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Equivariant Grothendieck-Riemann-Roch theorem via formal deformation theory

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We use the formalism of traces in higher categories to prove a common generalization of the holomorphic Atiyah-Bott fixed point formula and the Grothendieck-Riemann-Roch theorem. The proof is quite different from the original one proposed by Grothendieck et al.: it relies on the interplay between self dualities of quasi- and ind- coherent sheaves on X and formal deformation theory of Gaitsgory-Rozenblyum. In particular, we give a description of the Todd class in terms of the difference of two formal group structures on the derived loop scheme $\mathcal{L}X$. The equivariant case is reduced to the non-equivariant one by a variant of the Atiyah-Bott localization theorem.

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0. Introduction

Convention. For the rest of the document we will assume that we work over some base field k of characteristic zero.

In [15] the formalism of traces in symmetric monoidal $(\infty, 2)$-categories was used to prove the following classical result

Theorem (Holomorphic Atiyah-Bott fixed point formula)**.** Let X be a smooth proper k-scheme with an endomorphism $X \xrightarrow{g} X$ such that the graph of g intersects the diagonal in $X \times X$ transversely. Then for a lax g-equivariant perfect sheaf (E, t) (i.e. a sheaf $E \in \mathrm{QCoh}(X)^{\mathrm{perf}}$ equipped with a map $E \xrightarrow{t} g_*E$) there is an equality

$$\mathsf{Tr}_{\mathrm{Vect}_k}\left(\Gamma(X,E) \xrightarrow{\Gamma(X,t)} \Gamma(X,E) \right) = \sum_{x=g(x)} \frac{\mathsf{Tr}_{\mathrm{Vect}_k}\left(E_x \simeq E_{g(x)} \xrightarrow{t_x} E_x \right)}{\det(1 - d_x g)}$$

where $\mathbb{T}_{X,x} \xrightarrow{d_x g} \mathbb{T}_{X,g(x)} \simeq \mathbb{T}_{X,x}$ is the induced map on tangent spaces.

In turns out that the transversality assumption in the theorem above can be considerably weakened. For example, the following extreme opposite case (when the equivariant structure is trivial, i.e. $g = \mathrm{Id}_X$, $t = \mathrm{Id}_E$) was known a decade before the Atiyah-Bott formula

Theorem (Hirzebruch-Riemann-Roch)**.** Let X be a smooth proper scheme over k. Then for every perfect sheaf E on X there is an equality

$$\mathsf{Tr}\left(\Gamma(X,E) \xrightarrow{\mathrm{Id}_{\Gamma(X,E)}} \Gamma(X,E) \right) = \int_X \mathrm{ch}(E)\,\mathrm{td}_X$$

where $\mathrm{ch}(-)$ and td_X are Chern character and Todd class respectively.

The goal of this work is to provide a common generalization of the two theorems above as well as their relative versions naturally suggested by the formalism of traces. In order to state it we first need to introduce a bit of notations:

Notations 0.0.1. Let X be a smooth scheme equipped with an endomorphism $X \xrightarrow{g} X$ such that the reduced classical scheme $\overline{X^g} := \mathcal{H}^0(X^g)^{\mathsf{red}}$ is smooth (but not necessary connected). We will denote by $j\colon \overline{X^g} \longrightarrow X$ the canonical embedding and by $\mathcal{N}_g^\vee$ its conormal bundle. Note that the action of g on Ω^1_X in particular restricts to an endomorphism $g^*_{|\mathcal{N}_g}\colon \mathcal{N}_g^\vee \longrightarrow \mathcal{N}_g^\vee$.

We then have

Theorem (Equivariant Grothendieck-Riemann-Roch, Theorem 6.2.13)**.** Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[loop left, "g_X"] & Y \arrow[loop right, "g_Y"] \end{tikzcd}$$

be an equivariant morphism between smooth proper schemes such that

- Reduced fixed loci $\overline{X^{g_X}}$ and $\overline{Y^{g_Y}}$ are smooth.
- The induced morphisms on conormal bundles $1-(g_X^*)_{|\mathcal{N}^\vee_{g_X}}$ and $1-(g_Y^*)_{|\mathcal{N}^\vee_{g_Y}}$ are invertible.

Then for a lax g_X-equivariant perfect sheaf E on X there is an equality

$$(\overline{f^g})_*\left(\operatorname{ch}(E,t)\frac{\operatorname{td}_{\overline{X^{g_X}}}}{e_{g_X}}\right) = \operatorname{ch}(f_*(E,t))\frac{\operatorname{td}_{\overline{Y^{g_Y}}}}{e_{g_Y}} \quad \in \quad \bigoplus_p H^{p,p}(\overline{Y^{g_Y}}),$$

where $\overline{X^{g_X}} \xrightarrow{\overline{f^g}} \overline{Y^{g_Y}}$ is the induced map on reduced fixed loci, $\operatorname{ch}(-,-)$ is an equivariant Chern character (see Construction 1.1.4 and Proposition 1.3.3), td_- are usual Todd classes and e_{g_-} are equivariant Euler classes (see Definition 6.2.8 and Corollary 6.2.10).

If equivariant structures on X and Y are trivial (so $e_{g_X}=1, e_{g_Y}=1$), the theorem above reduces to the usual Grothendieck-Riemann-Roch theorem. On the other hand, if $Y=*$ and X^{g_X} is discrete, then $\operatorname{td}_{X^{g_X}}=1$ and we recover the holomorphic Atiyah Bott-formula (see corollaries 6.2.16 and 6.2.15 for more details).

Remark 0.0.2. Even in the case of trivial equivariant structures our proof of Grothendieck-Riemann-Roch is quite different from the original approach due to Grothendieck et al.: it is valid for arbitrary smooth proper k-schemes X and in particular does not rely on the trick of factoring a projective morphism into a composition of a closed embedding and a projection. On the other hand, due to heavy usage of deformation theory our proof works only in characteristic zero and we consider the Chern character and the Todd class as elements of Hodge cohomology $H^{*,*}(X)$, not of the Chow ring.

We now explain the key steps in the proof of the Equivariant Grothendieck-Riemann-Roch theorem above.

Non-equivariant part. Let $X \xrightarrow{f} Y$ be a morphism of smooth proper k-schemes and let E be a perfect sheaf on X. Recall that the Grothendieck-Riemann-Roch theorem asserts an equality

$$f_*(\operatorname{ch}(E)\operatorname{td}_X) = \operatorname{ch}(f_*(E))\operatorname{td}_Y \quad \in \quad \bigoplus_p H^p(Y,\Omega^p_Y).$$

Our first goal is to give a proof of the Grothendieck-Riemann-Roch theorem using the formalism of traces. Note that the formula above is equivalent to the commutativity of the diagram

$$(1)\qquad \begin{tikzcd} k \arrow[r, "\mathrm{ch}(E)"'] \arrow[rr, bend left, "\mathrm{ch}(f_*(E))"] & \bigoplus_p H^p(X,\Omega^p_X) \arrow[d, "\cdot\,\mathrm{td}_X", "\sim"'] & \bigoplus_p H^p(Y,\Omega^p_Y) \arrow[d, "\cdot\,\mathrm{td}_Y", "\sim"'] \\ & \bigoplus_p H^p(X,\Omega^p_X) \arrow[r, "f_*"'] & \bigoplus_p H^p(Y,\Omega^p_Y) \end{tikzcd}$$

of k-vector spaces. We now recall

Proposition 0.0.3 ([15, Definition 1.2.6])**.** Let $\mathcal{C},\mathcal{D} \in \mathrm{Cat}_k$ be a pair of dualizable k-linear presentable categories. Suppose we are given a (not necessarily commutative) diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, bend right, "\varphi"'] & \mathcal{C} \arrow[d, bend right, "\varphi"'] \arrow[dl, Rightarrow, "T"'] \\ \mathcal{D} \arrow[u, bend right, "\psi"'] \arrow[r, "F_{\mathcal{D}}"'] & \mathcal{D}, \arrow[u, bend right, "\psi"'] \end{tikzcd}$$

where φ is left adjoint to ψ and $\varphi \circ F_{\mathcal{C}} \xrightarrow{T} F_{\mathcal{D}} \circ \varphi$ is a (not necessary invertible) natural transformation. Then there exists a natural morphism

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(F_{\mathcal{C}}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\varphi,T)} \mathsf{Tr}_{\mathrm{Cat}_k}(F_{\mathcal{D}})$$

in the $(\infty,1)$-category $\mathrm{Hom}_{2\,\mathrm{Cat}_k}(\mathrm{Vect}_k,\mathrm{Vect}_k) \simeq \mathrm{Vect}_k$ called the *morphism of traces induced by* T (in the case when $T = \mathrm{Id}_{\varphi\circ F_{\mathcal{C}}}$ we will further frequently use the notation $\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\varphi) := \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\varphi,\mathrm{Id}_{\varphi\circ F_{\mathcal{C}}})$). Moreover the morphism of traces is functorial in an appropriate sense, see [15, Proposition 1.2.11.].

We refer the reader to [15, Section 1.2] for a detailed discussion of the formalism of traces (see also [13] for a more general treatment in the context of (∞,n)-categories and [5] for traces in the context of derived algebraic geometry). The main idea of our proof of the Grothendieck-Riemann-Roch theorem is that one can obtain commutativity of the diagram (1) above as a corollary of functoriality of the construction of the morphism of traces. Namely, let $2\,\mathrm{Cat}_k$ be the $(\infty,2)$-category of k-linear stable presentable categories and continuous functors. Recall that to each derived scheme Z we can associate the $(\infty,1)$-category $\mathrm{QCoh}(Z) \in 2\,\mathrm{Cat}_k$ of quasi-coherent sheaves on Z (see [10, Chapter 3]) and a closely related category $\mathrm{ICoh}(Z)$ of ind-coherent sheaves on Z (see [10, Chapter 4]). Then by applying functoriality of traces to the diagram

$$\begin{tikzcd}
\mathrm{Vect}_k \arrow[r, "E\otimes -"] & \mathrm{QCoh}(X) \arrow[r, "f_*"] \arrow[d, "\otimes\mathcal{O}_X", "\sim"'] & \mathrm{QCoh}(Y) \arrow[d, "\otimes\mathcal{O}_Y", "\sim"'] \\
& \mathrm{ICoh}(X) \arrow[r, "f_*"'] & \mathrm{ICoh}(Y)
\end{tikzcd}$$

in $2\,\mathrm{Cat}_k$ we obtain a commutative diagram of traces

$$\begin{tikzcd}
k \arrow[r, "\mathsf{Tr}_{2\,\mathrm{Cat}_k}(E\otimes -)"'] \arrow[rr, bend left, "\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*(E)\otimes -)"] & \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \arrow[r] \arrow[d, "\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_X)"] & \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(Y)}) \arrow[d, "\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_Y)"] \\
& \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \arrow[r, "\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*)"'] & \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)})
\end{tikzcd}$$

in Vect_k. The bulk of the paper will be devoted to identifying the morphism of traces in the diagram above with their classical counterparts. Namely

- Section 1: we prove

$$\pi_0\,\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \simeq \bigoplus_p H^p(X, \Omega_X^p)$$

$$\pi_0\,\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(Y)}) \simeq \bigoplus_p H^p(Y, \Omega_Y^p).$$

  Moreover, under the isomorphisms above $\mathsf{Tr}(E\otimes -)$ and $\mathsf{Tr}(f_*(E)\otimes -)$ will coincide with the usual Chern characters of E and $f_*(E)$ respectively. In fact, our description of $\mathsf{Tr}(E\otimes -)$ will be in terms of the Atiyah class of E and is closely related to the description given in [19].

- Section 2: we have

$$\pi_0 \,\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \simeq \bigoplus_p H^p(X, \Omega_X^p)^\vee$$

$$\pi_0 \,\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(Y)}) \simeq \bigoplus_p H^p(Y, \Omega_Y^p)^\vee.$$

  Moreover, under the two isomorphisms above the morphism of traces induced by the pushforward functor $\mathrm{ICoh}(X) \xrightarrow{f_*} \mathrm{ICoh}(Y)$ coincides with the usual pushforward in homology (defined as the Poincaré dual of the pullback).
- Sections 3, 4: under the Poincaré self-duality

$$\bigoplus_p H^p(X, \Omega_X^p) \simeq \bigoplus_p H^p(X, \Omega_X^p)^\vee$$

  the morphism $\mathsf{Tr}_{2\,\mathrm{Cat}_k}(- \otimes \mathcal{O}_X)$ is given by the multiplication with the Todd class td_X and analogously for Y.

Using these identifications and the commutative diagram of traces above, one immediately concludes the Grothendieck-Riemann-Roch theorem.

Equivariant part. We now discuss how one can get an equivariant version of the GRR theorem. Let X be a smooth proper scheme with an endomorphism g and (E, t) be a lax g-equivariant perfect sheaf on X. First, it turns out that if $g = \mathrm{Id}_X$ there is simple description of the equivariant Chern character $\mathrm{ch}(E, t)$ (see Construction 1.1.4) in the spirit of the Chern-Weil theory (see Remark 1.4.8):

Example 0.0.4. Let X be a smooth proper scheme with a trivial equivariant structure. Then under the identification

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \simeq \bigoplus_p H^p(X, \Omega_X^p)$$

the equivariant Chern character is equal to $\mathsf{Tr}(e^{\mathrm{At}(E)} \circ t)$, where $\mathrm{At}(E)$ is the Atiyah class of E (see Corollary 1.3.3).

Now if the equivariant structure on X is non-trivial, we in general do not have a convenient description of $\mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*)$ and therefore of $\mathrm{ch}(E, t)$. To circumvent this, we shall reduce the situation to the non-equivariant case by restricting along the (classical, reduced) fixed point locus $j\colon \overline{X^g} \longrightarrow X$ of the endomorphism g, like in the Atiyah-Bott localization theorem. Note that this indeed gives a good description of the Chern character: since the restriction of the lax g-equivariant sheaf (E,t) to $\overline{X^g}$ is naturally equivariant with respect to the trivial equivariant structure on $\overline{X^g}$, by the example above we have a grasp on

$$\mathsf{ch}(j^*(E,t)) \in \pi_0 \operatorname{\mathsf{Tr}}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(\overline{X^g})}) \simeq \bigoplus_p H^p(\overline{X^g}, \Omega^p_{\overline{X^g}}).$$

Moreover, we will show that under reasonable assumptions the morphism j^* in fact does not loose any information:

Theorem (Localization theorem, 6.2.2)**.** Assume that the reduced classical fixed locus $\overline{X^g}$ is smooth and let $\mathcal{N}_g^\vee$ be the conormal bundle of j. Then the induced map

$$\pi_0 \operatorname{\mathsf{Tr}}_{2\,\mathrm{Cat}_k}(g_*) \xrightarrow{\operatorname{\mathsf{Tr}}_{2\,\mathrm{Cat}_k}(j^*)} \pi_0 \operatorname{\mathsf{Tr}}(\mathrm{Id}_{\mathrm{QCoh}(\overline{X^g})}) \simeq \bigoplus_p H^p(\overline{X^g}, \Omega^p_{\overline{X^g}})$$

is an equivalence if and only if the determinant $\det(1 - g^*_{|\mathcal{N}_g^\vee})$ is invertible.

Let now

$$g_X \circlearrowright X \xrightarrow{f} Y \circlearrowleft g_Y$$

be an equivariant morphism from (X, g_X) to (Y, g_Y) satisfying assumptions of the Equivariant Grothendieck-Riemann-Roch theorem. Then by applying $\operatorname{\mathsf{Tr}}_{2\,\mathrm{Cat}_k}$ to the commutative diagram

$$\begin{tikzcd}
 & \mathrm{QCoh}(X) \arrow[loop above, "g_{X*}"] \arrow[r, "f_*"] \arrow[d, "\otimes\mathcal{O}_X", "\sim"'] & \mathrm{QCoh}(Y) \arrow[loop above, "g_{Y*}"] \arrow[d, "\otimes\mathcal{O}_Y", "\sim"'] \\
\mathrm{Vect}_k \arrow[ur, phantom] & \mathrm{ICoh}(X) \arrow[loop below, "g_{X*}"] \arrow[r, "f_*"'] & \mathrm{ICoh}(Y) \arrow[loop below, "g_{Y*}"]
\end{tikzcd}$$

$$\mathrm{Vect}_k \xrightarrow{E} \mathrm{QCoh}(X)$$

and using the identification from the localization theorem $\pi_0 \operatorname{\mathsf{Tr}}_{2\,\mathrm{Cat}_k}((g_X)_*) \simeq$

$\bigoplus_p H^{p,p}(\overline{X^{g_X}})$ and analogously for Y, we obtain a commutative diagram

$$\begin{tikzcd} k \arrow[r, "\mathrm{ch}(E,t)"'] \arrow[rr, bend left, "\mathrm{ch}(f_*(E,t))"] & \bigoplus_p H^{p,p}(\overline{X^{g_X}}) \arrow[r] \arrow[d, "\cdot\,\mathrm{td}_{g_X}", "\sim"'] & \bigoplus_p H^{p,p}(\overline{Y^{g_Y}}) \arrow[d, "\cdot\,\mathrm{td}_{g_Y}", "\sim"'] \\ & \bigoplus_p H^{p,p}(\overline{X^{g_X}}) \arrow[r, "(\overline{f^g})_*"'] & \bigoplus_p H^{p,p}(\overline{Y^{g_Y}}) \end{tikzcd}$$

in Vect_k for some equivalences td_{g_X} and td_{g_Y} (see Notation 6.1.5). Finally, by applying the commutative diagram above to the special case when the morphism is $\overline{X^{g_X}} \hookrightarrow X$ and using the non-equivariant part it is straightforward to identify td_{g_X} with $\frac{\mathrm{td}_{\overline{X^{g_X}}}}{e_{g_X}}$ (Proposition 6.2.12), where e_{g_X} is the Euler class (see Definition 6.2.8 and Corollary 6.2.10) and similarly for Y.

Remark 0.0.5. Even if one is only interested in the equivariant part, in order to reduce it to the non-equivariant version in our approach it is still necessary to identify the morphism of traces

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(\overline{X^g})}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_{\overline{X^g}})} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(\overline{X^g})})$$

in classical terms. In other words, we really need to reprove (non-equivariant) Grothendieck-Riemann-Roch theorem in the language of traces.

Relation to previous work

Our identification of the morphism of traces in the non-equivariant case is closely related to the work of Markarian [19], where the role of the canonical action of the Lie algebra $\mathbb{T}_X[-1]$ on any object of $\mathrm{QCoh}(X)$ (via Atiyah class) was emphasized. Moreover, he provides an interpretation of the Todd class td_X as an invariant volume form with respect to the Hopf algebra structure on Hochschild homology. The key difference of our approach is the systematic use of the formalism of traces, derived algebraic geometry, ind-coherent sheaves and deformation theory of Gaitsgory-Rozenblyum, allowing us to give a precise geometric interpretation of Markarian's ideas in terms of formal groups over X and to generalize them in some directions. The idea of such an interpretation of td_X was explained to us by Dennis Gaitsgory who in turn learned it from Maxim Kontsevich, according to whom this idea ultimately goes back to Boris Feigin.

Ideas similar to our trace formalism were used in a series of papers [4], [5], [6] to derive many interesting trace formulas in various contexts (such as D-modules on schemes or quasi-coherent sheaves on stacks). However, as explained in [5, Remark 1.6], some additional work is needed to fully recover the usual GRR-theorem from their results. Among other things, in this paper we provide a necessary identifications of morphisms of traces in classical context by studying the formal geometry of the derived loop scheme.

Localization theory relating equivariant cohomology and ordinary cohomology of the fixed locus was studied for a long time already, see e.g. [3] for the de Rham variant of the theory. For various reasons the methods of [3] don't apply directly in our context (e.g. since equivariant cohomology of a point is too small in the case of a plain endomorphism to invert something there), but the idea that equivariant cohomology and ordinary cohomology of the fixed locus agree up to localization of the Euler class is ultimately motivated by their work.

Formality of derived fibered products was thoroughly studied in [1]. As an application they prove ([1, Corollary 1.12]) that for a finite order automorphism the derived fixed locus is always formal. Hence our localization theorem 6.2.2 may be considered as a generalization of this result, providing formality criterion for an arbitrary endomorphism.

A similar result to that of ours was obtained in [8] over an arbitrary algebraically closed field but with an additional assumption that both X, Y are projective and that both g_X, g_Y are automorphisms of finite order coprime to the characteristic of the base field. Donovan's proof is quite different from ours and is much closer to the original approach due to Grothendieck: it relies on the fact that an equivariant projective morphism $f : (X, g_X) \longrightarrow (Y, g_Y)$ (with g_X, g_Y being automorphisms of finite orders) can be factored into an equivariant closed embedding into relative projective space followed by projection. We do not see how to generalize this approach to the case of a general endomorphism. On the other hand, Donovan's formula holds in the Chow ring, while our proof gives equality only in Hodge cohomology.

One can also extend the classical Grothendieck-Riemann-Roch theorem in a non-commutative direction by studying traces of maps on Hochschild homology induced by functors between nice enough (e.g. smooth and proper) categories (see [22], [16], [20], [7]). But while the Chern character makes perfect sense in the non-commutative context, it was already pointed out by Shklyarov that the Todd class seems to be of commutative nature and is missing in general non-commutative GRR-like theorems. Hence the Theorem 6.2.13 may be considered as a refinement of the more general non-commutative versions under additional geometricity assumptions on categories and endofunctors.

