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Geometric flows for the Type IIA string

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A geometric flow on 6-dimensional symplectic manifolds is introduced which is motivated by supersymmetric compactifications of the Type IIA string. The underlying structure turns out to be SU(3) holonomy, but with respect to the projected Levi-Civita connection of an almost-Hermitian structure. The short-time existence is established, and new identities for the Nijenhuis tensor are found which are crucial for Shi-type estimates. The integrable case can be completely solved, giving an alternative proof of Yau's theorem on Ricci-flat Kähler metrics. In the non-integrable case, models are worked out which suggest that the flow should lead to optimal almost-complex structures compatible with the given symplectic form.

1. Introduction

There has been a remarkable confluence in recent years between high energy physics, more specifically unified string theories, and special geometry. The earliest and particularly influential development was the 1985 recognition by Candelas, Horowitz, Strominger, and Witten [7] of Calabi-Yau manifolds as supersymmetric compactifications of the heterotic string. The importance of geometry in physical laws at their most fundamental level has of course been long recognized with electromagnetism, general relativity, and gauge theories. The key new feature here is the requirement that the 6-dimensional internal manifold have a special geometric structure, in this case a complex structure together with a holomorphic section of the canonical bundle. This requirement is equivalent to the manifold having SU(3) holonomy, and can be traced back to supersymmetry. Since then, the class of Calabi-Yau solutions has been enlarged in many directions. On one hand, the Calabi-Yau

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condition can be extended to the Hull-Strominger system for conformally balanced metrics [32, 44, 23, 24, 40, 41, 25, 21, 17]. On the other hand, it emerged from the mid 1990's that string theories, which are 10-dimensional and of which there are five, can be unified themselves into another theory, namely M Theory, one of whose limits is 11-dimensional supergravity [31, 48]. The compactification of an 11-dimensional space-time to a more familiar 4-dimensional space time results in a 7-dimensional internal space, and the role of Calabi-Yau manifolds is assumed in this case by manifolds with G_2 or Spin(7) holonomy [1, 28].

While the full string theories have been conjectured to merge ultimately into a single M Theory, this requires highly non-trivial dualities, and their low-energy approximations and geometric settings can be quite different. A common feature of their supersymmetric compactifications is a metric satisfying a curvature condition as well as a cohomological condition. In Kähler geometry, these are characteristic features of the notion of canonical metric, of which the Calabi-Yau condition is the prime example. Thus the general case can be viewed as a search for canonical metrics in non-Kähler geometry. The compactifications discussed above arise from three of the string theories, which are the Type I theory and the two heterotic string theories. The other two string theories are the Type IIA and the Type IIB theories. There is an immense literature on their supersymmetric compactifications, but some attractive mathematical formulations can be found in Grana et al. [27] and Tomasiello [45], and the study of the most basic examples was begun in Tseng and Yau [50, 51, 52]. The geometric realm for the Type IIB equation is that of complex geometry, albeit non-Kähler, and we described a geometric flow approach to it in [19]. The main goal of the present paper is to present a geometric flow approach to the Type IIA equation.

The Type IIA equation is of particular interest, because of all string theory compactifications, its geometric setting is unique in being that of symplectic geometry instead of complex geometry. More specifically, let M be a compact 6-dimensional manifold, equipped with a symplectic form ω, that is, a closed non-degenerate 2-form. Recall that on any oriented 6-manifold, Hitchin [30] had shown how to associate to a non-degenerate 3-form φ an almost-complex structure J_φ. In the Type IIA equation, a symplectic form ω is given, so it makes sense to consider the condition of primitiveness for φ with respect to ω. Explicitly, this is the condition $\Lambda\varphi = 0$, where

$$\Lambda : A^k(M) \to A^{k-2}(M) \tag{1.1}$$

is the usual Hodge operator of contracting with ω. As shown in §4.1 below, the symplectic form ω is invariant with respect to the almost-complex structure J_φ when φ is primitive. We obtain then a Hermitian form

$$g_\varphi(X,Y) = \omega(X, J_\varphi Y) \tag{1.2}$$

which becomes a Hermitian metric under the open condition that it be strictly positive. Thus we obtain an almost-Kähler 3-fold $(M, \omega, J_\varphi, g_\varphi)$ with the additional requirement that φ be primitive and J_φ arise from φ by the above construction. When φ is also closed, we shall refer to such a structure as a "Type IIA geometry".

Let ρ_A be now the Poincaré dual of a given finite linear combination of special Lagrangians calibrated by φ. Then the Type IIA equation is the following system of equations for a real-valued primitive 3-form φ

$$d\Lambda d(|\varphi|^2 \star\varphi) = \rho_A, \qquad d\varphi = 0, \qquad g_\varphi > 0. \tag{1.3}$$

Here $\star$ is the Hodge star operator and $|\varphi|$ the norm of φ with respect to the metric g_φ.

As in the case of the other string theories, the Type IIA equations as written in (1.3) involve, besides the open condition $g_\varphi > 0$, a curvature-type equation and the cohomological constraint $d\varphi = 0$. In order to enforce this cohomological constraint without invoking any particular Ansatz, we introduce the following geometric flow of 3-forms φ,

$$\partial_t\varphi = d\Lambda d(|\varphi|^2 \star\varphi) - \rho_A, \tag{1.4}$$

for any closed and primitive initial data φ_0 with $g_{\varphi_0} > 0$. Since the right hand side is closed, the flow preserves the closedness condition. It can also be verified to preserve the primitiveness property of φ. Thus it is a flow of Type IIA geometries, whose stationary points would give solutions of the Type IIA equation without recourse to any Ansatz. We shall refer to (1.4) as the Type IIA flow. The idea of preserving the closedness of a form by a flow was introduced by Bryant [5] in the Laplacian flow for G_2 structures. More recently, it was applied in [38, 39, 19, 18, 37] to the construction of geometric flows which preserve the conformally balanced condition in the Hull-Strominger system and the Type IIB equation. The geometric flow approach was particularly appropriate there, as it allowed to bypass the absence of a $\partial\bar{\partial}$-lemma in non-Kähler geometry.

The main goal of this paper is to start an in-depth study of the Type IIA flow. Except for the original formulation, we shall restrict to the most basic source-free case $\rho_A = 0$. Despite its very simple formulation (1.4), the flow turns out to be highly non-trivial and to present many new difficulties specific to symplectic geometry:

• The first is that the Type IIA equation is not elliptic. This difficulty was well-recognized in the works of Tseng and Yau [52] and Tseng and Wang [49], and led them to consider instead some 4th-order equations. However, 4th-order equations are complicated, and the closedness constraint on the 3-form φ would have to be imposed separately. Thus it appears still preferable to confront the specific difficulties of the Type IIA flow. They originate in any case from the geometric assumption of a given symplectic structure, which is fundamental in symplectic geometry.

• The second may be appreciated by comparing the flow of almost-complex structures J_φ in the Type IIA flow with the gradient flow of the Blair-Ianus functional on a symplectic manifold. The Blair-Ianus functional is the L^2 norm of the Nijenhuis tensor [4]. Its gradient flow was called the anti-complexified Ricci flow by Lê and Wang [33], who also established its short-time existence. However, this flow has proved to be difficult to use, because neither the corresponding Nijenhuis tensor nor curvature evolves there by parabolic equations. For the Type IIA flow to be viable, it has to overcome such difficulties.

• The third difficulty is more technical, but still serious. The Type IIA flow of 3-forms φ induces a flow of metrics g_φ which will be one of the main tasks of this paper to determine explicitly. The simplest case is when the initial almost-complex structure is integrable. It turns out that the Type IIA flow preserves the integrability condition, and becomes equivalent to the dual Anomaly flow introduced in [20]. Using the techniques there as well as in [42, 43], it gives a new proof of Yau's [53] celebrated theorem on the existence of Kähler Ricci-flat metrics. Thus the difficult case is the case of non-integrable almost-complex structures. There we shall see that the flow of metrics in the Type IIA flow is conformally equivalent to a perturbation of the Ricci flow by first-order terms involving the Nijenhuis tensor. In this respect, it is analogous to Bryant's G_2 Laplacian flow, which was shown relatively recently by Lotay and Wei [35] to be a perturbation of the Ricci flow by first-order terms involving the torsion tensor. However, the long-time behavior of the G_2 flow remains at this moment a subject of extensive research [2, 34].

Despite these difficulties, we shall find that the Type IIA flow is particularly rich, reflecting its unique position at the crossroads of symplectic geometry, complex geometry, and unified string theories. This rich structure will be much in evidence in the results described below. Furthermore, examples suggest that even when the flow develops singularities, it may be possible in some cases to continue the flow of the Nijenhuis tensor. Thus, besides the original motivation from string theory and interest in its stationary points, the flow should also be useful in finding optimal almost-complex structures.

2. Main results

We describe now our main results. Throughout this section, M is a compact 6-dimensional manifold equipped with a symplectic form ω. Given a primitive 3-form φ, we denote by J_φ the almost-complex structure defined by Hitchin [30], and by g_φ the corresponding Hermitian form, which we assume is a metric.

2.1. A Laplacian flow formulation

Our first result is an alternative formulation of the Type IIA flow:

Theorem 1. *The Type IIA flow defined in (1.4) can be rewritten as the following flow*

$$\partial_t \varphi = -dd^\dagger(|\varphi|^2\varphi) + 2\, d(|\varphi|^2 N^\dagger \cdot \varphi) - \rho_A \tag{2.1}$$

where $d^\dagger$ is the adjoint of the operator d with respect to the metric g_φ, and $N^\dagger$ is the operator from $\Lambda^3(M)$ to $\Lambda^2(M)$ defined by

$$(N^\dagger \cdot \varphi)_{kj} = N^\mu{}_j{}^\lambda \varphi_{\mu k\lambda} - N^\mu{}_k{}^\lambda \varphi_{\mu j\lambda}. \tag{2.2}$$

Here $N^m{}_{\gamma\beta}$ is the Nijenhuis tensor of J_φ, and indices are raised using the metric g_φ.

We note that, up to the factor of $|\varphi|^2$, the first terms on the right-hand side of the Type IIA flow are the same as in the standard heat equation. Up to sign, they are also reminiscent of Bryant's G_2 Laplacian flow. However, the terms involving the Nijenhuis tensor are also of leading order and account for a wide range of different phenomena.

Henceforth, we assume that the source ρ_A is 0, unless stated explicitly otherwise.

2.2. The short-time existence of the Type IIA flow

When a flow is not strictly parabolic, even its short-time existence can be a difficult question. Two powerful tools developed over the years for this issue have been the reparametrization method of DeTurck [14] and the Hamilton-Nash-Moser theorem [29], a combination of which has been applied successfully to many important flows, such as the Ricci flow [29], the G_2 Laplacian flow [5, 6], and the anti-complexified Ricci flow [33]. The fundamental new difficulty in the Type IIA flow is that there is a given symplectic form ω. It is not hard to see that reparametrizations by symplectomorphisms do not improve the parabolicity of the flow, while more general reparametrizations lead to a *coupled flow* of both metrics and symplectic forms. Thus a first major task in this paper is to establish the following theorem:

Theorem 2. *Let (M, ω) be a compact 6-dimensional symplectic manifold. Then for any φ_0 which is a smooth positive, primitive, and closed 3-form, the source-free Type IIA flow (1.4) with initial value φ_0 admits a unique and smooth solution on some time interval $[0, T)$ with $T > 0$. Furthermore, φ continues to be positive, primitive and closed at all times.*

While the theorem deals only with the short-time existence of the flow, the proof requires a rather deep probe of the structure of the flow and several new elements which are also useful elsewhere:

The first element is the behavior of the coupled flows mentioned above. It turns out that these coupled flows admit natural parabolic regularizations, which reduce to the desired flow for primitiveness initial data. While primitiveness was a requirement in the solution φ of the Type IIA equation, it may not have been anticipated that it would play such a central role for the very existence of the flow.

The second feature permeates the rest of the paper, and is the underlying Type IIA geometry. In the present context, it allows us to recapture the flow of the forms φ from the flow of the metrics $\tilde{g}_\varphi = |\varphi|^2 g_\varphi$, even though, pointwise, there is an ambiguity in determining φ from $\tilde{g}_\varphi$. Since we shall have to analyze in detail the flow of metrics in order to obtain Shi-type estimates and long-time existence criteria, it is simplest to deduce the uniqueness part of Theorem 2 from the corresponding uniqueness theorems for the flow of metrics. The flow of $\breve{g}_\varphi = |\varphi|^{-2} g_\varphi$ turns out to be a perturbation of the Ricci flow by terms of first order. In general, integrability operators are not stable under first-order terms perturbations. However, the underlying Type IIA geometry is what allows us to modify the Bianchi operator used by

Hamilton [29] for the Ricci flow into an integrability operator for the flow of $\breve{g}_\varphi$. From there we can establish the uniqueness for $\breve{g}_\varphi$, and from there the uniqueness in Theorem 2.

2.3. Type IIA geometry

We have stressed that the underlying structure for the Type IIA flow is Type IIA geometry, as defined in the Introduction, and which is more special than just a symplectic structure on a 6-manifold. The holonomy, curvature, and Nijenhuis tensor in Type IIA geometry have rather special properties, which play an important role in every aspect of the Type IIA flow. We list here some properties which are applied repeatedly in the paper and are the easiest to describe, but we expect others to emerge and prove their worth in time:

Theorem 3. *Let (M, ω, φ) be a Type IIA geometry, and g_φ the corresponding metric. Set*

$$\tilde{g}_\varphi = |\varphi|^2 g_\varphi. \tag{2.3}$$

Let $\mathfrak{D}$ and $\tilde{\mathfrak{D}}$ be the projected Levi-Civita connections of g_φ and $\tilde{g}_\varphi$ respectively, $\Omega = \varphi + iJ_\varphi\varphi$, and $|\Omega|_{\tilde{g}_\varphi}$ the norm of Ω with respect to $\tilde{g}_\varphi$. Then

(a) $\tilde{\mathfrak{D}}(\frac{\Omega}{|\Omega|_{\tilde{g}_\varphi}}) = 0$. *Thus $(M, \tilde{g}_\varphi)$ has holonomy in $SU(3)$, but with respect to the connection $\tilde{\mathfrak{D}}$.*

(b) $\mathfrak{D}^{0,1}\Omega = 0$, *so Ω is formally holomorphic, even when J_φ is not integrable.*

(c) *The Nijenhuis tensor has only 6 independent components.*

2.4. The flow of metrics in the Type IIA flow

Next, we can describe the flow of the 3-forms φ and metrics g_φ in terms of curvature:

Theorem 4. *Let φ be a smooth positive, primitive and closed 3-form evolving by the source-free Type IIA flow. Set $u = \log|\varphi|^2$.*

(a) *The flow of φ is given by*

$$\begin{aligned} \partial_t\varphi_{iab} \;=\; & e^{2u}\sum_{\mathrm{cyc}\ i,a,b}\Big(\varphi_{sab}(\tilde{\mathfrak{D}}_i + u_i)u^s + 2\varphi_{sta}(N^{st}{}_p\tilde{\mathfrak{T}}^p{}_{ib} \\ & -\frac{u^s}{2}\tilde{\mathfrak{T}}^t{}_{ib} + (\tilde{\mathfrak{D}}_i + u_i)N^{st}{}_b)\Big), \end{aligned} \tag{2.4}$$

where $\tilde{\mathfrak{T}}$ is the torsion tensor of the connection $\tilde{\mathfrak{D}}$.

(b) *The flow of $\tilde{g}_\varphi$ is given by*

$$(\partial_t \tilde{g}_\varphi)_{ij} = e^{2u}\Big[-2\tilde{R}_{ij} - 2(\tilde{\nabla}^2 u)_{ij} + 4u^s(N_{isj}+N_{jsi}) - 4(N_-^2)_{ij} \\ + u_i u_j - u_{Ji}u_{Jj} + (|du|^2_{\tilde{g}_\varphi} + |N|^2_{\tilde{g}})(\tilde{g}_\varphi)_{ij}\Big], \tag{2.5}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}_\varphi$.

Note that u is determined by $\tilde{g}_\varphi$, so the right hand side of (2.5) involves only tensors determined by $\tilde{g}$, and the equation is a self-contained flow for $\tilde{g}_\varphi$. Furthermore, the equation for φ can be viewed as a linear ODE of φ whose coefficients are tensors determined by $\tilde{g}_\varphi$. Thus φ is completely determined once $\tilde{g}_\varphi$ is determined. As we noted before, this is to be contrasted with the problem of having to resolve an ambiguity if we just try to recapture φ from g_φ at each fixed time. It may be worth observing that the ambiguity in recapturing φ from g_φ pointwise in time is reminiscent of the ambiguity in defining the angle in the special Lagrangian equation. It would be interesting to investigate if the angle in the special Lagrangian equation can be recaptured by a mechanism similar to the above Theorem 4 for the Type IIA flow.

2.5. An integrability operator for the flow of metrics

The flow (2.5) is reminiscent of the Ricci flow, except for the term $(\tilde{\nabla}^2 u)_{ij}$ which can normally be eliminated by a reparametrization. But as we noted in the above discussion of Theorem 2, a reparametrization would create other difficulties since it would change the given symplectic form ω. To bypass this difficulty, we make instead a conformal change

$$\breve{g}_{ij} = |\varphi|^{-2}(g_\varphi)_{ij} \tag{2.6}$$

and find that $\breve{g}_{ij}$ evolves by

$$\partial_t \breve{g}_{ij} = e^{\frac{3}{2}u}\Big[-2\breve{R}_{ij} + \frac{3}{2}u_i u_j - u_{Ji}u_{Jj} + 4u^k(N_{ikj}+N_{jki}) - 4(N_-^2)_{ij} \\ + \frac{1}{2}\left(|du|^2_{\breve{g}} + |N|^2_{\breve{g}}\right)\breve{g}_{ij}\Big]. \tag{2.7}$$

For the purpose of completing Theorem 2, we are particularly interested in the uniqueness of this flow. For the classical Ricci flow, both the short-time existence and uniqueness were established by Hamilton [29] using a version of the Nash-Moser theorem. This version, often referred to as the Hamilton-Nash-Moser theorem, requires an integrability condition, which was provided by the Bianchi identity in the case of the Ricci flow. More precisely, let L_0 be the operator defined by

$$(L_0(S))_j = 2\check{g}^{ik}\check{\nabla}_k S_{ik} - \check{g}^{ik}\check{\nabla}_j S_{ik} \in A^1(M) \tag{2.8}$$

for any $S_{ij} \in \mathrm{Sym}^2(TM)$. Then $L_0(-2\check{R}_{ij}) = 0$, which is the desired integrability condition. Now the flow (2.7) differs from the Ricci flow by first-order terms, so the Bianchi identity is no longer applicable as an integrability condition. In general, it is by no means clear whether a given first-order perturbation would still allow an integrability condition. So it is again a manifestation of the deep structure of Type IIA geometry that this can be done in this case:

Theorem 5. *Let S_{ij} be the symmetric 2-tensor defined by writing the flow (2.7) as*

$$\partial_t \check{g}_{ij} = e^{\frac{3}{2}u}(-2\check{R}_{ij} + S_{ij}). \tag{2.9}$$

(a) *For any $S_{ij} \in \mathrm{Sym}^2(TM)$, we define the operator Z by*

$$(Z(S))_j = -\frac{4}{3}u_j\check{g}^{ik}S_{ik} + 2u^s S_{js} - 4N^{st}{}_j S_{st} \in A^1(M) \tag{2.10}$$

If along the flow, the metric $\check{g}$ arises from a Type IIA geometry, then the following operator L is an integrability operator for the flow (2.7) in the sense that

$$\mathrm{Sym}^2(TM) \ni \check{g}_{ij} \;\to\; L(\check{g}_{ij}) := (L_0 + Z)(e^{\frac{3}{2}u}(-2\check{R}_{ij} + S_{ij})) \tag{2.11}$$

is of order 1 in $\check{g}_{ij}$.
(b) *As a consequence, the flow exists and is unique on some interval $[0,T)$ with $T > 0$.*

We stress that this theorem is used only to establish the uniqueness of the Type IIA flow, but not its existence. The reason is that, starting from the flow (2.7) at an initial Type IIA geometry, it is not a priori known whether the flow will remain a Type IIA geometry. Without this information, it is not known whether the above integrability condition (2.11) holds.

2.6. Formation of singularities and Shi-type estimates

In general, the fact that a flow may admit short-time existence does not imply that important geometric quantities will evolve by parabolic equations. For example, that is the case for the Ricci flow [29], but it is not the case for the anti-complexified Ricci flow [33]. The Type IIA flow shares common features with both the Ricci flow and the anti-complexified Ricci flow, and one may wonder which way it will behave when it comes to the evolution of the curvature and of the Nijenhuis tensor. It is a very attractive feature of the Type IIA flow that, in this particular respect, it is closer to the Ricci flow. Thus we find

Theorem 6. *Consider the source-free Type IIA flow with a smooth, positive, closed, and primitive initial value φ_0.*

(a) *Then the Nijenhuis tensor evolves by*

$$\begin{aligned}(2.12)\qquad & (\partial_t - e^u\Delta)|N|^2 \\ = \ & e^u\Big[-2|\nabla N|^2 + (\nabla^2 u) * N^2 + Rm * N^2 \\ & +N * \nabla N * (N + \nabla u) + N^4 + N^3 * \nabla u + N^2 * (\nabla u)^2\Big]\end{aligned}$$

(b) *The Riemann curvature tensor evolves by*

$$\begin{aligned}(2.13)\qquad & (\partial_t - e^u\Delta)|Rm|^2 \\ = \ & e^u\Big[-2|\nabla Rm|^2 + (\nabla Rm + \nabla^3 u + \nabla^2 N) * \mathcal{O}(Rm, \nabla u, N) \\ & +(\nabla N * \nabla N + \nabla^2 u * \nabla^2 u + 1) * \mathcal{O}(Rm, \nabla u, N)\Big].\end{aligned}$$

Here, $*$ denote the bilinear pairings (not to be confused with the Hodge star operator) and $\mathcal{O}(\nabla u, Rm, N)$ indicates terms which only depend on $\nabla u, Rm$ and N.

Other geometric quantities satisfy similar flows, which are written in greater detail in §8. Using these flows, we can establish the following Shi-type estimates and criteria for extending the flow:

Theorem 7. *Assume that we have a solution of the source-free Type* IIA *flow on some interval $[0, T)$, and that the bound*

$$|u| + |Rm| \leq A \tag{2.14}$$

holds for some finite constant A. Here Rm denotes the Riemann curvature tensor of the metric g_φ. Then for any multi-index α, we have

$$|\nabla^\alpha \varphi| \leq C(A, \alpha, T, \varphi(0)) \tag{2.15}$$

for some constant $C(A, \alpha, T, \varphi(0))$. In particular, the Type IIA *flow can be continued to an interval $[0, T + \varepsilon)$ for some $\varepsilon > 0$.*

It may be worth noting that, in this estimate, the estimates for the gradient $|\nabla u|$ are rather special, and make essential use of the underlying Type IIA geometry.

2.7. The stationary points in the case of no source

In the case $\rho_A = 0$, the stationary points of the flow can be identified, once we have developed Type IIA geometry:

Theorem 8. *A primitive and closed* 3*-form φ is a stationary point of the flow if and only if the corresponding almost-complex structure J_φ is integrable and the norm $|\varphi|$ is constant. Thus (M, J_φ, ω) is then a Kähler manifold, and the metric g_φ is Kähler and Ricci-flat.*

2.8. The integrable case and the Calabi conjecture

The simplest case is that of integrable almost-complex structures, and a complete description of the behavior of the Type IIA flow in this case is provided by the following theorem:

Theorem 9. *Assume that the initial value φ_0 of the source-free Type* IIA *flow is a positive, closed and primitive* 3*-form, and that the corresponding almost-complex structure $J_{\varphi_0} =: J_0$ is integrable. Then the source-free Type* IIA *flow exists for all time, the almost-complex structure J_φ corresponding to φ remains integrable along the flow, and the flow converges in C^∞ to a* 3*-form corresponding to a Kähler Ricci-flat metric.*

In fact, the corresponding flow of metrics g_φ turns out to reduce by diffeomorphisms to the dual Anomaly flow introduced in [20] which applies in all dimensions and gives another proof of the Calabi conjecture. This reduction of the Type IIA flow to the dual Anomaly flow, which was itself motivated by duality considerations for the Type IIB flow, can be viewed as a manifestation of the duality between the Type IIA and the Type IIB theories. We also observe that there are by now several known proofs of the Calabi conjecture, giving each a different sequence converging to the Kähler Ricci-flat metric: besides Yau's original proof, there are for example the proof by the Kähler-Ricci flow [8], by the Anomaly flow [42], by the dual Anomaly flow [20], by the inverse Monge-Ampère flow [9, 10], and by more general parabolic Monge-Ampère flows [43]. Nevertheless, new proofs from an independent geometric set-up remain of considerable interest, as they can detect different types of obstructions. Such a scenario is nicely illustrated in [10].

2.9. Examples

As we can see from the induced flow of metrics, the Type IIA flow is complicated. However, besides the integrable case which was solved above, there are non-integrable, geometrically interesting cases that can also be worked out completely and which exhibit varied and interesting behaviors. They suggest the possibility of a general phenomenon, namely that in all cases, the Type IIA flow leads to an optimal almost-complex structure with respect to the given symplectic form. A first example is provided by the torus:

Theorem 10. *Consider the source-free Type IIA flow on the torus* $\mathbf{R}^6/\mathbf{Z}^6$ *with the symplectic form* ω *as described in §9.3.1. Consider the Type IIA flow with non-integrable initial data of the form in (9.3.1). Then the flow exists for all time, and* φ *converges as* $t \to \infty$ *to a positive harmonic form.*

A rich class of models is the special generalized Calabi-Yau (or SGCY) manifolds introduced by de Bartholomeis [12, 13]. These are manifolds of Type IIA geometry with the additional property that $|\varphi|$ is constant. They are also sometimes referred to as symplectic half-flat structures. A large subclass is given by nilmanifolds, which are quotients of a nilpotent Lie group by a co-compact lattice, and the other subclass is the solvmanifolds, which are quotients of solvable Lie groups by a co-compact lattice. Details on the models which we consider can be found in §9.3.2, and we shall just state here the main conclusions.

Theorem 11. *Consider the source-free Type* IIA *flow on the nilmanifold and the solvmanifold described in §9.3.2, with the initial data described there.*

(a) *In the case of the nilmanifold, with initial data corresponding to (9.29), the flow exists for all time, and the Nijenhuis tensor tends to* 0 *as* $t \to \infty$.

(b) *In the case of the solvmanifold, with initial data corresponding to (9.32), the flow develops a singularity at a finite time* T. *However, the limit of* J_φ *as* $t \to T$ *exists, and is a harmonic almost-complex structure.*

3. The Type IIA flow as a Laplacian flow

We start now the proof of the results described in the previous section. For Theorem 1, we do not need any detailed information on the metric g_φ. Rather, we only need the main properties of the corresponding Hermitian connections, and how they differ from the Levi-Civita connection. These have been worked out by Gauduchon [26], and we begin with a brief review of the results from [26] that we require.

3.1. Gauduchon's formulas for connections on an almost-complex manifold

We revert momentarily to the general set-up of a smooth manifold M equipped with a Riemannian metric g, a compatible almost-complex structure J (not necessarily integrable), and the associated symplectic form ω. This means that g is a positive definite section of the bundle of quadratic forms on TM, ω is a 2-form on M, J is a section of the bundle of endomorphisms of TM satisfying $J^2 = -\mathrm{Id}$, and $g(X, Y) = \omega(X, JY)$ for any two vector fields X, Y on M. In local coordinates x^j, with X and Y given by their components X^i and Y^j, we shall write

$$g(X, Y) = g_{ij}X^iY^j,\ \omega(X, Y) = \omega_{ij}X^iY^j,\ (JX)^k = J^k{}_jX^j. \tag{3.1}$$

In particular g_{ij} and ω_{ij} are respectively symmetric and anti-symmetric in i and j, and the fact that J is a compatible almost-complex structure translates into

$$J^k{}_qJ^q{}_j = -\delta^k{}_j, \qquad g_{ij} = \omega_{iq}J^q{}_j = \omega_{jq}J^q{}_i. \tag{3.2}$$

Note that ω is invariant under the action of J, i.e. $\omega_{iq}J^q{}_jJ^i{}_\ell = \omega_{\ell j}$.

Clearly the structure defined by the triple (g, ω, J) is determined by any two of its components. Here we do not assume that ω is closed. For our purposes, we can assume that $d\omega$ has no $(3, 0)+(0, 3)$-components. Obviously this condition is satisfied when ω is closed, or conformally closed, or when J is integrable. When such a condition is satisfied, $d^c\omega = J^{-1}d\omega$ has no $(3, 0) + (0, 3)$-components either.

Associated to this setup are several important tensors:

- The first is the Nijenhuis tensor N, defined as

$$N(X, Y) = \frac{1}{4}([JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]). \tag{3.3}$$

By construction N is a 2-form valued in TM. Using the metric, we can lower the superscript to the first slot by $N_{X,Y,Z} = g(X, N(Y, Z))$.

• The second is the 3-form $d^c\omega$, where by our convention for differential forms, the form ω is defined from its coefficients ω_{ij} by

$$\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j.$$

The form $d^c\omega$ can also be written as $d^c\omega = J^{-1}dJ\omega = -Jd\omega$ since $d^c = J^{-1}dJ$ and ω is invariant under the action of J. In components, we have

$$d\omega = \frac{1}{3!}(\partial_i\omega_{jk} + \partial_j\omega_{ki} + \partial_k\omega_{ij})dx^i \wedge dx^j \wedge dx^k$$

and hence

$$\begin{aligned}
&(d\omega)_{ijk} = \partial_i\omega_{jk} + \partial_j\omega_{ki} + \partial_k\omega_{ij}, \\
&(d^c\omega)_{abc} = -J^k{}_c J^j{}_b J^i{}_a (d\omega)_{ijk} = -J^k{}_c J^j{}_b J^i{}_a (\partial_i\omega_{jk} + \partial_j\omega_{ki} + \partial_k\omega_{ij}).
\end{aligned}$$

As shown by Gauduchon [26], the construction of Hermitian connections associated to the structure (g, ω, J) is clearer if we also view $d^c\omega$ as a 2-form valued in TM, i.e. a section of $TM \otimes \Lambda^2 T^*M$, in analogy with the Nijenhuis tensor. This can be achieved by raising one index in $d^c\omega$, using the metric g_{ij}. Unless indicated otherwise, the TM-valued 2-form corresponding to $d^c\omega$ is obtained by raising the first index, i.e.,

$$(d^c\omega)^m{}_{jk} = g^{mi}(d^c\omega)_{ijk}.$$

• The third and the fourth tensors of interest are obtained by decomposing $d^c\omega$, viewed as a TM-valued 2-form, into components U and V which are respectively even and odd under the following involution $\mathcal{M}$ acting on the space of TM-valued 2-forms,

$$(\mathcal{M}\Psi)(X, Y) = \Psi(JX, JY), \qquad \Psi \in A^2(TM) \tag{3.4}$$

where we have denoted the space of TM-valued 2-forms by $A^2(TM)$. We can then define the TM-valued 2-forms U and V by

$$U = \frac{1}{4}(d^c\omega + \mathcal{M}(d^c\omega)), \qquad V = \frac{1}{4}(d^c\omega - \mathcal{M}(d^c\omega)). \tag{3.5}$$

In components,

$$(\mathcal{M}\Psi)^m{}_{bc} = \Psi^m{}_{jk} J^k{}_c J^j{}_b \tag{3.6}$$

and

$$\begin{aligned} U^m{}_{bc} &= \frac{1}{4}((d^c\omega)^m{}_{bc} + (d^c\omega)^m{}_{jk} J^k{}_c J^j{}_b), \\ V^m{}_{bc} &= \frac{1}{4}((d^c\omega)^m{}_{bc} - (d^c\omega)^m{}_{jk} J^k{}_c J^j{}_b). \end{aligned} \tag{3.7}$$

Note that $U^m{}_{bc}$ and $V^m{}_{bc}$ are still anti-symmetric in b and c, but if we let U_{abc} and V_{abc} the components of the T^*M-valued 2-form obtained by lowering the index m to an index a, then U_{abc} and V_{abc} are not anti-symmetric in a and b, unlike $(d^c\omega)_{abc}$. The tensors N, U, and V satisfy the following Bianchi-type identities

$$N_{ijk} + N_{jki} + N_{kij} = 0, \tag{3.8}$$

$$U_{ijk} + U_{jki} + U_{kij} = (d^c\omega)_{ijk}, \quad V_{ijk} + V_{jki} + V_{kij} = \frac{1}{2}(d^c\omega)_{ijk}. \tag{3.9}$$

Given an almost Hermitian structure (g, ω, J), the Gauduchon line of connections is a line of connections preserving all of (g, ω, J) which passes through the Chern connection and the projected Levi-Civita connection. If we denote the Levi-Civita connection by ∇, since $d^c\omega$ has only type $(2,1) + (1,2)$-components, the Gauduchon line can be parameterized by a real parameter t, and the corresponding connection $\mathfrak{D}^t$ is given by

$$\mathfrak{D}^t_i X^m = \nabla_i X^m + g^{mk}(-N_{ijk} - V_{ijk} + tU_{ijk}) X^j. \tag{3.10}$$

Equivalently, if we express a connection D in terms of its connection form $\Gamma(D)^m{}_{ij}$,

$$D_i X^m = \partial_i X^m + \Gamma(D)^m{}_{ij} X^j$$

then we have

$$\Gamma(\mathfrak{D}^t)^m{}_{ij} = \Gamma(\nabla)^m{}_{ij} + g^{mk}(-N_{ijk} - V_{ijk} + tU_{ijk}). \tag{3.11}$$

Since the torsion of a connection D is given by

$$T(D)^m{}_{ij} = \Gamma(D)^m{}_{ij} - \Gamma(D)^m{}_{ji}$$

and the Levi-Civita connection has zero torsion, it readily follows that

$$T(\mathfrak{D}^t)^m{}_{jk} = N^m{}_{jk} + (t-1)U^m{}_{jk} + 2tV^m{}_{jk}. \tag{3.12}$$

The two values of t of particular interest to us are:

- $t = 0$: this is the so-called projected Levi-Civita connection (a.k.a. the first canonical connection), which we shall henceforth denote by just $\mathfrak{D}^0 = \mathfrak{D}$.

- $t = 1$: this is the Chern connection ∇^C, also characterized by the condition that $\nabla^C_{\bar{U}} V = [\bar{U}, V]^{1,0}$, for any sections U, V of $T^{1,0}M$. Here we have set $\mathbf{C} \otimes TM = T^{1,0}M \oplus T^{0,1}M$ and used J to identify TM with $T^{1,0}M$. The expression $[\bar{U}, V]^{1,0}$ denotes the $(1,0)$-component of $[\bar{U}, V]$.

The value $t = -1$ gives the Bismut connection, but we shall not need it in this paper.

3.1.1. A convenient notation for the action of J. For the convenience of later use, we use the following abbreviations

$$(JV)^k = J^k_j V^j =: V^{Jk}, \qquad (JW)_m = W_j J^j{}_m =: W_{Jm}. \tag{3.13}$$

For example, the operator $\mathcal{M}$ acting on a TM-valued 2-form Ψ introduced earlier in (3.4) and (3.6) can now be expressed as

$$(\mathcal{M}\Psi)^m{}_{bc} = \Psi^m{}_{Jb,Jc} \tag{3.14}$$

and (3.7) as

$$\begin{aligned} U^m{}_{bc} &= \frac{1}{4}((d^c\omega)^m{}_{bc} + (d^c\omega)^m{}_{Jb,Jc}), \\ V^m{}_{bc} &= \frac{1}{4}((d^c\omega)^m{}_{bc} - (d^c\omega)^m{}_{Jb,Jc}). \end{aligned} \tag{3.15}$$

As another example, since $d^c\omega$ has no $(0,3)+(3,0)$ components, it satisfies

$$(d^c\omega)_{Ji,j,k} + (d^c\omega)_{i,Jj,k} + (d^c\omega)_{i,j,Jk} = (d^c\omega)_{Ji,Jj,Jk}. \tag{3.16}$$

When summing over repeated indices, J can be raised or lowered as follows

$$X^k \alpha_{Jk} = X^k J^l{}_k \alpha_l = X^{Jk}\alpha_k.$$

Moreover, we can insert J in summation indices at the cost of adding a minus sign:

$$X^k \alpha_k = -X^{Jk} \alpha_{Jk}.$$

In this short-hand notation, we have[1]

$$\omega_{jk} = g_{Jj,k} = -g_{j,Jk}, \qquad g_{jk} = \omega_{j,Jk} = \omega_{k,Jj},$$
$$\omega^{jk} = g^{Jj,k} = -g^{j,Jk}, \qquad g^{jk} = \omega^{j,Jk} = \omega^{k,Jj}.$$

3.1.2. The types of TM-valued 2-forms. All the tensors which we encountered above, namely the Nijenhuis tensor, the tensors $d^c\omega$, U, and V, and the torsion tensors can be viewed as TM-valued 2-forms, or equivalently 3-tensors which are antisymmetric in the last two slots. Denote this space by $A^2(TM)$ for simplicity. It is convenient to break up elements of $A^2(TM)$ into simpler components.

Recall the involution $\mathcal{M}$ on $A^2(TM)$ defined by (3.4) or (3.14) in components. Clearly $A^2(TM)$ splits into the direct sum of eigenspaces of $\mathcal{M}$ with eigenvalue ± 1. We shall call the eigenvalue 1 subspace the space of TM-valued (1,1)-forms, denoted by $A^{1,1}(TM)$. That is, $\Psi \in A^{1,1}(TM)$ if and only if

$$\Psi^p{}_{Jj,Jk} = \Psi^p{}_{jk} \quad \text{or equivalently} \quad \Psi_{i,Jj,k} + \Psi_{i,j,Jk} = 0.$$

The space of eigenvalue -1 can be decomposed further as follows. We say a TM-valued 2-form Ψ is of type (2,0) or (0,2) if[2]

$$\Psi(JX, Y) = J\Psi(X, Y) \quad \text{or} \quad \Psi(JX, Y) = -J\Psi(X, Y).$$

In this way we have a direct sum decomposition

$$A^2(TM) = A^{1,1}(TM) \oplus A^{2,0}(TM) \oplus A^{0,2}(TM).$$

In terms of indices we see that $\Psi \in A^{2,0}(TM)$ if

$$\Psi^p{}_{Jj,k} = \Psi^{Jp}{}_{jk} = \Psi^p{}_{j,Jk} \quad \text{or} \quad \Psi_{Ji,j,k} = -\Psi_{i,Jj,k} = -\Psi_{i,j,Jk};$$

[1] Here (ω^{jk}) denotes the inverse matrix of (ω_{jk}), $\omega^{jk}\omega_{kl} = \delta^j{}_l$. It is not the tensor one gets by raising indices using g_{jk}. In fact $\omega^{jk} = -g^{ja}\omega_{ab}g^{bk}$.

[2] To avoid confusion, we stress that this notion is specific to $A^2(TM)$ and is not the same as that of scalar-valued $(2,0)$ or $(0,2)$ forms.

and $\Psi \in A^{0,2}(TM)$ if

$$\Psi^p{}_{Jj,k} = -\Psi^{Jp}{}_{jk} = \Psi^p{}_{j,Jk} \quad \text{or} \quad \Psi_{Ji,j,k} = \Psi_{i,Jj,k} = \Psi_{i,j,Jk}.$$

Returning to the tensors which we have encountered, their types are as follows:

• It is readily seen that the Nijenhuis tensor is of type $(0, 2)$, therefore any contraction of N using either g or ω yields 0.

• It is easy to see that the tensor $U^m{}_{jk}$ is of type $(1, 1)$. As for V, we have by definition

$$\begin{aligned} V_{Ji,j,k} &= \frac{1}{4}((d^c\omega)_{Ji,j,k} - (d^c\omega)_{Ji,Jj,Jk}) \overset{(3.16)}{=} -\frac{1}{4}((d^c\omega)_{i,Jj,k} + (d^c\omega)_{i,j,Jk}) \\ &= V_{J(Ji),Jj,k} = -V_{i,Jj,k} \end{aligned}$$

and thus V is of type (2,0).

