mathbb{Q}_p)}^{L_P(\mathbb{Q}_p)} \chi'\delta_\lambda\delta_{B\cap L_P}\big)^\infty \longrightarrow \pi_P^\infty$$

and hence a nonzero morphism:

$$\big(\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)\cap L_P(\mathbb{Q}_p)}^{L_P(\mathbb{Q}_p)} \chi'\delta_\lambda\delta_{B\cap L_P}\big)^\infty \otimes_E L_P(\lambda) \longrightarrow \mathrm{Ord}_P(\pi_\wp \otimes_E L(\lambda)). \tag{7.21}$$

However, $\mathrm{Ord}_P(\pi_\wp \otimes_E L(\lambda))$ is unitary, while, by (7.16), (7.17) and (7.18), one can check that the left hand-side of (7.21) does not have any unitary subquotient (e.g. by considering the central characters, the key point being that, for w in the Weyl group of GL_n which does not lie in the Weyl group of L_P, if we replace the α_j by the $\alpha_j' := \alpha_{w^{-1}(j)}$ for $j = 1, \cdots, n$, then at least one of (7.16), (7.17) or (7.18) cannot hold). Consequently, any χ' as in (7.20) cannot inject into $J_{B\cap L_P}(\mathrm{Ord}_P(\pi_\wp \otimes_E L(\lambda)))$, and hence cannot appear in the semi-simplification of the latter (using that there does not exist nontrivial extension between different characters of $T(\mathbb{Q}_p)$). It follows that the natural injection induced by (7.14):

$$J_{B\cap L_P}\big((\otimes_{i=1,\cdots,k}\pi_i^\infty)\otimes_E L_P(\lambda)\big) \lhook\joinrel\longrightarrow J_{B\cap L_P}\big(\mathrm{Ord}_P(\pi_\wp \otimes_E L(\lambda))\big)$$

is bijective. Since $J_P(\pi_\wp)$ does not have cuspidal constituents and $J_{B\cap L_P}$ is an exact functor, we deduce that the injection (7.14) must be bijective. $\square$

Proposition 7.9. *With the notation of Proposition 7.6 and its proof, we have an $L_P(\mathbb{Q}_p)$-equivariant isomorphism:*

$$\operatorname{Ord}_P\big(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]^{\mathrm{lalg}}\big)\cong\Big(\big(\bigotimes_{i=1,\cdots,k}\pi_i^{\infty}\big)\otimes_E L_P(\lambda)\Big)^{\oplus m(x)}. \tag{7.22}$$

Proof. This is an immediate consequence of (7.12) and Lemma 7.8. □

Corollary 7.10. *(1) If x is benign, the representations ρ_{x_i} are crystalline for $i=1,\cdots,k$.*

(2) If x is benign, there exists an $L_P(\mathbb{Q}_p)$-equivariant injection:

$$\bigotimes_{i=1,\cdots,k}\big(\widehat{\pi}(\rho_{x_i})^{\mathrm{lalg}}\otimes\varepsilon^{s_i}\circ\det\big)\lhook\joinrel\longrightarrow\operatorname{Ord}_P\big(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\big). \tag{7.23}$$

Proof. (1) We use the notation of Proposition 7.6 and its proof. The first statement is clear when $n_i=1$ by Proposition 7.6(2). By *loc. cit.* and its proof, it is enough to prove that for $n_i=2$ we have $\alpha_{s_i+1}\alpha_{s_i+2}^{-1}\neq p^{\pm1}$. From the proof of Proposition 7.6, we have already seen $\alpha_{s_i+1}\alpha_{s_i+2}^{-1}\neq p^{-1}$. Assume there exists i such that $n_i=2$ and $\alpha_{s_i+1}\alpha_{s_i+2}^{-1}=p$, then π_i^{∞} is reducible and has a 1-dimensional quotient. Let π_j' be the (unique) irreducible quotient of π_j^{∞} for $j=1,\cdots,k$, we have $\otimes_{j=1}^k\pi_j^{\infty}\twoheadrightarrow\otimes_{j=1}^k\pi_j'$ where π_i' is 1-dimensional. By Lemma 7.8 and the fact that $\operatorname{Ord}_P(\pi_{\wp}\otimes_E L(\lambda))$ is a direct summand of $J_P\big(\pi_{\wp}\otimes_E L(\lambda)\big)(\delta_P^{-1})$ (which follows from (4.13)), we deduce an $L_P(\mathbb{Q}_p)$-equivariant surjection $J_P(\pi_{\wp})\longrightarrow\!\!\!\!\rightarrow(\otimes_{j=1}^k\pi_j')(\delta_P)$. By [69, Thm. 5.3(3)] this induces a nonzero morphism $\pi_{\wp}\to(\operatorname{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}(\otimes_{j=1}^k\pi_j')(\delta_P))^{\infty}$, which is an injection since $\pi_{\wp}$ is irreducible. However $(\operatorname{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}(\otimes_{j=1}^k\pi_j')(\delta_P))^{\infty}$ does not have any generic irreducible constituent since $\dim_{k(x)}\pi_i'=1$. This gives a contradiction and finishes the proof of (1). Note that we also obtain that π_i^{∞} is irreducible for $i=1,\cdots,k$.

(2) By well-known properties of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ we have:

$$\widehat{\pi}(\rho_{x_i})^{\mathrm{lalg}}\cong\begin{cases}\mathrm{unr}(\alpha_{s_i+1})x^{\lambda_{s_i+1}-s_i} & n_i=1\\ \big(\operatorname{Ind}_{\overline{B}_2}^{\mathrm{GL}_2}\mathrm{unr}(\alpha_{s_i+1})\otimes\mathrm{unr}(\alpha_{s_i+2}/p)\big)^{\infty}\otimes_E L_i(\underline{\lambda}_i-s_i) & n_i=2\end{cases}$$

where $\underline{\lambda}_i-s_i$ is by definition the weight $(\lambda_{s_i+1}-s_i,\lambda_{s_i+2}-s_i)$. Using $\varepsilon=$

$z\,\mathrm{unr}(p^{-1})$, we easily deduce:

$$\bigotimes_{i=1,\cdots,k}\big(\widehat{\pi}(\rho_{x_i})^{\mathrm{lalg}}\otimes\varepsilon^{s_i}\circ\det\big)\cong\Big(\bigotimes_{i=1,\cdots,k}\pi_i^{\infty}\Big)\otimes_E L_P(\lambda), \tag{7.24}$$

whence (2) by (7.10). □

7.1.3. *P*-ordinary eigenvarieties. We define and study P-ordinary Hecke eigenvarieties and use them to prove geometric properties of the scheme $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$.

We keep the notation and assumptions of the previous sections. We now consider the locally analytic representation of $T(\mathbb{Q}_p)$:

$$J_{B\cap L_P}\big(\mathrm{Ord}_P(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}})^{\mathrm{an}}\big)$$

where $J_{B\cap L_P}$ is Emerton's locally analytic Jacquet functor ([38, § 3.4]). This is an essentially admissible representation of $T(\mathbb{Q}_p)$ over E ([43, Def. 6.4.9]) which is equipped with an action of $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$ commuting with $T(\mathbb{Q}_p)$. Let $\mathcal{T}$ be the rigid analytic space over E parametrizing the locally analytic characters of $T(\mathbb{Q}_p)$ and $(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$ the generic rigid fiber (à la Raynaud-Berthelot) of the formal scheme $\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$ associated to the complete noetherian local ring $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$ (in particular the points of $(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$ are the closed points of $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$). Then, following [39, § 2.3] the continuous dual $J_{B\cap L_P}(\mathrm{Ord}_P(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}})^{\mathrm{an}})^{\vee}$ is the global sections of a coherent sheaf on the rigid analytic space

$$(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}\times_E\mathcal{T},$$

the schematic support of which defines a Zariski-closed immersion of rigid spaces:

$$\mathcal{E}^{P-\mathrm{ord}}\hookrightarrow(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}\times_E\mathcal{T}.$$

In particular $y=(x,\chi)\in\mathcal{E}^{P-\mathrm{ord}}$ if and only if there is a $T(\mathbb{Q}_p)$-equivariant embedding:

$$\chi\lhook\joinrel\longrightarrow J_{B\cap L_P}\big(\mathrm{Ord}_P(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}})^{\mathrm{an}}\big)[\mathfrak{m}_x].$$

By the same proof (in fact simpler) as for [12, Cor. 3.12] using Lemma 6.8(2) to ensure that the analogous results of the ones in [12, §§ 3.3 & 5.2] hold in our setting, we have the following proposition.

Proposition 7.11. *The rigid analytic space* $\mathcal{E}^{P-\mathrm{ord}}$ *is equidimensional of dimension* n*.*

Definition 7.12. *A point* $y = (x, \chi) \in \mathcal{E}^{P-\mathrm{ord}}$ *is* P*-ordinary classical if:*

- χ *is of the form* $\chi^\infty \delta_\lambda$ *where* χ^∞ *is smooth and* $\lambda = (\lambda_1, \cdots, \lambda_n)$ *is integral dominant*
- $J_{B\cap L_P}\big(\mathrm{Ord}_P(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}})^{\mathrm{lalg}}\big)[\mathfrak{m}_x, T(\mathbb{Q}_p) = \chi] \neq 0.$

Lemma 7.13. *Let* $y = (x, \chi) \in \mathcal{E}^{P-\mathrm{ord}}$ *be* P*-ordinary classical, then the point* x *is classical.*

Proof. This follows by the same argument as in the proof of Proposition 7.5(1) (except that we don't necessarily have $(\pi_x^\infty)^{L_P(\mathbb{Z}_p)} \neq 0$ anymore), using the adjunction property of the functor $J_{B\cap L_P}(\cdot)$ on locally algebraic representations and then applying Proposition 4.21. □

Lemma 7.14. *Let* $\lambda = (\lambda_1, \cdots, \lambda_n)$ *be an integral dominant weight,* $\chi^\infty = \chi_1^\infty \otimes \cdots \otimes \chi_n^\infty$ *be an unramified character of* $T(\mathbb{Q}_p)$*, and* $y = (x, \chi) \in \mathcal{E}^{P-\mathrm{ord}}$ *with* $\chi = \delta_\lambda \chi^\infty = \chi_1 \otimes \cdots \otimes \chi_n$*. If we have for all* $i = 1, \cdots, k$ *such that* $n_i = 2$*:*

$$\mathrm{val}_p(\chi_{s_i+1}(p)) < \lambda_{s_i+1} - \lambda_{s_i+2} \quad (\textit{equivalently } \mathrm{val}_p(\chi_{s_i+1}^\infty(p)) < -\lambda_{s_i+2}), \tag{7.25}$$

then y *is* P*-ordinary classical.*

Proof. As in § 3.3.1 we use without comment in this proof the theory of [63] (see [8, § 2] for a summary). For $i \in \{1, \cdots, k\}$ let $\pi_{s_i+1} := x^{\lambda_{s_i+1}} \chi_{s_i+1}^\infty$ if $n_i = 1$ and:

$$\pi_i := \mathcal{F}_{\overline{B}_2}^{\mathrm{GL}_2}\big(\overline{M}_i(-\underline{\lambda}_i), |\cdot|^{-1}\chi_{s_i+1}^\infty \otimes |\cdot|\chi_{s_i+2}^\infty\big)$$

if $n_i = 2$ where $-\underline{\lambda}_i$ is the algebraic weight $(-\lambda_{s_i+1}, -\lambda_{s_i+2})$ and $\overline{M}_i(-\underline{\lambda}_i) := \mathrm{U}(\mathfrak{gl}_2) \otimes_{\mathrm{U}(\overline{\mathfrak{b}}_2)} (-\underline{\lambda}_i)$ ($\overline{\mathfrak{b}}_2$ being the Lie algebra of $\overline{B}_2$). It follows from [7, Thm. 4.3] that the injection $\chi \hookrightarrow J_{B\cap L_P}(\mathrm{Ord}_P(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}})^{\mathrm{an}})[\mathfrak{m}_x]$ induces a nonzero continuous $L_P(\mathbb{Q}_p)$-equivariant morphism:

$$\widehat{\bigotimes}_{i=1,\ldots,k} \pi_i \longrightarrow \mathrm{Ord}_P(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}})^{\mathrm{an}}[\mathfrak{m}_x] \tag{7.26}$$

where the completed tensor product on the left hand side is with respect to the projective limit topology, or equivalently by [43, Prop. 1.1.31], the inductive limit topology, on $\pi_1\otimes_{k(x)}\cdots\otimes_{k(x)}\pi_k$. If $\mathrm{val}_p(p\chi^\infty_{s_i+1}(p)) < 1-\lambda_{s_i+2}$, by [8, Cor. 3.6] the representation:

$$\begin{aligned}&\mathcal{F}_{\overline{B}_2}^{\mathrm{GL}_2}\big(\overline{L}_i(-s\cdot\underline{\lambda}_i), |\cdot|^{-1}\chi^\infty_{s_i+1}\otimes|\cdot|\chi^\infty_{s_i+2}\big)\\ &\qquad\qquad\cong \big(\mathrm{Ind}_{\overline{B}_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} |\cdot|^{-1}\chi^\infty_{s_i+1}x^{\lambda_{s_i+2}-1}\otimes|\cdot|\chi^\infty_{s_i+2}x^{\lambda_{s_i+1}+1}\big)^{\mathrm{an}}\end{aligned}$$

does not have a $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant lattice, where $\overline{L}_i(-s\cdot\underline{\lambda}_i)$ is the unique simple subobject of $\overline{M}_i(-\underline{\lambda}_i)$. We then easily deduce that the map in (7.26) factors through a (nonzero) morphism:

$$\Big(\bigotimes_{i=1,\cdots,k}\pi_i^\infty\Big)\otimes_E L_P(\lambda)\longrightarrow \mathrm{Ord}_P(\widehat{S}(U^\wp,W^\wp)_{\overline{\rho}})^{\mathrm{an}}[\mathfrak{m}_x] \tag{7.27}$$

where $\pi_i^\infty:=\mathrm{unr}(\beta_i)$ if $n_i=1$ and $\pi_i^\infty:=(\mathrm{Ind}_{\overline{B}_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}|\cdot|^{-1}\chi^\infty_{s_i+1}\otimes|\cdot|\chi^\infty_{s_i+2})^\infty$ if $n_i=2$. By Proposition 4.21, the lemma follows. ☐

The proof of the following lemma is standard and we omit it (see e.g. the proof of [12, Thm. 3.19]).

Lemma 7.15. *The set of points satisfying the conditions in Lemma 7.14 is Zariski-dense in* $\mathcal{E}^{P-\mathrm{ord}}$.

Proposition 7.16. *The set of P-ordinary classical points is Zariski-dense in* $\mathcal{E}^{P-\mathrm{ord}}$.

Proof. This follows from Lemma 7.14 and Lemma 7.15. ☐

Replacing the locally analytic $T(\mathbb{Q}_p)$-representation

$$J_{B\cap L_P}(\mathrm{Ord}_P(\widehat{S}(U^\wp,W^\wp)_{\overline{\rho}})^{\mathrm{an}})$$

by $J_B(\widehat{S}(U^\wp,W^\wp)^{\mathrm{an}}_{\overline{\rho}})$, we obtain in the same way a rigid analytic variety $\mathcal{E}$ over E together with a Zariski-closed immersion:

$$\mathcal{E}\longhookrightarrow(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}})^{\mathrm{rig}}\times_E\mathcal{T}$$

such that $(x,\chi)\in\mathcal{E}$ if and only if there is a $T(\mathbb{Q}_p)$-equivariant embedding $\chi\hookrightarrow J_B(\widehat{S}(U^\wp,W^\wp)^{\mathrm{an}}_{\overline{\rho}})[\mathfrak{m}_x]$. Moreover $\mathcal{E}$ is also equidimensional of dimension n. Consider now the following closed immersion:

$$\iota^{P-\mathrm{ord}}:(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^\wp)^{P-\mathrm{ord}}_{\overline{\rho}})^{\mathrm{rig}}\times_E\mathcal{T}\quad\longhookrightarrow\quad(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}})^{\mathrm{rig}}\times_E\mathcal{T}$$

$$(x,\chi) \longmapsto (x,\chi\delta_P^{-1}).$$

Let $y \in \mathcal{E}^{P-\mathrm{ord}}$ be P-ordinary classical. By Lemma 7.13 and $J_{B\cap L_P}\circ \mathrm{Ord}_P \hookrightarrow J_B(\delta_P^{-1})$ (see § 4.3), we see $\iota^{P-\mathrm{ord}}(y)$ is a classical point in $\mathcal{E}$. Together with Proposition 7.16, we deduce that $\iota^{P-\mathrm{ord}}$ induces a closed immersion of reduced rigid analytic spaces:

$$\iota^{P-\mathrm{ord}} : \mathcal{E}_{\mathrm{red}}^{P-\mathrm{ord}} \longhookrightarrow \mathcal{E}_{\mathrm{red}} \tag{7.28}$$

where "red" means the reduced closed rigid subspace.

Corollary 7.17. *The rigid space $\mathcal{E}_{\mathrm{red}}^{P-\mathrm{ord}}$ is isomorphic to a union of irreducible components of $\mathcal{E}_{\mathrm{red}}$.*

Proof. This follows from (7.28) and the fact both $\mathcal{E}_{\mathrm{red}}^{P-\mathrm{ord}}$ and $\mathcal{E}_{\mathrm{red}}$ are equidimensional of dimension n. □

Recall that, for any $(x,\chi)\in\mathcal{E}$, the associated $\mathrm{Gal}_{\mathbb{Q}_p}$-representation $\rho_{x,\widetilde{\wp}}$ is trianguline (see [51] and also [58]) and that (x,χ) is called *noncritical* if $\chi\delta_B^{-1}(1\otimes\varepsilon^{-1}\otimes\cdots\otimes\varepsilon^{1-n})$ gives a parameter of the trianguline (φ,Γ)-module $D_{\mathrm{rig}}(\rho_{x,\widetilde{\wp}})$ associated to $\rho_{x,\widetilde{\wp}}$ (with the usual identification of the $T(\mathbb{Q}_p)$-character $\delta_1\otimes\cdots\otimes\delta_n$ and the parameter $(\delta_1,\cdots,\delta_n)$). We call a point $y=(x,\chi)$ of $\mathcal{E}^{P-\mathrm{ord}}$ *noncritical* if $\iota^{P-\mathrm{ord}}(y)$ is noncritical, or equivalently if $\chi\delta_{B\cap L_P}^{-1}(1\otimes\varepsilon^{-1}\otimes\cdots\otimes\varepsilon^{1-n})$ is a trianguline parameter of $D_{\mathrm{rig}}(\rho_{x,\widetilde{\wp}})$.

Lemma 7.18. *Let $y=(x,\chi)$ be a P-ordinary classical point with x benign, then y is noncritical.*

Proof. We use the notation of Definition 7.12 and of Lemma 7.8 and its proof. By Lemma 7.8 and (7.19), there exists $w\in S_2^{|I_P|}$ such that $\chi^\infty = \mathrm{unr}(\beta_{w,1})\otimes\cdots\otimes\mathrm{unr}(\beta_{w,n})$. It follows from Proposition 7.6(2) and its proof together with Corollary 7.10(1) that:

$$\big(\chi^\infty_{s_i+1}|\cdot|^{-1}x^{\lambda_{s_i+1}}\varepsilon^{-s_i}, \chi^\infty_{s_i+2}|\cdot|x^{\lambda_{s_i+2}}\varepsilon^{-s_i-1}\big)$$

is a trianguline parameter of $D_{\mathrm{rig}}(\rho_{x_i})$ if $n_i=2$ and $\chi^\infty_{s_i+1}x^{\lambda_{s_i+1}}\varepsilon^{-s_i}\cong\rho_{x_i}$ if $n_i=1$ (where $i\in\{1,\cdots,k\}$). Together with the fact $\rho_{x,\widetilde{\wp}}$ is isomorphic to a successive extension of the ρ_{x_i}, we deduce that $\chi\delta_{B\cap L_P}^{-1}(1\otimes\varepsilon^{-1}\otimes\cdots\otimes\varepsilon^{1-n})$ is a trianguline parameter of $\rho_{x,\widetilde{\wp}}$. □

We say that an r-dimensional crystalline representation V of $\mathrm{Gal}_{\mathbb{Q}_p}$ is *generic* if the eigenvalues $(\varphi_i)_{i=1,\cdots,r}$ of φ on $D_{\mathrm{cris}}(V)$ are such that $\varphi_i\varphi_j^{-1}\notin\{1,p,p^{-1}\}$ for $i\neq j$.

Lemma 7.19. *Let $y = (x, \chi) \in \mathcal{E}^{P-\mathrm{ord}}$ be as in Lemma 7.14 and assume moreover for all $i = 1, \cdots, k$ such that $n_i = 2$ (with the notation of loc. cit.):*

$$\mathrm{val}_p(\chi_{s_i+1}(p)) < \frac{\lambda_{s_i+1} - \lambda_{s_i+2}}{2} - 1. \tag{7.29}$$

Then x is a benign point and ρ_{x_i} is crystalline generic for $i = 1, \cdots, k$.

Proof. We use the notation of Lemma 7.14. Since $\chi_{s_i+1}(p) + \chi_{s_i+2}(p) = 0$ (as follows from (7.27)), we easily deduce from (7.29) that:

$$\chi_{s_i+1}^{\infty}(p)\chi_{s_i+2}^{\infty}(p)^{-1} \notin \{p^{-2}, p^{-1}, 1\}. \tag{7.30}$$

which implies that π_i^{∞} in the proof of Lemma 7.14 is irreducible. It then follows from (7.27) that x is benign. Hence the ρ_{x_i} are crystalline by Corollary 7.10(1). Moreover, by the proof of Proposition 7.6(2), the crystalline eigenvalues of φ on $D_{\mathrm{cris}}(\rho_{x_i})$ are given by $\{p^{s_i+1}\chi_{s_i+1}^{\infty}(p), p^{s_i}\chi_{s_i+2}^{\infty}(p)\}$ if $n_i = 2$ and $p^{s_i}\chi_{s_i+1}^{\infty}$ if $n_i = 1$. We deduce then from (7.30) that ρ_{x_i} is generic. $\square$

Denote by ω^1 the following composition:

$$\omega^1 : \mathcal{E}^{P-\mathrm{ord}} \hookrightarrow (\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}} \times_E \mathcal{T} \xrightarrow{\mathrm{pr}_1} (\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}.$$

Denote by Z_1' the set of P-ordinary classical points $y = (x, \chi) \in \mathcal{E}^{P-\mathrm{ord}}$ such that:

- y satisfies all the conditions in Lemma 7.14 and Lemma 7.19 (in particular x is benign)
- $\chi = \chi^{\infty}\delta_{\lambda}$ is such that $\lambda = (\lambda_1, \cdots, \lambda_n)$ is *strictly* dominant, i.e. $\lambda_j > \lambda_{j+1}$ for all j.

We let $Z_1 := \omega^1(Z_1') \subseteq (\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$, which we can also view as a subset of (closed) points of the scheme $\mathrm{Spec}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$.

Proposition 7.20. *(1) The set Z_1' is Zariski-dense in $\mathcal{E}^{P-\mathrm{ord}}$ and accumulates (see [12, Déf. 2.2]) at any point (x, χ) with χ locally algebraic such that χ^{∞} is unramified.*

(2) The set Z_1 is Zariski-dense in the scheme $\mathrm{Spec}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$.

Proof. (1) The proof is standard and we omit it.

(2) Let X_0 be the Zariski closure of Z_1 in $\mathrm{Spec}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$ and X be the associated closed subspace of $(\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$. Note that X contains

the Zariski closure of Z_1 in the rigid space $(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$. By Proposition 7.5(2) it is enough to show any benign point of $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$ belongs to X_0, or equivalently to X when seen in $(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$. Let x be a benign point and $y=(x,\chi)$ a P-ordinary classical point of $\mathcal{E}^{P-\mathrm{ord}}$ lying above x. The existence of y follows easily from Corollary 7.10(2) and its proof. By (1), Z_1' accumulates at y, in particular y lies in the Zariski closure of $(\omega^1)^{-1}(Z_1)$ in $\mathcal{E}^{P-\mathrm{ord}}$, from which we easily deduce that $\omega^1(y)=x$ lies in the Zariski closure of Z_1 in the rigid space $(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$. As the latter is contained in X, (2) follows. $\square$

Remark 7.21. We do not know if Z_1 is also Zariski-dense in the rigid analytic space $(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$.