Finally, there is a categorified version of the GRR-theorem conjectured in [23], where (among many other things) the role of the derived loops construction was empasized. See [24] and [14] for proofs and interesting applications.

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Conventions.
1) All the categories we work with are assumed to be $(\infty,1)$-categories. For an $(\infty,1)$-category $\mathcal{C}$ we will denote by $(\mathcal{C})^{\simeq}$ the underlying ∞-groupoid of $\mathcal{C}$ obtained by discarding all the non-invertible morphisms from $\mathcal{C}$.

2) We will denote by $\mathcal{S}$ the symmetric monoidal $(\infty,1)$-category of spaces. For a field k we will denote by Vect_k the stable symmetric monoidal $(\infty,1)$-category of unbounded cochain complexes over k up to quasi-isomorphism with the canonical $(\infty,1)$-enhancement. We will also denote by $\mathrm{Vect}_k^{\heartsuit}$ the ordinary category of k-vector spaces considered as an $(\infty,1)$-category.

3) We will denote by $\mathrm{Pr}^{\mathsf{L}}_{\infty}$ the $(\infty,1)$-category of presentable $(\infty,1)$-categories and continuous functors with the symmetric monoidal structure from [17, Proposition 4.8.1.15.]. Similarly, we will denote by $\mathrm{Pr}^{\mathsf{L},\mathsf{st}}_{\infty}$ the $(\infty,1)$-category of stable presentable $(\infty,1)$-categories and continuous functors considered as a symmetric monoidal $(\infty,1)$-category with the monoidal structure inherited from $\mathrm{Pr}^{\mathsf{L}}_{\infty}$.

4) Notice that Vect_k is a commutative algebra object in $\mathrm{Pr}^{\mathsf{L},\mathsf{st}}_{\infty}$. By [17, Theorem 4.5.2.1.] it follows that the presentable stable $(\infty,1)$-category of k-linear presentable $(\infty,1)$-categories and k-linear functors $\mathrm{Cat}_k := \mathrm{Mod}_{\mathrm{Vect}_k}(\mathrm{Pr}^{\mathsf{L},\mathsf{st}}_{\infty})$

admits natural symmetric monoidal structure. We will also denote by $2\,\mathrm{Cat}_k$, the symmetric monoidal $(\infty,2)$-category of k-linear presentable $(\infty,1)$-categories and continuous k-linear functors so that the underlying $(\infty,1)$-category of $2\,\mathrm{Cat}_k$ is precisely Cat_k.

5) We will denote by PreStack the $(\infty,1)$-category of functors $\mathsf{Funct}(\mathrm{CAlg}_k^{\leq 0}, \mathcal{S})$, where $\mathrm{CAlg}_k^{\leq 0} := \mathrm{CAlg}(\mathrm{Vect}_k)^{\leq 0}$ is the $(\infty,1)$-category of connective commutative algebras in Vect_k. For a prestack $X \in \mathrm{PreStack}_k$ we will denote the k-linear symmetric monoidal $(\infty,1)$-category of unbounded complexes of quasi-coherent sheaves on X by $\mathrm{QCoh}(X) \in \mathrm{CAlg}(\mathrm{Cat}_k)$. We refer the reader to [10, Part I] for an introduction to the basic concepts of derived algebraic geometry.

6) By 'scheme' we will always mean a derived schemes (in the sense of [10]) if not explicitly stated otherwise. For a smooth classical scheme X we will sometimes denote its Hodge cohomology $H^q(X, \Omega^p_X)$ by $H^{p,q}(X)$.

1. Categorical Chern character

Let X be an almost finite type scheme (see [10, Section 3.5.1]) with an endomorphism $X \xrightarrow{g} X$ and let $(E,t) \in \mathrm{QCoh}(X)$ be a lax g-equivariant compact quasi-coherent sheaf on X (i.e. a sheaf $E \in \mathrm{QCoh}(X)$ together with a morphism $E \xrightarrow{t} g_*E$). Our goal in this section is to describe the categorical Chern character $\mathrm{ch}(E,t) \in \mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*)$ obtained by applying the formalism of traces to the commutative diagram

$$\begin{tikzcd}
\mathrm{Vect}_k \arrow[rr, "\mathrm{Id}_{\mathrm{Vect}_k}"] \arrow[d, shift right, "-\otimes E"'] & & \mathrm{Vect}_k \arrow[d, shift right, "-\otimes E"'] \arrow[dll, Rightarrow, "T"] \\
\mathrm{QCoh}(X) \arrow[u, shift right, "\mathrm{Hom}(E,-)"'] \arrow[rr, "g_*"'] & & \mathrm{QCoh}(X) \arrow[u, shift right, "\mathrm{Hom}(E,-)"']
\end{tikzcd}$$

in more concrete terms. This will be done in several steps:

- First, using that the assignment $X \longmapsto \mathrm{QCoh}(X)$ lifts to a functor from an appropriate category of correspondences we will identify

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*) \simeq \Gamma(X^g, \mathcal{O}_{X^g})$$

where X^g is the derived fixed locus of g.

- Specializing to the case when $g = \mathrm{Id}_X$ so that $X^g \simeq \mathcal{L}X$ is the so-called *inertia group of* X we will show that E admits a canonical action of $\mathcal{L}X$ and compute $\mathrm{ch}(E, t)$ in terms of this action.
- Finally, under additional assumption that X is smooth (so that we can apply QCoh-version of formal deformation theory) we will show that the canonical action of the $\mathcal{L}X$ on E is closely related to the canonical action of the Lie algebra $\mathbb{T}_X[-1]$ given by the Atiyah class of E deducing that

$$\mathrm{ch}(E, t) = \mathsf{Tr}(\exp(\mathrm{At}(E)) \circ t).$$

In particular, if $t = \mathrm{Id}_E$ we will show that $\mathrm{ch}(E, \mathrm{Id}_E)$ coincides with the classical topological Chern character (defined using the splitting principle).

1.1. Self-duality of quasi-coherent sheaves and Chern character

We start with a short reminder of self-duality of QCoh. Recall the following

Theorem 1.1.1 ([10, Chapter 3, Proposition 3.4.2 and Chapter 6, Proposition 4.3.2])**.**

1. For any two $X, Y \in \mathrm{Sch}_{\mathsf{aft}}$ (for the definition see [10, Chapter 2, 3.5]) the morphism

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y) \longrightarrow \mathrm{QCoh}(X \times Y)$$

in Cat_k induced by the functor

$$\mathrm{QCoh}(X) \times \mathrm{QCoh}(Y) \xrightarrow{\boxtimes} \mathrm{QCoh}(X \times Y)$$

is an equivalence.

2. For any $X \in \mathrm{Sch}_{\mathsf{aft}}$ the morphisms

$$\mathrm{Vect}_k \xrightarrow{\Delta_* \mathcal{O}_X} \mathrm{QCoh}(X \times X) \simeq \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X)$$

and

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X \times X) \xrightarrow{\Gamma \circ \Delta^*} \mathrm{Vect}_k$$

exhibit $\mathrm{QCoh}(X)$ as a self-dual object in Cat_k.

Corollary 1.1.2. Let X be an almost finite type scheme with an endomorphism $X \xrightarrow{g} X$. Then there is an equivalence

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*) \simeq \Gamma(X, \Delta^*(\mathrm{Id}_X, g)_* \mathcal{O}_X) \simeq \Gamma(X^g, \mathcal{O}_{X^g}),$$

where X^g is the derived fixed locus of g defined by the pullback square

$$\begin{tikzcd} X^g \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "(\mathrm{Id}_X, g)"] \\ X \arrow[r, "\Delta"'] & X \times X. \end{tikzcd}$$

Proof. Unwinding the definitions and using the theorem above one finds that the trace $\mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*)$ is given by the composite

$$\mathrm{Vect}_k \xrightarrow{\Delta_* \mathcal{O}_X} \mathrm{QCoh}(X \times X) \xrightarrow{(\mathrm{Id}_X \times g)_*} \mathrm{QCoh}(X \times X) \xrightarrow{\Gamma \circ \Delta^*} \mathrm{Vect}_k$$

which is given by tensoring with $\Gamma(X, \Delta^*(\mathrm{Id}_X, g)_* \mathcal{O}_X)$. Now by applying the base change for quasi-coherent sheaves (see [10, Chapter 3, Proposition 2.2.2.]) to the diagram above we get

$$\Gamma(X, \Delta^*(\mathrm{Id}_X, g)_* \mathcal{O}_X) \simeq \Gamma(X, i_* i^* \mathcal{O}_X) \simeq \Gamma(X, i_* \mathcal{O}_{X^g}) \simeq \Gamma(X^g, \mathcal{O}_{X^g})$$

proving the claim. $\square$

Example 1.1.3. Note that in the case when $g = \mathrm{Id}_X$ we get an equivalence $X^g \simeq \mathcal{L}X$ where $\mathcal{L}X$ is the **derived loops scheme of** X defined as

$$\mathcal{L}X := \mathrm{Map}(S^1, X) \simeq X \underset{X \times X}{\times} X$$

In particular, in the case when X is smooth the Hochschild-Kostant-Rosenberg isomorphism (see Corollary 1.3.1) states that

$$\pi_i \Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \simeq \bigoplus_{p-q=i}^{\dim X} H^q(X, \Omega_X^p).$$

As we will in Theorem 6.2.2 under some reasonable assumptions one can use the Hochschild-Kostant-Rosenberg isomorphism to get some understanding of $\Gamma(X^g, \mathcal{O}_{X^g})$ for g which is not necessary an identity.

We now turn to the categorical Chern character. Recall from [15, Definition 1.2.9] the following

Construction 1.1.4. Given a compact sheaf $E \in \mathrm{QCoh}(X)$ together with an endomorphism $E \xrightarrow{t} g_* E$ we can form a diagram

$$\begin{tikzcd} \mathrm{Vect}_k \arrow[rr, "\mathrm{Id}_{\mathrm{Vect}_k}"] \arrow[d, bend right, "-\otimes E"'] & & \mathrm{Vect}_k \arrow[d, bend right, "-\otimes E"'] \arrow[dll, Rightarrow, "T"] \\ \mathrm{QCoh}(X) \arrow[rr, "g_*"'] \arrow[u, bend right, "\mathrm{Hom}(E,-)"'] & & \mathrm{QCoh}(X) \arrow[u, bend right, "\mathrm{Hom}(E,-)"'] \end{tikzcd}$$

in $2\,\mathrm{Cat}_k$ with the 2-morphism T induced by the morphism t. The corresponding element

$$\mathrm{ch}(E,t) \in \mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*) \simeq \Gamma(X^g, \mathcal{O}_{X^g})$$

obtained via the formalism of traces (see Proposition 0.0.3) is called a *categorical Chern character of E.*

Recall also the following result

Proposition 1.1.5 ([15, Proposition 2.2.3.])**.** Assume that X is a quasi-compact scheme (in this case $\mathrm{QCoh}(X)$ is compactly generated by dualizable objects). Given an endomorphism $X \xrightarrow{g} X$ and a dualizable sheaf $E \in \mathrm{QCoh}(X)$ together with a lax g-equivariant structure $E \xrightarrow{t} g_* E$ on E there is an equality

$$\mathrm{ch}(E,t) \simeq \mathsf{Tr}_{\mathrm{QCoh}(X^g)} \left(i^* E \xrightarrow[\sim]{\beta} i^* g^* E \xrightarrow{i^*(b)} i^* E \right)$$

in $\Gamma(X^g, \mathcal{O}_{X^g})$, where $b \in \mathrm{Hom}_{\mathrm{QCoh}(X)}(g^* E, E)$ is the morphism which corresponds to t using the adjunction between g^* and g_*, the morphism $X^g \overset{i}{\hookrightarrow} X$ is the inclusion of the derived fixed points and the equivalence β is induced by the equivalence $i \simeq g \circ i$.

It turns out that the equivalence β above is non-trivial even in the case $g = \mathrm{Id}_X$. In fact, it is closely related to the Atiyah class of E, as we will see further in this section.

1.2. Canonical $\mathcal{L}X$-equivariant structure

In this subsection we will construct a canonical action of $\mathcal{L}X$ on any sheaf in $\mathrm{QCoh}(X)$ and use this structure to give an alternative description to the morphism β from Proposition 1.1.5. We start with the following

Definition 1.2.1. Let $X \in \mathrm{PreStack}$ be a prestack and $Y \in \mathrm{PreStack}_{/X}$ be a prestack over X. Then for a group $G \in \mathrm{Grp}(\mathrm{PreStack}_{/X})$ which acts on Y define a prestack $Y/G \in \mathrm{PreStack}$ as the colimit

$$Y/G := \operatorname*{colim}_{\Delta^{\mathrm{op}}} \left(\dots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} G \times_X G \times_X Y \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} G \times_X Y \underset{q_2}{\overset{a}{\rightrightarrows}} Y \right)$$

in the category of prestacks.

Recall now the following

Definition 1.2.2. Let $X \in \mathrm{PreStack}$ be a prestack and $Y \in \mathrm{PreStack}_{/X}$ be a prestack over X. Then for a group $G \in \mathrm{Grp}(\mathrm{PreStack}_{/X})$ which acts on Y we define a **category of G-equivariant sheaves on** Y denoted by $\mathrm{Rep}_G(\mathrm{QCoh}(Y))$ simply as

$$\mathrm{Rep}_G(\mathrm{QCoh}(Y)) := \mathrm{QCoh}(Y/G).$$

We will further frequently abuse the notation by considering G-equivariant sheaves on X as objects of the category $\mathrm{QCoh}(Y)$ via the pullback functor

$$\mathrm{QCoh}(Y/G) \xrightarrow[\sim]{h^*} \mathrm{QCoh}(Y)$$

where $Y \xrightarrow{h} Y/G$ is the natural projection map.

Construction 1.2.3. Notice that by the definition of Y/G we have an equivalence of functors $a^*h^* \xrightarrow[\sim]{} q_2^*h^*$. Consequently, for every G-equivariant quasi-coherent sheaf $\mathcal{F} \in \mathrm{Rep}_G(\mathrm{QCoh}(Y))$ we get an equivalence $a^*\mathcal{F} \xrightarrow[\sim]{\alpha_{\mathcal{F}}} q_2^*\mathcal{F}$ which we will further call an *action morphism of G on $\mathcal{F}$.*

Remark 1.2.4. Suppose that the action of G on Y is trivial so that $a = q_2$ and therefore for any $\mathcal{F} \in \mathrm{Rep}_G(Y)$ the morphism $\alpha_{\mathcal{F}}$ is an automorphism. For dualizable $\mathcal{F} \in \mathrm{QCoh}(X)$ we then define a *character $\chi_{\mathcal{F}}$ of the representation $\mathcal{F}$* as the trace

$$\chi_{\mathcal{F}} := \mathsf{Tr}_{\mathrm{QCoh}(G\times_X Y)}(\alpha_{\mathcal{F}}) \in \pi_0\Gamma(G \times_X Y, \mathcal{O}_{G\times_X Y})$$

of the action morphism. To justify the notation above, suppose that $X = Y = \operatorname{Spec} k$ and G is an ordinary group scheme over a field k. In this case E corresponds to some representation $\rho \in \operatorname{Rep}_G(\operatorname{Vect}_k)$ and for every $g \in G(k)$ the pullback $g^*\alpha_{\mathcal{F}}$ is equal to $\rho(g)$. In particular, since the pullback functor is symmetric monoidal we get

$$\chi_{\mathcal{F}}(g) := g^* \operatorname{\mathsf{Tr}}_{\operatorname{QCoh}(G\times_X Y)}(\alpha_{\mathcal{F}}) = \operatorname{\mathsf{Tr}}(g^*\alpha_{\mathcal{F}}) = \operatorname{\mathsf{Tr}}(\rho(g))$$

i.e. $\chi_{\mathcal{F}}$ defined as above coincides with the usual character of representation ρ.

Notice now that for a morphism $Z \xrightarrow{p} X$ in PreStack together with a section $X \xrightarrow{s} Z$ of p the fiber product $G := X \times_Z X$ admits the structure of a group object over X: the morphisms p and s above realize Z as a pointed object in $\operatorname{PreStack}_{/X}$ and the fiber product $X \times_Z X$ is simply the loop object of $Z \in \operatorname{PreStack}_{/X}$ at the point s. Moreover, we can regard X as acted on by G by the (necessarily) trivial action and there is an induced map $X/G \longrightarrow Z$ (which is equivalence if and only if the section s is an effective epimorphism).

In this paper we are interested in the special case when $Z := X \times X$, the morphism $p := q_1$ is given by the projection to the first component and the section $s := \Delta$ is given by the diagonal morphism. In this case the resulting group

$$X \underset{X\times X}{\times} X \in \operatorname{Grp}(\operatorname{PreStack}_{/X})$$

is called the *inertia group* of X and by definition coincides with the derived loop stack $\mathcal{L}X$ of X from Example 1.1.3. In particular, by the discussion above we see that $\mathcal{L}X$ naturally acts on X and that we have a map $B_{/X}\mathcal{L}X \xrightarrow{c} X \times X$ (where $B_{/X}\mathcal{L}X \in \operatorname{PreStack}_{/X}$ is the delooping of $\mathcal{L}X$ over X). We are now ready to state the following

Proposition 1.2.5. Every $\mathcal{F} \in \operatorname{QCoh}(X)$ admits a natural $\mathcal{L}X$-equivariant structure.

Proof. The desired structure is given by the composite

$$\operatorname{QCoh}(X) \xrightarrow{q_2^*} \operatorname{QCoh}(X \times X) \xrightarrow{c^*} \operatorname{QCoh}(B_{/X}\mathcal{L}X) = \operatorname{Rep}_{\mathcal{L}X}(\operatorname{QCoh}(X))$$

where $X \times X \xrightarrow{q_2} X$ is the projection to the second component. $\square$

Following [11, Chapter 8, 6.2] we will further call this structure the *canonical $\mathcal{L}X$-equivariant structure* on $\mathcal{F}$.

Remark 1.2.6. Notice that if one would take the first projection q_1 in the proposition above instead the resulting $\mathcal{L}X$-equivariant structure on every quasi-coherent sheaf on X will be trivial (since we consider $X \times X$ as an object over X precisely via q_1). The canonical $\mathcal{L}X$-equivariant structure is, however, highly nontrivial as we will see just below.

We are now ready to describe how the categorical Chern character can be understood using the natural action of $\mathcal{L}X$ on X:

Proposition 1.2.7. Let $E \xrightarrow{t} E$ be a dualizable quasi-coherent sheaf on X together with an endomorphism. Then

$$\mathrm{ch}(E,t) = \mathsf{Tr}_{\mathrm{QCoh}(\mathcal{L}X)}(i^*(t) \circ \alpha_E) \in \pi_0\Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X})$$

where $\mathrm{ch}(E,t) \in \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \simeq \Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X})$ is the categorical Chern character and α_E is the action morphism of $\mathcal{L}X$ on E from Construction 1.2.3 (here E is considered as an $\mathcal{L}X$-equivariant quasi-coherent sheaf on X using the canonical $\mathcal{L}X$-equivariant structure from Proposition 1.2.5).

Remark 1.2.8. Note that in the case $t = \mathrm{Id}_E$ the right hand side coincides by definition with the character of the canonical representation of $\mathcal{L}X$ on E from Remark 1.2.4.

Proof. Let more generally g be arbitrary endomorphism of X and consider the pullback diagram defining X^g

$$\begin{tikzcd} X^g \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "(\mathrm{Id}_X, g)"] & \\ X \arrow[r, "\Delta"'] & X \times X \arrow[r, "q_2"'] & X. \end{tikzcd}$$

Unwinding the definitions, one finds that the morphism β from Proposition 1.1.5 can be rewritten as the composite

$$i^*E \simeq i^*\Delta^*q_2^*E \xrightarrow[\sim]{} i^*(\mathrm{Id}_X, g)^*q_2^*E \simeq i^*g^*E,$$

where the middle morphism is induced by the pullback diagram above. In particular, in the special case when $g := \mathrm{Id}_X$ and $b := t$ we get an equivalence

$$\mathrm{ch}(E,t) = \mathsf{Tr}_{\mathrm{QCoh}(\mathcal{L}X)}(i^*E \xrightarrow[\sim]{\beta} i^*E \xrightarrow{i^*(t)} i^*E)$$

and β is equivalent to the composite

$$i^*E \simeq i^*\Delta^*q_2^*E \xrightarrow[\sim]{} i^*\Delta^*q_2^*E \simeq i^*E$$

with the middle morphism above induced by the pullback diagram

$$\begin{tikzcd} \mathcal{L}X \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "\Delta"] & \\ X \arrow[r, "\Delta"'] & X \times X \arrow[r, "q_2"] & X. \end{tikzcd}$$

Rewriting this diagram as

$$\mathcal{L}X \underset{i}{\overset{i}{\rightrightarrows}} X \xrightarrow{\Delta} X \times X \xrightarrow{q_2} X$$

we see that β is precisely the action morphism of $\mathcal{L}X$ on $\Delta^*q_2^*E$, that is, an action morphism of $\mathcal{L}X$ on E with the canonical $\mathcal{L}X$-equivariant structure. □

1.3. Categorical Chern character as exponential

In this section we give a different description of $\mathcal{L}X$-equivariant structure on a sheaf $E \in \mathrm{QCoh}(X)$ on a smooth proper scheme X using the formal deformation theory developed in [11] (we invite the reader to look at Appendix A for a quick reminder). Recall that in order to conveniently work with deformation theory one needs to replace the category of quasi-coherent sheaves $\mathrm{QCoh}(X)$ with a closely related but different category of ind-coherent sheaves $\mathrm{ICoh}(X)$ (see section 2.1). However, for a smooth scheme X there is a natural symmetric monoidal equivalence $\mathrm{QCoh}(X) \xrightarrow[\sim]{\Upsilon_X} \mathrm{ICoh}(X)$ (see Example 2.1.5), so we will state all needed results of [11] using $\mathrm{QCoh}(X)$ instead of $\mathrm{ICoh}(X)$.