• Finally, tensors such as the torsions of connections on the Gauduchon line and consequently also differences of connections, correspond to forms of mixed types, whose decomposition in $\Lambda^{1,1}(TM) \oplus \Lambda^{2,0}(TM) \oplus \Lambda^{0,2}(TM)$ can be read off from formulas such as (3.11) and (3.12), since we know now the types of $N^m{}_{jk}$, $U^m{}_{jk}$, and $V^m{}_{jk}$.

3.2. Proof of Theorem 1

We can now give the proof of Theorem 1. For simplicity, we shall denote in the subsequent calculations g_φ and J_φ by just g and J. Recall that the operator d^c is defined by $d^c = J^{-1}dJ$, and that, for a compatible structure (ω, g, J), we have the identity

$$d^\Lambda := d\Lambda - \Lambda d = (d^c)^\dagger. \tag{3.17}$$

This identity holds even if J is not integrable. Since φ is primitive, we can replace in (1.4) Λd by $-d^\Lambda$ (since $d^2 = 0$) and rewrite the equation as

$$\begin{aligned} \partial_t\varphi &= d\Lambda d(|\varphi|^2 \star\varphi) - \rho_A = -d(d^c)^\dagger(|\varphi|^2 \star\varphi) - \rho_A \\ &= -d(J^{-1}dJ)^\dagger(|\varphi|^2 \star\varphi) - \rho_A = -dJ^{-1}d^\dagger J(|\varphi|^2 \star\varphi) - \rho_A. \end{aligned} \tag{3.18}$$

Here we have used the fact that the adjoint $J^\dagger$ of J with respect to g is J^{-1}, since J is an isometry. We shall see later that $\star\varphi = J\varphi$, so that the above equation can be rewritten as

$$\partial_t \varphi = dJd^\dagger(|\varphi|^2\varphi) - \rho_A. \tag{3.19}$$

Thus the theorem would be proved once we can establish the following lemma:

Lemma 1. *Let (M, g, J, ω) be a 6-dimensional compact almost-Hermitian manifold with $d\omega = 0$. We have*

$$Jd^\dagger\varphi = -d^\dagger\varphi + 2N^\dagger \cdot \varphi \tag{3.20}$$

where $N^\dagger$ is the operator defined in (2.2).

To establish this, we begin by recalling that the adjoint of the operator d on 3-forms is given by

$$(d^\dagger\varphi)_{\alpha\beta} = -\nabla^\gamma \varphi_{\gamma\alpha\beta} \tag{3.21}$$

where ∇ denotes the Levi-Civita connection of g, which has no torsion. We need to apply the operator J to both sides of this equation. For this, we need in turn the following lemma:

Lemma 2.

$$\nabla_k J^a{}_b = -2N_{Jk}{}^a{}_b. \tag{3.22}$$

The point of this lemma is that the Levi-Civita connection does not necessarily respect the almost-complex structure J. However, we can write it in terms of the projected Levi-Civita connection $\mathfrak{D}$ which does, at the cost of having to handle in addition terms coming from difference of connections, which gives us the Nijenhuis tensor. This lemma follows directly from (3.11), where $U = V = 0$ as ω is closed.

Returning to the proof of Lemma 1, since $\varphi_{\alpha,J\beta,J\gamma} = -\varphi_{\alpha\beta\gamma}$ by Lemma 2 and 6, we find

$$\begin{aligned}
(Jd^\dagger\varphi)_{kj} &= -J^\alpha{}_k J^\beta{}_j \nabla^\gamma \varphi_{\gamma\alpha\beta} \\
&= -\nabla^\gamma (J^\alpha{}_k J^\beta{}_j \varphi_{\gamma\alpha\beta}) - 2(N_{J\gamma}{}^\alpha{}_k J^\beta{}_j + N_{J\gamma}{}^\beta{}_j J^\alpha{}_k)\varphi^\gamma{}_{\alpha\beta} \\
&= -\nabla^\gamma \varphi_{\gamma,Jk,Jj} - 2(N^{\lambda\mu}{}_k \varphi_{\mu j\lambda} - N^{\lambda\mu}{}_j \varphi_{\mu k\lambda}) \\
&= -(d^\dagger\varphi)_{kj} + 2(N^\dagger \cdot \varphi)_{kj}.
\end{aligned} \tag{3.23}$$

This completes the proof of Lemma 1. Replacing φ by $|\varphi|^2\varphi$ in Lemma 1, we also obtain Theorem 1.

4. The principal symbol of the Type IIA equation

Our next task is to identify the symbol and the eigenvalues of the Type IIA equation.

4.1. Almost-complex structures and 3-forms

For this and for the rest of the paper, we make essential use of Hitchin's construction of an almost-complex structure J_φ and a metric g_φ from a 3-form φ on a 6-dimensional manifold [30]. We begin by recalling the results that we need.

4.1.1. Hitchin's construction. Let V be a 6-dimensional oriented vector space over $\mathbf{R}$. Following Hitchin [30][3], for any 3-form $\varphi \in \Lambda^3 V^*$, one can define a linear map $K_\varphi : V \to \Lambda^5 V^* \cong V \otimes \Lambda^6 V^*$ by

$$K_\varphi(v) = -\iota_v\varphi \wedge \varphi = -e_i \otimes e^i \wedge \iota_v\varphi \wedge \varphi, \tag{4.1}$$

where $\{e_i\}$ is an arbitrary basis of V and $\{e^i\}$ its dual basis in V^*. It follows that

$$\lambda_\varphi := \frac{1}{6}\mathrm{tr}_V K_\varphi^2 \in (\Lambda^6 V^*)^2 \tag{4.2}$$

is well-defined and it makes sense to talk about the sign of λ_φ. In general λ_φ is a homogeneous degree 4 polynomial in the components of φ. When $\lambda_\varphi < 0$, as V is oriented, one can take $\sqrt{-\lambda_\varphi} \in \Lambda^6 V^*$ to be the positive square root of $-\lambda_\varphi$. It is proved in [30]

$$J_\varphi := \frac{K_\varphi}{\sqrt{-\lambda_\varphi}} : V \to V \tag{4.3}$$

defines a complex structure on V. Note that $\lambda_\varphi < 0$ is an open condition. In fact, the set $\{\varphi \in \Lambda^3 V^* : \lambda_\varphi < 0\}$ forms an open orbit in $\Lambda^3 V^*$ of the natural $GL(V)$-action. Furthermore, there is a basis $\{e^i\}$ of V^* where φ takes the following "canonical form",

$$\varphi = \mathrm{Re}(e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) = e^{135} - e^{146} - e^{245} - e^{236} \tag{4.4}$$

and e^{123456} defines a positive volume form. In this basis, one can easily check that $J_\varphi e_{2k-1} = e_{2k}$ and $J_\varphi e_{2k} = -e_{2k-1}$ for $k = 1, 2, 3$. Therefore φ is the

[3] Here we adopt the convention used in [15, 16].

real part of a $(3,0)$-form with respect to the complex structure J_φ. It follows that the imaginary part

$$\hat\varphi = \mathrm{Im}(e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) = e^{136} + e^{145} + e^{235} - e^{246} \tag{4.5}$$

is also determined by φ through $\hat\varphi = J_\varphi \varphi$, meaning that

$$\hat\varphi(X, Y, Z) = \varphi(J_\varphi X, J_\varphi Y, J_\varphi Z)$$

for any $X, Y, Z \in V$. Furthermore two forms φ and $\tilde\varphi$ define the same complex structure J_φ if and only if they are related by $\mathbf{C}^*$-action

$$\tilde\varphi = \rho \cdot \mathrm{Re}(e^{-i\theta}(\varphi + i\hat\varphi)).$$

4.1.2. J_φ and symplectic structures. Now let us assume that V is equipped with a symplectic form $\omega \in \Lambda^2 V^*$ so V is canonically oriented by $\omega^3/3!$. A natural question is when the symplectic form ω is invariant under the induced complex structure J_φ. The answer is very simple:

Lemma 3. *ω is J_φ-invariant if and only if φ is primitive in the sense that $\omega \wedge \varphi = 0$.*

Proof. As stated above, we can choose a basis such that $\varphi = e^{135} - e^{146} - e^{245} - e^{236}$. In this coordinate, we may write $\omega = a_{ij} e^i \wedge e^j$ with $a_{ij} = -a_{ji}$. The condition that $\omega(e_i, e_j) = \omega(J_\varphi e_i, J_\varphi e_j)$ for any i and j is equivalent to the following system of linear equations

$$\begin{aligned} &a_{13} = a_{24}, \qquad a_{14} = -a_{23}, \qquad a_{15} = a_{26}, \\ &a_{16} = -a_{25}, \qquad a_{35} = a_{46}, \qquad a_{36} = -a_{45}. \end{aligned} \tag{4.6}$$

These are exactly the equations for φ being primitive in the sense that $\omega \wedge \varphi = 0$. Q.E.D.

For primitive φ, we can consider then the Hermitian form $g_\varphi(X, Y) = \omega(X, J_\varphi Y)$. We shall say that φ is positive if g_φ is positive, in which case g_φ is a metric, and the triple $(\omega, J_\varphi, g_\varphi)$ is compatible[4]. The positivity of φ is an open condition. Once we have a Riemannian metric and an orientation,

[4] This notion of positivity is different from the one defined in [30], which does not involve a symplectic form.

we have the associated Hodge star operator $\star$. It is straightforward to check that

$$\star\varphi = \hat{\varphi}, \quad \star\hat{\varphi} = -\varphi. \tag{4.7}$$

In particular we know that $\hat{\varphi}$ is primitive if φ is primitive.

Altogether, assuming the presence of ω and that φ is primitive and positive, we may upgrade the previous choice of orthonormal basis $\{e_j\}_{j=1}^6$ to the following useful statement:

Lemma 4. *(Normal form of φ)*
There exists an orthonormal basis $\{e_j\}_{j=1}^6$ of V (with respect to g) such that

$$\omega = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6, \tag{4.8}$$

$$\varphi = M\mathrm{Re}(e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6), \tag{4.9}$$

where $M = \frac{1}{2}|\varphi| > 0$. It follows that

$$\sqrt{-\lambda_\varphi} = \frac{1}{2}|\varphi|^2 \frac{\omega^3}{3!}. \tag{4.10}$$

Using this upgraded version of canonical form of φ, one can check that the following key formula for the metric g_φ holds:

Lemma 5. *In any coordinate system, we can write*

$$(g_\varphi)_{ij} = -|\varphi|^{-2}\varphi_{iab}\varphi_{jcd}\omega^{ac}\omega^{bd} = 2|\varphi|^{-2}\frac{\iota_i\varphi \wedge \iota_j\varphi \wedge \omega}{\omega^3/3!}. \tag{4.11}$$

The metric $\tilde{g}_\varphi$ introduced in (2.3) is then given by

$$(\tilde{g}_\varphi)_{ij} = -\varphi_{iab}\varphi_{jcd}\omega^{ac}\omega^{bd}. \tag{4.12}$$

Clearly $\tilde{g}_\varphi$ is conformal to g_φ and its associated Kähler form is $\tilde{\omega}_\varphi = |\varphi|^2\omega$. Since $|\varphi|^2$ is the square root of a complicated homogeneous degree 4 polynomial in components of φ, the metric $\tilde{g}_\varphi$ has the advantage that its expression is algebraic in φ, which makes it much easier to compute with. Also, the volume form of g_φ is just $\omega^3/3!$, but we can recapture $|\varphi|^2$ from the volume form of $\tilde{g}_\varphi$.

4.1.3. Basic identities. As always we fix a symplectic form ω. Let φ be a positive primitive 3-form so that it defines an almost complex structure J_φ compatible with ω. For simplicity, we shall denote in the subsequent calculations g_φ and J_φ by just g and J. Furthermore we can define another primitive 3-form $\hat\varphi = J\varphi = \star\varphi$ such that $\Omega = \varphi + i\hat\varphi$ is a nowhere vanishing (3,0)-form with respect to J, i.e. a complex volume form that trivializes the canonical bundle of (M, J).

Lemma 6. *The* 3*-forms* φ *and* $\hat\varphi$ *are related to each other by*

$$\begin{aligned}\varphi_{ijk} &= \hat\varphi_{Ji,j,k} = \hat\varphi_{i,Jj,k} = \hat\varphi_{i,j,Jk} = -\varphi_{Ji,Jj,k} = -\varphi_{Ji,j,Jk} \\ &= -\varphi_{i,Jj,Jk} = -\hat\varphi_{Ji,Jj,Jk}\end{aligned}$$

and

$$\begin{aligned}\hat\varphi_{ijk} &= -\varphi_{Ji,j,k} = -\varphi_{i,Jj,k} = -\varphi_{i,j,Jk} = -\hat\varphi_{Ji,Jj,k} = -\hat\varphi_{Ji,j,Jk} \\ &= -\hat\varphi_{i,Jj,Jk} = \varphi_{Ji,Jj,Jk}.\end{aligned}$$

Proof. Since $\varphi + i\hat\varphi$ is of type (3,0), we have $\iota_{\partial_k + iJ\partial_k}(\varphi + i\hat\varphi) = 0$. By taking the real and imaginary parts of the above equation and its iterations gives the desired identities. Q.E.D.

Since φ and $\hat\varphi$ are type $(3,0)+(0,3)$-forms, for any 1-form μ, we know that both $\mu\wedge\varphi$ and $\mu\wedge\hat\varphi$ are of type $(3,1)+(1,3)$, so

$$\mu\wedge\varphi = -J(\mu\wedge\varphi) = -J\mu\wedge\hat\varphi. \tag{4.13}$$

It is not hard to verify that wedging with φ or $\hat\varphi$ gives an isomorphism from the space of real 1-forms to the space of real $(3,1)+(1,3)$-forms. Note that the primitiveness of φ with respect to ω implies the primitiveness of φ with respect to $\tilde\omega$, and hence $\tilde\omega^{ji}\varphi_{ijk} = 0$, or equivalently,

$$\begin{aligned}&\tilde\omega_{ij}\varphi_{klm} - \tilde\omega_{kj}\varphi_{ilm} - \tilde\omega_{lj}\varphi_{kim} - \tilde\omega_{mj}\varphi_{kli} - \tilde\omega_{ik}\varphi_{jlm} \\ &-\tilde\omega_{il}\varphi_{kjm} - \tilde\omega_{im}\varphi_{klj} + \tilde\omega_{kl}\varphi_{ijm} + \tilde\omega_{ml}\varphi_{kji} + \tilde\omega_{mk}\varphi_{jli} = 0.\end{aligned} \tag{4.14}$$

We also have the following simple lemma:

Lemma 7. *The following are equivalent:*

(a) $d\hat\varphi = 0$*;*

(b) *The almost-complex structure* J *is integrable.*

In both cases, the $(3,0)$*-form* Ω *is holomorphic and the form* φ *is harmonic.*

Proof. If J is integrable, then $\partial\Omega = 0$ since it is a $(4,0)$-form in a 3-dimensional complex manifold. Similarly for $\bar{\partial}\bar{\Omega}$. But $\varphi = \frac{1}{2}(\Omega + \bar{\Omega})$, so $d\varphi = 0$ implies $0 = \bar{\partial}\Omega + \partial\bar{\Omega}$, and hence Ω is holomorphic and $d\Omega = 0$. This implies that $d\hat{\varphi} = 0$.

Conversely, if $d\hat{\varphi} = 0$, then we know that $\Omega = \varphi + \sqrt{-1}\hat{\varphi}$ is also closed. Therefore for any $(1,0)$-form λ, we know $\lambda \wedge \Omega = 0$, hence

$$0 = d(\lambda \wedge \Omega) = d\lambda \wedge \Omega,$$

so we deduce that $d\lambda$ has no $(0,2)$-components, which implies that J is integrable by Frobenius theorem. Q.E.D.

The defining equation for $\tilde{g}$ tells us the effect of contracting twice with ω^{ij} a quadratic polynomial in φ. It actually follows from a stronger identity with only one contraction, and which can be verified explicitly using the normal form of φ in Lemma 4. Using the fact that $g^{ij} = \omega^{i,Jj}$, we can readily deduce the effect of contracting with g^{ij}. We summarize these contractions in the following lemma:

Lemma 8. *The following quadratic identities hold:*

$$\omega^{ij}\varphi_{iab}\varphi_{jcd} = \frac{1}{4}(\omega_{ac}\tilde{g}_{bd} - \omega_{bc}\tilde{g}_{ad} - \omega_{ad}\tilde{g}_{bc} + \omega_{bd}\tilde{g}_{ac}) \tag{4.15}$$

$$g^{ij}\varphi_{iab}\varphi_{jcd} = \frac{1}{4}(g_{ac}\tilde{g}_{bd} - g_{bc}\tilde{g}_{ad} + \omega_{ad}\tilde{\omega}_{bc} - \omega_{bd}\tilde{\omega}_{ac}) \tag{4.16}$$

$$g^{ac}g^{bd}\varphi_{iab}\varphi_{jcd} = \tilde{g}_{ij}. \tag{4.17}$$

4.2. The variation $\delta(\hat{\varphi})$

The key variational formula for $\hat{\varphi}$ is given by the following lemma:

Lemma 9.

$$\delta(\hat{\varphi}) = -J_\varphi(\delta\varphi) + 2\frac{\delta\varphi \wedge \varphi}{\varphi \wedge \hat{\varphi}}\varphi + 2\frac{\delta\varphi \wedge \hat{\varphi}}{\varphi \wedge \hat{\varphi}}\hat{\varphi}. \tag{4.18}$$

When φ is primitive and positive with respect to a symplectic form ω, the above formula can be written as

$$\delta(\hat{\varphi}) = -J_\varphi(\delta\varphi) - \frac{2(\delta\varphi, \hat{\varphi})}{|\varphi|^2}\varphi + \frac{2(\delta\varphi, \varphi)}{|\varphi|^2}\hat{\varphi}. \tag{4.19}$$

Proof. This is a purely linear algebra problem. We break its proof into two steps.

• Step 1: The primitive case.

Choose a nondegenerate (1,1)-form ω compatible with J_φ. We shall first prove Lemma 9 under the assumption that $\delta\varphi$ is primitive with respect to ω. By Lemma 4, one can find an orthonormal basis $\{e^i\}_{i=1}^6$ of V^* such that

$$\begin{aligned}
\omega &= e^1\wedge e^2 + e^3\wedge e^4 + e^5\wedge e^6,\\
\varphi &= M(e^1\wedge e^3\wedge e^5 - e^1\wedge e^4\wedge e^6 - e^2\wedge e^4\wedge e^5 - e^2\wedge e^3\wedge e^6),\\
\star\varphi &= M(e^1\wedge e^3\wedge e^6 + e^1\wedge e^4\wedge e^5 + e^2\wedge e^3\wedge e^5 - e^2\wedge e^4\wedge e^6),
\end{aligned}$$

where $M = \frac{1}{2}|\varphi| > 0$. For simplicity, we denote $e^{123456} = \omega^3/3!$ by ϵ. Let $\delta\varphi = \mu = \frac{1}{3!}\mu_{ijk}e^i\wedge e^j\wedge e^k$. Straightforward computation gives us

$$\delta(K_\varphi) = 2M\epsilon \times
\begin{bmatrix}
A_1 & \mu_{135}-\mu_{146} & -\mu_{125} & \mu_{126} & \mu_{123} & -\mu_{124}\\
\mu_{236}+\mu_{245} & -A_1 & \mu_{126} & \mu_{125} & -\mu_{124} & -\mu_{123}\\
\mu_{345} & -\mu_{346} & A_2 & \mu_{135}-\mu_{236} & -\mu_{134} & \mu_{234}\\
-\mu_{346} & -\mu_{345} & \mu_{245}+\mu_{146} & -A_2 & \mu_{234} & \mu_{134}\\
-\mu_{356} & \mu_{456} & \mu_{156} & -\mu_{256} & A_3 & \mu_{135}-\mu_{245}\\
\mu_{456} & \mu_{356} & -\mu_{256} & -\mu_{156} & \mu_{236}+\mu_{146} & -A_3
\end{bmatrix},$$

where

$$\begin{aligned}
A_1 &= \frac{\mu_{246}+\mu_{136}+\mu_{145}-\mu_{235}}{2},\\
A_2 &= \frac{\mu_{246}+\mu_{136}-\mu_{145}+\mu_{235}}{2},\\
A_3 &= \frac{\mu_{246}-\mu_{136}+\mu_{145}+\mu_{235}}{2}.
\end{aligned}$$

It follows that

$$\delta(K_\varphi^2) = 4M^3\epsilon^2(\mu_{236}+\mu_{245}-\mu_{135}+\mu_{146})\mathrm{Id}_V = 4M^2\epsilon^2\frac{\hat\varphi\wedge\mu}{\epsilon}\mathrm{Id}_V. \tag{4.20}$$

Therefore

$$\delta(-\lambda_\varphi) = 4M^2\epsilon^2\frac{\mu\wedge\hat\varphi}{\epsilon} = 4M^2\epsilon^2(\mu,\varphi) \tag{4.21}$$

$$\delta\sqrt{-\lambda_\varphi} = \frac{1}{2}\delta(|\varphi|^2)\epsilon = \mu\wedge\hat\varphi = (\delta\varphi,\varphi)\epsilon, \tag{4.22}$$

which agrees with Hitchin's formula [30, Proposition 4]. As a consequence, we have

$$\delta(J_\varphi)=\frac{\delta(K_\varphi)}{2M^2\epsilon}-\frac{\mu\wedge\hat{\varphi}}{2M^2\epsilon}J_\varphi. \tag{4.23}$$

For simplicity of notation, we introduce the following

$$\begin{aligned} F &:= \frac{\delta(K_\varphi)}{2M\epsilon},\\ \varphi_0 &:= e^1\wedge e^3\wedge e^5-e^1\wedge e^4\wedge e^6-e^2\wedge e^4\wedge e^5-e^2\wedge e^3\wedge e^6,\\ \hat{\varphi}_0 &:= e^1\wedge e^3\wedge e^6+e^1\wedge e^4\wedge e^5+e^2\wedge e^3\wedge e^5-e^2\wedge e^4\wedge e^6,\\ dz^k &:= e^{2k-1}+ie^{2k},\ k=1,2,3. \end{aligned}$$

Again by straightforward calculation, we get

$$\begin{aligned} F.dz^1 &= iB_0dz^1+(A_1+iB_1)d\bar{z}^1+C_3d\bar{z}^2+C_2d\bar{z}^3,\\ F.dz^2 &= C_3d\bar{z}^1+iB_0dz^2+(A_2+iB_2)d\bar{z}^2+C_1d\bar{z}^3,\\ F.dz^3 &= C_2d\bar{z}^1+C_1d\bar{z}^2+iB_0dz^3+(A_3+iB_3)d\bar{z}^3, \end{aligned}$$

where

$$\begin{aligned} B_0 &= \frac{1}{2}(\mu_{135}-\mu_{146}-\mu_{236}-\mu_{245})=\frac{\mu\wedge\hat{\varphi}_0}{2\epsilon},\\ B_1 &= \frac{1}{2}(\mu_{135}-\mu_{146}+\mu_{236}+\mu_{245}),\\ B_2 &= \frac{1}{2}(\mu_{135}+\mu_{146}-\mu_{236}+\mu_{245}),\\ B_3 &= \frac{1}{2}(\mu_{135}+\mu_{146}+\mu_{236}-\mu_{245}),\\ C_1 &= \mu_{156}-i\mu_{256}=-\mu_{134}+i\mu_{234},\\ C_2 &= -\mu_{356}+i\mu_{456}=\mu_{123}-i\mu_{124},\\ C_3 &= \mu_{345}-i\mu_{346}=-\mu_{125}+i\mu_{126}. \end{aligned}$$

Here to obtain expressions of C_j one makes use of primitiveness of μ. For completeness we also introduce

$$A_0=\frac{\mu_{246}-\mu_{235}-\mu_{136}-\mu_{145}}{2}=\frac{\mu\wedge\varphi_0}{2\epsilon}.$$

Collecting all these together, we get

$$F.(\varphi_0+i\hat{\varphi}_0)=F.(dz^1\wedge dz^2\wedge dz^3) \tag{4.24}$$

$$
\begin{aligned}
&= F.(dz^1)\wedge dz^2\wedge dz^3 + dz^1\wedge F.(dz^2)\wedge dz^3 + dz^1\wedge dz^2\wedge F.(dz^3)\\
&= 3iB_0 dz^1\wedge dz^2\wedge dz^3 + (A_1+iB_1)d\bar z^1\wedge dz^2\wedge dz^3\\
&\quad +(A_2+iB_2)dz^1\wedge d\bar z^2\wedge dz^3 + (A_3+iB_3)dz^1\wedge dz^2\wedge d\bar z^3\\
&\quad +C_1 dz^1\wedge(d\bar z^3\wedge dz^3 + dz^2\wedge d\bar z^2) - C_2 dz^2\wedge(d\bar z^3\wedge dz^3 + dz^1\wedge d\bar z^1)\\
&\quad +C_3 dz^3\wedge(dz^1\wedge d\bar z^1 + d\bar z^2\wedge dz^2).
\end{aligned}
$$

Notice we can express μ as

$$
\begin{aligned}
\mu \;=\; & -\frac{i}{4}\Big[(C_1 dz^1 + \overline{C_1} d\bar z^1)\wedge(d\bar z^3\wedge dz^3 + dz^2\wedge d\bar z^2)\\
& -(C_2 dz^2 + \overline{C_2} d\bar z^2)\wedge(d\bar z^3\wedge dz^3 + dz^1\wedge d\bar z^1)\\
& +(C_3 dz^3 + \overline{C_3} d\bar z^3)\wedge(dz^1\wedge d\bar z^1 + d\bar z^2\wedge dz^2)\Big] + \frac{B_0}{2}\varphi_0 - \frac{A_0}{2}\hat\varphi_0\\
& -\frac{i}{4}(A_1+iB_1)d\bar z^1\wedge dz^2\wedge dz^3 + \frac{i}{4}(A_1-iB_1)dz^1\wedge d\bar z^2\wedge d\bar z^3\\
& -\frac{i}{4}(A_2+iB_2)dz^1\wedge d\bar z^2\wedge dz^3 + \frac{i}{4}(A_2-iB_2)d\bar z^1\wedge dz^2\wedge d\bar z^3\\
& -\frac{i}{4}(A_3+iB_3)dz^1\wedge dz^2\wedge d\bar z^3 + \frac{i}{4}(A_3-iB_3)d\bar z^1\wedge d\bar z^2\wedge dz^3.
\end{aligned}
$$

By taking real part of (4.24) we get

$$F.\varphi_0 \;=\; 2J_\varphi\mu - A_0\varphi_0 - 4B_0\hat\varphi_0.$$

Since

$$
\begin{aligned}
J_\varphi.\varphi_0 &= \varphi_0(J_\varphi\cdot,\cdot,\cdot) + \varphi_0(\cdot,J_\varphi\cdot,\cdot) + \varphi_0(\cdot,\cdot,J_\varphi\cdot) = -3\hat\varphi_0,\\
\delta(J_\varphi) &= \frac{1}{M}(F - B_0 J_\varphi),
\end{aligned}
$$

it follows that

$$\delta(J_\varphi).\varphi_0 \;=\; \frac{1}{M}(2J_\varphi\mu - A_0\varphi_0 - B_0\hat\varphi_0). \tag{4.25}$$

Consequently we obtain

$$\delta\hat\varphi = J_\varphi\delta\varphi - \delta(J_\varphi).\varphi = -J_\varphi\mu + A_0\varphi_0 + B_0\hat\varphi_0.$$

Rewrite this equation in a coordinate-free manner, we obtain the desired formula.

- Step 2: The general case.

Choosing ω as before, a general variation $\delta\varphi$ takes the form

$$\delta\varphi = \mu + \omega \wedge \lambda,$$

where μ is primitive with respect to ω and λ is a 1-form. By linearity, we only need to prove our formula for $\delta\varphi = \omega \wedge \lambda$. By symmetry, we may assume that $\lambda = Ne^1$ for some number N. Therefore $\omega \wedge \lambda = Ne^1 \wedge (e^{34} + e^{56})$ is a linear combination of $e^1 \wedge (e^{34} - e^{56})$ and $e^1 \wedge (Ke^{34} - e^{56})$ for some constant $K > 1$. Notice that $e^1 \wedge (e^{34} - e^{56})$ is primitive with respect to ω, and $e^1 \wedge (Ke^{34} - e^{56})$ is primitive with respect to another J_φ-compatible $(1,1)$-form $\omega' = e^{12} + Ke^{34} + e^{56}$. By linearity we reduce Step 2 to Step 1 with ω replaced by ω'. Q.E.D.

4.3. The eigenvalues of the principal symbol

With all preparations from last section, we are now ready to compute the principal symbol of the Type IIA flow (1.4) without source. When φ is primitive, the right hand side of the Type IIA flow is also primitive, so we only need to consider primitive variations in Lemma 9, which takes the form

$$\delta(|\varphi|^2\hat{\varphi}) = -|\varphi|^2 J(\delta\varphi) - 2(\delta\varphi, \hat{\varphi})\varphi + 4(\delta\varphi, \varphi)\hat{\varphi}. \tag{4.26}$$

Thus the symbol of the leading term in the Type IIA flow is given by

$$\delta\varphi \mapsto \xi \wedge \Lambda\big\{\xi \wedge (-|\varphi|^2 J(\delta\varphi) - 2(\delta\varphi, \hat{\varphi})\varphi + 4(\delta\varphi, \varphi)\hat{\varphi})\big\} \tag{4.27}$$

and whether the flow is parabolic or not, depends on the eigenvalues of this operator. Since by our assumption, the right hand side of the flow is primitive and admits an integrability operator d. Thus by the Hamilton-Nash-Moser theorem [29], we can restrict $\delta\varphi$ to the space

$$W = \{\delta\varphi \in \Lambda^3 V^* : \xi \wedge \delta\varphi = 0, \Lambda(\delta\varphi) = 0\}.$$

As before, we may choose an orthonormal basis $\{e^i\}_{i=1}^6$ of V^* such that

$$\begin{aligned}
\omega &= e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6, \\
\varphi &= M(e^1 \wedge e^3 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^4 \wedge e^5 - e^2 \wedge e^3 \wedge e^6), \\
\star\varphi &= M(e^1 \wedge e^3 \wedge e^6 + e^1 \wedge e^4 \wedge e^5 + e^2 \wedge e^3 \wedge e^5 - e^2 \wedge e^4 \wedge e^6).
\end{aligned}$$

As we only care about the sign of the eigenvalues of the principal symbol, we may assume that $|\xi| = 2M = 1$. Moreover by rotational symmetry we may assume that $\xi = e^1$. Under such reduction, it is easy to see that

$$W = \{e^1 \wedge \gamma : \gamma \in \Lambda^2(V')^*, \Lambda'\gamma = 0\},$$

where $V' = \mathrm{span}\{e_j\}_{j=3}^6$ equipped with the symplectic form $\omega' = e^3 \wedge e^4 + e^5 \wedge e^6$ and Λ' is the contraction with respect to ω'. In this way we can also simplify the operator to

$$\begin{aligned} e^1 \wedge \gamma \quad \mapsto \quad & e^1 \wedge \Big[J\gamma + \frac{1}{2}(\gamma, e^3 \wedge e^6 + e^4 \wedge e^5)(e^3 \wedge e^6 + e^4 \wedge e^5) \\ & + (\gamma, e^{35} - e^{46})(e^{35} - e^{46}) \Big], \end{aligned}$$

which is equivalent to

$$\gamma \mapsto J\gamma + \frac{1}{2}(\gamma, e^3 \wedge e^6 + e^4 \wedge e^5)(e^3 \wedge e^6 + e^4 \wedge e^5) + (\gamma, e^{35} - e^{46})(e^{35} - e^{46}). \tag{4.28}$$

Then it is clear that the eigenvalues are $\lambda = 1$ (multiplicity 4) with eigenvectors $\gamma = e^3 \wedge e^4 - e^5 \wedge e^6, e^3 \wedge e^5 + e^4 \wedge e^6, e^3 \wedge e^6 - e^4 \wedge e^5$ and $e^3 \wedge e^5 - e^4 \wedge e^6$, $\lambda = 0$ with eigenvector $\gamma = e^3 \wedge e^6 + e^4 \wedge e^5$. We summarize our findings in the following lemma:

Lemma 10. *The leading symbol in the Type IIA flow, restricted to closed and primitive forms, is only weakly parabolic. More precisely, it has an eigenvalue $\lambda = 1$ with multiplicity 4, and an eigenvalue $\lambda = 0$ with multiplicity 1.*

5. Proof of Theorem 2: existence

In this section we establish the short-time existence of the Type IIA flow. As we saw in §4, the flow is not strictly parabolic, and the presence of the symplectic form prevents a direct application of either the reparametrization arguments of [14] or the Hamilton-Nash-Moser theorem of [29]. Rather, we proceed as follows: first we do apply a reparametrization, but we have to accompany it at the same time with a flow of the symplectic form. This new coupled flow of (φ, ω) is still not strictly parabolic, but one of its key properties is that it admits a strictly regularization with integrability condition, to which the Hamilton-Nash-Moser theorem can apply. Thus we obtain the short-time existence for a regularized version of the Type IIA flow. Next, we show that the regularized flow preserves the primitiveness of the data, and reduces to the Type IIA flow if the form φ is known to be primitive. Altogether, we obtain the desired short-time existence of the Type IIA flow for primitive data. The uniqueness of the solution will be shown later in §7.5, as a consequence of the uniqueness of the flows of the corresponding metrics.

5.1. A coupled flow for (φ, ω)

More precisely, we consider a reparametrization of the Type IIA flow by the following time-dependent vector field

$$V^k = e^u \left(g^{pq}(\Gamma^k_{pq} - (\Gamma_0)^k_{pq}) - g^{lk} u_l \right), \tag{5.1}$$

where $|\varphi|$, u and g are defined by

$$|\varphi|^2 \frac{\omega^3}{3!} = \varphi \wedge \hat{\varphi}, \quad u = \log|\varphi|^2, \quad g_{ij} = -|\varphi|^{-2} \varphi_{iab} \varphi_{jcd} \omega^{ac} \omega^{bd},$$

and Γ and Γ_0 are Christoffel symbols associated to the evolving metric g and the initial metric g_0. Under a reparametrization by the diffeomorphisms generated by the vector field V^k, the given symplectic form in the Type IIA flow would become time-dependent and evolve by its Lie derivative. It is convenient to change notation slightly, and denote the given symplectic form by ω_0 while reserving $\omega = \omega(t)$ for the evolving symplectic form. This consideration inspires us to consider the following coupled flow for the pair (φ, ω),

$$\begin{cases} \partial_t \varphi = d\Lambda d(|\varphi|^2 \hat{\varphi}) + d(\iota_V \varphi) \\ \partial_t \omega = d(\iota_V \omega) \end{cases} \tag{5.2}$$

with initial data $\varphi(0) = \varphi_0$, $\omega(0) = \omega_0$, where φ_0 would be a closed primitive positive 3-form with respect to ω_0. Although the initial metric g_0 is almost Kähler, a priori we should not assume that $\varphi(t)$ is primitive with respect to $\omega(t)$, hence $g(t)$ a priori may not even be almost Hermitian.

Our first task is to work out the eigenvalues of the principal symbol for this coupled flow. Note that because of the coupling, the principal symbol of (5.2) is now a linear operator acting on both $\delta\varphi$ and $\delta\omega$, and not just on

$\delta\varphi$, and we may no longer assume that $\delta\varphi$ is primitive. It is easy to see that the principal symbol of the linearization of (5.2) is determined by

$$(\delta\varphi,\delta\omega)\ \to\ (d\Lambda d\,\delta(|\varphi|^2\hat\varphi)+d(\iota_{\delta V}\varphi),\ d(\iota_{\delta V}\omega)). \tag{5.3}$$

Now the leading order term in δV is

$$e^u g^{pq}g^{kl}(\nabla_p(\delta g)_{lq}-\frac{1}{2}\nabla_l(\delta g)_{pq})-g^{lk}\nabla_l\delta(e^u),$$

so if we define the vector field W_ξ by

$$W_\xi^k=|\varphi|^2g^{pq}g^{kl}(\xi_p(\delta g)_{lq}-\frac{1}{2}\xi_l(\delta g)_{pq})-g^{lk}\xi_l\delta|\varphi|^2,$$

it follows immediately that the principal symbol of the linearized operator is

$$(\delta\varphi,\delta\omega)\mapsto(\xi\wedge\Lambda(\xi\wedge\delta(|\varphi|^2\hat\varphi))+\xi\wedge\iota_{W_\xi}\varphi,\xi\wedge\iota_{W_\xi}\omega) \tag{5.4}$$

with integrability conditions $\xi\wedge\delta\varphi=\xi\wedge\delta\omega=0$.

We work out more explicitly the symbol at a point (φ,ω) where φ is primitive with respect to ω. In this case, we may choose an orthonormal basis of g such that

$$\begin{aligned}\omega &= e^{12}+e^{34}+e^{56},\\ \varphi &= \frac{|\varphi|}{2}(e^{135}-e^{146}-e^{245}-e^{236}),\quad \hat\varphi=\frac{|\varphi|}{2}(e^{136}+e^{145}+e^{235}-e^{246}).\end{aligned}$$

Without loss of generality, we may further assume that $\xi=e^1$ and $|\varphi|=1$. In this case, we can write $\delta\varphi=e^1\wedge\gamma$ and $\delta\omega=e^1\wedge\alpha$ for some 2-form γ and 1-form α such that $\alpha,\gamma\in\bigwedge^*\{e^2,\dots,e^6\}$. It is straightforward to check that

$$\begin{aligned}\delta|\varphi|^2 &= 2(\delta\varphi,\varphi)-|\varphi|^2(\delta\omega,\omega)=(\gamma,e^{35}-e^{46})-(\alpha,e^2),\\ \delta(|\varphi|^2\hat\varphi) &= -|\varphi|^2J(\delta\varphi)-2(\delta\varphi,\hat\varphi)\varphi+4(\delta\varphi,\varphi)\hat\varphi-|\varphi|^2(\delta\omega,\omega)\hat\varphi\\ &= e^2\wedge J\gamma-(\gamma,e^{36}+e^{45})\varphi+(2(\gamma,e^{35}-e^{46})-(\alpha,e^2))\hat\varphi\\ W_\xi^k &= (\delta g)_{k1}-\delta_1^k\left(\frac{1}{2}\mathrm{tr}_g\delta g+(\gamma,e^{35}-e^{46})-(\alpha,e^2)\right).\end{aligned} \tag{5.5}$$

We see that the key is to compute δg, especially $(\delta g)_{k1}$. By definition of $\tilde{g}$ and straightforward calculation, we have

$$
\begin{aligned}
&(\delta\tilde{g})_{11} = 2(\gamma, e^{35} - e^{46}), \quad (\delta\tilde{g})_{22} = 0, \quad (\delta\tilde{g})_{12} = -(\gamma, e^{36} + e^{45}),\\
&(\delta\tilde{g})_{33} = (\delta\tilde{g})_{55} = 2(\gamma, e^{35}) - (\alpha, e^2), \ (\delta\tilde{g})_{44} = (\delta\tilde{g})_{66} = -2(\gamma, e^{46}) - (\alpha, e^2),\\
&(\delta\tilde{g})_{13} = (\gamma, e^{26}) + \frac{1}{2}(\alpha, e^4), \quad (\delta\tilde{g})_{14} = (\gamma, e^{25}) - \frac{1}{2}(\alpha, e^3),\\
&(\delta\tilde{g})_{15} = -(\gamma, e^{24}) + \frac{1}{2}(\alpha, e^6), \quad (\delta\tilde{g})_{16} = -(\gamma, e^{23}) - \frac{1}{2}(\alpha, e^5),\\
&(\delta g)_{ij} = (\delta\tilde{g})_{ij} - ((\gamma, e^{35} - e^{46}) - (\alpha, e^2))\delta_{ij}.
\end{aligned}
$$

It follows that $\mathrm{tr}_g \delta g = 2(\alpha, e^2)$, hence

$$
\begin{aligned}
W_\xi^1 &= (\alpha, e^2), & W_\xi^2 &= -(\gamma, e^{36} + e^{45})\\
W_\xi^3 &= (\gamma, e^{26}) + \frac{1}{2}(\alpha, e^4), & W_\xi^4 &= (\gamma, e^{25}) - \frac{1}{2}(\alpha, e^3),\\
W_\xi^5 &= -(\gamma, e^{24}) + \frac{1}{2}(\alpha, e^6), & W_\xi^6 &= -(\gamma, e^{23}) - \frac{1}{2}(\alpha, e^5).
\end{aligned}
$$

Consequently we find that

$$
\begin{aligned}
\xi \wedge \iota_{W_\xi}\varphi &= \frac{1}{2}e^1 \wedge \Big[(\gamma_{36} + \gamma_{45})(e^{36} + e^{45}) + (\gamma_{26} + \frac{\alpha_4}{2})e^{26} + (\gamma_{25} - \frac{\alpha_3}{2})e^{25}\\
&\quad + (\gamma_{24} - \frac{\alpha_6}{2})e^{24} + (\gamma_{23} + \frac{\alpha_5}{2})e^{23} + \alpha_2(e^{35} - e^{46})\Big],\\
\xi \wedge \iota_{W_\xi}\omega &= \frac{1}{2}e^1 \wedge (\alpha + \alpha_2 e^2 - 2(\gamma_{25}e^3 - \gamma_{26}e^4 - \gamma_{23}e^5 + \gamma_{24}e^6)).
\end{aligned}
$$

If we further write $\gamma = e^2 \wedge \beta + \lambda$, $\alpha = \alpha_2 e^2 + \mu$, where $\beta, \mu, \lambda \in \bigwedge^*\{e^3, \ldots, e^6\}$, we have

$$
\begin{aligned}
\xi \wedge \iota_{W_\xi}\varphi &= \frac{1}{2}e^1 \wedge \Big[(\lambda_{36} + \lambda_{45})(e^{36} + e^{45}) + e^2 \wedge \beta\\
&\quad - \frac{1}{2}e^2 \wedge \iota_\mu(e^{35} - e^{46}) + \alpha_2(e^{35} - e^{46})\Big],\\
\xi \wedge \iota_{W_\xi}\omega &= \frac{1}{2}e^1 \wedge (\mu + 2\alpha_2 e^2 + 2\iota_\beta(e^{35} - e^{46})),\\
\xi \wedge \Lambda(\xi \wedge \delta(|\varphi|^2\hat{\varphi}) &= e^1 \wedge \Big[J\lambda + \frac{\lambda_{36} + \lambda_{45}}{2}(e^{36} + e^{45})\\
&\quad + (\lambda_{35} - \lambda_{46} - \frac{\alpha_2}{2})(e^{35} - e^{46})\Big].
\end{aligned}
$$

It follows that the principal symbol is the linear map

$$(\beta, \lambda, \alpha_2, \mu) \mapsto \left(\frac{\beta}{2} - \frac{\iota_\mu}{4}(e^{35} - e^{46}), \lambda, \alpha_2, \frac{\mu}{2} + \iota_\beta(e^{35} - e^{46}) \right) \tag{5.6}$$

This matrix is only positive semi-definite. The part $(\lambda, \alpha_2) \mapsto (\lambda, \alpha_2)$ is the identity map. However the other part

$$(\beta, \mu) \mapsto \left(\frac{\beta}{2} - \frac{\iota_\mu}{4}(e^{35} - e^{46}), \frac{\mu}{2} + \iota_\beta(e^{35} - e^{46}) \right)$$

has eigenvalues 0 and 1, both of multiplicities 4. So the coupled flow (5.2) for (φ, ω) is still not strictly parabolic.