Lemma 7.22. *Let $y=(x,\chi)\in\mathcal{E}^{P-\mathrm{ord}}$ such that Z_1' accumulates at y. Let $i\in\{1,\cdots,k\}$ and assume $\chi_{s_i+1}\chi_{s_i+2}^{-1}\neq x^m|\cdot|^2$ for any $m\in\mathbb{Z}$ if $n_i=2$. Then ρ_{x_i} is trianguline and there exists an injection of (φ,Γ)-modules over $\mathcal{R}_{k(x)}$:*

$$\begin{cases}\mathcal{R}_{k(x)}(\chi_{s_i+1}\varepsilon^{-s_i})\xrightarrow{\sim}D_{\mathrm{rig}}(\rho_{x_i}) & n_i=1\\ \mathcal{R}_{k(x)}(\chi_{s_i+1}\varepsilon^{-s_i}|\cdot|^{-1})\hookrightarrow D_{\mathrm{rig}}(\rho_{x_i}) & n_i=2.\end{cases}\tag{7.31}$$

Proof. Let $i\in\{1,\cdots,k\}$, by Lemma 7.18 we have (7.31) for any point in Z_1'. The result then follows from the global triangulation theory ([51], [58]), and we leave the (standard) details to the reader. $\square$

Proposition 7.23. *Let $y=(x,\chi)\in\mathcal{E}^{P-\mathrm{ord}}$ be a P-ordinary classical point with x benign. Then any injection as in (7.23) extends to an injection of locally analytic representations of $L_P(\mathbb{Q}_p)$ over $k(x)$:*

$$\big(\mathrm{Ind}_{\overline{B}\cap L_P(\mathbb{Q}_p)}^{L_P(\mathbb{Q}_p)}\chi\delta_{B\cap L_P}^{-1}\big)^{\mathrm{an}}\hookrightarrow\mathrm{Ord}_P\big(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\big)^{\mathrm{an}}.\tag{7.32}$$

Proof. We use the notation in the proofs of Proposition 7.6 and Lemma 7.14. Let V be an irreducible constituent of $(\mathrm{Ind}_{\overline{B}\cap L_P(\mathbb{Q}_p)}^{L_P(\mathbb{Q}_p)}\chi\delta_{B\cap L_P}^{-1})^{\mathrm{an}}$. By a dévissage using [13, Cor. 2.2, Lem. 2.8 & Lem. 2.10] together with [63, Thm. 5.8], we deduce that $V\cong\widehat{\otimes}_{i=1,\cdots,k}V_i$ where $V_i\cong\pi_i^{\infty}\otimes_E L_i(\underline{\lambda}_i)$ if $n_i=1$ and $V_i\cong\pi_i^{\infty}\otimes_E L_i(\underline{\lambda}_i)$ or $\mathcal{F}_{\overline{B}_2}^{\mathrm{GL}_2}(\overline{L}_i(-s\cdot\underline{\lambda}_i),|\cdot|^{-1}\chi_{s_i+1}^{\infty}\otimes|\cdot|\chi_{s_i+2}^{\infty})$ if $n_i=2$. Assume that we have an injection:

$$V\hookrightarrow\mathrm{Ord}_P\big(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\big)^{\mathrm{an}}\tag{7.33}$$

for a constituent V such that there exists $i \in \{1, \cdots, k\}$ with V_i *not* locally algebraic (so $n_i = 2$ and $V_i = \mathcal{F}_{\overline{B}_2}^{\mathrm{GL}_2}(\overline{L}_i(-s \cdot \underline{\lambda}_i), |\cdot|^{-1}\chi_{s_i+1}^{\infty} \otimes |\cdot|\chi_{s_i+2}^{\infty}))$. Applying the functor $J_{B\cap L_P}(\cdot)$ to (7.33) gives a point $y' = (x, \chi') \in \mathcal{E}^{P-\mathrm{ord}}$ with $(\chi')^{\infty} = \chi^{\infty}$ (which is unramified) and $\chi'_{s_i+1} = \chi_{s_i+1}x^{\lambda_{s_i+2}-\lambda_{s_i+1}-1}$, $\chi'_{s_i+2} = \chi_{s_i+2}x^{\lambda_{s_i+1}-\lambda_{s_i+2}+1}$. If $\chi_{s_i+1}\chi_{s_i+2}^{-1} \neq x^{\lambda_{s_i+1}-\lambda_{s_i+2}}|\cdot|^2$ (thus $\chi_{s_i+1}\chi_{s_i+2}^{-1} \neq x^m|\cdot|^2$ for any $m \in \mathbb{Z}$, and hence also $\chi'_{s_i+1}(\chi'_{s_i+2})^{-1} \neq x^m|\cdot|^2$ for any $m \in \mathbb{Z}$), applying Lemma 7.22 to the point y' (via Proposition 7.20(1)), we easily deduce a contradiction with the fact the 2-dimensional crystalline $\mathrm{Gal}_{\mathbb{Q}_p}$-representation ρ_{x_i} is nonsplit. Hence such a point y' doesn't exist on $\mathcal{E}^{P-\mathrm{ord}}$ (and we can't have (7.33)). If $\chi_{s_i+1}\chi_{s_i+2}^{-1} = x^{\lambda_{s_i+1}-\lambda_{s_i+2}}|\cdot|^2$, we have $\mathrm{val}_p(\chi_{s_i+1}(p)) = \frac{\lambda_{s_i+1}-\lambda_{s_i+2}}{2} - 1 < \lambda_{s_i+1} - \lambda_{s_i+2}$. As in the proof of Lemma 7.14, we then see by [8, Cor. 3.6] that V_i does not admit a $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant lattice, a contradiction with (7.33). Using [7, Cor. 4.5] we deduce that y' again doesn't exist on $\mathcal{E}^{P-\mathrm{ord}}$. The proposition then follows by the same arguments as in [4, § 6.4 Cas $i = 1$] (or as in [10, § 5.6] when $k = 1$) using Lemma 6.8(2) as a replacement for [4, Lem. 6.3.1] and the above discussion as a replacement for [4, Prop. 6.3.4]. ☐

Corollary 7.24. *Let x be a benign point, then any injection as in (7.23) extends to a closed injection of Banach representations of $L_P(\mathbb{Q}_p)$ over $k(x)$:*

$$\widehat{\bigotimes}_{i=1,\cdots,k}\big(\widehat{\pi}(\rho_{x_i}) \otimes \varepsilon^{s_i} \circ \det\big) \lhook\joinrel\longrightarrow \mathrm{Ord}_P\big(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\big). \tag{7.34}$$

Proof. (a) We use the notation in the proofs of Proposition 7.6 or Corollary 7.10. When $n_i = 2$, by exchanging α_{s_i+1} and α_{s_i+2} if necessary, we can assume $\mathrm{val}_p(\alpha_{s_i+1}) \geq \mathrm{val}_p(\alpha_{s_i+2})$. Let $\chi := \delta_{\lambda}\chi^{\infty}$ with $\chi_{s_i+1}^{\infty} := \mathrm{unr}(p^{-s_i}\alpha_{s_i+1})$ if $n_i = 1$ and $\chi_{s_i+1}^{\infty} := \mathrm{unr}(p^{-s_i-1}\alpha_{s_i+1})$, $\chi_{s_i+2}^{\infty} := \mathrm{unr}(p^{-s_i}\alpha_{s_i+2})$ if $n_i = 2$. We have $(x, \chi) \in \mathcal{E}^{P-\mathrm{ord}}$. From Proposition 7.23, we deduce a continuous $L_P(\mathbb{Q}_p)$-equivariant injection:

$$\widehat{\bigotimes}_{i=1,\cdots,k}\pi_i^{\mathrm{an}} \cong \big(\mathrm{Ind}_{\overline{B}\cap L_P(\mathbb{Q}_p)}^{L_P(\mathbb{Q}_p)}\chi\delta_{B\cap L_P}^{-1}\big)^{\mathrm{an}} \lhook\joinrel\longrightarrow \mathrm{Ord}_P\big(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\big) \tag{7.35}$$

where $\pi_i^{\mathrm{an}} := \chi_{s_i+1}$ if $n_i = 1$ and $\pi_i^{\mathrm{an}} := (\mathrm{Ind}_{\overline{B}_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_{s_i+1}|\cdot|^{-1} \otimes \chi_{s_i+2}|\cdot|)^{\mathrm{an}}$ if $n_i = 2$. By the above condition on α_{s_i+1}, α_{s_i+2}, we know that $\widehat{\pi}(\rho_{x_i}) \otimes \varepsilon^{s_i} \circ \det$ is isomorphic to the universal unitary completion of π_i^{an} (see [3] for the case where $\alpha_{s_i+1} \neq \alpha_{s_i+2}$ and [64] for the case where $\alpha_{s_i+1} = \alpha_{s_i+2}$). It then follows from [13, Lem. 3.4] that the universal unitary completion of $\widehat{\otimes}_{i=1,\cdots,k}\pi_i^{\mathrm{an}}$ is isomorphic to $\widehat{\otimes}_{i=1,\cdots,k}(\widehat{\pi}(\rho_{x_i}) \otimes \varepsilon^{s_i} \circ \det)$. We deduce that

(7.35) induces a continuous $L_P(\mathbb{Q}_p)$-equivariant morphism:

$$\widehat{\bigotimes}_{i=1,\cdots,k}\big(\widehat{\pi}(\rho_{x_i})\otimes_{k(x)}\varepsilon^{s_i}\circ\det\big)\longrightarrow \mathrm{Ord}_P\,\big(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\big) \tag{7.36}$$

which restricts to (7.35) on the left hand side. Since (7.35) is injective and the two Banach representations in (7.36) are admissible (for the left hand side, this follows by induction e.g. from [13, Lem. 2.14]), it follows from [75, § 7] that (7.36) is also injective, and from [73, § 3] that it is automatically closed. □

We now give a lower bound on the Krull dimension of the scheme $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$. We denote by $\mathcal{W}$ the rigid analytic space over E parametrizing the locally analytic characters of $T(\mathbb{Z}_p)$, by ω^2 the composition:

$$\omega^2:\mathcal{E}_{\mathrm{red}}^{P-\mathrm{ord}}\hookrightarrow(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}\times_E\mathcal{T}\xrightarrow{\mathrm{pr}_2}\mathcal{T}$$

and by ω_0^2 the composition of ω^2 with the natural surjection $\mathcal{T}\twoheadrightarrow\mathcal{W}$.

We fix $x\in Z_1$ and use the notation in the proof of Lemma 7.8. For $J\subseteq I_P$, we let $w_J:=(w_{J,i})_{i\in I_P}\in S_2^{|I_P|}$ with $w_{J,i}\neq 1$ if and only if $i\in J$, and put $\chi_J:=\delta_\lambda(\otimes_{j=1}^n\mathrm{unr}(\beta_{w_J,j}))$ with the notation of (7.19). By definition, λ is strictly dominant. By (7.19) and the proof of Lemma 7.18, we have that $y_J:=(x,\chi_J)\in\mathcal{E}^{P-\mathrm{ord}}$ and y_J is noncritical. By Proposition 7.6, the second part of Remark 7.7 and Lemma 7.19, we easily deduce that $\rho_{x,\widetilde{\wp}}$ is crystalline generic. Recall we have assumed Hypothesis 6.9. We now assume one more condition *till the end of the paper.*

Hypothesis 7.25. *If $n>3$, we have U_v maximal hyperspecial at all inert places v.*

It then follows from [19, Thm. 4.8 & 4.10] and the smoothness of $\mathcal{W}$ that the rigid variety $\mathcal{E}_{\mathrm{red}}$ is smooth at the point $\iota^{P-\mathrm{ord}}(y_J)$ (see (7.28)), which therefore belongs to only one irreducible component of $\mathcal{E}_{\mathrm{red}}$. Combining [19, Thm. 4.8 & 4.10] with Corollary 7.17, we deduce the following result.

Proposition 7.26. *The morphism ω_0^2 is étale at the point y_J.*

For $i\in\{1,\cdots,k\}$, we denote by $\omega_i:R_{\overline{\rho}_i}\to\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$ the i-th factor of ω in (6.20) and we still denote by ω_i the induced morphism on the respective $(\mathrm{Spf}\,\cdot)^{\mathrm{rig}}$. We fix $i\in\{1,\cdots,k\}$ and denote by ω_i^1 the following composition:

$$\omega_i^1:\mathcal{E}_{\mathrm{red}}^{P-\mathrm{ord}}\xrightarrow{\omega^1}(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}\xrightarrow{\omega_i}(\mathrm{Spf}\,R_{\overline{\rho}_i})^{\mathrm{rig}}. \tag{7.37}$$

Recall we have $\widehat{\mathcal{O}}_{(\mathrm{Spf}\, R_{\overline{\rho}_i})^{\mathrm{rig}},x_i} \cong R_{\rho_{x_i}}$ ([52, § 2.3] and see § 5.1 for $R_{\rho_{x_i}}$), hence the tangent space:

$$V_{(\mathrm{Spf}\, R_{\overline{\rho}_i})^{\mathrm{rig}},x_i} = \mathrm{Hom}_{k(x_i)-\mathrm{alg}}(\widehat{\mathcal{O}}_{(\mathrm{Spf}\, R_{\overline{\rho}_i})^{\mathrm{rig}},x_i}, k(x_i)[\epsilon]/\epsilon^2)$$

of the rigid analytic variety $(\mathrm{Spf}\, R_{\overline{\rho}_i})^{\mathrm{rig}}$ at x_i is naturally isomorphic to $\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i})$. Extending scalars if necessary, we can see everything over the finite extension $k(x)$ of $k(x_i)$. Assume first $n_i = 1$, then we have

$$\dim_{k(x)} \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) = 2$$

and we denote by $\mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i})$ the 1-dimensional $k(x)$-vector subspace of de Rham (or equivalently crystalline) deformations. Assume $n_i = 2$, since ρ_{x_i} is crystalline, generic and nonsplit, we have $\dim_{k(x)}\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) = 5$ (e.g. by similar arguments as in Lemma 3.5). For each refinement $(\alpha_{s_i+w_i(1)}, \alpha_{s_i+w_i(2)})$ on the Frobenius eigenvalues $\{\alpha_{s_i+1}, \alpha_{s_i+2}\}$ of $D_{\mathrm{cris}}(\rho_{x_i})$ with $w_i \in S_2$, one can proceed as in (3.5) and Lemma 3.6 and define a $k(x)$-vector subspace $\mathrm{Ext}^1_{w_i}(\rho_{x_i}, \rho_{x_i})$ of $\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) \cong \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_{\mathrm{rig}}(\rho_{x_i}), D_{\mathrm{rig}}(\rho_{x_i}))$, analogous to the subspace $\mathrm{Ext}^1_{\mathrm{tri}}(D, D)$ of $\mathrm{Ext}^1_{(\varphi,\Gamma)}(D, D)$ in Lemma 3.6, consisting of trianguline deformations of ρ_{x_i} over $k(x)[\epsilon]/\epsilon^2$ with respect to the triangulation on $D_{\mathrm{rig}}(\rho_{x_i})$ associated to the refinement $(\alpha_{s_i+w_i(1)}, \alpha_{s_i+w_i(2)})$. We denote by $\mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i}) \subseteq \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i})$ the $k(x)$-vector subspace of de Rham deformations, or equivalently of crystalline deformations (since ρ_{x_i} is crystalline generic).

Lemma 7.27. *Let $i \in \{1, \cdots, k\}$ such that $n_i = 2$.*

(1) For any $w_i \in S_2$, we have

$$\dim_{k(x)} \mathrm{Ext}^1_{w_i}(\rho_{x_i}, \rho_{x_i}) = 4, \ \dim_{k(x)} \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i}) = 2$$

and $\mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i}) \subseteq \mathrm{Ext}^1_{w_i}(\rho_{x_i}, \rho_{x_i})$.

(2) We have $\sum_{w_i \in S_2} \mathrm{Ext}^1_{w_i}(\rho_{x_i}, \rho_{x_i}) = \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i})$.

Proof. (1) follows by arguments similar to the ones in the proofs of Lemma 3.6 and Lemma 3.11. (2) easily follows from $\dim_{k(x)} \mathrm{Ext}^1_{w_i}(\rho_{x_i}, \rho_{x_i}) = 4$ and $\dim_{k(x)} \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) = 5$. □

For a morphism $f : X \to Y$ of rigid analytic varieties and a point $x \in X$, we denote by $df_x : V_{X,x} \to V_{Y,f(x)}$ the $k(x)$-linear map induced by f on the respective tangent spaces of X and Y at x and $f(x)$.

We fix $J \subseteq I_P$ and denote by $V_J = V_{\mathcal{E}_{\mathrm{red}}^{P-\mathrm{ord}}, y_J}$ the tangent space of $\mathcal{E}_{\mathrm{red}}^{P-\mathrm{ord}}$ at the point y_J. We let $\overline{d}\omega^1_{i,y_J}$ be the composition:

$$\overline{d}\omega^1_{i,y_J} : V_J \xrightarrow{d\omega^1_{i,y_J}} \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) / \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i})$$

where we recall that $V_{(\mathrm{Spf}\, R_{\overline{\rho}_i})^{\mathrm{rig}}, x_i} \cong \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i})$. We set:

$$\overline{d}\omega^1_{y_J} := (\overline{d}\omega^1_{i,y_J})_{i=1,\cdots,k} : V_J \longrightarrow \bigoplus_{i=1,\cdots,k} \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) / \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i}).$$

Proposition 7.28. *(1) Let $i \in \{1, \cdots, k\}$, we have*

$$\mathrm{Im}(d\omega^1_{i,y_J}) \subseteq \mathrm{Ext}^1_{w_{J,i}}(\rho_{x_i}, \rho_{x_i})$$

where $\mathrm{Ext}^1_{w_{J,i}}(\rho_{x_i}, \rho_{x_i}) := \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i})$ *if $n_i = 1$.*

(2) The morphism $\overline{d}\omega^1_{y_J}$ induces a bijection (using the fact $\mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i}) \subseteq \mathrm{Ext}^1_{w_{J,i}}(\rho_{x_i}, \rho_{x_i})$):

$$\overline{d}\omega^1_{y_J} : V_J \xrightarrow{\sim} \bigoplus_{i=1,\cdots,k} \mathrm{Ext}^1_{w_{J,i}}(\rho_{x_i}, \rho_{x_i}) / \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i}).$$

Proof. (1) Let $v \in V_J$, set $\widetilde{\rho}_{x_i} := d\omega^1_{i,y_J}(v)$, which we view as a deformation of ρ_{x_i} over $k(x)[\epsilon]/\epsilon^2$, and let $\widetilde{\chi}_J := d\omega^2_{y_J}(v)$, which we view as a deformation of χ_J over $k(x)[\epsilon]/\epsilon^2$. From the global triangulation theory (see for instance [58, Prop. 5.13]) and Lemma 7.22, we derive:

$$\begin{cases} \mathcal{R}_{k(x)[\epsilon]/\epsilon^2}(\widetilde{\chi}_{J,s_i+1}\varepsilon^{-s_i}) \xrightarrow{\sim} D_{\mathrm{rig}}(\widetilde{\rho}_{x_i}) & n_i = 1 \\ \mathcal{R}_{k(x)[\epsilon]/\epsilon^2}(\widetilde{\chi}_{J,s_i+1}\varepsilon^{-s_i}|\cdot|^{-1}) \hookrightarrow D_{\mathrm{rig}}(\widetilde{\rho}_{x_i}) & n_i = 2. \end{cases} \tag{7.38}$$

Then (1) follows by definition of $\mathrm{Ext}^1_{w_{J,i}}(\rho_{x_i}, \rho_{x_i})$.

(2) By Proposition 7.26, we have $\dim_{k(x)} V_J = n$. By Lemma 7.27(1) and the discussion before it we have:

$$\begin{cases} \dim_{k(x)} \mathrm{Ext}^1_{w_{J,i}}(\rho_{x_i}, \rho_{x_i}) / \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i}) = 1 & n_i = 1 \\ \dim_{k(x)} \mathrm{Ext}^1_{w_{J,i}}(\rho_{x_i}, \rho_{x_i}) / \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i}) = 2 & n_i = 2. \end{cases} \tag{7.39}$$

Hence it is enough to prove that $\overline{d}\omega^1_{y_J}$ is injective. If $0 \neq v \in V_J$ then we have $d\omega^2_{0,y_J}(v) \neq 0$ by Proposition 7.26, and hence there exists $j \in \{1, \cdots, n\}$

such that the character $\widetilde{\chi}_{J,j}$ is *not* locally algebraic (i.e. doesn't come from an extension of $\chi_{J,j}$ by $\chi_{J,j}$ given by $E\,\mathrm{val}_p$). It then follows from (7.38) and (1.12) that $\widetilde{\rho}_{x_i} \notin \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i})$ if $j \in \{s_i+1, s_i+n_i\}$, whence $\overline{d}\omega^1_{y_J}(v) \neq 0$. □

We denote by V_x the tangent space of the rigid variety $(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})^{P-\mathrm{ord}}_{\overline{\rho}})^{\mathrm{rig}}$ at the point x.

Corollary 7.29. *We have* $\dim_{k(x)} V_x \geq n + (n-k)$.

Proof. For any $J \subseteq I_P$ the morphism $\overline{d}\omega^1_{y_J}$ factors as:

$$\begin{aligned}\overline{d}\omega^1_{y_J} : V_J \xrightarrow{d\omega^1_{y_J}} V_x \xrightarrow{\oplus_i d\omega_{i,x}} & \bigoplus_{i=1,\cdots,k} \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) \\ & \twoheadrightarrow \bigoplus_{i=1,\cdots,k} \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) / \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i}).\end{aligned} \tag{7.40}$$

This implies an inclusion of $k(x)$-vector spaces:

$$\sum_{J \subseteq I_P} \mathrm{Im}(\overline{d}\omega^1_{y_J}) \subseteq \mathrm{Im}\Big(V_x \longrightarrow \bigoplus_{i=1,\cdots,k} \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) / \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i})\Big). \tag{7.41}$$

But, by Proposition 7.28 we have:

$$\begin{aligned}\bigoplus_{J \subseteq I_P} \mathrm{Im}(\overline{d}\omega^1_{y_J}) & \cong \bigoplus_{J \subseteq I_P} \Big(\bigoplus_{i=1,\cdots,k} \mathrm{Ext}^1_{w_{J,i}}(\rho_{x_i}, \rho_{x_i}) / \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i}) \Big) \\ & \cong \bigoplus_{i=1,\cdots,k} \Big(\bigoplus_{J \subseteq I_P} \big(\mathrm{Ext}^1_{w_{J,i}}(\rho_{x_i}, \rho_{x_i}) / \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i}) \big) \Big) \\ & \twoheadrightarrow \bigoplus_{i=1,\cdots,k} \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) / \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i})\end{aligned}$$

where the last morphism is surjective by Lemma 7.27(2). Together with (7.41) it follows that the morphism

$$V_x \longrightarrow \oplus_{i=1,\cdots,k} \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) / \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i})$$

is in fact surjective. Since the right hand side has dimension $n + |I_P| = n + (n-k)$ by Lemma 7.27(1) and the discussion before it, the corollary follows. □

Proposition 7.30. *Each irreducible component of* $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$ *has (Krull) dimension* $\geq n+(n-k)$.

Proof. By Lemma 6.7 $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$ is a reduced scheme and by Proposition 7.20(2) the set of closed points Z_1 is Zariski-dense in the scheme $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$. Thus for each irreducible component X of the scheme $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$, there exists a closed point of X which is in Z_1 and such that X is smooth at x. Since the completed local rings of the scheme $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$ and of the rigid space $(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$ at x are isomorphic (see e.g. [29, Lem. 7.1.9]), the tangent space of $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$ at the point x is isomorphic to V_x. The result then follows from Corollary 7.29. □

7.1.4. Local-global compatibility.

We prove local-global compatibility results for the $L_P(\mathbb{Q}_p)$-representation $\mathrm{Ord}_P(\widehat{S}(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}})$ by generalizing Emerton's method ([42]).

We keep the notation and assumptions of §§ 6.3, 7.1.1, 7.1.2, 7.1.3 (in particular we assume Hypothesis 6.9 & 7.25) and we assume moreover that the $\mathrm{GL}_2(\mathbb{Q}_p)$-representations $\overline{\rho}_i$ satisfy the assumption (A.2) in the appendix when $n_i=2$. We denote by $\overline{\pi}_i$ the representation of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$ over k_E associated to $\overline{\rho}_i$ by the modulo p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ normalized as in [6, § 3.1] when $n_i=2$ and by local class field theory for $\mathrm{GL}_1(\mathbb{Q}_p)$ normalized as in § 1 when $n_i=1$. We denote by π_i^{univ} the universal deformation of $\overline{\pi}_i$ over $R_{\overline{\rho}_i}$ (see for example § A.2, where we consider deformations in the sense of [42, Def. 3.3.7]). We set:

$$\pi_i(U^{\wp}) := \pi_i^{\mathrm{univ}}\widetilde{\otimes}_{R_{\overline{\rho}_i}}\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}} \tag{7.42}$$

where $\widetilde{\otimes}$ means the $\mathfrak{m}_{\overline{\rho}}$-adic completion of the tensor product (still denoting by $\mathfrak{m}_{\overline{\rho}}$ the maximal ideal of $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$). One can check that this is an orthonormalizable admissible representation of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$ over $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$ in the sense of [42, Def. 3.1.11]. We set:

$$\pi_P^{\otimes}(U^{\wp}) := \widetilde{\bigotimes}_{i=1,\cdots,k}\big(\pi_i(U^{\wp})\otimes\varepsilon^{s_i}\circ\det\big) \tag{7.43}$$

(the $\mathfrak{m}_{\overline{\rho}}$-completed tensor product being over $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$) which is an orthonormalizable admissible representation of $L_P(\mathbb{Q}_p)$ over $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$. We have:

$$\pi_P^{\otimes}(U^{\wp}) \otimes_{\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}} \big(\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}/\mathfrak{m}_{\overline{\rho}}\big) \cong \bigotimes_{i=1,\cdots,k} \big((\pi_i^{\mathrm{univ}} \otimes_{R_{\overline{\rho}_i}} k_E) \otimes \overline{\varepsilon}^{s_i} \circ \det\big) \cong \bigotimes_{i=1,\cdots,k} (\overline{\pi}_i \otimes \overline{\varepsilon}^{s_i} \circ \det).$$

As in [42, Def. 6.3.4], we define the $\mathcal{O}_E$-module:

$$X_P(U^{\wp}) := \mathrm{Hom}^{\mathrm{cts}}_{\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[L_P(\mathbb{Q}_p)]} \big(\pi_P^{\otimes}(U^{\wp}), \mathrm{Ord}_P(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}})\big) \tag{7.44}$$

where "cts" denotes the continuous maps for the $\mathfrak{m}_{\overline{\rho}}$-adic topology on the source and the ϖ_E-adic topology on the target. Note that $X_P(U^{\wp})$ is equipped with a natural action of $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$.