As we have seen in Proposition 1.2.7 the Chern character can be described using the group structure (over X) on the inertia group $\mathcal{L}X$ of X. Note that the underlying quasi-coherent sheaf of the Lie algebra $\mathrm{Lie}_X(\mathcal{L}X)$ that corresponds to $\mathcal{L}X$ via formal groups-Lie algebras correspondence (see

Theorem A.0.2) is easy to understand: because of the pullback square

$$\begin{tikzcd} \mathcal{L}X \arrow[r, "i"] \arrow[d, "i"] & X \arrow[d, "\Delta"] \\ X \arrow[r, "\Delta"'] \arrow[u, bend left, "e"] & X \times X \end{tikzcd}$$

where e is obtained by pulling back q_1 we get an equivalence

$$e^*\mathbb{T}_{\mathcal{L}X/X} \simeq e^* i^* \mathbb{T}_{X/X\times X} \simeq \mathbb{T}_{X/X\times X} \simeq \mathbb{T}_X[-1]$$

in $\mathrm{QCoh}(X)$.

Corollary 1.3.1 (Hochschild-Kostant-Rosenberg)**.** For a smooth scheme X we have

$$i_*\mathcal{O}_{\mathcal{L}X} \simeq \mathrm{Sym}_{\mathrm{QCoh}(X)}(\Omega^1_X[1]) \simeq \bigoplus_{p=0}^{\dim X} \Omega^p_X[p]$$

$$HH_i(X) \simeq \bigoplus_{p-q=i}^{\dim X} H^q(X, \Omega^p_X).$$

Proof. Since by definition $HH_i(X) := \pi_i\Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \simeq \pi_i\Gamma(X, i_*\mathcal{O}_{\mathcal{L}X})$ the second equivalence immediately follows from the first one. To get the first equivalence, note that applying the exponent map (see Theorem A.0.6) we get an equivalence

$$\mathbb{V}(\mathbb{T}_X[-1]) \xrightarrow[\sim]{\exp_{\mathcal{L}X}} \mathcal{L}X$$

of formal schemes over X. But by smoothness of X we have $(\mathbb{T}_X[-1])^\vee \simeq \Omega^1_X[1] \in \mathrm{Coh}^{<0}(X)$ and hence by Example A.0.5 we get

$$\mathbb{V}(\mathbb{T}_X[-1]) \simeq \mathrm{Spec}_{/X}\left(\mathrm{Sym}_{\mathrm{QCoh}(X)}(\Omega_X[1])\right)$$

thus obtaining an equivalence $i_*\mathcal{O}_{\mathcal{L}X} \simeq i_*\mathcal{O}_{\mathbb{V}(\mathbb{T}_X[-1])} \simeq \mathrm{Sym}_{\mathrm{QCoh}(X)}(\Omega_X[1])$ in $\mathrm{QCoh}(X)$. $\square$

Now the Hochschild-Kostant-Rosenberg theorem above shows us that for a perfect sheaf with an endomorphism (E,t) the categorical Chern character $\mathrm{ch}(E,t)$ is in fact an element of

$$\pi_0\Gamma(\mathcal{L}X,\mathcal{O}_{\mathcal{L}X})\simeq\bigoplus_{p=0}^{\dim X}H^p(X,\Omega^p_X).$$

In particular, it makes sense to ask for a more concrete description of $\mathrm{ch}(E,t)$. In order to do this, let us pick a sheaf $E\in\mathrm{QCoh}(X)$ and analyze the canonical $\mathcal{L}X$-equivariant structure on E. Since action of a formal group correspond to the action of the corresponding Lie algebra (see Theorem A.0.2) in the case when our formal group is the inertia group $\mathcal{L}X$ we get an equivalence

$$\mathrm{Rep}_{\mathcal{L}X}\left(\mathrm{QCoh}(X)\right)\simeq\mathrm{Mod}_{\mathbb{T}_X[-1]}\left(\mathrm{QCoh}(X)\right).$$

Consequently, we see that the canonical $\mathcal{L}X$-equivariant structure on $E\in\mathrm{QCoh}(X)$ can be equivalently understood in terms of the corresponding action of $\mathbb{T}_X[-1]\in\mathrm{LAlg}(\mathrm{QCoh}(X))$ on E (which we will further also call *canonical*).

In order to understand the canonical action of $\mathbb{T}_X[-1]$, let us for a second discuss a more general situation. Suppose that we have a Lie algebra $\mathfrak{g}\in\mathrm{QCoh}(X)$ which acts on some $\mathcal{F}\in\mathrm{QCoh}(X)$. The associative algebra structure on $\mathcal{E}nd_{\mathrm{QCoh}(X)}(\mathcal{F})\in\mathrm{Alg}(\mathrm{QCoh}(X))$ endows it with the structure of a Lie algebra (we will further denote by $\mathfrak{gl}_{\mathcal{F}}\in\mathrm{LAlg}(\mathrm{QCoh}(X))$ the corresponding Lie algebra) and the action of $\mathfrak{g}$ on $\mathcal{F}$ induces a map $\mathfrak{g}\longrightarrow\mathfrak{gl}_{\mathcal{F}}$ in $\mathrm{LAlg}(\mathrm{QCoh}(X))$. Now by Lie algebras-formal groups correspondence from theorem A.0.2, we can "integrate" this map to obtain a map $\widehat{G}\xrightarrow{\rho}\mathrm{GL}_{\mathcal{F}}$ of formal groups, where $\mathrm{Lie}_X(\widehat{G})\simeq\mathfrak{g}$ and $\mathrm{Lie}_X(\widehat{\mathrm{GL}}_{\mathcal{F}})\simeq\mathfrak{gl}_{\mathcal{F}}$. It particular, by functoriality this gives a commutative diagram

$$\begin{tikzcd}
\mathbb{V}(\mathfrak{g}) \arrow[r, "\mathbb{V}(\mathrm{Lie}_X(\rho))"] \arrow[d, "\exp_{\widehat{G}}"', "\sim"] & \mathbb{V}(\mathfrak{gl}_{\mathcal{F}}) \arrow[d, "\sim"', "\exp_{\widehat{\mathrm{GL}}_{\mathcal{F}}}"] \\
\widehat{G} \arrow[r, "\rho"'] & \widehat{\mathrm{GL}}_{\mathcal{F}}
\end{tikzcd}$$

of formal moduli problems over X.

Now applying the above procedure to the canonical action of $\mathbb{T}_X[-1]$ on E we arrive at the following

Definition 1.3.2. Let X be a smooth scheme. Define *Atiyah class* $\mathrm{At}(E)$ of $E \in \mathrm{QCoh}(X)$ as the top horizontal map in the commutative diagram

$$\begin{tikzcd} \mathbb{V}\big(\mathbb{T}_X[-1]\big) \arrow[r, "\mathrm{At}(E)"] \arrow[d, "\exp_{\mathcal{L}X}"', "\sim"] & \mathbb{V}(\mathfrak{gl}_E) \arrow[d, "\sim"', "\exp_{\widehat{\mathrm{GL}}_E}"] \\ \mathcal{L}X \arrow[r, "\alpha_E"'] & \widehat{\mathrm{GL}}_E. \end{tikzcd}$$

By the Lie algebras–formal groups correspondence Atiyah class of E corresponds by definition to the canonical $\mathbb{T}_X[-1]$-module structure on E. By taking dual of the Lie-module structure map $\mathbb{T}_X[-1] \longrightarrow \mathcal{E}nd_{\mathrm{QCoh}(X)}(E)$ we see that $\mathrm{At}(E)$ corresponds to some class in $H^1(X, \mathcal{E}nd_{\mathrm{QCoh}(X)}(E) \otimes \mathbb{L}_X)$.

Combining Proposition 1.2.7 with the notation above we obtain:

Corollary 1.3.3. Let X be a smooth proper scheme and $E \in \mathrm{QCoh}(X)$ is a perfect sheaf with an endomorphism $E \xrightarrow{t} E$. Then under the Hochschild-Kostant-Rosenberg identification we have an equality

$$\mathrm{ch}(E,t) = \mathsf{Tr}_{\mathrm{QCoh}(\mathcal{L}X)}\left(i^*E \xrightarrow[\sim]{\exp(\mathrm{At}(E))} i^*E \xrightarrow{\quad i^*(t)\quad} i^*E \right)$$

of elements of $\bigoplus_p H^p(X, \Omega^p_X)$.

Example 1.3.4. Recall that for a perfect sheaf $E \in \mathrm{QCoh}(X)$ we have an equality

$$\mathrm{ch}_0(E) = \mathrm{rk}(E) = \mathsf{Tr}_{\mathrm{QCoh}(X)}(\mathrm{Id}_E)$$

where $\mathrm{ch}(E)$ here is the classical Chern character. Now as we will see below (Proposition 1.4.9), the categorical Chern character $\mathrm{ch}(E, \mathrm{Id}_E)$ in fact coincides with the classical one $\mathrm{ch}(E)$. In particular, the corollary above allows as to prove a generalization of this statement: for a perfect sheaf (E,t) with an endomorphism we get an equality

$$\mathrm{ch}(E,t) = \mathsf{Tr}\left(t + t \circ \mathrm{At}(E) + t \circ \frac{\mathrm{At}(E)^{\wedge 2}}{2} + \dots \right)$$

and therefore since $\mathsf{Tr}(\mathrm{At}(E)^{\wedge n}) \in H^n(X, \Omega^n_X)$ we obtain $\mathrm{ch}_0(E,t) = \mathsf{Tr}(t)$.

1.4. Comparison with the classical Chern character

We will now compare our definition of Atiyah class with a more classical one. These results are well-known to experts and we include them mostly for reader's convenience.

Definition 1.4.1. The prestack Perf of *perfect sheaves* is defined as

$$\mathrm{Perf}(R) := (\mathrm{Mod}_R^{\mathrm{perf}})^{\simeq}.$$

Note that since the operation of taking full subcategory of dualizable objects commutes with limits of symmetric monoidal categories and the operation of taking maximal subgroupoid commutes with limits of categories for any prestack $Y \in \mathrm{PreStack}$ we get a natural equivalence

$$\mathrm{Hom}_{\mathrm{PreStack}}(Y, \mathrm{Perf}) \simeq \lim_{\mathrm{Spec}\, R \in \mathrm{Aff}_{/Y}} \mathrm{Hom}_{\mathrm{PreStack}}(\mathrm{Spec}\, R, \mathrm{Perf}) =$$

$$= \lim_{\mathrm{Spec}\, R \in \mathrm{Aff}_{/Y}} (\mathrm{Mod}_R^{\mathrm{perf}})^{\simeq} \simeq \left(\mathrm{QCoh}(Y)^{\mathrm{perf}}\right)^{\simeq}$$

Let us now denote by $\mathcal{E} \in \mathrm{QCoh}(\mathrm{Perf})$ the universal perfect sheaf classified by the identity morphism $\mathrm{Id}_{\mathrm{Perf}}$. We then have

Proposition 1.4.2 ([11, Chapter 8, Proposition 3.3.4.])**.** There is a canonical equivalence

$$\mathbb{T}_{\mathrm{Perf}}[-1] \xrightarrow[\sim]{} \mathcal{E}nd_{\mathrm{QCoh}(\mathrm{Perf})}(\mathcal{E})$$

of Lie algebras.

Remark 1.4.3. It is not hard to show that the underlying sheaves of $\mathbb{L}_{\mathrm{Perf}}$ and $\mathcal{E}nd_{\mathrm{QCoh}(\mathrm{Perf})}(\mathcal{E})[-1]$ are equivalent. Indeed, the pullback diagram

$$\begin{tikzcd}
\mathcal{L}\,\mathrm{Perf} \arrow[r, "i"] \arrow[d, "i"'] & \mathrm{Perf} \arrow[d, "\Delta"] \\
\mathrm{Perf} \arrow[r, "\Delta"'] \arrow[u, bend left, dashed, "e"] & \mathrm{Perf} \times \mathrm{Perf}
\end{tikzcd}$$

induces an equivalence

$$\mathbb{L}_{\mathrm{Perf}} \simeq \mathbb{L}_{\mathrm{Perf}\,/\,\mathrm{Perf} \times \mathrm{Perf}}[-1] \simeq e^* i^* \mathbb{L}_{\mathrm{Perf}\,/\,\mathrm{Perf} \times \mathrm{Perf}}[-1] \simeq e^* \mathbb{L}_{\mathcal{L}\,\mathrm{Perf}\,/\,\mathrm{Perf}}[-1].$$

Now to calculate $e^*\mathbb{L}_{\mathcal{L}\,\mathrm{Perf}\,/\,\mathrm{Perf}}[-1]$ we note that by definition for any morphism

$$\operatorname{Spec} R \xrightarrow{\eta} \mathrm{Perf}$$

classifying $E \in \mathrm{Mod}_R^{\mathrm{perf}}$ and an R-module $M \in \mathrm{Mod}_R^{\leq 0}$ the space

$$\mathrm{Hom}_{\mathrm{Mod}_R}(\eta^* e^* \mathbb{L}_{\mathcal{L}\,\mathrm{Perf}\,/\,\mathrm{Perf}}, M)$$

is equivalent to the space of lifts

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, "\eta"] \arrow[d, "t"'] & \mathrm{Perf} \arrow[r, "e"] & \mathcal{L}\,\mathrm{Perf} \arrow[d, "i"] \\ \operatorname{Spec} R[M] \arrow[urr, dashed] \arrow[rr, "\zeta"'] & & \mathrm{Perf}\,. \end{tikzcd}$$

Now the morphism ζ classifies the module u^*E where $\operatorname{Spec} R[M] \xrightarrow{u} \operatorname{Spec} R$ is the projection map (so that $t^*u^* \simeq \mathrm{Id}_{\mathrm{Mod}_R}$) and $\mathcal{L}X$ classifies a perfect sheaf together with an automorphism we get

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Mod}_R}(\eta^* e^* \mathbb{L}_{\mathcal{L}\,\mathrm{Perf}\,/\,\mathrm{Perf}}, M) \simeq \mathrm{Aut}_{\mathrm{Mod}_{R[M]}}(u^*E) \underset{\mathrm{Aut}_{\mathrm{Mod}_R}(E)}{\times} \{\mathrm{Id}_E\} \simeq \\ &\simeq \mathrm{Hom}_{\mathrm{Mod}_{R[M]}}(u^*E, u^*E) \underset{\mathrm{Aut}_{\mathrm{Mod}_R}(E)}{\times} \{\mathrm{Id}_E\} \simeq \\ &\simeq \mathrm{Hom}_{\mathrm{Mod}_R}(E, u_*u^*E) \underset{\mathrm{Aut}_{\mathrm{Mod}_R}(E)}{\times} \{\mathrm{Id}_E\} \simeq \\ &\simeq \mathrm{Hom}_{\mathrm{Mod}_R}(E, E \oplus E \otimes_R M) \underset{\mathrm{Aut}_{\mathrm{Mod}_R}(E)}{\times} \{\mathrm{Id}_E\} \simeq \mathrm{Hom}_{\mathrm{Mod}_R}(E, E \otimes_R M) \simeq \\ &\simeq \mathrm{Hom}_{\mathrm{Mod}_R}\big(\mathcal{E}nd_{\mathrm{Mod}_R}(E), M\big) \simeq \mathrm{Hom}_{\mathrm{Mod}_R}\big(\eta^* \mathcal{E}nd_{\mathrm{QCoh(Perf)}}(\mathcal{E}), M\big) \end{aligned}$$

proving the claim, where above we use that any endomorphism of $u^*E \in \mathrm{Mod}_{R[M]}$ lying over Id_E is automatically invertible.

The following proposition provides a convenient description of the Atiyah class as a map of underlying sheaves:

Proposition 1.4.4. Let $E \in \mathrm{Perf}(X)$ be a perfect sheaf on a smooth scheme X classified by the map $X \xrightarrow{e} \mathrm{Perf}$. Then the induced map of tangent spaces

$$\mathbb{T}_X \longrightarrow e^*\mathbb{T}_{\mathrm{Perf}} \simeq \mathcal{E}nd_{\mathrm{QCoh}(X)}(E)[1]$$

is equal to $\mathrm{At}(E)[1]$.

Proof. Note that given two stacks $\mathcal{X}, \mathcal{Y}$ admitting deformation theory together with formal groups $H \in \mathrm{Grp}(\widehat{\mathrm{Moduli}_{\mathcal{X}}})$ and $G \in \mathrm{Grp}(\widehat{\mathrm{Moduli}_{\mathcal{Y}}})$ over $\mathcal{X}$ and $\mathcal{Y}$ respectively the datum of commutative square

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d] & \mathcal{Y} \arrow[d] \\ \widehat{B}_{/\mathcal{X}}H \arrow[r] & \widehat{B}_{/\mathcal{Y}}G \end{tikzcd}$$

determines a morphism

$$H \longrightarrow G_{\mathcal{X}} := \mathcal{X} \times_{\mathcal{Y}} G$$

of formal groups over $\mathcal{X}$. Unwinding the definitions, one finds that under the identifications

$$\mathrm{Lie}_{\mathcal{X}}(H) \simeq \mathbb{T}_{\mathcal{X}/(\widehat{B}_{/\mathcal{X}}H)} \qquad \mathrm{Lie}_{\mathcal{X}}(G_{\mathcal{X}}) \simeq f^* \,\mathrm{Lie}_{\mathcal{Y}}(G) \simeq f^*\mathbb{T}_{\mathcal{Y}/(\widehat{B}_{/\mathcal{Y}}G)}$$

the induced map of Lie algebras

$$\mathbb{T}_{\mathcal{X}/(\widehat{B}_{/\mathcal{X}}H)} \simeq \mathrm{Lie}_X(H) \longrightarrow f^* \,\mathrm{Lie}_Y(G) \simeq f^*\mathbb{T}_{\mathcal{Y}/(\widehat{B}_{/\mathcal{Y}}G)}$$

coincides with the natural map of relative tangent sheaves. Applying this observation to the diagram

(2)
$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "\Delta"'] & \mathrm{Perf} \arrow[d, "\Delta"] \\ (X \times X)_{\widehat{\Delta}} \arrow[r, "e\times e"'] & (\mathrm{Perf} \times \mathrm{Perf})_{\widehat{\Delta}} \end{tikzcd}$$

and using the equivalence $\mathbb{T}_X \simeq \mathbb{T}_{X/(X\times X)_{\widehat{\Delta}}}[1]$ (and similarly for Perf) we deduce that the map of tangent sheaves $\mathbb{T}_X \longrightarrow e^*\mathbb{T}_{\mathrm{Perf}} \simeq \mathcal{E}nd_{\mathrm{QCoh}(X)}(E)[1]$ we are interested in is the shift of the morphism that underlies the map of Lie algebras induced by the morphism $\mathcal{L}X \longrightarrow X \times_{\mathrm{Perf}} \widehat{\mathcal{L}}\,\mathrm{Perf}$ of formal groups over X, where $\widehat{\mathcal{L}}\,\mathrm{Perf} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}_{\mathrm{Perf}}})$ is the completion of $\mathcal{L}\,\mathrm{Perf}$ along the constant loops. Since $\mathcal{L}\,\mathrm{Perf}$ classifies a perfect sheaf with an automorphism we get an equivalence $X \times_{\mathrm{Perf}} \widehat{\mathcal{L}}\,\mathrm{Perf} \simeq \widehat{\mathrm{GL}}_{/X}(E)$ of formal groups over X. Moreover, by commutativity of the diagram (2), the induced group morphism $\mathcal{L}X \xrightarrow{\alpha} \widehat{\mathrm{GL}}_{/X}(E)$ is given precisely by the canonical action. The result now follows from the definition 1.3.2 of the Atiyah class. □

Recall now the following

Construction 1.4.5 (Classical algebraic Chern character)**.** Let X be a smooth proper scheme and $\mathcal{L}$ be a line bundle on X. We define

$$\mathrm{ch}(\mathcal{L}) := \exp(c_1(\mathcal{L})) \quad \in \quad \bigoplus_p H^p(X, \Omega^p_X),$$

where $c_1(\mathcal{L}) \in H^1(X, \Omega^1_X)$ is the algebraic (Hodge version of the) first Chern class defined via the map $\mathcal{O}^*_X \xrightarrow{d\log} \Omega^1_X$. For a general perfect sheaf E on X we define $\mathrm{ch}(E) \in \bigoplus_p H^p(X, \Omega^p_X)$ by additivity and the splitting principle.