5.2. A parabolic regularization of the coupled flow

To solve this problem, we add an extra term on the right hand side of the evolution equation of φ in (5.2). This term takes the form

$$-BdJd(|\varphi|^2 \Lambda \hat{\varphi}),$$

where B is a constant to be determined. In fact, $\Lambda(\hat{\varphi})$ is expected to be zero along the flow as φ should always be primitive. Again let us consider the linearization of $|\varphi|^2 \Lambda \hat{\varphi}$ at a primitive pair (ω, φ) and we may assume that $|\varphi|^2 = 1$ at the point of linearization. It follows that

$$\delta(|\varphi|^2 \Lambda \hat{\varphi}) = (\delta\Lambda)(\hat{\varphi}) - \Lambda(J\delta\varphi). \tag{5.7}$$

As before, we may assume that $\xi = e^1$ and $\delta\varphi = e^1 \wedge (e^2 \wedge \beta + \lambda)$ and $\delta\omega = e^1 \wedge (\alpha_2 e^2 + \mu)$. The principal symbol for the extra term is

$$Be^{12} \wedge J\big[(\delta\Lambda)(\hat{\varphi}) - \Lambda(J\delta\varphi)\big]. \tag{5.8}$$

The second term is easy to compute:

$$-Be^{12} \wedge J(\Lambda(J\delta\varphi)) = Be^{12} \wedge \beta. \tag{5.9}$$

The first term is more complicated, notice that

$$B(\delta\Lambda)(\hat{\varphi})_k = -\frac{B}{2}\omega^{js}(\delta\omega)_{st}\omega^{ti}\hat{\varphi}_{ijk} = B\sum_{t=3}^{6} \mu_t \omega^{ti} \hat{\varphi}_{2ik},$$

therefore by straightforward calculation, this part of the principal symbol is equivalent to the linear map

$$\mu \mapsto \frac{B}{2}\iota_\mu(e^{35} - e^{46}).$$

Therefore the principal symbol for the full evolution equation is equivalent to the linear map

$$(\beta, \lambda, \alpha_2, \mu) \mapsto \left(\frac{\beta}{2}(1 + 2B) + \frac{2B - 1}{4}\iota_\mu(e^{35} - e^{46}), \lambda, \alpha_2, \frac{\mu}{2} + \iota_\beta(e^{35} - e^{46})\right).$$

If $B > 0$, then all the eigenvalues of the above matrix are positive. In this sense, the coupled flow with the additional B term is parabolic.

Lemma 11. *Consider the flow*

$$\begin{aligned} &\partial_t\varphi = d\Lambda d(|\varphi|^2\hat{\varphi}) - BdJd(|\varphi|^2\Lambda\hat{\varphi}) + d(\iota_V\varphi), \\ &\partial_t\omega = d(\iota_V\omega), \end{aligned} \tag{5.10}$$

for any fixed, strictly positive constant B. Then for any initial value φ_0 which is a closed, positive, and primitive form with respect to the initial symplectic form ω_0, the flow exists and is smooth at least on some interval $[0, T)$ with $T > 0$. Clearly the flow preserves the closedness of both the forms φ and ω.

Proof. Let d be the exterior derivative. The preceding fact that the eigenvalues of the principal symbol of the flow (5.10) when restricted to closed and primitive forms are positive means that the flow (5.10) together with d as the integrability operator satisfies the condition of the Hamilton-Nash-Moser theorem ([29], Theorem 5.1). This theorem implies the short-time existence and uniqueness of the flow (5.10). Q.E.D.

It should be noted that we need to treat φ and ω as tensors evolving independently at this moment, therefore we cannot assume that φ is primitive with respect to ω (though we shall prove it is indeed the case later). Consequently the metric g defined above is not necessarily compatible with J or ω: we only know it is a Riemannian metric. As (5.10) preserves the closedness of φ and ω, by performing the reverse reparametrization, we obtain immediately

Lemma 12. *Fix any positive constant B. Then the flow of 3-forms φ*

$$\partial_t\varphi = d\Lambda d(|\varphi|^2\hat{\varphi}) - BdJd(|\varphi|^2\Lambda\hat{\varphi}) \tag{5.11}$$

admits a closed smooth solution φ *on some interval* $[0, T)$ *with* $T > 0$*, for any initial value* φ_0 *which is a smooth closed, positive, and primitive form with respect to the symplectic form* ω_0*.*

5.3. Preservation of the primitiveness condition

Next we shall show that, if the initial data φ_0 is primitive in the flow (5.11), then $\varphi(t)$ remains primitive for all time. Since φ primitive implies that $\hat{\varphi}$ is also primitive, it follows that the terms with coefficient B in (5.11) all drop out, and the flow reduces to the Type IIA flow, establishing Theorem 2 in the case of no sources.

From now on, we take $B = 1$. Let $\varphi(t)$ be a solution to (5.11) on $M \times [0, T)$ with $\varphi(0)$ being closed, positive, and primitive. Clearly for any t, $\varphi(t)$ stays closed. Let

$$\varphi = P + \beta \wedge \omega \tag{5.12}$$

be the primitive decomposition of φ, where P is a primitive 3-form. It follows that

$$\beta = \frac{\Lambda\varphi}{2}.$$

We wish to show that $\beta = 0$ by the maximum principle. To do so, we need to compute the evolution equation of β. We fix a background metric $\bar{g} = g(0)$ which is compatible with ω. We denote by $\bar{\nabla}$ the covariant derivatives with respect to $\bar{g}$. Since φ is closed, $d\beta$ is primitive, and thus

$$\omega^{jk}\bar{\nabla}_j\beta_k = 0. \tag{5.13}$$

Furthermore,

$$(\Lambda\hat{\varphi})_k = \frac{\omega^{ji}}{2}\hat{\varphi}_{ijk} = -\frac{\omega^{ji}}{2}\varphi_{i,j,Jk} = -(\Lambda\varphi)_{Jk},$$

hence there exists a primitive 3-form $\hat{P}$ such that the primitive decomposition for $\hat{\varphi}$ is

$$\hat{\varphi} = \hat{P} - J\beta \wedge \omega. \tag{5.14}$$

Using this decomposition for $\hat{\varphi}$, we can derive the evolution equation for β,

$$\partial_t\beta = -d\Lambda d(|\varphi|^2 J\beta) + \Lambda(dJd(|\varphi|^2 J\beta)). \tag{5.15}$$

We know that φ, J, and all their covariant derivatives are bounded in $M \times [0,\tau]$ for any $\tau < T$, therefore we can write (5.15) in the form

$$\partial_t \beta = |\varphi|^2(-d\Lambda d(J\beta) + \Lambda(dJdJ\beta) + \bar{\nabla}\beta * S_1 + \beta * S_2), \tag{5.16}$$

where S_1 and S_2 are bounded tensors on $M \times [0,\tau]$ and $*$ represents certain contraction of indices. We need to compute the leading term of β in (5.16). Notice that $(J\beta)_j = J^p{}_j\beta_p$, so

$$\begin{aligned}
d(J\beta)_{jk} &= J^p{}_k\bar{\nabla}_j\beta_p - J^p{}_j\bar{\nabla}_k\beta_p + O(\beta),\\
\Lambda d(J\beta) &= \omega^{kj}J^p{}_k\bar{\nabla}_j\beta_p + O(\beta),\\
(d\Lambda d(J\beta))_l &= \omega^{Jp,j}\bar{\nabla}_l\bar{\nabla}_j\beta_p + O(\beta,\bar{\nabla}\beta),\\
(JdJ\beta)_{jk} &= J^s{}_jJ^t{}_k(J^p{}_t\bar{\nabla}_s\beta_p - J^p{}_s\bar{\nabla}_t\beta_p + O(\beta))\\
&= J^t{}_k\bar{\nabla}_t\beta_j - J^t{}_j\bar{\nabla}_t\beta_k + O(\beta),\\
(dJdJ\beta)_{jkl} &= J^t{}_k(\bar{\nabla}_l\bar{\nabla}_t\beta_j - \bar{\nabla}_j\bar{\nabla}_t\beta_l) - J^t{}_j(\bar{\nabla}_l\bar{\nabla}_t\beta_k - \bar{\nabla}_k\bar{\nabla}_t\beta_l)\\
&\quad + J^t{}_l(\bar{\nabla}_j\bar{\nabla}_t\beta_k - \bar{\nabla}_k\bar{\nabla}_t\beta_j) + O(\beta,\bar{\nabla}\beta),\\
\Lambda(dJdJ\beta)_l &= \omega^{kj}J^t{}_k(\bar{\nabla}_l\bar{\nabla}_t\beta_j - \bar{\nabla}_j\bar{\nabla}_t\beta_l) + J^t{}_l\bar{\nabla}_t(\omega^{kj}\bar{\nabla}_j\beta_k) + O(\beta,\bar{\nabla}\beta)\\
&= \omega^{Jj,p}\bar{\nabla}_l\bar{\nabla}_j\beta_p + \omega^{j,Jt}\bar{\nabla}_j\bar{\nabla}_t\beta_l + O(\beta,\bar{\nabla}\beta).
\end{aligned}$$

It follows that

$$\partial_t\beta = |\varphi|^2(L\beta + O(\beta,\bar{\nabla}\beta)), \tag{5.17}$$

where L is defined by

$$(L\beta)_l = (\omega^{Jj,p} + \omega^{j,Jp})\bar{\nabla}_l\bar{\nabla}_j\beta_p + \omega^{j,Jt}\bar{\nabla}_j\bar{\nabla}_t\beta_l.$$

We further notice that $\omega^{Jj,p} + \omega^{j,Jp} = O(\beta)$, $\omega^{j,Jt} - \delta^{jt} = O(\beta)$, and as β is a smooth function of φ, we also have $|\bar{\nabla}^2\beta|$ is uniformly bounded. Therefore one can also write

$$\partial_t\beta = |\varphi|^2(\bar{\Delta}\beta + O(\beta,\bar{\nabla}\beta)). \tag{5.18}$$

It follows that

$$\begin{aligned}
\partial_t|\beta|^2_{\bar{g}} &= 2|\varphi|^2(\bar{\Delta}\beta,\beta)_{\bar{g}} + O(\beta,\bar{\nabla}\beta) * \beta * S\\
&\leq |\varphi|^2\bar{\Delta}(|\beta|^2_{\bar{g}}) - 2|\varphi|^2|\bar{\nabla}\beta|^2 + O(\beta,\bar{\nabla}\beta) * \beta * S\\
&\leq |\varphi|^2\bar{\Delta}(|\beta|^2_{\bar{g}}) + C|\beta|^2.
\end{aligned} \tag{5.19}$$

Since $\beta = 0$ initially, by maximum principle we know that $\beta = 0$ on $[0, \tau]$ for any $\tau < T$. Therefore $\varphi(t)$ is primitive for as long as the flow exists, and J is always compatible with ω. The existence part of Theorem 2 is proved in the case of no sources.

6. Type IIA geometry: proof of Theorem 3

The goal of this section is to work out some properties specific to Type IIA geometry. What is crucial is that the almost complex structure J_φ in Type IIA strings comes from a closed primitive positive 3-form φ via Hitchin's construction. In fact, the closedness of φ imposes subtle "higher integrability" conditions on J_φ which in turn distinguish J_φ from a generic almost complex structure. This feature gives rise to various identities that are not available in the more general almost-Kähler setting. We begin with the curvature and Nijenhuis tensor on general almost-complex manifolds, and gradually specialize to almost-Kähler manifolds, and then to Type IIA geometry.

6.1. Curvature tensors on general almost-complex manifolds

For any affine connection D, we define its curvature tensor $R(D)$ and torsion $T(D)$ by

$$[D_i, D_j]X^m = R(D)_{ij}{}^m{}_l X^l - T(D)^l{}_{ij} D_l X^m. \tag{6.1}$$

The curvature tensor with four lower indices is defined in the usual way

$$R(D)_{ijkl} = R(D)_{ij}{}^p{}_l g_{pk}. \tag{6.2}$$

As in the case for Levi-Civita connection, we define the Ricci curvature of D, also denoted by $R(D)$, by

$$R(D)_{ik} = g^{jl} R(D)_{ijkl}. \tag{6.3}$$

Let now J be an almost-complex structure on the Riemannian manifold M. In subsequent developments, we shall need both the Levi-Civita connection ∇ and the projected Levi-Civita connection $\mathfrak{D} = \mathfrak{D}^0$. Therefore we will reserve the Latin letter R for various curvature tensors associated to ∇ and the German letters $\mathfrak{d}$, $\mathfrak{T}$, and $\mathfrak{R}$ for Christoffel symbol, torsion, and curvature tensors associated to $\mathfrak{D}$. When we have other Hermitian metrics with decoration like $\tilde{g}$ or $\hat{g}$, we shall decorate the corresponding connections and curvature tensors with the same symbol. Identities for the curvature and torsion of the Chern connection have been worked out in the paper of Tosatti, Weinkove, and Yau [47]. However, they are expressed there in complex frames, and it is difficult for us to apply their formulas, as we shall have to let the almost-complex structure evolve. Thus we develop here a formalism for curvature and torsion identities with the action of J in real coordinate systems.

To pass back and forth from ∇ to $\mathfrak{d}$, we note that from (3.11) and (3.12) that

$$\mathfrak{d}^m{}_{ij} = \Gamma^m{}_{ij} - N_{ij}{}^m - V_{ij}{}^m =: \Gamma^m{}_{ij} - A_{ij}{}^m, \tag{6.4}$$

$$\mathfrak{T}^m{}_{ij} = N^m{}_{ij} - U^m{}_{ij}, \tag{6.5}$$

where $A = V + N$ is of type $(2,0)+(0,2)$, and hence their curvature tensors are related by

$$\begin{aligned} R_{ijkl} = \; & \mathfrak{R}_{ijkl} - (\mathfrak{D}_i A_{jkl} - \mathfrak{D}_j A_{ikl} + \mathfrak{T}^p{}_{ij} A_{pkl} \\ & + A_{ik}{}^p A_{jlp} - A_{jk}{}^p A_{ilp}). \end{aligned} \tag{6.6}$$

As R_{ijkl} is the curvature tensor of the Levi-Civita connection, it has various symmetries and satisfies the Bianchi identities. On the other hand, since $\mathfrak{D}J = 0$, its curvature $\mathfrak{R}$ satisfies

$$\mathfrak{R}_{ijkl} = \mathfrak{R}_{i,j,Jk,Jl}. \tag{6.7}$$

It is easy to deduce from the preceding relation between R_{ijkl} and $\mathfrak{R}_{ijkl}$ how to modify the identity for each curvature if it is replaced by the other.

The projected Levi-Civita connection $\mathfrak{D}$ induces a connection on the canonical bundle of M, whose curvature represents the first Chern class (up to a constant) of the almost complex manifold (M, J). To be precise, if we use small Greek letter to denote the index for "holomorphic" tangent bundle $T^{1,0}M$, then

$$\frac{1}{4\pi}\mathfrak{R}_{ijkl}\omega^{lk} = \frac{\sqrt{-1}}{2\pi}\mathfrak{R}_{ij}{}^{\gamma}{}_{\gamma} \in [c_1(M,J)]. \tag{6.8}$$

Since A is of type $(2,0)+(0,2)$, the contraction of its last two indices using ω or g vanishes, therefore by (6.6) we see that

$$\mathfrak{R}_{ijkl}\omega^{lk} = R_{ijkl}\omega^{lk} + 2A_{ik}{}^p A_{jlp}\omega^{lk} \tag{6.9}$$

is a closed 2-form. We end this subsection by deriving the following formula for Ricci curvature

$$R_{ij} = -2g^{kl}(\mathfrak{D}_i A_{kjl} - \mathfrak{D}_k A_{ijl} + \mathfrak{T}^p{}_{ik} A_{pjl}) + \frac{1}{2}\omega^{lk} R_{i,Jj,k,l}. \tag{6.10}$$

Indeed, by (6.6) we see that

$$\begin{aligned} R_{i,j,Jk,Jl} - R_{ijkl} &= (\mathfrak{D}_i A_{jkl} - \mathfrak{D}_j A_{ikl} + \mathfrak{T}^p{}_{ij} A_{pkl} + A_{ik}{}^p A_{jlp} - A_{jk}{}^p A_{ilp}) \\ &\quad -(\mathfrak{D}_i A_{j,Jk,Jl} - \mathfrak{D}_j A_{i,Jk,Jl} + \mathfrak{T}^p{}_{ij} A_{p,Jk,Jl} \\ &\quad + A_{i,Jk}{}^p A_{j,Jl,p} - A_{j,Jk}{}^p A_{i,Jl,p}). \end{aligned}$$

Recall that A is of type $(2,0)+(0,2)$, so $A_{ijk} = -A_{i,Jj,Jk}$, therefore we get

$$R_{i,j,Jk,Jl} - R_{ijkl} = 2(\mathfrak{D}_i A_{jkl} - \mathfrak{D}_j A_{ikl} + \mathfrak{T}^p{}_{ij} A_{pkl}). \tag{6.11}$$

Let us denote the right hand side of the above equation by B_{ijkl}. Then the above equation is equivalent to

$$\begin{aligned} -R_{i,j,Jk,l} - R_{i,j,k,Jl} &= 2(\mathfrak{D}_i A_{j,k,Jl} - \mathfrak{D}_j A_{i,k,Jl} + \mathfrak{T}^p{}_{ij} A_{p,k,Jl}) \\ &= B_{i,j,Jk,l} = B_{i,j,k,Jl}. \end{aligned} \tag{6.12}$$

Let $\{e_a\}$ be an orthonormal frame for the given Riemannian metric, and so is the frame $\{Je_a\}$. By definition of Ricci curvature, we have

$$\begin{aligned} R_{i,Jj} &= \sum_a R(i, e_a, Jj, e_a) = \sum_a R(i, Je_a, Jj, Je_a) \\ &= \sum_a (R(i, Je_a, j, e_a) + B(i, Je_a, j, e_a)) \\ &= \sum_a (-R(j, e_a, J(Ji), Je_a) - B(i, Je_a, Jj, Je_a)) \\ &= \sum_a (-R(j, e_a, Ji, e_a) - B(j, e_a, Ji, e_a) - B(i, e_a, Jj, e_a)) \\ &= -R_{j,Ji} - g^{kl}(B_{i,k,Jj,l} + B_{j,k,Ji,l}). \end{aligned} \tag{6.13}$$

On the other hand, by taking trace of (6.12) and using Bianchi identity of R, we have

$$\begin{aligned} g^{jl} B_{i,j,Jk,l} &= -g^{jl}(R_{i,j,Jk,l} + R_{i,j,k,Jl}) \\ &= -R_{i,Jk} + g^{jl}(R_{j,k,i,Jl} + R_{k,i,j,Jl}) \end{aligned}$$

$$= -R_{i,Jk} + g^{jl}(R_{k,j,Ji,l} + B_{k,j,Ji,l}) + R_{kijl}\omega^{lj}$$

$$(6.14) \qquad = -R_{i,Jk} + R_{k,Ji} + g^{jl}B_{k,j,Ji,l} + R_{kijl}\omega^{lj}.$$

(6.13) and (6.14) can be rewritten as

$$\begin{aligned} R_{i,Jj} + R_{j,Ji} &= -g^{kl}(B_{i,k,Jj,l} + B_{j,k,Ji,l}), \\ R_{i,Jj} - R_{j,Ji} &= -g^{kl}(B_{i,k,Jj,l} - B_{j,k,Ji,l}) - R_{ijkl}\omega^{lk}. \end{aligned}$$

Adding these two equations up we get

$$R_{i,Jj} = -g^{kl}B_{i,k,Jj,l} - \frac{1}{2}R_{ijkl}\omega^{lk}$$

which is equivalent to

$$\begin{aligned} R_{ij} &= -g^{kl}B_{ikjl} + \frac{1}{2}\omega^{lk}R_{i,Jj,k,l} \\ &= -2g^{kl}(\mathfrak{D}_i A_{kjl} - \mathfrak{D}_k A_{ijl} + \mathfrak{T}^p{}_{ik}A_{pjl}) + \frac{1}{2}\omega^{lk}R_{i,Jj,k,l}. \end{aligned}$$

This gives the desired formula.

6.2. Quadratic expressions in the Nijenhuis tensor

We shall encounter frequently later quadratic expressions of the Nijenhuis tensor. It is convenient to introduce the following two symmetric tensors quadratic in N:

$$\begin{aligned} (N_+^2)_{ij} &:= N^{pk}{}_i N_{pkj} \geq 0, \\ (N_-^2)_{ij} &:= N^{kp}{}_i N_{pkj}. \end{aligned}$$

Since N is skew-symmetric in the last two slots and it satisfies the Bianchi identity (3.8), all the other similar tensors can be expressed as a linear combination of N_+^2 and N_-^2. For example

$$0 \leq N_{ipk}N_j{}^{pk} = (N_{pki} - N_{kpi})(N^{pk}{}_j - N^{kp}{}_j) = 2(N_+^2)_{ij} - 2(N_-^2)_{ij}.$$

Obviously $g^{ij}N_{ipk}N_j{}^{pk} = |N|^2 = g^{ij}N^{pk}{}_i N_{pkj}$, so we find that

$$(6.15) \qquad |N|^2 = \mathrm{tr}N_+^2 = g^{ij}(N_+^2)_{ij} = 2g^{ij}(N_-^2)_{ij} = 2\mathrm{tr}N_-^2.$$

Also we observe that both N_+^2 and N_-^2 are J-invariant in the sense that $(N_\pm^2)_{ij} = (N_\pm^2)_{Ji,Jj}$. In general, for any symmetric 2-tensor $A = A_{ij}$, we define its J-invariant and J-anti-invariant parts respectively by

$$(A^J)_{ij} := \frac{1}{2}(A_{ij} + A_{Ji,Jj}), \qquad (A^{-J})_{ij} := \frac{1}{2}(A_{ij} - A_{Ji,Jj}).$$

In this notation, we have $N_\pm^2 = (N_\pm^2)^J$.

Clearly we have $A = A^J + A^{-J}$ and this decomposition is orthogonal with respect to the inner product induced by the metric g. Later such a decomposition will play an important role in our calculations.

6.3. Curvature tensors in almost-Kähler geometry

In this subsection we restrict ourselves to the case $d\omega = 0$, namely the case (M, J, g) is an almost-Kähler manifold. Since ω is a symplectic form, we know $d^c\omega = 0$, hence both U and V defined in (3.5) are zero. Therefore (6.4) and (6.5) specialize to $A = N = \mathfrak{T}$. Therefore the previously deduced formula (6.6) becomes

$$\begin{aligned} R_{ijkl} \;&=\; \mathfrak{R}_{ijkl} - (\mathfrak{D}_i N_{jkl} - \mathfrak{D}_j N_{ikl} \\ &\quad + N^p{}_{ij} N_{pkl} + N_{ik}{}^p N_{jlp} - N_{jk}{}^p N_{ilp}), \end{aligned} \tag{6.16}$$

thus we have

$$R_{ij} = \mathfrak{R}_{ij} + \mathfrak{D}^k N_{ijk} - (N_-^2)_{ij}. \tag{6.17}$$

Combining (6.10) with (6.16), we also obtain

$$\begin{aligned} R_{ij} \;&=\; 2\mathfrak{D}^k N_{ijk} - 2(N_+^2)_{ij} + \frac{1}{2}\omega^{lk} R_{i,Jj,k,l} \\ &=\; 2\mathfrak{D}^k N_{ijk} - 2(N_+^2)_{ij} + \frac{1}{2}\omega^{lk}(\mathfrak{R}_{i,Jj,k,l} - (\mathfrak{D}_i N_{Jj,k,l} - \mathfrak{D}_{Jj} N_{ikl} \\ &\quad + N^p{}_{i,Jj} N_{pkl} + N_{ik}{}^p N_{Jj,l,p} - N_{Jj,k}{}^p N_{ilp})) \\ &=\; 2\mathfrak{D}^k N_{ijk} - 2(N_-^2)_{ij} + \frac{1}{2}\omega^{lk}\mathfrak{R}_{i,Jj,k,l}. \end{aligned} \tag{6.18}$$

Alternatively

$$\begin{aligned} R_{ij} \;&=\; 2\mathfrak{D}^k N_{ijk} - 2(N_+^2)_{ij} + \frac{1}{2}\omega^{lk} R_{k,l,i,Jj} \\ &=\; 2\mathfrak{D}^k N_{ijk} - 2(N_+^2)_{ij} + \frac{1}{2}\omega^{lk}(\mathfrak{R}_{k,l,i,Jj} - (\mathfrak{D}_k N_{l,i,Jj} \end{aligned}$$

$$
\begin{aligned}
&+N^p{}_{kl}N_{p,i,Jj} - \mathfrak{D}_l N_{k,i,Jj} + N_{ki}{}^p N_{l,Jj,p} - N_{li}{}^p N_{k,Jj,p}))\\
&= \mathfrak{D}^k(N_{ijk}+N_{jik}) - (N_+^2)_{ij} + \frac{1}{2}\omega^{lk}\mathfrak{R}_{k,l,i,Jj}. \qquad (6.19)
\end{aligned}
$$

From (6.19) we can immediately read off that

$$
(R^J)_{ij} = -(N_+^2)_{ij} + \frac{1}{2}\omega^{lk}\mathfrak{R}_{k,l,i,Jj}, \qquad (6.20)
$$

$$
(R^{-J})_{ij} = \mathfrak{D}^k(N_{ijk}+N_{jik}). \qquad (6.21)
$$

Taking the trace of (6.18) and plugging in (6.16), we see that

$$
R = \frac{1}{2}\omega^{ji}\omega^{lk}\mathfrak{R}_{ijkl} - |N|^2 = \frac{1}{2}\omega^{ji}\omega^{lk}R_{ijkl} - 2|N|^2. \qquad (6.22)
$$

In the literature, the expression $\frac{1}{2}\omega^{ji}\omega^{lk}R_{ijkl}$ is sometimes known as the $\star$-scalar curvature. This relation (6.22) was first discovered by Blair-Ianus [4], and Blair [3] together with Oproiu [36].

Combining (6.16) with the symmetry of R, we can derive that

$$
\begin{aligned}
\mathfrak{R}_{ijkl} - \mathfrak{R}_{klij} &= \mathfrak{D}_i N_{jkl} - \mathfrak{D}_j N_{ikl} - \mathfrak{D}_k N_{lij} + \mathfrak{D}_l N_{kij}\\
&\quad + N_{ik}{}^p N_{jlp} - N_{jk}{}^p N_{ilp} - N_{ki}{}^p N_{ljp} + N_{li}{}^p N_{kjp}. \qquad (6.23)
\end{aligned}
$$

As $\mathfrak{R}_{ijkl} = \mathfrak{R}_{i,j,Jk,Jl}$, by making use of (6.23), we get

$$
\begin{aligned}
\mathfrak{R}_{Ji,Jj,k,l} - \mathfrak{R}_{ijkl} &= \mathfrak{R}_{Ji,Jj,k,l} - \mathfrak{R}_{k,l,Ji,Jj} + \mathfrak{R}_{klij} - \mathfrak{R}_{ijkl}\\
&= \mathfrak{D}_{Ji}N_{Jj,k,l} - \mathfrak{D}_{Jj}N_{Ji,k,l} - \mathfrak{D}_i N_{jkl}\\
&\quad + \mathfrak{D}_j N_{ikl} + 2\mathfrak{D}_k N_{lij} - 2\mathfrak{D}_l N_{kij}. \qquad (6.24)
\end{aligned}
$$

Notice that the LHS of (6.24) does not change if one replace k and l by Jk and Jl respectively, so we get an interesting identity satisfied by $\mathfrak{D}N$

$$
\begin{aligned}
&\mathfrak{D}_{Ji}N_{Jj,k,l} - \mathfrak{D}_{Jj}N_{Ji,k,l} - \mathfrak{D}_{Jk}N_{Jl,i,j} + \mathfrak{D}_{Jl}N_{Jk,i,j}\\
&= \mathfrak{D}_i N_{jkl} - \mathfrak{D}_j N_{ikl} - \mathfrak{D}_k N_{lij} + \mathfrak{D}_l N_{kij}, \qquad (6.25)
\end{aligned}
$$

which allows us to rewrite one covariant derivative of N in terms of some other combination of covariant derivatives.

In the same vein we can derive the Bianchi-type identity for $\mathfrak{R}$

$$
\begin{aligned}
\mathfrak{R}_{ijkl} + \mathfrak{R}_{jkil} + \mathfrak{R}_{kijl} &= -\mathfrak{D}_i N_{ljk} - \mathfrak{D}_j N_{lki} - \mathfrak{D}_k N_{lij}\\
&\quad + N^p{}_{ij}N_{lkp} + N^p{}_{jk}N_{lip} + N^p{}_{ki}N_{ljp}. \qquad (6.26)
\end{aligned}
$$

Equation (6.24) accounts for the (2,0)+(0,2)-part of the curvature tensor $\mathfrak{R}$. If we use Greek letters for barred and unbarred directions, then (6.24) can be translated into

$$\begin{aligned}\mathfrak{R}_{\alpha\beta\bar{\gamma}\delta} &= \mathfrak{D}_\alpha N_{\beta\bar{\gamma}\delta} - \mathfrak{D}_\beta N_{\alpha\bar{\gamma}\delta} - \mathfrak{D}_{\bar{\gamma}} N_{\delta\alpha\beta} + \mathfrak{D}_\delta N_{\bar{\gamma}\alpha\beta} \\ &= -\mathfrak{D}_{\bar{\gamma}} N_{\delta\alpha\beta}, \\ \mathfrak{R}_{**\gamma\delta} &= \mathfrak{R}_{**\bar{\gamma}\bar{\delta}} = 0,\end{aligned}$$

which is the content of (2.17) in [47]. Replace i, j, k, l by $\bar{\alpha}, \beta, \bar{\gamma}, \delta$ respectively in the Bianchi-type identity (6.26), we get

$$\mathfrak{R}_{\bar{\alpha}\beta\bar{\gamma}\delta} - \mathfrak{R}_{\bar{\gamma}\beta\bar{\alpha}\delta} = \mathfrak{R}_{\bar{\alpha}\beta\bar{\gamma}\delta} + \mathfrak{R}_{\beta\bar{\gamma}\bar{\alpha}\delta} + \mathfrak{R}_{\bar{\gamma}\bar{\alpha}\beta\delta} = N^p{}_{\bar{\gamma}\bar{\alpha}} N_{\delta\beta p} = N^\lambda{}_{\bar{\gamma}\bar{\alpha}} N_{\delta\beta\lambda}.$$

This is the content of (2.16) in [47].

6.4. The holonomy of Type IIA geometry

We now restrict ourselves further to Type IIA geometry, namely a triple (M, ω, φ) where (M, ω) is a symplectic 6-manifold and φ is a closed positive ω-primitive 3-form.

Our first task is to prove Theorem 3(a). Recall that $|\varphi|$ is the norm of φ with respect to the metric g_φ, and that we have defined the metric $\tilde{g}_\varphi$ by $\tilde{g}_\varphi = |\varphi|^2 g_\varphi$. It is not hard to see that

$$|\varphi|_{\tilde{g}} = |\varphi|^{-2}.$$

Henceforth we shall denote g_φ, $\tilde{g}_\varphi$, and J_φ by just g, $\tilde{g}$, and J for simplicity. It is clear that J is compatible with $\tilde{g}$ and the corresponding Kähler form $\tilde{\omega} = |\varphi|^2\omega$ satisfies

$$d\tilde{\omega} = -\alpha \wedge \tilde{\omega}, \qquad d^c\tilde{\omega} = J\alpha \wedge \tilde{\omega},$$

where

$$\alpha = d\log|\varphi|_{\tilde{g}} = -d\log|\varphi|^2. \tag{6.27}$$

It follows from (3.16) that

$$U_{ijk} = \frac{1}{4}(2\alpha_{Ji}\tilde{\omega}_{jk} + \alpha_{Jj}\tilde{\omega}_{ki} + \alpha_{Jk}\tilde{\omega}_{ij} + \alpha_j\tilde{g}_{ki} - \alpha_k\tilde{g}_{ij}), \tag{6.28}$$

$$V_{ijk} = \frac{1}{4}(\alpha_{Jj}\tilde{\omega}_{ki} + \alpha_{Jk}\tilde{\omega}_{ij} - \alpha_j\tilde{g}_{ki} + \alpha_k\tilde{g}_{ij}). \tag{6.29}$$

We need now the following lemmas for computational purposes.

Lemma 13.
Let μ be any differential form, D any affine connection, $T = T(D)$ the torsion tensor associated to D. Then we have the following formula

$$d\mu = dx^j \wedge D_j\mu + T \boxtimes \mu, \tag{6.30}$$

where $\boxtimes$ is a multiplication operation linear in both factors. We only need the explicit expression of $\boxtimes$ when μ is a 3-form, in which case $T \boxtimes \mu$ is a 4-form given by

$$(T \boxtimes \mu)_{ijkl} = T^p{}_{ij}\mu_{pkl} + T^p{}_{kl}\mu_{pij} - T^p{}_{ik}\mu_{pjl} - T^p{}_{jl}\mu_{pik} + T^p{}_{il}\mu_{pjk} + T^p{}_{jk}\mu_{pil}, \tag{6.31}$$

as well as the case μ is a 2-form, where $T \boxtimes \mu$ is a 3-form of the form

$$(T \boxtimes \mu)_{ijk} = T^p{}_{ij}\mu_{pk} + T^p{}_{jk}\mu_{pi} + T^p{}_{ki}\mu_{pj}. \tag{6.32}$$

Proof: We give the proof of (6.32) and leave (6.31) to the reader. For $\mu = \frac{1}{2}\mu_{ij}dx^i \wedge dx^j$, we have

$$d\mu = \frac{1}{2}\partial_\alpha\mu_{ij}\, dx^\alpha \wedge dx^i \wedge dx^j. \tag{6.33}$$

We write $D_iW_j = \partial_iW_j - \Gamma(D)^k{}_{ij}W_k$, and obtain

$$d\mu = \frac{1}{2}(D_k\mu_{ij} + \Gamma(D)^\beta{}_{ki}\mu_{\beta j} + \Gamma(D)^\beta{}_{kj}\mu_{i\beta})\, dx^k \wedge dx^i \wedge dx^j. \tag{6.34}$$

This becomes

$$\begin{aligned} d\mu &= dx^k \wedge D_k\mu \\ &+ \frac{1}{3!}\Big(\Gamma(D)^\beta{}_{ki}\mu_{\beta j} + \Gamma(D)^\beta{}_{jk}\mu_{\beta i} + \Gamma(D)^\beta{}_{ij}\mu_{\beta k}\Big)\, dx^k \wedge dx^i \wedge dx^j \\ &+ \frac{1}{3!}\Big(\Gamma(D)^\beta{}_{kj}\mu_{i\beta} + \Gamma(D)^\beta{}_{ik}\mu_{j\beta} + \Gamma(D)^\beta{}_{ji}\mu_{k\beta}\Big)\, dx^k \wedge dx^i \wedge dx^j \end{aligned} \tag{6.35}$$

which leads to

$$d\mu = dx^k \wedge D_k\mu + \frac{1}{3!}(T^\beta{}_{ki}\mu_{\beta j} + T^\beta{}_{jk}\mu_{\beta i} + T^\beta{}_{ij}\mu_{\beta k})dx^k \wedge dx^i \wedge dx^j. \tag{6.36}$$

Q.E.D.

Lemma 14.
In the notation in Lemma 13, the 4-form $N \boxtimes \varphi$ is of type $(2,2)$.

Proof of the Lemma: By Lemma 13, we know that

$$\begin{aligned}(N\boxtimes\varphi)_{ijkl} &= N^p{}_{ij}\varphi_{pkl} + N^p{}_{kl}\varphi_{pij} - N^p{}_{ik}\varphi_{pjl} - N^p{}_{jl}\varphi_{pik}\\ &\quad + N^p{}_{il}\varphi_{pjk} + N^p{}_{jk}\varphi_{pil}.\end{aligned}$$

Since $N \in A^{0,2}(TM)$ and φ satisfies Lemma 6, we find

$$\begin{aligned}(J(N\boxtimes\varphi))_{ijkl} &= N^p{}_{Ji,Jj}\varphi_{p,Jk,Jl} + N^p{}_{Jk,Jl}\varphi_{p,Ji,Jj} - N^p{}_{Ji,Jk}\varphi_{p,Jj,Jl}\\ &\quad - N^p{}_{Jj,Jl}\varphi_{p,Ji,Jk} + N^p{}_{Ji,Jl}\varphi_{p,Jj,Jk} + N^p{}_{Jj,Jk}\varphi_{p,Ji,Jl}\\ &= N^p{}_{ij}\varphi_{pkl} + N^p{}_{kl}\varphi_{pij} - N^p{}_{ik}\varphi_{pjl} - N^p{}_{jl}\varphi_{pik}\\ &\quad + N^p{}_{il}\varphi_{pjk} + N^p{}_{jk}\varphi_{pil}\\ &= (N\boxtimes\varphi)_{ijkl}.\end{aligned}$$

As J acts on $(3,1)+(1,3)$-forms as -1 and acts on $(2,2)$-forms as 1, we deduce that $N \boxtimes \varphi$ is a $(2,2)$-form. Q.E.D.