We fix a point x of $(\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$ and let x_i for $i \in \{1, \cdots, k\}$ the associated closed points of $\mathrm{Spec}\, R_{\overline{\rho}_i}[1/p]$ as in § 7.1.2. For $i \in \{1, \cdots, k\}$ we let $\widehat{\pi}(\rho_{x_i})^0$ be the open bounded $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$-invariant $\mathcal{O}_{k(x)}$-lattice of $\widehat{\pi}(\rho_{x_i})$ given by $\widehat{\pi}(\rho_{x_i})^0 := \pi_i^{\mathrm{univ}} \otimes_{R_{\overline{\rho}_i}} \mathcal{O}_{k(x)}$ where the morphism $R_{\overline{\rho}_i} \to \mathcal{O}_{k(x)}$ is given by x_i. We can deduce then (note that the $\mathfrak{m}_{\overline{\rho}}$-adic topology on $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}/\mathfrak{p}_x$ coincides with the p-adic topology):

$$\pi_P^{\otimes}(U^{\wp}) \otimes_{\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}} \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}/\mathfrak{p}_x \cong \widehat{\bigotimes}_{i=1,\cdots,k} \big(\widehat{\pi}(\rho_{x_i})^0 \otimes \varepsilon^{s_i} \circ \det\big), \tag{7.45}$$

from which we easily get:

$$X_P(U^{\wp})[\mathfrak{p}_x] \cong \mathrm{Hom}_{\mathcal{O}_{k(x)}[L_P(\mathbb{Q}_p)]} \Big(\widehat{\bigotimes}_{i=1,\cdots,k} \big(\widehat{\pi}(\rho_{x_i})^0 \otimes \varepsilon^{s_i} \circ \det\big), \mathrm{Ord}_P(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}})[\mathfrak{p}_x]\Big) \tag{7.46}$$

(where $\widehat{\otimes}$ means the p-adic completion of the tensor product). We refer to [42, Def. C.1] for the definition of a cofinitely generated $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$-module.

Lemma 7.31. *The $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$-module $X_P(U^{\wp})$ is cofinitely generated.*

Proof. We verify the conditions in [42, Def. C.1]. The first three conditions are easy to check from the definition (7.44). We have an injection of k_E-vector spaces:

$$X_P(U^{\wp})/\varpi_E \hookrightarrow \mathrm{Hom}_{\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[L_P(\mathbb{Q}_p)]} \Big(\bigotimes_{i=1,\cdots,k} (\overline{\pi}_i \otimes \overline{\varepsilon}^{s_i} \circ \det), \mathrm{Ord}_P(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}})/\varpi_E\Big) \tag{7.47}$$

from which we deduce an injection of k_E-vector spaces:

$$(7.48)\quad (X_P(U^{\wp})/\varpi_E)[\mathfrak{m}_{\overline{\rho}}] \\ \hookrightarrow \mathrm{Hom}_{k_E[L_P(\mathbb{Q}_p)]}\Big(\bigotimes_{i=1,\cdots,k}(\overline{\pi}_i\otimes\overline{\varepsilon}^{s_i}\circ\det), \big(\mathrm{Ord}_P(\widehat{S}(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}})/\varpi_E\big)[\mathfrak{m}_{\overline{\rho}}]\Big).$$

By Lemma 6.8(1) and its proof (see the isomorphism (6.16)), we have an isomorphism:

$$(7.49)\qquad \big(\mathrm{Ord}_P(\widehat{S}(U^{\wp},\mathbb{W}^{\wp})_{\overline{\rho}})/\varpi_E\big)[\mathfrak{m}_{\overline{\rho}}] \cong \mathrm{Ord}_P(S(U^{\wp},\mathbb{W}^{\wp}/\varpi_E)_{\overline{\rho}})[\mathfrak{m}_{\overline{\rho}}]$$

which is a smooth admissible representation of $L_P(\mathbb{Q}_p)$ over k_E. Together with the fact that $\otimes_{i=1,\cdots,k}(\overline{\pi}_i\otimes\overline{\varepsilon}^{s_i}\circ\det)$ can be generated over $L_P(\mathbb{Q}_p)$ by a finite dimensional k_E-vector subspace, we easily deduce that the right hand side of (7.48) is finite dimensional over k_E. The lemma follows. ☐

Theorem 7.32. *(1) The $\widetilde{\mathbb{T}}(U^{\wp})^{P-\mathrm{ord}}_{\overline{\rho}}$-module $X_P(U^{\wp})$ is faithful.*

(2) For any point $x\in(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})^{P-\mathrm{ord}}_{\overline{\rho}})^{\mathrm{rig}}$, we have $X_P(U^{\wp})[\mathfrak{p}_x]\neq 0$, equivalently by (7.46) there exists a nonzero morphism of admissible Banach representations of $L_P(\mathbb{Q}_p)$ over $k(x)$:

$$(7.50)\qquad \widehat{\bigotimes}_{i=1,\cdots,k}\big(\widehat{\pi}(\rho_{x_i})\otimes_{k(x)}\varepsilon^{s_i}\circ\det\big)\longrightarrow \mathrm{Ord}_P\big(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}\big)[\mathfrak{m}_x].$$

Proof. By [42, Prop. C.36], (1) and (2) are equivalent, hence it is enough to prove (1). By Corollary 7.24, if x is a benign point we have $X_P(U^{\wp})[\mathfrak{p}_x]\neq 0$. By Proposition 7.5(2) the benign points are Zariski-dense in the scheme $\mathrm{Spec}\,\widetilde{\mathbb{T}}(U^{\wp})^{P-\mathrm{ord}}_{\overline{\rho}}[1/p]$. The theorem then follows by the same argument as in the proof of [42, Prop. C.36] (see also [6, Prop. 4.7]). ☐

Corollary 7.33. *Let $x\in(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})^{P-\mathrm{ord}}_{\overline{\rho}})^{\mathrm{rig}}$, there exists a nonzero morphism of admissible Banach representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $k(x)$:*

$$(7.51)\quad \Big(\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}\widehat{\bigotimes}_{i=1,\cdots,k}\big(\widehat{\pi}(\rho_{x_i})\otimes\varepsilon^{s_i}\circ\det\big)\Big)^{\mathcal{C}^0}\longrightarrow\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x].$$

Proof. This follows from (7.50) and [40, Thm. 4.4.6]. ☐

Corollary 7.34. *Let $x\in(\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})^{P-\mathrm{ord}}_{\overline{\rho}})^{\mathrm{rig}}$ and assume:*

- *for any $i\in\{1,\cdots,k\}$, the $\mathrm{Gal}_{\mathbb{Q}_p}$-representation ρ_{x_i} is irreducible de Rham with distinct Hodge-Tate weights $\{-\mu_{s_i+1},-\mu_{s_i+n_i}\}$*

- $-\mu_1 > -\mu_2 > \cdots > -\mu_n$.

Then the point x is classical.

Proof. Let $\lambda_j := -\mu_j+(j-1)$, thus $\lambda := (\lambda_1, \cdots, \lambda_n)$ is a dominant weight. It follows from [24, Thm. VI.5.7 & VI.6.18] that there exists a nonzero smooth representation π_i^∞ of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$ over $k(x)$ such that $\widehat{\pi}(\rho_{x_i})^{\mathrm{lalg}} \cong \pi_i^\infty \otimes_{k(x)} L_i(\underline{\lambda}_i)$ where $\underline{\lambda}_i := (\lambda_{s_i+1}, \lambda_{s_i+n_i})$. Moreover, since ρ_{x_i} is irreducible, we know that $\widehat{\pi}(\rho_{x_i})$ is also irreducible as a continuous representation of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$. We claim that the morphism (7.50) restricts to a non-zero $L_P(\mathbb{Q}_p)$-equivariant morphism:

$$\begin{aligned}(7.52)\quad &\Big(\bigotimes_{i=1,\cdots,k} \pi_i^\infty\Big) \otimes_{k(x)} L_P(\lambda)\\ &\qquad \cong \bigotimes_{i=1,\cdots,k} \big(\widehat{\pi}(\rho_{x_i})^{\mathrm{lalg}} \otimes \varepsilon^{s_i} \circ \det\big) \longrightarrow \mathrm{Ord}_P\big(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_x]\big).\end{aligned}$$

Indeed, let $0 \neq v = v_1 \otimes \cdots \otimes v_k \in \otimes_{i=1,\cdots,k}\big(\widehat{\pi}(\rho_{x_i})^{\mathrm{lalg}} \otimes \varepsilon^{s_i} \circ \det\big) \subseteq \widehat{\otimes}_{i=1,\cdots,k}\big(\widehat{\pi}(\rho_{x_i}) \otimes_{k(x)} \varepsilon^{s_i} \circ \det\big)$, if (7.52) is zero, we see the morphism (7.50) sends v to zero. However, since $\widehat{\pi}(\rho_{x_i})$ is irreducible for all i, it is not difficult to see that $\widehat{\otimes}_{i=1,\cdots,k}(\widehat{\pi}(\rho_{x_i}) \otimes \varepsilon^{s_i} \circ \det)$ can be topologically generated by v under the $L_P(\mathbb{Q}_p)$-action. We deduce hence (7.50) is zero, a contradiction. By Proposition 4.21, the morphism (7.52) induces a nonzero $\mathrm{GL}_n(\mathbb{Q}_p)$-equivariant morphism:

$$\Big(\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \bigotimes_{i=1,\cdots,k} \pi_i^\infty\Big)^\infty \otimes_{k(x)} L(\lambda) \longrightarrow \widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_x]$$

which implies $\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_x]^{\mathrm{lalg}} \neq 0$, whence the result. $\square$

Remark 7.35. For $x \in (\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$, by passing to a smaller parabolic subgroup, it should be possible to prove that Corollary 7.34 still holds when ρ_{x_i} is reducible for some i.

We set (where $\mathrm{Hom}_{\mathcal{O}_E} = \mathcal{O}_E$-linear homomorphisms):

$$(7.53)\qquad M_P(U^\wp) := \mathrm{Hom}_{\mathcal{O}_E}(X_P(U^\wp), \mathcal{O}_E)$$

which, by [42, Prop. C.5], is a finitely generated $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}}$-module which is $\mathcal{O}_E$-torsion free. Moreover by [42, Lem. C.14], for any $x \in (\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$, the $\mathcal{O}_{k(x)}$-modules $M_P(U^\wp)/\mathfrak{p}_x$ and $X_P(U^\wp)[\mathfrak{p}_x]$ are finitely generated free of the same rank, that we denote by $m_P(x)$.

Lemma 7.36. *Let x be a benign point, then $m_P(x) = m(x)$ (see (7.13) for $m(x)$).*

Proof. Consider the following composition:

$$\begin{aligned}
&\mathrm{Hom}_{L_P(\mathbb{Q}_p)}\Big(\widehat{\bigotimes}_{i=1,\cdots,k}\big(\widehat{\pi}(\rho_{x_i})\otimes\varepsilon^{s_i}\circ\det\big),\mathrm{Ord}_P(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}})[\mathfrak{m}_x]\Big)\\
&\quad\longrightarrow \mathrm{Hom}_{L_P(\mathbb{Q}_p)}\Big(\widehat{\bigotimes}_{i=1,\cdots,k}(\pi_i^{\mathrm{an}}\otimes\varepsilon^{s_i}\circ\det),\mathrm{Ord}_P(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}})[\mathfrak{m}_x]\Big)\\
&\longrightarrow \mathrm{Hom}_{L_P(\mathbb{Q}_p)}\Big(\bigotimes_{i=1,\cdots,k}\big(\widehat{\pi}(\rho_{x_i})^{\mathrm{lalg}}\otimes\varepsilon^{s_i}\circ\det\big),\mathrm{Ord}_P(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}})[\mathfrak{m}_x]\Big)
\end{aligned}$$

where π_i^{an} is as in the proof of Corollary 7.24. The first map is injective since $\widehat{\otimes}_{i=1,\cdots,k}\pi_i^{\mathrm{an}}$ is dense in $\widehat{\otimes}_{i=1,\cdots,k}\widehat{\pi}(\rho_{x_i})$ (see the proof of Corollary 7.24). By Corollary 7.24, the composition is surjective. By the proof of Proposition 7.23, the second map is injective. We deduce then that all these maps are bijective. From Proposition 4.21, we deduce an isomorphism:

$$\begin{aligned}
&\mathrm{Hom}_{L_P(\mathbb{Q}_p)}\Big(\bigotimes_{i=1,\cdots,k}\big(\widehat{\pi}(\rho_{x_i})^{\mathrm{lalg}}\otimes\varepsilon^{s_i}\circ\det\big),\mathrm{Ord}_P\left(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]^{\mathrm{lalg}}\right)\Big)\\
&\xrightarrow{\sim}\mathrm{Hom}_{L_P(\mathbb{Q}_p)}\Big(\bigotimes_{i=1,\cdots,k}\big(\widehat{\pi}(\rho_{x_i})^{\mathrm{lalg}}\otimes\varepsilon^{s_i}\circ\det\big),\mathrm{Ord}_P(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}})[\mathfrak{m}_x]\Big).
\end{aligned}$$

The lemma follows from these isomorphisms together with (7.46), (7.24), Proposition 7.9. $\square$

Let $S_{\mathrm{gl}}^{P-\mathrm{ord}}(\overline{\rho}_{\widetilde{\wp}})$ be the set of (isomorphism classes of) irreducible $L_P(\mathbb{Z}_p)$-representations $\sigma=\otimes_{i=1}^k\sigma_i$ over k_E such that:

$$\mathrm{Hom}_{L_P(\mathbb{Z}_p)}\left(\sigma,\mathrm{Ord}_P\left(S(U^{\wp},\mathbb{W}^{\wp}/\varpi_E)_{\overline{\rho}}[\mathfrak{m}_{\overline{\rho}}]\right)\right)\neq 0.$$

For $i\in\{1,\cdots,k\}$ let $S_{\mathrm{gl}}(\overline{\rho}_i)$ be the set of (isomorphism classes of) irreducible $\mathrm{GL}_{n_i}(\mathbb{Z}_p)$-representations σ_i such that there exist irreducible $\mathrm{GL}_{n_j}(\mathbb{Z}_p)$-representations σ_j over k_E for $j\neq i$ such that $\otimes_{j=1}^k(\sigma_j\otimes\overline{\varepsilon}^{s_j}\circ\det)\in S_{\mathrm{gl}}^{P-\mathrm{ord}}(\overline{\rho}_{\widetilde{\wp}})$. Finally let $S(\overline{\rho}_i)$ be the set of Serre weights attached to $\overline{\rho}_i$, that is the set of irreducible summands in $\mathrm{soc}(\overline{\pi}_i|_{\mathrm{GL}_{n_i}(\mathbb{Z}_p)})$, and let $S^{P-\mathrm{ord}}$ be the set of (isomorphism classes of) irreducible $L_P(\mathbb{Z}_p)$-representations $\sigma\cong\otimes_{i=1}^k\,(\sigma_i\otimes\overline{\varepsilon}^{s_i}\circ\det)$ with $\sigma_i\in S(\overline{\rho}_i)$.

Proposition 7.37. *We have $S_{\mathrm{gl}}^{P-\mathrm{ord}}(\overline{\rho}_{\widetilde{\wp}})\subseteq S^{P-\mathrm{ord}}(\overline{\rho}_{\widetilde{\wp}})$, hence $S_{\mathrm{gl}}(\overline{\rho}_i)\subseteq S(\overline{\rho}_i)$ for any $i\in\{1,\cdots,k\}$.*

Proof. The proposition follows by similar arguments as in the proof of [42, Thm. 5.7.7(1)]. For $\sigma = \otimes_{i=1}^{k} (\sigma_i \otimes \overline{\varepsilon}^{s_i} \circ \det) \in S_{\mathrm{gl}}^{P-\mathrm{ord}}(\overline{\rho}_{\widetilde{\wp}})$, we lift σ to an algebraic representation $\Theta \cong \otimes_{i=1,\cdots,k}\Theta_i$ of $L_P(\mathbb{Z}_p)$ over $\mathcal{O}_E$ of (dominant) weight λ such that $\lambda_{s_i} \geq \lambda_{s_i+1}$ and $0 \leq \lambda_{s_i+n_i} - \lambda_{s_i+1} \leq p-1$ for $i = 1, \cdots, k$. Since $\mathrm{Ord}_P(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}})$ is isomorphic to a direct factor of $\mathcal{C}(L_P(\mathbb{Z}_p), \mathcal{O}_E)^{\oplus r}$ (cf. Lemma 6.8(2)), we have an isomorphism (e.g. by [68, Lem. 2.14]):

$$\begin{aligned}\mathrm{Hom}_{L_P(\mathbb{Z}_p)}&\big(\Theta, \mathrm{Ord}_P(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}})\big)/\varpi_E \\ &\xrightarrow{\sim} \mathrm{Hom}_{L_P(\mathbb{Z}_p)}\big(\sigma, \mathrm{Ord}_P(S(U^{\wp}, \mathbb{W}^{\wp}/\varpi_E)_{\overline{\rho}})\big).\end{aligned}$$

We deduce, using that λ is dominant:

$$\begin{aligned}0 \neq \mathrm{Hom}_{L_P(\mathbb{Z}_p)}&\big(\Theta, \mathrm{Ord}_P(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}})\big) \\ &\cong \mathrm{Hom}_{L_P(\mathbb{Z}_p)}\big(\Theta, \mathrm{Ord}_P(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}})_{+}^{L_P(\mathbb{Z}_p)-\mathrm{alg}}\big).\end{aligned} \tag{7.54}$$

By (6.3) and the same argument as in the proof of Proposition 7.5(2), it follows that there exists a nonempty finite set C of benign points such that the $\mathcal{O}_E$-module (7.54) is isomorphic to:

$$\bigoplus_{x\in C} \mathrm{Hom}_{L_P(\mathbb{Z}_p)}\big(\Theta, \mathrm{Ord}_P(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}})[\mathfrak{p}_x]\big)$$

with each factor in the direct sum being nonzero. Let $x \in C$ and consider (recall Θ is an $\mathcal{O}_E$-lattice in $L_P(\lambda)$ stable by $L_P(\mathbb{Z}_p)$):

$$\begin{aligned}\pi_x^{\infty} := &\Big(\mathrm{Ord}_P\big(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\big) \otimes_{\mathcal{O}_E} L_P(\lambda)^{\vee}\Big)^{\mathrm{sm}} \\ &\supseteq \Big(\mathrm{Ord}_P\big(\widehat{S}(U^{\wp}, \mathbb{W}^{\wp})_{\overline{\rho}}[\mathfrak{p}_x]^{\mathrm{lalg}}\big) \otimes_{\mathcal{O}_E} \Theta^{\vee}\Big)^{L_P(\mathbb{Z}_p)}.\end{aligned} \tag{7.55}$$

By assumption we have $(\pi_x^{\infty})^{L_P(\mathbb{Z}_p)} \neq 0$, so that (picking up $0 \neq v \in (\pi_x^{\infty})^{L_P(\mathbb{Z}_p)}$) we can define smooth irreducible $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$-representations π_i^{∞} as in the proofs of Proposition 7.6 and Corollary 7.10(1). In particular we have $\otimes_{i=1,\cdots,k}\pi_i^{\infty} \hookrightarrow \pi_x^{\infty}$ and $(\pi_i^{\infty})^{\mathrm{GL}_{n_i}(\mathbb{Z}_p)} \neq 0$, and from (7.24) we also have $\pi_i^{\infty} \otimes L_i(\underline{\lambda}_i) \cong \widehat{\pi}(\rho_{x_i})^{\mathrm{lalg}} \otimes \varepsilon^{s_i} \circ \det$ where $\underline{\lambda}_i = (\lambda_{s_i+1}, \lambda_{s_i+n_i})$. But the latter isomorphism together with $(\pi_i^{\infty})^{\mathrm{GL}_{n_i}(\mathbb{Z}_p)} \neq 0$ easily imply, using that Θ_i is up to scaling the only $\mathcal{O}_E$-lattice in $L_i(\underline{\lambda}_i)$ which is stable by $\mathrm{GL}_{n_i}(\mathbb{Z}_p)$:

$$\sigma_i = \overline{\Theta_i} \otimes \overline{\varepsilon}^{-s_i} \circ \det \in S(\overline{\rho}_i).$$

The proposition follows. $\square$

Theorem 7.38. *If there exists i such that $n_i = 2$ and $\overline{\rho}_i$ is peu ramifié (up to twist), assume that any subrepresentation $\pi = \otimes_{i=1,\cdots,k}\pi_i$ of $\mathrm{Ord}_P(S(U^\wp, \mathbb{W}^\wp/\varpi_E)_{\overline{\rho}})$ is such that π_i is infinite dimensional. Then the evaluation map:*

$$\mathrm{ev} : X_P(U^\wp)\widehat{\otimes}_{\widetilde{\mathbb{T}}(U^\wp)^{P-\mathrm{ord}}_{\overline{\rho}}}\pi_P^{\otimes}(U^\wp) \longrightarrow \mathrm{Ord}_P(\widehat{S}(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}}) \tag{7.56}$$

is an isomorphism where $\widehat{\otimes}$ denotes the ϖ_E-adic completion of the usual tensor product.

Proof. (a) By [42, Lem. C.46], the map ev is injective with saturated image (see [42, Def. C.6]) if and only if the induced morphism:

$$(X_P(U^\wp)/\varpi_E)[\mathfrak{m}_{\overline{\rho}}] \otimes_{k_E} \Big(\bigotimes_{i=1,\cdots,k} (\overline{\pi}_i \otimes \overline{\varepsilon}^{s_i} \circ \det)\Big) \\ \longrightarrow \mathrm{Ord}_P(S(U^\wp, \mathbb{W}^\wp/\varpi_E)_{\overline{\rho}})[\mathfrak{m}_{\overline{\rho}}] \tag{7.57}$$

is injective. By (7.48), it is enough to prove that the evaluation map:

(7.58)
$$\mathrm{Hom}_{k_E[L_P(\mathbb{Q}_p)]}\Big(\bigotimes_{i=1,\cdots,k} (\overline{\pi}_i \otimes_{k_E} \overline{\varepsilon}^{s_i} \circ \det), \big(\mathrm{Ord}_P(\widehat{S}(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}})/\varpi_E\big)[\mathfrak{m}_{\overline{\rho}}]\Big) \\ \otimes_{k_E} \Big(\bigotimes_{i=1,\cdots,k} (\overline{\pi}_i \otimes_{k_E} \overline{\varepsilon}^{s_i} \circ \det)\Big) \longrightarrow \mathrm{Ord}_P(S(U^\wp, \mathbb{W}^\wp/\varpi_E)_{\overline{\rho}})[\mathfrak{m}_{\overline{\rho}}]$$

is injective. By [42, Lem. 6.4.15], it is enough to show that any nonzero homomorphism in:

$$\mathrm{Hom}_{k_E[L_P(\mathbb{Q}_p)]}\Big(\bigotimes_{i=1,\cdots,k} (\overline{\pi}_i \otimes_{k_E} \overline{\varepsilon}^{s_i} \circ \det), \big(\mathrm{Ord}_P(\widehat{S}(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}})/\varpi_E\big)[\mathfrak{m}_{\overline{\rho}}]\Big)$$

is injective. But this follows from the same argument as in the proof of [42, Thm. 6.4.16] (using Proposition 7.37 and the assumption to deal with those $\overline{\pi}_i$ which are reducible).

(b) We show that the map ev is surjective. Since its image is saturated, it is enough to prove the surjection after inverting p. By [42, Lem. 3.1.16] and the proof of [42, Prop. 3.1.3], $\mathrm{Im}(\mathrm{ev} \otimes E)$ is a closed $L_P(\mathbb{Q}_p)$-subrepresentation of $\mathrm{Ord}_P(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}})$ which is preserved by $\widetilde{\mathbb{T}}(U^\wp)^{P-\mathrm{ord}}_{\overline{\rho}}$. By Lemma 6.8, Corollary 7.3 and the same argument as in the proof of

Proposition 7.5(2), it is enough to prove that for any benign point x, we have:

$$\mathrm{Ord}_P\left(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\right)_+^{L_P(\mathbb{Z}_p)-\mathrm{alg}} \subset \mathrm{Im}(\mathrm{ev}\otimes E). \tag{7.59}$$

Using the adjunction formula of Proposition 4.21, we can deduce (see the proof of Proposition 7.5(2)):

$$\mathrm{Ord}_P\left(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]\right)_+^{L_P(\mathbb{Z}_p)-\mathrm{alg}} \subseteq \mathrm{Ord}_P\left(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_x]^{\mathrm{lalg}}\right). \tag{7.60}$$

Then (7.59) easily follows from Proposition 7.9, (7.24), Corollary 7.24 and (7.46). □

Remark 7.39. The assumption in Theorem 7.38 when $\overline{\rho}_i$ is peu ramifié is in the style of "Ihara's lemma" (see e.g. the proof of [42, Thm. 5.7.7(3)]) and one can conjecture that it is always satisfied.

Corollary 7.40. *Keep the assumption of Theorem 7.38. There exists an integer $s \geq 1$ such that we have an isomorphism of smooth admissible $L_P(\mathbb{Q}_p)$-representations over k_E:*

$$\Big(\bigotimes_{i=1,\cdots,k}\big(\overline{\pi}_i \otimes_{k_E} \overline{\varepsilon}^{s_i}\circ \det\big)\Big)^{\oplus s} \xrightarrow{\sim} \mathrm{Ord}_P\left(S(U^{\wp}, \mathbb{W}^{\wp}/\varpi_E)_{\overline{\rho}}[\mathfrak{m}_{\overline{\rho}}]\right).$$

Consequently, $S_{\mathrm{gl}}^{P-\mathrm{ord}}(\overline{\rho}_{\widetilde{\wp}}) = S^{P-\mathrm{ord}}(\overline{\rho}_{\widetilde{\wp}})$.