Warning 1.4.6. Let $k = \mathbb{C}$, X a smooth proper scheme over k and E a vector bundle on X. Then one has a priori two different notions of the Chern character: the one constructed above and the topological Chern character $\mathrm{ch}^{\mathrm{top}}(E) \in H^*(X(\mathbb{C}), \mathbb{Z}) \hookrightarrow H^*(X(\mathbb{C}), \mathbb{C}) \simeq \bigoplus_{p,q} H^{q,p}(X)$. It turns out these two notions *do not* coincide, but are very closely related to each other: a simple computation on $\mathbb{P}^1$ shows that $c_1^{\mathrm{top}}(\mathcal{O}_{\mathbb{P}^1}(1)) = -2\pi i \cdot c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ and hence in general $\mathrm{ch}^{\mathrm{top}}_k(E) = (-2\pi i)^k \, \mathrm{ch}_k(E)$.

In particular, as a well-known corollary we recover the fact that the classical Chern character of a sheaf can be expressed in terms of the Atiyah class:

Corollary 1.4.7. Let X be a smooth proper scheme.

1. Let $\mathcal{M}$ be a line bundle on X. Then the Atiyah class $\mathrm{At}(\mathcal{M}) \in H^1(X, \mathcal{E}nd_{\mathrm{QCoh}(X)}(\mathcal{M}) \otimes \Omega^1_X) \simeq H^1(X, \Omega^1_X)$ of $\mathcal{M}$ coincides with the first Chern class $c_1(\mathcal{M})$ of $\mathcal{M}$.
2. Let E be a perfect sheaf on X. Then the classical Chern character of E is equal to $\mathsf{Tr}\exp(\mathrm{At}(E))$.

Proof. For the first claim, note that by Proposition 1.4.4 the Atiyah class of any perfect sheaf $E \in \mathrm{QCoh}(X)^{\mathrm{perf}}$ can be equivalently described as the shift of the induced map on tangent spaces

$$\mathbb{T}_X \longrightarrow e^*\mathbb{T}_{\mathrm{Perf}} \simeq \mathcal{E}nd_{\mathrm{QCoh}(X)}(E)[1] \,,$$

where the map $X \xrightarrow{e} \mathrm{Perf}$ classifies $E \in \mathrm{QCoh}(X)^{\mathrm{perf}}$. Now if $E = \mathcal{M}$ is a line bundle, the classifying map $X \longrightarrow \mathrm{Perf}$ factors through $B\mathbb{G}_m$ and so it is enough to prove the statement for the universal line bundle $\mathfrak{L}$ on $B\mathbb{G}_m$. But the canonical map $B\mathbb{G}_m \xrightarrow{u} \mathrm{Perf}$ induces an equivalence $\mathbb{T}_{B\mathbb{G}_m} \simeq u^*\mathbb{T}_{\mathrm{Perf}}$ and hence the Atiyah class obtained as the shift of

$$\mathrm{At}(\mathfrak{L})\colon \mathcal{O}_{B\mathbb{G}_m}[1] \simeq \mathbb{T}_{B\mathbb{G}_m} \longrightarrow \mathcal{E}nd_{\mathrm{QCoh}(B\mathbb{G}_m)}(\mathfrak{L})[1] \simeq \mathcal{O}_{B\mathbb{G}_m}[1]$$

is just the identity map, which corresponds to $c_1(\mathfrak{L})$.

For the second claim, as both $\mathrm{ch}(E)$ and $\mathsf{Tr}(\exp(\mathrm{At}(E)))$ are additive in triangles and commute with pullbacks by the splitting principle it is sufficient to prove the equality for $E = \mathcal{M}$ being a line bundle. But since by the previous part $\mathrm{At}(\mathcal{M}) = c_1(\mathcal{M})$, we get

$$\mathsf{Tr}(\exp(\mathrm{At}(\mathcal{M}))) = \exp(\mathrm{At}(\mathcal{M})) = \exp(c_1(\mathcal{M})) = \mathrm{ch}(\mathcal{M})$$

as claimed. □

Remark 1.4.8 (Chern-Weil theory and the Atiyah class)**.** Since $\mathrm{ch}(E,t)$ is k-linear in the second argument, one expects it to be an element of some k-linear cohomology theory of X like de Rham or Hodge cohomology. Our description of $\mathrm{ch}(E,t)$ is closer to the differential-geometric approach to characteristic classes. Namely, recall that if X is a smooth proper scheme over the field $\mathbb{C}$ of complex numbers and E is a vector bundle over X one can give the following description of the algebraic Chern character of E (see Warning 1.4.6 for the difference between algebraic and topological Chern characters): choose a smooth connection ∇ on E and let $F_\nabla \in \mathcal{E}nd(E) \otimes \Omega^2_X$ be the corresponding curvature form. Then

$$\mathrm{ch}(E) = \mathsf{Tr}\, e^{F_\nabla} \in H^*_{\mathrm{dR}}(X, \mathbb{C}).$$

Now assume additionally that ∇ is of type $(1,0)$. Then the curvature F_∇ splits into a sum $F^{2,0}_\nabla + F^{1,1}_\nabla$. It follows that $\mathsf{Tr}\, e^{F^{1,1}_\nabla}$ is a representative of $\mathrm{ch}(E)$ in Hodge cohomology $\bigoplus_p H^{p,p}(X) \simeq \bigoplus_p F^p H^p_{\mathrm{dR}}(X,\mathbb{C})/F^{p+1}H^p_{\mathrm{dR}}(X,\mathbb{C})$. Finally, it turns out the class $F^{1,1}_\nabla \in H^1(X, \Omega^1_X \otimes \mathcal{E}nd(E))$ is independent of a choice of ∇ and in fact coincides with the Atiyah class $\mathrm{At}(E)$ (see [2, Proposition 4]), linking the previous corollary and the Chern-Weil theory.

Finally we obtain a concrete description of the categorical Chern character of a sheaf with the trivial equivariant structure

Proposition 1.4.9. Let E be a dualizable object of $\mathrm{QCoh}(X)$. Then under the Hochschild-Kostant-Rosenberg isomorphism 1.3.1

$$\pi_0 \,\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \simeq \pi_0\Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \xrightarrow[\sim]{\mathrm{HKR}} \bigoplus_{p=0}^{\dim X} H^p(X, \Omega^p_X)$$

the categorical Chern character $\mathrm{ch}(E, \mathrm{Id}_E)$ coincides with the classical one $\mathrm{ch}(E)$.

Proof. By Proposition 1.2.7 we have $\mathrm{ch}(E, \mathrm{Id}_E) = \mathsf{Tr}_{\mathrm{QCoh}(\mathcal{L}X)}(\alpha_E)$ and by definition of the Atiyah class we have $\exp^*_{\mathcal{L}X}(\alpha_E) = \exp(\mathrm{At}(E))$. Since by construction $\mathrm{HKR} = \exp^*_{\mathcal{L}X}$, we conclude by the second part of Corollary 1.4.7. □

2. Trace of pushforward functor via ind-coherent sheaves

Recall from [10, Chapter 5, 5.3] the quasi-coherent sheaves functor in fact has an appropriate 2-categorical functoriality, in a sense that it can be lifted to a symmetric monoidal functor from a symmetric monoidal $(\infty, 2)$-category of correspondences (see Appendix B for a discussion of traces and correspondences). This allows us to reformulate many 2-categorical questions about quasi-coherent sheaves to questions about the category of correspondences, where they can be in most cases answered by direct diagram chasing. This observation, for example, gives a direct proof that the morphism of traces $\mathsf{Tr}(f^*, \mathrm{Id}_{f^*})$ induced by the diagram

$$\begin{tikzcd}
\mathrm{QCoh}(X) \arrow[r, "\mathrm{Id}_{\mathrm{QCoh}(X)}"] \arrow[d, bend right, "f^*"'] & \mathrm{QCoh}(X) \arrow[d, bend right, "f^*"'] \\
\mathrm{QCoh}(Y) \arrow[r, "\mathrm{Id}_{\mathrm{QCoh}(Y)}"'] \arrow[u, bend right, "f_*"'] & \mathrm{QCoh}(Y) \arrow[u, bend right, "f_*"']
\end{tikzcd}$$

in Cat_k coincides with the classical pullback of global sections (see Remark 2.2.6). However, the same argument does not apply to the morphism of traces

$$\bigoplus_{p=0}^{\dim X} H^p(X, \Omega^p_X) \simeq \mathsf{Tr}(\mathrm{Id}_{\mathrm{QCoh}(Y)}) \xrightarrow{\mathsf{Tr}(f_*, \mathrm{Id}_{f_*})} \mathsf{Tr}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \simeq \bigoplus_{p=0}^{\dim Y} H^p(Y, \Omega^p_Y)$$

we are interested in. Indeed, due to the post factum knowledge that the answer should involve the Todd class, one should not expect to obtain a concrete description of it in a purely formal way.

But as is mentioned in previously, apart from $\mathrm{QCoh}(X)$ there is another important $(\infty,1)$-category we can associate to X: the $(\infty,1)$-category $\mathrm{ICoh}(X) \in 2\,\mathrm{Cat}_k$ of ind-coherent sheaves on X. As a toy example, for a smooth classical scheme X there is a functor $\mathrm{ICoh}(X) \xrightarrow[\sim]{\Psi_X} \mathrm{QCoh}(X)$ which identifies the category of ind-coherent sheaves with the category of quasi-coherent sheaves (see Example 2.1.5) as a plain category, but does not preserve monoidal structure: via this equivalence the natural $\overset{!}{\otimes}$-monoidal structure on $\mathrm{ICoh}(X)$ is given by

$$\mathcal{F} \overset{!}{\otimes} \mathcal{G} \simeq \mathcal{F} \otimes \mathcal{G} \otimes \omega_X^{-1}.$$

where $\omega_X \in \mathrm{QCoh}(X)$ is the dualizing sheaf.

In this section we will prove that the morphism of traces induced by pushforward could be understood relatively easy in the setting of ind-coherent sheaves:

- In subsection 2.1 we will review relevant facts about the category of ind-coherent sheaves.
- Similar to that of QCoh, using (Serre) self-duality of ICoh we will prove that there is an equivalence

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*) \simeq \Gamma(X^g, \omega_{X^g}^{\mathrm{ICoh}}).$$

- Then we will show that the morphism of traces

$$\Gamma(X^g, \omega_{X^g}^{\mathrm{ICoh}}) \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_{X*}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_{Y*}) \simeq \Gamma(Y^g, \omega_{Y^g}^{\mathrm{ICoh}})$$

  induced by the diagram

$$\begin{tikzcd} \mathrm{ICoh}(X) \arrow[r, "(g_X)_*"] \arrow[d, "f_*"'] & \mathrm{ICoh}(X) \arrow[d, "f_*"] \\ \mathrm{ICoh}(Y) \arrow[r, "(g_Y)_*"'] & \mathrm{ICoh}(Y) \end{tikzcd}$$

  in $2\,\mathrm{Cat}_k$ coincides with the natural pushforward of distributions. In particular if both X, Y are smooth and proper with trivial equivariant structure, from this we will deduce that the morphism of traces

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*)} \mathsf{Tr}(\mathrm{Id}_{\mathrm{ICoh}(Y)})$$

$$\left(\bigoplus_{p=0}^{\dim X} H^p(X, \Omega_X^p)\right)^\vee \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*)} \mathsf{Tr}(\mathrm{Id}_{\mathrm{ICoh}(Y)}) \left(\bigoplus_{p=0}^{\dim Y} H^p(Y, \Omega_Y^p)\right)^\vee$$

coincides with the pushforward in homology.

After dealing with this, we will use further sections to investigate how one can describe the relation between the morphism of traces induced by pushforward in the setting of ICoh and in the setting of QCoh.

2.1. Reminder on Ind-coherent sheaves

In this subsection we review some basic facts and constructions related to ind-coherent sheaves. We refer reader to [10, Part II] and [9] for further details. We start with the following

Definition 2.1.1. For $X \in \mathrm{Sch}_{\mathsf{aft}}$ (see [10, Chapter 4, 1.1.1]) define the category of **ind-coherent sheaves on** X denoted by $\mathrm{ICoh}(X)$ simply as

$$\mathrm{ICoh}(X) := \mathrm{Ind}(\mathrm{Coh}(X)),$$

where we denote by $\mathrm{Coh}(X)$ the category of coherent sheaves on X, i.e. the full subcategory of $\mathrm{QCoh}(X)$ consisting of those $\mathcal{F} \in \mathrm{QCoh}(X)$ such that $\mathcal{H}^i(\mathcal{F})$ are non-zero only for finitely many i and are coherent over the sheaf of algebras $\mathcal{H}^0(\mathcal{O}_X)$ in the usual sense.

Properties of the ind-coherent sheaves construction we need in this paper can be summarized by the following

Proposition 2.1.2.
1) ([10, Chapter 4, Proposition 2.1.2, Proposition 2.2.3]) The assignment of ind-coherent sheaves can be lifted to a functor

$$\mathrm{Sch}_{\mathsf{aft}} \xrightarrow{\mathrm{ICoh}_*} \mathrm{Cat}_k$$

such that, moreover, for every morphism $X \xrightarrow{f} Y$ in $\mathrm{Sch}_{\mathsf{aft}}$ the diagram

$$\begin{tikzcd} \mathrm{ICoh}(X) \arrow[r, "\Psi_X"] \arrow[d, "f_*"'] & \mathrm{QCoh}(X) \arrow[d, "f_*"] \\ \mathrm{ICoh}(Y) \arrow[r, "\Psi_Y"'] & \mathrm{QCoh}(Y) \end{tikzcd}$$

commutes, where $\mathrm{ICoh}(X) \xrightarrow{\Psi_X} \mathrm{QCoh}(X)$ is obtained by ind-extending the natural inclusion $\mathrm{Coh}(X) \subseteq \mathrm{QCoh}(X)$ (and similar for Y).

2) ([10, Chapter 4, Corollary 5.1.12]) The assignment of ind-coherent sheaves can be lifted to a functor

$$\mathrm{Sch}^{\mathsf{op}}_{\mathsf{aft,proper}} \xrightarrow{\mathrm{ICoh}^!} \mathrm{Cat}_k,$$

such that, moreover, given a proper morphism $X \xrightarrow{f} Y$ in $\mathrm{Sch}_{\mathsf{aft}}$ the induced pullback functor $f^! := \mathrm{ICoh}^!(f)$ is right adjoint to f_*.

3) ([10, Chapter 4, Proposition 6.3.7; Chapter 5, Theorem 4.2.5]) For every $X \in \mathrm{Sch}_{\mathsf{aft}}$ the category $\mathrm{ICoh}(X)$ is symmetric monoidal and self-dual as an object of Cat_k (see Theorem 2.2.1 below for a concrete description of duality maps). Moreover, for every proper $X \xrightarrow{f} Y$ the induced functor $f^!$ is symmetric monoidal. We will further denote the monoidal structure on $\mathrm{ICoh}(X)$ by $\overset{!}{\otimes}$ and the monoidal unit, the so-called ICoh-*dualizing sheaf*, by $\omega_X^{\mathrm{ICoh}} \in \mathrm{ICoh}(X)$. It is straightforward to see that there is in fact an equivalence $\omega_X^{\mathrm{ICoh}} \simeq p^! k$, where $X \xrightarrow{p} *$ is the projection and $k \in \mathrm{ICoh}(*) \simeq \mathrm{Vect}_k$.

4) ([10, Chapter 6, 0.3.5, 3.2.5]) The functor $\mathrm{QCoh}(X) \xrightarrow{\Upsilon_X} \mathrm{ICoh}(X)$ obtained from Ψ_X using self-dualities of $\mathrm{QCoh}(X)$ and $\mathrm{ICoh}(X)$ is symmetric monoidal and for every proper morphism $X \xrightarrow{f} Y$ in $\mathrm{Sch}_{\mathsf{aft}}$ the diagram

$$\begin{tikzcd} \mathrm{QCoh}(X) \arrow[r, "\Upsilon_X"] & \mathrm{ICoh}(X) \\ \mathrm{QCoh}(Y) \arrow[r, "\Upsilon_Y"'] \arrow[u, "f^*"] & \mathrm{ICoh}(Y) \arrow[u, "f^!"'] \end{tikzcd}$$

commutes.

In our main case of interest the categories of quasi-coherent sheaves and ind-coherent sheaves are not that far away to each other. Recall first the following

Construction 2.1.3. Since $X \in \mathrm{Sch}_{\mathsf{aft}}$ is quasi-compact, the global sections functor

$$\mathrm{QCoh}(X) \xrightarrow{p_*} \mathrm{Vect}_k$$

is continuous, hence admits a right adjoint $p^!$. We define QCoh-*dualizing sheaf* $\omega_X \in \mathrm{QCoh}(X)$ by setting $\omega_X := p^!(k)$.

Example 2.1.4. If X is smooth, then by classical Serre duality there is an equivalence

$$\omega_X \simeq \Omega_X^{\dim X}[\dim X]$$

in $\mathrm{QCoh}(X)$. In particular, ω_X is dualizable in this case.

Example 2.1.5. Let X be a smooth classical scheme. Since X is quasi-compact and separated there is an equivalence $\mathrm{QCoh}(X) \simeq \mathrm{Ind}(\mathrm{QCoh}(X)^{\mathrm{perf}})$ of categories (see e.g. [4, Proposition 3.19]), and by smoothness we also get $\mathrm{Coh}(X) \simeq \mathrm{QCoh}(X)^{\mathrm{perf}}$. Consequently, it follows that the canonical functor $\mathrm{ICoh}(X) \xrightarrow{\Psi_X} \mathrm{QCoh}(X)$ is an equivalence of categories and thus the functor $\mathrm{QCoh}(X) \xrightarrow{\Upsilon_X} \mathrm{ICoh}(X)$ is a symmetric monoidal equivalence of categories. In particular we can identify $\mathrm{ICoh}(X)$ with $\mathrm{QCoh}(X)$ with the twisted monoidal structure

$$\mathcal{F} \overset{!}{\otimes} G \simeq \mathcal{F} \otimes \mathcal{G} \otimes \omega_X^{-1}.$$

We will further need comparison between ω_X and ω_X^{ICoh}:

Proposition 2.1.6. Let X be a proper derived scheme. Then there is an equivalence $\omega_X \simeq \Psi_X(\omega_X^{\mathrm{ICoh}})$ in $\mathrm{QCoh}(X)$. In particular $\Gamma(X, \omega_X^{\mathrm{ICoh}}) \simeq \Gamma(X, \omega_X)$ where here $\Gamma = p_*$ is the pushforward along projection morphism $X \xrightarrow{p} *$ in the setting of ind-coherent and quasi-coherent sheaves respectively.

Proof. The first statement follows from the fact that due to [9, Proposition 7.2.2., Proposition 7.2.9(a)] the diagram

$$\begin{tikzcd}
\mathrm{ICoh}(X) \arrow[r, "\Psi_X"] & \mathrm{QCoh}(X) \\
\mathrm{Vect}_k \arrow[u, "p^{!,\mathrm{ICoh}}"] \arrow[r, "\Psi_*", "\sim"'] & \mathrm{Vect}_k \arrow[u, "p^{!,\mathrm{QCoh}}"']
\end{tikzcd}$$

commutes. The assertion about global sections follows from commutativity of the square

$$\begin{tikzcd} \mathrm{ICoh}(X) \arrow[r, "\Psi_X"] \arrow[d, "p_*"'] & \mathrm{QCoh}(X) \arrow[d, "p_*"] \\ \mathrm{Vect}_k \arrow[r, "\Psi_*", "\sim"'] & \mathrm{Vect}_k . \end{tikzcd}$$

□

Remark 2.1.7. Note that unlike the QCoh-dualizing sheaf, the ICoh-dualizing sheaf is defined for much bigger class of prestacks (the quasi-coherent dualizing sheaf $\omega_X \in \mathrm{QCoh}(X)$ exists only if $\mathcal{O}_X \in \mathrm{QCoh}(X)$ is compact). However for the comparison of the morphism of traces induced by pushforward in QCoh-setting and ICoh-setting it is more convenient to work with QCoh-version of dualizing sheaf.

2.2. Computing the trace of pushforward

In this subsection we discuss morphism of traces in the setting of ind-coherent sheaves. We first note that similar to quasi-coherent sheaves, ind-coherent sheaves are self-dual as an object of Cat_k:

Theorem 2.2.1 ([10, Chapter 4, Proposition 6.3.4; Chapter 5, Theorem 4.2.5])**.**

1. For any two $X, Y \in \mathrm{Sch}_{\mathsf{aft}}$ (for the definition see [10, Chapter 2, 3.5]) the morphism

$$\mathrm{ICoh}(X) \otimes \mathrm{ICoh}(Y) \longrightarrow \mathrm{ICoh}(X \times Y)$$

in Cat_k induced by the functor

$$\mathrm{ICoh}(X) \times \mathrm{ICoh}(Y) \xrightarrow{\overset{!}{\boxtimes}} \mathrm{ICoh}(X \times Y)$$

is an equivalence.
2. For any $X \in \mathrm{Sch}_{\mathsf{aft}}$ the morphisms

$$\mathrm{Vect}_k \xrightarrow{\Delta_* \omega_X^{\mathrm{ICoh}}} \mathrm{ICoh}(X \times X) \simeq \mathrm{ICoh}(X) \otimes \mathrm{ICoh}(X)$$

and

$$\mathrm{ICoh}(X) \otimes \mathrm{ICoh}(X) \simeq \mathrm{ICoh}(X \times X) \xrightarrow{\Gamma \circ \Delta^!} \mathrm{Vect}_k$$

exhibit $\mathrm{ICoh}(X)$ as a self-dual object in Cat_k.