Lemma 15.
Using the notation in Lemma 13, we have

$$d^c\tilde\omega \boxtimes \varphi = \mathcal{M}(d^c\tilde\omega)\boxtimes\varphi = 2\alpha\wedge\varphi, \tag{6.37}$$

$$d^c\tilde\omega \boxtimes \hat\varphi = \mathcal{M}(d^c\tilde\omega)\boxtimes\hat\varphi = 2\alpha\wedge\hat\varphi. \tag{6.38}$$

Proof. As we have seen $d^c\tilde\omega = J\alpha\wedge\tilde\omega$, so the first term in $(d^c\tilde\omega)\boxtimes\varphi$ is

$$\begin{aligned}(d^c\tilde\omega)^p{}_{ij}\varphi_{pkl} &= \tilde g^{pq}((J\alpha)_q\tilde\omega_{ij} + (J\alpha)_i\tilde\omega_{jq} + (J\alpha)_j\tilde\omega_{qi})\varphi_{pkl}\\ &= \tilde g^{pq}\alpha_{Jq}\tilde\omega_{ij}\varphi_{pkl} + \alpha_{Ji}\varphi_{Jj,k,l} - \alpha_{Jj}\varphi_{Ji,k,l}\\ &= \tilde g^{pq}\alpha_{Jq}\tilde\omega_{ij}\varphi_{pkl} - \alpha_{Ji}\hat\varphi_{jkl} + \alpha_{Jj}\hat\varphi_{ikl}.\end{aligned}$$

Hence

$$\begin{aligned}&(d^c\tilde\omega\boxtimes\varphi)_{ijkl}\\ =\ & \tilde g^{pq}\alpha_{Jq}(\tilde\omega_{ij}\varphi_{pkl} + \tilde\omega_{kl}\varphi_{pij} - \tilde\omega_{ik}\varphi_{pjl} - \tilde\omega_{jl}\varphi_{pik} + \tilde\omega_{il}\varphi_{pjk} + \tilde\omega_{jk}\varphi_{pil})\\ & - \alpha_{Ji}\hat\varphi_{jkl} + \alpha_{Jj}\hat\varphi_{ikl} - \alpha_{Jk}\hat\varphi_{lij} + \alpha_{Jl}\hat\varphi_{kij} + \alpha_{Ji}\hat\varphi_{kjl} - \alpha_{Jk}\hat\varphi_{ijl}\\ & + \alpha_{Jj}\hat\varphi_{lik} - \alpha_{Jl}\hat\varphi_{jik} - \alpha_{Ji}\hat\varphi_{ljk} + \alpha_{Jl}\hat\varphi_{ijk} - \alpha_{Jj}\hat\varphi_{ilk} + \alpha_{Ji}\hat\varphi_{jlk}\\ =\ & \tilde g^{pq}\alpha_{Jq}(\tilde\omega_{ip}\varphi_{jkl} - \tilde\omega_{jp}\varphi_{ikl} + \tilde\omega_{kp}\varphi_{ijl} - \tilde\omega_{lp}\varphi_{ijk}) - 3(J\alpha\wedge\hat\varphi)_{ijkl}\\ =\ & \alpha_{Jq}(J^q{}_i\varphi_{jkl} - J^q{}_j\varphi_{ikl} + J^q{}_k\varphi_{ijl} - J^q{}_l\varphi_{ijk}) + 3(\alpha\wedge\varphi)_{ijkl}\\ =\ & 2(\alpha\wedge\varphi)_{ijkl}.\end{aligned}$$

In this proof we only used the fact that φ is primitive (4.14) so in the same manner we have

$$d^c\tilde{\omega} \boxtimes \hat{\varphi} = 2\alpha \wedge \hat{\varphi}.$$

The other identities can be proved similarly. Q.E.D.

Now we are ready to prove Theorem 3 (a).

Since $\tilde{\mathfrak{D}}J = 0$, there exists a complex-valued 1-form $\theta = \alpha + \sqrt{-1}\beta$ such that

$$\tilde{\mathfrak{D}}\Omega = \theta \otimes \Omega.$$

Taking its real and imaginary parts, we get

$$\tilde{\mathfrak{D}}\varphi = \alpha \otimes \varphi - \beta \otimes \hat{\varphi}, \tag{6.39}$$

$$\tilde{\mathfrak{D}}\hat{\varphi} = \beta \otimes \varphi + \alpha \otimes \hat{\varphi}. \tag{6.40}$$

The 1-form α is very easy to find: as $\tilde{\mathfrak{D}}\tilde{g} = 0$, we know that

$$d|\varphi|^2_{\tilde{g}} = \tilde{\mathfrak{D}}\tilde{g}(\varphi, \varphi) = 2\tilde{g}(\tilde{\mathfrak{D}}\varphi, \varphi) = 2|\varphi|^2_{\tilde{g}}\alpha,$$

hence we conclude that

$$\alpha = \frac{1}{2}d\log|\varphi|^2_{\tilde{g}} = d\log|\varphi|_{\tilde{g}} = -d\log|\varphi|^2, \tag{6.41}$$

which is the exactly same expression we assigned to α in (6.27). To find β, we plug (6.39) in (6.30) to get

$$0 = d\varphi = \alpha \wedge \varphi - \beta \wedge \hat{\varphi} + \tilde{\mathfrak{T}} \boxtimes \varphi. \tag{6.42}$$

Apply (3.12) to the Hermitian metric $\tilde{g}$ with $t = 0$, we get

$$\tilde{\mathfrak{T}} = N - U, \tag{6.43}$$

where $U = \frac{1}{4}(d^c\tilde{\omega} + \mathcal{M}(d^c\tilde{\omega}))$. According to Lemma 15 we have

$$\begin{aligned} \tilde{\mathfrak{T}} \boxtimes \varphi &= N \boxtimes \varphi - \frac{1}{4}d^c\tilde{\omega} \boxtimes \varphi - \frac{1}{4}\mathcal{M}(d^c\tilde{\omega}) \boxtimes \varphi \\ &= N \boxtimes \varphi - \alpha \wedge \varphi. \end{aligned}$$

Consequently (6.42) can be simplified to

$$N \boxtimes \varphi = \beta \wedge \hat{\varphi}.$$

By Lemma 14, the LHS of the above equation is a $(2,2)$-form while the RHS is a $(3,1)+(1,3)$-form. Therefore we conclude that $N \boxtimes \varphi = 0$ and $\beta = 0$. As a result,

$$\tilde{\mathfrak{D}}\Omega = \alpha \otimes \Omega, \tag{6.44}$$

which implies immediately that $\tilde{\mathfrak{D}}\left(\frac{\Omega}{|\Omega|_{\tilde{g}}}\right) = 0$. Q.E.D.

Remark: Heuristically we can argue as follows. Since N accounts for the non-integrability of J, the form $N \boxtimes \varphi$ is responsible for the "exotic" component of $d\varphi$ which vanishes automatically in the integrable case. Because φ is a $(3,0)+(0,3)$-form, $d\varphi$ would be a $(3,1)+(1,3)$-form if J is integrable. As a result

$$N \boxtimes \varphi = \text{(2,2) component of } d\varphi = 0.$$

Corollary 1.
The pair (N,φ) satisfies

$$N^p{}_{ij}\varphi_{pkl} + N^p{}_{kl}\varphi_{pij} = 0, \tag{6.45}$$

$$N^p{}_{ij}\hat{\varphi}_{pkl} - N^p{}_{kl}\hat{\varphi}_{pij} = 0. \tag{6.46}$$

Proof: In the proof of Theorem (3) Part (a), we showed that $N \boxtimes \varphi = 0$, namely

$$N^p{}_{ij}\varphi_{pkl} + N^p{}_{kl}\varphi_{pij} - N^p{}_{ik}\varphi_{pjl} - N^p{}_{jl}\varphi_{pik} + N^p{}_{il}\varphi_{pjk} + N^p{}_{jk}\varphi_{pil} = 0. \tag{6.47}$$

Replace i and j in (6.47) by Ji and Jj, by using symmetry of N and φ, we get instead

$$-N^p{}_{ij}\varphi_{pkl} - N^p{}_{kl}\varphi_{pij} - N^p{}_{ik}\varphi_{pjl} - N^p{}_{jl}\varphi_{pik} + N^p{}_{il}\varphi_{pjk} + N^p{}_{jk}\varphi_{pil} = 0. \tag{6.48}$$

By combining (6.47) and (6.48) we prove the corollary. Equation (6.46) follows from (6.45) and Lemma 6. Q.E.D.

6.5. The curvature in Type IIA geometry

Next, we prove Theorem 3 (b).

As we have seen in Theorem 3 (a), in Type IIA geometry, the nowhere vanishing (3, 0)-form $\Omega/|\Omega|_{\tilde g}$ is parallel under the connection $\tilde{\mathfrak{D}}$. A direct consequence is that the first Chern form associated to $\tilde{\mathfrak{D}}$ is identically zero, that is,

$$-\frac{\sqrt{-1}}{2}\tilde{\mathfrak{R}}_{ijkl}\tilde{\omega}^{lk} = \tilde{\mathfrak{R}}_{ij}{}^{\beta}{}_{\beta} = 0. \tag{6.49}$$

As $\tilde g = |\varphi|^2 g$, one can relate $\tilde{\mathfrak{D}}$ with $\mathfrak{D}$ by the conformal change formula. Combining it with (6.44), it is not hard to see that

$$\mathfrak{D}\Omega = -\frac{1}{2}(\alpha - \sqrt{-1}J\alpha)\otimes\Omega. \tag{6.50}$$

As a consequence, the curvature tensor $\mathfrak{R}$ satisfies

$$\begin{aligned} -\frac{\sqrt{-1}}{2}\mathfrak{R}_{ijkl}\omega^{lk} &= \mathfrak{R}_{ij}{}^{\beta}{}_{\beta} = \frac{1}{2}d(\alpha - \sqrt{-1}J\alpha)_{ij} \\ &= -\sqrt{-1}(dd^c \log|\varphi|)_{ij}, \end{aligned} \tag{6.51}$$

and we recover the well-known formula for Ricci curvature in the Kähler case. In fact, (6.50) implies that $\mathfrak{D}^{0,1}\Omega = 0$. For an almost Kähler manifold, the Gauduchon line of connections [26] collapses to a point, so $\mathfrak{D}$ is also the Chern connection (in the almost complex setting), hence $\mathfrak{D}^{0,1} = \bar\partial$, and we conclude that Ω is a holomorphic section of the canonical bundle associated to (M, J). Theorem 3 (b) is proved.

We complete this section with some identities linking the curvature and Nijenhuis tensor. Recall the globally defined function $u = \log|\varphi|^2$. In this notation we have $\tilde g = e^u g$ and $\alpha = -du$. Furthermore (6.51) can be rewritten as

$$\begin{aligned} \mathfrak{R}_{ijkl}\omega^{lk} &= (dd^c u)_{ij} = -(\mathfrak{D}_i(du)_{Jj} - \mathfrak{D}_j(du)_{Ji} + N^k{}_{ij}u_{Jk}) \\ &= -(\nabla^2 u)_{i,Jj} + (\nabla^2 u)_{j,Ji} - 2N^k{}_{ij}u_{Jk}. \end{aligned} \tag{6.52}$$

Substitute (6.52) back to (6.18), we get

$$R_{ij} = 2\mathfrak{D}^k N_{ijk} - 2(N_-^2)_{ij} + \frac{1}{2}(\nabla^2 u)_{ij} + \frac{1}{2}(\nabla^2 u)_{Ji,Jj} - u_k N^k{}_{ij}.$$

Since R_{ij} is symmetric, we conclude

$$
\begin{aligned}
R_{ij} & \\
(6.53) \quad &= \mathfrak{D}^k(N_{ijk}+N_{jik}) - 2(N_-^2)_{ij} + \frac{1}{2}(\nabla^2 u)_{ij} + \frac{1}{2}(\nabla^2 u)_{Ji,Jj}, \\
(6.54) \quad &= \nabla^k(N_{ijk}+N_{jik}) + 2(N_-^2)_{ij} - 2(N_+^2)_{ij} + \frac{1}{2}(\nabla^2 u)_{ij} + \frac{1}{2}(\nabla^2 u)_{Ji,Jj}
\end{aligned}
$$

and that N satisfies

$$
\mathfrak{D}^k N_{kij} = \nabla^k N_{kij} = -u^k N_{kij}. \tag{6.55}
$$

Therefore the J-invariant and J-anti-invariant components of the Ricci curvature are given by (6.21) and the following refinement of (6.20)

$$
(R^J)_{ij} = -2(N_-^2)_{ij} + \left((\nabla^2 u)^J\right)_{ij}, \; (R^{-J})_{ij} = \mathfrak{D}^k(N_{ijk}+N_{jik}). \tag{6.56}
$$

The scalar curvature is

$$
R = \Delta u - |N|^2. \tag{6.57}
$$

(6.52) then implies that

$$
\frac{1}{2}\mathfrak{R}_{ijkl}\omega^{ji}\omega^{lk} = \Delta u.
$$

It follows from (6.22) that the $\star$-scalar curvature is given by

$$
\frac{1}{2}R_{ijkl}\omega^{ji}\omega^{lk} = \Delta u + |N|^2. \tag{6.58}
$$

Similarly we can derive the formulae for $\tilde{R}$, the curvature tensor associated to the conformal metric $\tilde{g}$:

$$
\begin{aligned}
\tilde{R}_{ij} \;=\; & -(\tilde{\mathfrak{D}}^s - \frac{1}{2}u^s)(N_{isj}+N_{jsi}) + \frac{1}{2}((\tilde{\nabla}^2 u)_{Ji,Jj} - 3(\tilde{\nabla}^2 u)_{ij} - \tilde{\Delta} u \tilde{g}_{ij}) \\
(6.59) \quad & -2(N_-^2)_{ij} - \frac{1}{2}u_i u_j + \frac{1}{2}u_{Ji}u_{Jj} + \frac{1}{2}|du|^2_{\tilde{g}}\tilde{g}_{ij},
\end{aligned}
$$

and

$$
\tilde{R} \;=\; -4\tilde{\Delta} u + 3|du|^2_{\tilde{g}} - |N|^2_{\tilde{g}}, \tag{6.60}
$$

with (6.55) becoming

$$
2\tilde{\mathfrak{D}}^k N_{kij} = u^k N_{kij}, \qquad 2\tilde{\nabla}^k N_{kij} = 3u^k N_{kij}.
$$

6.6. The Nijenhuis tensor in Type IIA geometry

As we have seen in previous sections, on an almost Kähler manifold, the Nijenhuis tensor N is a $(0,2)$-type TM-valued 2-form satisfying the Bianchi identity (3.8). Moreover, one can define two J-invariant symmetric tensors N_+^2 and N_-^2 satisfying

$$\mathrm{tr}N_+^2 = |N|^2 = 2\mathrm{tr}N_-^2.$$

When an almost Kähler structure is enhanced to a Type IIA structure, the integrability of J is improved, hence one should expect more identities satisfied by N. For example, we have already seen that certain divergences of N are actually terms of lower order term (6.55). In this subsection, we shall derive more identities and differential equations satisfied by N, showing that an almost-complex structure coming from a Type IIA geometry is more "integrable" than a generic one. We shall also complete the proof of Theorem 3 by proving Part (c).

First, we show that N_+^2 and N_-^2 are related to each other:

Proposition 1.
For any Type IIA structure (M,ω,φ), the Nijenhuis tensor N satisfies

$$N_-^2 = 2N_+^2 - \frac{1}{4}|N|^2 g. \tag{6.61}$$

Proof: In view of (6.45), we notice that

$$\varphi_{iap}N^{st}{}_b N^p{}_{st} = -\varphi_{pst}N^{st}{}_b N^p{}_{ia} = \varphi_{stb}N^{st}{}_p N^p{}_{ia},$$

so

$$|\varphi|^{-2}N^{st}{}_b N^p{}_{st}\varphi_{iap}\varphi_{jcd}\omega^{ac}\omega^{bd} = |\varphi|^{-2}N^{st}{}_p N^p{}_{ia}\varphi_{stb}\varphi_{jcd}\omega^{ac}\omega^{bd}.$$

Applying Lemma 8 to both sides, we get

$$2N^{st}{}_i N_{jst} - N^{stk}N_{kst}g_{ij} = 2N_j{}^{st}N_{tsi} + 2N^{st}{}_j N_{tsi}.$$

Converting everything into N_+^2 and N_-^2, we get

$$-2N_+^2 + 2N_-^2 - N^{stk}N_{kst}g = 2N_+^2.$$

By Bianchi identity, we know that

$$-N^{stk}N_{kst} = N^{stk}(N_{stk} - N_{tsk}) = \mathrm{tr}N_+^2 - \mathrm{tr}N_-^2 = \frac{1}{2}|N|^2.$$

Consequently

$$N_-^2 = 2N_+^2 - \frac{1}{4}|N|^2 g.$$

As a corollary, we obtain the inequality

$$0 \le N_+^2 \le \frac{1}{4}|N|^2 g \tag{6.62}$$

since $0 \le N_+^2 - N_-^2$. Q.E.D.

We now come to the proof of Theorem 3 (c), which is a very powerful tool for proving identities involving N:

Let us choose a frame at a given point as in Lemma 4. Since N is a $(0,2)$-type TM-valued 2-form satisfying the Bianchi identity, we get the following relations:

$$\begin{aligned}
&0 = N_{*jj} = N_{*12} = N_{*21} = N_{*34} = N_{*43} = N_{*56} = N_{*65},\\
&N_{135} = -N_{153} = -N_{146} = N_{164} = -N_{236} = N_{263} = -N_{245} = N_{254},\\
&N_{136} = -N_{163} = N_{145} = -N_{154} = N_{235} = -N_{253} = -N_{246} = N_{264},\\
&N_{315} = -N_{351} = -N_{326} = N_{362} = -N_{416} = N_{461} = -N_{425} = N_{452},\\
&N_{316} = -N_{361} = N_{325} = -N_{352} = N_{415} = -N_{451} = -N_{426} = N_{462},\\
&N_{513} = -N_{531} = -N_{524} = N_{542} = -N_{614} = N_{641} = -N_{623} = N_{632},\\
&N_{514} = -N_{541} = N_{523} = -N_{532} = N_{613} = -N_{631} = -N_{624} = N_{642},
\end{aligned}$$

and

$$\begin{aligned}
&N_{113} = -N_{131} = -N_{124} = N_{142} = -N_{214} = N_{241} = -N_{223} = N_{232},\\
&N_{114} = -N_{141} = N_{123} = -N_{132} = N_{213} = -N_{231} = -N_{224} = N_{242},\\
&N_{115} = -N_{151} = -N_{126} = N_{162} = -N_{216} = N_{261} = -N_{225} = N_{252},\\
&N_{116} = -N_{161} = N_{125} = -N_{152} = N_{215} = -N_{251} = -N_{226} = N_{262},\\
&N_{331} = -N_{313} = -N_{342} = N_{324} = -N_{432} = N_{423} = -N_{441} = N_{414},\\
&N_{332} = -N_{323} = N_{341} = -N_{314} = N_{431} = -N_{413} = -N_{442} = N_{424},\\
&N_{335} = -N_{353} = -N_{346} = N_{364} = -N_{436} = N_{463} = -N_{445} = N_{454},
\end{aligned}$$

$$
\begin{aligned}
N_{336} &= -N_{363} = N_{345} = -N_{354} = N_{435} = -N_{453} = -N_{446} = N_{464},\\
N_{551} &= -N_{515} = -N_{562} = N_{526} = -N_{652} = N_{625} = -N_{661} = N_{616},\\
N_{552} &= -N_{525} = N_{561} = -N_{516} = N_{651} = -N_{615} = -N_{662} = N_{626},\\
N_{553} &= -N_{535} = -N_{564} = N_{546} = -N_{654} = N_{645} = -N_{663} = N_{636},\\
N_{554} &= -N_{545} = N_{563} = -N_{536} = N_{653} = -N_{635} = -N_{664} = N_{646},
\end{aligned}
$$

with constraints

$$
\begin{aligned}
N_{135} + N_{351} + N_{513} &= 0,\\
N_{136} + N_{361} + N_{613} &= 0.
\end{aligned}
$$

Furthermore, by evaluating (6.45) at the given point, we get

$$
\begin{aligned}
0 &= N_{135} = N_{136} = N_{315} = N_{316} = N_{513} = N_{514},\\
0 &= N_{331} - N_{551} = N_{113} - N_{553} = N_{115} - N_{335},\\
0 &= N_{114} + N_{554} = N_{116} + N_{336} = N_{332} + N_{552}.
\end{aligned}
$$

Therefore N has only 6 independent components at the given point. Q.E.D.

We can choose and name such independent components as

$$
a := N_{331},\quad b := N_{332},\quad c := N_{113},\quad d := N_{114},\quad e := N_{115},\quad f := N_{116}.
$$

It follows that

$$
|N|^2 = 16(a^2 + b^2 + c^2 + d^2 + e^2 + f^2).
$$

We can further express N_+^2 and N_-^2 in terms of these components. For instance, it is straightforward to verify that

$$
N_+^2 = 2 \times
\begin{bmatrix}
r^2+a^2+b^2 & 0 & ac+bd & -ad+bc & ae-bf & -af-be\\
0 & r^2+a^2+b^2 & ad-bc & ac+bd & af+be & ae-bf\\
ac+bd & ad-bc & r^2+c^2+d^2 & 0 & ce+df & cf-de\\
-ad+bc & ac+bd & 0 & r^2+c^2+d^2 & -cf+de & ce+df\\
ae-bf & af+be & ce+df & -cf+de & r^2+e^2+f^2 & 0\\
-af-be & ae-bf & cf-de & ce+df & 0 & r^2+e^2+f^2
\end{bmatrix},
$$

where $r^2 = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = \frac{1}{16}|N|^2$. Similarly we can find N^2_- as well. This normal form also allows us to quickly prove that

$$|N^2_+|^2 = 48r^4 = \frac{3}{16}|N|^4. \tag{6.63}$$

As an application of Theorem 3 (c), we prove that N satisfies the following differential equation:

Lemma 16.
Given a Type IIA structure (M, ω, φ), the Nijenhuis tensor N satisfies

$$8N^{sti}\nabla_i N_{stj} = 8N^{sti}\mathfrak{D}_i N_{stj} = \mathfrak{D}_j|N|^2 + u_j|N|^2. \tag{6.64}$$

Proof: By (6.25), we have

$$\begin{aligned} & \mathfrak{D}_i N_{sjt} - \mathfrak{D}_s N_{ijt} - \mathfrak{D}_j N_{tis} + \mathfrak{D}_t N_{jis} \\ = \quad & \mathfrak{D}_{Ji} N_{Js,j,t} - \mathfrak{D}_{Js} N_{Ji,j,t} - \mathfrak{D}_{Jj} N_{Jt,i,s} + \mathfrak{D}_{Jt} N_{Jj,i,s}. \end{aligned}$$

Contracting this equation with N^{sti}, we get

$$2N^{sti}(\mathfrak{D}_i N_{sjt} - \mathfrak{D}_s N_{ijt} + \mathfrak{D}_t N_{jis}) = N^{sti}(\mathfrak{D}_j N_{tis} - \mathfrak{D}_{Jj} N_{Jt,i,s}).$$

The LHS can be simplified as follows

$$\begin{aligned} \text{LHS} &= -2N^{sti}\mathfrak{D}_i N_{stj} + 2N^{its}\mathfrak{D}_i N_{stj} + 2N^{sit}\mathfrak{D}_i N_{jts} \\ &= 2N^{tis}(\mathfrak{D}_i N_{stj} + \mathfrak{D}_i N_{jst}) \\ &= -2N^{sti}\mathfrak{D}_i N_{stj}. \end{aligned}$$

On the other hand, we see that

$$N^{sti}\mathfrak{D}_j N_{tis} = -N^{sti}\mathfrak{D}_j N_{tsi} = -\frac{1}{2}\mathfrak{D}_j(N^{sti}N_{tsi}) = -\frac{1}{4}\mathfrak{D}_j|N|^2.$$

Therefore to prove the lemma, we only need to show that

$$N^{sti}\mathfrak{D}_{Jj} N_{Jt,i,s} = \frac{1}{4}u_j|N|^2,$$

or equivalently

$$N^{sti}\mathfrak{D}_j N_{t,s,Ji} = \frac{1}{4}u_{Jj}|N|^2. \tag{6.65}$$

We only need to verify (6.65) pointwise. To do so, at any given point, we expand the LHS of (6.65) using the normal form of φ in Lemma 4. For simplicity of notation, let us write $B = \mathfrak{D}_j N$. Clearly B has the same symmetry as N, namely it is a TM-valued type $(0,2)$-form and it satisfies the Bianchi identity. By Lemma 4, we get

$$\begin{aligned} \text{LHS} &= \sum_{s,t,i} N_{sti}(\mathfrak{D}_j N)_{t,s,Ji} \\ &= N_{331}(B_{332} - B_{134} + B_{431} - B_{233} + B_{341} - B_{244} - B_{442} - B_{143}) \\ &\quad + N_{331}(B_{552} - B_{156} + B_{651} - B_{255} + B_{561} + B_{266} - B_{662} - B_{165}) \\ &\quad + N_{332}(-B_{331} - B_{234} + B_{432} + B_{133} + B_{342} - B_{144} + B_{441} - B_{243}) \\ &\quad - N_{332}(-B_{551} - B_{256} + B_{652} + B_{155} + B_{562} - B_{166} + B_{661} - B_{265}) \\ &\quad + \ldots \\ &= 4N_{331}(B_{332} + B_{552}) + 4N_{113}(B_{114} + B_{554}) + 4N_{115}(B_{116} + B_{336}) \\ &\quad + 4N_{332}(B_{551} - B_{331}) + 4N_{114}(B_{553} - B_{113}) + 4N_{116}(B_{335} - B_{115}). \end{aligned}$$

Since

$$\mathfrak{D}_j \varphi = \frac{1}{2}(u_j \varphi + u_{Jj} \hat{\varphi}),$$

by taking derivative of (6.45), we get

$$\mathfrak{D}_j N^p{}_{ab} \varphi_{pcd} + \mathfrak{D}_j N^p{}_{cd} \varphi_{pab} = -u_{Jj} N^p{}_{ab} \hat{\varphi}_{pcd},$$

or equivalently

$$B^p{}_{ab} \varphi_{pcd} + B^p{}_{cd} \varphi_{pab} = -u_{Jj} N^p{}_{ab} \hat{\varphi}_{pcd}.$$

Evaluating the above equation at the given point using Lemma 4, we get the following relations

$$\begin{aligned} B_{331} - B_{551} &= -u_{Jj} N_{332}, & B_{332} + B_{552} &= u_{Jj} N_{331}, \\ B_{113} - B_{553} &= -u_{Jj} N_{114}, & B_{114} + B_{554} &= u_{Jj} N_{113}, \\ B_{115} - B_{335} &= -u_{Jj} N_{116}, & B_{116} + B_{336} &= u_{Jj} N_{115}. \end{aligned}$$

It follows that

$$\begin{aligned} \text{LHS of (6.65)} &= 4u_{Jj}(N_{331}^2 + N_{332}^2 + N_{113}^2 + N_{114}^2 + N_{115}^2 + N_{116}^2) \\ &= \frac{1}{4} u_{Jj} |N|^2. \end{aligned}$$

Q.E.D.

7. The flow of the metric in the Type IIA flow

The main task of this section is to prove Theorem 4, which gives explicit formulas for the flows of φ and $\tilde{g}_\varphi$ in terms of the curvature and Nijenhuis tensors.

7.1. A tensor coefficients ODE for φ: proof of Theorem 4(a)

We begin with the proof of Theorem 4 (a), which gives the flow of φ. Since we will be mainly working with the metric $\tilde{g}$, we shall use $\tilde{g}$ to raise or lower indices in this subsection.

It is clear that $\varphi(t)$ is closed and primitive for any t. We can also assume that φ is positive, since this is an open condition and later estimates (7.25) will show that this property is preserved along the flow. Therefore we get a family of Type IIA structures $(M, \omega, \varphi(t))$. So we can apply formulae in Type IIA geometry to expand the right hand side of the flow equation. Now the right hand side is given by $d\Lambda d(|\varphi|^2\hat{\varphi}) = d\Lambda d(e^u\hat{\varphi})$. Combining (6.40), (6.30), and $\beta = 0$, we obtain

$$d\hat{\varphi} = \alpha \wedge \hat{\varphi} + \tilde{\mathfrak{T}} \boxtimes \hat{\varphi} \tag{7.1}$$

where $\tilde{\mathfrak{T}} = N - \frac{1}{4}(d^c\tilde{\omega} + \mathcal{M}(d^c\tilde{\omega}))$ by (6.43). Applying Lemma 15, we obtain $d\hat{\varphi} = N \boxtimes \hat{\varphi}$ and

$$d(e^u\hat{\varphi}) = e^u(du \wedge \hat{\varphi} + N \boxtimes \hat{\varphi}).$$

To proceed, we need to compute $\Lambda(du \wedge \hat{\varphi})$ and $\Lambda(N \boxtimes \hat{\varphi})$, which are 2-forms of type $(2, 0) + (0, 2)$ and of type $(1, 1)$ respectively. We have the following lemmas:

Lemma 17.

$$(\Lambda(du \wedge \hat{\varphi}))_{kl} = g^{ji}u_i\varphi_{jkl}. \tag{7.2}$$

Proof: Since $(du \wedge \hat{\varphi})_{ijkl} = u_i\hat{\varphi}_{jkl} - u_j\hat{\varphi}_{ikl} + u_k\hat{\varphi}_{ijl} - u_l\varphi_{ijk}$, by definition of Λ, we have

$$(\Lambda(du \wedge \hat{\varphi}))_{kl} = \frac{1}{2}\omega^{ji}(u_i\hat{\varphi}_{jkl} - u_j\hat{\varphi}_{ikl} + u_k\hat{\varphi}_{ijl} - u_l\hat{\varphi}_{ijk}).$$

The last two terms in the above expression are zero since $\hat{\varphi}$ is primitive. In addition, the first two terms are identical due to the symmetry in switching

i and j, so it follows that

$$\begin{aligned}(\Lambda(du\wedge\hat{\varphi}))_{kl} &= \omega^{ji}u_i\hat{\varphi}_{jkl}=g^{Jj,i}u_i\hat{\varphi}_{jkl}=g^{ji}u_i\hat{\varphi}_{Jj,k,l}\\ &= g^{ji}u_i\varphi_{jkl},\end{aligned}$$

where in the last step we make use of Lemma 6. Q.E.D.

Lemma 18.

$$\Lambda(N\boxtimes\hat{\varphi})_{kl}=2g^{ji}N^p{}_{il}\varphi_{pjk}=-2g^{ji}N^p{}_{ik}\varphi_{pjl}. \tag{7.3}$$

Proof: By (6.31) and (6.46) we see that

$$\Lambda(N\boxtimes\hat{\varphi})_{kl}=\omega^{ji}(N^p{}_{kl}\hat{\varphi}_{pij}-N^p{}_{ik}\hat{\varphi}_{pjl}+N^p{}_{il}\hat{\varphi}_{pjk}).$$

Notice that the first term above vanishes due to the primitiveness of $\hat{\varphi}$, and again, the last two terms are identical because of the symmetry of switching i and j, so we conclude that

$$\Lambda(N\boxtimes\hat{\varphi})_{kl}=2\omega^{ji}N^p{}_{il}\hat{\varphi}_{pjk}=-2\omega^{ji}N^p{}_{il}\varphi_{p,Jj,k}=2g^{ji}N^p{}_{il}\varphi_{pjk}.$$

Here we again used Lemma 6 to simplify our expression. Q.E.D.

Combining Lemma 17 and Lemma 18, we see immediately that

$$\begin{aligned}\mu_{kl} &:= (\Lambda d(e^u\hat{\varphi}))_{kl}=e^ug^{ji}(u_i\varphi_{jkl}+2N^p{}_{il}\varphi_{pjk})\\ &= e^{2u}(u^s\varphi_{skl}+2N^{st}{}_l\varphi_{stk})=e^{2u}(u^s\varphi_{skl}-2N^{st}{}_k\varphi_{stl}).\end{aligned} \tag{7.4}$$

To compute $d\Lambda d(e^u\hat{\varphi})=d\mu$, we make use of Lemma 13 to get

$$(d\mu)_{iab} = (\tilde{\mathfrak{T}}\boxtimes\mu)_{iab}+\sum_{\text{cyc } i,a,b}\tilde{\mathfrak{D}}_i\mu_{ab}. \tag{7.5}$$

The first term in (7.5) is already in good shape, since by (6.32) we get

$$\begin{aligned}(\tilde{\mathfrak{T}}\boxtimes\mu)_{iab} &= \sum_{\text{cyc } i,a,b}\tilde{\mathfrak{T}}^p{}_{ia}\mu_{pb}=e^{2u}\sum_{\text{cyc } i,a,b}\tilde{\mathfrak{T}}^p{}_{ia}(u^s\varphi_{spb}-2N^{st}{}_p\varphi_{stb})\\ &= e^{2u}\sum_{\text{cyc } i,a,b}\varphi_{sta}(2N^{st}{}_p\tilde{\mathfrak{T}}^p{}_{ib}-u^s\tilde{\mathfrak{T}}^t{}_{ib}),\end{aligned} \tag{7.6}$$

which is linear in φ. For the second term in (7.5), we need

Lemma 19.

$$\tilde{\mathfrak{D}}_i \mu_{ab} = e^{2u}(\varphi_{sab}(\tilde{\mathfrak{D}}_i + u_i)u^s + 2\varphi_{sta}(\tilde{\mathfrak{D}}_i + u_i)N^{st}{}_b). \tag{7.7}$$

Proof: Plugging in (7.4), we see that

$$\begin{aligned}
\tilde{\mathfrak{D}}_i \mu_{ab} &= \tilde{\mathfrak{D}}_i(e^{2u}(u^s \varphi_{sab} + 2N^{st}{}_b \varphi_{sta})) \\
&= e^{2u}(2u_i(u^s \varphi_{sab} + 2N^{st}{}_b \varphi_{sta}) + u^s \tilde{\mathfrak{D}}_i \varphi_{sab} + 2N^{st}{}_b \tilde{\mathfrak{D}}_i \varphi_{sta} \\
&\quad + \varphi_{sab} \tilde{\mathfrak{D}}_i u^s + 2\varphi_{sta} \tilde{\mathfrak{D}}_i N^{st}{}_b) \\
&\overset{(6.44)}{=} e^{2u}(\varphi_{sab}(\tilde{\mathfrak{D}}_i + u_i)u^s + 2\varphi_{sta}(\tilde{\mathfrak{D}}_i + u_i)N^{st}{}_b).
\end{aligned}$$

Q.E.D.

Combining (7.6) and (7.7) we obtain the evolution equation for φ stated in Theorem 4 (a).

Next we justify the remark made after Theorem 4, to the effect that the function u is determined by $\tilde{g}$. Indeed $\tilde{g} = e^u g$, and thus to prove the second part of the statement, we notice that $\tilde{g} = e^u g$, therefore the volume element associated to $\tilde{g}$ satisfies

$$\mathrm{dvol}_{\tilde{g}} = e^{3u} \mathrm{dvol}_g = e^{3u} \frac{\omega^3}{3!}.$$

Therefore (in Darboux coordinate) we may write the global function u as

$$u = \frac{1}{6} \log \det \tilde{g}, \tag{7.8}$$

which is entirely determined by $\tilde{g}$. Hence the metric g is also determined by $\tilde{g}$, and so is the almost complex structurer J since ω is fixed. It follows that the Nijenhuis tensor N, the projected Levi-Civita connection $\tilde{\mathfrak{D}}$ and its torsion $\tilde{\mathfrak{T}}$ are also determined by $\tilde{g}$.

For the convenience of later calculations, we derive a more explicit evolution equation for φ than what we have in Theorem 4 Part (a). The starting point is (7.5), which can be expanded as

$$(d\mu)_{iab} = (N \boxtimes \mu)_{iab} - (U \boxtimes \mu)_{iab} + \sum_{\text{cyc } i,a,b} \tilde{\mathfrak{D}}_i \mu_{ab} \tag{7.9}$$

by using (6.43). The first two terms in (7.9) can be expressed as follows:

Lemma 20.

$$(N \boxtimes \mu)_{iab} = e^{2u} \sum_{\mathrm{cyc}\ i,a,b} \varphi_{iap}(u^s N^p{}_{sb} - 2N^{st}{}_b N^p{}_{st}). \tag{7.10}$$

Proof of the Lemma: By (6.32), we have

$$\begin{aligned}(N \boxtimes \mu)_{iab} &= \sum_{\mathrm{cyc}\ i,a,b} N^p{}_{ia}\mu_{pb} = e^{2u} \sum_{\mathrm{cyc}\ i,a,b} N^p{}_{ia}(u^s\varphi_{spb} + 2N^{st}{}_b\varphi_{stp}) \\ &= e^{2u} \sum_{\mathrm{cyc}\ i,a,b} N^p{}_{ia}(-u^s\varphi_{psb} + 2N^{st}{}_b\varphi_{pst}) \\ &\overset{(6.45)}{=} e^{2u} \sum_{\mathrm{cyc}\ i,a,b} \varphi_{iap}(u^s N^p{}_{sb} - 2N^{st}{}_b N^p{}_{st}).\end{aligned}$$

Q.E.D.