Proof. By Theorem 7.38, (7.49) and (7.45), (7.57) is actually an isomorphism. The corollary follows since $(X_P(U^{\wp})/\varpi_E)[\mathfrak{m}_{\overline{\rho}}]$ is a finite dimensional k_E-vector space. □

Corollary 7.41. *Keep the assumption of Theorem 7.38. Let $x \in (\mathrm{Spf}\,\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$, then:*

$$\mathrm{Ord}_P\left(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}\right)[\mathfrak{m}_x] \cong \left(\widehat{\otimes}_{i=1,\cdots,r}\big(\widehat{\pi}(\rho_{x_i}) \otimes_{k(x)} \varepsilon^{s_i}\circ\det\big)\right)^{\oplus m_P(x)}.$$

Proof. The corollary follows from Theorem 7.38, [42, Lem. 3.1.17] (applied with $A = \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}$ and $M = \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}/\mathfrak{p}_x$), (7.45) and the definition of $m_P(x)$. □

7.2. $\mathcal{L}$-invariants

We prove Conjecture 6.2 when $\rho_{\widetilde{\wp}}$ has consecutive Hodge-Tate weights assuming weak genericity conditions.

7.2.1. Preliminaries. We start with easy preliminaries.

Throughout § 7.2 we keep the notation and assumptions of § 6.3 and of all the subsections of § 7.1, in particular we assume Hypothesis 6.9 and that the open compact subgroup $U^\wp$ is such that U^p is sufficiently small, U_v is maximal for $v|p$, $v \neq \wp$, and U_v is maximal hyperspecial at all inert places v if $n > 3$ (Hypothesis 7.25). We assume moreover that $\overline{\rho}$ is such that $\overline{\rho}_{\widetilde{\wp}}$ is a successive extension of characters $\overline{\chi}_i$ for $i = 1, \cdots, n$ with $\overline{\chi}_i\overline{\chi}_{i+1}^{-1} = \overline{\varepsilon}$ (so in particular all the n_i are 1 and $k = n$) and that $\overline{\rho}_{\widetilde{\wp}}$ is strictly B-ordinary (Definition 5.8). This implies $\overline{\chi}_i = \overline{\varepsilon}^{1-i}\overline{\chi}_1$ for $i = 1, \cdots, n$ and $p > n$. We fix $\rho\colon \mathrm{Gal}_F \to \mathrm{GL}_n(E)$ a continuous representation such that ρ is unramified outside $S(U^p)$ and such that:

- $\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{lalg}} \neq 0$
- $\rho_{\widetilde{\wp}}$ is semi-stable noncrystalline and is isomorphic to a successive extension of characters $\chi_i : \mathrm{Gal}_{\mathbb{Q}_p} \to E^\times$ such that $\chi_i\chi_{i+1}^{-1} = \varepsilon$.

The first assumption implies that ρ is absolutely irreducible (since $\overline{\rho}$ is), is automorphic (by (6.3)) and satisfies $\rho^\vee \circ c \cong \rho \otimes \varepsilon^{1-n}$, and then the second implies that the monodromy operator on $D_{\mathrm{st}}(\rho_{\widetilde{\wp}})$ satisfies $N^{n-1} \neq 0$ (use [17] together with the fact that the automorphic representation associated to ρ has a generic local component at $\wp$ by base change to GL_n and the irreducibility of ρ, see the proof of Proposition 7.6(1)). In particular $(\chi_1, \cdots, \chi_n)$ is the unique parameter of the (φ, Γ)-module $D := D_{\mathrm{rig}}(\rho_{\widetilde{\wp}})$ and it is moreover special (see Definition 2.1 and use [2, Thm. A]) and such that $\chi_i = \varepsilon^{1-i}\chi_1$ for $i = 1, \cdots, n$. We also easily deduce that $\rho_{\widetilde{\wp}}$ is strictly P-ordinary for any parabolic subgroup P of GL_n containing B.

Using [17], we see that there exists $m(\rho)$ such that:

$$\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{lalg}} \cong (\mathrm{St}_n^\infty \otimes \chi_1 \circ \det)^{\oplus m(\rho)} \tag{7.61}$$

where St_n^∞ denotes the standard smooth Steinberg representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over E. As in (7.13), we have by (6.3) and our assumptions on U_v for $v|p$, $v \neq \wp$:

$$m(\rho) = \sum_\pi m(\pi) \dim_{\overline{\mathbb{Q}_p}} (\pi^{\infty,\wp})^{U^\wp} = \sum_\pi m(\pi) \dim_{\overline{\mathbb{Q}_p}} (\pi^{\infty,p})^{U^p} \tag{7.62}$$

where π runs through the automorphic representations of $G(\mathbb{A}_{F_+})$ which contribute to the locally algebraic representation $\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{lalg}}$. We easily check that:

$$\mathrm{Ord}_B(\mathrm{St}_n^\infty \otimes \chi_1 \circ \det) \cong J_B(\mathrm{St}_n^\infty \otimes \chi_1 \circ \det)(\delta_B^{-1}) \cong \chi_1 \circ \det\,. \tag{7.63}$$

Lemma 7.42. *We have an isomorphism of $T(\mathbb{Q}_p)$-representations:*

$$\mathrm{soc}_{T(\mathbb{Q}_p)}\, J_B\big(\widehat{S}(U^\wp, W^\wp)^{\mathrm{an}}_{\overline{\rho}}[\mathfrak{m}_\rho]\big) \cong ((\chi_1 \circ \det) \otimes \delta_B)^{\oplus m(\rho)}.$$

Proof. From the global triangulation theory ([51], [58]) applied to the eigenvariety $\mathcal{E}$ (see § 7.1.3), exactly the same proof as the one of [4, Prop. 6.3.4] gives:

$$\mathrm{Hom}_{T(\mathbb{Q}_p)}\big(\delta, J_B\big(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]\big)\big) \neq 0 \Rightarrow \delta\delta_B^{-1} = \chi_1 \circ \det. \tag{7.64}$$

There exists thus an integer $m' \geq 1$ such that the isomorphism in the statement holds with $m(\rho)$ replaced by m'. By (7.61) and (7.63), we have $m' \geq m(\rho)$. Using [7, Thm. 4.3] together with (7.64), we see that an "extra" copy of $(\chi_1 \circ \det) \otimes \delta_B$ in the socle would yield an extra copy of $\mathrm{St}_n^\infty \otimes \chi_1 \circ \det$ in $\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{lalg}}$, hence $m' = m(\rho)$. □

We denote by x the point of $(\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}})^{\mathrm{rig}}$ associated to ρ (thus $\mathfrak{m}_x = \mathfrak{m}_\rho$). By (7.63) and Lemma 4.12, we obtain that $x \in (\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^\wp)^{P-\mathrm{ord}}_{\overline{\rho}})^{\mathrm{rig}}$ for all $P \supseteq B$.

For $1 \leq i < i' \leq n$, we denote by $\rho_i^{i'}$ the (unique) subquotient of $\rho_{\widetilde{\wp}}$ which is isomorphic to a successive extension of the characters χ_j for $i \leq j \leq i'$. We have $D_{\mathrm{rig}}(\rho_i^{i'}) = D_i^{i'} =$ the (unique) subquotient of D isomorphic to a successive extension of the $\mathcal{R}_E(\chi_j)$ for $i \leq j \leq i'$ (see the beginning of § 2 for this notation).

7.2.2. Simple $\mathcal{L}$-invariants.

For L_P with only one factor being GL_2, we show that one can recover the corresponding simple $\mathcal{L}$-invariant in $\mathrm{Ord}_P(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}})[\mathfrak{m}_\rho]$ (Corollary 7.47). We work in arbitrary dimension.

We keep the notation and assumptions of § 7.2.1. By Theorem 7.38, we have an isomorphism (note that the assumption in *loc. cit.* is here automatic since $n_i = 1$ for all i):

$$X_B(U^\wp)\widehat{\otimes}_{\widetilde{\mathbb{T}}(U^\wp)^{B-\mathrm{ord}}_{\overline{\rho}}} \pi_B^{\otimes}(U^\wp) \xrightarrow{\sim} \mathrm{Ord}_B(\widehat{S}(U^\wp, \mathbb{W}^\wp)_{\overline{\rho}}). \tag{7.65}$$

Recall we defined the integer $m_B(x)$ just before Lemma 7.36.

Lemma 7.43. *We have $m_B(x) = m(\rho)$.*

Proof. By Corollary 7.41 combined with (4.18), (7.61) and (7.63), we have $m_B(x) \geq m(\rho)$. By [40, Thm. 4.4.6], we have:

$$\begin{aligned}(7.66)\quad &\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} 1)^{\mathcal{C}^0} \otimes \chi_1 \circ \det, \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]\big)\\ &\qquad\xrightarrow{\sim} \mathrm{Hom}_{T(\mathbb{Q}_p)}\big(\chi_1 \circ \det, \mathrm{Ord}_B(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho])\big).\end{aligned}$$

We have an obvious injection:

$$\begin{aligned}(7.67)\quad &\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} 1)^{\mathcal{C}^0} \otimes \chi_1 \circ \det, \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]\big)\\ &\hookrightarrow \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} 1)^{\mathrm{an}} \otimes \chi_1 \circ \det, \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{an}}\big).\end{aligned}$$

From the description of irreducible constituents of $(\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} 1)^{\mathrm{an}}$ for $B \subseteq P$ (see [63, § 6]), Lemma 7.42 and [8, Cor. 3.4], we obtain if $P \supsetneq B$:

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} 1)^{\mathrm{an}} \otimes \chi_1 \circ \det, \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{an}}\big) = 0,$$

from which we deduce:

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} 1)^{\mathrm{an}} \otimes \chi_1 \circ \det, \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{an}}\big)\\ &\qquad\xleftarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_p)}\big(\mathrm{St}_n^{\mathrm{an}} \otimes \chi_1 \circ \det, \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{an}}\big)\\ &\qquad\qquad\hookrightarrow \mathrm{Hom}_{T(\mathbb{Q}_p)}\big(\mathrm{St}_n^{\infty} \otimes \chi_1 \circ \det, \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{an}}\big)\end{aligned}$$

where $\mathrm{St}_n^{\mathrm{an}} := (\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} 1)^{\mathrm{an}} / \sum_{P \supsetneq B} (\mathrm{Ind}_{\overline{P}(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} 1)^{\mathrm{an}}$. Together with (7.61), (7.67), (7.66) and Corollary 7.41, we deduce then $m_B(x) \leq m(\rho)$. The lemma follows. □

Lemma 7.44. *The $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{B-\mathrm{ord}}[1/p]$-module $M_B(U^{\wp})[1/p]$ is locally free at the point x.*

Proof. (a) Let $X := \mathrm{Spec}\, A := \mathrm{Spec}(\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{B-\mathrm{ord}}[1/p]/\mathfrak{p})$ be any irreducible component containing the closed point x. We show that the A-module $M_B(U^{\wp})[1/p]/\mathfrak{p}$ (see (7.53)) is locally free at x, from which the lemma follows by [48, Ex. II.5.8(c)] (recall that $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{B-\mathrm{ord}}$, hence $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{B-\mathrm{ord}}[1/p]$ and X, are reduced by Lemma 6.7). We define:

$$Z := \{\text{benign points in } X\} \cup \{x\}.$$

By Proposition 7.5(2), we know that Z is Zariski-dense in X. By Lemma 7.36 (resp. by Lemma 7.43), we know $m_B(x') = m(x')$ for $x' \in Z \setminus \{x\}$ (resp. $m_B(x) = m(\rho)$).

(b) For any finite place $\mathfrak{l} \nmid p$ of F, we deduce from (6.17) a continuous representation $\rho_{A,\mathfrak{l}} : \mathrm{Gal}_{F_{\mathfrak{l}}} \to \mathrm{GL}_n(A)$. By [44, Prop. 4.1.6], we can associate to $\rho_{A,\mathfrak{l}}$ a Weil-Deligne representation over A. Then the statement of [1, Prop. 7.8.19] (with "open affinoid" replaced by "open affine") still holds where the rigid space X of *loc. cit.* is replaced by the scheme X in (a) and the Weil-Deligne representation in [1, Prop. 7.8.14] is replaced by the one above (the argument of the proof of [1, Prop. 7.8.19] is then analogous, and even easier since we are in the setting of affine schemes). An examination of their proofs then shows that [19, Lem. 4.5] (for any n) and [19, Lem. 4.6] (for $n \leq 3$) both hold *verbatim* with $(\rho, \mathcal{O}(X))$ of *loc. cit.* replaced by $(\rho_{A,\mathfrak{l}}|_{W_{F_{\mathfrak{l}}}}, A)$. From (7.13), (7.62) together with $m(\pi) = 1$ (which follows from [71] and [55]), we then deduce $m(x') = m(\rho)$ for all $x' \in Z$, and hence $m_B(x') = m(\rho)$ by (a) for all $x' \in Z$.

(c) Denote by $\mathcal{M}$ the coherent sheaf on X attached to the A-module $M_B(U^{\wp})[1/p]/\mathfrak{p}$. For any prime ideal $\mathfrak{p}'$ of A, set:

$$m_B(\mathfrak{p}') := \dim_{\mathrm{Frac}(A/\mathfrak{p}')} \big((M_B(U^{\wp})[1/p]/\mathfrak{p}') \otimes_{A/\mathfrak{p}'} \mathrm{Frac}(A/\mathfrak{p}')\big)$$

which is upper semi-continuous on $\mathrm{Spec}\, A$ by [48, Ex. II.5.8(a)]. In particular, the sets:

$$U_m := \{\mathfrak{p}' \in \mathrm{Spec}\, A,\ m_B(\mathfrak{p}') \leq m\} = \{\mathfrak{p}' \in \mathrm{Spec}\, A,\ m_B(\mathfrak{p}') < m + 1\}$$

are Zariski-open for $m \in \mathbb{Z}_{\geq 0}$. It follows from (b) that we have $Z \subseteq U_{m(\rho)}$ and $Z \cap U_{m(\rho)-1} = \emptyset$. Since Z is Zariski-dense in X, this implies $U_{m(\rho)-1} = \emptyset$, and thus the function $\mathfrak{p}' \mapsto m(\mathfrak{p}')$ is constant of value $m(\rho)$ on the open set $U_{m(\rho)}$ which contains the point x. By [48, Ex. II.5.8(c)], we deduce that $\mathcal{M}$ is locally free on $U_{m(\rho)}$, which finishes the proof. $\square$

Denote by V_x the tangent space of $(\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{B-\mathrm{ord}})^{\mathrm{rig}}$ at x. Recall that we have a natural morphism (see (7.37)):

$$\omega = (\omega_i)_{i=1,\cdots,n} : (\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{B-\mathrm{ord}})^{\mathrm{rig}} \longrightarrow \prod_{i=1}^{n} (\mathrm{Spf}\, R_{\overline{\rho}_i})^{\mathrm{rig}}$$

where $\overline{\rho}_i = \overline{\chi}_i$. Recall also that we uniformly (in $i = 1, \cdots, n$) identify the tangent space of $(\mathrm{Spf}\, R_{\overline{\rho}_i})^{\mathrm{rig}}$ at $\omega_i(x)$ with $\mathrm{Hom}(\mathbb{Q}_p^{\times}, E)$ via:

$$\begin{aligned} \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) &= \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\chi_i, \chi_i) \\ &\cong \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\chi_i), \mathcal{R}_E(\chi_i)) \overset{(1.11)}{\cong} \mathrm{Hom}(\mathbb{Q}_p^{\times}, E). \end{aligned}$$

Lemma 7.45. *The morphism $d\omega_x : V_x \longrightarrow \oplus_{i=1,\cdots,n} \operatorname{Hom}(\mathbb{Q}_p^\times, E)$ induced by ω on the tangent spaces is injective. Moreover, the induced morphism:*

$$\overline{d}\omega_x : V_x \longrightarrow \bigoplus_{i=1,\cdots,n} \big(\operatorname{Hom}(\mathbb{Q}_p^\times, E)/\operatorname{Hom}_\infty(\mathbb{Q}_p^\times, E)\big) \tag{7.68}$$

is bijective.

Proof. By Proposition 7.30, we have $\dim_E V_x \geq n$. Since this is also the dimension of the right hand side of (7.68), it is enough to prove that $\overline{d}\omega_x$ is injective. Let $0 \neq v \in V_x$ such that $\overline{d}\omega_x(v) = 0$ and denote by $\mathcal{I}_v := \operatorname{Ker}(\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{B-\mathrm{ord}} \to E[\epsilon]/\epsilon^2)$ the ideal attached to v (so $\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{B-\mathrm{ord}}/\mathcal{I}_v \cong \mathcal{O}_E[\epsilon]/\epsilon^2$). From [42, Prop. C.11] applied with $M = \widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{B-\mathrm{ord}}/\mathcal{I}_v$ we obtain:

$$\begin{aligned} & (M_B(U^\wp)/\mathcal{I}_v)[1/p] \\ \cong\ & \operatorname{Hom}_{\mathcal{O}_E}\big(\operatorname{Hom}_{\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{B-\mathrm{ord}}}(\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{B-\mathrm{ord}}/\mathcal{I}_v, X_B(U^\wp)), \mathcal{O}_E\big)[1/p] \\ \cong\ & \operatorname{Hom}_{\mathcal{O}_E}\big(X_B(U^\wp)[\mathcal{I}_v], \mathcal{O}_E\big)[1/p]. \end{aligned} \tag{7.69}$$

From (7.69) and Lemma 7.44 it easily follows that $(X_B(U^\wp)[\mathcal{I}_v])[1/p]$ is free of rank $m_B(x)$ over $(\widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{B-\mathrm{ord}}/\mathcal{I}_v)[1/p] \cong E[\epsilon]/\epsilon^2$. For $i = 1, \cdots, n$ denote by $\widetilde{\chi}_i$ the extension of χ_i by χ_i associated to $d\omega_{i,x}(v) \in \operatorname{Hom}(\mathbb{Q}_p^\times, E)$. From (7.42) we get $(\pi_i(U^\wp)/\mathcal{I}_v)[1/p] \cong \widetilde{\chi}_i$ (since $n_i = 1$). Let $\chi := \otimes_{i=1}^n(\chi_i \otimes \varepsilon^{s_i})$ and $\widetilde{\chi} := \otimes_{i=1}^n(\widetilde{\chi}_i \otimes \varepsilon^{s_i})$ where the tensor product $\otimes_{i=1}^n$ on the latter is over $E[\epsilon]/\epsilon^2$. By (7.65) together with [42, Lem. 3.1.17] applied with $M = \widetilde{\mathbb{T}}(U^\wp)_{\overline{\rho}}^{B-\mathrm{ord}}/\mathcal{I}_v$, we obtain a commutative diagram:

$$\begin{tikzcd} \chi^{\oplus m_B(x)} \arrow[r, "\sim"] \arrow[d] & \operatorname{Ord}_B(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}})[\mathfrak{m}_\rho] \arrow[d] \\ \widetilde{\chi}^{\oplus m_B(x)} \arrow[r, "\sim"] & \operatorname{Ord}_B(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}})[\mathcal{I}_v]. \end{tikzcd} \tag{7.70}$$

Since $\overline{d}\omega_x(v) = 0$, the character $\widetilde{\chi}$ is locally algebraic by (1.12). It then follows from $\mathfrak{m}_\rho^2 \subseteq \mathcal{I}_v[1/p]$ and Proposition 6.13 that the bottom horizontal map in (7.70) factors through $\operatorname{Ord}_B(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}})[\mathfrak{m}_\rho]$, which contradicts (7.70). The lemma follows. $\square$

Recall that for $i = 1, \cdots, n-1$ the (φ, Γ)-module D_i^{i+1} was defined at the end of § 7.2.1, and that $\mathcal{L}_{\mathrm{FM}}(D_i^{i+1} : \mathcal{R}_E(\chi_i))$ is the line in

$$\operatorname{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\chi_i), \mathcal{R}_E(\chi_i)) \cong \operatorname{Hom}(\mathbb{Q}_p^\times, E)$$

defined as the orthogonal of $ED_i^{i+1} \subseteq \mathrm{Ext}^1_{(\varphi,\Gamma_L)}(\mathcal{R}_E(\chi_{i+1}), \mathcal{R}_E(\chi_i))$ via the pairing as in (2.1), see § 2.

Proposition 7.46. *For $i = 1, \cdots, n-1$, the morphism $d\omega_{i,x} - d\omega_{i+1,x}$ factors through a surjection:*

$$d\omega_{i,x} - d\omega_{i+1,x} : V_x \twoheadrightarrow \mathcal{L}_{\mathrm{FM}}(D_i^{i+1} : \mathcal{R}_E(\chi_i)) \subsetneq \mathrm{Hom}(\mathbb{Q}_p^\times, E).$$

Proof. Recall we have a morphism of rigid spaces (see (6.20)):

$$\omega' : (\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^\wp)^{B-\mathrm{ord}}_{\overline{\rho}})^{\mathrm{rig}} \longrightarrow (\mathrm{Spf}\, R^{B-\mathrm{ord}}_{\overline{\rho}_{\widetilde{\wp}}})^{\mathrm{rig}}.$$

For any nonzero v in V_x, let $\widetilde{\rho}$ (resp. $\widetilde{\chi}_i$) be the $\mathrm{Gal}_{\mathbb{Q}_p}$-representation over $E[\epsilon]/\epsilon^2$ attached to $d\omega'_x(v)$ (resp. $d\omega_{i,x}(v)$). We know that $\widetilde{\rho}$ (resp. $\widetilde{\chi}_i$) is a deformation of $\rho_{\widetilde{\wp}}$ (resp. χ_i) over $E[\epsilon]/\epsilon^2$. It follows from Proposition 5.7(2) that v can be seen as an $E[\epsilon]/\epsilon^2$-valued point of $\mathrm{Spec}\, R^{B-\mathrm{ord}}_{\rho_{\widetilde{\wp}},\{\chi_i\}}$, hence that $\widetilde{\rho}$ is isomorphic to a successive extension of the $\widetilde{\chi}_i$ as $\mathrm{Gal}_{\mathbb{Q}_p}$-representation over $E[\epsilon]/\epsilon^2$. Then from Theorem 2.7 we easily deduce $(d\omega_{i,x} - d\omega_{i+1,x})(v) \in \mathcal{L}_{\mathrm{FM}}(D_i^{i+1} : \mathcal{R}_E(\chi_i))$ for all $i = 1, \cdots, n-1$. If $d\omega_{i,x} - d\omega_{i+1,x} = 0$ for some i (equivalently $d\omega_{i,x} - d\omega_{i+1,x}$ is not surjective), then the morphism in Lemma 7.45 cannot be surjective, a contradiction. □

For $r \in \{1, \cdots, n-1\}$, we denote by P_r the parabolic subgroup as in (5.1) with $k = n-1$, $n_i = 1$ for $i \in \{1, \cdots, n-1\} \backslash \{r\}$ and $n_i = 2$ for $i = r$ (note that this implies $n \geq 3$). We have isomorphisms of smooth representations of $L_{P_r}(\mathbb{Q}_p)$ over E:

$$\begin{aligned} (7.71) \quad \mathrm{Ord}_{P_r}(\mathrm{St}_n^\infty \otimes \chi_1 \circ \det) &\cong J_{P_r}(\mathrm{St}_n^\infty \otimes \chi_1 \circ \det)(\delta_{P_r}^{-1}) \\ &\cong \Big(\big(\bigotimes_{\substack{i=1,\cdots,n-1 \\ i \neq r}} 1 \big) \otimes \mathrm{St}_2^\infty \Big) \otimes (\chi_1 \circ \det) \end{aligned}$$

where the first isomorphism follows from the second (see § 4.3 for Ord_{P_r}) and where the second easily follows from $J_{B \cap L_{P_r}}(J_{P_r}(\mathrm{St}_n^\infty)) \cong J_B(\mathrm{St}_n^\infty) \cong \delta_B$ and the usual adjunction for $J_{B \cap L_{P_r}}(\cdot)$.