The proof of the following corollary is similar to that of Corollary 1.1.2 (where we use [10, Chapter 4, Proposition 5.2.2] to perform base change for ind-coherent sheaves)

Corollary 2.2.2. Let X be an almost finite type scheme with an endomorphism g. Then

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*) \simeq \Gamma(X, \Delta^!(\mathrm{Id}_X, g)_*\omega_X^{\mathrm{ICoh}}) \simeq \Gamma(X^g, \omega_{X^g}^{\mathrm{ICoh}}).$$

In particular, in the case when X is proper by Proposition 2.1.6 we get an equivalence

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*) \simeq \Gamma(X^g, \omega_{X^g}^{\mathrm{ICoh}}) \simeq \Gamma(X^g, \omega_{X^g}).$$

Remark 2.2.3. Let $Z \xrightarrow{p} *$ be an almost finite type scheme. Note that then

$$\Gamma(Z, \omega_Z) \simeq \mathrm{Hom}_{\mathrm{QCoh}(Z)}(\mathcal{O}_Z, p^!k) \simeq \mathrm{Hom}_{\mathrm{Vect}_k}(p_*\mathcal{O}_Z, k) \simeq \Gamma(Z, \mathcal{O}_Z)^\vee.$$

In particular, using the previous proposition we obtain an equivalence

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*) \simeq \Gamma(X^g, \omega_{X^g}) \simeq \Gamma(X^g, \mathcal{O}_{X^g})^\vee.$$

As a corollary, we get

Corollary 2.2.4. Let X be smooth and proper scheme. Then

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \simeq \Gamma(\mathcal{L}X, \omega_{\mathcal{L}X}) \simeq \Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X})^\vee \overset{\mathrm{HKR}^\vee}{\simeq} \left(\bigoplus_{p=0}^{\dim X} H^p(X, \Omega_X^p)\right)^\vee$$

where HKR is the Hochschild-Kostant-Rosenberg equivalence (see Corollary 1.3.1).

We now turn to the computation of morphism of traces. Main result of this section is the following

Proposition 2.2.5. Let $(X, g_X) \xrightarrow{f} (Y, g_Y)$ be an equivariant proper morphism in $\mathrm{Sch}_{\mathsf{aft}}$. Then the induced morphism of traces

$$\Gamma(X^{g_X}, \omega_{X^{g_X}}^{\mathrm{ICoh}}) \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_{X*}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_{Y*}) \simeq \Gamma(Y^{g_Y}, \omega_{Y^{g_Y}}^{\mathrm{ICoh}})$$

can be obtained by applying the global sections functor $\Gamma(Y^{g_Y}, -)$ to the morphism

$$(f^g)_*\omega^{\mathrm{ICoh}}_{X^{g_X}} \simeq (f^g)_*(f^g)^!\omega^{\mathrm{ICoh}}_{Y^{g_Y}} \longrightarrow \omega^{\mathrm{ICoh}}_{Y^{g_Y}}$$

in $\mathrm{ICoh}(X)$ induced by the counit of the adjunction $(f^g)_* \dashv (f^g)^!$, where $X^{g_X} \xrightarrow{f^g} Y^{g_Y}$ is the induced by f morphism on derived fixed points.

Proof. The proof is a direct consequence of the fact that the self-duality of ind-coherent sheaves arise from the category of correspondences. Namely, by [10, Chapter 5, Theorem 4.1.2] the ind-coherent sheaves functor can be lifted to a symmetric monoidal functor

$$\mathsf{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}} \longrightarrow 2\,\mathrm{Cat}_k,$$

where $\mathsf{Corr}(\mathrm{Sch}_{\mathsf{aft}})$ is a symmetric monoidal $(\infty, 2)$-category whose objects are $X \in \mathrm{Sch}_{\mathsf{aft}}$, morphisms from X to Y are spans

$$X \longleftarrow W \longrightarrow Y$$

in $\mathrm{Sch}_{\mathsf{aft}}$ (with the composition given by pullbacks), 2-morphisms are commutative diagrams

$$\begin{tikzcd} & W_1 \arrow[dl] \arrow[dd, "h"] \arrow[dr] & \\ X & & Y \\ & W_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where h is proper and the monoidal structure is given by the cartesian product. Informally speaking, the extension of the ind-coherent sheaves to the category of correspondences is given by mapping the span $X \xleftarrow{s} W \xrightarrow{t} Y$ to the morphism $\mathrm{ICoh}(X) \xrightarrow{s^!} \mathrm{ICoh}(W) \xrightarrow{t_*} \mathrm{ICoh}(Y)$ in $2\,\mathrm{Cat}_k$. We refer to [10, Chapter 7, Chapter 5] for a throughout discussion of the category of correspondences and to Corollary B.1.7 for a complete proof of the proposition. $\square$

Remark 2.2.6. One can similarly show that the morphism of traces

$$\Gamma(Y^{g_Y}, \mathcal{O}_{Y^{g_Y}}) \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_{Y*}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f^*)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_{X*}) \simeq \Gamma(X^{g_X}, \mathcal{O}_{X^{g_X}})$$

is induced by the natural map $\mathcal{O}_{Y^{g_Y}} \longrightarrow (f^g)_*\mathcal{O}_{X^{g_X}}$. It follows that for X and Y smooth and proper with trivial equivariant structure, under the Hochschild-Kostant-Rosenberg equivalence

$$\pi_i\Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \simeq \bigoplus_{p-q=i} H^q(X, \Omega^p_X) \qquad \pi_i\Gamma(\mathcal{L}Y, \mathcal{O}_{\mathcal{L}Y}) \simeq \bigoplus_{p-q=i} H^q(Y, \Omega^p_Y)$$

the morphism of traces is exactly the pullback in cohomology.

Note, however, that the strategy above does not give directly the description of the morphism of traces

$$\Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(Y)}) \simeq \Gamma(\mathcal{L}Y, \mathcal{O}_{\mathcal{L}Y})$$

on quasi-coherent sheaves. The reason is that in this case the formalism of traces uses the right adjoint to f_* which is $f^!$. However, the functor $f^!$ does not come from the QCoh functor out of the category of correspondences (which uses the adjoint pair $f^* \dashv f_*$ instead).

Finally, using the identification of Corollary 2.2.4 we obtain

Corollary 2.2.7. Let X, Y be smooth and proper with the trivial equivariant structure. Then under the HKR-identification the morphism of traces

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(Y)})$$

$$\left(\bigoplus_{p=0}^{\dim X} H^p(X, \Omega^p_X)\right)^\vee \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*)} \left(\bigoplus_{p=0}^{\dim Y} H^p(Y, \Omega^p_Y)\right)^\vee$$

coincides with the pushforward in homology (which we define as the map dual to the pullback of cohomology).

Proof. Immediately follows from the fact that for a map of schemes $Z \xrightarrow{f} W$ the induced map

$$\Gamma(Z, \mathcal{O}_Z)^\vee \simeq \Gamma(Z, \omega_Z) \longrightarrow \Gamma(W, \omega_W) \simeq \Gamma(W, \mathcal{O}_W)^\vee$$

coincides by construction of dualizing sheaves with the map dual to the pullback of functions. □

3. Orientations and traces

Let $X \xrightarrow{f} Y$ be a morphism between smooth proper schemes with trivial equivariant structure. Our goal in this section is to give some description of the morphism of traces induced by the pushforward functor $\mathrm{QCoh}(X) \xrightarrow{f_*} \mathrm{QCoh}(Y)$ (which is automatically equivariant). Since the diagram

$$\begin{tikzcd} \mathrm{QCoh}(X) \arrow[r, "f_*"] \arrow[d, "-\otimes\mathcal{O}_X"'] & \mathrm{QCoh}(Y) \arrow[d, "-\otimes\mathcal{O}_Y"] \\ \mathrm{ICoh}(X) \arrow[r, "f_*"'] & \mathrm{ICoh}(Y) \end{tikzcd}$$

commutes and we already have a description of the morphism of traces induced by the functor $\mathrm{ICoh}(X) \xrightarrow{f_*} \mathrm{ICoh}(Y)$ (where we take identity endomorphisms on both $\mathrm{ICoh}(X)$ and $\mathrm{ICoh}(Y)$), it is enough to understand the morphism of traces induced by $\mathrm{QCoh}(X) \xrightarrow{-\otimes\mathcal{O}_X} \mathrm{ICoh}(X)$ and analogously for Y. We start by introducing the following

Definition 3.0.1. For an almost finite type scheme Z an *orientation* on Z is a choice of an equivalence $\mathcal{O}_Z \simeq \omega_Z$ in $\mathrm{QCoh}(Z)$.

Remark 3.0.2. Let $\mathrm{u}\colon \mathcal{O}_Z \simeq \omega_Z$ be an orientation on Z. Then in particular we obtain a self-duality equivalence

$$\Gamma(Z, \mathcal{O}_Z) \xrightarrow[\sim]{\mathrm{u}} \Gamma(Z, \omega_Z) \simeq \Gamma(Z, \mathcal{O}_Z)^\vee$$

which is moreover a morphism of $\Gamma(Z, \mathcal{O}_Z)$-modules. Note that the space of orientations on Z is a torsor over $\Gamma(Z, \mathcal{O}_Z)^\times$, i.e. after a choice of particular orientation, all other orientations are in bijection with invertible functions on Z.

The relevance of the constructions above to the comparison of traces is explained by

Remark 3.0.3. If X is smooth proper scheme, then any orientation t on $\mathcal{L}X$ induces an equivalence

$$\pi_0 \operatorname{\mathsf{Tr}}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \simeq \pi_0\Gamma(\mathcal{L}X, \omega_{\mathcal{L}X}) \xrightarrow[\sim]{\mathrm{u}^{-1}} \pi_0\Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \overset{\mathrm{HKR}}{\simeq} \bigoplus_{p=0}^{\dim X} H^p(X, \Omega_X^p).$$

3.1. Serre orientation

In this and the next subsections we will introduce several orientations on the derived fixed schemes and discuss some of their properties. We start by recalling that there is a well-known isomorphism

$$\bigoplus_{p,q} H^q(X, \Omega_X^p) \simeq \left(\bigoplus_{p,q} H^q(X, \Omega_X^p)\right)^\vee$$

given by the Poincaré duality. The next construction shows that this equivalence is in fact induced by an orientation:

Construction 3.1.1 (Serre orientation)**.** Let X be a smooth proper scheme and denote by $\mathbb{V}(\mathbb{T}_X[-1]) \xrightarrow{j} X$ the evident projection map so that in particular we have an equivalence of sheaves

$$j_* \mathcal{O}_{\mathbb{V}(\mathbb{T}_X[-1])} \simeq \mathrm{Sym}_{\mathrm{QCoh}(X)}(\Omega_X[1])$$

(see Corollary 1.3.1). By projecting this equivalence to the top exterior summand we obtain a map

$$j_* \mathcal{O}_{\mathbb{V}(\mathbb{T}_X[-1])} \longrightarrow \omega_X$$

in $\mathrm{QCoh}(X)$ which using the adjunction $j_* \dashv j^!$ (as the morphism j is proper) induces an equivalence

$$\mathcal{O}_{\mathbb{V}(\mathbb{T}_X[-1])} \xrightarrow[\sim]{} j^! \omega_X \simeq \omega_{\mathbb{V}(\mathbb{T}_X[-1])}$$

and hence endows $\mathbb{V}(\mathbb{T}_X[-1]) \simeq \mathcal{L}X$ with an orientation. We will further call this equivalence the *Serre orientation.*

The following proposition is a formal consequence of the construction

Proposition 3.1.2. Let X be smooth proper scheme. Then the equivalence

$$\bigoplus_{p-q=i} H^q(X, \Omega_X^p) \xrightarrow[\sim]{\mathrm{HKR}} \pi_i \Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \xrightarrow[\sim]{\mathrm{u}_S} \pi_i \Gamma(\mathcal{L}X, \omega_{\mathcal{L}X})$$

where u_S is induced by the Serre orientation, followed by

$$\pi_i \Gamma(\mathcal{L}X, \omega_{\mathcal{L}X}) \xrightarrow[\sim]{} \pi_i \Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X})^\vee \xrightarrow[\sim]{\mathrm{HKR}^\vee} \bigoplus_{q-p=i} H^q(X, \Omega_X^p)^\vee$$

coincides with the one induced by the classical Serre duality. In particular, it sends a form $\eta \in \bigoplus_{p,q}^{\dim X} H^q(X, \Omega_X^p)$ to the functional $\int_X - \wedge \eta$, i.e. it is given by the usual Poincaré pairing.

Using the proposition above we obtain

Proposition 3.1.3. Let $X \xrightarrow{f} Y$ be a morphism of smooth proper schemes. Then the induced morphism of traces

(3)
$$\bigoplus_{p=0}^{\dim X} H^p(X, \Omega_X^p) \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(Y)}) \simeq \bigoplus_{p=0}^{\dim Y} H^p(Y, \Omega_Y^p)$$

coincides with the pushforward in cohomology (the Poincaré dual of the pullback map), where the first and the last equivalences are obtained from 3.0.3 using the Serre orientation 3.1.1.

Proof. By Corollary 2.2.7 the morphism of traces

$$\left(\bigoplus_{p=0}^{\dim X} H^p(X, \Omega_X^p)\right)^\vee \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(f_*)} \left(\bigoplus_{p=0}^{\dim Y} H^p(Y, \Omega_Y^p)\right)^\vee$$

is dual to the pullback in cohomology. It follows from the Proposition 3.1.2 that the composition (3) first takes Poincaré dual of the form, then pushes it forward in homology and then again takes the Poincaré dual. $\square$

3.2. Canonical orientation

Another important CY-structure is given by the

Construction 3.2.1 (Canonical orientation)**.** Let X be a smooth proper scheme. Given an endomorphism $X \xrightarrow{g} X$ the *derived fixed-points of* g defined as the pullback

$$\begin{tikzcd} X^g \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "(\mathrm{Id}_X, g)"] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

admits an orientation which is given by the series of equivalences

$$\mathcal{O}_{X^g} \simeq i^*\omega_X \otimes i^*\omega_X^{-1} \simeq i^*\omega_X \otimes i^*\omega_{X/X\times X} \simeq i^*\omega_X \otimes \omega_{X^g/X} \simeq i^!\omega_X \simeq \omega_{X^g}.$$

We will further call this orientation on X^g *canonical.*

To see why the canonical orientation is relevant, we prove the following

Proposition 3.2.2. For a classical smooth scheme X the morphism of traces

$$\Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \xrightarrow[\sim]{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_X)} \Gamma(\mathcal{L}X, \omega_{\mathcal{L}X})$$

induced by the diagram

$$\begin{tikzcd}
\mathrm{QCoh}(X) \arrow[r, "\mathrm{Id}_{\mathrm{QCoh}(X)}"] \arrow[d, "-\otimes\mathcal{O}_X"'] & \mathrm{QCoh}(X) \arrow[d, "-\otimes\mathcal{O}_X"] \\
\mathrm{ICoh}(X) \arrow[r, "\mathrm{Id}_{\mathrm{ICoh}(X)}"'] & \mathrm{ICoh}(X)
\end{tikzcd}$$

is obtained by applying the global sections functor $\Gamma(\mathcal{L}X, -)$ to the canonical orientation $\mathcal{O}_{\mathcal{L}X} \xrightarrow[\sim]{\mathrm{u}_C} \omega_{\mathcal{L}X}$ on $\mathcal{L}X$.

Proof. The proof essentially boils down to coherence of various operations, see Theorem B.2.7 in appendix for a full proof. □

3.3. Group orientations

The previous subsection raises the question of how one can understand the canonical orientation more explicitly. Our goal now is to show how one can obtain Serre and canonical orientations on $\mathcal{L}X$ using various formal group structures on $\mathcal{L}X$. To begin, we first need to fix some concrete way how we trivialize the tangent sheaf to a group:

Construction 3.3.1. Let $\widehat{G} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$ be a formal group over smooth proper scheme X such that the corresponding Lie algebra $\mathfrak{g} := \mathrm{Lie}_X(\widehat{G})$ lies in $\mathrm{Coh}(X)^{<0}$. Consider the pullback diagram

$$\begin{tikzcd}
\widehat{G} \arrow[r, "i"] \arrow[d, shift left, "i"] & X \arrow[d] \\
X \arrow[u, bend left, "e"] \arrow[r] & \widehat{B}_{/X}\widehat{G}
\end{tikzcd}$$

where $\widehat{B}_{/X}\widehat{G}$ is the completion along X of $B_{/X}\widehat{G}$. Since the relative tangent sheaf by its universal property is stable under pullbacks we get a trivialization

$$\mathbb{T}_{\widehat{G}/X} \simeq i^*\mathbb{T}_{X/\widehat{B}_{/X}\widehat{G}} \simeq i^*e^*i^*\mathbb{T}_{X/\widehat{B}_{/X}\widehat{G}} \simeq i^*e^*\mathbb{T}_{\widehat{G}/X} \simeq i^*\mathfrak{g}.$$

Remark 3.3.2. Intuitively, the trivialization $\mathbb{T}_{\widehat{G}/X} \simeq i^*\mathfrak{g}$ above is given by the left-invariant vector fields.

Now trivialization of the tangent sheaf to a group sometimes allows to construct orientations. To explain how, recall that for an eventually coconnective almost of finite type morphism of derived schemes $Z \xrightarrow{f} W$, there is a natural equivalence $f^! - \simeq \omega_f \otimes f^* -$ of functors, and, moreover, if the morphism f is a regular embedding one can explicitly identify ω_f with the shifted determinant of the normal bundle of f (see e.g. [9, Corollary 7.2.5.] and [11, Chapter 9, Section 7] respectively). We will need not only the existence of the equivalences from *loc. cit.* but also their constructions, so we review here relevant parts of the theory in QCoh-language. We start with the following general

Proposition 3.3.3. Let $\mathcal{C} = \mathrm{QCoh}(X)$ and let $V \in \mathcal{C}$ be a dualizable object such that $\mathrm{Sym}^{d+1}_{\mathcal{C}}(V) \simeq 0$ and $\mathrm{Sym}^d_{\mathcal{C}}(V) \not\simeq 0$ for some $d \geq 0$. Then:

1. We have $\mathrm{Sym}^{>d}_{\mathcal{C}}(V) \simeq 0$ and $\mathrm{Sym}_{\mathcal{C}}(V) \in \mathcal{C}$ is dualizable.
2. The top symmetric power $\mathrm{Sym}^d(V)$ is an invertible object of $\mathcal{C}$ with the inverse equivalent to $\mathrm{Sym}^d_{\mathcal{C}}(V^\vee)$.
3. The multiplication map followed by the projection on the top summand

$$\mathrm{Sym}_{\mathcal{C}}(V) \otimes \mathrm{Sym}_{\mathcal{C}}(V) \longrightarrow \mathrm{Sym}_{\mathcal{C}}(V) \longrightarrow \mathrm{Sym}^d_{\mathcal{C}}(V)$$

is a perfect pairing, i.e. the induced map $\mathrm{Sym}_{\mathcal{C}}(V) \otimes \mathrm{Sym}^d_{\mathcal{C}}(V)^{-1} \to \mathrm{Sym}_{\mathcal{C}}(V)^\vee$ is an equivalence.

Proof. 1. Assume that $\mathrm{Sym}^n_{\mathcal{C}}(V) \simeq 0$ for some $n \in \mathbb{Z}_{>0}$. Since we are in characteristic zero we have an equivalence $\mathrm{Sym}^n_{\mathcal{C}}(V) \simeq (V^{\otimes n})^{\Sigma_n}$ and the natural map $(V^{\otimes n+1})^{\Sigma_{n+1}} \longrightarrow (V^{\otimes n})^{\Sigma_n}$ admits a section $\mathrm{Nm}^{\Sigma_{n+1}}_{\Sigma_n}/[\Sigma_{n+1} : \Sigma_n]$. Hence $\mathrm{Sym}^{n+1}_{\mathcal{C}}(V)$ is a direct summand in $(V^{\otimes n+1})^{\Sigma_n} \simeq \mathrm{Sym}^n_{\mathcal{C}}(V) \otimes V \simeq 0$, so $\mathrm{Sym}^{n+1}_{\mathcal{C}}(V) \simeq 0$. Finally, using characteristic zero assumption again we deduce that $\mathrm{Sym}^k_{\mathcal{C}}$ is a colimit over a finite diagram and hence $\mathrm{Sym}_{\mathcal{C}}(V) \simeq \bigoplus_{k=0}^d \mathrm{Sym}^k_{\mathcal{C}}(V)$ is dualizable as a finite colimit of dualizable objects.