Lemma 21. *For any real 2-form μ, we may write $\mu = \mu^+ + \mu^-$, where μ^+ and μ^- are the $(1,1)$ and $(2,0)+(0,2)$ components of μ respectively. Then*

$$(U \boxtimes \mu)_{iab} = -\frac{1}{2}\tilde{\omega}^{qp}u_q(\tilde{\omega} \wedge \mu)_{piab} - \frac{1}{2}\sum_{\mathrm{cyc}\ i,a,b} u_{Ji}(\mu^-)_{Ja,b}. \tag{7.11}$$

In particular, for the specific μ in (7.4), we get

$$\begin{aligned}(U \boxtimes \mu)_{iab} &= \frac{1}{2}(\tilde{\omega} \wedge \iota_W\mu)_{iab} + \frac{e^{2u}}{2}|du|^2_{\tilde{g}}\varphi_{iab} \\ &\quad -e^{2u} \sum_{\mathrm{cyclic}\ i,a,b} u_i(u^s\varphi_{sab} + N^{st}{}_b\varphi_{sta}).\end{aligned} \tag{7.12}$$

Proof of the Lemma: By (6.32) and the definition of U (6.28), we have

$$\begin{aligned}(U \boxtimes \mu)_{iab} &= \tilde{g}^{pq} \sum_{\mathrm{cyc}\ i,a,b} U_{qia}\mu_{pb} \\ &= \frac{1}{4}\tilde{g}^{pq} \sum_{\mathrm{cyc}\ i,a,b} (2\alpha_{Jq}\tilde{\omega}_{ia} + \alpha_{Ji}\tilde{\omega}_{aq} + \alpha_{Ja}\tilde{\omega}_{qi} + \alpha_i\tilde{g}_{aq} - \alpha_a\tilde{g}_{qi})\mu_{pb} \\ &= \frac{1}{4} \sum_{\mathrm{cyc}\ i,a,b} 2\tilde{\omega}^{qp}\alpha_q\tilde{\omega}_{ia}\mu_{pb} + \alpha_{Ji}\mu_{Ja,b} - \alpha_{Ja}\mu_{Ji,b} + \alpha_i\mu_{ab} - \alpha_a\mu_{ib} \\ &= \frac{1}{2}(\alpha \wedge \mu)_{iab} + \frac{1}{4}\sum_{\mathrm{cyc}\ i,a,b} 2\tilde{\omega}^{qp}\alpha_q\tilde{\omega}_{ia}\mu_{pb} + \alpha_{Ji}(\mu_{Ja,b} - \mu_{Jb,a}).\end{aligned}$$

Notice that $(\tilde{\omega} \wedge \mu)_{piab} = \tilde{\omega}_{ia}\mu_{pb} + \tilde{\omega}_{ab}\mu_{pi} + \tilde{\omega}_{bi}\mu_{pa} - \tilde{\omega}_{pa}\mu_{ib} - \tilde{\omega}_{pb}\mu_{ai} - \tilde{\omega}_{pi}\mu_{ba}$, therefore

$$\tilde{\omega}^{qp}\alpha_q(\tilde{\omega} \wedge \mu)_{piab} = (\alpha \wedge \mu)_{iab} + \tilde{\omega}^{qp}\alpha_q \sum_{\text{cyc } i,a,b} \tilde{\omega}_{ia}\mu_{pb}.$$

Hence we find out that

$$(U \boxtimes \mu)_{iab} = \frac{1}{2}\tilde{\omega}^{qp}\alpha_q(\tilde{\omega} \wedge \mu)_{piab} + \frac{1}{4}\sum_{\text{cyc } i,a,b} \alpha_{Ji}(\mu_{Ja,b} - \mu_{Jb,a}).$$

Write $\mu = \mu^+ + \mu^-$, where μ^+ and μ^- are the $(1,1)$ and $(2,0)+(0,2)$ components. Then by definition, we have

$$(\mu^+)_{Ja,b} = (\mu^+)_{Jb,a}, \quad (\mu^-)_{Ja,b} = -(\mu^-)_{Jb,a}.$$

So we conclude that

$$(U \boxtimes \mu)_{iab} = -\frac{1}{2}\tilde{\omega}^{qp}u_q(\tilde{\omega} \wedge \mu)_{piab} - \frac{1}{2}\sum_{\text{cyc } i,a,b} u_{Ji}(\mu^-)_{Ja,b}.$$

Now let us apply this to $\mu = \Lambda d(e^u\hat{\varphi})$. It is clear from (7.4) that

$$(\mu^-)_{ab} = e^{2u}u^s\varphi_{sab} = e^{2u}(\iota_{\tilde{\nabla}u}\varphi)_{ab}. \tag{7.13}$$

Let $W = W^p\partial_p$ be the vector field defined by $W^p = -\tilde{\omega}^{qp}u_q = (\tilde{\nabla}u)^{Jp}$, we see that

$$-\frac{1}{2}\tilde{\omega}^{qp}u_q(\tilde{\omega} \wedge \mu)_{piab} = \frac{1}{2}(\iota_W(\tilde{\omega} \wedge \mu))_{iab} = \frac{1}{2}(\tilde{\omega} \wedge \iota_W\mu)_{iab} - \frac{1}{2}(du \wedge \mu)_{iab},$$

hence

$$\begin{aligned}(U \boxtimes \mu)_{iab} &\overset{(7.13)}{=} \frac{1}{2}(\tilde{\omega} \wedge \iota_W\mu)_{iab} - \frac{1}{2}(du \wedge \mu)_{iab} - \frac{1}{2}e^{2u}\sum_{\text{cyclic } i,a,b} u_{Ji}u^{Js}\varphi_{sab} \\ &\overset{(7.4)}{=} \frac{1}{2}(\tilde{\omega} \wedge \iota_W\mu)_{iab} - \frac{e^{2u}}{2}(du \wedge \iota_{\tilde{\nabla}u}\varphi + Jdu \wedge \iota_{J\tilde{\nabla}u}\varphi) \\ &\quad - e^{2u}\sum_{\text{cyclic } i,a,b} u_i N^{st}{}_b\varphi_{sta}.\end{aligned}$$

From (4.13) we know that

$$du \wedge \varphi = -Jdu \wedge J\varphi,$$

by taking interior product with $\tilde{\nabla} u$, we get

$$|du|^2_{\tilde{g}}\varphi - du \wedge \iota_{\tilde{\nabla} u}\varphi = \iota_{\tilde{\nabla} u}(du \wedge \varphi) = -\iota_{\tilde{\nabla} u}(Jdu \wedge J\varphi) = -Jdu \wedge \iota_{J\tilde{\nabla} u}\varphi.$$

Substitute the RHS of the above equation back to the previous one, we get

$$(U \boxtimes \mu)_{iab} = \frac{1}{2}(\tilde{\omega} \wedge \iota_W \mu)_{iab} + \frac{e^{2u}}{2}|du|^2_{\tilde{g}}\varphi_{iab} - e^{2u} \sum_{\text{cyclic } i,a,b} u_i(u^s\varphi_{sab} + N^{st}{}_b\varphi_{sta}).$$

Q.E.D.

Combining (7.10), (7.13) and (7.7), we conclude that

$$\begin{aligned}
(7.14)\ \partial_t\varphi_{iab} &= (d\mu)_{iab} \\
&= -\frac{1}{2}(\tilde{\omega} \wedge \iota_W \mu)_{iab} - \frac{e^{2u}}{2}|du|^2_{\tilde{g}}\varphi_{iab} + e^{2u} \sum_{\text{cyc } i,a,b} \varphi_{iap}(u^s N^p{}_{sb} - 2N^{st}{}_b N^p{}_{st}) \\
&\quad + e^{2u} \sum_{\text{cyc } i,a,b} (\varphi_{sab}(\tilde{\mathfrak{D}}_i + 2u_i)u^s + \varphi_{sta}(2\tilde{\mathfrak{D}}_i + 3u_i)N^{st}{}_b).
\end{aligned}$$

7.2. The flow of $\tilde{g}_\varphi$: proof of Theorem 4(b)

By definition of $\tilde{g}$ (4.12), we know that

$$\begin{aligned}
\partial_t\tilde{g}_{ij} &= -\partial_t\varphi_{iab}\varphi_{jcd}\omega^{ac}\omega^{bd} - \varphi_{iab}\partial_t\varphi_{jcd}\omega^{ac}\omega^{bd} \\
&= -\partial_t\varphi_{iab}\varphi_{jcd}\omega^{ac}\omega^{bd} + (i \leftrightarrow j).
\end{aligned} \tag{7.15}$$

We only need to compute the first term in (7.15) as the full expression is the symmetrization of the first term there. This term can be calculated using (7.14). It is useful to observe the following:

Lemma 22. *Suppose λ is a 3-form that can be factorized as the product of a 1-form with ω, i.e. $\lambda = \nu \wedge \omega$ for some 1-form ν. Then*

$$(\varphi_{iab}\lambda_{jcd} + \lambda_{iab}\varphi_{jcd})\omega^{ac}\omega^{bd} = 0.$$

Proof of the lemma. By our assumption, $\lambda_{iab} = \nu_i\omega_{ab} + \nu_a\omega_{bi} + \nu_b\omega_{ia}$. Therefore

$$\begin{aligned}
\lambda_{iab}\varphi_{jcd}\omega^{ac}\omega^{bd} &= (\nu_i\omega_{ab} + \nu_a\omega_{bi} + \nu_b\omega_{ia})\varphi_{jcd}\omega^{ac}\omega^{bd} \\
&= \nu_i\varphi_{jcd}\omega^{dc} - \nu_a\varphi_{jci}\omega^{ac} + \nu_b\varphi_{jid}\omega^{bd}
\end{aligned}$$

$$= \quad -2\nu_a\omega^{ac}\varphi_{ijc},$$

where the primitiveness of φ is used. After symmetrization in i and j, the outcome is zero. Q.E.D.

By Lemma 22, we do not need to worry about the first term in (7.14). For simplicity of notation, let F be the 3-form defined by

$$\begin{aligned} F_{iab} &= \sum_{\text{cyc } i,a,b}\Big(\varphi_{iap}(u^s N^p{}_{sb} - 2N^{st}{}_b N^p{}_{st}) + \varphi_{sab}(\tilde{\mathfrak{D}}_i + 2u_i)u^s \\ &\qquad + \varphi_{sta}(2\tilde{\mathfrak{D}}_i + 3u_i)N^{st}{}_b\Big) \\ &= \sum_{\text{cyc } i,a,b}\Big(\varphi_{pab}(u^s N^p{}_{si} - 2N^{st}{}_i N^p{}_{st} + (\tilde{\mathfrak{D}}_i + 2u_i)u^p) \\ &\qquad + \varphi_{sta}(2\tilde{\mathfrak{D}}_i + 3u_i)N^{st}{}_b\Big), \end{aligned} \tag{7.16}$$

and hence (7.14) can be rewritten as

$$\partial_t\varphi = -\frac{1}{2}\tilde{\omega}\wedge\iota_W\mu - \frac{e^{2u}}{2}|du|^2_{\tilde{g}}\varphi + e^{2u}F,$$

and we have that

$$e^{-2u}\partial_t\tilde{g}_{ij} + |du|^2_{\tilde{g}}\tilde{g}_{ij} = -F_{iab}\varphi_{jcd}\omega^{ac}\omega^{bd} + (i \leftrightarrow j). \tag{7.17}$$

The goal is to compute $F_{iab}\varphi_{jcd}\omega^{ac}\omega^{bd}$. By (7.16) we know

$$\begin{aligned} & F_{iab}\varphi_{jcd}\omega^{ac}\omega^{bd} \\ = \;& \varphi_{jcd}\omega^{ac}\omega^{bd}\sum_{\text{cyc } i,a,b}\Big(\varphi_{pab}(u^s N^p{}_{si} - 2N^{st}{}_i N^p{}_{st} + (\tilde{\mathfrak{D}}_i + 2u_i)u^p) \\ & + \varphi_{sta}(2\tilde{\mathfrak{D}}_i + 3u_i)N^{st}{}_b\Big) \\ = \;& \varphi_{pab}\varphi_{jcd}\omega^{ac}\omega^{bd}(u^s N^p{}_{si} - 2N^{st}{}_i N^p{}_{st} + (\tilde{\mathfrak{D}}_i + 2u_i)u^p) \\ & + 2\varphi_{iap}\varphi_{jcd}\omega^{ac}\omega^{bd}(u^s N^p{}_{sb} - 2N^{st}{}_b N^p{}_{st} + (\tilde{\mathfrak{D}}_b + 2u_b)u^p) \\ & + \varphi_{stb}\varphi_{jcd}\omega^{ac}\omega^{bd}((4\tilde{\mathfrak{D}}_a + 6u_a)N^{st}{}_i - (2\tilde{\mathfrak{D}}_i + 3u_i)N^{st}{}_a)) \\ =: \;& (A) + (B) + (C), \end{aligned} \tag{7.18}$$

where (A), (B), and (C) denote the first, the second, and the third line in (7.18) respectively. (A) can be computed using definition of $\tilde{g}$ (4.12) directly:

$$\begin{aligned}(A) &= -(u^s N_{jsi} - 2N^{st}{}_i N_{jst} + (\tilde{\mathfrak{D}}_i + 2u_i)u_j) \\ &= -u^s N_{jsi} + 2(N_-^2)_{ij} - 2(N_+^2)_{ij} - \tilde{\mathfrak{D}}_i u_j - 2u_i u_j. \end{aligned} \tag{7.19}$$

To compute (B) and (C), we need to invoke Lemma 8. It follows that

$$\begin{aligned}(B) &= 2\varphi_{iap}\varphi_{jcd}\omega^{ac}\omega^{bd}(u^s N^p{}_{sb} - 2N^{st}{}_b N^p{}_{st} + (\tilde{\mathfrak{D}}_b + 2u_b)u^p) \\ &= \frac{1}{2}\omega^{bd}(\omega_{ij}\tilde{g}_{pd} - \omega_{pj}\tilde{g}_{id} - \omega_{id}\tilde{g}_{p_j} + \omega_{pd}\tilde{g}_{ij}) \times \\ &\quad (u^s N^p{}_{sb} - 2N^{st}{}_b N^p{}_{st} + (\tilde{\mathfrak{D}}_b + 2u_b)u^p) \\ &= \frac{1}{2}(\tilde{\omega}_{ij}J^b{}_p - \tilde{\omega}_{pj}J^b{}_i + \tilde{g}_{pj}\delta^b{}_i - \tilde{g}_{ij}\delta^b{}_p) \times \\ &\quad (u^s N^p{}_{sb} - 2N^{st}{}_b N^p{}_{st} + (\tilde{\mathfrak{D}}_b + 2u_b)u^p) \\ &= \frac{1}{2}(-4N^{st}{}_i N_{jst} + 2N^{stp}N_{pst}\tilde{g}_{ij} + \tilde{\omega}_{ij}\tilde{\omega}^{bp}\tilde{\mathfrak{D}}_b u_p \\ &\quad + (\tilde{\mathfrak{D}}_i + 2u_i)u_j + (\tilde{\mathfrak{D}}_{Ji} + 2u_{Ji})u_{Jj} - (\tilde{\mathfrak{D}}^s u_s + 2|du|^2_{\tilde{g}})\tilde{g}_{ij}) \\ &= 2(N_+^2)_{ij} - 2(N_-^2)_{ij} - \frac{1}{2}(|N|^2_{\tilde{g}} + \tilde{\mathfrak{D}}^s u_s + 2|du|^2_{\tilde{g}})\tilde{g}_{ij} \\ &\quad + \frac{1}{2}((\tilde{\mathfrak{D}}_i + 2u_i)u_j + (\tilde{\mathfrak{D}}_{Ji} + 2u_{Ji})u_{Jj} + \tilde{\omega}_{ij}\tilde{\omega}^{bp}\tilde{\mathfrak{D}}_b u_p) \end{aligned} \tag{7.20}$$

and

$$\begin{aligned}(C) &= \varphi_{stb}\varphi_{jcd}\omega^{ac}\omega^{bd}((4\tilde{\mathfrak{D}}_a + 6u_a)N^{st}{}_i - (2\tilde{\mathfrak{D}}_i + 3u_i)N^{st}{}_a)) \\ &= \frac{1}{4}\omega^{ac}(\omega_{sj}\tilde{g}_{tc} - \omega_{tj}\tilde{g}_{sc} - \omega_{sc}\tilde{g}_{tj} + \omega_{tc}\tilde{g}_{sj}) \times \\ &\quad ((4\tilde{\mathfrak{D}}_a + 6u_a)N^{st}{}_i - (2\tilde{\mathfrak{D}}_i + 3u_i)N^{st}{}_a)) \\ &= \frac{1}{4}(\tilde{\omega}_{sj}J^a{}_t - \tilde{\omega}_{tj}J^a{}_s + \tilde{g}_{tj}\delta^a{}_s - \tilde{g}_{sj}\delta^a{}_t) \times \\ &\quad ((4\tilde{\mathfrak{D}}_a + 6u_a)N^{st}{}_i - (2\tilde{\mathfrak{D}}_i + 3u_i)N^{st}{}_a)) \\ &= -(2\tilde{\mathfrak{D}}^s + 3u^s)N_{isj}. \end{aligned} \tag{7.21}$$

Combining (7.19), (7.20), (7.21), and (7.17), we get

$$\begin{aligned}\partial_t \tilde{g}_{ij} = e^{2u}\Big[& 2(\tilde{\mathfrak{D}}^k + 2u^k)(N_{ikj} + N_{jki}) + 2u_i u_j - 2u_{Ji}u_{Jj} \\ & + \frac{1}{2}(\tilde{\mathfrak{D}}_i u_j + \tilde{\mathfrak{D}}_j u_i - \tilde{\mathfrak{D}}_{Ji}u_{Jj} - \tilde{\mathfrak{D}}_{Jj}u_{Ji}) + (\tilde{\mathfrak{D}}^s u_s + |du|^2_{\tilde{g}} + |N|^2_{\tilde{g}})\tilde{g}_{ij}\Big]. \end{aligned} \tag{7.22}$$

Equation (7.22) is self-contained in the sense that its RHS is entirely determined by the metric $\tilde{g}$ and the φ-dependence is fully eliminated. However, to study its analytic behavior, we need to rewrite it in a more familiar form as we have in the case of Ricci flow or the Lê-Wang flow [33]. Moreover, we would like to replace all the $\tilde{\mathfrak{D}}$-derivatives to $\tilde{\nabla}$-derivatives as it is more convenient for us to apply the conformal change technique.

Notice that

$$\tilde{\mathfrak{D}}_i u_j = \tilde{\nabla}_i u_j - u^k N_{ikj} + \frac{1}{4}(u_i u_j + u_{Ji} u_{Jj} - |du|^2_{\tilde{g}} \tilde{g}_{ij}),$$

therefore (7.22) can be rephrased as

$$\begin{aligned}\partial_t \tilde{g}_{ij} &= e^{2u}\Big[(2\tilde{\mathfrak{D}}^k + 3u^k)(N_{ikj} + N_{jki}) + (\tilde{\nabla}^2 u)_{ij} - (\tilde{\nabla}^2 u)_{Ji,Jj} \\ &\quad +2u_i u_j - 2u_{Ji} u_{Jj} + (\tilde{\Delta} u + |N|^2_{\tilde{g}})\tilde{g}_{ij}\Big].\end{aligned}$$

Taking (6.59) into account, we obtain the desired formula

$$\begin{aligned}\partial_t \tilde{g}_{ij} &= e^{2u}\Big[-2\tilde{R}_{ij} - 2(\tilde{\nabla}^2 u)_{ij} + 4u^k(N_{ikj} + N_{jki}) - 4(N^2_-)_{ij} \\ &\quad +u_i u_j - u_{Ji} u_{Jj} + (|du|^2_{\tilde{g}} + |N|^2_{\tilde{g}})\tilde{g}_{ij}\Big]. \qquad (7.23)\end{aligned}$$

Q.E.D.

Recalling that $u = (1/6)\log \det \tilde{g}$, we can derive from (7.23) that

$$\begin{aligned}\partial_t u &= \frac{e^{2u}}{3}\left(-\tilde{\Delta} u - \tilde{R} + 3|du|^2_{\tilde{g}} + 2|N|^2_{\tilde{g}}\right) \\ &\overset{(6.60)}{=} e^{2u}(\tilde{\Delta} u + |N|^2_{\tilde{g}}). \qquad (7.24)\end{aligned}$$

By the maximum principle, we immediately prove the following estimate

Lemma 23. *Suppose $\varphi(t)$ is a solution to the source-free Type IIA flow on $M \times [0,T]$. Then*

$$|\varphi(t)|^2 \geq \min_M |\varphi_0|^2 \qquad (7.25)$$

for any $t \in [0,T]$.

This lemma has the important consequence that if $\varphi(t)$ is a solution to the source-free Type IIA flow on $M \times [0, T]$ with primitive closed initial data, then $\varphi(t)$ remains positive on $M \times [0, T]$, which allows us to define the almost-complex structure J and the metric g. Indeed, the lemma implies that $\sqrt{-\lambda_\varphi} = \frac{1}{2}|\varphi|^2 \frac{\omega^3}{3!}$ cannot pass through zero.

7.3. Conformal transformation to a perturbed Ricci flow

In this subsection, we wish to establish the uniqueness of the flow (7.23). Besides the Ricci curvature, the right hand side of (7.23) also contains the 2nd order term $\tilde{\nabla}^2 u$, which cannot be reparametrized away since this would change the symplectic structure and the reparametrized flow would be non-local. Therefore we need a different technique to deal with the Hessian term, namely we absorb it in the Ricci tensor by a conformal change of metric.

More specifically, we consider a family of conformal Hermitian metrics $g^{(s)} = e^{su} g$, where we have $g^{(0)} = g$ and $g^{(1)} = \tilde{g}$. Notice that all the metrics $g^{(s)}$ are equivalent except for $s = 0$, in which case we need the pair (g, u). This is because one can solve u from $g^{(s)}$ when $s \neq 0$ by

$$u = \frac{1}{6s} \log \det\, g^{(s)}.$$

Thus we only need to show the short-time existence and uniqueness of any of the flows satisfied by $g^{(s)}$ with $s \neq 0$, or that for the coupled flow (g, u). To begin with, we first compute the evolution equation satisfied by the pair (g, u).

$$\begin{aligned}
& \partial_t g_{ij} \\
= \quad & \partial_t(e^{-u}\tilde{g}_{ij}) = e^{-u}(\partial_t \tilde{g}_{ij} - \partial_t u \cdot \tilde{g}_{ij}) \\
= \quad & e^u\Big[-2\tilde{R}_{ij} - 2(\tilde{\nabla}^2 u)_{ij} + 4u^k(N_{ikj} + N_{jki}) - 4(N_-^2)_{ij} \\
& + u_i u_j - u_{Ji}u_{Jj} + (|du|^2_{\tilde{g}} - \tilde{\Delta} u)\tilde{g}_{ij}\Big] \\
= \quad & e^u\Big[-2R_{ij} + 2(\nabla^2 u)_{ij} + u_i u_j - u_{Ji}u_{Jj} \\
& \quad + 4u^k(N_{ikj} + N_{jki}) - 4(N_-^2)_{ij}\Big] \qquad (7.26) \\
\overset{(6.53)}{=} \quad & 2e^u\Big[-(R^{-J})_{ij} + ((\nabla^2 u)^{-J})_{ij} + ((du \otimes du)^{-J})_{ij}
\end{aligned}$$

$$+2u^s(N_{isj}+N_{jsi})\Big]. \tag{7.27}$$

Meanwhile (7.24) can be rewritten as

$$\partial_t u = e^u(\Delta u + 2|du|^2 + |N|^2). \tag{7.28}$$

Using the same method, we can derive that

$$\begin{aligned}\partial_t g^{(s)}_{ij} &= e^{(s+1)u}\Big[-2R^{(s)}_{ij} + (2-4s)((\nabla^{(s)})^2 u)_{ij} + (1+2s-2s^2)u_iu_j \\ &\quad -u_{Ji}u_{Jj} + 4u^k(N_{ikj}+N_{jki}) - 4(N^2_-)_{ij} + s(|du|^2_{g^{(s)}} + |N|^2_{g^{(s)}})g^{(s)}_{ij}\Big] \qquad (7.29)\\ &= e^{(s+1)u}\Big[2\left((-R^{(s)} + (1-2s)(\nabla^{(s)})^2u + (1+s-s^2)du\otimes du)^{-J}\right)_{ij} \\ &\quad +4u^k(N_{ikj}+N_{jki}) + s\left(\Delta^{(s)}u + 2(1-s)|du|^2_{g^{(s)}} + |N|^2_{g^{(s)}}\right)g^{(s)}_{ij}\Big]. \qquad (7.30)\end{aligned}$$

Formulae (6.54) and (6.55) now take the form

$$\begin{aligned}R^{(s)}_{ij} &= -\nabla^{(s)}_k(N_i{}^k{}_j + N_j{}^k{}_i) + (\frac{1}{2}-2s)((\nabla^{(s)})^2u)_{ij} + \frac{1}{2}((\nabla^{(s)})^2u)_{Ji,Jj} \\ &\quad +\frac{s}{2}(1-2s)u_iu_j + \frac{s}{2}u_{Ji}u_{Jj} + \frac{3s}{2}u^k(N_{ikj}+N_{jki}) \\ &\quad +2(N^2_-)_{ij} - 2(N^2_+)_{ij} - \frac{s}{2}\Delta^{(s)}u g^{(s)}_{ij} - \frac{s}{2}(1-2s)|du|^2_{g^{(s)}}g^{(s)}_{ij}, \qquad (7.31)\end{aligned}$$

and

$$\nabla^{(s)}_k N^k{}_{ij} = \frac{5s-2}{2}u^kN_{kij}. \tag{7.32}$$

In particular for $s=\frac{1}{2}$, from (7.29) we know that the metric $\breve{g} := g^{(\frac{1}{2})}$ evolves by

$$\begin{aligned}\partial_t\breve{g}_{ij} &= e^{\frac{3}{2}u}\Big[-2\breve{R}_{ij} + \frac{3}{2}u_iu_j - u_{Ji}u_{Jj} + 4u^k(N_{ikj}+N_{jki}) - 4(N^2_-)_{ij} \\ &\quad +\frac{1}{2}\left(|du|^2_{\breve{g}} + |N|^2_{\breve{g}}\right)\breve{g}_{ij}\Big], \qquad (7.33)\end{aligned}$$

where the only 2nd order term on RHS is the Ricci curvature term. We stress that it is important to keep in mind the fact that $\breve{g}_{ij}$ arises from a conformal change from a Type IIA geometry.

7.4. An integrability condition: proof of Theorem 5

We now prove Theorem 5, which provides the key integrability condition needed later to establish the uniqueness of the Type IIA solutions to the flow (7.33) of the metrics $\check{g}_{ij}$.

We know that for the Ricci flow, the integrability operator L comes from the contracted Bianchi identity. Since our flow (7.33) can be viewed as a deformation of the Ricci flow, our L should be a deformation of the contracted Bianchi operator. Let us simplify our notation in (7.33) by introducing the tensor S defined as

$$S_{ij} : \;=\; \frac{3}{2}u_i u_j - u_{Ji}u_{Jj} + 4u^k(N_{ikj} + N_{jki}) - 4(N_-^2)_{ij} + \frac{1}{2}\left(|du|_{\check{g}}^2 + |N|_{\check{g}}^2\right)\check{g}_{ij}, \tag{7.34}$$

so (7.33) can be written as

$$\partial_t \check{g}_{ij} \;=\; e^{\frac{3}{2}u}(-2\check{R}_{ij} + S_{ij}), \tag{7.35}$$

where we can think of S as the lower order deformation term of the Ricci curvature. Let L_0 denote the contracted Bianchi identity operator defined by

$$L_0(P)_j := 2\check{g}^{ik}\check{\nabla}_k P_{ij} - \check{g}^{ik}\check{\nabla}_j P_{ik}$$

for any symmetric 2-tensor P. We know that $L_0(-2\check{R}) = 0$. Now we would like to look for a zeroth order linear operator Z such that $(L_0+Z)(-2\check{R}+S)$ is of degree 1 in the metric $\check{g}$. To do so we need to compute $L_0(S)$ first.

Proposition 2.

$$L_0(S)_j \;=\; 4u^s\check{R}_{sj} - 8\check{R}_{sk}N^{ks}{}_j - \frac{8}{3}u_j\check{R} - 16u^s(N_-^2)_{sj} + 2u_s u_k N^{ks}{}_j + \frac{1}{3}u_j(2|du|_{\check{g}}^2 - 5|N|_{\check{g}}^2). \tag{7.36}$$

Proof of the Proposition: We apply L_0 to each term of S in (7.34) to get

$$\frac{3}{2}L_0(du\otimes du)_j = 3\check{\Delta}u\cdot u_j,$$
$$-L_0(J^*(du\otimes du))_j = 2\check{g}^{ik}u_k(\check{\nabla}^2 u)_{Ji,Jj} + \check{\nabla}_j|du|_{\check{g}}^2 - 2\check{g}^{ik}u_s u_t\check{\nabla}_i(J^s{}_j J^t{}_k)$$

$$= 2\check{g}^{ik}u_k(\check{\nabla}^2 u)_{Ji,Jj} + \check{\nabla}_j|du|^2_{\check{g}} + 4u_s u_k N^{ks}{}_j,$$

$$4L_0(u^k(N_{*k*} + N_{*k*}))_j = 8(\check{\nabla}^2 u)_{sk}N^{ks}{}_j + 8u^s\check{\nabla}^i(N_{isj} + N_{jsi}),$$

$$-4L_0(N^2_-)_j = -8\check{g}^{ik}\check{\nabla}_k(N^2_-)_{ij} + 2\check{\nabla}_j|N|^2_{\check{g}},$$

$$\frac{1}{2}L_0\left((|du|^2_{\check{g}} + |N|^2_{\check{g}})\check{g}\right)_j = -2\check{\nabla}_j|du|^2_{\check{g}} - 2\check{\nabla}_j|N|^2_{\check{g}}.$$

To obtain these expressions, we need to use that

$$\begin{aligned}
\nabla_i J^s{}_t &= -2N_{Ji}{}^s{}_t,\\
\check{\nabla}_i J^s{}_t &= -2N_{Ji}{}^s{}_t + \frac{1}{4}(u_{Jt}\delta^s{}_i + \check{g}_{it}u^{Js} - \check{\omega}_{it}u^s - u_t J^s{}_i)
\end{aligned}$$

and we raise and lower indices using the metric $\check{g}$. Combining the calculation above, we get

$$\begin{aligned}
L_0(S)_j &= 3u_j\check{\Delta}u + 2u^i(\check{\nabla}^2 u)_{Ji,Jj} + 8(\check{\nabla}^2 u)_{sk}N^{ks}{}_j + 8u^s\check{\nabla}^i(N_{isj} + N_{jsi})\\
&\quad -8\check{\nabla}^i(N^2_-)_{ij} - \check{\nabla}_j|du|^2_{\check{g}} + 4u_s u_k N^{ks}{}_j. \qquad (7.37)
\end{aligned}$$

Take $s = \frac{1}{2}$ in (7.31) and (7.32), we get

$$\begin{aligned}
\check{R}_{ij} &= -(\check{\nabla}^k - \frac{3}{4}u^k)(N_{ikj} + N_{jki}) - \frac{1}{2}(\check{\nabla}^2 u)_{ij} + \frac{1}{2}(\check{\nabla}^2 u)_{Ji,Jj}\\
&\quad -\frac{1}{4}\check{\Delta}u\cdot\check{g}_{ij} + \frac{1}{4}u_{Ji}u_{Jj} + 2(N^2_-)_{ij} - 2(N^2_+)_{ij}, \qquad (7.38)
\end{aligned}$$

$$\check{R} = -\frac{3}{2}\check{\Delta}u + \frac{1}{4}|du|^2_{\check{g}} - |N|^2_{\check{g}}, \qquad (7.39)$$

$$\check{\nabla}^k N_{kij} = \frac{1}{4}u^k N_{kij}. \qquad (7.40)$$

Using (7.38) and (7.40), Equation (7.37) can be rearranged as

$$\begin{aligned}
L_0(S)_j &= 3u_j\check{\Delta}u + 2u^i((\check{\nabla}^2 u)_{Ji,Jj} - (\check{\nabla}^2 u)_{ij}) - 4u^s\check{\nabla}^i(N_{jis} + N_{sij} + 3N_{ijs})\\
&\quad +8(\check{\nabla}^2 u)_{sk}N^{ks}{}_j - 8\check{\nabla}^i(N^2_-)_{ij} + 4u_s u_k N^{ks}{}_j\\
&= 4u^s\check{R}_{sj} + 4u_j\check{\Delta}u + 8(\check{\nabla}^2 u)_{sk}N^{ks}{}_j - 8\check{\nabla}^i(N^2_-)_{ij} + 4u_s u_k N^{ks}{}_j\\
&\quad +8u^s((N^2_+)_{sj} - (N^2_-)_{sj})\\
&= 4u^s\check{R}_{sj} - \frac{8}{3}u_j\check{R} + 8(\check{\nabla}^2 u)_{sk}N^{ks}{}_j - 8\check{\nabla}^i(N^2_-)_{ij} + 4u_s u_k N^{ks}{}_j\\
&\quad +8u^s((N^2_+)_{sj} - (N^2_-)_{sj}) + \frac{2}{3}u_j(|du|^2_{\check{g}} - 4|N|^2_{\check{g}}). \qquad (7.41)
\end{aligned}$$

By (7.38), we also know that

$$\begin{aligned}
& (\check{\nabla}^2 u)_{sk} N^{ks}{}_j \\
= \;& -(\check{\nabla}^p - \frac{3}{4}u^p)(N_{spk} + N_{kps})N^{ks}{}_j - \check{R}_{sk}N^{ks}{}_j - \frac{1}{4}u^k u^s N_{ksj} \\
= \;& -\check{R}_{sk}N^{ks}{}_j - \check{\nabla}^p(2N_{kps} + N_{psk})N^{ks}{}_j - \frac{1}{4}u^k u^s N_{ksj} \\
& -\frac{3}{4}u^p(N_+^2 + N_-^2)_{pj} \\
= \;& -\check{R}_{sk}N^{ks}{}_j + 2N^{ks}{}_j\check{\nabla}^p N_{ksp} - \frac{1}{2}u^p(N_-^2 + 2N_+^2)_{pj} - \frac{1}{4}u^k u^s N_{ksj} \\
= \;& -\check{R}_{sk}N^{ks}{}_j + 2\check{\nabla}^p(N_+^2)_{pj} - 2N^{ksp}\check{\nabla}_p N_{ksj} - \frac{1}{4}u^k u^s N_{ksj} \\
& -\frac{1}{2}u^p(N_-^2 + 2N_+^2)_{pj} \\
= \;& -\check{R}_{sk}N^{ks}{}_j + \check{\nabla}^i(N_-^2)_{ij} - 2N^{ksp}\check{\nabla}_p N_{ksj} + \frac{1}{4}\check{\nabla}_j|N|^2_{\check{g}} - \frac{1}{4}u^k u^s N_{ksj} \\
& -\frac{1}{2}u^s(N_-^2 + 2N_+^2)_{sj}. \qquad (7.42)
\end{aligned}$$

Plugging (7.42) in (7.41), we get

$$\begin{aligned}
L_0(S)_j \;=\; & 4u^s\check{R}_{sj} - 8\check{R}_{sk}N^{ks}{}_j - \frac{8}{3}u_j\check{R} - 16N^{ksp}\check{\nabla}_p N_{ksj} + 2\check{\nabla}_j|N|^2_{\check{g}} \\
& -12u^s(N_-^2)_{sj} + 2u_s u_k N^{ks}{}_j + \frac{2}{3}u_j(|du|^2_{\check{g}} - 4|N|^2_{\check{g}}). \qquad (7.43)
\end{aligned}$$

To deal with the remaining second order terms in (7.43), we rewrite (6.64) using $\check{\nabla}$-derivatives as

$$8N^{ksp}\check{\nabla}_p N_{ksj} + 2u_j|N|^2_{\check{g}} - 2u^i(N_-^2)_{ij} = \frac{3}{2}u_j|N|^2_{\check{g}} + \check{\nabla}_j|N|^2_{\check{g}}.$$

Incorporating this identity, we see (7.43) becomes

$$\begin{aligned}
L_0(S)_j \;=\; & 4u^s\check{R}_{sj} - 8\check{R}_{sk}N^{ks}{}_j - \frac{8}{3}u_j\check{R} - 16u^s(N_-^2)_{sj} + 2u_s u_k N^{ks}{}_j \\
& +\frac{1}{3}u_j(2|du|^2_{\check{g}} - 5|N|^2_{\check{g}}).
\end{aligned}$$

Q.E.D.

Let Z be the zeroth order linear operator defined by

$$Z(P)_j := 2u^i P_{ij} - 4N^{st}{}_j P_{st} - \frac{4}{3}u_j \breve{g}^{st} P_{st},$$

then (7.36) says

$$L_0(S)_j = Z(2\check{R}_{**})_j - 16u^s(N_-^2)_{sj} + 2u_s u_k N^{ks}{}_j + \frac{1}{3}u_j(2|du|^2_{\breve{g}} - 5|N|^2_{\breve{g}}).$$

Consider the first linear operator $L_1 = L_0 + Z$, then

$$\begin{aligned} L_1(-2\check{R}_{**} + S)_j &= L_0(-2\check{R}_{**} + S)_j + Z(-2\check{R}_{**} + S)_j \\ &= L_0(S)_j - Z(2\check{R}_{**})_j + Z(S)_j \\ &= -16u^s(N_-^2)_{sj} + 2u_s u_k N^{ks}{}_j + \frac{1}{3}u_j(2|du|^2_{\breve{g}} - 5|N|^2_{\breve{g}}) \\ &\quad +8u^s(2N_+^2 + N_-^2)_{sj} - 2u_s u_k N^{ks}{}_j - \frac{2}{3}u_j(|du|^2_{\breve{g}} + 2|N|^2_{\breve{g}}) \\ &= -u_j|N|^2_{\breve{g}} \end{aligned}$$

is of first order in $\breve{g}$. Therefore if we define the first order linear operator L by

$$L(P) = L_1(e^{-\frac{3}{2}u}P),$$

then L is an integrability condition for the flow (7.33). Theorem 5 is proved. Q.E.D.

7.5. Return to the proof of Theorem 2: uniqueness

It is now easy to establish the uniqueness part in Theorem 2.

Assume that we have two closed, primitive, and positive solutions $\varphi(t)$ and $\varphi'(t)$ of the Type IIA flow on some time interval $[0, T)$ for some $T > 0$, with the same initial data $\varphi(0) = \varphi'(0)$. By Theorem 4, the corresponding pairs $(\varphi(t), \tilde{g}_\varphi(t))$ and $(\varphi'(t), \tilde{g}_{\varphi'}(t))$ satisfy the flows in Theorem 4. Since the geometries $(\omega, J_\varphi, g_\varphi)$ and $(\omega, J_{\varphi'}, g_{\varphi'})$ are by definition Type IIA geometries, the corresponding flows for $\breve{g}_\varphi(t)$ and $\breve{g}_{\varphi'}(t)$ satisfy the integrability condition in Theorem 5. Since the principal symbols in the flow of $\breve{g}_{ij}$ and the integrability condition L are the same up to a multiplicative factor as their counterparts in the Ricci flow, it follows that the flow of $\breve{g}_{ij}$ together with the integrability condition L satisfy all the conditions in the Hamilton-Nash-Moser theorem ([29], Theorem 5.1). By the uniqueness part in this theorem, we conclude that $\check{g}_{ij}(t)$ and $\check{g}'_{ij}(t)$ must be equal. But then φ and $\varphi'(t)$ satisfy the same ODE with the same initial data and hence must be equal. Q.E.D.

7.6. Monotonicity formulas

Recall that the function u evolves by

$$\partial_t u = e^u(\Delta u + 2|\nabla u|^2 + |N|^2). \tag{7.44}$$

From (7.44) one can derive a number of things.

Proposition 3.

$$\min_M u(t) \geq \min_M u(0), \tag{7.45}$$

Proof of the Proposition: Apply maximum principle to (7.44). Q.E.D.

This proposition can be interpreted that if φ_0 is initially positive, then φ stays positive as long as the flow exists. This is because that the only possibility for φ leaving the positive cone is that it first hits the wall of degeneracy defined $|\varphi| = 0$, which contradicts the above C^0-estimate.

Like its Type IIB counterpart [20], one has the following monotonicity formulas for the dilaton functional along the flow.

Proposition 4.

$$\partial_t \int_M e^{pu}\frac{\omega^3}{3!} = p\int_M e^{(p+1)u}\left((1-p)|\nabla u|^2 + |N|^2\right)\frac{\omega^3}{3!}. \tag{7.46}$$

If we denote $\int e^{pu}$ by E_p, then it follows that E_p is monotonely non-increasing along the flow for $p < 0$ and it is monotonely non-decreasing along the flow for $0 < p \leq 1$. In particular, the Hitchin's functional E_1 [30] is monotonely non-decreasing along the source-free Type IIA flow.

8. Estimates for the Type IIA flow

Recall that the Type IIA flow becomes the following flow for the pair $(g(t), u(t))$:

$$\partial_t g_{ij} \;=\; e^u\big[-2R_{ij} + 2\nabla_i\nabla_j u - 4(N_-^2)_{ij} + u_i u_j - u_{Ji}u_{Jj}$$

$$
\begin{aligned}
& \qquad +4u_p(N_i{}^p{}_j + N_j{}^p{}_i)\big] \\
\partial_t u \quad &= \quad e^u\left[\Delta u + 2|\nabla u|^2 + |N|^2\right] \qquad (8.1)
\end{aligned}
$$

In this section, we show that if $|u|+|Rm(g)| \le C$ remains bounded on $[0,T)$, then the flow can be extended to $[0,T+\epsilon)$ for some $\epsilon > 0$.

To start, we examine some consequences of the boundedness of the Riemann curvature tensor. We note that by equation (6.22), a bound on Rm implies a bound on $|N|^2$. Therefore we may assume

$$
|u| + |N|^2 + |Rm| \le C. \qquad (8.2)
$$

Next, since the Ricci curvature Ric is also bounded, so are its J-invariant and J-anti-invariant parts. From (6.56), we know that the J-invariant part of the Ricci curvature $(R^J)_{ij}$ is given by $(\mathrm{Ric}^J) = (\nabla^2 u)^J - 2N_-^2$. As $|N|^2$ is already bounded, we conclude that $\frac{1}{2}(\nabla^2 u)_{ij} + \frac{1}{2}(\nabla^2 u)_{Ji,Jj}$ is bounded, namely, the J-invariant part of $\nabla^2 u$ is bounded. Consequently Δu is also bounded.

Our goal will be to obtain bounds on all derivatives of u, N, Rm. For this, we must first compute the evolution equations of $\nabla^k u$, $\nabla^k N$ and $\nabla^k Rm$.

8.1. The evolution of the derivatives of u

In this section, we compute the evolution of $|\nabla u|^2$ and $|\nabla\nabla u|^2$.