Corollary 7.47. *For $r = 1, \cdots, n-1$, the restriction morphism:*

(7.72)

$$\mathrm{Hom}_{L_{P_r}(\mathbb{Q}_p)}\Big(\big(\bigotimes_{\substack{i=1,\ldots,n-1 \\ i \neq r}} \chi_1 \big) \otimes (\widehat{\pi}(\rho_r^{r+1}) \otimes \varepsilon^{r-1} \circ \det), \mathrm{Ord}_{P_r}(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}})[\mathfrak{m}_\rho] \Big)$$

$$\longrightarrow \mathrm{Hom}_{L_{P_r}(\mathbb{Q}_p)}\Big(\Big(\Big(\bigotimes_{\substack{i=1,\cdots,n-1\\ i\neq r}} 1\Big)\otimes \mathrm{St}_2^{\infty}\Big)\otimes(\chi_1\circ\det), \mathrm{Ord}_{P_r}(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}})[\mathfrak{m}_{\rho}]\Big)$$

is an isomorphism. In particular, we have (see (7.61) for $m(\rho)$):

$$\dim_E \mathrm{Hom}_{L_{P_r}(\mathbb{Q}_p)}\Big(\Big(\bigotimes_{\substack{i=1,\ldots,n-1\\ i\neq r}} \chi_1\Big)\otimes(\widehat{\pi}(\rho_r^{r+1})\otimes\varepsilon^{r-1}\circ\det), \mathrm{Ord}_{P_r}(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}})[\mathfrak{m}_{\rho}]\Big) = m(\rho).$$

Proof. Note first that $\chi_i\otimes\varepsilon^{s_i}=\chi_1$ for $i=1,\cdots,n$. Let $0\neq\psi\in\mathcal{L}_{\mathrm{FM}}(D_r^{r+1}:\mathcal{R}_E(\chi_r))$, then we have the following restriction maps (writing $\mathrm{Ord}_{P_r}[\mathfrak{m}_x]$ for $\mathrm{Ord}_{P_r}(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}})[\mathfrak{m}_x]$):

$$\begin{aligned} &\mathrm{Hom}_{L_{P_r}(\mathbb{Q}_p)}\Big(\Big(\bigotimes_{\substack{i=1,\ldots,n-1\\ i\neq r}} \chi_1\Big)\otimes(\widehat{\pi}(\rho_r^{r+1})\otimes\varepsilon^{r-1}\circ\det), \mathrm{Ord}_{P_r}[\mathfrak{m}_x]\Big)\\ \xrightarrow{\sim}\ &\mathrm{Hom}_{L_{P_r}(\mathbb{Q}_p)}\Big(\Big(\bigotimes_{\substack{i=1,\ldots,n-1\\ i\neq r}} \chi_1\Big)\otimes(\widehat{\pi}(\rho_r^{r+1})^{\mathrm{an}}\otimes\varepsilon^{r-1}\circ\det), \mathrm{Ord}_{P_r}[\mathfrak{m}_x]^{\mathrm{an}}\Big)\\ \to\ &\mathrm{Hom}_{L_{P_r}(\mathbb{Q}_p)}\Big(\Big(\bigotimes_{\substack{i=1,\ldots,n-1\\ i\neq r}} \chi_1\Big)\otimes(\pi(0,\psi)^-\otimes(\chi_1\circ\det)), \mathrm{Ord}_{P_r}[\mathfrak{m}_x]^{\mathrm{an}}\Big)\\ \to\ &\mathrm{Hom}_{L_{P_r}(\mathbb{Q}_p)}\Big(\Big(\Big(\bigotimes_{\substack{i=1,\cdots,n-1\\ i\neq r}} 1\Big)\otimes\mathrm{St}_2^{\infty}\Big)\otimes(\chi_1\circ\det), \mathrm{Ord}_{P_r}[\mathfrak{m}_x]^{\mathrm{an}}\Big)\end{aligned}$$

where the first isomorphism follows from the fact that the universal completion of $\widehat{\pi}(\rho_r^{r+1})^{\mathrm{an}}\cong\pi(0,\psi)\otimes\chi_1\circ\det$ is $\widehat{\pi}(\rho_r^{r+1})$ ([26] and see § 3.2.2 for $\pi(0,\psi)$ and $\pi(0,\psi)^-$). Using [8, Cor. 3.4], (7.28) and Lemma 7.42, we deduce that the second and third morphisms are injective by the same type of argument as in the proof of Proposition 7.23. By the same arguments as in [4, § 6.4 Cas $i=1$] using Lemma 6.8(2) and Lemma 7.42, one can prove that the second morphism is moreover surjective (see also the end of the proof of Proposition 7.23 for analogous considerations). By Proposition 7.46 and an easier variation of step (c) in the proof of Theorem 7.52 below, it follows that the third morphism is also surjective (see also the proof of [32, Prop. 12] for similar arguments). The last assertion follows from (7.61), (7.71) and Proposition 4.21. □

Remark 7.48. (1) Corollary 7.47 would actually be an easy consequence of Theorem 7.38, but we prove it here *without* the assumption in Theorem 7.38. This is important as it is used in the proof of the main result.

(2) Applying [40, Thm. 4.4.6] to (7.72), one can in fact (re)prove [31, Thm. 1.2] in the case where $L = \mathbb{Q}_p$ and $\rho_{\widetilde{\wp}}$ is ordinary.

7.2.3. Higher $\mathcal{L}$-invariants.

The main result of this section is Proposition 7.51, which can be seen as a version of Proposition 7.46 for higher $\mathcal{L}$-invariants. We still work in arbitrary dimension.

We keep the notation and assumptions of §§ 7.2.1 & 7.2.2. We fix $r \in \{1, \cdots, n-1\}$ and set $P := P_r$ (so $p > n \geq 3$). Since $\overline{\rho}_{\widetilde{\wp}}$ is strictly B-ordinary, one can check that $\overline{\rho}_{\widetilde{\wp}}$ is strictly P-ordinary. With the notation of § 5.1 we have $k = n-1$ and:

$$\overline{\rho}_i = \begin{cases} \overline{\chi}_i & i \in \{1, \cdots, r-1\} \\ \text{nonsplit extension of } \overline{\chi}_{r+1} \text{ by } \overline{\chi}_r & i = r \\ \overline{\chi}_{i+1} & i \in \{r+1, \cdots, n-1\} \end{cases}$$

with $\overline{\rho}_r$ satisfying (A.2).

Lemma 7.49. *The $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}[1/p]$-module $M_P(U^{\wp})[1/p]$ is locally free at the point x.*

Proof. By (7.46) and the last statement in Corollary 7.47 we have $m_P(x) = m(\rho)$. Together with Lemma 7.36, the lemma then follows by the same argument as in the proof of Lemma 7.44. □

We denote by V_x the tangent space of $(\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$ at x. Let $\overline{d\omega}_x$ be the induced morphism:

$$\overline{d\omega}_x : V_x \longrightarrow \bigoplus_{i=1,\cdots,n-1} \big(\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_{x_i}, \rho_{x_i}) / \mathrm{Ext}^1_g(\rho_{x_i}, \rho_{x_i})\big) \tag{7.73}$$

where $\omega = (\omega_i)_{i=1,\cdots,n-1} : (\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}} \longrightarrow \prod_{i=1,\cdots,n-1} (\mathrm{Spf}\, R_{\overline{\rho}_i})^{\mathrm{rig}}$ (there will be no confusion with the tangent space V_x and the map ω in § 7.2.2 and note that $\rho_{x_i} = \chi_i$ if $i < r$, $\rho_{x_i} = \chi_{i+1}$ if $i > r$ and $\rho_{x_r} = \rho_r^{r+1}$). The following lemma is analogous to Lemma 7.45.

Lemma 7.50. *The morphism $\overline{d\omega}_x$ is bijective.*

Proof. By Proposition 7.30, $\dim_E V_x \geq n+1$. By Lemma 3.5 and Lemma 3.11(3), we see the right hand side of (7.73) has dimension $(n-2)+(5-2) = n+1$. It is thus enough to prove that $\overline{d\omega}_x$ is injective.

(a) Let $v \in V_x$, $\mathcal{I}_v := \mathrm{Ker}(\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}} \to E[\epsilon]/\epsilon^2)$ be the ideal attached to v (so $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}/\mathcal{I}_v \cong \mathcal{O}_E[\epsilon]/\epsilon^2$) and $\widetilde{\rho}_i$ for $i = 1, \cdots, n-1$ the extension of ρ_{x_i} by ρ_{x_i} associated to $d\omega_{i,x}(v)$. Denote by $\widetilde{\pi}_i := (\pi_i(U^{\wp}) \otimes_{\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}} \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}/\mathcal{I}_v)[1/p]$ (cf. (7.42)), which is isomorphic to the unitary Banach representation of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$ over $E[\epsilon]/\epsilon^2$ attached to $\widetilde{\rho}_i$ via (3.52), Proposition 3.30 and Remark 3.31(2). Note that for $i \neq r$ we have $\widetilde{\pi}_i \cong \widetilde{\rho}_i$ as characters of $\mathbb{Q}_p^{\times}$ over $E[\epsilon]/\epsilon^2$. We set (cf. (7.43)):

$$\begin{aligned}\pi &:= \big(\pi_P^{\otimes}(U^{\wp}) \otimes_{\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}} \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}/\mathfrak{p}_x\big)[1/p] \\ &\cong \Big(\bigotimes_{\substack{i=1,\cdots,n-1 \\ i \neq r}} \rho_{x_i} \otimes \varepsilon^{s_i}\Big) \otimes_E \big(\widehat{\pi}(\rho_{x_r}) \otimes \varepsilon^{r-1} \circ \det\big)\end{aligned}$$

$$\begin{aligned}\widetilde{\pi} &:= \big(\pi_P^{\otimes}(U^{\wp}) \otimes_{\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}} \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}/\mathcal{I}_v\big)[1/p] \\ &\cong \Big(\bigotimes_{\substack{i=1,\cdots,n-1 \\ i \neq r}} \widetilde{\rho}_i \otimes \varepsilon^{s_i}\Big) \otimes_{E[\epsilon]/\epsilon^2} \big(\widetilde{\pi}_r \otimes \varepsilon^{r-1} \circ \det\big)\end{aligned}$$

where the tensor product of the $\widetilde{\rho}_i$ in the last term is over $E[\epsilon]/\epsilon^2$. Since $\widetilde{\pi}_i$ is free of rank one over $E[\epsilon]/\epsilon^2$ for $i \neq r$, we see that $\widetilde{\pi}$ is isomorphic to an extension of π by π. Since $M_P(U^{\wp})[1/p]$ is locally free at x by Lemma 7.49, by (7.69) and the discussion that follows we see that the evaluation map (7.56) induces a commutative diagram:

$$\begin{tikzcd} \pi^{m_P(x)} \arrow[r] \arrow[d, "\iota"'] & \mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}})[\mathfrak{m}_{\rho}] \arrow[d] \\ \widetilde{\pi}^{m_P(x)} \arrow[r] & \mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}})[\mathcal{I}_v] \end{tikzcd} \tag{7.74}$$

where the vertical maps are the natural injections (coming from $\pi \subseteq \widetilde{\pi}[\epsilon]$ for the first) and where the top horizontal map is also injective by Corollary 7.47 (and its proof).

(b) We prove that the injection $\pi \xrightarrow{\iota} \widetilde{\pi}$ has image exactly $\epsilon\widetilde{\pi}$. It is enough to prove that ι induces $\pi \xrightarrow{\sim} \widetilde{\pi}[\epsilon]$ (since then we have a short exact sequence $0 \to \pi \xrightarrow{\iota} \widetilde{\pi} \to \epsilon\widetilde{\pi} \to 0$ and we use that $\widetilde{\pi}$ is an extension of π by π). From [43, Lem. 3.1.17] we deduce isomorphisms:

$$\Big(\big(X_P(U^{\wp}) \widehat{\otimes}_{\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}}} \pi_P^{\otimes}(U^{\wp})\big)[\mathcal{I}_v]\Big)[1/p] \cong (X_P(U^{\wp})[\mathcal{I}_v])[1/p] \otimes_{E[\epsilon]/\epsilon^2} \widetilde{\pi} \cong \widetilde{\pi}^{m_P(x)}$$

$$\Big(\big(X_P(U^{\wp})\widehat{\otimes}_{\widetilde{\mathbb{T}}(U^{\wp})^{P-\mathrm{ord}}_{\overline{\rho}}}\pi_P^{\otimes}(U^{\wp})\big)[\mathfrak{p}_x]\Big)[1/p]\cong(X_P(U^{\wp})[\mathfrak{p}_x])[1/p]\otimes_{E[\epsilon]/\epsilon^2}\pi\cong\pi^{m_P(x)}$$

using that $(X_P(U^{\wp})[\mathcal{I}_v])[1/p]$ is free of rank $m_P(x)$ over $E[\epsilon]/\epsilon^2$ by the same argument as in the proof of Lemma 7.45 (and using Lemma 7.49). The result follows using $(\cdot)[\mathcal{I}_v][\epsilon]=(\cdot)[\mathfrak{p}_x]$.

(c) Suppose now that we have $\overline{d\omega}_x(v)=0$. Then it follows that $\widetilde{\pi}_i^{\mathrm{lalg}}=\widetilde{\pi}_i=\widetilde{\rho}_i$ is locally algebraic when $i\neq r$ (use (1.12)) and that $\widetilde{\pi}_r^{\mathrm{lalg}}$ is an extension of $\widehat{\pi}(\rho_{x_r})^{\mathrm{lalg}}$ by $\widehat{\pi}(\rho_{x_r})^{\mathrm{lalg}}$ when $i=r$ (use (3.55)). In particular we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \pi^{\mathrm{lalg}} \arrow[r, "\iota"] \arrow[d] & \widetilde{\pi}^{\mathrm{lalg}} \arrow[r] \arrow[d] & \pi^{\mathrm{lalg}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi \arrow[r, "\iota"] & \widetilde{\pi} \arrow[r] & \pi \arrow[r] & 0
\end{tikzcd} \tag{7.75}$$

where the vertical maps are the natural inclusions. By (b), the multiplication by ϵ on $\widetilde{\pi}$ factors as $\widetilde{\pi}\twoheadrightarrow\pi\xrightarrow{\sim}\epsilon\widetilde{\pi}\hookrightarrow\widetilde{\pi}$. It follows from (7.75) that the multiplication by ϵ on $\widetilde{\pi}^{\mathrm{lalg}}$ also factors as $\widetilde{\pi}^{\mathrm{lalg}}\twoheadrightarrow\pi^{\mathrm{lalg}}\xrightarrow{\sim}\epsilon\widetilde{\pi}^{\mathrm{lalg}}\hookrightarrow\widetilde{\pi}^{\mathrm{lalg}}$, in particular we have $\iota(\pi^{\mathrm{lalg}})=\epsilon\widetilde{\pi}^{\mathrm{lalg}}$ inside $\widetilde{\pi}^{\mathrm{lalg}}$. From $\mathfrak{m}_\rho^2\subseteq\mathcal{I}_v[1/p]$ and Proposition 6.13, any morphism $\widetilde{\pi}^{\mathrm{lalg}}\to\mathrm{Ord}_P(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}})[\mathcal{I}_v]$ factors through $\widetilde{\pi}^{\mathrm{lalg}}\to\mathrm{Ord}_P(\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}^{\mathrm{lalg}})[\mathfrak{m}_\rho]$. It then follows that the bottom horizontal morphism in (7.74), which is $E[\epsilon]/\epsilon^2$-linear, sends $(\epsilon\widetilde{\pi}^{\mathrm{lalg}})^{m_P(x)}=\iota(\pi^{\mathrm{lalg}})^{m_P(x)}$ to 0, which contradicts the injections in (7.74). The lemma follows. ☐

We consider the E-linear injection

$$\xi:\mathrm{Hom}(\mathbb{Q}_p^\times,E)\hookrightarrow\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_r^{r+1},\rho_r^{r+1}),\ \ \psi\mapsto\rho_r^{r+1}\otimes_E(1+\psi\epsilon)$$

and set $d\omega_{r,x}^+:=d\omega_{r,x}-\xi\circ d\omega_{r+1,x}$ (if $r<n-1$) and $d\omega_{r,x}^-:=d\omega_{r,x}-\xi\circ d\omega_{r-1,x}$ (if $r>1$). The following result is somewhat analogous to Proposition 7.46 (see § 2 for $\mathcal{L}_{\mathrm{FM}}(\cdot)$ and $\ell_{\mathrm{FM}}(\cdot)$).

Proposition 7.51. *(1) Inside* $\mathrm{Ext}^1_{(\varphi,\Gamma)}(D_r^{r+1},D_r^{r+1})\cong\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_r^{r+1},\rho_r^{r+1})$ *we have* $\mathrm{Im}(d\omega_{r,x}^+)\subseteq\mathcal{L}_{\mathrm{FM}}(D_r^{r+2}:D_r^{r+1})$ *(if* $r<n-1$*) and* $\mathrm{Im}(d\omega_{r,x}^-)\subseteq\mathcal{L}_{\mathrm{FM}}(D_{r-1}^{r+1}:D_r^{r+1})$ *(if* $r>1$*).*

(2) If $r<n-1$ *(resp. if* $r>1$*) the composition:*

$$\mathrm{Im}(d\omega_{r,x}^+)\lhook\joinrel\longrightarrow\mathrm{Ext}^1_{(\varphi,\Gamma)}(D_r^{r+1},D_r^{r+1})\longrightarrow\mathrel{\mkern-14mu}\rightarrow\mathrm{Ext}^1_{(\varphi,\Gamma)}(D_r^{r+1},\mathcal{R}_E(\chi_{r+1}))$$

$$\textit{(resp. }\mathrm{Im}(d\omega_{r,x}^-)\lhook\joinrel\longrightarrow\mathrm{Ext}^1_{(\varphi,\Gamma)}(D_r^{r+1},D_r^{r+1})\longrightarrow\mathrel{\mkern-14mu}\rightarrow\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\chi_r),D_r^{r+1})\textit{)}$$

induces a surjective map

$$\mathrm{Im}(d\omega_{r,x}^+) \twoheadrightarrow \ell_{\mathrm{FM}}(D_r^{r+2} : D_r^{r+1}) \subseteq \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_r^{r+1}, \mathcal{R}_E(\chi_{r+1}))$$

(resp. $\mathrm{Im}(d\omega_{r,x}^-) \twoheadrightarrow \ell_{\mathrm{FM}}(D_{r-1}^{r+1} : D_r^{r+1}) \subseteq \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(\chi_r), D_r^{r+1})$*).*

Proof. (1) From (6.20) we have $\omega' : (\mathrm{Spf}\, R_{\overline{\rho}_{\widetilde{\wp}}}^{P-\mathrm{ord}})^{\mathrm{rig}} \to (\mathrm{Spf}\, \widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P-\mathrm{ord}})^{\mathrm{rig}}$. Let $0 \neq v \in V_x$ and $\widetilde{\rho}$ (resp. $\widetilde{\rho}_i$) the $\mathrm{Gal}_{\mathbb{Q}_p}$-representation over $E[\epsilon]/\epsilon^2$ attached to $d\omega'_x(v)$ (resp. $d\omega_{i,x}(v)$). We know that $\widetilde{\rho}$ (resp. $\widetilde{\rho}$) is a deformation of $\rho_{\widetilde{\wp}}$ (resp. ρ_{x_i}) over $E[\epsilon]/\epsilon^2$, and using Proposition 5.7(2) we see as in the proof of Proposition 7.46 that $\widetilde{\rho}$ is isomorphic to a successive extension of $\widetilde{\rho}_i$ as representations over $E[\epsilon]/\epsilon^2$. (1) follows then from Theorem 2.7.

(2) We prove the statement for $\mathrm{Im}(d\omega_{r,x}^+)$ (and $r < n-1$), the other case being similar. By Corollary 2.4(2) and Lemma 3.5 (and the assumptions on $\rho_{\widetilde{\wp}}$ in § 7.2.1), we have $\dim_E \ell_{\mathrm{FM}}(D_r^{r+2} : D_r^{r+1}) = 2$. Recall that we have by Lemma 3.5 and Lemma 3.11(3):

$$\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_r^{r+1}, D_r^{r+1}) / \mathrm{Ext}^1_g(D_r^{r+1}, D_r^{r+1}) = 5 - 2 = 3.$$

By Lemma 7.50, it is then not difficult to deduce $\dim_E \mathrm{Im}(d\omega_{r,x}^+) \geq 3$. By (1) and (2.4), we have an exact sequence (see (2.2) for the morphism κ):

$$0 \longrightarrow \mathrm{Im}(d\omega_{r,x}^+) \cap \mathrm{Ker}(\kappa) \longrightarrow \mathrm{Im}(d\omega_{r,x}^+) \longrightarrow \ell_{\mathrm{FM}}(D_r^{r+2} : D_r^{r+1}).$$

We have $\dim_E \mathrm{Ker}(\kappa) = 2$ by (2.4), Corollary 2.4(2) and $\dim_E \ell_{\mathrm{FM}}(D_r^{r+2} : D_r^{r+1}) = 2$. If $\dim_E \mathrm{Im}(d\omega_{r,x}^+) \geq 4$, the result is thus clear. Assume $\dim_E \mathrm{Im}(d\omega_{r,x}^+) = 3$, it is enough to prove $\dim_E \mathrm{Im}(d\omega_{r,x}^+) \cap \mathrm{Ker}(\kappa) \leq 1$. From Lemma 3.7 and Lemma 3.11(1)&(2), we deduce $\dim_E \mathrm{Ker}(\kappa) \cap \mathrm{Ext}^1_g(D_r^{r+1}, D_r^{r+1}) = 1$. It easily follows from Lemma 7.50 that the morphism $\overline{d\omega}_{r,x}^+ := \overline{d\omega}_{r,x} - \xi \circ \overline{d\omega}_{r+1,x}$ is surjective (note that it is well-defined since ξ sends $\mathrm{Hom}_\infty(\mathbb{Q}_p^\times, E)$ to $\mathrm{Ext}^1_g(\rho_r^{r+1}, \rho_r^{r+1})$). This implies that the composition:

$$\begin{aligned}(7.76)\quad \mathrm{Im}(d\omega_{r,x}^+) &\hookrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_r^{r+1}, D_r^{r+1}) \\ &\twoheadrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_r^{r+1}, D_r^{r+1}) / \mathrm{Ext}^1_g(D_r^{r+1}, D_r^{r+1})\end{aligned}$$

is also surjective, hence bijective as source and target have dimension 3. If $\dim_E \mathrm{Im}(d\omega_{r,x}^+) \cap \mathrm{Ker}(\kappa) = 2$, we have $\mathrm{Ker}(\kappa) \subseteq \mathrm{Im}(d\omega_{r,x}^+)$ since $\dim_E \mathrm{Ker}(\kappa) = 2$, and thus $\mathrm{Im}(d\omega_{r,x}^+) \cap \mathrm{Ext}^1_g(D_r^{r+1}, D_r^{r+1}) \neq 0$ as $\mathrm{Ker}(\kappa) \cap \mathrm{Ext}^1_g(D_r^{r+1}, D_r^{r+1}) \neq 0$, which contradicts the fact (7.76) is bijective. This concludes the proof. □

7.2.4. Local-global compatibility for $\mathbf{GL_3}(\mathbb{Q}_{\boldsymbol{p}})$. In dimension 3, we finally use most of the previous material to prove our main local-global compatibility result (Corollary 7.54).

We keep all the notation of §§ 7.2.1, 7.2.2, 7.2.3 and now assume $n = 3$ (and thus $p > 3$). For $r = 1, 2$, we let $\mathcal{L}_r \in E$ such that:

$$\psi_{\mathcal{L}_r} := \log_p - \mathcal{L}_r \operatorname{val}_p \in \mathcal{L}_{\mathrm{FM}}(D_r^{r+1} : \mathcal{R}_E(\chi_r)) \subsetneq \mathrm{Hom}(\mathbb{Q}_p^\times, E).$$

We set $\lambda := (\mathrm{wt}(\chi_1), \mathrm{wt}(\chi_1), \mathrm{wt}(\chi_1)) \in \mathbb{Z}^3$ and let $\alpha \in E^\times$ such that $\chi_1 = \mathrm{unr}(\alpha) x^{\mathrm{wt}(\chi_1)}$. We define $v_{\overline{P}_r}^\infty(\lambda)$ for $r = 1, 2$ as in § 3.3.1, $\widetilde{\Pi}^1(\lambda, \psi_{\mathcal{L}_1})$ as in (3.76) and set $v_{\overline{P}_r}^\infty(\alpha, \lambda) := v_{\overline{P}_r}^\infty(\lambda) \otimes \mathrm{unr}(\alpha) \circ \det$, $\widetilde{\Pi}^1(\alpha, \lambda, \psi_{\mathcal{L}_1}) := \widetilde{\Pi}^1(\lambda, \psi_{\mathcal{L}_1}) \otimes \mathrm{unr}(\alpha) \circ \det$ and $\mathcal{L}_{\mathrm{aut}}(D : D_1^2) \subseteq \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v_{\overline{P}_2}^\infty(\alpha, \lambda), \widetilde{\Pi}^1(\alpha, \lambda, \psi))$ as in (3.100) (tensoring by $\mathrm{unr}(\alpha) \circ \det$). The assumptions on $\rho_{\widetilde{\wp}}$ imply in particular that D is sufficiently generic in the sense of (the end of) § 6.1. We set $\widetilde{\Pi}^1(D)^- := \mathscr{E}(\widetilde{\Pi}^1(\alpha, \lambda, \psi_{\mathcal{L}_1}), v_{\overline{P}_2}^\infty(\alpha, \lambda)^{\oplus 2}, \mathcal{L}_{\mathrm{aut}}(D : D_1^2))$ as in Notation 3.4 (see (3.102) when $\alpha = 1$). Likewise we define $\widetilde{\Pi}^2(\alpha, \lambda, \psi_{\mathcal{L}_2}) := \widetilde{\Pi}^2(\lambda, \psi_{\mathcal{L}_2}) \otimes \mathrm{unr}(\alpha) \circ \det$ (see before § 3.3.4), $\mathcal{L}_{\mathrm{aut}}(D : D_2^3) \subseteq \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v_{\overline{P}_1}^\infty(\alpha, \lambda), \widetilde{\Pi}^2(\alpha, \lambda, \psi))$ (see (3.107)) and we set (see (3.109) when $\alpha = 1$):

$$\widetilde{\Pi}^2(D)^- := \mathscr{E}\big(\widetilde{\Pi}^2(\alpha, \lambda, \psi_{\mathcal{L}_2}), v_{\overline{P}_1}^\infty(\alpha, \lambda)^{\oplus 2}, \mathcal{L}_{\mathrm{aut}}(D : D_2^3)\big).$$

Theorem 7.52. *For $r \in \{1, 2\}$, the following restriction morphism is bijective:*

$$\begin{aligned}(7.77)\quad & \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\widetilde{\Pi}^r(D)^-, \widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]\big) \\ & \qquad \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\mathrm{St}_3^\infty \otimes_E \chi_1 \circ \det, \widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]\big).\end{aligned}$$

Proof. We only prove the case $r = 1$, the case $r = 2$ being symmetric.