2. Let us show that the evaluation map $e : \mathrm{Sym}^d_{\mathcal{C}}(V) \otimes \mathrm{Sym}^d_{\mathcal{C}}(V^\vee) \longrightarrow I_{\mathcal{C}}$ is an equivalence. Since $\mathcal{C} = \mathrm{QCoh}(X)$ is a limit of module categories, it is enough to assume $\mathcal{C} \simeq \mathrm{Mod}_R$ for some connective k-algebra R. Moreover, by derived Nakayama's lemma we can assume R is discrete. Further, by the usual Nakayama's lemma it is enough to prove the statement for all residue fields of R, in which case the statement is clear.
3. Similar to the previous point. □

Let now $A \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra object in a presentably symmetric monoidal k-linear category $\mathcal{C}$. Note that the forgetful functor $\mathrm{Mod}_A(\mathcal{C}) \xrightarrow{i} \mathcal{C}$ admits both left i^* and right $i^!$ adjoints explicitly given by

$$i^*(\mathcal{F}) \simeq A \otimes \mathcal{F} \qquad\qquad i^!(\mathcal{F}) \simeq \mathcal{H}om_{\mathcal{C}}(A, \mathcal{F}),$$

where $\mathcal{H}om_{\mathcal{C}}(-,-)$ is the inner hom in $\mathcal{C}$. We then have

Corollary 3.3.4 (Grothendieck's formula)**.** In the notations above let $A := \mathrm{Sym}_{\mathcal{C}}(V)$ where $V \in \mathcal{C} = \mathrm{QCoh}(X)$ is a dualizable object such that moreover $\mathrm{Sym}^{d+1}_{\mathcal{C}}(V) \simeq 0$ and $\mathrm{Sym}^d_{\mathcal{C}}(V) \not\simeq 0$ for some $d \geq 0$. Then for any $\mathcal{F} \in \mathcal{C}$ there is a natural equivalence

$$i^!(\mathcal{F}) \simeq \omega \otimes_A i^*(\mathcal{F})$$

in $\mathrm{Mod}_A(\mathcal{C})$, where $\omega := A \otimes \mathrm{Sym}^d(V[1])^{-1}$.

Proof. By the previous proposition A is dualizable and $A^\vee \simeq \omega$. Hence

$$i^!(\mathcal{F}) \simeq \mathcal{H}om_{\mathcal{C}}(A, \mathcal{F}) \simeq A^\vee \otimes \mathcal{F} \simeq A^\vee \otimes_A (A \otimes \mathcal{F}) \simeq \omega \otimes_A i^*(\mathcal{F})$$

as claimed. □

By applying The Grothendieck's formula to the case $\mathcal{F} = \mathcal{O}_X$, $V = \Omega_X[1]$ we thus obtain an equivalence

$$T^{ab}\colon \omega_{\mathbb{V}(\mathbb{T}_X[-1])/X} \simeq i^* \left(\mathrm{Sym}^{\mathrm{top}}(\Omega_X[1])^{-1}\right) \simeq$$

$$\simeq \mathrm{Sym}^{\mathrm{top}}\left(i^*(\mathbb{T}_X[-1])\right) \simeq \mathrm{Sym}^{\mathrm{top}}(\mathbb{T}_{\mathbb{V}(\mathbb{T}_X[-1])/X}),$$

where in the last equivalence we use the abelian group structure on $\mathbb{V}(\mathbb{T}_X[-1])$ to identify $i^*(\mathbb{T}_X[-1]) \simeq \mathbb{T}_{\mathbb{V}(\mathbb{T}_X[-1])/X}$. Consequently, given any other trivialization of the relative tangent bundle $\alpha\colon \mathbb{T}_{\mathbb{V}(\mathbb{T}_X[-1])/X} \simeq i^*(\mathbb{T}_X[-1])$ we can precompose $\mathrm{Sym}^{\mathrm{top}}(\alpha)$ with T^{ab} to obtain an equivalence $\omega_{\mathbb{V}(\mathbb{T}_X[-1])/X} \simeq i^*\omega_X^{-1}$. This suggests the following

Construction 3.3.5 (Loop group and abelian orientations)**.** Let X be a smooth proper scheme. For any group structure on $\mathbb{V}(\mathbb{T}_X[-1])$ by Construction 3.3.1 we obtain a trivialization of the relative tangent sheaf $\mathbb{T}_{\mathbb{V}(\mathbb{T}_X[-1])/X}$ and hence by discussion above an equivalence $\omega_{\mathbb{V}(\mathbb{T}_X[-1])/X} \simeq i^*\omega_X^{-1}$. By multiplying both sides with $i^*\omega_X$ we obtain an orientation

$$\omega_{\mathbb{V}(\mathbb{T}_X[-1])} \simeq \omega_{\mathbb{V}(\mathbb{T}_X[-1])/X} \otimes i^*\omega_X \simeq i^*\omega_X^{-1} \otimes i^*\omega_X \simeq \mathcal{O}_{\mathbb{V}(\mathbb{T}_X[-1])}.$$

If group structure on $\mathbb{V}(\mathbb{T}_X[-1])$ is pulled back from $\mathcal{L}X$ via the exponent map $\exp_{\mathcal{L}X}$, we will call the orientation above *loop group orientation*. In the case when group structure on $\mathbb{V}(\mathbb{T}_X[-1])$ is abelian, we will call the corresponding orientation *abelian orientation*.

The orientations on $\mathbb{V}(\mathbb{T}_X[-1])$ from above are in fact not new ones:

Proposition 3.3.6. Let X be smooth proper scheme. Then:

1. The abelian group orientation on $\mathcal{L}X$ coincides with the Serre orientation from construction 3.1.1.
2. The loop group orientation on $\mathcal{L}X$ coincides with the canonical structure from construction 3.2.1.

Proof. 1. Since $\mathbb{V}(\mathbb{T}_X[-1])$ is affine over X, the space of orientations $\mathcal{O}_{\mathbb{V}(\mathbb{T}_X[-1])} \simeq \omega_{\mathbb{V}(\mathbb{T}_X[-1])}$ on $\mathbb{V}(\mathbb{T}_X[-1])$ is equivalent to the space of $i_*\mathcal{O}_{\mathbb{V}(\mathbb{T}_X[-1])}$-linear equivalences $i_*\mathcal{O}_{\mathbb{V}(\mathbb{T}_X[-1])} \simeq i_*\omega_{\mathbb{V}(\mathbb{T}_X[-1])}$ in $\mathrm{QCoh}(X)$. Also note that

$$i_*\mathcal{O}_{\mathbb{V}(\mathbb{T}_X[-1])} \simeq \mathrm{Sym}_{\mathrm{QCoh}(X)}(\Omega_X[1])$$

and

$$i_*\omega_{\mathbb{V}(\mathbb{T}_X[-1])} \simeq \mathcal{H}om_{\mathrm{QCoh}(X)}\big(\mathrm{Sym}(\Omega_X[1]), \omega_X\big).$$

In particular, unwinding definitions one finds that the Serre orientation is induced by the non-degenerate pairing given by the multiplication

$$\mathrm{Sym}_{\mathrm{QCoh}(X)}(\Omega_X[1]) \otimes \mathrm{Sym}_{\mathrm{QCoh}(X)}(\Omega_X[1]) \longrightarrow \mathrm{Sym}_{\mathrm{QCoh}(X)}(\Omega_X[1])$$

followed by the projection to the top summand

$$\mathrm{Sym}_{\mathrm{QCoh}(X)}(\Omega_X[1]) \longrightarrow \mathrm{Sym}^{\mathrm{top}}_{\mathrm{QCoh}(X)}(\Omega_X[1]) \simeq \omega_X.$$

But this is precisely the same pairing which we used to construct an equivalence $\omega_{\mathbb{V}(\mathbb{T}_X[-1])/X} \simeq \mathrm{Sym}^{\mathrm{top}}\left(i^*(\mathbb{T}_X[-1])\right)$ (see Proposition 3.3.4).

2. Consider more generally arbitrary formal group structure on $\mathbb{V}(\mathbb{T}_X[-1])$ induced by a pullback diagram

$$\begin{tikzcd} \mathbb{V}(\mathbb{T}_X[-1]) \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d] \\ X \arrow[r] & \widehat{B} \end{tikzcd}$$

for some $\widehat{B} \in \widehat{\mathrm{Moduli}}_{X//X}$. This diagram induces an equivalence

$$C\colon \omega_{\mathbb{V}(\mathbb{T}_X[-1])/X} \simeq i^*(\omega_{X/\widehat{B}})$$

of relative dualizing sheaves. Since $\omega_{X/\widehat{B}} \simeq \omega_X^{-1}$, by multiplying C with $i^*\omega_X$ we obtain an orientation $\mathrm{u}_{\widehat{B}}$ on $\mathbb{V}(\mathbb{T}_X[-1])$. As an example, if we consider $B = (X \times X)_{\widehat{\Delta}}$ the orientation obtained this way by definition coincides with the canonical one. Let now $\alpha\colon \mathbb{T}_{\mathbb{V}(\mathbb{T}_X[-1])/X} \simeq i^*(\mathbb{T}_X[-1])$ be the trivialization of the tangent sheaf obtained from the pullback diagram above. Unwinding the definitions, one finds that the composite equivalence

$$\omega_{\mathbb{V}(\mathbb{T}_X[-1])/X} \xrightarrow[\sim]{T^{ab}} \mathrm{Sym}^{\mathrm{top}}(\mathbb{T}_{\mathbb{V}(\mathbb{T}_X[-1])/X}) \xrightarrow[\sim]{\mathrm{Sym}^{\mathrm{top}}(\alpha)} \mathrm{Sym}^{\mathrm{top}}(i^*\mathbb{T}_X[-1]) \xrightarrow[\sim]{} i^*(\omega_X^{-1})$$

coincides with C. But by definition this equivalence tensored by $i^*(\omega_X)$ is the group orientation and $C \otimes i^*(\omega_X) = \mathrm{u}_{\widehat{B}}$, hence the group orientation coincides with $\mathrm{u}_{\widehat{B}}$ as claimed. $\square$

4. The Todd class

From Proposition 3.2.2 and Proposition 3.3.6 we know that the morphism of traces

$$\Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes \mathcal{O}_X)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \simeq \Gamma(\mathcal{L}X, \omega_{\mathcal{L}X})$$

is given by $\Gamma(\mathcal{L}X, \mathrm{u}_C)$, where u_C is the canonical orientation from Construction 3.2.1. In Proposition 4.1.3 we will prove that the composite equivalence

$$\mathcal{O}_{\mathcal{L}X} \xrightarrow[\sim]{\mathrm{u}_C} \omega_{\mathcal{L}X} \xrightarrow[\sim]{\mathrm{u}_S^{-1}} \mathcal{O}_{\mathcal{L}X}$$

is given by the determinant (we refer the reader to [21, Section 3.1] for the construction of the determinant map $\det : \mathrm{Perf}(X) \longrightarrow \mathrm{Pic}(X)$) of the derivative of the exponential map

$$d\exp_{\mathcal{L}X} : i^*\mathbb{T}_X[-1] \simeq \mathbb{T}_{\mathbb{V}(\mathbb{T}_X[-1])/X} \simeq \exp_{\mathcal{L}X}^* \mathbb{T}_{\mathcal{L}X/X} \simeq i^*\mathbb{T}_X[-1],$$

where the first equivalence above is via abelian group structure on $\mathbb{V}(\mathbb{T}_X[-1])$, and the second one uses exponent and loop group structure on $\mathcal{L}X$. In this and next sections we will prove that the determinant of the morphism above coincides with the classical Todd class td_X which is defined as a multiplicative characteristic class. Our proof is motivated by the following

Example 4.0.1. Let G be a real Lie group with the corresponding Lie algebra $\mathfrak{g}$. In a small enough neighborhood of 0 we then have two trivializations of $\mathbb{T}_G$ induced by the group structure on G and abelian group structure on $\mathfrak{g}$ (via the exponential map $\mathfrak{g} \xrightarrow{\exp_G} G$). One can then compute (see Lemma 5.3.3 for the proof in the formal power series setting) that for $x \in \mathfrak{g}$ close enough to 0 and $e(x) := \exp_G(x)$ the change of trivialization isomorphism

$$\mathfrak{g} \simeq \mathbb{T}_{\mathfrak{g},0} \xrightarrow[\sim]{+x} \mathbb{T}_{\mathfrak{g},x} \xrightarrow[\sim]{(d\exp_G)_x} \mathbb{T}_{G,e(x)} \xrightarrow[\sim]{(dL_{e(x)^{-1}})_{e(x)}} \mathbb{T}_{\mathfrak{g},0} \simeq \mathfrak{g}\ ,$$

(where for $g \in G$ we denote by $G \xrightarrow{L_g} G$ the left translations maps by g) is given by the linear operator $(1 - e^{-\mathrm{ad}_{\mathfrak{g}}(x)})/\,\mathrm{ad}_{\mathfrak{g}}(x)$. Note that in this way we obtain an $\mathcal{E}nd(\mathfrak{g})$-valued function on $\mathfrak{g}$

$$x \longmapsto \frac{1 - e^{-\mathrm{ad}_{\mathfrak{g}}(x)}}{\mathrm{ad}_{\mathfrak{g}}(x)}.$$

In this section we will imitate the example above:

- Given a map $\mathfrak{g} \xrightarrow{\rho} \mathcal{E}nd_{\mathcal{C}}(E)$ in a k-linear presentably symmetric monoidal category $\mathcal{C}$ and a power series $f \in k[[t]]$ we will construct an $\mathcal{E}nd_{\mathcal{C}}(E)$-valued "formal function on $\mathfrak{g}$" (which is by definition simply a map $\mathrm{Sym}_{\mathcal{C}}(\mathfrak{g}) \xrightarrow{f(\rho)} \mathcal{E}nd_{\mathcal{C}}(E)$ in $\mathcal{C}$) which informally sends an element $x \in \mathfrak{g}$ to $f(\rho(x))$. In the special case $\mathcal{C} = \mathrm{QCoh}(X)$ and $\mathfrak{g} = \mathbb{T}_X[-1]$ we will give an interpretation of multiplicative characteristic classes in these terms.

- Using the interpretation from the previous step the problem of comparing Todd classes reduces to proving that

$$\det\left(\frac{1-e^{-\operatorname{ad}_{\mathbb{T}_X[-1]}}}{\operatorname{ad}_{\mathbb{T}_X[-1]}}\right) = \det(d\exp_{\mathcal{L}X}),$$

  where the left hand side is obtained by applying determinant to the formal function constructed from $f(t) = (1-e^{-t})/t$ and $\rho = \operatorname{ad}_{\mathbb{T}_X[-1]}$. In order to prove this, we show that both sides make sense for any Lie algebra $\mathfrak{g} \in \operatorname{LAlg}(\mathcal{C})$. Moreover, since both sides are functorial with respect to continuous symmetric monoidal functors the equivalence above can be checked in the classifying category for Lie algebras $\mathcal{U}_{\mathrm{Lie}}$ (see Construction 5.2.1). We will show that $\mathcal{U}_{\mathrm{Lie}}$ admits a set of functors to Vect_k which detects non-zero morphisms, hence reducing the problem to ordinary $\mathfrak{gl}_n$ in Vect_k for which the statement is well-known.

4.1. Group-theoretic Todd class

We return our discussion to the trivialization of tangent bundle to a group: note that if a formal moduli problem over X has two different structures of a formal group, Construction 3.3.1 gives two a priori different trivializations of the tangent sheaf. In order to conveniently measure the difference between these trivializations we introduce the following

Construction 4.1.1. Let $\mathcal{Y} \in (\widehat{\operatorname{Moduli}}_{/X})_*$ be a pointed formal moduli problem over smooth proper scheme X such that the pullback of the tangent sheaf $(\mathbb{T}_{\mathcal{Y}/X})_{|X} \in \operatorname{QCoh}(X)$ is perfect (so that $\mathbb{T}_{\mathcal{Y}/X} \in \operatorname{QCoh}(\mathcal{Y})$ is itself perfect) and let s_1, s_2 be two formal group structures on $\mathcal{Y}$. Then by Construction 3.3.1 we get two trivialization of the tangent sheaf $\mathbb{T}_{\mathcal{Y}/X}$ and thus by applying the first trivialization and then inverse of the second trivialization we get an automorphism $\gamma_{s_1,s_2} : \mathbb{T}_{\mathcal{Y}/X} \simeq \mathbb{T}_{\mathcal{Y}/X}$ in $\operatorname{QCoh}(\mathcal{Y})$. We define the *group-theoretic Todd class of* $\mathcal{Y}$ denoted by $\operatorname{td}_{\mathcal{Y},s_1,s_2} \in \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})^\times$ as the determinant $\operatorname{td}_{\mathcal{Y},s_1,s_2} := \det(\gamma_{s_1,s_2})$.

Construction 4.1.2. In the case when $\widehat{G}$ is a formal group over X it has two distinguished group structures: the abelian one s_1 coming from the exponent map and the given one s_2. In this case we will further use the notation

$$d\exp_{\widehat{G}} := \gamma_{s_1,s_2} \qquad \text{and} \qquad \operatorname{td}_{\widehat{G}} := \det(d\exp_{\widehat{G}}) \in \Gamma(\widehat{G}, \mathcal{O}_{\widehat{G}})^\times$$

for the corresponding automorphism and its determinant.

Note that from Proposition 3.3.6 we get

Proposition 4.1.3. Let $\mathcal{O}_{\mathcal{L}X} \xrightarrow[\sim]{\mathrm{u}_S} \omega_{\mathcal{L}X}$ be the Serre orientation (construction 3.1.1) and $\mathcal{O}_{\mathcal{L}X} \xrightarrow[\sim]{\mathrm{u}_C} \omega_{\mathcal{L}X}$ be the canonical orientation (construction 3.2.1). Then the composite automorphism

$$\mathcal{O}_{\mathcal{L}X} \xrightarrow[\sim]{\mathrm{u}_C} \omega_{\mathcal{L}X} \xrightarrow[\sim]{\mathrm{u}_S^{-1}} \mathcal{O}_{\mathcal{L}X}$$

is given by $\mathrm{td}_{\mathcal{L}X}$, where we consider $\mathcal{L}X$ as a formal group over X using the derived loops group structure.

Proof. By construction and Proposition 3.3.6 under the equivalence

$$\mathrm{End}_{\mathrm{QCoh}(\mathcal{L}X)}(\mathcal{O}_{\mathcal{L}X}) \simeq \mathrm{End}_{\mathrm{QCoh}(\mathcal{L}X)}(i^*\omega_X^{-1}) \simeq \mathrm{End}_{\mathrm{QCoh}(\mathcal{L}X)}\left(\mathrm{Sym}^{\mathrm{top}}(i^*\mathbb{T}_X[-1])\right)$$

the composite $\mathrm{u}_C^{-1} \circ \mathrm{u}_S$ is given by the induced map on $\mathrm{Sym}^{\mathrm{top}}$ applied to the composite autoequivalence

$$d\exp_{\mathcal{L}X} \colon\; i^*\mathbb{T}_X[-1] \xrightarrow[\sim]{} \mathbb{T}_{\mathcal{L}X/X} \xrightarrow[\sim]{} i^*\mathbb{T}_X[-1]\;,$$

which under the equivalence $\mathrm{Sym}^d(V[-1]) \simeq \Lambda^d(V)[-d]$ is given by the map $\Lambda^{\mathrm{top}}(d\exp_{\mathcal{L}X}[1])$. On the other hand

$$\mathrm{td}_{\mathcal{L}X} = \det(d\exp_{\mathcal{L}X}) = \det(d\exp_{\mathcal{L}X}[1])^{-1} = \Lambda^{\mathrm{top}}(d\exp_{\mathcal{L}X}[1])^{-1}.$$

It follows $\mathrm{td}_{\mathcal{L}X} = (\mathrm{u}_C^{-1} \circ \mathrm{u}_S)^{-1} = \mathrm{u}_S^{-1} \circ \mathrm{u}_C$. $\square$

Thus using Propositions 3.2.2 and 3.1.3 we conclude

Corollary 4.1.4. Let X be a smooth proper scheme. Then under Serre orientation equivalence u_S^{-1} the morphism of traces

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes \mathcal{O}_X)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)})$$

$$\Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes \mathcal{O}_X)} \Gamma(\mathcal{L}X, \omega_{\mathcal{L}X}) \overset{\mathrm{u}_S^{-1}}{\simeq} \Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X})$$

is given by multiplication with $\mathrm{td}_{\mathcal{L}X}$.

4.2. Formal functions

In this subsection we will introduce a construction of formal functions mentioned in the introduction to this section. We first recall the following

Construction 4.2.1. Let $\mathcal{C}$ be a presentably symmetric monoidal category and let $V \in \mathcal{C}$ be an object of $\mathcal{C}$. We define *symmetric algebra with divided powers* $\mathrm{Sym}_{\mathrm{dp},\mathcal{C}}(V)$ *on* V as

$$\mathrm{Sym}_{\mathrm{dp},\mathcal{C}}(V) := \bigoplus_{n=0}^{\infty} (V^{\otimes n})^{\Sigma_n}.$$

Note that there exists a canonical map $\mathrm{Sym}_{\mathrm{dp},\mathcal{C}}(V) \longrightarrow \mathrm{Free}_{\mathcal{C}}^{\mathbb{E}_1}(V)$ where $\mathrm{Free}_{\mathcal{C}}^{\mathbb{E}_1}$ denotes the free associative algebra functor in $\mathcal{C}$.