8.1.1. The evolution of the gradient of u. We start with

$$
\partial_t |\nabla u|^2 = 2g^{ij}\nabla_i \dot{u} \nabla_j u - g^{ia}\dot{g}_{ab}g^{bj}u_i u_j. \qquad (8.3)
$$

The differentiated evolution of u is

$$
\begin{aligned}
\nabla_i \dot{u} \quad &= \quad e^u\left(\nabla_i \Delta u + 2\nabla_i|\nabla u|^2 + \nabla_i|N|^2\right) \\
&\qquad + e^u\left(\Delta u + 2|\nabla u|^2 + |N|^2\right) u_i
\end{aligned} \qquad (8.4)
$$

Commuting derivatives

$$
\nabla_i \Delta u = g^{pq}\nabla_i\nabla_p\nabla_q u = \Delta\nabla_i u - g^{pq}R_{ip}{}^{\lambda}{}_q u_\lambda \qquad (8.5)
$$

Therefore the first term in (8.3)

$$
\begin{aligned}
2g^{ij}\nabla_i\dot{u}\nabla_j u &= 2e^u\left(g^{ij}\Delta\nabla_i u\nabla_j u - g^{pq}R_p{}^{i\lambda}{}_q u_\lambda u_i + 2g^{ij}\nabla_i|\nabla u|^2 u_j\right)\\
&\quad +g^{ij}\nabla_i|N|^2 u_j + 2e^u\left(\Delta u + 2|\nabla u|^2 + |N|^2\right)|\nabla u|^2\\
&= e^u\big(\Delta|\nabla u|^2 - 2|\nabla\nabla u|^2 - 2R^{i\lambda}u_\lambda u_i + 8(\nabla\nabla u)^{ij}u_i u_j\\
&\quad +2g^{ij}\nabla_i|N|^2 u_j + 2\Delta u|\nabla u|^2 + 4|\nabla u|^4 + 2|N|^2|\nabla u|^2\big) \qquad (8.6)
\end{aligned}
$$

The second term in (8.3) is

$$
\begin{aligned}
-g^{ia}\dot{g}_{ab}g^{bj}u_i u_j &= e^u\Big[2R^{ij} - 2\nabla^i\nabla^j u + 4(N_-^2)^{ij} - 4u_p(N^{ipj}+N^{jpi})\Big]u_i u_j\\
&\quad -e^u|\nabla u|^4 + e^u[\omega^{ai}u_a u_i][\omega^{bj}u_b u_j]\\
&= e^u\Big[2R^{ij} - 2(\nabla\nabla u)^{ij} + 4(N_-^2)^{ij}\Big]u_i u_j - e^u|\nabla u|^4 \qquad (8.7)
\end{aligned}
$$

The term $(N^{ipj}+N^{jpi})u_i u_j u_p$ vanishes since N^{ipj} is anti-symmetric in p and j, and N^{jpi} is anti-symmetric in p and i. Altogether, (8.3) becomes

$$
\begin{aligned}
(\partial_t - e^u\Delta)|\nabla u|^2 &= e^u\Big[-2|\nabla\nabla u|^2 + 2g^{ij}\nabla_i|N|^2 u_j + 2\Delta u|\nabla u|^2 + 3|\nabla u|^4\\
&\quad +2|N|^2|\nabla u|^2 + 6(\nabla^2 u)^{ij}u_i u_j + 4(N_-^2)^{ij}u_i u_j\Big] \qquad (8.8)
\end{aligned}
$$

The identity $N_-^2 = 2N_+^2 - \frac{1}{4}|N|^2 g$ implies

$$
\begin{aligned}
(\partial_t - e^u\Delta)|\nabla u|^2 &= e^u\Big[-2|\nabla\nabla u|^2 + 2g^{ij}\nabla_i|N|^2 u_j + 2\Delta u|\nabla u|^2 + 3|\nabla u|^4\\
&\quad +|N|^2|\nabla u|^2 + 6(\nabla^2 u)^{ij}u_i u_j + 8(N_+^2)^{ij}u_i u_j\Big]. \qquad (8.9)
\end{aligned}
$$

8.1.2. The evolution of the Hessian of u. We use as usual the notation $u_{ij} = \nabla_i\nabla_j u = (\nabla^2 u)_{ij}$. The variation of the Hessian is

$$\partial_t u_{ip} = \nabla_i\nabla_p\dot{u} - \dot{\Gamma}^\lambda_{ip}u_\lambda. \qquad (8.10)$$

Differentiating (8.4)

$$\nabla_p\nabla_i\dot{u} = e^u\Big[\nabla_p\nabla_i\Delta u + 2\nabla_p\nabla_i|\nabla u|^2 + \nabla_p\nabla_i|N|^2\Big]$$

$$+e^u\left[\nabla_i\Delta u+2\nabla_i|\nabla u|^2+\nabla_i|N|^2\right]u_p$$

$$+e^u\left[\Delta u+2|\nabla u|^2+|N|^2\right]u_{ip}+(i\leftrightarrow p) \tag{8.11}$$

Commuting derivatives, we see that

$$\begin{aligned}\nabla_p\nabla_i\Delta u &= g^{ab}\nabla_p\nabla_a\nabla_i\nabla_b u-g^{ab}\nabla_p(R_{ia}{}^{\lambda}{}_b u_\lambda)\\ &= g^{ab}\nabla_a\nabla_p\nabla_i\nabla_b u-g^{ab}R_{ap}{}^{\lambda}{}_i u_{\lambda b}-g^{ab}R_{ap}{}^{\lambda}{}_b u_{i\lambda}\\ &\quad -g^{ab}\nabla_p(R_{ia}{}^{\lambda}{}_b u_\lambda)\\ &= \Delta\nabla_p\nabla_i u-g^{ab}\nabla_a(R_{pb}{}^{\lambda}{}_i u_\lambda)-g^{ab}R_{ap}{}^{\lambda}{}_i u_{\lambda b}-g^{ab}R_{ap}{}^{\lambda}{}_b u_{i\lambda}\\ &\quad -g^{ab}\nabla_p(R_{ia}{}^{\lambda}{}_b u_\lambda)\end{aligned}$$

Therefore

$$\begin{aligned}\nabla_p\nabla_i\dot u &= e^u\Delta u_{ip}+e^u\left[4g^{ab}u_{ia}u_{pb}\right]\\ &\quad +e^u\left[\nabla Rm*\nabla u+\nabla^2N*N+\nabla^3u*\nabla u\right]\\ &\quad +e^u\left[\nabla^2u*\mathcal{O}(\nabla u,Rm,N)+\nabla N*\nabla N+\nabla N*N*\nabla u\right]\end{aligned}$$

Here we used the identity $R=\Delta u-|N|^2$ on the term $e^u\Delta u\, u_{ip}$.

Before further proceeding the computation, we explain the notations used in the above formula. The terms written as $\alpha*\beta$ represent contractions of the tensors α and β which are linear in both α,β. In later computation, we will also use $(\alpha+\gamma)*(\beta+\eta)$ to represent the linear contractions among the tensors α,β,γ and η. The notation $\mathcal{O}(\nabla u,Rm,N)$ indicates terms which only depend on $\nabla u, Rm$ and N (but the dependence may be nonlinear). We will soon prove a gradient estimate $|\nabla u|\leq C$, so that $\mathcal{O}(\nabla u,Rm,N)$ will be treated as bounded terms.

Next,

$$-\dot\Gamma^{\lambda}_{ip}u_\lambda=-\frac{g^{\lambda\mu}}{2}(-\nabla_\mu\dot g_{ip}+\nabla_p\dot g_{\mu i}+\nabla_i\dot g_{\mu p})u_\lambda \tag{8.12}$$

Since

(8.13)

$$\nabla_\mu\dot g_{ij}=\nabla_\mu\left[e^u(-2R_{ij}+2\nabla_i\nabla_j u-4(N_-^2)_{ij}+u_iu_j-u_{Ji}u_{Jj}+4u_p(N_i{}^p{}_j+N_j{}^p{}_i))\right]$$

we get

$$
\begin{aligned}
-\dot{\Gamma}_{ip}^{\lambda}u_{\lambda} &= e^u\Big[\nabla Rm * \nabla u + \nabla^3 u * \nabla u\Big] + \mathcal{O}(\nabla u, Rm, N)\\
&\quad + e^u\Big[\nabla^2 u * \mathcal{O}(\nabla u, Rm, N) + \nabla N * (N + \nabla u) * \nabla u\Big]
\end{aligned} \tag{8.14}
$$

Therefore

$$
\begin{aligned}
&(\partial_t - e^u\Delta)u_{ip}\\
=\;& e^u\Big[4g^{ab}u_{ia}u_{pb}\Big] + \mathcal{O}(\nabla u, Rm, N)\\
&+e^u\Big[\nabla Rm * \nabla u + \nabla^2 N * N + \nabla^3 u * \nabla u\Big]\\
&+e^u\Big[\nabla^2 u * \mathcal{O}(\nabla u, Rm, N) + \nabla N * \nabla N + \nabla N * (N + \nabla u) * \nabla u\Big]
\end{aligned}
$$

Next, we compute

$$
\begin{aligned}
(\partial_t - e^u\Delta)|\nabla^2 u|^2 &= 2g^{ij}g^{pq}(\partial_t - e^u\Delta)u_{ip}u_{jq} - 2e^u|\nabla^3 u|^2\\
&\quad -g^{ia}\dot{g}_{ab}g^{bj}g^{pq}u_{ip}u_{jq} - g^{ij}g^{pa}\dot{g}_{ab}g^{bq}u_{ip}u_{jq}.
\end{aligned} \tag{8.15}
$$

The first term is then

$$
\begin{aligned}
&2g^{ij}g^{pq}(\partial_t - e^u\Delta)u_{ip}u_{jq}\\
=\;& e^u\Big[8g^{ab}u_{ia}u_{pb}(\nabla^2 u)^{ip}\Big] + \mathcal{O}(\nabla u, Rm, N) * \nabla^2 u\\
&+e^u\Big[\nabla Rm * \nabla u + \nabla^2 N * N + \nabla^3 u * \nabla u\Big] * \nabla^2 u\\
&+e^u\Big[\nabla^2 u * \mathcal{O}(\nabla u, Rm, N) + \nabla N * \nabla N + \nabla N * (N + \nabla u) * \nabla u\Big] * \nabla^2 u
\end{aligned}
$$

Since $\dot{g}_{ab} = 2e^u u_{ab} + \mathcal{O}(\nabla u, Rm, N)$

$$
\begin{aligned}
&(\partial_t - e^u\Delta)|\nabla^2 u|^2 \qquad (8.16)\\
=\;& e^u\Big[4g^{ab}u_{ia}u_{pb}(\nabla^2 u)^{ip}\Big] - 2e^u|\nabla^3 u|^2 + \mathcal{O}(\nabla u, Rm, N) * \nabla^2 u\\
&+e^u\Big[\nabla Rm * \nabla u + \nabla^2 N * N + \nabla^3 u * \nabla u\Big] * \nabla^2 u
\end{aligned}
$$

$$+e^u\left[\nabla^2 u * \mathcal{O}(\nabla u, Rm, N) + \nabla N * \nabla N + \nabla N * (N+\nabla u) * \nabla u\right] * \nabla^2 u$$

8.2. The evolution of the Nijenhuis tensor: proof of Theorem 6(a)

8.2.1. Rewriting the flow of the complex structure. The almost complex structure is given by $J^k{}_j = \omega^{ki} g_{ij}$. Therefore, $\partial_t J^k{}_j = \omega^{ki}\partial_t g_{ij}$. By substituting equation (6.53) for the Ricci curvature R_{ij} into the flow of metric $\partial_t g_{ij}$ (8.1), we obtain

$$\begin{aligned}\partial_t J^k{}_j &= e^u \omega^{ki}\Big\{ 4\mathfrak{D}_p N_i{}^p{}_j - J^p{}_i J^q{}_j \nabla_q \nabla_p u + \nabla_i \nabla_j u \\ &\qquad + u_i u_j - u_{Ji} u_{Jj} + 2u_p N_j{}^p{}_i + 6 u_p N_i{}^p{}_j \Big\}. \end{aligned} \tag{8.17}$$

This simplifies to

$$\begin{aligned}\partial_t J^k{}_j &= e^u \Big\{ 4J^k{}_q \mathfrak{D}_p N^{qp}{}_j - J^q{}_j \nabla_q \nabla^k u + J^k{}_q \nabla^q \nabla_j u \\ &\qquad + u^{Jk} u_j - u^k u_{Jj} + 2u_p N_j{}^{p,Jk} + 6 u_p N^{Jk,p}{}_j \Big\}. \end{aligned} \tag{8.18}$$

Converting covariant derivatives using $\nabla_\ell V^p = \mathfrak{D}_\ell V^p + N_{\ell\lambda}{}^p V^\lambda$, we obtain

$$\begin{aligned} 4J^k{}_q \mathfrak{D}_p N^{qp}{}_j &= -4J^k{}_q \mathfrak{D}_p N^q{}_j{}^p \\ &= -4J^k{}_q (\nabla_p N^q{}_j{}^p - N_{p\lambda}{}^q N^\lambda{}_j{}^p + N_{pj}{}^\lambda N^q{}_\lambda{}^p - N_{p\lambda}{}^p N^q{}_j{}^\lambda) \end{aligned}$$

Using the symmetries of the Nijenhuis tensor, this is

$$\begin{aligned} 4J^k{}_q \mathfrak{D}_p N^{qp}{}_j &= -4J^k{}_q \nabla_p N^q{}_j{}^p - 4N_{p\lambda}{}^{Jk} N^{\lambda p}{}_j + 4N^{p\lambda}{}_j N^{Jk}{}_{\lambda p} \\ &= -4J^k{}_q \nabla_p N^q{}_j{}^p - 4N_{p\lambda}{}^{Jk} N^{\lambda p}{}_j + 4N^{p\lambda}{}_j(-N_p{}^{Jk}{}_\lambda - N_{\lambda p}{}^{Jk}) \\ &= -4J^k{}_q \nabla_p N^q{}_j{}^p - 8(N_-^2)^{Jk}{}_j + 4(N_+^2)^{Jk}{}_j \end{aligned} \tag{8.19}$$

using $(N_+^2)_{ij} = N^{p\lambda}{}_i N_{p\lambda j}$ and $(N_-^2)_{ij} = N^{p\lambda}{}_i N_{\lambda p j}$. Thus

$$\begin{aligned}\partial_t J^k{}_j &= e^u \Big\{ -4J^k{}_q \nabla_p N^q{}_j{}^p - J^q{}_j \nabla_q \nabla^k u + J^k{}_q \nabla^q \nabla_j u \\ &\qquad + u^{Jk} u_j - u^k u_{Jj} + 2u_p N_j{}^{p,Jk} + 6 u_p N^{Jk,p}{}_j \\ &\qquad - 8(N_-^2)^{Jk}{}_j + 4(N_+^2)^{Jk}{}_j \Big\}. \end{aligned} \tag{8.20}$$

8.2.2. A first formulation of the evolution of the Nijenhuis tensor. We start with the identity

$$\nabla_i J^k{}_j = -2N_{ij}{}^{Jk}, \tag{8.21}$$

which follows from the formula relating ∇ to $\mathfrak{D}$ and $N_{i,Jj}{}^k = -N_{ij}{}^{Jk}$. Indeed,

$$\nabla_i J^k{}_j = \mathfrak{D}_i J^k{}_j + N_{i\lambda}{}^k J^\lambda{}_j - J^k{}_\lambda N_{ij}{}^\lambda = -2N_{ij}{}^{Jk}. \tag{8.22}$$

We can expand (8.21) and obtain

$$J^k{}_p N_{ij}{}^p = -\frac{1}{2}\nabla_i J^k{}_j = -\frac{1}{2}\partial_i J^k{}_j - \frac{1}{2}(\Gamma^k_{i\lambda} J^\lambda{}_j - J^k{}_\lambda \Gamma^\lambda_{ij}). \tag{8.23}$$

Differentiating this gives

$$\dot{J}^k{}_p N_{ij}{}^p + J^k{}_p \dot{N}_{ij}{}^p = -\frac{1}{2}\nabla_i \dot{J}^k{}_j - \frac{1}{2}(\dot{\Gamma}^k_{i\lambda} J^\lambda{}_j - J^k{}_\lambda \dot{\Gamma}^\lambda_{ij}), \tag{8.24}$$

which leads to

$$\partial_t N_{ij}{}^\ell = \frac{1}{2} J^\ell{}_k \nabla_i \dot{J}^k{}_j + J^\ell{}_k \dot{J}^k{}_p N_{ij}{}^p + \frac{1}{2}(J^\ell{}_k \dot{\Gamma}^k_{i\lambda} J^\lambda{}_j + \dot{\Gamma}^\ell_{ij}). \tag{8.25}$$

We will introduce some notation to group terms. We first introduce the tensor Z given by

$$Z_{ij}{}^{J\ell} = J^\ell{}_r \dot{\Gamma}^r_{in} J^n{}_j + \dot{\Gamma}^\ell{}_{ij}. \tag{8.26}$$

Next, we denote $\dot{J}^k{}_j = e^u E^k{}_j$, where by (8.20),

$$\begin{aligned} E^k{}_j \;=\;& -4J^k{}_q \nabla_p N^q{}_j{}^p - J^q{}_j \nabla_q \nabla^k u + J^k{}_q \nabla^q \nabla_j u \\ & + J^k{}_p u^p u_j - J^p{}_j u^k u_p + 2J^k{}_\ell u_p N_j{}^{p\ell} + 6J^k{}_\ell u_p N^{\ell p}{}_j \\ & - 8J^k{}_\ell (N^2_-)^\ell{}_j + 4J^k{}_\ell (N^2_+)^\ell{}_j. \end{aligned} \tag{8.27}$$

We write

$$\begin{aligned} \nabla_i E^k{}_j \;=\;& -4J^k{}_q \nabla_i \nabla_p N^q{}_j{}^p - J^q{}_j \nabla_i \nabla_q \nabla^k u + J^k{}_q \nabla_i \nabla^q \nabla_j u \\ & + J^k{}_p \nabla_i (u^p u_j) - J^p{}_j \nabla_i (u^k u_p) + Y_i{}^k{}_j, \end{aligned} \tag{8.28}$$

where

$$Y_r{}^k{}_j \;=\; -4\nabla_r J^k{}_q \nabla_p N^q{}_j{}^p - \nabla_r J^q{}_j \nabla_q \nabla^k u + \nabla_r J^k{}_q \nabla^q \nabla_j u$$

$$+\nabla_r J^k{}_p u^p u_j - \nabla_r J^p{}_j u^k u_p + 2\nabla_r (J^k{}_\ell u_p N_j{}^{p\ell}) + 6\nabla_r (J^k{}_\ell u_p N^{\ell p}{}_j)$$
$$-8\nabla_r (J^k{}_\ell (N_-^2)^\ell{}_j) + 4\nabla_r (J^k{}_\ell (N_+^2)^\ell{}_j) \tag{8.29}$$

Therefore

$$\begin{aligned} J^\ell{}_k \nabla_i \dot{J}^k{}_j &= J^\ell{}_k \nabla_i e^u E^k{}_j + J^\ell{}_k e^u \nabla_i E^k{}_j \\ &= e^u \Big[4\nabla_i \nabla_p N^\ell{}_j{}^p - J^\ell{}_k J^q{}_j \nabla_i \nabla_q \nabla^k u - \nabla_i \nabla^\ell \nabla_j u \\ &\quad -\nabla_i (u^\ell u_j) - J^\ell{}_k J^p{}_j \nabla_i (u^k u_p) + J^\ell{}_k Y_i{}^k{}_j + u_i E^{J\ell}{}_j \Big]. \end{aligned} \tag{8.30}$$

Substituting this,

$$\begin{aligned} \partial_t N_{ij}{}^\ell &= e^u \Big[-2\nabla_i \nabla_p N^{\ell p}{}_j - \frac{1}{2} J^\ell{}_k J^q{}_j \nabla_i \nabla_q \nabla^k u - \frac{1}{2} \nabla_i \nabla^\ell \nabla_j u \\ &\quad -\frac{1}{2} \nabla_i (u^\ell u_j) - \frac{1}{2} J^\ell{}_k J^p{}_j \nabla_i (u^k u_p) + \frac{1}{2} J^\ell{}_k Y_i{}^k{}_j \\ &\quad +\frac{1}{2} u_i E^{J\ell}{}_j + J^\ell{}_k E^k{}_p N_{ij}{}^p \Big] + \frac{1}{2} Z_{ij}{}^{J\ell}. \end{aligned} \tag{8.31}$$

To interpret the highest order terms, we will need the following identity. We claim:

$$\begin{aligned} \Delta N_{ij\ell} &= -2\nabla_i \nabla_p N_\ell{}^p{}_j - \nabla_i R_{\ell j} - \frac{1}{2} \nabla^p (R_{pij\ell} - R_{p,i,Jj,J\ell}) \\ &\quad +[\nabla_i, \nabla_p] N^p{}_{\ell j} + \frac{1}{2} \nabla_i \nabla_\ell \nabla_j u + \frac{1}{2} \nabla_i (J^p{}_\ell J^q{}_j \nabla_p \nabla_q u) \\ &\quad +2\Big[\nabla^p (N_{pj}{}^r N_{i\ell r}) - \nabla^p (N_{ij}{}^r N_{p\ell r}) + \nabla_i (N_-^2)_{\ell j} - \nabla_i (N_+^2)_{\ell j} \Big]. \end{aligned} \tag{8.32}$$

We assume identity (8.32) for now and give the proof in §8.2.3. The evolution of N becomes

$$\begin{aligned} \partial_t N_{ij}{}^\ell &= e^u \Big[\Delta N_{ij}{}^\ell - \nabla_i \nabla^\ell \nabla_j u + \nabla_i R_{\ell j} + \frac{1}{2} \nabla^p (R_{pij\ell} - R_{p,i,Jj,J\ell}) \\ &\quad -\frac{1}{2} (u_i u_j u^\ell + u_i u_{Jj} u^{J\ell}) + (\mathrm{IIa})_{ij}{}^\ell + \frac{e^{-u}}{2} Z_{ij}{}^{J\ell} \\ &\quad +Rm * N + \nabla N * (N + \nabla u) + N^3 + N^2 * \nabla u + N * (\nabla u)^2 \Big] \end{aligned} \tag{8.33}$$

where terms involving $\nabla\nabla u$ will need to be tracked for future use, and are given explicitly by

$$
\begin{aligned}
(\mathrm{IIa})_{ij}{}^{\ell} \;=\; & -\frac{1}{2}\nabla_i(u^\ell u_j) - \frac{1}{2}J^\ell{}_k J^p{}_j \nabla_i(u^k u_p) - \frac{1}{2}g^{r\ell}\nabla_i(J^p{}_r J^q{}_j)(\nabla^2 u)_{pq} \\
& +\frac{1}{2}J^\ell{}_k\Big[-\nabla_i J^q{}_j (\nabla^2 u)_q{}^k + \nabla_i J^k{}_q (\nabla^2 u)^q{}_j + 2J^k{}_r(\nabla^2 u)_{ip}N_j{}^{pr} \\
& +6J^k{}_r(\nabla^2 u)_{ip}N^{rp}{}_j\Big] + \frac{1}{2}u_i J^\ell{}_k\Big[-J^q{}_j(\nabla^2 u)^k{}_q + J^k{}_q(\nabla^2 u)^q{}_j\Big] \\
& +J^\ell{}_k\Big[-J^q{}_p(\nabla^2 u)_q{}^k + J^k{}_q(\nabla^2 u)^q{}_p\Big]N_{ij}{}^p
\end{aligned}
\tag{8.34}
$$

which, using $\nabla_i J^k{}_j = -2N_{ij}{}^{Jk}$ and simplifying, become

$$
\begin{aligned}
(\mathrm{IIa})_{ij}{}^{\ell} = & -\frac{1}{2}\nabla_i(u^\ell u_j) - \frac{1}{2}J^\ell{}_k J^p{}_j \nabla_i(u^k u_p) + N_i{}^{\ell Jp}(\nabla^2 u)_{p,Jj} - N_{ij}{}^{Jq}(\nabla^2 u)^{J\ell}{}_q \\
& +N_{ij}{}^{Jq}(\nabla^2 u)_q{}^{J\ell} + N_{iq}{}^{\ell}(\nabla^2 u)^q{}_j - (\nabla^2 u)_{ip}N_j{}^{p\ell} - 3(\nabla^2 u)_{ip}N^{\ell p}{}_j \\
& -\frac{1}{2}u_i\Big[(\nabla^2 u)^{J\ell}{}_{Jj} + (\nabla^2 u)^\ell{}_j\Big] - \Big[(\nabla^2 u)_{Jp}{}^{J\ell} + (\nabla^2 u)^\ell{}_p\Big]N_{ij}{}^p.
\end{aligned}
\tag{8.35}
$$

Next, we claim that

$$
\begin{aligned}
Z_{ij}{}^{Jp} \;=\; & \nabla_i \dot{g}^p{}_j + \frac{1}{2}(-\nabla^p \dot{g}_{ij} + \nabla_j \dot{g}^p{}_i) + \frac{1}{2}(\omega^{rp}J^n{}_j - \omega^{np}J^r{}_j)\nabla_r \dot{g}_{in} \\
& +(N_i{}^{pr}\dot{g}_{jr} + N_{ij}{}^r \dot{g}^p{}_r).
\end{aligned}
\tag{8.36}
$$

We assume identity (8.36) for now and give the proof later in §8.2.3. Substituting the evolution of g_{ij} (8.1) into this expression for $Z_{ij}{}^{Jp}$ and then in our expression for $\partial_t N_{ij}{}^\ell$, we obtain

$$
\begin{aligned}
\partial_t N_{ij}{}^\ell \;=\; & e^u\Big[\Delta N_{ij}{}^\ell - \nabla_i\nabla^\ell\nabla_j u + \nabla_i R^\ell{}_j + \frac{1}{2}\nabla^p(R_{pij}{}^\ell - g^{\ell r}R_{p,i,Jj,Jr}) \\
& -\nabla_i R^\ell{}_j - \frac{1}{2}(-\nabla^\ell R_{ij} + \nabla_j R^\ell{}_i) - \frac{1}{2}(\omega^{r\ell}J^n{}_j - \omega^{n\ell}J^r{}_j)\nabla_r R_{in} \\
& +\frac{1}{2}(-\nabla^\ell\nabla_i\nabla_j u + \nabla_j\nabla^\ell\nabla_i u) + \frac{1}{2}(\omega^{r\ell}J^n{}_j - \omega^{n\ell}J^r{}_j)\nabla_r\nabla_i\nabla_n u \\
& +\nabla_i\nabla^\ell\nabla_j u + Ric * \nabla u + (\mathrm{IIa})_{ij}{}^\ell + (\mathrm{IIb})_{ij}{}^\ell \\
& +Rm * N + \nabla N * (N + \nabla u) + N^3 + N^2 * \nabla u + N * (\nabla u)^2\Big]
\end{aligned}
\tag{8.37}
$$

where terms of order $(\nabla u)^3$ (e.g. $u_i u_j u^\ell$) have cancelled and the additional terms involving $\nabla^2 u$ are

$$\begin{aligned}(\mathrm{IIb})_{ij}{}^{\ell} &= \frac{1}{2}[\nabla_i(u^\ell u_j - u^{J\ell}u_{Jj}) + 4(\nabla^2 u)_{pi}(N^{\ell p}{}_j + N_j{}^{p\ell}) + 2u_i(\nabla^2 u)^\ell{}_j] \\ &-\frac{1}{4}[\nabla^\ell(u_i u_j - u_{Ji}u_{Jj}) + 4(\nabla^2 u)_p{}^\ell(N_i{}^p{}_j + N_j{}^p{}_i) + 2u^\ell(\nabla^2 u)_{ij}] \\ &+\frac{1}{4}[\nabla_j(u_i u^\ell - u_{Ji}u^{J\ell}) + 4(\nabla^2 u)_{pj}(N_i{}^{p\ell} + N^{\ell p}{}_i) + 2u_j(\nabla^2 u)_i{}^\ell] \\ &+\frac{1}{4}(\omega^{r\ell}J^n{}_j - \omega^{n\ell}J^r{}_j)[\nabla_r(u_i u_n - u_{Ji}u_{Jj}) + 2u_r(\nabla^2 u)_{in}] \\ &+4(\nabla^2 u)_{pr}(N_i{}^p{}_n + N_n{}^p{}_i) + (N_i{}^{\ell r}(\nabla^2 u)_{jr} + N_{ij}{}^r(\nabla^2 u)^\ell{}_r). \end{aligned} \tag{8.38}$$

Since we can commute $\nabla_\ell\nabla_j\nabla_i u = \nabla_j\nabla_\ell\nabla_i u - R_{\ell j}{}^p{}_i u_p$, the terms of order $\nabla^3 u$ in (8.37) cancel. We are left with

$$\begin{aligned}\partial_t N_{ij}{}^\ell &= e^u\Big[\Delta N_{ij}{}^\ell + \frac{1}{2}\nabla^p R_{pij}{}^\ell - \frac{1}{2}\omega^{n\ell}J^r{}_j\nabla^p R_{pirn} + Ric * \nabla u \\ &-\frac{1}{2}(-\nabla^\ell R_{ij} + \nabla_j R^\ell{}_i) - \frac{1}{2}(\omega^{r\ell}J^n{}_j - \omega^{n\ell}J^r{}_j)\nabla_r R_{in} \\ &+\frac{1}{2}(-R_j{}^{\ell p}{}_i u_p + R^{J\ell}{}_i{}^p{}_{Jj}u_p - R_{Jj,i}{}^{p,J\ell}u_p) + (\mathrm{IIa})_{ij}{}^\ell + (\mathrm{IIb})_{ij}{}^\ell \\ &+Rm * N + \nabla N * (N + \nabla u) + N^3 + N^2 * \nabla u + N * (\nabla u)^2\Big]. \end{aligned} \tag{8.39}$$

The terms of order ∇Rm also cancel. Indeed, the Bianchi identity is

$$\nabla^p R_j{}^\ell{}_{pi} + \nabla^\ell R^p{}_{jpi} + \nabla_j R^{\ell p}{}_{pi} = 0$$

and hence

$$\frac{1}{2}\nabla^p R_{pij}{}^\ell = \frac{1}{2}\nabla^p R_j{}^\ell{}_{pi} = \frac{1}{2}(-\nabla^\ell R_{ij} + \nabla_j R^\ell{}_i). \tag{8.40}$$

For the terms involving ω, J, the same argument gives

$$-\frac{1}{2}\omega^{n\ell}J^r{}_j\nabla^p R_{pirn} = -\frac{1}{2}\omega^{n\ell}J^r{}_j\nabla^p R_{nrip} = \frac{1}{2}\omega^{n\ell}J^r{}_j(\nabla_n R_{ri} - \nabla_r R_{in}) \tag{8.41}$$

This is the same thing as

$$-\frac{1}{2}\omega^{n\ell}J^r{}_j\nabla^p R_{pirn} = \frac{1}{2}(\omega^{r\ell}J^n{}_j - \omega^{n\ell}J^r{}_j)\nabla_r R_{in} \tag{8.42}$$

Putting these identities back to (8.39), we get the cancellation of the ∇Rm terms. Thus

$$\begin{aligned}\partial_t N_{ij}{}^k &= e^u\Big[\Delta N_{ij}{}^k + (Rm*\nabla u)_{ij}{}^k + (\mathrm{IIa})_{ij}{}^k + (\mathrm{IIb})_{ij}{}^k \\ &\quad + Rm*N + \nabla N*(N+\nabla u) + N^3 + N^2*\nabla u + N*(\nabla u)^2\Big],\end{aligned} \tag{8.43}$$

where the (II) terms involve $\nabla^2 u$ and are explicitly given in (8.35) and (8.38) and $Rm*\nabla u$ is of the form

$$(Rm*\nabla u)_{ij}{}^k = \frac{1}{2}(-R_j{}^{kp}{}_i u_p + R^{Jk}{}_i{}^p{}_{Jj} u_p - R_{Jj,i}{}^{p,Jk} u_p) + (Ric*\nabla u)_{ij}{}^k. \tag{8.44}$$

8.2.3. Proof of identity (8.32) and identity (8.36). *Proof of identity (8.32):* The starting point is the identity (6.11) for the action of J on the Riemann curvature tensor. Recall that in the case $d\omega = 0$, (6.4) and (6.5) specialize to $A = N = \mathfrak{T}$, and so the identity (6.11) becomes

$$\begin{aligned}R_{i,j,Jk,Jl} - R_{ijkl} &= 2(\mathfrak{D}_i N_{jkl} - \mathfrak{D}_j N_{ikl} + N^r{}_{ij} N_{rkl}) \\ &= 2(\mathfrak{D}_i N_{jkl} - \mathfrak{D}_j N_{ikl} + N_{ji}{}^r N_{rkl} - N_{ij}{}^r N_{rkl}),\end{aligned} \tag{8.45}$$

using $N_{ijk} + N_{kij} + N_{jki} = 0$. We can convert $\mathfrak{D}N$ to ∇N. For example,

$$\mathfrak{D}_i N_{jkl} = \nabla_i N_{jkl} + N_{ij}{}^r N_{rkl} + N_{ik}{}^r N_{jrl} + N_{il}{}^r N_{jkr}. \tag{8.46}$$

After converting $\mathfrak{D}$ to ∇, (8.45) becomes

$$\nabla_j N_{i\ell k} = \nabla_i N_{j\ell k} + \frac{1}{2}(R_{jik\ell} - R_{j,i,Jk,J\ell}) - 2N_{jk}{}^r N_{i\ell r} + 2N_{ik}{}^r N_{j\ell r}. \tag{8.47}$$

Differentiating this identity, we obtain

$$\begin{aligned}\nabla_q\nabla_p N_{kij} &= \nabla_q\nabla_k N_{pij} + \frac{1}{2}\nabla_q(R_{pkji} - R_{p,k,Jj,Ji}) \\ &\quad -2\nabla_q(N_{pj}{}^r N_{kir}) + 2\nabla_q(N_{kj}{}^r N_{pir}).\end{aligned} \tag{8.48}$$

By the Bianchi identity,

$$\Delta N_{ijk} = -\Delta N_{kij} + \Delta N_{jik}$$

$$= -\nabla_p\nabla_k N^p{}_{ij} - \frac{1}{2}\nabla^p(R_{pkji} - R_{p,k,Jj,Ji})$$
$$+2\nabla^p(N_{pj}{}^r N_{kir}) - 2\nabla^p(N_{kj}{}^r N_{pir}) + \Delta N_{jik}. \tag{8.49}$$

Therefore

$$\begin{aligned} -\Delta N_{jik} + \Delta N_{ijk} &= -\nabla_k\nabla_p N^p{}_{ij} - \frac{1}{2}\nabla^p(R_{pkji} - R_{p,k,Jj,Ji}) \\ &\quad +[\nabla_k, \nabla_p]N^p{}_{ij} + 2\nabla^p(N_{pj}{}^r N_{kir}) - 2\nabla^p(N_{kj}{}^r N_{pir}) \\ &= \nabla_k\nabla_p N_j{}^p{}_i - \nabla_k\nabla_p N_i{}^p{}_j - \frac{1}{2}\nabla^p(R_{pkji} - R_{p,k,Jj,Ji}) \\ &\quad +[\nabla_k, \nabla_p]N^p{}_{ij} + 2\nabla^p(N_{pj}{}^r N_{kir}) - 2\nabla^p(N_{kj}{}^r N_{pir}) \end{aligned} \tag{8.50}$$

The formula for Ricci curvature in Type IIA geometry given in (6.54) is

$$\nabla_p N_j{}^p{}_i = -\nabla_p N_i{}^p{}_j - R_{ij} + 2(N_-^2)_{ij} - 2(N_+^2)_{ij} + \frac{1}{2}(\nabla^2 u)_{ij} + \frac{1}{2}(\nabla^2 u)_{Ji,Jj}. \tag{8.51}$$

Substituting this into (8.50),

$$\begin{aligned} &-\Delta N_{jik} + \Delta N_{ijk} \\ = \; &-2\nabla_k\nabla_p N_i{}^p{}_j - \nabla_k R_{ij} - \frac{1}{2}\nabla^p(R_{pkji} - R_{p,k,Jj,Ji}) \\ &+[\nabla_k, \nabla_p]N^p{}_{ij} + \frac{1}{2}\nabla_k\nabla_i\nabla_j u + \frac{1}{2}\nabla_k(J^p{}_i J^q{}_j \nabla_p\nabla_q u) \\ &+2\nabla^p(N_{pj}{}^r N_{kir}) - 2\nabla^p(N_{kj}{}^r N_{pir}) + 2\nabla_k(N_-^2)_{ij} - 2\nabla_k(N_+^2)_{ij} \end{aligned} \tag{8.52}$$

This proves the identity after using $N_{ijk} + N_{kij} + N_{jki} = 0$. Q.E.D.

Proof of Identity (8.36): The variation of the Christoffel symbol is given by

$$\dot\Gamma^p_{in} = \frac{g^{ps}}{2}(-\nabla_s \dot g_{in} + \nabla_i \dot g_{sn} + \nabla_n \dot g_{is}). \tag{8.53}$$

Thus

$$\begin{aligned} Z_{ij}{}^p = \dot\Gamma^p_{in} J^n{}_j - J^p{}_n \dot\Gamma^n{}_{ij} &= \frac{g^{ps}}{2}(-\nabla_s \dot g_{in} J^n{}_j + \nabla_i \dot g_{sn} J^n{}_j + J^n{}_j \nabla_n \dot g_{is}) \\ &\quad -\frac{g^{ns}}{2} J^p{}_n(-\nabla_s \dot g_{ij} + \nabla_i \dot g_{sj} + \nabla_j \dot g_{is}). \end{aligned} \tag{8.54}$$

Therefore

$$Z_{ij}{}^{J\ell} = \frac{g^{ps}}{2} J^\ell{}_p(-\nabla_s \dot{g}_{in} J^n{}_j + \nabla_i \dot{g}_{sn} J^n{}_j + J^n{}_j \nabla_n \dot{g}_{is}) \\ -\frac{g^{ns}}{2} J^\ell{}_p J^p{}_n(-\nabla_s \dot{g}_{ij} + \nabla_i \dot{g}_{sj} + \nabla_j \dot{g}_{is}), \tag{8.55}$$

which becomes

$$Z_{ij}{}^{Jp} = -\frac{g^{ps}}{2} J^r{}_s \nabla_i \dot{g}_{rn} J^n{}_j + \frac{1}{2}(-\nabla^p \dot{g}_{ij} + \nabla_i \dot{g}^p{}_j + \nabla_j \dot{g}^p{}_i) \\ +\frac{1}{2}(\omega^{rp} J^n{}_j \nabla_r \dot{g}_{in} - \omega^{np} J^r{}_j \nabla_r \dot{g}_{in}). \tag{8.56}$$

From the evolution equation of g, we have the identity $J^r{}_i \dot{g}_{rn} J^n{}_j = -\dot{g}_{ij}$. Therefore the first term

$$-\frac{g^{ps}}{2} J^r{}_s \nabla_i \dot{g}_{rn} J^n{}_j \tag{8.57}$$

$$= -\frac{g^{ps}}{2} \nabla_i (J^r{}_s \dot{g}_{rn} J^n{}_j) + \frac{g^{ps}}{2} \nabla_i J^r{}_s \dot{g}_{rn} J^n{}_j + \frac{g^{ps}}{2} J^r{}_s \dot{g}_{rn} \nabla_i J^n{}_j$$

simplifies to

$$-\frac{g^{ps}}{2} J^r{}_s \nabla_i \dot{g}_{rn} J^n{}_j = \frac{1}{2} \nabla_i \dot{g}^p{}_j + N_i{}^{pr} \dot{g}_{rj} + \dot{g}^p{}_n N_{ij}{}^n. \tag{8.58}$$

Therefore

$$Z_{ij}{}^{Jp} = \nabla_i \dot{g}^p{}_j + \frac{1}{2}(-\nabla^p \dot{g}_{ij} + \nabla_j \dot{g}^p{}_i) + \frac{1}{2}(\omega^{rp} J^n{}_j \nabla_r \dot{g}_{in} - \omega^{np} J^r{}_j \nabla_r \dot{g}_{in}) \\ +N_i{}^{pr} \dot{g}_{rj} + \dot{g}^p{}_n N_{ij}{}^n. \tag{8.59}$$

Q.E.D.