(a) It follows from (7.64) that (7.77) is injective (by the usual argument: if (7.77) is not injective, there exists an irreducible constituent V of $\widetilde{\Pi}^1(D)^-$ such that $V \hookrightarrow \widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]$, hence $J_B(V) \hookrightarrow J_B(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho])$, which contradicts (7.64) using [8, Cor. 3.4]).

(b) We have natural morphisms:

$$\begin{aligned}(7.78)\quad & \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\mathrm{St}_3^\infty \otimes (\chi_1 \circ \det), \widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]\big) \\ \xrightarrow{\sim}\ & \mathrm{Hom}_{L_{P_1}(\mathbb{Q}_p)}\big((\mathrm{St}_2^\infty \otimes 1) \otimes (\chi_1 \circ \det), \mathrm{Ord}_{P_1}(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho])\big) \\ \xrightarrow{\sim}\ & \mathrm{Hom}_{L_{P_1}(\mathbb{Q}_p)}\big(\widehat{\pi}(\rho_1^2) \otimes \chi_1, \mathrm{Ord}_{P_1}(\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho])\big) \\ \xrightarrow{\sim}\ & \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \widehat{\pi}(\rho_1^2) \otimes \chi_1)^{\mathcal{C}^0}, \widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]\big)\end{aligned}$$

$$\begin{aligned}&\hookrightarrow \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big((\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\widehat{\pi}(\rho_1^2)^{\mathrm{an}}\otimes\chi_1)^{\mathrm{an}}, \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{an}}\big)\\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\widetilde{\Pi}^1(\alpha,\lambda,\psi_{\mathcal{L}_1}), \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{an}}\big)\end{aligned}$$

where the first map is given by Lemma 4.18 together with (7.71) and is bijective by Proposition 4.21 together with (7.61), the second isomorphism follows from Corollary 7.47, the third from [40, Thm. 4.4.6], the fourth map is injective since the locally analytic vectors are dense in the corresponding Banach representation, and where the last bijection follows from the fact that any irreducible constituent of the kernel W of the surjection $(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\widehat{\pi}(\rho_1^2)^{\mathrm{an}}\otimes\chi_1)^{\mathrm{an}} \twoheadrightarrow \widetilde{\Pi}^1(\alpha,\lambda,\psi_{\mathcal{L}_1})$ does not occur in $\mathrm{soc}_{\mathrm{GL}_3(\mathbb{Q}_p)}\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{an}}$ (see the discussion below (3.80) and argue as in (a)). One can check by using the functor $J_B(\cdot)$ that the composition in (7.78) gives a section of the restriction morphism:

$$\begin{aligned}(7.79)\quad &\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\widetilde{\Pi}^1(\alpha,\lambda,\psi_{\mathcal{L}_1}), \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]\big)\\ &\qquad\longrightarrow \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\mathrm{St}_3^\infty\otimes(\chi_1\circ\det), \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]\big),\end{aligned}$$

which is therefore surjective. Since (7.79) is injective by (again) the same argument as in (a), it follows that (7.79) is bijective. Consequently, the fourth injection in (7.78) is also bijective.

(c) By (a) and (b), it is enough to prove that, for any line $Ew \subseteq \mathcal{L}_{\mathrm{aut}}(D : D_1^2)$, setting $\Pi := \mathscr{E}(\widetilde{\Pi}^1(\alpha,\lambda,\psi_{\mathcal{L}_1}), v_{\overline{P}_2}^\infty(\alpha,\lambda), Ew)$ (see Notation 3.4, in fact this is just here the representation associated to the extension w), the following restriction morphism is surjective:

$$\begin{aligned}(7.80)\quad &\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\Pi, \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]\big)\\ &\qquad\longrightarrow \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\mathrm{St}_3^\infty\otimes(\chi_1\circ\det), \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]\big).\end{aligned}$$

As in (7.78), we have:

$$\begin{aligned}(7.81)\quad &\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\mathrm{St}_3^\infty\otimes(\chi_1\circ\det), \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]\big)\\ &\xrightarrow{\sim} \mathrm{Hom}_{L_{P_1}(\mathbb{Q}_p)}\big(\widehat{\pi}(\rho_1^2)\otimes\chi_1, \mathrm{Ord}_{P_1}(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho])\big) \cong X_P(U^{\wp})[\mathfrak{p}_x]\otimes_{\mathcal{O}_E} E\end{aligned}$$

where we use the notation in § 7.2.3. Let $0 \neq w \in \mathcal{L}_{\mathrm{aut}}(D : D_1^2) \cong \ell_{\mathrm{FM}}(D : D_1^2)$ (cf. (3.101)), by Proposition 7.51(2) there exists $v \in V_x$ such that $d\omega_{1,x}^+(v) \mapsto w \in \mathrm{Ext}^1_{(\varphi,\Gamma)}(D_1^2, \mathcal{R}_E(\chi_2))$. Denote by $\mathcal{I}_v$ the ideal of $\widetilde{\mathbb{T}}(U^{\wp})_{\overline{\rho}}^{P_1-\mathrm{ord}}$ attached to v (see e.g. the beginning of the proof of Lemma 7.50). Recall we have (see e.g. (b) in the proof of Lemma 7.50):

$$(X_P(U^{\wp})[\mathcal{I}_v])[1/p] \text{ is free over } E[\epsilon]/\epsilon^2. \tag{7.82}$$

Let f_0 be a nonzero element in the right hand side of (7.80), and let $f : \widehat{\pi}(\rho_1^2)\otimes\chi_1 \hookrightarrow \mathrm{Ord}_{P_1}(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathfrak{m}_{\rho}])$ and $e \in (X_P(U^{\wp})[\mathfrak{p}_x])[1/p]$ be the corresponding elements via (7.81). By (7.82) and $X_P(U^{\wp})[\mathfrak{p}_x] = X_P(U^{\wp})[\mathcal{I}_v][\epsilon]$, there exists $\tilde{e} \in (X_P(U^{\wp})[\mathcal{I}_v])[1/p]$ such that $\epsilon\tilde{e} = e$. As in the proof of Lemma 7.50 (see (7.74)), letting $\widetilde{\pi}_1$ (resp. $\widetilde{\chi}_1$) be the deformation of $\widehat{\pi}(\rho_1^2)$ (resp. χ_1) over $E[\epsilon]/\epsilon^2$ attached to $d\omega_{1,x}(v)$ (resp. $d\omega_{2,x}(v)$), we have a commutative diagram:

$$\begin{tikzcd}
\pi := \widehat{\pi}(\rho_1^2) \boxtimes_E \chi_1 \arrow[r, "f"] \arrow[d, "\iota"'] & \mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}})[\mathfrak{m}_{\rho}] \arrow[d] \\
\widetilde{\pi} := \widetilde{\pi}_1 \boxtimes_{E[\epsilon]/\epsilon^2} \widetilde{\chi}_1 \arrow[r, "\tilde{f}"] & \mathrm{Ord}_P(\widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}})[\mathcal{I}_v]
\end{tikzcd} \tag{7.83}$$

where $\tilde{f}$ is the morphism corresponding to $\tilde{e}$ and where we write $\boxtimes$ instead of $\otimes$ to emphasize that it is an *exterior* tensor product of representations ($\mathrm{GL}_2(\mathbb{Q}_p)$ acting on the left and $\mathbb{Q}_p^{\times}$ on the right). Using Proposition 7.51(1), let $w_0 := d\omega_{1,x}^{+}(v) \in \mathcal{L}_{\mathrm{FM}}(D : D_1^2)$ and $\widetilde{\pi}_{w_0}$ the associated deformation of $\widehat{\pi}(\rho_1^2)$ over $E[\epsilon]/\epsilon^2$ via (3.52). From the definition of $d\omega_{1,x}^{+}$ we have:

$$\begin{aligned}
\widetilde{\pi} &\cong ((\chi_1^{-1}\widetilde{\chi}_1) \circ \det\nolimits_{\mathrm{GL}_2} \otimes_{E[\epsilon]/\epsilon^2} \widetilde{\pi}_{w_0}) \boxtimes_{E[\epsilon]/\epsilon^2} \widetilde{\chi}_1 \\
&\cong (\chi_1^{-1}\widetilde{\chi}_1) \circ \det\nolimits_{L_{P_1}} \otimes_{E[\epsilon]/\epsilon^2} (\widetilde{\pi}_{w_0} \boxtimes_E \chi_1).
\end{aligned} \tag{7.84}$$

By [40, Thm. 4.4.6], taking $(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \cdot)^{\mathcal{C}^0}$ and then locally analytic vectors, the maps ι and $\widetilde{f}$ in (7.83) induce morphisms of locally analytic representations of $\mathrm{GL}_3(\mathbb{Q}_p)$ over E:

$$\big(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \pi^{\mathrm{an}}\big)^{\mathrm{an}} \hookrightarrow \big(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \widetilde{\pi}^{\mathrm{an}}\big)^{\mathrm{an}} \longrightarrow \widehat{S}(U^{\wp}, W^{\wp})_{\overline{\rho}}[\mathcal{I}_v]^{\mathrm{an}}. \tag{7.85}$$

Let $\widetilde{\pi}_0 := \widetilde{\pi}_{w_0} \boxtimes_E \chi_1$, from (7.84) we deduce:

$$\big(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \widetilde{\pi}^{\mathrm{an}}\big)^{\mathrm{an}} \cong (\chi_1^{-1}\widetilde{\chi}_1) \circ \det\nolimits_{\mathrm{GL}_3} \otimes_{E[\epsilon]/\epsilon^2} \big(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)} \widetilde{\pi}_0^{\mathrm{an}}\big)^{\mathrm{an}}. \tag{7.86}$$

As in (b), (7.85) factors as (see (a) for W):

$$(7.87)\qquad \widetilde{\Pi}^1(\alpha,\lambda,\psi_{\mathcal{L}_1}) \hookrightarrow \big(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\widetilde{\pi}^{\mathrm{an}}\big)^{\mathrm{an}}/W \longrightarrow \widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathcal{I}_v]^{\mathrm{an}}.$$

Since $w_0 \in \mathcal{L}_{\mathrm{FM}}(D:D_1^2) \mapsto w \in \ell_{\mathrm{FM}}(D:D_1^2)$, it follows from (3.89) and Remark 3.47 that $\Pi = \mathscr{E}(\widetilde{\Pi}^1(\lambda,\psi_{\mathcal{L}_1}), v_{\overline{P}_2}^{\infty}(\lambda), Ew)\otimes \mathrm{unr}(\alpha)\circ\det$ is a subrepresentation of $(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\widetilde{\pi}_0^{\mathrm{an}})^{\mathrm{an}}/W$. By Lemma 3.2 and (7.86), we deduce that Π is also a subrepresentation of $(\mathrm{Ind}_{\overline{P}_1(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\widetilde{\pi}^{\mathrm{an}})^{\mathrm{an}}/W$. Hence (7.87) induces $\mathrm{GL}_3(\mathbb{Q}_p)$-equivariant morphisms:

$$(7.88)\qquad \widetilde{\Pi}^1(\alpha,\lambda,\psi_{\mathcal{L}_1}) \hookrightarrow \Pi \xrightarrow{\tilde{f}} \widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathcal{I}_v]^{\mathrm{an}}.$$

As the composition in (7.88) restricts to f_0 via (7.79), we see it has image in $\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]$ (using that the analogue of (7.79) with $\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathcal{I}_v]$ instead of $\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]$ is still an injection). If $\mathfrak{m}_\rho \,\mathrm{Im}(\tilde{f}) \neq 0$, we deduce that

$$\mathfrak{m}_\rho \,\mathrm{Im}(\tilde{f}) \cong v_{\overline{P}_2}^{\infty}(\lambda)\otimes \mathrm{unr}(\alpha)\circ\det$$

is a subrepresentation of $\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathcal{I}_v]^{\mathrm{an}}$, a contradiction. Thus we have $\mathfrak{m}_\rho \,\mathrm{Im}(\tilde{f}) = 0$, i.e. $\tilde{f}$ also has image in $\widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]^{\mathrm{an}}$. The map $f \mapsto \tilde{f}$ gives a section to (7.80), which concludes the proof. ☐

We refer to § 3.3.3, § 3.3.4 for the definition of the subrepresentations $\Pi^r(\lambda,\psi_{\mathcal{L}_r})_0$, $\Pi^r(\lambda,\psi_{\mathcal{L}_r})$, $\Pi^r(\lambda,\psi_{\mathcal{L}_r})^+$ of $\widetilde{\Pi}^r(\lambda,\psi_{\mathcal{L}_r})$. We set $\Pi^r(\alpha,\lambda,\psi_{\mathcal{L}_r})_0 := \Pi^r(\lambda,\psi_{\mathcal{L}_r})_0\otimes\mathrm{unr}(\alpha)\circ\det$, $\Pi^r(\alpha,\lambda,\psi_{\mathcal{L}_r}) := \Pi^r(\lambda,\psi_{\mathcal{L}_r})\otimes\mathrm{unr}(\alpha)\circ\det$, and $\Pi^r(\alpha,\lambda,\psi_{\mathcal{L}_r})^+ := \Pi^r(\lambda,\psi_{\mathcal{L}_r})^+\otimes\mathrm{unr}(\alpha)\circ\det$.

Corollary 7.53. *Let $r\in\{1,2\}$.*

(1) Let $\psi\in\mathrm{Hom}(\mathbb{Q}_p^{\times},E)$, an injection

$$f : \mathrm{St}_3^{\infty}\otimes(\chi_1\circ\det) \hookrightarrow \widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]$$

extends to $\tilde{f}_1 : \Pi^r(\alpha,\lambda,\psi)^+ \to \widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]$ if and only if $\psi\in E\psi_{\mathcal{L}_r}$.

(2) Let $s\in\{1,2\}$, $s\neq r$, and let $v\in\mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v_{\overline{P}_s}^{\infty}(\alpha,\lambda),\Pi^r(\alpha,\lambda,\psi_{\mathcal{L}_r})^+)$. An injection $\Pi^r(\alpha,\lambda,\psi_{\mathcal{L}_r})^+ \hookrightarrow \widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]$ extends to:

$$\mathscr{E}\big(\Pi^r(\alpha,\lambda,\psi_{\mathcal{L}_r})^+, v_{\overline{P}_s}^{\infty}(\alpha,\lambda), Ev\big) \longrightarrow \widehat{S}(U^{\wp},W^{\wp})_{\overline{\rho}}[\mathfrak{m}_\rho]$$

if and only if $v\in\mathcal{L}_{\mathrm{aut}}(D:D_r^{r+1})$.

Proof. For $i \in \{1, 2\}$, denote by $\pi^i(\alpha, \lambda) := \mathrm{St}_3^\infty \otimes (\chi_1 \circ \det) - v_{\overline{P}_i}^\infty(\alpha, \lambda)$ the unique nonsplit locally algebraic extension of $v_{\overline{P}_i}^\infty(\alpha, \lambda)$ by $\mathrm{St}_3^\infty \otimes (\chi_1 \circ \det)$.

(1) By Theorem 7.52, f extends to $\tilde{f}_0 : \Pi^r(\alpha, \lambda, \psi_{\mathcal{L}_r})^+ \hookrightarrow \widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]$, the "if" part follows. If $\psi \notin E\psi_{\mathcal{L}_r}$, we have $E\psi + E\psi_{\mathcal{L}_r} = \mathrm{Hom}(\mathbb{Q}_p^\times, E)$ and an injection induced by $\tilde{f}_0$, $\tilde{f}_1$ (where $S_{s,0}$ is defined as in § 3.3.3 with $\lambda = 0$):

$$\Pi^r(\alpha, \lambda, \psi_{\mathcal{L}_r})_0 \oplus_{S_{s,0} \otimes (\chi_1 \circ \det)} \Pi^r(\alpha, \lambda, \psi)_0 \longrightarrow \widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho].$$

By Proposition 3.35, we deduce that the left hand side is isomorphic to:

$$\mathscr{E}\big(S_{s,0}, (v_{\overline{P}_r}^\infty)^{\oplus 2}, \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v_{\overline{P}_r}^\infty, S_{s,0})\big) \otimes_E (\chi_1 \circ \det)$$

and hence contains $\pi^r(\alpha, \lambda) \subset \mathscr{E}(S_{s,0}, v_{\overline{P}_r}^\infty, E\,\mathrm{val}_p) \otimes_E (\chi_1 \circ \det)$ as a subrepresentation. However $\pi^r(\alpha, \lambda)$ is not a subrepresentation of $\widehat{S}(U^\wp, W^\wp)_{\overline{\rho}}[\mathfrak{m}_\rho]$ by (7.61), a contradiction.

(2) follows by the same argument. Indeed, if $v \notin \mathcal{L}_{\mathrm{aut}}(D : D_r^{r+1})$, then $Ev + \mathcal{L}_{\mathrm{aut}}(D : D_r^{r+1}) = \mathrm{Ext}^1_{\mathrm{GL}_3(\mathbb{Q}_p)}(v_{\overline{P}_s}^\infty(\alpha, \lambda), \Pi^r(\alpha, \lambda, \psi_{\mathcal{L}_r})^+)$. One deduces from Lemma 3.42 that $\pi^s(\alpha, \lambda)$ is a subrepresentation of:

$$\mathscr{E}(\Pi^r(\alpha, \lambda, \psi_{\mathcal{L}_r})^+, v_{\overline{P}_s}^\infty(\alpha, \lambda), Ev) \oplus_{\Pi^r(\alpha, \lambda, \psi_{\mathcal{L}_r})^+} \Pi^r(D)^-$$

(where $\Pi^r(D)^- \subseteq \widetilde{\Pi}^r(D)^-$ is defined in (3.103) and (3.108) modulo the twist by $\mathrm{unr}(\alpha) \circ \det$), a contradiction. $\square$

We can now state our main result. We fix $\wp | p$, $\widetilde{\wp} | \wp$, $U^\wp = \prod_{v \neq \wp} U_v$ and $W^\wp$ as in § 6.1.

Corollary 7.54. *Assume $n = 3$, $F_\wp^+ \cong F_{\widetilde{\wp}} = \mathbb{Q}_p$, $p \geq 5$ and U_v maximal if $v | p$, $v \neq \wp$. Let $\rho \colon \mathrm{Gal}_F \to \mathrm{GL}_3(E)$ be a continuous representation which is unramified at the places of $\Sigma(U^p)$ and such that:*

- *$\overline{\rho}$ is absolutely irreducible*
- $\widehat{S}(U^\wp, W^\wp)[\mathfrak{m}_\rho]^{\mathrm{lalg}} \neq 0$
- *$\rho_{\widetilde{\wp}}$ is semi-stable with consecutive Hodge-Tate weights and $N^2 \neq 0$ on $D_{\mathrm{st}}(\rho_{\widetilde{\wp}})$*
- *any dimension 2 subquotient of $\overline{\rho}_{\widetilde{\wp}} = \overline{\rho}|_{\mathrm{Gal}_{F_{\widetilde{\wp}}}}$ is nonsplit.*

Then we have the following results.

(1) The statement in Conjecture 6.2 is true, i.e. the restriction morphism is bijective:

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\Pi(\rho_{\widetilde{\wp}}), \widehat{S}(U^{\wp}, W^{\wp})[\mathfrak{m}_\rho]\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\Pi(\rho_{\widetilde{\wp}})^{\mathrm{lalg}}, \widehat{S}(U^{\wp}, W^{\wp})[\mathfrak{m}_\rho]\big).$$

(2) The representation $\rho_{\widetilde{\wp}}$ *of* $\mathrm{Gal}_{\mathbb{Q}_p}$ *is determined by the locally analytic representation* $\widehat{S}(U^{\wp}, W^{\wp})[\mathfrak{m}_\rho]^{\mathrm{an}}$ *of* $\mathrm{GL}_3(\mathbb{Q}_p)$ *(hence also by the continuous representation* $\widehat{S}(U^{\wp}, W^{\wp})[\mathfrak{m}_\rho]$*).*

Proof. (1) By the same argument as in [4, § 6.2 Étape 1], we can assume that U^p is sufficiently small. Define $\Pi(D)^-$ as at the end of § 3.3.4, then it follows from Theorem 7.52 and (a) in its proof (and arguing e.g. as in [4, § 6.2 Étape 2]) that the statement holds with $\Pi(D)^-$ instead of $\Pi(\rho_{\widetilde{\wp}}) = \Pi(D)$. By [4, § 6.4 Cas $i \geq 3$], we have:

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\Pi(D), \widehat{S}(U^{\wp}, W^{\wp})[\mathfrak{m}_\rho]\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{Q}_p)}\big(\Pi(D)^-, \widehat{S}(U^{\wp}, W^{\wp})[\mathfrak{m}_\rho]\big)$$

and (1) follows. (2) is a direct consequence of Corollary 7.53 (which *a fortiori* still holds when U^p is not sufficiently small). ☐

Appendix A. Appendix

The aim of this appendix is to give a complete proof of Proposition 3.32, for which we couldn't find precise references in the existing literature.

A.1. Notation and preliminaries

We recall some notation and results of Emerton and Colmez.

As in [40], we denote by $\mathrm{Comp}(\mathcal{O}_E)$ the category of complete noetherian local $\mathcal{O}_E$-algebras with finite residue field. For G a topological group which is locally pro-p and $A \in \mathrm{Comp}(\mathcal{O}_E)$, we denote by $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ the category of smooth representations of G over A in the sense of [40, § 2.2], $\mathrm{Mod}_G^{\mathrm{fin}}(A)$ the full subcategory of smooth representations of finite length and $\mathrm{Mod}_G^{\mathrm{lfin}}(A)$ the full subcategory of smooth representations locally of finite length (i.e. the subrepresentation generated by v is of finite length for any vector v). We denote by $(\cdot)^\vee := \mathrm{Hom}_{\mathcal{O}_E}(\cdot, E/\mathcal{O}_E)$ = Pontryagin duality.

We let $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ be the category of profinite augmented representations of G over A in the sense of [40, Def. 2.1.6]. By [40, (2.2.8)], the functor $\pi \mapsto \pi^\vee$ induces an anti-equivalence of categories:

$$\mathrm{Mod}_G^{\mathrm{sm}}(A) \xrightarrow{\sim} \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A). \tag{A.1}$$

As in [40, § 2.1], we denote by $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ the full subcategory of $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ consisting of augmented G-representations that are finitely generated over $A[[H]]$ for some (equivalently any) compact open subgroup H of G. We denote by $\mathrm{Mod}_G^{\mathrm{ortho}}(A)$ the category of orthonormalizable admissible representations of G over A in the sense of [42, Def. 3.1.11]. By [42, Prop. 3.1.12], the functor $\pi \mapsto \mathrm{Hom}_A(\pi, A)$ induces an anti-equivalence of categories between $\mathrm{Mod}_G^{\mathrm{ortho}}(A)$ and the full subcategory of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ consisting of G-representations which are moreover pro-free A-modules.

We denote by $\mathrm{Rep}^{\mathrm{fin}}_{\mathrm{Gal}_{\mathbb{Q}_p}}(\mathcal{O}_E)$ the category of continuous representations of $\mathrm{Gal}_{\mathbb{Q}_p}$ on finite length (hence torsion) $\mathcal{O}_E$-modules equipped with the discrete topology. Recall that Colmez defined a covariant exact functor (called Colmez's functor, see [24]):

$$\mathbf{V} : \mathrm{Mod}^{\mathrm{fin}}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathcal{O}_E) \longrightarrow \mathrm{Rep}^{\mathrm{fin}}_{\mathrm{Gal}_{\mathbb{Q}_p}}(\mathcal{O}_E).$$

For a continuous character $\zeta : \mathbb{Q}_p^\times \to \mathcal{O}_E^\times$ (which we view as a continuous character of $\mathrm{Gal}_{\mathbb{Q}_p}$), we denote by $\mathbf{V}_\zeta$ the functor $\pi \mapsto \mathbf{V}(\pi) \otimes \zeta$. As in [42, § 3.2], for $A \in \mathrm{Comp}(\mathcal{O}_E)$, $\mathbf{V}$ (resp. $\mathbf{V}_\zeta$) extends to a covariant and exact functor, still denoted by $\mathbf{V}$ (resp. by $\mathbf{V}_\zeta$), from the full subcategory of $\mathrm{Mod}^{\mathrm{ortho}}_{\mathrm{GL}_2(\mathbb{Q}_p)}(A)$ consisting of A-representations π such that $\pi \otimes_A A/\mathfrak{m}_A \in \mathrm{Mod}^{\mathrm{fin}}_{\mathrm{GL}_2(\mathbb{Q}_p)}(k_E)$ to the category of continuous $\mathrm{Gal}_{\mathbb{Q}_p}$-representations on finite rank free A-modules.

A.2. Deformations I

The main results of this section are Corollary A.2 and Corollary A.7 below.

We keep the notation of § A.1. We fix $\overline{\rho}\colon \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_2(k_E)$ a continuous representation and let $\pi(\overline{\rho})$ be the smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over k_E associated to $\overline{\rho}$ by the mod p Langlands correspondence normalized so that $\mathbf{V}_{\varepsilon^{-1}}(\pi(\overline{\rho})) \cong \overline{\rho}$ (this is the normalization of [6, § 3.1]). We assume:

$$\overline{\rho} \ncong \begin{pmatrix} 1 & * \\ 0 & \varepsilon \end{pmatrix} \text{ up to twist by a character (with $*$ zero or not).} \tag{A.2}$$

Note that the assumption implies that $\pi(\overline{\rho})$ has length ≤ 3.