Remark 4.2.2. There is also a canonical norm map $\mathrm{Sym}_{\mathcal{C}}(V) \longrightarrow \mathrm{Sym}_{\mathrm{dp},\mathcal{C}}(V)$ which is an equivalence in characteristic zero.

Construction 4.2.3. Let $\mathcal{C}$ be a k-linear presentably symmetric monoidal category and $\mathfrak{g} \in \mathcal{C}$ be an object together with a map $\mathfrak{g} \xrightarrow{\rho} \mathcal{E}nd_{\mathcal{C}}(E)$. Also let

$$f(t) = \sum_{n=0}^{\infty} f_n t^n \in k[[t]]$$

be a power series with $f_0 = 1$. Define then an "$\mathcal{E}nd_{\mathcal{C}}(E)$-valued formal function $f(\rho)$ on $\mathfrak{g}$" as the composite

$$f(\rho)\colon \mathrm{Sym}_{\mathrm{dp},\mathcal{C}}(\mathfrak{g}) \xrightarrow{\mathrm{Sym}_{\mathrm{dp}}(\rho)} \mathrm{Sym}_{\mathrm{dp},\mathcal{C}}(\mathcal{E}nd_{\mathcal{C}}(E)) \longrightarrow \mathrm{Free}_{\mathcal{C}}^{\mathbb{E}_1}(\mathcal{E}nd_{\mathcal{C}}(E)) \xrightarrow{f} \mathcal{E}nd_{\mathcal{C}}(E)$$

where the map $\mathrm{Free}_{\mathcal{C}}^{\mathbb{E}_1}(\mathcal{E}nd_{\mathcal{C}}(E)) \xrightarrow{f} \mathcal{E}nd_{\mathcal{C}}(E)$ is defined on the n-th component as the composition map $\circ : \mathcal{E}nd_{\mathcal{C}}(E)^{\otimes n} \longrightarrow \mathcal{E}nd_{\mathcal{C}}(E)$ followed by the multiplication by the coefficient f_n.

Variant 4.2.4. By adjunction

$$\mathrm{Hom}_{\mathcal{C}}\big(\mathrm{Sym}_{\mathrm{dp},\mathcal{C}}(\mathfrak{g}), \mathcal{E}nd_{\mathcal{C}}(E)\big) \simeq \mathrm{Hom}_{\mathcal{C}}\big(\mathrm{Sym}_{\mathrm{dp},\mathcal{C}}(\mathfrak{g}) \otimes E, E\big)$$

the morphism $f(\rho)$ corresponds to some map $\mathrm{Sym}_{\mathrm{dp},\mathcal{C}}(\mathfrak{g}) \otimes E \longrightarrow E$ which we by abuse of notations will also denote by $f(\rho)$.

Example 4.2.5. Let $\mathfrak{g}$ be a Lie algebra in Vect_k and let $\mathfrak{g} \xrightarrow{\rho} \mathcal{E}nd_{\mathrm{Vect}_k}(E)$ be a representation of $\mathfrak{g}$. Then by definition for any power series $f(t) \in k[[t]]$ as above and $x \in \mathfrak{g}$ we have $f(\rho)(x^{\otimes n}) = f_n \rho(x)^{\circ n}$.

We now give a more geometric description of $f(\rho)$ in the case $\mathcal{C} = \mathrm{QCoh}(X)$.

Construction 4.2.6. Let $\mathfrak{g}$ be a Lie algebra such that $\mathfrak{g} \in \mathrm{Coh}(X)^{<0}$ (where X is smooth proper scheme) and let $\mathfrak{g} \xrightarrow{\rho} \mathcal{E}nd_{\mathrm{QCoh}(X)}(E)$ be some representation where $E \in \mathrm{QCoh}(X)^{\mathrm{perf}}$. Then the composite

$$\mathrm{Sym}_{\mathrm{QCoh}(X)}(\mathfrak{g}) \xrightarrow[\sim]{} \mathrm{Sym}_{\mathrm{dp},\mathrm{QCoh}(X)}(\mathfrak{g}) \xrightarrow{\rho(g)} \mathcal{E}nd_{\mathrm{QCoh}(X)}(E)$$

induces a map

$$\mathcal{O}_X \longrightarrow \mathrm{Sym}_{\mathrm{QCoh}(X)}(\mathfrak{g}^\vee) \otimes \mathcal{E}nd_{\mathrm{QCoh}(X)}(E)$$

and hence an element in

$$\mathrm{Hom}_{\mathrm{QCoh}(X)}\big(\mathcal{O}_X, \mathrm{Sym}_{\mathrm{QCoh}(X)}(\mathfrak{g}^\vee) \otimes \mathcal{E}nd_{\mathrm{QCoh}(X)}(E)\big) \simeq$$

$$\simeq \Gamma\big(X, j_* j^* \mathcal{E}nd_{\mathrm{QCoh}(X)}(E)\big) \simeq \Gamma\big(\mathbb{V}(\mathfrak{g}), \mathcal{E}nd_{\mathrm{QCoh}(\mathbb{V}(\mathfrak{g}))}(j^*E)\big) \simeq$$

$$\simeq \mathrm{End}_{\mathrm{QCoh}(\mathbb{V}(\mathfrak{g}))}(j^*E),$$

which we will by abuse of notation denote by the same symbol $f(\rho) \in \mathrm{Aut}_{\mathrm{QCoh}(\mathbb{V}(\mathfrak{g}))}(j^*E)$ (where $\mathbb{V}(\mathfrak{g}) \xrightarrow{j} X$ is the projection map and $f(\rho)$ is invertible since $f_0 \neq 0$ and $\mathbb{V}(\mathfrak{g})$ is a nil-thickening of X). Moreover, since pullbacks preserve perfect objects, the sheaf $j^*E \in \mathrm{QCoh}(\mathbb{V}(\mathfrak{g}))$ is also perfect. Consequently, we can take the determinant of the automorphism $f(\rho)$ above to obtain an element

$$c^f_{\mathfrak{g}}(E) := \det\big(f(\rho)\big) \in \mathrm{Aut}_{\mathrm{QCoh}(\mathbb{V}(\mathfrak{g}))}(\mathcal{O}_{\mathbb{V}(\mathfrak{g})}) \simeq \Gamma(\mathbb{V}(\mathfrak{g}), \mathcal{O}_{\mathbb{V}(\mathfrak{g})})^\times.$$

4.3. $\mathbf{td}_{\mathcal{L}X}$ and multiplicative characteristic classes

We now show that the construction above is closely related to the theory of multiplicative characteristic classes. Namely, recall the following

Definition 4.3.1. For a power series $f \in 1 + tk[[t]]$ define a multiplicative characteristic class

$$K_0(X) \xrightarrow{c^f} \bigoplus_{p=0}^{\dim X} H^p(X, \Omega_X^p)$$

by setting it to be $f(c_1(\mathcal{M}))$ on line bundles and extending to all vector bundles by multiplicativity and the splitting principle.

We now prove the following

Proposition 4.3.2. Let X be a smooth algebraic variety, $E \in \mathrm{QCoh}(X)$ be a perfect sheaf considered as a $\mathbb{T}_X[-1] \simeq \mathrm{Lie}_X(\mathcal{L}X)$-module via the canonical $\mathcal{L}X$-equivariant structure from 1.2.5. Then for any power series f as above the determinant

$$c^f_{\mathbb{T}_X[-1]}(E) \in \Gamma\left(\mathbb{V}(\mathbb{T}_X[-1]), \mathcal{O}_{\mathbb{V}(\mathbb{T}_X[-1])}\right) \simeq \bigoplus_{p=0}^{\dim X} H^p(X, \Omega_X^p)$$

is equal to $c^f(E)$.

Proof. Let us denote the canonical action of $\mathbb{T}_X[-1]$ on E by a. Since both $c^f_{\mathbb{T}_X[-1]}(E) = \det(f(a))$ and $c^f(E)$ commute with pullbacks and map direct sums to products, by the splitting principle it is enough to prove the statement in the case when $E := \mathcal{M}$ is a line bundle. In this case via the equivalence $\widehat{\mathrm{GL}}(\mathcal{M}) \simeq \widehat{\mathbb{G}_m}$ the morphism $\det(f(a))$ corresponds to $f(a)$ and so it is left to show that the map

$$\mathcal{O}_X \xrightarrow{f(a)^\vee} \mathrm{Sym}_{\mathrm{QCoh}(X)}(\mathbb{T}_X[-1])^\vee \simeq \mathrm{Sym}_{\mathrm{QCoh}(X)}(\Omega_X[1]).$$

dual to the morphism $f(a)$

$$\mathrm{Sym}_{\mathrm{QCoh}(X)}(\mathbb{T}_X[-1]) \xrightarrow{\mathrm{Sym}_{\mathrm{QCoh}(X)}(a)} \mathrm{Sym}_{\mathrm{QCoh}(X)}(\mathcal{O}_X) \xrightarrow{\sim} \mathrm{Free}^{\mathbb{E}_1}(\mathcal{O}_X) \xrightarrow{f} \mathcal{O}_X$$

coincides with $c^f(\mathcal{M})$, where we use above that $\mathcal{E}nd_{\mathrm{QCoh}(X)}(\mathcal{M}) \simeq \mathcal{O}_X$ as $\mathcal{M}$ is a line bundle. Since by Proposition 1.4.7 the representation $\mathbb{T}_X[-1] \xrightarrow{a} \mathcal{M}$ classifies $\mathrm{At}(\mathcal{M}) \simeq c_1(\mathcal{M})$ (see Corollary 1.4.7) we have $a^\vee = c_1(\mathcal{M})$, and so unwinding the construction we find $f(a)^\vee = f(c_1(\mathcal{M}))$ which is by definition $c^f(\mathcal{M})$. $\square$

4.4. Comparison with the classical Todd class

Recall by Corollary 4.1.4 that the morphism of traces

$$\mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_X)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)})$$

$$(4) \qquad \Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \xrightarrow{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_X)} \Gamma(\mathcal{L}X, \omega_{\mathcal{L}X}) \overset{\mathrm{u}_S^{-1}}{\simeq} \Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X})$$

is closely related to $d\exp_{\mathcal{L}X}$. The following theorem provides a description of $d\exp_{\widehat{G}}$ for an arbitrary formal group $\widehat{G}$ over X

Theorem 4.4.1. Let $\widehat{G}$ be a formal group over X such that $\mathfrak{g} := \mathrm{Lie}_X(\widehat{G}) \in \mathrm{Coh}^{<0}$. Then

$$d\exp_{\widehat{G}} = \frac{1 - e^{-\operatorname{ad}_{\mathfrak{g}}}}{\operatorname{ad}_{\mathfrak{g}}}.$$

Remark 4.4.2. In the theorem above one can drop any assumptions on $\mathfrak{g}$ and smoothness assumption on X if instead of QCoh-version one considers $\mathfrak{g}$ as a Lie algebra in $\mathrm{ICoh}(X)$.

The proof of (a generalization of) this theorem is the content of the next section. Here we will only use this theorem to deduce that the group theoretic Todd class $\mathrm{td}_{\mathcal{L}X}$ (see Construction 4.1.2) coincides with the classical one and thus will give a concrete description of the morphism of traces (4). Namely, recall that the classical Todd class td_X of X is defined as

$$\mathrm{td}_X := c^{1/f}(\mathbb{T}_X) = c^f(\mathbb{T}_X[-1]) \qquad \text{where} \qquad f(t) = \frac{1-e^{-t}}{t}.$$

Corollary 4.4.3. Let X be a smooth proper scheme. Then $\mathrm{td}_{\mathcal{L}X} = \mathrm{td}_X$.

Proof. Note that by Proposition 4.3.2 above we have

$$\mathrm{td}_X = \det\left(\frac{1 - e^{-\operatorname{ad}_{\mathbb{T}_X[-1]}}}{\operatorname{ad}_{\mathbb{T}_X[-1]}}\right),$$

where $\operatorname{ad}_{\mathbb{T}_X[-1]}$ is the adjoint representation of $\mathbb{T}_X[-1]$. Consequently, since the group-theoretic Todd class $\mathrm{td}_{\mathcal{L}X}$ was defined as the determinant of $d\exp_{\mathcal{L}X}$ it is enough to prove that

$$d\exp_{\mathcal{L}X} = \frac{1 - e^{-\operatorname{ad}_{\mathbb{T}_X[-1]}}}{\operatorname{ad}_{\mathbb{T}_X[-1]}},$$

which is a special case of the Theorem 4.4.1 above. □

Finally, we can describe the morphism of traces (4) in the classical terms

Corollary 4.4.4. Let X be a smooth proper scheme. Then under the Serre orientation and HKR identifications the morphism of traces given by the composite of

$$\bigoplus_{p,q} H^{q,p}(X) \overset{\mathrm{HKR}}{\simeq} \pi_* \operatorname{\mathsf{Tr}}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)})$$

$$\pi_* \operatorname{\mathsf{Tr}}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{QCoh}(X)}) \xrightarrow{\operatorname{\mathsf{Tr}}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_X)} \pi_* \operatorname{\mathsf{Tr}}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)})$$

$$\pi_* \operatorname{\mathsf{Tr}}_{2\,\mathrm{Cat}_k}(\mathrm{Id}_{\mathrm{ICoh}(X)}) \overset{\mathrm{HKRou}_S^{-1}}{\simeq} \bigoplus_{p,q} H^{q,p}(X)$$

is given by multiplication with td_X.

Proof. By Corollary 4.1.4 we know that the morphism of traces is given by multiplication with the group theoretic Todd class $\mathrm{td}_{\mathcal{L}X}$, and by the previous Corollary $\mathrm{td}_{\mathcal{L}X} = \mathrm{td}_X$. □

5. Abstract exponential

The goal of this section is to prove Theorem 4.4.1 by first extending it to arbitrary Lie algebras in any stable presentably symmetric monoidal k-linear category and then by reducing this more general statement to the case of $\mathfrak{gl}_V$ for $V \in \mathrm{Vect}_k^{\heartsuit}$. More concretely, let $\widehat{G}$ be a formal group over X with the corresponding Lie algebra $\mathfrak{g}$. Note that the morphism $i^*\mathfrak{g} \xrightarrow{d\exp_{\widehat{G}}} i^*\mathfrak{g}$ via a series of adjunctions similar to that in Construction 4.2.6 corresponds to some morphism

$$\widetilde{d\exp}_{\widehat{G}} \colon \mathrm{Sym}_{\mathrm{QCoh}(X)}(\mathfrak{g}) \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

and the same is true for $(1 - e^{-\operatorname{ad}_{\mathfrak{g}}})/\operatorname{ad}_{\mathfrak{g}}$ by Variant 4.2.4. We will show in this section that these two morphisms are actually equal. Before proceeding to the proof, we will first discuss theory of tangent comodules of arbitrary cocommutative coalgebra (which plays the role of tangent space to a formal moduli problem) and review some generalities about operads.

5.1. Tangent comodule

In this subsection we describe the construction of relative tangent sheaf in an abstract setting. Informally, the construction will be dual to the cotangent complex formalism developed in [17, Section 7.3]. However, since in [17, Section 7.3] for most of the statements the category $\mathcal{C}$ is assumed to be presentable and we are rather interested in the case of $\mathcal{C}^{\mathsf{op}}$ (so that $\mathrm{CAlg}(\mathcal{C}^{\mathsf{op}}) = \mathrm{coCAlg}(\mathcal{C})^{\mathsf{op}}$ is the opposite to the category of cocommutative coalgebras in $\mathcal{C}$) we explain the construction here in full details. We start by introducing the following formal

Definition 5.1.1. Let $\mathcal{C}$ be a finitely cocomplete category (i.e. $\mathcal{C}$ admits all finite colimits). Define then a **costabilization of** $\mathcal{C}$ denoted by $\mathsf{coStab}(\mathcal{C})$ as

$$\mathsf{coStab}(\mathcal{C}) := \mathsf{Stab}(\mathcal{C}^{\mathsf{op}})^{\mathsf{op}}.$$

Remark 5.1.2. Notice that the category $\mathsf{coStab}(\mathcal{C})$ is always stable. Moreover, since by definition the costabilization $\mathsf{coStab}(\mathcal{C})$ of $\mathcal{C}$ can be concretely described as the limit of the diagram

$$\dots \xrightarrow{\Sigma} \mathcal{C}_{/\emptyset} \xrightarrow{\Sigma} \mathcal{C}_{/\emptyset} \xrightarrow{\Sigma} \mathcal{C}_{/\emptyset}$$

in Cat_∞, where $\emptyset \in \mathcal{C}$ is the initial object, we see that if the category $\mathcal{C}$ is presentable then so is $\mathsf{coStab}(\mathcal{C})$. In particular, in this case the evident projection functor

$$\Sigma^\infty_{\mathsf{co}} : \mathsf{coStab}(\mathcal{C}) = \mathsf{Stab}(\mathcal{C}^{\mathsf{op}})^{\mathsf{op}} \xrightarrow{(\Omega^\infty_{\mathcal{C}^{\mathsf{op}}})^{\mathsf{op}}} (\mathcal{C}^{\mathsf{op}})^{\mathsf{op}} \simeq \mathcal{C}$$

by the adjoint functor theorem admits a right adjoint which we will further denote by $\Omega^\infty_{\mathsf{co}}$.

Remark 5.1.3. Let $\mathcal{C}$ be a finitely cocomplete category and $A \xrightarrow{f} B$ be a morphism in $\mathcal{C}$. Then the induced functor $\mathcal{C}_{A/} \xrightarrow{-\sqcup_A B} \mathcal{C}_{B/}$ preserves colimits and consequently induces a colimit-preserving functor

$$\mathsf{coStab}(\mathcal{C}_{A/}) \xrightarrow{f_*} \mathsf{coStab}(\mathcal{C}_{B/}) .$$

Moreover, in the case $\mathcal{C}$ is presentable by the adjoint functor theorem we see that the functor f_* admits a right adjoint $f_* \dashv f^*$.

Example 5.1.4. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathsf{L,st}}_{\infty})$ be a stable presentably symmetric monoidal category and let $C \in \mathrm{coCAlg}(\mathcal{C})$ be a cocommutative coalgebra object in $\mathcal{C}$. We then get an equivalence

$$\mathsf{coStab}(\mathrm{coCAlg}(\mathcal{C})_{C/}) = \mathsf{Stab}\left((\mathrm{coCAlg}(\mathcal{C})_{C/})^{\mathsf{op}}\right)^{\mathsf{op}} =$$

$$= \mathsf{Stab}\left(\mathrm{CAlg}(\mathcal{C}^{\mathsf{op}})_{/C}\right)^{\mathsf{op}} \simeq \mathrm{Mod}_C(\mathcal{C}^{\mathsf{op}})^{\mathsf{op}} = \mathrm{coMod}_C(\mathcal{C})$$

where the middle equivalence follows from [17, Theorem 7.3.4.13].

Example 5.1.4 and Remark 5.1.2 above motivate the following

Definition 5.1.5. Let $\mathcal{C}$ be a presentable category and $A \in \mathcal{C}$ be an object. Define then a **tangent complex to** A denoted by $\mathbb{T}_A \in \mathsf{coStab}(\mathcal{C}_{A/})$ as the image of $(A \xrightarrow{\mathrm{Id}_A} A) \in \mathcal{C}_{A/}$ under the functor

$$\mathcal{C}_{A/} \xrightarrow{\Omega^{\infty}_{\mathsf{co}}} \mathsf{coStab}(\mathcal{C}_{A/}).$$

Remark 5.1.6. Let $\mathcal{C}$ be a presentable category and $A \xrightarrow{f} B$ be a morphism in $\mathcal{C}$. Since the diagram

$$\begin{tikzcd}
\mathsf{coStab}(\mathcal{C}_{A/}) \arrow[r, "f_*"] \arrow[d, "\Sigma^{\infty}_{\mathsf{co}}"'] & \mathsf{coStab}(\mathcal{C}_{B/}) \arrow[d, "\Sigma^{\infty}_{\mathsf{co}}"] \\
\mathcal{C}_{A/} \arrow[r, "-\sqcup_A B"'] & \mathcal{C}_{B/}
\end{tikzcd}$$

by construction commutes, we see that the diagram of right adjoints

$$\begin{tikzcd}
\mathsf{coStab}(\mathcal{C}_{A/}) & \mathsf{coStab}(\mathcal{C}_{B/}) \arrow[l, "f^*"'] \\
\mathcal{C}_{A/} \arrow[u, "\Omega^{\infty}_{\mathsf{co}}"] & \mathcal{C}_{B/} \arrow[l] \arrow[u, "\Omega^{\infty}_{\mathsf{co}}"']
\end{tikzcd}$$

also commutes. In particular, we get an equivalence $\Omega^{\infty}_{\mathsf{co}}(A \xrightarrow{f} B) \simeq f^*\mathbb{T}_B$ in $\mathsf{coStab}(\mathcal{C}_{A/})$.