8.2.4. The evolution of the norm of *N*. The norm of the Nijenhuis tensor, which is $|N|^2 = g^{ij} g^{pq} g^{k\ell} N_{ipk} N_{jq\ell}$, evolves by

$$(\partial_t - e^u \Delta)|N|^2 = 2N^{ij}{}_k (\partial_t - e^u \Delta) N_{ij}{}^k - 2e^u |\nabla N|^2 \\ -g^{ir} \dot{g}_{rs} g^{sj} N_{ipk} N_j{}^{pk} - 2g^{kr} \dot{g}_{rs} g^{s\ell} N_{ipk} N^{ip}{}_\ell. \tag{8.60}$$

By the equation for the evolution of g_{ij} (8.1), this is of the form

$$(\partial_t - e^u \Delta)|N|^2 = 2N^{ij}{}_k (\partial_t - e^u \Delta) N_{ij}{}^k - 2e^u |\nabla N|^2$$

$$+e^u\Big[-2(\nabla^2 u)^{ij}N_{ipk}N_j{}^{pk}-4(\nabla^2 u)^{k\ell}N_{ipk}N^{ip}{}_\ell$$

$$+N^2*(Rm+N*N+\nabla u*\nabla u+\nabla u*N)\Big] \tag{8.61}$$

Substituting (8.43), we obtain

$$\begin{aligned}
&(\partial_t-e^u\Delta)|N|^2\\
=\;& e^u\Big[-2|\nabla N|^2+2N^{ij}{}_k(\mathrm{IIa}+\mathrm{IIb})_{ij}{}^k-2(\nabla^2 u)^{ij}N_{ipk}N_j{}^{pk}\\
&-4(\nabla^2 u)^{k\ell}N_{ipk}N^{ip}{}_\ell+2N^{ij}{}_k(Rm*\nabla u)_{ij}{}^k+Rm*N^2\\
&+\nabla N*N*(N+\nabla u)+N^4+N^3*\nabla u+N^2*(\nabla u)^2\Big].
\end{aligned}\tag{8.62}$$

The expressions for the terms (IIa+IIb) are given in (8.35) and (8.38). A calculation can be done to verify that each term $N^{ij}{}_k(\mathrm{IIa})_{ij}{}^k$ and $N^{ij}{}_k(\mathrm{IIb})_{ij}{}^k$ only contributes terms of the type $(\nabla^2 u)*N^2$ since the others vanish by symmetry. For example, if we denote the terms on each line of (IIb) (8.38) by $(i)+(ii)+(iii)+(iv)+(v)$, we have

$$\begin{aligned}
(i)\;&=\;\frac{1}{2}N^{ij}{}_\ell[\nabla_i(u^\ell u_j)-J^\ell{}_rJ^s{}_j\nabla_i(u^r u_s)+4u_{pi}(N^{\ell p}{}_j+N_j{}^{p\ell})+2u_iu^\ell{}_j]\\
&=\;-2u_{pi}N^{ij}{}_\ell N^\ell{}_{jp}-2u_{pi}N^{ij}{}_\ell N_j{}^\ell{}_p\\
&=\;2u^p{}_i(N^{\ell ij}+N^{j\ell i})N_{\ell jp}+2u_{pi}(N^{\ell ij}+N^{j\ell i})N_{j\ell p}=0
\end{aligned}\tag{8.63}$$

where we used the symmetry $N^{ij}{}_\ell=-N^i{}_\ell{}^j$, the identity $N^{i,Jj}{}_{Jr}=N^{ij}{}_r$, and the Bianchi identity $N_{ijk}+N_{kij}+N_{jki}=0$. The symmetry $N^{ij}{}_\ell=-N^i{}_\ell{}^j$ allows us to combine the $(ii)+(iii)$ terms:

$$\begin{aligned}
&(ii)+(iii)\\
=\;&-\frac{1}{2}N^{ij}{}_\ell[\nabla^\ell(u_iu_j)-J^r{}_iJ^s{}_j\nabla^\ell(u_ru_s)+4u_p{}^\ell(N_i{}^p{}_j+N_j{}^p{}_i)+2u^\ell u_{ij}]\\
=\;&-\frac{1}{2}N^{ij}{}_\ell[2u^\ell{}_iu_j+4u_p{}^\ell(N_i{}^p{}_j+N_j{}^p{}_i)+2u^\ell u_{ij}]\\
=\;&2u_p{}^\ell(N^{ij}{}_\ell N_{ij}{}^p)+2u_p{}^\ell(N^{ij}{}_\ell N_{ji}{}^p)
\end{aligned}\tag{8.64}$$

where we used $N^{Ji,Jj}{}_\ell=-N^{ij}{}_\ell$. Next,

$$(iv)\;=\;\frac{1}{2}N^{ij}{}_\ell\omega^{r\ell}J^n{}_j[\nabla_r(u_iu_n)-J^p{}_iJ^q{}_n\nabla_r(u_pu_q)$$

$$
\begin{aligned}
& +4u_{pr}(N_i{}^p{}_n + N_n{}^p{}_i) + 2u_r u_{in}] \\
= \ & N^{ij}{}_\ell \omega^{r\ell} J^n{}_j [u_{ri}u_n + u_i u_{rn} + 2u_{pr}(N_i{}^p{}_n + N_n{}^p{}_i) + u_r u_{in}] \\
= \ & -N^{i,Jn}{}_{J\ell} g^{r\ell}[u_{ri}u_n + u_i u_{rn} + 2u_{pr}(N_i{}^p{}_n + N_n{}^p{}_i) + u_r u_{in}] \\
= \ & -N^{in}{}_\ell [u^\ell{}_i u_n + u_i u^\ell{}_n + 2u^\ell{}_p (N_i{}^p{}_n + N_n{}^p{}_i) + u^\ell u_{in}] \\
= \ & 2u^\ell{}_p(N^{in}{}_\ell N_{in}{}^p) + 2u^\ell{}_p(N^{in}{}_\ell N_{ni}{}^p). \qquad (8.65)
\end{aligned}
$$

The computations of the other terms is similar. The result is then

$$
\begin{aligned}
(8.66) \quad & (\partial_t - e^u \Delta)|N|^2 \\
= \ & e^u \Big[-2|\nabla N|^2 + (\nabla^2 u) * N^2 + 2N^{ij}{}_k (Rm * \nabla u)_{ij}{}^k \\
& + Rm * N^2 + \nabla N * N * (N + \nabla u) + N^4 + N^3 * \nabla u + N^2 * (\nabla u)^2 \Big]
\end{aligned}
$$

where $(\nabla^2 u) \star N^2$ is an expression involving $(\nabla u)^{ij}(N^2_+)_{ij}$ and $(\nabla u)^{ij}(N^2_-)_{ij}$. The expression for the term $(Rm * \nabla u)_{ij}{}^k$ is given in (8.44), and using $N^{i,Jj}{}_{Jk} = N^{ij}{}_k$ and symmetries of the curvature tensor, it becomes

$$
N^{ij}{}_k (Rm * \nabla u)_{ij}{}^k = -R^p{}_{ijk} u_p N^{ijk} + N^{ij}{}_k (Ric * \nabla u)_{ij}{}^k. \qquad (8.67)
$$

By (6.16), we can convert $R^p{}_{ijk} = \mathfrak{R}^p{}_{ijk} + \nabla N + N^2$. Since $\mathfrak{R}^p{}_{i,Jj,Jk} = \mathfrak{R}^p{}_{ijk}$ and $N^{i,Jj,Jk} = -N^{ijk}$, the term $\mathfrak{R}^p_{ijk} N^{ijk} = 0$ by symmetry. Similarly, we can reduce terms of the form $N^{ij}{}_k u^k R_{ij} = N^{ij}{}_k u^k R^{-J}_{ij}$ where R^{-J}_{ij} is the J-anti-invariant part of the Ricci tensor given in (6.56) by $Ric^{-J} = \nabla N + N * N$. Absorbing these terms gives the expression

$$
\begin{aligned}
(\partial_t - e^u \Delta)|N|^2 = \ & e^u \Big[-2|\nabla N|^2 + (\nabla^2 u) * N^2 + Rm * N^2 \\
& + \nabla N * N * (N + \nabla u) + N^4 + N^3 * \nabla u + N^2 * (\nabla u)^2 \Big]. \qquad (8.68)
\end{aligned}
$$

We remark that from this expression, we see that if $|N|^2 = 0$ at the initial time, that $|N|^2 \equiv 0$ along the flow. To see this, we assume $|N|^2 \leq 1$ and $|Rm| + |\nabla^2 u| + |\nabla u| \leq C$ on $[0, \epsilon)$ and apply the maximum principle to $e^{-At}|N|^2$ for $A \gg 1$.

8.2.5. The evolution of the gradient of N. We compute the evolution of ∇N.

$$
\partial_t \nabla_\ell N_{ij}{}^k = \nabla_\ell \dot{N}_{ij}{}^k - N_{\lambda j}{}^k \dot{\Gamma}^\lambda_{\ell i} - N_{i\lambda}{}^k \dot{\Gamma}^\lambda_{\ell j} + \dot{\Gamma}^k_{\ell\lambda} N_{ij}{}^\lambda \qquad (8.69)
$$

Using the equation (8.1) for $\dot g_{ij}$, we can compute the time derivative of the Christoffel symbol as

$$\begin{aligned}\dot\Gamma^k_{ij} &= -\frac{g^{k\mu}}{2}(-\nabla_\mu \dot g_{ij} + \nabla_j \dot g_{\mu i} + \nabla_i \dot g_{\mu p})\\ &= e^u(\nabla Rm * g + \nabla^3 u * g + (\nabla N + \nabla^2 u + 1) * \mathcal{O}(\nabla u, N, Rm)).\end{aligned} \tag{8.70}$$

Substituting (8.43),

$$\begin{aligned}&\partial_t \nabla_\ell N_{ij}{}^k\\ = \ & e^u\Big[\nabla_\ell \Delta N_{ij}{}^k + (\nabla Rm + \nabla^2 N + \nabla^3 u) * (N + \nabla u)\\ &+(\nabla N + \nabla^2 u) * (\nabla N + \nabla^2 u) + (\nabla N + \nabla^2 u + 1) * \mathcal{O}(\nabla u, N, Rm)\Big]\end{aligned}$$

We can commute the derivatives $\nabla_\ell \Delta N_{ij}{}^k$ up to lower order terms, and so

$$\begin{aligned}&(\partial_t - e^u\Delta)\nabla_\ell N_{ij}{}^k\\ = \ & e^u\Big[(\nabla Rm + \nabla^2 N + \nabla^3 u) * (N + \nabla u)\\ &+(\nabla N + \nabla^2 u) * (\nabla N + \nabla^2 u) + (\nabla N + \nabla^2 u + 1) * \mathcal{O}(\nabla u, N, Rm)\Big]\end{aligned}$$

The evolution of the norm $|\nabla N|^2$ is

$$(\partial_t - e^u\Delta)|\nabla N|^2 = 2\langle(\partial_t - e^u\Delta)\nabla N, \nabla N\rangle - 2e^u|\nabla^2 N|^2 + \partial_t g * \nabla N * \nabla N$$

Altogether,

$$\begin{aligned}&(\partial_t - e^u\Delta)|\nabla N|^2\\ \leq \ & -2e^u|\nabla^2 N|^2 + Ce^u\Big[|\nabla N|^3 + |\nabla^2 u||\nabla N|^2 + |\nabla^2 u|^2|\nabla N|\\ &+|\nabla N|(|\nabla Rm| + |\nabla^2 N| + |\nabla^3 u|)(|N| + |\nabla u|)\\ &+\mathcal{O}(\nabla u, N, Rm)(|\nabla N|^2 + |\nabla N||\nabla^2 u| + 1)\Big].\end{aligned} \tag{8.71}$$

8.3. The evolution of the curvature: proof of Theorem 6(b)

The general formula for the variation of the curvature tensor (see e.g. [29]) is

$$\begin{aligned} \frac{d}{dt}R_{jikl} &= \frac{1}{2}(\nabla_i\nabla_k\dot{g}_{jl} - \nabla_i\nabla_l\dot{g}_{jk} - \nabla_j\nabla_k\dot{g}_{il} + \nabla_j\nabla_l\dot{g}_{ik}) \\ &\quad +\frac{1}{2}\dot{g}_{k\lambda}R_{ji}{}^{\lambda}{}_l - \frac{1}{2}\dot{g}_{l\lambda}R_{ji}{}^{\lambda}{}_k. \end{aligned} \tag{8.72}$$

In our case, we write the evolution of g_{ij} (8.1) as

$$\begin{aligned} \dot{g}_{ij} &= e^u(-2R_{ij} + 2\nabla_i\nabla_j u + E_{ij}), \\ E_{ij} &= -4(N_-^2)_{ij} + u_i u_j - u_{Ji}u_{Jj} + 4u_p(N_i{}^p{}_j + N_j{}^p{}_i). \end{aligned} \tag{8.73}$$

Differentiating once gives

$$\nabla_k\dot{g}_{j\ell} = e^u(-2\nabla_k R_{j\ell} + 2\nabla_k\nabla_j\nabla_\ell u + \nabla_k E_{j\ell}) + e^u(-2R_{j\ell} + 2\nabla_j\nabla_\ell u + E_{j\ell})\nabla_k u. \tag{8.74}$$

Differentiating twice gives

$$\begin{aligned} \nabla_i\nabla_k\dot{g}_{j\ell} &= e^u(-2\nabla_i\nabla_k R_{j\ell} + 2\nabla_i\nabla_k\nabla_j\nabla_\ell u + \nabla_i\nabla_k E_{j\ell}) \\ &\quad +e^u(-2\nabla_k R_{j\ell} + 2\nabla_k\nabla_j\nabla_\ell u + \nabla_k E_{j\ell})\nabla_i u + (i \leftrightarrow k) \\ &\quad +e^u(-2R_{j\ell} + 2\nabla_j\nabla_\ell u + E_{j\ell})\nabla_i\nabla_k u \\ &\quad +e^u(-2R_{j\ell} + 2\nabla_j\nabla_\ell u + E_{j\ell})\nabla_k u\nabla_i u. \end{aligned}$$

We can group this as

$$\begin{aligned} \nabla_i\nabla_k\dot{g}_{j\ell} &= e^u(-2\nabla_i\nabla_k R_{j\ell} + 2\nabla_i\nabla_k\nabla_j\nabla_\ell u) \\ &\quad +e^u(\nabla Rm + \nabla^3 u + \nabla^2 N) * \mathcal{O}(\nabla u, N) + \nabla N * \nabla N \\ &\quad +\nabla N * (\nabla^2 u + \mathcal{O}(\nabla u, N)) + \mathcal{O}(Rm, N, \nabla^2 u, \nabla u, u). \end{aligned} \tag{8.75}$$

We have

$$\nabla_i\nabla_k\nabla_j\nabla_\ell u - \nabla_i\nabla_j\nabla_k\nabla_\ell u = \nabla_i(-R_{kj}{}^{\lambda}{}_\ell u_\lambda) \tag{8.76}$$

and (Lemma 7.2 in [29])

$$\nabla_i\nabla_k R_{j\ell} - \nabla_i\nabla_\ell R_{jk} - \nabla_j\nabla_k R_{i\ell} + \nabla_j\nabla_\ell R_{ik} = -\Delta R_{jik\ell} + Rm * Rm. \tag{8.77}$$

Substituting all this into (8.72), we obtain

$$(\partial_t - e^u\Delta)Rm = e^u\Big[(\nabla Rm + \nabla^3 u + \nabla^2 N) * \mathcal{O}(\nabla u, N) + \mathcal{O}(Rm, N, \nabla u)$$
$$+\nabla N * (\nabla^2 u + \mathcal{O}(\nabla u, N)) + \nabla N * \nabla N + \nabla^2 u * \nabla^2 u\Big]. \tag{8.78}$$

The norm evolves by

$$(\partial_t - e^u\Delta)|Rm|^2 = 2\langle(\partial_t - e^u\Delta)Rm, Rm\rangle - 2e^u|\nabla Rm|^2 + \partial_t g * Rm * Rm. \tag{8.79}$$

Therefore

$$(\partial_t - e^u\Delta)|Rm|^2 \tag{8.80}$$
$$= e^u\Big[-2|\nabla Rm|^2 + (\nabla Rm + \nabla^3 u + \nabla^2 N) * \mathcal{O}(Rm, \nabla u, N)$$
$$+(\nabla N * \nabla N + \nabla^2 u * \nabla^2 u + 1) * \mathcal{O}(Rm, \nabla u, N)\Big].$$

8.4. Lower order estimates

8.4.1. Gradient estimate. In this section, we estimate the gradient of u.

Proposition 5. *Suppose over a finite interval* $[0, T)$ *the flow exists and that* $|u| + |Rm| \le C$*, then there exists a constant* C' *such that* $|\nabla u|^2 \le C'$ *in the time interval* $[0, T)$*.*

We recall that by our work so far, we know that $|N|^2$ is bounded and the J-invariant part of $\nabla^2 u$ is bounded.

Equation (8.9) for the evolution of $|\nabla u|^2$ together with the evolution $(\partial_t - e^u\Delta)u = e^u(2|\nabla u|^2 + |N|^2)$ of u imply

$$(\partial_t - e^u\Delta)(e^{pu}|\nabla u|^2) \tag{8.81}$$
$$= e^{(p+1)u}\Big(-2|\nabla^2 u|^2 + (6-4p)(\nabla^2 u)_{ij}u^i u^j + 8(N_+^2)_{ij}u^i u^j$$
$$+2u^2\nabla_s|N|^2 + |\nabla u|^2(2\Delta u + (3+2p-p^2)|\nabla u|^2 + (1+p)|N|^2)\Big).$$

Let $V = e^{pu}(|N|^2 + |\nabla u|^2)$ for some constant p. From (2.12) and (8.81), we see that

$$\begin{aligned} & e^{-(p+1)u}(\partial_t - e^u\Delta)V \\ = & \Big(- 2|\nabla N|^2 + (\nabla^2 u) * N^2 + N * (Rm + \nabla N) * (N + \nabla u) \\ & + N * (N + \nabla u)^3 - 2pu^s\nabla_s|N|^2 - (p^2 - 2p)|\nabla u|^2|N|^2 + p|N|^4\Big) \\ & + \Big(- 2|\nabla^2 u|^2 + (6 - 4p)(\nabla^2 u)_{ij}u^iu^j + 8(N^2_+)_{ij}u^iu^j + 2u^s\nabla_s|N|^2 \\ & + |\nabla u|^2(2\Delta u + (3 + 2p - p^2)|\nabla u|^2 + (1 + p)|N|^2)\Big). \end{aligned}$$

We note that $(\nabla^2 u) * N^2$ represent terms of the form $a(\nabla^2 u)^{ij}(N^2_+)_{ij} + b(\nabla^2 u)^{ij}(N^2_-)_{ij}$ for some constant a and b. Those terms are also bounded since $N^2_\pm$ is J-invariant hence only the J-invariant part of $\nabla^2 u$ contributes to this term, which is bounded. Also, we can control all the terms linear in ∇N by the good term $-|\nabla N|^2$. Therefore

$$\begin{aligned} & e^{-(p+1)u}(\partial_t - e^u\Delta)V \\ \leq & -2|\nabla^2 u|^2 + (6 - 4p)(\nabla^2 u)_{ij}u^iu^j + (3 + 2p - p^2)|\nabla u|^4 \\ & +C(p)|\nabla u|^3 + C(p). \end{aligned} \tag{8.82}$$

To handle the term $(\nabla^2 u)_{ij}u^iu^j$, we need to make use of the fact that the J-invariant part of $\nabla^2 u$ is bounded. To do so, let us denote the J-invariant and the J-anti-invariant parts of $\nabla^2 u$ by $\nabla^2_J u$ and $\nabla^2_{-J}u$ respectively. Under this notation, we see that

$$\begin{aligned} & -2|\nabla^2 u|^2 + (6 - 4p)(\nabla^2 u)_{ij}u^iu^j + (3 + 2p - p^2)|\nabla u|^4 \\ = & -2|\nabla^2_J u|^2 - 2|\nabla^2_{-J}u|^2 + (6 - 4p)(\nabla^2_J u + \nabla^2_{-J}u, \nabla u \otimes \nabla u) \\ & +(3 + 2p - p^2)|\nabla u|^4 \\ \leq & -2|\nabla^2_{-J}u|^2 + (6 - 4p)(\nabla^2_{-J}u, \nabla u \otimes \nabla u) + C|\nabla u|^2 \\ & +(3 + 2p - p^2)|\nabla u|^4. \end{aligned} \tag{8.83}$$

The advantage of this consideration is that only the J-anti-invariant part of $\nabla u \otimes \nabla u$, namely $\frac{1}{2}(u_iu_j - u_{Ji}u_{Jj})$, contributes to the inner product term. Therefore

$$-2|\nabla^2 u|^2 + (6 - 4p)(\nabla^2 u)_{ij}u^iu^j + (3 + 2p - p^2)|\nabla u|^4$$

$$
\begin{aligned}
&\leq -2|\nabla^2_{-J}u|^2 + (3-2p)(\nabla^2_{-J}u)^{ij}(u_iu_j - u_{Ji}u_{Jj}) + C|\nabla u|^2 \\
&\quad +(3+2p-p^2)|\nabla u|^4 \\
&= -2|(\nabla^2_{-J}u)_{ij} + \frac{1}{2}(p-\frac{3}{2})(u_iu_j - u_{Ji}u_{Jj})|^2 + \frac{1}{2}(p-\frac{3}{2})^2|u_iu_j - u_{Ji}u_{Jj}|^2 \\
&\quad +(3+2p-p^2)|\nabla u|^4 + C|\nabla u|^2 \\
&\leq \frac{1}{2}(p-\frac{3}{2})^2|u_iu_j - u_{Ji}u_{Jj}|^2 + (3+2p-p^2)|\nabla u|^4 + C|\nabla u|^2.
\end{aligned}
$$

The J-anti-invariant part of $\nabla u \otimes \nabla u$ has half of the norm square compared to the full $\nabla u \otimes \nabla u$:

$$
|u_iu_j - u_{Ji}u_{Jj}|^2 = 2|\nabla u \otimes \nabla u|^2 = 2|\nabla u|^4. \tag{8.84}
$$

So the conclusion is that

$$
\begin{aligned}
&-2|\nabla^2 u|^2 + (2-4p)(\nabla^2 u)_{ij}u^iu^j + (3+2p-p^2)|\nabla u|^4 \\
\leq\ & ((p-\frac{3}{2})^2 - p^2 + 2p + 3)|\nabla u|^4 + C|\nabla u|^2 \\
=\ & (-p + \frac{9}{4} + 3)|\nabla u|^4 + C|\nabla u|^2 \\
\leq\ & -|\nabla u|^4 + C|\nabla u|^2
\end{aligned}
$$

for $p = (9/4) + 4$. Thus

$$
e^{-(p+1)u}(\partial_t - e^u\Delta)V \leq -|\nabla u|^4 + C(p)|\nabla u|^3 + C(p). \tag{8.85}
$$

Then by maximum principle and the boundedness of u, we prove the proposition.

8.4.2. Second order estimate. In this section, we obtain estimates on $|\nabla N| + |\nabla^2 u|$. We refer to ∇N and $\nabla^2 u$ as second order terms since they involve two derivatives of φ.

Proposition 6. *Let $(g_{ij}(t), u(t))$ evolve by Type IIA flow on $M \times [0,T]$. Suppose*

$$
\sup_{M\times[0,T]} \Big(|Rm| + |N| + |\nabla u| + |u| \Big) \leq \Lambda. \tag{8.86}
$$

Then there exists a constant C depending on Λ and $(g_{ij}(0), u(0))$ such that

$$
\sup_{M\times[0,T]} \Big(|\nabla N| + |\nabla^2 u| \Big) \leq C. \tag{8.87}
$$

A basic building block in the construction of our test function for this estimate will be

$$\tau(z) = |N|^2 + |\nabla u|^2. \tag{8.88}$$

It satisfies $\tau \leq C$, and using our work so far, its evolution can be estimated by

$$(\partial_t - e^u \Delta)\tau \leq -|\nabla N|^2 - |\nabla^2 u|^2 + C. \tag{8.89}$$

We start with the test function

$$Q = \frac{|\nabla^2 u|^2 + |\nabla N|^2}{K - \tau} \tag{8.90}$$

where K is a large constant to be determined. We can compute its evolution

$$\begin{aligned}
&(\partial_t - e^u \Delta)Q \\
= \ & \frac{1}{K-\tau}(\partial_t - e^u\Delta)(|\nabla^2 u|^2 + |\nabla N|^2) + \frac{|\nabla^2 u|^2 + |\nabla N|^2}{(K-\tau)^2}(\partial_t - e^u \Delta)\tau \\
& - \frac{2e^u}{(K-\tau)^2}\nabla_i(|\nabla^2 u|^2 + |\nabla N|^2)\nabla^i \tau - 2e^u \frac{|\nabla^2 u|^2 + |\nabla N|^2}{(K-\tau)^3}|\nabla \tau|^2.
\end{aligned}$$

By our evolution equations (8.71), (8.16), (8.89) for $|\nabla N|^2$, $|\nabla^2 u|^2$, and τ, we obtain the estimate

$$\begin{aligned}
(\partial_t - e^u \Delta)Q \leq \ & \frac{e^u}{(K-\tau)}\Big[- |\nabla^3 u|^2 - |\nabla^2 N|^2 + C|\nabla N|^3 + C|\nabla^2 u|^3 \\
& + C|\nabla Rm||\nabla N| + C|\nabla Rm||\nabla^2 u| + C|\nabla N||\nabla^3 u| \\
& + C|\nabla N|^2 |\nabla^2 u| + C|\nabla^2 N||\nabla^2 u| + C \\
& - \frac{|\nabla N|^4}{(K-\tau)} - \frac{|\nabla^2 u|^4}{(K-\tau)} - 2\frac{|\nabla^2 u|^2 |\nabla N|^2}{(K-\tau)} \\
& + C\frac{|\nabla^2 u|^2 + |\nabla N|^2}{(K-\tau)} - 2\frac{|\nabla^2 u|^2 + |\nabla N|^2}{(K-\tau)^2}|\nabla \tau|^2 \\
& - \frac{2}{(K-\tau)}\nabla_i(|\nabla^2 u|^2 + |\nabla N|^2)\nabla^i \tau \Big]
\end{aligned}$$

By the bound on $|N|$ and $|\nabla u|$, we can choose $K - \tau \geq \frac{K}{2}$ large. The terms $|\nabla N|^4$ and $|\nabla^2 u|^4$ can absorb lower order terms. We also drop the last term.

$$(\partial_t - e^u \Delta)Q \leq \frac{e^u}{(K-\tau)}\Big[- |\nabla^3 u|^2 - |\nabla^2 N|^2 + \frac{1}{10}|\nabla Rm|^2 + C(K)$$

$$-\frac{|\nabla N|^4}{2K} - \frac{|\nabla^2 u|^4}{2K} - \frac{|\nabla^2 u|^2|\nabla N|^2}{K}$$
$$-\frac{2}{(K-\tau)}\nabla_i(|\nabla^2 u|^2 + |\nabla N|^2)\nabla^i\tau\Big] \tag{8.91}$$

Using $|\nabla\tau| \leq C(|\nabla^2 u| + |\nabla N| + 1)$, we can estimate

$$\begin{aligned} & -\frac{2}{(K-\tau)}\nabla_i(|\nabla^2 u|^2 + |\nabla N|^2)\nabla^i\tau \\ \leq\ & \frac{C}{K}|\nabla|\nabla^2 u|^2 + \nabla|\nabla N|^2||\nabla\tau| \\ \leq\ & \frac{C}{K}\Big(|\nabla^3 u|(|\nabla^2 u|^2 + |\nabla^2 u||\nabla N|) + |\nabla^2 N|(|\nabla N|^2 + |\nabla^2 u||\nabla N|)\Big) \\ \leq\ & \frac{1}{2}|\nabla^3 u|^2 + \frac{1}{2}|\nabla^2 N|^2 + \frac{C_0}{K^2}|\nabla^2 u|^4 + \frac{C_0}{K^2}|\nabla^2 u|^2|\nabla N|^2 + \frac{C_0}{K^2}|\nabla N|^4 \end{aligned} \tag{8.92}$$

Choose K large such that $K \geq 4C_0 \gg 1$. Then the main inequality becomes

$$\begin{aligned} (\partial_t - e^u\Delta)Q \ \leq\ & \frac{e^u}{(K-\tau)}\Big[-\frac{1}{2}|\nabla^3 u|^2 - \frac{1}{2}|\nabla^2 N|^2 - \frac{|\nabla N|^4}{4K} - \frac{|\nabla^2 u|^4}{4K} \\ & + \frac{1}{10}|\nabla Rm|^2 + C(K)\Big]. \end{aligned} \tag{8.93}$$

We can now prove that if $|u| + |\nabla u| + |N| + |Rm| \leq C$ along the flow, then we can bound $|\nabla N|$ and $|\nabla^2 u|$. Consider the test function

$$S = Q + |Rm|^2. \tag{8.94}$$

By (8.93) and (2.13), we can estimate the evolution of Q and $|Rm|^2$.

$$\begin{aligned} (\partial_t - e^u\Delta)S \ \leq\ & \frac{e^u}{(K-\tau)}\Big[-\frac{1}{2}|\nabla^3 u|^2 - \frac{1}{2}|\nabla^2 N|^2 - \frac{|\nabla N|^4}{4K} - \frac{|\nabla^2 u|^4}{4K} \\ & + \frac{1}{10}|\nabla Rm|^2 + C(K)\Big] - e^u|\nabla Rm|^2 \\ & + C|\nabla^3 u| + C|\nabla^2 N| + C|\nabla N|^2 + C|\nabla^2 u|^2 + C \end{aligned} \tag{8.95}$$

As long as K is large enough such that $K - \tau \geq 1$, it follows that at a maximum point (x_0, t_0) of S with $t_0 > 0$, then

$$|\nabla N|^2(x_0,t_0) + |\nabla^2 u|^2(x_0,t_0) \leq C(K). \tag{8.96}$$

Since $Q = \frac{|\nabla^2 u|^2 + |\nabla N|^2}{K-\tau}$, it follows that $Q(x_0, t_0) \leq C$ and hence $S(x_0, t_0) \leq C$.

Therefore S is bounded on $M \times [0, T]$. It follows that if $|u| + |\nabla u| + |N| + |Rm| \leq C_0$ on $M \times [0, T]$, then

$$|\nabla N| + |\nabla^2 u| \leq C \tag{8.97}$$

where C depends on C_0 and the initial data.

8.5. Higher order estimates

In this section, we prove the following estimate.

Proposition 7. *Let $(g_{ij}(t), u(t))$ evolve by Type IIA flow on $M \times [0, T]$. Suppose*

$$\sup_{M \times [0,T]} \Big(|Rm| + |\nabla N| + |N| + |\nabla^2 u| + |\nabla u| + |u| \Big) \leq \Lambda. \tag{8.98}$$

Then for each integer $k \geq 1$, there exists a constant C_k depending on k, Λ and $(g_{ij}(0), u(0))$ such that

$$\sup_{M \times [0,T]} \Big(|\nabla^k Rm| + |\nabla^{k+1} N| + |\nabla^{k+2} u| \Big) \leq C_k. \tag{8.99}$$

Note: in earlier work, we showed that the estimate $|u| + |Rm| \leq C$ implies the estimate $|\nabla u| + |\nabla^2 u| + |N| + |\nabla N| \leq C$. Combining these two results, we conclude that if $|u| + |Rm| \leq C$ remains bounded along the flow, then all geometric terms remain bounded.

Let I_k denote any combination of geometric terms of derivative order $\leq k$ in the metric. For example,

$$\begin{aligned} I_2 &= f(u, \nabla u, \nabla^2 u, N, \nabla N, Rm), \\ I_3 &= f(u, \nabla u, \nabla^2 u, \nabla^3 u, N, \nabla N, \nabla^2 N, Rm, \nabla Rm). \end{aligned} \tag{8.100}$$

In this section, we will evolve all higher order geometric terms appearing in the equation of the metric Type IIA flow.

8.5.1. The evolution of $|\nabla^k Rm|^2$. We write the evolution of the curvature as

$$(\partial_t - e^u\Delta)Rm = E(Rm), \tag{8.101}$$

where

$$E(Rm) = (\nabla^3 u + \nabla^2 N + \nabla Rm) * I_1 + I_2. \tag{8.102}$$

We have for example

$$\nabla E(Rm) = (\nabla^4 u+\nabla^3 N+\nabla^2 Rm)*I_1+(\nabla^3 u+\nabla^2 N+\nabla Rm)*I_2+I_2 \tag{8.103}$$

and in general

$$\begin{aligned}\nabla^k E(Rm) &= (\nabla^{k+3}u + \nabla^{k+2}N + \nabla^{k+1}Rm) * I_1 \\ &\quad +(\nabla^{k+2}u + \nabla^{k+3}N + \nabla^{k+1}Rm) * I_2 + I_{k+1}\end{aligned} \tag{8.104}$$

Then

$$\begin{aligned}\partial_t \nabla^k Rm &= \partial_t(\partial + \Gamma)^k Rm \\ &= \nabla^k(\partial_t Rm) + \sum_{i=0}^{k-1} \nabla^i \partial_t \Gamma\, \nabla^{k-1-i} Rm \\ &= \nabla^k E(Rm) + \nabla^k(e^u \Delta Rm) + \sum_{i=0}^{k-1} \nabla^i \partial_t \Gamma\, \nabla^{k-1-i} Rm\end{aligned} \tag{8.105}$$

We have the general commutator formula

$$\nabla^k \Delta A = \Delta \nabla^k A + \nabla^k (Rm * A) \tag{8.106}$$

which implies

$$\begin{aligned}(\partial_t - e^u\Delta)\nabla^k Rm &= \nabla^k(Rm * Rm) + \nabla^k E(Rm) + \sum_{i=1}^{k} \nabla^i e^u * \nabla^{k-i}\Delta Rm \\ &\quad + \sum_{i=0}^{k-1} \nabla^i \partial_t \Gamma * \nabla^{k-1-i} Rm\end{aligned} \tag{8.107}$$

We note

$$\partial_t \Gamma = (\nabla^3 u + \nabla^2 N + \nabla Rm) * I_1 + I_2, \tag{8.108}$$

and

$$\nabla^k \partial_t \Gamma = (\nabla^{k+3} u + \nabla^{k+2} N + \nabla^{k+1} Rm) * I_1 + (\nabla^{k+2} u + \nabla^{k+3} N + \nabla^{k+1} Rm) * I_2 + I_{k+1}. \tag{8.109}$$

Therefore

$$(\partial_t - e^u \Delta)\nabla^k Rm = I_1 * (\nabla^{k+3} u + \nabla^{k+2} N + \nabla^{k+1} Rm) + I_2 * (\nabla^{k+2} u + \nabla^{k+1} N + \nabla^k Rm) + I_{k+1} \tag{8.110}$$

The norm is evolving by

$$(\partial_t - e^u \Delta)|\nabla^k Rm|^2 = 2\langle (\partial_t - e^u \Delta)\nabla^k Rm, \nabla^k Rm\rangle - 2e^u |\nabla^{k+1} Rm|^2 + \partial_t g * \nabla^k Rm * \nabla^k Rm. \tag{8.111}$$

Thus

$$\begin{aligned} &(\partial_t - e^u \Delta)|\nabla^k Rm|^2 \\ = & -2e^u|\nabla^{k+1} Rm|^2 + I_1 * (\nabla^{k+3} u + \nabla^{k+2} N + \nabla^{k+1} Rm) * \nabla^k Rm \\ & + I_2 * (\nabla^{k+2} u + \nabla^{k+1} N + \nabla^k Rm)^2 + I_{k+1} * \nabla^k Rm. \end{aligned} \tag{8.112}$$

8.5.2. The evolution of $|\nabla^k N|^2$. We will evolve $\nabla^k N$ in this section for all $k \geq 2$. We write $(\partial_t - e^u \Delta)N = E(N)$. Higher order terms evolve by

$$\begin{aligned} \partial_t \nabla^k N &= \partial_t (\partial + \Gamma)^k N \\ &= \nabla^k(\partial_t N) + \sum_{i=0}^{k-1} \nabla^i \partial_t \Gamma * \nabla^{k-1-i} N \\ &= \nabla^k E(N) + e^u \nabla^k \Delta N + \sum_{i=1}^{k} \nabla^i e^u * \nabla^{k-i} \Delta N \\ &\quad + \sum_{i=0}^{k-1} \nabla^i \partial_t \Gamma * \nabla^{k-1-i} N. \end{aligned} \tag{8.113}$$

Using (8.106) to commute derivatives gives

$$\begin{aligned} &(\partial_t - e^u \Delta)\nabla^k N \\ = & \nabla^k E(N) + \sum_{i=1}^{k} \nabla^i e^u * \nabla^{k-i} \Delta N + \nabla^k (Rm * N) + \sum_{i=0}^{k-1} \nabla^i \partial_t \Gamma * \nabla^{k-1-i} N. \end{aligned} \tag{8.114}$$

By (8.43),

$$E(N) = (\nabla^2 u + \nabla N + Rm) * I_1 + I_1. \tag{8.115}$$

Differentiating this once gives

$$\nabla E(N) = (\nabla^3 u + \nabla^2 N + \nabla Rm) * I_1 + I_2. \tag{8.116}$$

Differentiating again, we obtain

$$\begin{aligned} \nabla^2 E(N) &= (\nabla^4 u + \nabla^3 N + \nabla^2 Rm) * I_1 \\ &\quad + (\nabla^3 u + \nabla^2 N + \nabla Rm) * I_2 + I_2. \end{aligned} \tag{8.117}$$

Higher order derivatives are

$$\begin{aligned} \nabla^k E(N) &= (\nabla^{k+2} u + \nabla^{k+1} N + \nabla^k Rm) * I_1 \\ &\quad + (\nabla^{k+1} u + \nabla^k N + \nabla^{k-1} Rm) * I_2 + I_k \end{aligned} \tag{8.118}$$

for $k \geq 2$. Substituting this and (8.108) into (8.114)

$$\begin{aligned} (\partial_t - e^u \Delta)\nabla^k N &= (\nabla^{k+2} u + \nabla^{k+1} N + \nabla^k Rm) * I_1 \\ &\quad + (\nabla^{k+1} u + \nabla^k N + \nabla^{k-1} Rm) * I_2 + I_k. \end{aligned} \tag{8.119}$$

The norm evolves by

$$\begin{aligned} (\partial_t - e^u \Delta)|\nabla^k N|^2 &= -2e^u |\nabla^{k+1} N|^2 + 2\langle (\partial_t - e^u \Delta)\nabla^k N, \nabla^k N \rangle \\ &\quad + \partial_t g * \nabla^k N * \nabla^k N. \end{aligned} \tag{8.120}$$

Therefore

$$\begin{aligned} &(\partial_t - e^u \Delta)|\nabla^k N|^2 \\ = \; &-2e^u |\nabla^{k+1} N|^2 + I_1 * (\nabla^{k+2} u + \nabla^{k+1} N + \nabla^k Rm) * \nabla^k N \\ &+ I_2 * (\nabla^{k+1} u + \nabla^k N + \nabla^{k-1} Rm)^2 + I_k * \nabla^k N. \end{aligned} \tag{8.121}$$

8.5.3. The evolution of $|\nabla^k u|^2$. Denote as before $(\partial_t - e^u \Delta)u = E(u)$. We will compute the evolution of $\nabla^k u$ for $k \geq 3$.

$$\begin{aligned} \partial_t \nabla^k u &= \partial_t (\partial + \Gamma)^{k-1} \partial u \\ &= \nabla^k (\partial_t u) + \sum_{i=0}^{k-2} \nabla^i \partial_t \Gamma * \nabla^{k-1-i} u \end{aligned}$$

$$
\begin{aligned}
&= e^u\nabla^k\Delta u + \nabla^k E(u) + \sum_{i=1}^{k}\nabla^i e^u\nabla^{k-i}\Delta u + \sum_{i=0}^{k-2}\nabla^i\partial_t\Gamma * \nabla^{k-1-i}u \\
&= e^u\Delta\nabla^k u + \sum_{i=0}^{k-1}\nabla^i Rm * \nabla^{k-i}u + \nabla^k E(u) \\
&\quad + \sum_{i=1}^{k}\nabla^i e^u\nabla^{k-i}\Delta u + \sum_{i=0}^{k-2}\nabla^i\partial_t\Gamma * \nabla^{k-1-i}u
\end{aligned} \tag{8.122}
$$

The evolution of u is of the form $E(u) = I_1$. We will differentiate this 3 times before it becomes linear enough to use in our general argument. Differentiating once

$$
\nabla E(u) = (\nabla^2 u + \nabla N + Rm) * I_1 + I_1, \tag{8.123}
$$

twice

$$
\nabla^2 E(u) = (\nabla^3 u + \nabla^2 N + \nabla Rm) * I_1 + I_2 \tag{8.124}
$$

and three times

$$
\nabla^3 E(u) = (\nabla^4 u + \nabla^3 N + \nabla^2 Rm) * I_1 + (\nabla^3 u + \nabla^2 N + \nabla Rm) * I_2 + I_2. \tag{8.125}
$$

Higher order derivatives are

$$
\begin{aligned}
\nabla^k E(u) &= (\nabla^{k+1}u + \nabla^k N + \nabla^{k-1}Rm) * I_1 \\
&\quad + (\nabla^k u + \nabla^{k-1}N + \nabla^{k-2}Rm) * I_2 + I_{k-1},
\end{aligned} \tag{8.126}
$$

for $k \geq 3$. Substituting this and (8.108) into (8.122)

$$
\begin{aligned}
(\partial_t - e^u\Delta)\nabla^k u &= (\nabla^{k+1}u + \nabla^k N + \nabla^{k-1}Rm) * I_1 \\
&\quad + (\nabla^k u + \nabla^{k-1}N + \nabla^{k-2}Rm) * I_2 + I_{k-1}.
\end{aligned} \tag{8.127}
$$

Using the evolution of the norm

$$
\begin{aligned}
(\partial_t - e^u\Delta)|\nabla^k u|^2 &= -2e^u|\nabla^{k+1}u|^2 + 2\langle(\partial_t - e^u\Delta)\nabla^k u, \nabla^k u\rangle \\
&\quad + \partial_t g * \nabla^k u * \nabla^k u,
\end{aligned} \tag{8.128}
$$

we conclude

$$
\begin{aligned}
&(\partial_t - e^u\Delta)|\nabla^k u|^2 \\
&= -2e^u|\nabla^{k+1}u|^2 + I_1 * (\nabla^{k+1}u + \nabla^k N + \nabla^{k-1}Rm) * \nabla^k u \\
&\quad + I_2 * (\nabla^k u + \nabla^{k-1}N + \nabla^{k-2}Rm)^2 + I_{k-1} * \nabla^k u.
\end{aligned} \tag{8.129}
$$

8.5.4. Estimates: proof of Theorem 7. Putting everything together, we obtain

$$\begin{aligned} & (\partial_t - e^u\Delta)(|\nabla^k u|^2 + |\nabla^{k-1}N|^2 + |\nabla^{k-2}Rm|^2) \\ = & -2e^u(|\nabla^{k+1}u|^2 + |\nabla^k N|^2 + |\nabla^{k-1}Rm|^2) \\ & + I_1 * (\nabla^{k+1}u + \nabla^k N + \nabla^{k-1}Rm) * (\nabla^k u + \nabla^{k-1}N + \nabla^{k-2}Rm) \\ & + I_2 * (\nabla^k u + \nabla^{k-1}N + \nabla^{k-2}Rm)^2 \\ & + I_{k-1} * (\nabla^k u + \nabla^{k-1}N + \nabla^{k-2}Rm) \end{aligned} \tag{8.130}$$

Let $k \geq 3$. Suppose $I_{k-1} \leq C$. Then

$$\begin{aligned} & (\partial_t - e^u\Delta)(|\nabla^k u|^2 + |\nabla^{k-1}N|^2 + |\nabla^{k-2}Rm|^2) \\ \leq & -e^u(|\nabla^{k+1}u|^2 + |\nabla^k N|^2 + |\nabla^{k-1}Rm|^2) \\ & + C|\nabla^k u|^2 + C|\nabla^{k-1}N|^2 + C|\nabla^{k-2}Rm|^2 + C \end{aligned} \tag{8.131}$$

and

$$\begin{aligned} & (\partial_t - e^u\Delta)(|\nabla^{k-1}u|^2 + |\nabla^{k-2}N|^2 + |\nabla^{k-3}Rm|^2) \\ \leq & -e^u(|\nabla^k u|^2 + |\nabla^{k-1}N|^2 + |\nabla^{k-2}Rm|^2) + C. \end{aligned} \tag{8.132}$$

It follows that the test function

$$\begin{aligned} & (\partial_t - e^u\Delta)\Big[|\nabla^k u|^2 + |\nabla^{k-1}N|^2 + |\nabla^{k-2}Rm|^2 \\ & + \Lambda(|\nabla^{k-1}u|^2 + |\nabla^{k-2}N|^2 + |\nabla^{k-3}Rm|^2)\Big] \\ \leq & -|\nabla^k u|^2 - |\nabla^{k-1}N|^2 - |\nabla^{k-2}Rm|^2 + \Lambda C \end{aligned} \tag{8.133}$$

for $\Lambda \gg 1$ large. By the maximum principle, we conclude that if $I_{k-1} \leq K$ then

$$|\nabla^k u|^2 + |\nabla^{k-1}N|^2 + |\nabla^{k-2}Rm|^2 \leq C(K, g(0)) \tag{8.134}$$

and hence I_k is bounded along the flow. This argument shows that if I_2 is bounded, then I_k is bounded for all k.