We denote by $\mathrm{Def}_{\overline{\rho}}$ the groupoid over $\mathrm{Comp}(\mathcal{O}_E)$ of deformations of $\overline{\rho}$ (see [42, Def. 3.3.6]) and by $\mathrm{Def}_{\pi(\overline{\rho}),\mathrm{ortho}}$ the groupoid over $\mathrm{Comp}(\mathcal{O}_E)$ of orthonormalizable admissible deformations of $\pi(\overline{\rho})$ (see [42, Def. 3.3.7]). Following [42, Def. 3.3.9] we denote by $\mathrm{Def}^*_{\pi(\overline{\rho}),\mathrm{ortho}} \subseteq \mathrm{Def}_{\pi(\overline{\rho}),\mathrm{ortho}}$ the subgroupoid of deformations π such that the center of G acts on π by the character $\det(\mathbf{V}_{\varepsilon^{-1}}(\pi))\varepsilon$. The following theorem follows from work of Kisin and Paškūnas (see [42, Thm. 3.3.13 & Rem. 3.3.14]).

Theorem A.1. *The functor* $\boldsymbol{V}_{\varepsilon^{-1}}$ *induces an isomorphism of groupoids:*

$$\mathrm{Def}^*_{\pi(\overline{\rho}),\mathrm{ortho}} \xrightarrow{\sim} \mathrm{Def}_{\overline{\rho}}\,. \tag{A.3}$$

Let $\xi = (\rho^0_\xi, \iota_\xi) \in \mathrm{Def}(\overline{\rho})(\mathcal{O}_E)$ and $\rho_\xi := \rho^0_\xi \otimes_{\mathcal{O}_E} E$ (recall ι_ξ is a $\mathrm{Gal}_{\mathbb{Q}_p}$-equivariant isomorphism $\iota_\xi : \rho^0_\xi \otimes_{\mathcal{O}_E} k_E \xrightarrow{\sim} \overline{\rho}$). We still denote by $\xi := (\pi^0_\xi, \iota'_\xi) \in \mathrm{Def}^*(\pi(\overline{\rho}))(\mathcal{O}_E)$ the inverse image of ξ via the isomorphism (A.3) and set $\widehat{\pi}(\rho_\xi) := \pi^0_\xi \otimes_{\mathcal{O}_E} E$. The map $\rho_\xi \mapsto \widehat{\pi}(\rho_\xi)$ is the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ (normalized as in [6, § 3.1]).

Corollary A.2. *The functor* $\boldsymbol{V}_{\varepsilon^{-1}}$ *induces a natural surjection:*

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\widehat{\pi}(\rho_\xi), \widehat{\pi}(\rho_\xi)) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_\xi, \rho_\xi) \tag{A.4}$$

where the extension group on the left is in the category of (admissible) unitary Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Proof. Let $\widetilde{\pi} \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\widehat{\pi}(\rho_\xi), \widehat{\pi}(\rho_\xi))$ and $\widetilde{\pi}_0$ a $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant open lattice. Using that two open lattices in the Banach space $\widetilde{\pi}$ are commensurable and the exactness $\mathbf{V}_{\varepsilon^{-1}}$, one easily checks that $\mathbf{V}_{\varepsilon^{-1}}(\widetilde{\pi}_0)[1/p]$ is in $\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho_\xi, \rho_\xi)$ and doesn't depend on the choice of $\widetilde{\pi}_0$. This defines the morphism (A.4). We prove (A.4) is surjective. Let $\widetilde{\rho}_\xi$ be a deformation of ρ_ξ over $E[\epsilon]/\epsilon^2$. By the proof of [52, Prop. 2.3.5], one can find a finite $\mathcal{O}_E$-subalgebra $A \subseteq E[\epsilon]/\epsilon^2$ such that $A[1/p] \cong E[\epsilon]/\epsilon^2$ and a deformation $\rho_{A,\xi}$ of $\overline{\rho}$ over A such that $\rho_{A,\xi} \otimes_A \mathcal{O}_E \cong \rho^0_\xi$ (via the natural surjection $A \twoheadrightarrow \mathcal{O}_E$ induced by $E[\epsilon]/\epsilon^2 \twoheadrightarrow E$) and $\rho_{A,\xi} \otimes_A E[\epsilon]/\epsilon^2 \cong \widetilde{\rho}_\xi$. By (A.3), there exists a deformation $\widehat{\pi}_A$ of $\pi(\overline{\rho})$ over A such that $\mathbf{V}_{\varepsilon^{-1}}(\widehat{\pi}_A) \cong \rho_{A,\xi}$. It is straightforward to check that $\widehat{\pi}_A[1/p] \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\widehat{\pi}(\rho_\xi), \widehat{\pi}(\rho_\xi))$ (using (A.3) again) and that $\widehat{\pi}_A[1/p]$ is sent to $\widetilde{\rho}_\xi$ via (A.4). ☐

From now on, we assume moreover $\mathrm{End}_{\mathrm{Gal}_{\mathbb{Q}_p}}(\overline{\rho}) \cong k_E$. By [53, Lem. 2.1.2], we also have $\mathrm{End}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\overline{\rho})) \cong k_E$. We now still denote by $\mathrm{Def}_{\overline{\rho}}$ (resp. $\mathrm{Def}^*_{\pi(\overline{\rho}),\mathrm{ortho}}$, $\mathrm{Def}_{\pi(\overline{\rho}),\mathrm{ortho}}$) the (usual) deformation functor (e.g. as in § 5.1) attached to the groupoid $\mathrm{Def}_{\overline{\rho}}$ (resp. $\mathrm{Def}^*_{\pi(\overline{\rho}),\mathrm{ortho}}$, $\mathrm{Def}_{\pi(\overline{\rho}),\mathrm{ortho}}$). We know that $\mathrm{Def}_{\overline{\rho}}$ is representable, hence so is $\mathrm{Def}^*_{\pi(\overline{\rho}),\mathrm{ortho}}$ by Theorem A.1.

Let $\overline{\zeta} := \wedge^2_{k_E}\overline{\rho}$ be the determinant of $\overline{\rho}$. Recall that any element in $\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\overline{\rho}, \overline{\rho})$ (resp. $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\overline{\rho}), \pi(\overline{\rho}))$) can be viewed as a deformation

$\widetilde{\rho}$ (resp. $\widetilde{\pi}$) of $\overline{\rho}$ (resp. $\pi(\overline{\rho})$) over $k_E[\epsilon]/\epsilon^2$. In particular we have a k_E-linear morphism:

$$\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\overline{\rho},\overline{\rho}) \longrightarrow \mathrm{Hom}(\mathrm{Gal}_{\mathbb{Q}_p},k_E) \cong \mathrm{Hom}(\mathbb{Q}_p^\times,k_E) \tag{A.5}$$

(= group homomorphisms to the additive group k_E) sending $\widetilde{\rho}$ to $\big(\overline{\zeta}'\overline{\zeta}^{-1}-1\big)/\epsilon$ where $\overline{\zeta}' := \wedge^2_{k_E[\epsilon]/\epsilon^2}\widetilde{\rho}$. We define $\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p},\overline{\zeta}}(\overline{\rho},\overline{\rho})$ as the kernel of (A.5). By the assumptions on $\overline{\rho}$, each irreducible constituent π of $\pi(\overline{\rho})$ has multiplicity one in $\pi(\overline{\rho})$. Using the same arguments as in the proof of Lemma 3.15, we can then show that there exists $\overline{\zeta}' : \mathbb{Q}_p^\times \to (k_E[\epsilon]/\epsilon^2)^\times$ such that the center $Z(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times$ acts on $\widetilde{\pi}$ by $\overline{\zeta}'\overline{\varepsilon}$. We thus deduce another k_E-linear morphism:

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\overline{\rho}),\pi(\overline{\rho})) \longrightarrow \mathrm{Hom}(\mathbb{Q}_p^\times,k_E),\ \widetilde{\pi} \longmapsto \big(\overline{\zeta}'\overline{\zeta}^{-1}-1\big)/\epsilon \tag{A.6}$$

and we define $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),\overline{\zeta}\overline{\varepsilon}}(\pi(\overline{\rho}),\pi(\overline{\rho}))$ as the kernel of (A.6), which is the k_E-vector subspace of extensions with central character $\overline{\zeta}\overline{\varepsilon}$.

Lemma A.3. *We have short exact sequences of k_E-vector spaces:*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p},\overline{\zeta}}(\overline{\rho},\overline{\rho}) \longrightarrow \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\overline{\rho},\overline{\rho}) \xrightarrow{(A.5)} \mathrm{Hom}(\mathbb{Q}_p^\times,E) \longrightarrow 0$$

$$0 \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),\overline{\zeta}\overline{\varepsilon}}(\pi(\overline{\rho}),\pi(\overline{\rho})) \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\overline{\rho}),\pi(\overline{\rho})) \xrightarrow{(A.6)} \mathrm{Hom}(\mathbb{Q}_p^\times,k_E) \to 0.$$

Proof. It is enough to prove that (A.5) (resp. (A.6)) is surjective. The map $\psi \mapsto \overline{\rho} \otimes (1+\psi/2\epsilon)$ (resp. $\psi \mapsto \pi(\overline{\rho}) \otimes (1+\psi/2\epsilon) \circ \det$) gives a section of (A.5) (resp. of (A.6)). $\square$

As in [67], we call $\overline{\rho}$ *generic* if either $\overline{\rho}$ is irreducible or $\overline{\rho} \cong \begin{pmatrix} \delta_1 & * \\ 0 & \delta_2 \end{pmatrix}$ for $\delta_1\delta_2^{-1} \notin \{\overline{\varepsilon},1\}$ and we call $\overline{\rho}$ *nongeneric* if $\overline{\rho} \cong \begin{pmatrix} \delta\overline{\varepsilon} & * \\ 0 & \delta \end{pmatrix}$ for some $\delta : \mathrm{Gal}_{\mathbb{Q}_p} \to k_E^\times$ (recall we have $* \neq 0$ since $\mathrm{End}_{\mathrm{Gal}_{\mathbb{Q}_p}}(\overline{\rho}) \cong k_E$).

Proposition A.4. *We have:*

$$\dim_{k_E}\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p},\overline{\zeta}}(\overline{\rho},\overline{\rho}) = \dim_{k_E}\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),\overline{\zeta}\overline{\varepsilon}}(\pi(\overline{\rho}),\pi(\overline{\rho})) = 3,$$

$$\dim_{k_E}\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\overline{\rho},\overline{\rho}) = \dim_{k_E}\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\overline{\rho}),\pi(\overline{\rho})) = 5.$$

Proof. By Lemma A.3, it is enough to prove the result for $\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p},\overline{\zeta}}$ and $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),\overline{\zeta\varepsilon}}$. By our assumptions on $\overline{\rho}$, we easily check that $\dim_{k_E}\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p},\overline{\zeta}}(\overline{\rho},\overline{\rho}) = 3$. The result for $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),\overline{\zeta\varepsilon}}$ follows from [67, Prop. 6.1] (in the supersingular case), [67, Cor. 8.5] (in the generic nonsupersingular case) and [68, Thm. 6.10] together with $\dim_{k_E}\mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p},\overline{\zeta}}(\overline{\rho},\overline{\rho}) = 3$ (in the nongeneric case). □

Since $\mathrm{End}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\overline{\rho}))\cong k_E$ and $\dim_{k_E}\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi(\overline{\rho}),\pi(\overline{\rho}))<\infty$ by the last equality in Proposition A.4, it follows from Schlessinger's criterion that the functor $\mathrm{Def}_{\pi(\overline{\rho}),\mathrm{ortho}}$ is representable. Using Theorem A.1, the third equality in Proposition A.4 and [45, Lem. 2.1] (and the representability of $\mathrm{Def}_{\overline{\rho}}$, $\mathrm{Def}^*_{\pi(\overline{\rho}),\mathrm{ortho}}$, $\mathrm{Def}_{\pi(\overline{\rho}),\mathrm{ortho}}$), we easily deduce that we have in fact isomorphisms:

$$\mathrm{Def}^*_{\pi(\overline{\rho}),\mathrm{ortho}} \xrightarrow{\sim} \mathrm{Def}_{\pi(\overline{\rho}),\mathrm{ortho}} \xrightarrow{\sim} \mathrm{Def}_{\overline{\rho}}\,. \tag{A.7}$$

Recall $\mathrm{Art}(\mathcal{O}_E)$ is the category of local artinian $\mathcal{O}_E$-algebras with residue field k_E and let $\mathcal{C}(\mathcal{O}_E)$ be the subcategory of $\mathrm{Mod}^{\mathrm{pro\,aug}}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathcal{O}_E)$ dual to $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathcal{O}_E)$ via (A.1). Denote by $\mathrm{Def}_{\pi(\overline{\rho})^\vee,\mathcal{C}(\mathcal{O}_E)}$ the functor from $\mathrm{Art}(\mathcal{O}_E)$ to (isomorphism classes of) deformations of $\pi(\overline{\rho})^\vee$ in the category $\mathcal{C}(\mathcal{O}_E)$ in the sense of [67, Def. 3.21] (since we only deal with commutative rings here, we drop the subscript "ab" of [67, § 3.1]). As $\mathrm{Hom}_{\mathcal{C}(\mathcal{O}_E)}(\pi(\overline{\rho})^\vee,\pi(\overline{\rho})^\vee)=k_E$ and $\dim_{k_E}\mathrm{Ext}^1_{\mathcal{C}(\mathcal{O}_E)}(\pi(\overline{\rho})^\vee,\pi(\overline{\rho})^\vee)<\infty$, Schlessinger's criterion again implies that $\mathrm{Def}_{\pi(\overline{\rho})^\vee,\mathcal{C}(\mathcal{O}_E)}$ is pro-representable by a complete local noetherian $\mathcal{O}_E$-algebra $R_{\pi(\overline{\rho})^\vee}$ of residue field k_E.

When considering a deformation, we now do not write anymore the reduction morphism ι (which is understated).

Lemma A.5. *(1) Let A in $\mathrm{Art}(\mathcal{O}_E)$ and $M_A\in\mathrm{Def}_{\pi(\overline{\rho})^\vee,\mathcal{C}(\mathcal{O}_E)}(A)$, then $M_A\in\mathrm{Mod}^{\mathrm{fg\,aug}}_{\mathrm{GL}_2(\mathbb{Q}_p)}(A)$ and M_A is a pro-free A-module.*

(2) Let A in $\mathrm{Art}(\mathcal{O}_E)$ and $\pi_A\in\mathrm{Def}_{\pi(\overline{\rho}),\mathrm{ortho}}(A)$, then $\mathrm{Hom}_A(\pi_A,A)\in\mathrm{Def}_{\pi(\overline{\rho})^\vee,\mathcal{C}(\mathcal{O}_E)}(A)$.

Proof. (1) Since A is in $\mathcal{C}(\mathcal{O}_E)$, it is profinite. By definition (see [67, Def. 3.21]), M_A is a flat A-module and by [30, Exp. $\mathrm{VII}_B(0.3.8)$], the second part of (1) follows. It is straightforward to see M_A is in $\mathrm{Mod}^{\mathrm{pro\,aug}}_{\mathrm{GL}_2(\mathbb{Q}_p)}(A)$. Let H be a pro-p compact open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, the algebra $A[[H]]$ is (noncommutative) local. Since $\pi(\overline{\rho})$ is admissible, we know $M_A\otimes_{A[[H]]}k_E\cong\pi(\overline{\rho})^\vee\otimes_{k_E[[H]]}k_E$ is a finite dimensional k_E-vector space. By Nakayama's lemma (see e.g. [56, Lem. 4.22]), we deduce M_A is finitely generated over $A[[H]]$.

(2) By [42, Prop. 3.1.12] and its proof, we have that $M_A := \mathrm{Hom}_A(\pi_A, A)$ is flat over A and $M_A \otimes_A k_E \cong \pi(\overline{\rho})^\vee$. Since π_A is admissible and A is artinian, π_A is locally finite by [40, Thm. 2.3.8]. The lemma follows by definition of $\mathcal{C}(\mathcal{O}_E)$. ☐

Proposition A.6. *We have an isomorphism of deformation functors:*

$$\begin{aligned} \mathrm{Def}_{\pi(\overline{\rho}),\mathrm{ortho}} &\xrightarrow{\sim} \mathrm{Def}_{\pi(\overline{\rho})^\vee,\mathcal{C}(\mathcal{O}_E)}, \\ [A \mapsto \{\pi_A\}_{/\sim}] &\longmapsto [A \mapsto \{M_A = \mathrm{Hom}_A(\pi_A, A)\}_{/\sim}]. \end{aligned}$$

Proof. This follows from [42, Prop. 3.1.12] and Lemma A.5. ☐

Proposition A.6 together with (A.3) and (A.7) imply an isomorphism of deformation functors:

$$\mathrm{Def}_{\pi(\overline{\rho})^\vee,\mathcal{C}(\mathcal{O}_E)} \xrightarrow{\sim} \mathrm{Def}_{\overline{\rho}} \tag{A.8}$$

and hence $R_{\pi(\overline{\rho})^\vee} \cong R_{\overline{\rho}}$. Let ρ^{univ} be the universal deformation of $\overline{\rho}$ over $R_{\overline{\rho}}$ (for $\mathrm{Def}_{\overline{\rho}}$), $\mathcal{N} \in \mathcal{C}(\mathcal{O}_E)$ the universal deformation of $\pi(\overline{\rho})^\vee$ over $R_{\overline{\rho}}$ (for $\mathrm{Def}_{\pi(\overline{\rho})^\vee,\mathcal{C}(\mathcal{O}_E)}$) and $\pi^{\mathrm{univ}}(\overline{\rho}) \in \mathrm{Mod}^{\mathrm{ortho}}_{\mathrm{GL}_2(\mathbb{Q}_p)}(R_{\overline{\rho}})$ the universal deformation of $\pi(\overline{\rho})$ (for $\mathrm{Def}_{\pi(\overline{\rho}),\mathrm{ortho}}$).

Corollary A.7. *We have* $\mathcal{N} \cong \mathrm{Hom}_{R_{\overline{\rho}}}(\pi^{\mathrm{univ}}(\overline{\rho}), R_{\overline{\rho}})$.

Proof. This easily follows from Proposition A.6. ☐

Corollary A.8. *Let $\mathcal{I}$ be an ideal of $R_{\overline{\rho}}$, then we have:*

$$\mathcal{N} \otimes_{R_{\overline{\rho}}} R_{\overline{\rho}}/\mathcal{I} \cong \mathrm{Hom}_{R_{\overline{\rho}}/\mathcal{I}}\big(\pi^{\mathrm{univ}}(\overline{\rho}) \otimes_{R_{\overline{\rho}}} R_{\overline{\rho}}/\mathcal{I}, R_{\overline{\rho}}/\mathcal{I}\big).$$

Proof. This follows from the isomorphism in Corollary A.7 and [42, Lem. B.7]. ☐

Remark A.9. Recall the isomorphism in Corollary A.7 and the isomorphism in Corollary A.8 are topological isomorphisms where the left hand side is equipped with the profinite topology and the right hand side with the topology of pointwise convergence (see [42, Prop. B.11(2)]).

For any $\zeta : \mathbb{Q}_p^\times \to \mathcal{O}_E^\times$ we denote by $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{GL}_2(\mathbb{Q}_p),\zeta}(\mathcal{O}_E)$ the full subcategory of $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathcal{O}_E)$ of representations on which $Z(\mathbb{Q}_p)$ acts by ζ, and by $\mathcal{C}_\zeta(\mathcal{O}_E)$ the full subcategory of $\mathcal{C}(\mathcal{O}_E)$ dual to $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{GL}_2(\mathbb{Q}_p),\zeta}(\mathcal{O}_E)$ via (A.1).

For any $\zeta : \mathbb{Q}_p^\times \to \mathcal{O}_E^\times$ such that $\zeta \equiv \overline{\zeta} \bmod \varpi_E$, we denote by $\mathrm{Def}_{\overline{\rho}}^{\zeta}$ the subfunctor of $\mathrm{Def}_{\overline{\rho}}$ of deformations with fixed determinant ζ and by $R_{\overline{\rho}}^{\zeta}$ the universal deformation ring for $\mathrm{Def}_{\overline{\rho}}^{\zeta}$. We denote by $\mathrm{Def}_{\pi(\overline{\rho})^\vee, \mathcal{C}_{\zeta\varepsilon}(\mathcal{O}_E)}$ the deformation functor on $\mathrm{Art}(\mathcal{O}_E)$ defined in the same way as $\mathrm{Def}_{\pi(\overline{\rho})^\vee, \mathcal{C}(\mathcal{O}_E)}$ replacing $\mathcal{C}(\mathcal{O}_E)$ by the subcategory $\mathcal{C}_{\zeta\varepsilon}(\mathcal{O}_E)$. By the second equality in Proposition A.4 and Schlessinger's criterion, $\mathrm{Def}_{\pi(\overline{\rho})^\vee, \mathcal{C}_{\zeta\varepsilon}(\mathcal{O}_E)}$ is pro-representable by a complete local noetherian $\mathcal{O}_E$-algebra $R_{\pi(\overline{\rho})^\vee}^{\zeta\varepsilon}$ of residue field k_E. It is not difficult to see that the isomorphism in (A.8) induces a natural isomorphism (so that $R_{\overline{\rho}}^{\zeta} \xrightarrow{\sim} R_{\pi(\overline{\rho})^\vee}^{\zeta\varepsilon}$):

$$\mathrm{Def}_{\pi(\overline{\rho})^\vee, \mathcal{C}_{\zeta\varepsilon}(\mathcal{O}_E)} \xrightarrow{\sim} \mathrm{Def}_{\overline{\rho}}^{\zeta}. \tag{A.9}$$

We denote by $\mathcal{N}^{\zeta\varepsilon}$ the universal deformation of $\pi(\overline{\rho})^\vee$ over $R_{\pi(\overline{\rho})^\vee}^{\zeta\varepsilon} \cong R_{\overline{\rho}}^{\zeta}$ for $\mathrm{Def}_{\pi(\overline{\rho})^\vee, \mathcal{C}_{\zeta\varepsilon}(\mathcal{O}_E)}$ (note that $\mathcal{N}^{\zeta\varepsilon}$ is denoted by N in [49]) and by $\rho^{\mathrm{univ},\zeta}$ the universal deformation of $\overline{\rho}$ over $R_{\overline{\rho}}^{\zeta}$. Let Λ be the universal deformation ring of the trivial 1-dimensional representation of $\mathrm{Gal}_{\mathbb{Q}_p}$ over k_E and 1^{univ} the corresponding universal deformation (which is thus a free Λ-module of rank 1). We have $R_{\overline{\rho}} \cong R_{\overline{\rho}}^{\zeta} \widehat{\otimes}_{\mathcal{O}_E} \Lambda$ and $\rho^{\mathrm{univ}} \cong \rho^{\mathrm{univ},\zeta} \widehat{\otimes}_{\mathcal{O}_E} 1^{\mathrm{univ}}$ where $\widehat{\otimes}$ denotes the ϖ_E-adic completion of the usual tensor product. We equip 1^{univ} with a natural action of $\mathrm{GL}_2(\mathbb{Q}_p)$ via $\det : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$. One easily sees $1^{\mathrm{univ}} \in \mathcal{C}(\mathcal{O}_E)$.

Proposition A.10. *We have* $\mathcal{N} \cong \mathcal{N}^{\zeta\varepsilon} \widehat{\otimes}_{\mathcal{O}_E} 1^{\mathrm{univ}}$.

Proof. We have that $\mathcal{N}^{\zeta\varepsilon} \widehat{\otimes}_{\mathcal{O}_E} 1^{\mathrm{univ}}$ is a deformation of $\pi(\overline{\rho})^\vee$ over $R_{\overline{\rho}}^{\zeta} \widehat{\otimes}_{\mathcal{O}_E} \Lambda$ in $\mathcal{C}(\mathcal{O}_E)$, from which we deduce a morphism of local $\mathcal{O}_E$-algebras $R_{\overline{\rho}} \longrightarrow R_{\overline{\rho}}^{\zeta} \widehat{\otimes}_{\mathcal{O}_E} \Lambda$. One can easily check this is an isomorphism (e.g. by proving the tangent map is bijective). The proposition follows. □

A.3. Deformations II

We prove here a key projectivity property of $\mathcal{N}$.

We keep the previous notation and assumption (in particular $\overline{\rho}$ satisfies (A.2) and is such that $\mathrm{End}_{\mathrm{Gal}_{\mathbb{Q}_p}}(\overline{\rho}) \cong k_E$). We assume moreover $p \geq 5$ if $\overline{\rho}$ is nongeneric.

Proposition A.11. *There exist* x, $y \in R_{\overline{\rho}}$ *such that* $S := \mathcal{O}_E[[x, y]]$ *is a subring of* $R_{\overline{\rho}}$ *and* $\mathcal{N}$ *is a finitely generated projective* $S[[\mathrm{GL}_2(\mathbb{Z}_p)]]$*-module.*

Proof. We fix K a pro-p compact open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ such that $K \cong K/Z_0 \times Z_0$ with $Z_0 := K \cap Z(\mathbb{Q}_p)$ isomorphic to $\mathbb{Z}_p$. If R is a (non-commutative) ring, we denote by $\mathrm{Mod}_R^{\mathrm{fg}}$ the category of finitely generated R-modules. It is enough to prove the statement with $\mathrm{GL}_2(\mathbb{Z}_p)$ replaced by K.

(a) By [49, Thm. 3.3] (in the generic case) and [49, Thm. 3.5] and its proof (in the nongeneric case), there exists x in the maximal ideal of $R_{\overline{\rho}}^{\zeta}$ such that $\mathcal{N}^{\zeta\varepsilon} \otimes_{R_{\overline{\rho}}^{\zeta}} R_{\overline{\rho}}^{\zeta}/x$ is a finitely generated $\mathcal{O}_E[[K]]$-module and is projective in the category $\mathrm{Mod}^{\mathrm{fg}}_{\mathcal{O}_E[[K]],\zeta\varepsilon} :=$ the full subcategory of $\mathrm{Mod}^{\mathrm{fg}}_{\mathcal{O}_E[[K]]}$ on which Z_0 acts by $\zeta\varepsilon$. In particular $\mathcal{N}^{\zeta\varepsilon}$ is a finitely generated $S_1[[K]]$-module for $S_1 := \mathcal{O}_E[[x]]$. We first want to prove that $\mathcal{N}^{\zeta\varepsilon}$ is moreover projective in $\mathrm{Mod}^{\mathrm{fg}}_{S_1[[K]],\zeta\varepsilon}$ (with obvious notation, as $\mathcal{N}^{\zeta\varepsilon}$ is a $R_{\overline{\rho}}^{\zeta}$-module note this will also imply $S_1 \hookrightarrow R_{\overline{\rho}}^{\zeta}$). Let $\chi : Z_0 \to \mathcal{O}_E^{\times}$ such that $\chi^2 = \zeta$ (enlarging E if necessary and using $Z_0 \cong \mathbb{Z}_p$), we deduce an isomorphism of $\mathcal{O}_E[[K/Z_0]]$-modules (using that $\mathcal{O}_E[[K/Z_0]]$ is a local ring):

$$(\mathcal{N}^{\zeta\varepsilon} \otimes \chi^{-1} \circ \det) \otimes_{R_{\overline{\rho}}^{\zeta}} R_{\overline{\rho}}^{\zeta}/x \cong \mathcal{O}_E[[K/Z_0]]^{\oplus r}$$

and it is enough to prove that $\mathcal{N}_1 := \mathcal{N}^{\zeta\varepsilon} \otimes \chi^{-1} \circ \det$ is projective in $\mathrm{Mod}^{\mathrm{fg}}_{S_1[[K/Z_0]]}$.

(b) As at the beginning of [68, § 2.5], it is enough to prove $\mathrm{Tor}_1^{S_1[[K/Z_0]]}(\mathcal{N}_1, k_E) = 0$ where $\mathrm{Tor}_i^{S_1[[K/Z_0]]}(-, k_E)$ denotes the i-th derived functor of $(\cdot)\otimes_{S_1[[K/Z_0]]} k_E$ in $\mathrm{Mod}^{\mathrm{fg}}_{S_1[[K/Z_0]]}$ (recall $S_1[[K/Z_0]]$ is a local ring of residue field k_E). Indeed, let $\mathcal{P}$ be a projective envelope of $\mathcal{N}_1$ in $\mathrm{Mod}^{\mathrm{fg}}_{S_1[[K/Z_0]]}$ (whose existence follows from [56, § 23 & Prop. 24.12]), and consider a short exact sequence $0 \to \mathcal{M}_1 \to \mathcal{P} \to \mathcal{N}_1 \to 0$ in $\mathrm{Mod}^{\mathrm{fg}}_{S_1[[K/Z_0]]}$. If $\mathrm{Tor}_1^{S_1[[K/Z_0]]}(\mathcal{N}_1, k_E) = 0$, we get:

$$0 \longrightarrow \mathcal{M}_1 \otimes_{S_1[[K/Z_0]]} k_E \longrightarrow \mathcal{P} \otimes_{S_1[[K/Z_0]]} k_E \longrightarrow \mathcal{N}_1 \otimes_{S_1[[K/Z_0]]} k_E \longrightarrow 0.$$

Since $\mathcal{P}$ is the projective envelope of $\mathcal{N}_1$, we have

$$\mathcal{P} \otimes_{S_1[[K/Z_0]]} k_E \xrightarrow{\sim} \mathcal{N}_1 \otimes_{S_1[[K/Z_0]]} k_E,$$

whence $\mathcal{M}_1 \otimes_{S_1[[K/Z_0]]} k_E = 0$, and $\mathcal{M}_1 = 0$ by Nakayama's lemma ([56, Lem. 4.22]). Now, the exact sequence $0 \longrightarrow \mathcal{N}_1 \xrightarrow{x} \mathcal{N}_1 \longrightarrow \mathcal{N}_1/x \longrightarrow 0$ (recall $\mathcal{N}^{\zeta\varepsilon}$ is flat over $R_{\overline{\rho}}^{\zeta}$) induces:

(A.10)

$$\begin{aligned}&\mathrm{Tor}_1^{S_1[[K/Z_0]]}(\mathcal{N}_1, k_E) \xrightarrow{x} \mathrm{Tor}_1^{S_1[[K/Z_0]]}(\mathcal{N}_1, k_E) \to \mathrm{Tor}_1^{S_1[[K/Z_0]]}(\mathcal{N}_1/x, k_E)\\ &\quad \to \mathcal{N}_1 \otimes_{S_1[[K/Z_0]]} k_E \to \mathcal{N}_1 \otimes_{S_1[[K/Z_0]]} k_E \to (\mathcal{N}_1/x) \otimes_{S_1[[K/Z_0]]} k_E \to 0.\end{aligned}$$

Let $r := \dim_{k_E} \mathcal{N}_1 \otimes_{S_1[[K/Z_0]]} k_E = \dim_{k_E} (\mathcal{N}_1/x) \otimes_{S_1[[K/Z_0]]} k_E$. By the argument as at end of the proof of [68, Prop. 2.34], $\mathrm{Tor}_1^{S_1[[K/Z_0]]}(\mathcal{N}_1, k_E)$ is a finitely generated S_1-module (even a finite dimensional k_E-vector space). Using the exact sequence:

$$0 \longrightarrow S_1[[K/Z_0]]^{\oplus r} \xrightarrow{x} S_1[[K/Z_0]]^{\oplus r} \longrightarrow \mathcal{O}_E[[K/Z_0]]^{\oplus r} (\cong \mathcal{N}_1/x) \longrightarrow 0,$$

we easily deduce $\mathrm{Tor}_1^{S_1[[K/Z_0]]}(\mathcal{N}_1/x, k_E) \xrightarrow{\sim} k_E^{\oplus r}$, which implies with (A.10) that the morphism $\mathrm{Tor}_1^{S_1[[K/Z_0]]}(\mathcal{N}_1, k_E) \xrightarrow{x} \mathrm{Tor}_1^{S_1[[K/Z_0]]}(\mathcal{N}_1, k_E)$ is surjective. But since $x \mapsto 0 \in k_E$, we deduce $\mathrm{Tor}_1^{S_1[[K/Z_0]]}(\mathcal{N}_1, k_E) = 0$ and hence $\mathcal{N}_1$ is projective, and even isomorphic to $S_1[[K/Z_0]]^{\oplus r}$.

(c) We now finish the proof. Let $\Gamma := 1 + p\mathbb{Z}_p$, the pro-$p$ completion of $\mathbb{Q}_p^\times$ is isomorphic to $\Gamma \times \mathbb{Z}_p$, from which we deduce $\Lambda \cong \mathcal{O}_E[[\Gamma \times \mathbb{Z}_p]]$. There exists thus $y \in \Lambda$ such that $\Lambda \cong S_2[[\Gamma]]$ with $S_2 := \mathcal{O}_E[[y]]$. Since Z_0 is a subgroup of finite index of Γ, we deduce Λ is finite étale over $S_2[[Z_0]]$, and hence 1^{univ} is a finite projective $S_2[[Z_0]]$-module. Together with (b), Proposition A.10 and $K \cong K/Z_0 \times Z_0$, we obtain that $\mathcal{N} \cong \mathcal{N}^{\zeta\varepsilon} \widehat{\otimes}_{\mathcal{O}_E} 1^{\mathrm{univ}}$ is a finitely generated projective $S[[K]]$-module with $S := \mathcal{O}_E[[x, y]]$. This concludes the proof. $\square$

A.4. Proof of Proposition 3.32

We finally prove Proposition 3.32.

We keep the previous notation. We assume $p \geq 5$ and fix $\rho\colon \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ as in Proposition 3.30, so that we have $D_{\mathrm{rig}}(\rho) \cong D(\alpha, \lambda, \psi)$ with $D(\alpha, \lambda, \psi)$ as in Lemma 3.29. It is enough to prove the proposition with $D(p, \lambda, \psi)$, $\pi(\lambda^\flat, \psi)$ replaced by $D(\alpha, \lambda, \psi)$, $\pi(p^{-1}\alpha, \lambda^\flat, \psi)$ respectively (as in the proof of Proposition 3.30). We fix a mod p reduction $\overline{\rho}$ of ρ satisfying (A.2) and $\mathrm{End}_{\mathrm{Gal}_{\mathbb{Q}_p}}(\overline{\rho}) = k_E$, and we define using Corollary A.7 and Remark A.9:

$$\Pi := \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_E}\big(\mathcal{N}, \mathcal{O}_E\big) \otimes_{\mathcal{O}_E} E \cong \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_E}\big(\mathrm{Hom}_{R_{\overline{\rho}}}(\pi^{\mathrm{univ}}(\overline{\rho}), R_{\overline{\rho}}), \mathcal{O}_E\big) \otimes_{\mathcal{O}_E} E$$

where "cts" means the continuous morphisms. It follows from [73, Thm. 1.2] and Proposition A.11 that the Banach space Π (equipped with the supremum norm) is an $R_{\overline{\rho}}$-admissible continuous representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ in the sense of [12, Déf. 3.1].

Lemma A.12. *We have an isomorphism of Banach spaces:*

$$\Pi \cong \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_E}(R_{\overline{\rho}}, \mathcal{O}_E)\widehat{\otimes}_{R_{\overline{\rho}}}\pi^{\mathrm{univ}}(\overline{\rho})[1/p]$$

where $R_{\overline{\rho}}$ *(in* $\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_E}(R_{\overline{\rho}}, \mathcal{O}_E)$*) is equipped with its* $\mathfrak{m}_{R_{\overline{\rho}}}$*-adic topology and* $\widehat{\otimes}$ *is the* ϖ_E*-adic completion of the usual tensor product.*

Proof. Note that $\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_E}(R_{\overline{\rho}}, \mathcal{O}_E)$ is a cofinitely generated $R_{\overline{\rho}}$-module by [42, Prop. C.5]. By [73, Thm. 1.2], it is enough to prove

$$\mathrm{Hom}_{\mathcal{O}_E}(\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_E}(R_{\overline{\rho}}, \mathcal{O}_E)\widehat{\otimes}_{R_{\overline{\rho}}}\pi^{\mathrm{univ}}(\overline{\rho}), \mathcal{O}_E) \cong \mathcal{N}.$$

But:

$$\begin{aligned} & \mathrm{Hom}_{\mathcal{O}_E}\left(\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_E}(R_{\overline{\rho}}, \mathcal{O}_E)\widehat{\otimes}_{R_{\overline{\rho}}}\pi^{\mathrm{univ}}(\overline{\rho}), \mathcal{O}_E\right) \\ \cong\ & \mathrm{Hom}_{\mathcal{O}_E}\left(\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_E}(R_{\overline{\rho}}, \mathcal{O}_E)\otimes_{R_{\overline{\rho}}}\pi^{\mathrm{univ}}(\overline{\rho}), \mathcal{O}_E\right) \\ \cong\ & \mathrm{Hom}_{R_{\overline{\rho}}}\left(\pi^{\mathrm{univ}}(\overline{\rho}), \mathrm{Hom}_{\mathcal{O}_E}(\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_E}(R_{\overline{\rho}}, \mathcal{O}_E), \mathcal{O}_E)\right) \\ \cong\ & \mathrm{Hom}_{R_{\overline{\rho}}}\left(\pi^{\mathrm{univ}}(\overline{\rho}), R_{\overline{\rho}}\right) \cong \mathcal{N} \end{aligned}$$

where the first two isomorphisms are easy, the third one comes from [42, Prop. C.5] and the last one from Corollary A.7. ☐

Any $\widetilde{\rho} \in \mathrm{Ext}^1_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho, \rho)$ gives rise to an $E[\epsilon]/\epsilon^2$-valued point of $R_{\overline{\rho}}$, hence to an ideal $\mathcal{I}_{\widetilde{\rho}} \subseteq R_{\overline{\rho}}$ with $R_{\overline{\rho}}/\mathcal{I}_{\widetilde{\rho}} \cong \mathcal{O}_E[\epsilon]/\epsilon^2$.

Lemma A.13. *Let* $\pi(\widetilde{\rho})^{\mathrm{an}}$ *be the image of* $\widetilde{\rho}$ *via (3.59), then we have an isomorphism* $\pi(\widetilde{\rho})^{\mathrm{an}} \cong \Pi[\mathcal{I}_{\widetilde{\rho}}]^{\mathrm{an}}$ *of locally analytic representations of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *over* E*.*

Proof. By the same proof as for Lemma A.12 using

$$\Pi[\mathcal{I}_{\widetilde{\rho}}] \cong \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_E}(\mathcal{N}/\mathcal{I}_{\widetilde{\rho}}, \mathcal{O}_E) \otimes_{\mathcal{O}_E} E$$

and Corollary A.8, we deduce

$$\Pi[\mathcal{I}_{\widetilde{\rho}}] \cong \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_E}(R_{\overline{\rho}}/\mathcal{I}_{\widetilde{\rho}}, \mathcal{O}_E)\widehat{\otimes}_{R_{\overline{\rho}}/\mathcal{I}_{\widetilde{\rho}}}(\pi^{\mathrm{univ}}(\overline{\rho})/\mathcal{I}_{\widetilde{\rho}})[1/p].$$

The result follows then from Remark 3.31(2) and the fact $\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_E}(R_{\overline{\rho}}/\mathcal{I}_{\widetilde{\rho}}, \mathcal{O}_E)$ is free of rank one over $R_{\overline{\rho}}/\mathcal{I}_{\widetilde{\rho}} \cong \mathcal{O}_E[\epsilon]/\epsilon^2$. ☐

As in [12, Déf. 3.2], we denote by $\Pi^{R_{\overline{\rho}}-\mathrm{an}}$ the subspace of locally $R_{\overline{\rho}}$-analytic vectors of Π and consider the locally analytic $T(\mathbb{Q}_p)$-representation

$J_B(\Pi^{R_{\overline{\rho}}-\mathrm{an}})$ (T, B as in § 3.2.2). As in [12, § 3.2], the strong dual $J_B(\Pi^{R_{\overline{\rho}}-\mathrm{an}})^\vee$ corresponds to a coherent sheaf $\mathcal{M}$ over $(\mathrm{Spf}\, R_{\overline{\rho}})^{\mathrm{rig}} \times \mathcal{T}$ ($\mathcal{T}$ as in § 7.1.3) and we let X denote the schematic support of $\mathcal{M}$. A point $x = (\rho_x, \delta_x) \in (\mathrm{Spf}\, R_{\overline{\rho}})^{\mathrm{rig}} \times \mathcal{T}$ lies in X if and only if there is a $T(\mathbb{Q}_p)$-embedding $\delta_x \hookrightarrow J_B(\Pi^{R_{\overline{\rho}}-\mathrm{an}}[\mathfrak{p}_{\rho_x}]) = J_B(\widehat{\pi}(\rho_x)^{\mathrm{an}})$ where $\mathfrak{p}_{\rho_x} \subseteq R_{\overline{\rho}}$ is the prime ideal attached to ρ_x and the isomorphism $\widehat{\pi}(\rho_x)^{\mathrm{an}} \cong \Pi^{R_{\overline{\rho}}-\mathrm{an}}[\mathfrak{p}_{\rho_x}] = \Pi[\mathfrak{p}_{\rho_x}]^{\mathrm{an}}$ is proven as for the one in Lemma A.13.

Consider the Zariski-closed trianguline variety $X_{\mathrm{tri}}(\overline{\rho})$ of $(\mathrm{Spf}\, R_{\overline{\rho}})^{\mathrm{rig}} \times \mathcal{T}$ defined exactly as in [12, § 2.2] (for $K = \mathbb{Q}_p$ and $n = 2$) but without the framing, i.e. replacing $R_{\overline{\rho}}^{\square}$ by $R_{\overline{\rho}}$. As in [12, Thm. 2.6] the rigid variety $X_{\mathrm{tri}}(\overline{\rho})$ is equidimensional of dimension 4 and contains a Zariski-open Zariski-dense subspace $U_{\mathrm{tri}}(\overline{\rho})^{\mathrm{reg}}$ which we define in the same way but removing the framing. Arguing inside $(\mathrm{Spf}\, R_{\overline{\rho}})^{\mathrm{rig}} \times \mathcal{T}$, it easily follows from [25], [59] (and the above characterization of points of X) that there is an embedding $U^{\mathrm{tri}}(\overline{\rho})^{\mathrm{reg}} \hookrightarrow X$ (be careful that there is a shift on the $\mathcal{T}$-part between the two sides analogous to (the inverse of) [11, (3.2)]), and hence we deduce a closed embedding (note that $X_{\mathrm{tri}}(\overline{\rho})$ is reduced) $j : X_{\mathrm{tri}}(\overline{\rho}) \longhookrightarrow X$. The pull-back $\mathcal{M}_1 := j^*\mathcal{M}$ is thus a coherent sheaf on $X_{\mathrm{tri}}(\overline{\rho})$.

It follows from Proposition A.11 that $\mathcal{N}$ is finitely generated and *projective* as $S[[\mathrm{GL}_2(\mathbb{Z}_p)]]$-module where $S = \mathcal{O}_E[[x, y]] \hookrightarrow R_{\overline{\rho}}$. In this case, by the same argument as in [12, §§ 3.3, 3.4 & 3.5] (see especially [12, Thm. 3.19]), one can prove that the set Z of points $(\rho, \delta) \in X$ such that ρ is crystalline generic (see before Lemma 7.19) and δ is noncritical (see before Lemma 7.18) is Zariski-dense and accumulation in X. Since such points are in $U_{\mathrm{tri}}(\overline{\rho})^{\mathrm{reg}}$ (modulo the aforementioned shift) we deduce that j induces an isomorphism $X_{\mathrm{tri}}(\overline{\rho}) \xrightarrow{\sim} X_{\mathrm{red}}$. In particular, the noncritical point $x := (\rho, \delta_\lambda(|\cdot| \otimes 1)\, \mathrm{unr}(\alpha) \circ \det)$ is in $X_{\mathrm{tri}}(\overline{\rho})$ (indeed, as j is an isomorphism, all the trianguline representations with mod p reduction isomorphic to $\overline{\rho}$ appear on $X_{\mathrm{tri}}(\overline{\rho})$ since they do on X using [25], [59]).

Using the isomorphism $X_{\mathrm{tri}}(\overline{\rho}) \xrightarrow{\sim} X_{\mathrm{red}}$ and the above characterization of points of X together with [25], [59] and [37, Ex. 5.1.9], it easily follows that there exists a sufficiently small affinoid neighborhood $U \subseteq X_{\mathrm{tri}}(\overline{\rho})$ of x such that the special fiber of the coherent sheaf $\mathcal{M}_1$ at each point $x' \in U$ is one dimensional over the residue field of x'. Since U is reduced, we deduce $\mathcal{M}_1$ is locally free of rank 1 over U by [48, Ex. II.5.8(c)] (which is there in the scheme setting, but the rigid setting is analogous). We denote by V_x the tangent space of $X_{\mathrm{tri}}(\overline{\rho})$ at x and we identify the tangent space of $\mathcal{T}$ at $\delta_x := \delta_\lambda(|\cdot| \otimes 1)\, \mathrm{unr}(\alpha) \circ \det$ with $\mathrm{Hom}(T(\mathbb{Q}_p), E)$. By the global triangulation theory ([51], [58]) and using similar arguments as in [11, §4.1], we have the following facts:

- the morphism $X_{\mathrm{tri}}(\overline{\rho}) \longrightarrow (\mathrm{Spf}\, R_{\overline{\rho}})^{\mathrm{rig}}$ induces an isomorphism:

$$j_x : V_x \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{tri}}(\rho, \rho) \tag{A.11}$$

- for $v \in V_x$, denote by $\Psi_v = (\psi_{v,1}, \psi_{v,2}) \in \mathrm{Hom}(T(\mathbb{Q}_p), E)$ the image of v in the tangent space of $\mathcal{T}$ at δ_x induced by $X_{\mathrm{tri}}(\overline{\rho}) \to \mathcal{T}$, then the $\mathrm{Gal}_{\mathbb{Q}_p}$-representation $j_x(v)$ is trianguline of parameter $(x^{k_1}|\cdot|(1+\psi_{v,1}\epsilon)\,\mathrm{unr}(\alpha), x^{k_2}(1+\psi_{v,2}\epsilon)\,\mathrm{unr}(\alpha))$.

Now let $0 \neq v : \mathrm{Spec}\, E[\epsilon]/\epsilon^2 \hookrightarrow U$ be a nonzero element in V_x. Since $\mathcal{M}_1$ is locally free at x, we have that $W_v := v^*\mathcal{M}_1$ is a free $E[\epsilon]/\epsilon^2$-module of rank 1. The action of $R_{\overline{\rho}}$ on W_v is induced by $v : \mathrm{Spec}\, E[\epsilon]/\epsilon^2 \hookrightarrow U \to (\mathrm{Spf}\, R_{\overline{\rho}})^{\mathrm{rig}}$ and we denote as usual by $\mathcal{I}_v$ the corresponding ideal of $R_{\overline{\rho}}$. Moreover $T(\mathbb{Q}_p)$ acts on the E-dual of W_v by $\delta_\lambda(|\cdot|\otimes 1)(1+\Psi_v\epsilon)\,\mathrm{unr}(\alpha)\circ\det$. Note that it is possible that $\Psi_v = 0$, but we always have $\mathcal{I}_v \neq \mathfrak{m}_\rho$ by (A.11). Since the rigid space $(\mathrm{Spf}\, R_{\overline{\rho}})^{\mathrm{rig}} \times \mathcal{T}$ is nested ([1, Def. 7.2.10]), so are its closed subspaces X and $X_{\mathrm{tri}}(\overline{\rho})$. It follows that the composition:

$$v : \mathrm{Spec}\, E[\epsilon]/\epsilon^2 \hookrightarrow U \hookrightarrow X_{\mathrm{tri}}(\overline{\rho}) \xrightarrow{\sim} X$$

induces a surjection $\Gamma(X, \mathcal{M}) \twoheadrightarrow v^*\mathcal{M} \cong W_v$ (using that the image of the composition $\Gamma(X,\mathcal{M}) \to \Gamma(U,\mathcal{M}) \twoheadrightarrow v^*\mathcal{M} \cong W_v$ is dense as a composition of continuous maps with dense images, hence is surjective since W_v is finite dimensional). Taking duals and keeping track of the shift, we obtain an $R_{\overline{\rho}} \times T(\mathbb{Q}_p)$-equivariant injection $\delta_{\lambda^\flat}(|\cdot|\otimes|\cdot|^{-1})(1+\Psi_v\epsilon)\,\mathrm{unr}(p^{-1}\alpha)\circ\det \hookrightarrow J_B(\Pi^{R_{\overline{\rho}}-\mathrm{an}})$. Since the E-dual of W_v is killed by $\mathcal{I}_v$, we see that this map factors through an $E[\epsilon]/\epsilon^2$-linear embedding of locally analytic representations of $T(\mathbb{Q}_p)$:

$$\delta_{\lambda^\flat}(|\cdot|\otimes|\cdot|^{-1})(1+\Psi_v\epsilon)\,\mathrm{unr}(p^{-1}\alpha) \hookrightarrow J_B(\Pi^{R_{\overline{\rho}}-\mathrm{an}}[\mathcal{I}_v]) \tag{A.12}$$

(note that the left hand side of (A.12) always has dimension 2 over E even if $\Psi_v = 0$).

We can now finally prove Proposition 3.32. By Proposition 3.30, Lemma 3.6 and Proposition 3.22(1), it is enough to prove that (3.59) maps $\mathrm{Ext}^1_{\mathrm{tri}}$ to $\mathrm{Ext}^1_{\mathrm{tri}}$ in such a way that Hypothesis 3.26(3) holds (up to twist by $\mathrm{unr}(p^{-1}\alpha)$ on both sides). Fix an extension in $\mathrm{Ext}^1_{\mathrm{tri}}(\rho,\rho)$, i.e. a trianguline deformation $\widetilde{\rho}$ of ρ over $E[\epsilon]/\epsilon^2$, by (A.11) and what is below (A.11), we have that $(x^{k_1}|\cdot|(1+\psi_{v,1}\epsilon)\,\mathrm{unr}(\alpha), x^{k_2}(1+\psi_{v,2}\epsilon)\,\mathrm{unr}(\alpha))$ is a parameter for $D_{\mathrm{rig}}(\widetilde{\rho})$ where $v \in V_x$ is the associated vector. Let $\pi(\widetilde{\rho})^{\mathrm{an}}$ be the image of $\widetilde{\rho}$ via (3.59), by Lemma A.13 we have $\pi(\widetilde{\rho})^{\mathrm{an}} \cong \Pi[\mathcal{I}_v]^{\mathrm{an}} = \Pi^{R_{\overline{\rho}}-\mathrm{an}}[\mathcal{I}_v]$ and by (A.12) together with Proposition 3.22(2), we finally deduce the result.

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