8.6. Long-time existence

Let $(u(t), g(t))$ be a solution to the Type IIA flow on $[0, T)$. Suppose $|u| + |Rm| \leq C$ on $M \times [0, T)$. We have shown that in this case $|\nabla^k u| + |\nabla^k N| +$

$|\nabla^k Rm| \leq C$ for all $k \geq 1$. We now give the standard argument (see e.g. [29]) which shows that the flow can be extended past $t = T$. Denote $\partial_t g_{ij} = E_{ij}$. Our estimates imply that

$$|E| + |\nabla^k E| \leq C \tag{8.135}$$

where ∇ is with respect to the evolving metric $g(t)$. If we take $x \in M$, $v \in T_x M$ and $t_1, t_2 \in (0, T)$, then

$$\left| \log g(t_2)(v,v) - \log g(t_1)(v,v) \right| = \left| \int_{t_1}^{t_2} \frac{\dot{g}(\tau)(v,v)}{g(\tau)(v,v)} d\tau \right| \leq C|t_2 - t_1|. \tag{8.136}$$

It follows that $g(t)$ is a Cauchy sequence as $t \to T$ and

$$e^{-CT} g(0) \leq g(t) \leq e^{CT} g(0) \tag{8.137}$$

and the metrics g_{ij} do not degenerate on $[0, T)$. Let $\bar{\nabla}$ denote the covariant derivative with respect to $\bar{g} = g(0)$. We have

$$\partial_t \bar{\nabla}_k g_{ij} = \bar{\nabla}_k \dot{g}_{ij} = \nabla_k E_{ij} + (\bar{\Gamma} - \Gamma) * E_{ij} \tag{8.138}$$

The difference between two connections is

$$\Gamma^k_{ij} - \bar{\Gamma}^k_{ij} = \frac{1}{2} g^{kp} (-\bar{\nabla}_p g_{ij} + \bar{\nabla}_i g_{pj} + \bar{\nabla}_j g_{pi}), \tag{8.139}$$

and hence

$$\partial_t \bar{\nabla} g = \bar{\nabla} g * E + \mathcal{O}(1). \tag{8.140}$$

Therefore

$$\partial_t |\bar{\nabla} g|^2_{\bar{g}} \leq C |\bar{\nabla} g|^2_{\bar{g}} + C |\bar{\nabla} g|_{\hat{g}} \tag{8.141}$$

and hence $|\bar{\nabla} g|_{\bar{g}} \leq C(T)$ on $[0, T)$. Higher order derivatives are similar: indeed, let $k \geq 1$ and suppose that $|\bar{\nabla}^\ell g|_{\bar{g}} \leq C$ for all $\ell \leq k$. Then a similar calculation gives

$$\partial_t \bar{\nabla}^{k+1} g = \bar{\nabla}^{k+1} g * E + \mathcal{O}(1), \tag{8.142}$$

from which it follows that

$$|\bar{\nabla}^{k+1} g|_{\bar{g}} \leq C(T). \tag{8.143}$$

Therefore, the evolving metrics g_{ij} and all their derivatives are bounded on $[0,T)$. Since we showed $g(t)$ is Cauchy, it follows that $g(t) \to g(T)$ smoothly as $t \to T$. A similar argument shows that $u(t) \to u(T)$.

This produces a limiting pair $(g(T), u(T))$. The linear ODE for φ given in Theorem 4 has coefficients which only depend on $\tilde{g}_{ij} = e^u g_{ij}$, thus these coefficients are smoothly defined on $[0,T]$. It follows that $\varphi(t)$ has a smooth solution on $[0,T]$. By the non-degeneracy estimate (7.25), we have that $\varphi(T)$ is closed, primitive, and in the positive cone $(-\lambda_\varphi) > 0$. By the short-time existence theorem, the flow can be extended to $[0, T+\epsilon)$ for some $\epsilon > 0$. The discussion here also implies that $|\nabla^\alpha \varphi| \leq C$ on $[0,T)$ for any multi-index α. This completes the proof of Theorem 7.

9. Examples and applications

In this section, we discuss a range of examples and applications of the Type IIA flow with no sources.

9.1. The stationary points: proof of Theorem 8

We begin with the proof of Theorem 8. First, we note that $-d^\Lambda(|\varphi|^2\hat{\varphi}) = \Lambda d(|\varphi|^2\hat{\varphi}) = \partial_-(|\varphi|^2\hat{\varphi})$, where ∂_- is a first order differential operator introduced in [50] such that

$$d(|\varphi|^2\hat{\varphi}) = \omega \wedge \partial_-(|\varphi|^2\hat{\varphi})$$

and $\partial_-(|\varphi|^2\hat{\varphi})$ is a primitive 2-form. Therefore the stationary point equation $d\Lambda d(|\varphi|^2 \star \varphi) = 0$ can be expressed as $d\partial_-(|\varphi|^2\hat{\varphi}) = 0$. In particular,

$$0 = \int_M d\partial_-(|\varphi|^2\hat{\varphi}) \wedge \hat{\varphi} = -\int_M \partial_-(|\varphi|^2\hat{\varphi}) \wedge d\hat{\varphi}. \tag{9.1}$$

Combining (6.40), (6.30), and $\beta = 0$ implies $d\hat{\varphi} = \alpha \wedge \hat{\varphi} + \tilde{\mathfrak{T}} \boxtimes \hat{\varphi}$. By (6.43), we have $\tilde{\mathfrak{T}} = N - \frac{1}{4}(d^c\tilde{\omega} + \mathcal{M}(d^c\tilde{\omega}))$. By Lemma 15, we see that

$$d\hat{\varphi} = N \boxtimes \hat{\varphi} \tag{9.2}$$

which is a (2,2)-form (by the argument in the proof of Lemma 14). Therefore

$$d(|\varphi|^2\hat{\varphi}) = |\varphi|^2(-\alpha \wedge \hat{\varphi} + N \boxtimes \hat{\varphi}), \tag{9.3}$$

where the first term is a $(3,1)+(1,3)$-form, the second is a $(2,2)$-form, and $\alpha = -d|\varphi|^2$. It follows that

$$\begin{aligned}\partial_-\hat{\varphi} &= \Lambda(N \boxtimes \hat{\varphi}),\\ \partial_-(|\varphi|^2\hat{\varphi}) &= |\varphi|^2(-\Lambda(\alpha\wedge\hat{\varphi}) + \partial_-\hat{\varphi}),\end{aligned}$$

where $\Lambda(N \boxtimes \hat{\varphi})$ is a $(1,1)$-form and $\Lambda(\alpha\wedge\hat{\varphi})$ is a $(2,0)+(0,2)$-form. Thus (9.1) becomes

$$\begin{aligned} 0 &= \int_M |\varphi|^2(-\Lambda(\alpha\wedge\hat{\varphi}) + \partial_-\hat{\varphi})\wedge\omega\wedge\partial_-\hat{\varphi} = \int_M |\varphi|^2\omega\wedge\partial_-\hat{\varphi}\wedge\partial_-\hat{\varphi}\\ &= -\int_M |\varphi|^2|\partial_-\hat{\varphi}|^2\frac{\omega^3}{3!}. \end{aligned} \tag{9.4}$$

Consequently we conclude that $\partial_-\hat{\varphi} = 0$ and $d\hat{\varphi} = \omega\wedge\partial_-\hat{\varphi} = 0$. In view of Lemma 7, the almost-complex structure J is integrable and the form φ is harmonic. Now we use the integration by parts argument again to get

$$\begin{aligned} 0 &= \int_M d\partial_-(|\varphi|^2\hat{\varphi})\wedge|\varphi|^2\hat{\varphi} = -\int_M \omega\wedge\partial_-(|\varphi|^2\hat{\varphi})\wedge\partial_-(|\varphi|^2\hat{\varphi})\\ &= -\int_M |\varphi|^4|\Lambda(\alpha\wedge\hat{\varphi})|^2\frac{\omega^3}{3!}. \end{aligned}$$

So we deduce that $\Lambda(\alpha\wedge\hat{\varphi}) = 0$, hence $\alpha\wedge\hat{\varphi} = 0$ and $\alpha = 0$, so $|\varphi|$ is a constant. Q.E.D.

9.2. Integrable almost-complex structures: proof of Theorem 9

Next, we give the proof of Theorem 9. The following identity for any smooth function f and any differential form is well-known:

$$d^\dagger(f\mu) = fd^\dagger\mu - \iota_{\nabla f}\mu. \tag{9.5}$$

Indeed, it can be quickly verified by using $d^\dagger\mu = -\iota_k(\nabla^k\mu)$.

Back to the proof of Theorem 9, we apply Theorem 1 and the identity (9.5) to rewrite the Type IIA flow without sources as

$$\begin{aligned} \partial_t\varphi &= -d(|\varphi|^2d^\dagger\varphi - \iota_{\nabla|\varphi|^2}\varphi) + 2d(|\varphi|^2N^\dagger\cdot\varphi)\\ &= \mathcal{L}_{\nabla|\varphi|^2}\varphi - d(|\varphi|^2d^\dagger\varphi) + 2d(|\varphi|^2N^\dagger\cdot\varphi). \end{aligned} \tag{9.6}$$

On any orbit of the diffeomorphism group which contains a form φ with an integrable almost-complex structure, we have $N = 0$ and, in view of Lemma 7, $d^\dagger\varphi = 0$. Thus the flow reduces to

$$\partial_t\varphi = \mathcal{L}_{\nabla|\varphi|^2}\varphi \tag{9.7}$$

and a solution is given by the reparametrizations of φ along the time-dependent vector field $\nabla|\varphi|^2$. By the uniqueness part of Theorem 2, this is the unique solution.

We now re-express the Type IIA flow in an equivalent formulation, but with a fixed complex structure. For this, let f_t be the flow generated by the time-dependent vector field $-\nabla|\varphi|^2$ in the sense that

$$\frac{d}{dt}f_t(x) = -\nabla|\varphi|^2(t,x) \tag{9.8}$$

for any $x \in X$ and time t. It follows that

$$\frac{d}{dt}(f_t^*\varphi_t) = f_t^*\partial_t\varphi_t + f_t^*\mathcal{L}_{-\nabla|\varphi|^2}\varphi_t = 0,$$

hence $f_t^*\varphi_t \equiv \varphi_0$ is a constant 3-form on X. We see immediately that if we reparametrize M by the time-dependent diffeomorphism f_t, then along the flow, the 3-form $f_t^*\varphi_t$ and hence the complex structure $f_t^*J_t$ are fixed. In this new gauge, the Kähler metric $\omega_t = f_t^*\omega$ evolves by the equation

$$\partial_t\omega_t = f_t^*\mathcal{L}_{-\nabla|\varphi|^2}\omega. \tag{9.9}$$

Notice that

$$\mathcal{L}_{-\nabla|\varphi|^2}\omega = -d\iota_{\nabla|\varphi|^2}\omega = dJd|\varphi|^2 = -dd^c|\varphi|^2 \tag{9.10}$$

so the flow of ω_t can be written as

$$\partial_t\omega_t = -dd^c|\varphi|^2_{\omega_t}. \tag{9.11}$$

We remark that this equation can be viewed as a T-dual of the Anomaly flow for conformally Kähler data. In that case, we are given a fixed holomorphic $(3,0)$ form $\check\Omega$ on a Calabi-Yau threefold and the evolving Kähler metrics $\check\omega_t$ satisfy (see equation (4.10) in [20])

$$\partial_t\check\omega_t = dd^c|\check\Omega|^{-2}_{\check\omega_t}. \tag{9.12}$$

We see that $\check{R} = |\check{\Omega}|^{-2}_{\check{\omega}}$ is exchanged with $1/R = |\varphi|^2_\omega$.

This type of duality was observed in [20] on semi-flat Calabi-Yau three-folds. To connect with the work there, we can consider the conformally changed metric $\eta_t = |\varphi|^{-2}_{\omega_t}\omega_t$. It follows that $|\varphi|_{\eta_t} = |\varphi|^4_{\omega_t}$, hence η_t satisfies the conformally balanced equation $d(|\varphi|_{\eta_t}\eta_t^2) = 0$ since

$$|\varphi|_{\eta_t}\eta_t^2 = |\varphi|^4_{\omega_t}(|\varphi|^{-2}_{\omega_t}\omega_t)^2 = \omega_t^2$$

is closed. Moreover its evolution equation is

$$\begin{aligned}\partial_t(|\varphi|_{\eta_t}\eta_t^2) &= \partial_t(\omega_t^2) = 2\omega_t \wedge \partial_t\omega_t \\ &= 2\omega_t \wedge (-dd^c|\varphi|^2_{\omega_t}) = -4i\partial\bar{\partial}(|\varphi|^2_{\omega_t}\omega_t) \\ &= -4i\partial\bar{\partial}(|\varphi|_{\eta_t}\eta_t),\end{aligned} \tag{9.13}$$

which is exactly (up to a positive constant) the dual Anomaly flow in complex dimension 3 firstly introduced in [20]. Since we are in the conformally Kähler case, by the results of [20], we know the flow (9.11) is equivalent to the inverse MA-flow introduced by Cao-Keller [9] and Collins-Hisamoto-Takahashi [10], which converges to the unique Ricci-flat Kähler metric in the cohomology class $[\omega_0]$.

9.3. Symplectic manifolds with non-integrable almost complex structures

Next, we work out the Type IIA flow on some model symplectic manifolds with non-integrable almost-complex structures, more specifically tori, symplectic half-flat manifolds, and nilmanifolds.

9.3.1. The Type IIA flow on a torus. Consider the 6-torus $M = (\mathbf{R}/\mathbf{Z})^6$, with coordinates $\{x^j\}_{j=1}^6$ and the standard symplectic form $\omega = dx^{12} + dx^{34} + dx^{56}$. Let $\alpha, \beta, \gamma, \delta : \mathbf{R}/\mathbf{Z} \to \mathbf{R}$ be smooth functions depending only on the variable x^1. Consider

$$\varphi = e^\alpha dx^{135} - e^\beta dx^{146} - dx^{245} - dx^{236} + \gamma dx^{136} + \delta dx^{145}. \tag{9.14}$$

Clearly φ is closed and primitive. It is straightforward to find out that

$$|\varphi|^2 = 2\sqrt{4e^{\alpha+\beta} - (\gamma-\delta)^2}$$

and

$$\begin{aligned}\hat{\varphi} &= \frac{2}{|\varphi|^2}(-e^{\alpha}(\gamma+\delta)dx^{135} + (2e^{\alpha+\beta} - \gamma(\gamma-\delta))dx^{136} \\ &\quad +(2e^{\alpha+\beta} + \delta(\gamma-\delta))dx^{145} + e^{\beta}(\gamma+\delta)dx^{146} + 2e^{\alpha}dx^{235} \\ &\quad +(\gamma-\delta)dx^{236} - (\gamma-\delta)dx^{245} - 2e^{\beta}dx^{246}).\end{aligned}$$

Consequently we see that

$$\begin{aligned}d(|\varphi|^2\hat{\varphi}) &= 2dx^{12}\wedge(2(e^{\alpha})'dx^{35} + (\gamma-\delta)'(dx^{36} - dx^{45}) - 2(e^{\beta})'dx^{46}), \\ \Lambda d(|\varphi|^2\hat{\varphi}) &= 2(2(e^{\alpha})'dx^{35} + (\gamma-\delta)'(dx^{36} - dx^{45}) - 2(e^{\beta})'dx^{46}), \\ d\Lambda d(|\varphi|^2\hat{\varphi}) &= 4(e^{\alpha})''dx^{135} + 2(\gamma-\delta)''(dx^{136} - dx^{145}) - 4(e^{\beta})''dx^{146}.\end{aligned}$$

So the Type IIA flow in this case reduces to

$$\partial_t(e^{\alpha}) = 4(e^{\alpha})'', \qquad \partial_t(e^{\beta}) = 4(e^{\beta})'', \tag{9.15}$$

$$\partial_t\gamma = 2(\gamma-\delta)'', \qquad \partial_t\delta = -2(\gamma-\delta)''. \tag{9.16}$$

For calculations, it is convenient to introduce $a = 2e^{\alpha}, b = 2e^{\beta}, c = \gamma-\delta, d = \gamma+\delta$ and

$$|\varphi|^2 = 2\sqrt{4e^{\alpha+\beta} - (\gamma-\delta)^2} = 2\sqrt{ab - c^2}.$$

It follows that d is a constant along the flow, while a, b, c satisfy the standard heat equation:

$$\partial_t \begin{bmatrix} a & c \\ c & b \end{bmatrix} = 4\begin{bmatrix} a & c \\ c & b \end{bmatrix}''. \tag{9.17}$$

Obviously the matrix $\begin{bmatrix} a & c \\ c & b \end{bmatrix}$ converges to a constant matrix as t goes to infinity. Moreover along the flow the positive-definiteness is preserved and the limiting matrix is also positive definite. Thus φ_t converges to a positive primitive harmonic form.

Now let us analyze the behavior of $|N|^2$ along the flow. The easiest way to find $|N|^2$ is to use (7.28), which says that

$$|N|^2 = e^{-u}\partial_t u - (\Delta u + 2|du|^2). \tag{9.18}$$

The key is to compute Δu. Observe that the metric g can be expressed as

$$g = e^{-u} \begin{bmatrix} ab + d^2 - c^2 & -2d & & & & \\ -2d & 4 & & & & \\ & & 2a & -2c & & \\ & & -2c & 2b & & \\ & & & & 2a & 2c \\ & & & & 2c & 2b \end{bmatrix} \tag{9.19}$$

As $\Delta u = g^{ij}(u_{ij} - \Gamma_{ij}^k u_k) = g^{11}u'' - g^{ij}\Gamma_{ij}^1 u'$, and the Christoffel symbol term can be simplified to

$$\begin{aligned} g^{ij}\Gamma_{ij}^1 &= \frac{1}{2} g^{ij} g^{1l}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \\ &= g^{1j} g^{1l} g'_{jl} - \frac{1}{2} g^{11} g^{ij} g'_{ij} \\ &= \frac{1}{2}(g^{11})^2 g'_{11} + g^{11} g^{12} g'_{12} + ((g^{12})^2 - \frac{1}{2} g^{11} g^{22}) g'_{22} \\ &\quad - g^{11}(g^{33} g'_{33} + 2g^{34} g'_{34} + g^{44} g'_{44}) \\ &= 8e^{-3u}(4u'(ab - c^2) - (a'b + ab' - 2cc')) \\ &= 8e^{-3u} v', \end{aligned} \tag{9.20}$$

where $v = ab - c^2$ and $u = \log 2 + \dfrac{1}{2}\log v$. Therefore

$$\Delta u = 4e^{-u} u'' - 8e^{-3u} u' v' = 4e^{-u}(u'' - (u')^2). \tag{9.21}$$

Consequently

$$\begin{aligned} |N|^2 &= e^{-u}\partial_t u - (\Delta u + 2|du|^2) \\ &= 2e^{-u}\frac{a''b + ab'' - 2cc''}{v} - 4e^{-u}(u'' + (u')^2) \\ &= 16e^{-5u}\left(ab\left(2c' - \frac{cb'}{b} - \frac{ca'}{a}\right)^2 + \frac{ab - c^2}{ab}(ab' - a'b)^2 \right). \end{aligned} \tag{9.22}$$

This calculation suggests that J is integrable if and only if a, b, c are proportional to each other. In summary we have proved

Proposition 8. *Under our ansatze (9.14), the Type IIA flow on* $(\mathbf{R}/\mathbf{Z})^6$ *reduces to the standard heat equation on* $\mathbf{R}/\mathbf{Z}$. *If initially* φ *is of the form (9.14) whose associated almost complex structure is not integrable, the Type IIA flow still converges to Kähler Calabi-Yau geometry.*

9.3.2. The Type IIA flow on homogeneous symplectic half-flat manifolds. Because of Theorem 8, the convergence of the Type IIA flow is only possible when the underlying manifold is Kähler. We shall see in this subsection and the next that the Type IIA flow can be used to find optimal almost complex structures compatible with a given symplectic form, even when the underlying manifold does not admit any Kähler structure.

In order to run the Type IIA flow, we first need compact symplectic 6-manifolds with Type IIA structures. A special case of Type IIA structures can be found on the so-called symplectic half-flat manifolds (firstly introduced by de Bartholomeis [12], also known as special generalized Calabi-Yau manifolds [13]). In our terminology, a symplectic half-flat manifold is simply a symplectic manifold with Type IIA structure (M, ω, φ) and the extra condition that $|\varphi|^2$ is constant. Many compact symplectic half-flat manifolds can be constructed as quotients of Lie groups by co-compact lattices, where all the structures are homogeneous under the natural group action. Therefore we shall call symplectic half-flat manifolds constructed in this way *homogeneous*. It is clear that for homogeneous symplectic half-flat manifolds, their geometry up to covering is fully characterized by the underlying Lie algebra, or equivalently the exterior differential system defined by invariant 1-forms. Moreover, homogeneous symplectic half-flat structures have been fully classified by [11] and [22] when the Lie group is nilpotent or solvable respectively.

It is clear that if we run the Type IIA flow on a homogeneous symplectic half-flat manifold with homogeneous initial data, the homogeneity is preserved and the Type IIA flow reduces to a polynomial ODE system. Moreover, in the homogeneous setting, the function u and $|N|^2$ are constants on the manifold, therefore we have the following monotonicity formulas

Proposition 9. *Along the Type IIA flow on homogeneous symplectic half-flat manifolds, the following monotonicity formulae hold*

$$\partial_t u = e^u |N|^2 \geq 0, \tag{9.23}$$

$$\partial_t |N|^2 = -2e^u |(R^{-J})_{ij}|^2 \leq 0. \tag{9.24}$$

Proof: The first formula follows directly from (7.28). For the second formula, we note that Blair-Ianus [4] proved that

$$\partial_t \int_M |N|^2 \frac{\omega^3}{3!} = \int_M (\partial_t g_{ij}, (R^{-J})_{ij}) \frac{\omega^3}{3!}. \tag{9.25}$$

In our case u is a constant, hence (7.27) becomes $\partial_t g_{ij} = -2e^u (R^{-J})_{ij}$, and (9.25) simplifies to

$$\partial_t \int_M |N|^2 \frac{\omega^3}{3!} = -2 \int_M e^u |(R^{-J})_{ij}|^2 \frac{\omega^3}{3!}. \tag{9.26}$$

As everything is homogeneous, so all the scalars must be constant, consequently (9.26) still holds without integration, and (9.24) is proved. Q.E.D.

Corollary 2. *Let (M, ω) be a compact 6-dimensional homogeneous symplectic manifold. If (M, ω) admits a homogeneous symplectic half-flat structure (M, ω, φ_0) with which the Type IIA flow exists for all time, then there exist homogeneous almost complex structures compatible with ω and with arbitrary small Nijenhuis tensor.*

Proof of the Corollary: We run the Type IIA flow with initial data φ_0. By monotonicity formulas above, we know that

$$\frac{d}{dt} e^{-u} = -|N|^2 \leq 0, \tag{9.27}$$

$$\frac{d^2}{dt^2} e^{-u} = 2e^u |(R^{-J})_{ij}|^2 \geq 0. \tag{9.28}$$

So e^{-u} is a monotone non-increasing and convex function with lower bound. If the flow exists for all time, then we must have

$$\lim_{t\to\infty} |N|^2 = -\lim_{t\to\infty} \frac{d}{dt} e^{-u} = 0,$$

as was to be shown. Q.E.D.

The Type IIA flow on a nilmanifold. Now let us consider some explicit examples.

Consider the homogeneous symplectic half-flat structure in [13, Example 5.2], where the Lie algebra of the nilpotent Lie group is characterized by invariant 1-forms $\{e^1, \ldots, e^6\}$ satisfying

$$de^1 = de^2 = de^3 = de^5 = 0,$$
$$de^4 = e^{15}, \qquad de^6 = e^{13}.$$

Clearly $\omega = e^{12} + e^{34} + e^{56}$ defines an invariant symplectic structure. Moreover, this nilpotent Lie group admits co-compact lattices so all the constructions descend to compact nilmanifolds. Consider the ansatze

$$\varphi = \varphi_{a,b} = (1+a)e^{135} - e^{146} - e^{245} - e^{236} + b(e^{134} - e^{156}), \tag{9.29}$$

it is straightforward to check that $\varphi_{a,b}$ is primitive and closed for any a, b. The positivity condition for $\varphi_{a,b}$ is that $\frac{1}{16}|\varphi|^4 = 1+a-b^2 > 0$. By straightforward calculations, we get

$$\hat{\varphi} = 4|\varphi|^{-2}((1+a-b^2)e^1 \wedge (e^{36} + e^{45}) + e^2 \wedge (be^{34} + (1+a)e^{35} - e^{46} - be^{56})).$$

It follows that

$$\begin{aligned} &d(|\varphi|^2\hat{\varphi}) = 4e^{12}(e^{34} + 2be^{35} - e^{56}), \\ &\Lambda d(|\varphi|^2\hat{\varphi}) = 4(e^{34} + 2be^{35} - e^{56}), \quad d\Lambda d(|\varphi|^2\hat{\varphi}) = 8e^{135}. \end{aligned}$$

Therefore under our ansatze the Type IIA flow reduces to the following ODE system

$$\dot{a}(t) = 8, \qquad \dot{b}(t) = 0.$$

Hence the unique solution to the Type IIA flow is

$$\varphi(t) = (1 + a_0 + 8t)e^{135} - e^{146} - e^{245} - e^{236} + b_0(e^{134} - e^{156}), \tag{9.30}$$

which exists for all time $t \geq 0$.

One can easily verify that $\lim_{t\to\infty} J_t$ does not exist and

$$|N|^2 = (1 + a - b^2)^{-3/2} = (1 + a_0 + 8t - b_0^2)^{-3/2} \tag{9.31}$$

is decreasing to zero as $t \to \infty$. This is an explicit example where Corollary 2 applies.

The Type IIA flow on a solvmanifold. Consider the symplectic half-flat structure on the solvmanifold M constructed by Tomassini and Vezzoni in [46, Theorem 3.5]. The geometry of this solvmanifold is characterized by invariant 1-forms $\{e^j\}_{j=1}^6$ satisfying

$$de^1 = -\lambda e^{15}, \quad de^2 = \lambda e^{25}, \quad de^3 = -\lambda e^{36},$$

$$de^4 = \lambda e^{46}, \quad de^5 = 0, \quad de^6 = 0,$$

where $\lambda = \log \dfrac{3+\sqrt{5}}{2}$. One can easily check that $\omega = e^{12} + e^{34} + e^{56}$ is an invariant symplectic form on M. A particular symplectic half-flat structure on M takes the form

$$\begin{aligned}\varphi &= \frac{\sqrt{2}}{2}(e^{135} + e^{136} + e^{145} - e^{146} + e^{235} - e^{236} - e^{245} - e^{246}) \\ &= \frac{\sqrt{2}}{2\lambda} d(e^{13} + e^{14} - e^{23} + e^{24}),\end{aligned}$$

so $[\varphi] = 0 \in H^3(M; \mathbf{R})$.

Consider the ansatze

$$\varphi = \alpha(e^{135} + e^{136}) + \beta(e^{145} - e^{146}) + \gamma(e^{235} - e^{236}) - \delta(e^{245} + e^{246}). \tag{9.32}$$

A direct calculation gives

$$\begin{aligned}|\varphi|^2\hat{\varphi} &= 8(-\alpha\beta\gamma(e^{135} - e^{136}) + \alpha\beta\delta(e^{145} + e^{146}) \\ &\quad + \alpha\gamma\delta(e^{235} + e^{236}) + \beta\gamma\delta(e^{245} - e^{246})).\end{aligned}$$

The nondegenerate condition is that $|\varphi|^4 = 64\alpha\beta\gamma\delta > 0$. It follows that

$$\begin{aligned}d(|\varphi|^2\hat{\varphi}) &= 16\lambda(\alpha\beta\gamma e^{1356} + \alpha\beta\delta e^{1456} - \alpha\gamma\delta e^{2356} + \beta\gamma\delta e^{2456}), \\ \Lambda d(|\varphi|^2\hat{\varphi}) &= 16\lambda(\alpha\beta\gamma e^{13} + \alpha\beta\delta e^{14} - \alpha\gamma\delta e^{23} + \beta\gamma\delta e^{24}), \\ d\Lambda d(|\varphi|^2\hat{\varphi}) &= 16\lambda^2(\alpha\beta\gamma(e^{135} + e^{136}) + \alpha\beta\delta(e^{145} - e^{146}) \\ &\quad + \alpha\gamma\delta(e^{235} - e^{236}) - \beta\gamma\delta(e^{245} + e^{246})).\end{aligned}$$

After time rescaling, the Type IIA flow under our ansatze reduces to

$$\begin{aligned}\partial_t\alpha &= \alpha\beta\gamma, \qquad \partial_t\beta = \alpha\beta\delta, \\ \partial_t\gamma &= \alpha\gamma\delta, \qquad \partial_t\delta = \beta\gamma\delta.\end{aligned}$$

It is easy to see that there exist time-independent nonzero constants C_1 and C_2 such that $\alpha(t) = C_1\delta(t)$ and $\beta(t) = C_2\gamma(t)$. The ODE system simplifies to

$$\partial_t\gamma = C_1\gamma\delta^2, \qquad \partial_t\delta = C_2\gamma^2\delta.$$

Integrate these equations we know there is a constant C such that $C_2\gamma^2 - C_1\delta^2 = C$, hence

$$\partial_t\gamma = \gamma(C_2\gamma^2 - C).$$

One can solve explicitly

$$\gamma^2(t) = \frac{C\gamma_0^2}{C_2\gamma_0^2(1-e^{2Ct}) + Ce^{2Ct}}, \quad \delta^2(t) = \frac{C\delta_0^2 e^{2Ct}}{C_1\delta_0^2 + C - C_1\delta_0^2 e^{2Ct}}. \tag{9.33}$$

Assuming φ is initially positive, we know that C_1, C_2 are positive constants and γ_0, δ_0 are initial values of γ and δ satisfying $\beta\gamma - \alpha\delta = C_2\gamma_0^2 - C_1\delta_0^2 = C$. When $C = 0$, the above formula should be understood as

$$\gamma^2 = \frac{\gamma_0^2}{1 - 2C_2\gamma_0^2 t}, \qquad \delta^2 = \frac{\delta_0^2}{1 - 2C_1\delta_0^2 t}. \tag{9.34}$$

From the above explicit formulas, we can deduce that, no matter what C is, the flow has finite time singularity. A more symmetric expression for the solution (without time rescaling) is

$$\begin{aligned}
\alpha(t) &= \alpha_0\sqrt{\frac{(\beta_0\gamma_0 - \alpha_0\delta_0)e^{32\lambda^2\beta_0\gamma_0 t}}{\beta_0\gamma_0 e^{32\lambda^2\alpha_0\delta_0 t} - \alpha_0\delta_0 e^{32\lambda^2\beta_0\gamma_0 t}}},\\
\beta(t) &= \beta_0\sqrt{\frac{(\beta_0\gamma_0 - \alpha_0\delta_0)e^{32\lambda^2\alpha_0\delta_0 t}}{\beta_0\gamma_0 e^{32\lambda^2\alpha_0\delta_0 t} - \alpha_0\delta_0 e^{32\lambda^2\beta_0\gamma_0 t}}},\\
\gamma(t) &= \gamma_0\sqrt{\frac{(\beta_0\gamma_0 - \alpha_0\delta_0)e^{32\lambda^2\alpha_0\delta_0 t}}{\beta_0\gamma_0 e^{32\lambda^2\alpha_0\delta_0 t} - \alpha_0\delta_0 e^{32\lambda^2\beta_0\gamma_0 t}}},\\
\delta(t) &= \delta_0\sqrt{\frac{(\beta_0\gamma_0 - \alpha_0\delta_0)e^{32\lambda^2\beta_0\gamma_0 t}}{\beta_0\gamma_0 e^{32\lambda^2\alpha_0\delta_0 t} - \alpha_0\delta_0 e^{32\lambda^2\beta_0\gamma_0 t}}},
\end{aligned}$$

and in the critical case when $\alpha_0\delta_0 = \beta_0\gamma_0 = S > 0$, one has

$$\begin{aligned}
\alpha(t) &= \frac{\alpha_0}{\sqrt{1 - 32\lambda^2 St}}, \qquad \beta(t) = \frac{\beta_0}{\sqrt{1 - 32\lambda^2 St}},\\
\gamma(t) &= \frac{\gamma_0}{\sqrt{1 - 32\lambda^2 St}}, \qquad \delta(t) = \frac{\delta_0}{\sqrt{1 - 32\lambda^2 St}}.
\end{aligned}$$

From these explicit expressions, we see that the maximal existence time T is given by

$$T = \frac{1}{32\lambda^2}\frac{\log(\alpha_0\delta_0) - \log(\beta_0\gamma_0)}{\alpha_0\delta_0 - \beta_0\gamma_0}. \tag{9.35}$$

All of $\alpha, \beta, \gamma, \delta$ tend to infinity as $t \to T$, therefore $|\varphi|^2 = 8(\alpha\beta\gamma\delta)^{1/2} \to \infty$. To compute $|N|^2$, the quickest way is to use (9.23):

$$\begin{aligned}|N|^2 &= e^{-u}\partial_t u = -\partial_t e^{-u} = \frac{(\alpha\beta\gamma\delta)^{-3/2}}{16}\partial_t(\alpha\beta\gamma\delta) = 2\lambda^2\frac{\alpha\delta+\beta\gamma}{(\alpha\beta\gamma\delta)^{1/2}} \\ &= \frac{2\lambda^2}{(\alpha_0\beta_0\gamma_0\delta_0)^{1/2}}\left(\alpha_0\delta_0 e^{16\lambda^2(\beta_0\gamma_0-\alpha_0\delta_0)t} + \beta_0\gamma_0 e^{-16\lambda^2(\beta_0\gamma_0-\alpha_0\delta_0)t}\right) \\ &\geq 4\lambda^2.\end{aligned}$$

Now let us analyze the behavior of φ in detail. We shall see that the Type IIA flow naturally leads us to optimal almost-complex structures compatible with ω.

1. No matter what the integral constant $C = \beta_0\gamma_0 - \alpha_0\delta_0$ is, the flow of φ blows up when $t \to T$, and the same is true for the metric $\tilde{g}$. However the expressions of g, J, and N extend smoothly to $t = T$. In fact, the limit $|\varphi|^{-1}\varphi$ as $t \to T$ exists.
2. In the critical case $C = \beta_0\gamma_0 - \alpha_0\delta_0 = 0$, the flow of φ (as well as $\tilde{g}$) is a self-expander in the sense that

$$\varphi(t) = \frac{\varphi_0}{\sqrt{1 - 32\lambda^2 St}} \tag{9.36}$$

for a positive constant S determined by initial data. In this case, all of g, J and N are stationary with $|N|^2 = 4\lambda^2$. In fact such J provide examples of harmonic almost-complex structures in the sense of Blair-Ianus [4], namely almost-complex structures compatible with ω and satisfying

$$(R^{-J})_{ij} = \frac{1}{2}(R_{ij} - R_{Ji,Jj}) = 0. \tag{9.37}$$

These harmonic almost-complex structures are critical points of the energy functional studied by Blair-Ianus [4] and Lê-Wang [33].
3. When the integral constant $C = \beta_0\gamma_0 - \alpha_0\delta_0$ is not zero, all of g, J and N are evolving. When t approaches T, the limit $\lim_{t\to T} J(t)$ exists and is a harmonic almost-complex structure, which is also a minimizer of $|N|^2$ among all almost complex structures associated to our ansatze (9.32).

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