Using the remark 5.1.6 we can introduce the following

Definition 5.1.7. Let $\mathcal{C}$ be a presentable category and $A \xrightarrow{f} B$ be a morphism in $\mathcal{C}$. Define then the **relative tangent complex** denoted by $\mathbb{T}_{A/B} \in \mathsf{coStab}(\mathcal{C}_{A/})$ as the fiber

$$\mathbb{T}_{A/B} := \mathsf{fib}(\mathbb{T}_A \longrightarrow f^*\mathbb{T}_B),$$

where the morphism $\mathbb{T}_A \longrightarrow f^*\mathbb{T}_B$ is obtained by applying the functor $\mathcal{C}_{A/} \xrightarrow{\Omega^\infty_{\mathrm{co}}} \mathsf{coStab}(\mathcal{C}_{A/})$ to the morphism $A \xrightarrow{f} B$ in $\mathcal{C}_{A/}$.

We end this subsection with the following

Proposition 5.1.8. Let $\mathcal{C}$ be a presentable category and

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "t"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

be a pullback square in $\mathcal{C}$. Then there is a canonical equivalence

$$\mathbb{T}_{A/C} \simeq f^*\mathbb{T}_{B/D}$$

in $\mathsf{coStab}(\mathcal{C}_{A/})$.

Proof. Since the diagram above can be also considered as a pullback square in $\mathcal{C}_{A/}$ and the functor $\mathcal{C}_{A/} \xrightarrow{\Omega^\infty_{\mathrm{co}}} \mathsf{coStab}(\mathcal{C}_{A/})$ preserves limits (being right adjoint) using the remark 5.1.6 we get a pullback

$$\begin{tikzcd} \mathbb{T}_A \arrow[r] \arrow[d] & f^*\mathbb{T}_B \arrow[d] \\ g^*\mathbb{T}_C \arrow[r] & f^*t^*\mathbb{T}_D \end{tikzcd}$$

in $\mathsf{coStab}(\mathcal{C}_{A/})$. In particular, passing to the fibers of the vertical morphisms we get an equivalence

$$\mathbb{T}_{A/C} = \mathsf{fib}(\mathbb{T}_A \longrightarrow g^*\mathbb{T}_C) \simeq \mathsf{fib}(f^*\mathbb{T}_B \longrightarrow f^*t^*\mathbb{T}_D) \simeq$$

$$\simeq f^*\mathsf{fib}(\mathbb{T}_B \longrightarrow t^*\mathbb{T}_D) = f^*\mathbb{T}_{B/D}$$

as claimed. $\square$

5.2. Monoidal category built from an operad

In this subsection we describe the construction of the universal category one can assign to an operad we need in the proof of Proposition 5.3.2.

Proposition 5.2.1. Let $\mathcal{O}$ be an ∞-operad in Vect_k (we refer the reader to [11, Chapter 6, Section 1] and [17, Chapter 2] for a discussion of ∞-operads). Then there exists a symmetric monoidal k-linear category $\mathcal{U}_{\mathcal{O}} \in \mathrm{CAlg}(\mathrm{Cat}_k)$ such that for any symmetric monoidal k-linear category $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Cat}_k)$ there is a natural equivalence

$$\mathsf{Funct}_{\mathrm{CAlg}(\mathrm{Cat}_k)}(\mathcal{U}_{\mathcal{O}}, \mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}),$$

where $\mathsf{Funct}_{\mathrm{CAlg}(\mathrm{Cat}_k)}(\mathcal{U}_{\mathcal{O}}, \mathcal{C})$ is the full subcategory of $\mathsf{Funct}^{\otimes}(\mathcal{U}_{\mathcal{O}}, \mathcal{C})$ spanned by k-linear symmetric monoidal functors.

Proof. The category $\mathcal{U}_{\mathcal{O}}$ is simply the category of Vect_k-presheaves $\mathrm{Psh}_k(\mathsf{Env}_k(\mathcal{O}))$ on the Vect_k-enriched analogue $\mathsf{Env}_k(\mathcal{O})$ of the monoidal envelope of $\mathcal{O}$ from [17, Proposition 2.2.4.1] with the Day convolution monoidal structure (see [17, Example 2.2.6.17] and [12]). The desired universal property follows from Vect_k-enriched version of [17, Proposition 2.2.4.9] and [12, Lemma 2.13]. $\square$

Remark 5.2.2. Suppose that the operad $\mathcal{O}$ has only one color. Then the category $\mathcal{U}_{\mathcal{O}}$ can be informally constructed as follows: first, one considers the $(\infty,1)$-category $\mathsf{Env}_k(\mathcal{O})$ whose objects are natural numbers and morphisms are described as

$$\mathrm{Hom}_{\mathsf{Env}_k(\mathcal{O})}(n,m) \simeq \bigoplus_{f:n\to m} \bigotimes_{i=1}^{m} \mathcal{O}(f^{-1}(i)).$$

Moreover, the category $\mathsf{Env}_k(\mathcal{O})$ has a natural symmetric monoidal structure given by addition. The category $\mathcal{U}_{\mathcal{O}}$ is then the category of Vect_k-presheaves on $\mathsf{Env}_k(\mathcal{O})$ with the symmetric monoidal structure determined by the fact that it preserves colimits and that the Yoneda's embedding $\mathsf{Env}_k(\mathcal{O}) \longrightarrow \mathcal{U}_{\mathcal{O}}$ is symmetric monoidal. The equivalence

$$\mathsf{Funct}_{\mathrm{CAlg}(\mathrm{Cat}_k)}(\mathcal{U}_{\mathcal{O}}, \mathcal{C}) \xrightarrow[\sim]{} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$$

for $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Cat}_k)$ is then simply given by the evaluation at $\mathrm{Hom}_{\mathsf{Env}_k(\mathcal{O})}(-,1) \in \mathcal{U}_{\mathcal{O}}$.

5.3. Proof of Theorem 4.4.1

In this section we prove a generalization of the Theorem 4.4.1 to an arbitrary k-linear presentably symmetric monoidal category $\mathcal{C}$ with the monoidal unit $I \in \mathcal{C}$. Our first step is to define $d\exp_{\widehat{G}}$ in this context. Note that given a formal group $\widehat{G} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$ one can trivialize its tangent sheaf by applying the base change to the pullback diagram

$$\begin{tikzcd}
\widehat{G} \arrow[rr, "i"] \arrow[d, "\Delta"'] & & X \arrow[d, "e"] \\
\widehat{G} \times \widehat{G} \arrow[r, "\mathrm{Id}_{\widehat{G}} \times \mathrm{inv}_G"'] & \widehat{G} \times \widehat{G} \arrow[r, "m"'] & \widehat{G}
\end{tikzcd} \tag{5}$$

where the lower horizontal map informally sends a pair (g, h) to $g \cdot h^{-1}$. The morphism $d\exp_{\widehat{G}}$ is then defined by comparing two trivializations of the tangent sheaf that come from two formal group structures on $\widehat{G}$ (the initial one and the abelian one).

Using the formalism of tangent comodules one can emulate the same construction in algebraic setting. Note first that by Yoneda's lemma the diagram analogous to (5) is fibered for any group object in any category admitting finite limits. Consequently, we can introduce the following

Construction 5.3.1. Let $\mathcal{C}$ be a k-linear symmetric presentably monoidal category and $\mathfrak{g} \in \mathrm{LAlg}(\mathcal{C})$ be a Lie algebra in $\mathcal{C}$ (so that we get a group object $U(\mathfrak{g})$ in the category $\mathrm{coCAlg}(\mathrm{ICoh}(X))$ of cocommutative coalgebras in $\mathcal{C}$). Consider the pullback square of cocommutative coalgebras

$$\begin{tikzcd}
U(\mathfrak{g}) \arrow[r] \arrow[d] & I \arrow[d] \\
U(\mathfrak{g}) \otimes U(\mathfrak{g}) \arrow[r] & U(\mathfrak{g}).
\end{tikzcd}$$

Using Proposition 5.1.8 this diagram induces an equivalence of $U(\mathfrak{g})$-comodules $\mathbb{T}_{U(\mathfrak{g})} \simeq U(\mathfrak{g}) \otimes \mathfrak{g}$. As in Construction 4.1.2 by comparing the trivial Lie algebra structure on $\mathfrak{g}$ with the given one we obtain an autoequivalence $d\exp_{\mathfrak{g}} : U(\mathfrak{g}) \otimes \mathfrak{g} \xrightarrow[\sim]{} U(\mathfrak{g}) \otimes \mathfrak{g}$ of $U(\mathfrak{g})$-comodules and hence by adjunction a morphism $\widetilde{d\exp_{\mathfrak{g}}} : U(\mathfrak{g}) \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ in $\mathcal{C}$.

If we take $\mathcal{C} = \mathrm{QCoh}(X)$ and $\mathfrak{g} = \mathrm{Lie}_X \widehat{G}$ in the construction above the morphism $\widetilde{d\exp}_{\mathfrak{g}}$ is precisely $\widetilde{d\exp}_{\widehat{G}}$ from the introduction to this section. Consequently, to conclude it is left to prove

Proposition 5.3.2. Let $\mathcal{C}$ be a stable symmetric presentably monoidal k-linear category, $\mathfrak{g} \in \mathrm{LAlg}(\mathcal{C})$ be a Lie algebra in $\mathcal{C}$. Then

$$\widetilde{d\exp}_{\mathfrak{g}} = \frac{1 - e^{-\operatorname{ad}_{\mathfrak{g}}}}{\operatorname{ad}_{\mathfrak{g}}},$$

where the morphism on the right is obtained from Variant 4.2.4 by applying $f = (1 - e^{-t})/t$ to the adjoint representation of $\mathfrak{g}$.

Proof. We first argue that it is sufficient to prove the equality holds in $\widetilde{\mathrm{Vect}_k}$ for discrete free Lie algebras. Indeed, since both of the morphisms $\widetilde{d\exp}_{\mathfrak{g}}$ and $(1 - e^{-\operatorname{ad}_{\mathfrak{g}}})/\operatorname{ad}_{\mathfrak{g}}$ are functorial with respect to continuous monoidal functors, to show that $\widetilde{d\exp}_{\mathfrak{g}} - (1 - e^{-\operatorname{ad}_{\mathfrak{g}}})/\operatorname{ad}_{\mathfrak{g}} = 0$ it is sufficient to prove the statement for the Lie algebra $\mathfrak{g} := \mathrm{Hom}_{\mathsf{Env}_k(\mathrm{Lie})}(-, I) \in \mathrm{Lie}(\mathcal{U}_{\mathrm{Lie}})$, where $\mathsf{Env}_k(\mathrm{Lie})$ and $\mathcal{U}_{\mathrm{Lie}}$ are the universal categories from Proposition 5.2.1. Now since

$$\mathrm{Hom}_{\mathcal{U}_{\mathrm{Lie}}}(U(\mathfrak{g}) \otimes \mathfrak{g}, \mathfrak{g}) \simeq \mathrm{Hom}_{\mathcal{U}_{\mathrm{Lie}}}(\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n+1}, \mathfrak{g}) \simeq$$

$$\simeq \prod_{n \geq 0} \mathrm{Hom}_{\mathcal{U}_{\mathrm{Lie}}}(\mathfrak{g}^{\otimes n+1}, \mathfrak{g}) \simeq \prod_{n \geq 0} \mathrm{Lie}(n+1)$$

we see that if $\widetilde{d\exp}_{\mathfrak{g}} - (1 - e^{-\operatorname{ad}_{\mathfrak{g}}})/\operatorname{ad}_{\mathfrak{g}} \neq 0$, then there exists $n \in \mathbb{N}$ such that $\mathsf{pr}_{\mathrm{Lie}(n+1)}(\widetilde{d\exp}_{\mathfrak{g}} - (1 - e^{-\operatorname{ad}_{\mathfrak{g}}})/\operatorname{ad}_{\mathfrak{g}}) \neq 0$. In particular, since the evident map

$$\mathrm{Lie}(n+1) \longrightarrow \mathrm{Hom}_{\mathrm{Vect}_k}\big(\mathsf{Free}(n+1)^{\otimes n+1}, \mathsf{Free}(n+1)\big)$$

is injective, where $\mathsf{Free}(n+1) \in \mathrm{LAlg}(\mathrm{Vect}_k^{\heartsuit}) \subset \mathrm{LAlg}(\mathrm{Vect}_k)$ is the discrete free Lie algebra in Vect_k on $(n+1)$-dimensional vector space, we see that $\widetilde{d\exp}_{\mathfrak{g}} - (1 - e^{-\operatorname{ad}_{\mathfrak{g}}})/\operatorname{ad}_{\mathfrak{g}}$ should be nonzero for $\mathsf{Free}(n+1)$. Consequently, to conclude the statement of the proposition it is sufficient to prove that $\widetilde{d\exp}_{\mathfrak{g}} = (1 - e^{-\operatorname{ad}_{\mathfrak{g}}})/\operatorname{ad}_{\mathfrak{g}}$ for discrete free Lie algebras in Vect_k.

Now since for a free, discrete Lie algebra the adjoint representation is faithful, we can further assume that $\mathfrak{g} = \mathfrak{gl}_V$ for $V \in \mathrm{Vect}_k^{\heartsuit}$. In geometric terms, we want to compute the derivative $d\exp_{\widehat{\mathrm{GL}}_V} \colon \exp_{\widehat{\mathrm{GL}}_V *} \mathbb{T}_{\mathbb{V}(\mathfrak{gl}_V)} \longrightarrow \mathbb{T}_{\widehat{\mathrm{GL}}_V}$

of the exponential map $\exp_{\widehat{\mathrm{GL}}_V} : \mathbb{V}(\mathfrak{gl}_V) \longrightarrow \widehat{\mathrm{GL}}_V$. Since by construction tangent complexes are determined by their restriction to A-points, it is enough to compute the induced map for any $\operatorname{Spec} A \xrightarrow{x} \mathbb{V}(\mathfrak{gl}_V)$. Moreover, since formal moduli are determined by their restriction to Artinian local k-algebras, we can assume that this is the case. Since by construction the ICoh-tangent sheaf is defined as the Serre dual of Pro(QCoh)-cotangent sheaf $\widehat{\mathbb{L}}_{-}$ (see [11, Chapter 1, Section 4.4]), it is enough to compute the induced map on pro-cotangent sheaves (we need to work with the pro-categories since $\mathfrak{gl}_V$ is an ind-scheme for infinite dimensional V). Let M be a connective A-module and let $A \oplus M$ be the corresponding trivial square zero extension. By definition the space $\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Mod}_A)}(x^*\widehat{\mathbb{L}}_{\mathbb{V}(\mathfrak{gl}_V)}, M)$ classifies lifts in the diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "x"] \arrow[d, hook] & \mathbb{V}(\mathfrak{gl}_V) \\ \operatorname{Spec} A \oplus M \arrow[ur, dashed] & \end{tikzcd}$$

and the differential map

$$\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Mod}_A)}(x^*\widehat{\mathbb{L}}_{\mathbb{V}(\mathfrak{gl}_V)}, M) \longrightarrow \mathrm{Hom}_{\mathrm{Pro}(\mathrm{Mod}_A)}(x^* \exp^*_{\widehat{\mathrm{GL}}_V} \widehat{\mathbb{L}}_{\widehat{\mathrm{GL}}_V}, M)$$

correspond to the postcomposition with $\exp_{\widehat{\mathrm{GL}}_V}$. Unwinding the definitions, one finds that $x^*\widehat{\mathbb{L}}_{\mathbb{V}(\mathfrak{gl}_V)} \simeq {}``\varprojlim" A \otimes_k V_i^\vee \in \mathrm{Pro}(\mathsf{Mod}_A)$ where $\{V_i\}$ is the diagram of finite dimensional k-vector subspaces of V and analogously for $x^*\widehat{\mathbb{L}}_{\widehat{\mathrm{GL}}_V}$. Hence both $x^*\widehat{\mathbb{L}}_{\mathbb{V}(\mathfrak{gl}_V)}$ and $x^* \exp^*_{\widehat{\mathrm{GL}}_V} \widehat{\mathbb{L}}_{\widehat{\mathrm{GL}}_V}$ are pro-free of finite rank A-modules (as $V \in \mathrm{Vect}_k^\heartsuit$), and so by Yoneda's lemma it is enough to understand the morphism $\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Mod}_A)}(\exp^*_{\widehat{\mathrm{GL}}_V,x}, M)$ for all free of finite rank modules M. Further, since each such module is a direct sum of A, we can assume $M = A$ (in this case $A \oplus M \simeq A[\varepsilon] := A \otimes_k k[\varepsilon]$, where $k[\varepsilon] := k[\varepsilon]/(\varepsilon^2)$ with $\deg(\varepsilon) = 0$).

Now unwinding the definitions one finds that for a local Artinian augmented k-algebra A we have

$$\mathbb{V}(\mathfrak{gl}_V)(A) \simeq \mathfrak{gl}_V(\mathfrak{m}_A) := \mathrm{End}_{\mathrm{Mod}_A}(V \otimes_k A) \underset{\mathrm{End}_k(V)}{\times} \{0\}$$

$$\widehat{\mathrm{GL}}_V(A) \simeq \widehat{\mathrm{GL}}_V(\mathfrak{m}_A) := \mathrm{Aut}_{\mathrm{Mod}_A}(V \otimes_k A) \underset{\mathrm{Aut}_k(V)}{\times} \{\mathrm{Id}_V\}$$

and the exponential map $\exp_{\widehat{\mathrm{GL}}_V}(A)$ sends a matrix X to $\sum_{n=0}^{\infty} \frac{X^n}{n!}$. Hence the general case follows from the following well-known lemma:

Lemma 5.3.3. Let V be a discrete k-vector space and let A be a local Artinian augmented k-algebra. Then for each $X, Y \in \mathfrak{gl}_V(\mathfrak{m}_A)$ we have an equality

$$e^{-X}e^{X+\varepsilon Y} = 1 + \varepsilon \cdot \frac{1-e^{-\operatorname{ad}_X}}{\operatorname{ad}_X}(Y)$$

in $\widehat{\mathrm{GL}}_V(A[\varepsilon])$.

Proof. It is sufficient to prove the statement in the universal case when $A = \mathbb{Q}[\varepsilon]\langle\langle X, Y\rangle\rangle$ the free ring of non-commutative power series on two variables over $\mathbb{Q}[\varepsilon]$. We have

$$e^{-X}e^{X+\varepsilon Y} = e^{-X}\sum_{n=0}^{\infty}\frac{(X+\varepsilon Y)^n}{n!} = \tag{6}$$

$$= e^{-X}\cdot\sum_{n=0}^{\infty}\frac{1}{n!}\left(X^n + \varepsilon\sum_{k=0}^{n-1}X^kYX^{n-1-k}\right) = 1 + e^{-X}\cdot\sum_{n=0}^{\infty}\frac{\varepsilon}{n!}\sum_{k=0}^{n-1}X^kYX^{n-1-k}.$$

Note that

$$\operatorname{ad}_X\left(\sum_{k=0}^{n-1}X^kYX^{n-1-k}\right) = \sum_{k=0}^{n-1}\left(X^{k+1}YX^{n-1-k} - X^kYX^{n-k}\right) = X^nY - YX^n.$$

Hence by applying ad_X to both sides of (6) we obtain

$$\operatorname{ad}_X\left(e^{-X}e^{X+\varepsilon Y}\right) = \operatorname{ad}_X\left(e^{-X}\cdot\sum_{n=0}^{\infty}\frac{\varepsilon}{n!}\sum_{k=0}^{n-1}X^kYX^{n-1-k}\right) =$$

$$= e^{-X}\sum_{n=0}^{\infty}\frac{\varepsilon}{n!}(X^nY - YX^n) = \varepsilon\cdot e^{-X}\cdot\left(e^XY - Ye^X\right) =$$

$$= \varepsilon\cdot(Y - \operatorname{Ad}_{e^{-X}}(Y)) = \varepsilon\cdot\left(1 - e^{-\operatorname{ad}_X}\right)(Y)$$

It follows that $e^{-X}e^{X+\varepsilon Y}$ and $(1+\varepsilon\cdot(1-e^{-\operatorname{ad}_X})/\operatorname{ad}_X)(Y)$ differ by something commuting with X, i.e. there exists a formal power series $f(X)$ such that

$$e^{-X}e^{X+\varepsilon Y} - \left(1 + \varepsilon\cdot\frac{1-e^{-\operatorname{ad}_X}}{\operatorname{ad}_X}(Y)\right) = f(X).$$

Setting $Y = 0$ in the last equality we find that $f(X) = 0$. □

□

6. Equivariant Grothendieck-Riemann-Roch

In this section we finally bring the results of previous sections together to give a proof of the classical Grothendieck-Riemann-Roch theorem as well as its equivariant analogue.

6.1. Abstract GRR theorem

We start by introducing the context we are interested in:

Definition 6.1.1. Let (X, g_X) and (Y, g_Y) be a pair of derived schemes with endomorphisms. An *equivariant morphism* $(X, g_X) \xrightarrow{f} (Y, g_Y)$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g_Y"'] & Y \end{tikzcd}$$

where $X \xrightarrow{f} Y$ is some morphism of schemes. In this setting we will further denote by $X^{g_X} \xrightarrow{f^g} Y^{g_Y}$ the induced map on fixed points.

Remark 6.1.2. Note that for a lax g_X-equivariant sheaf $\mathcal{F} \xrightarrow{t} (g_X)_*\mathcal{F}$ its pushforward $f_*\mathcal{F} \in \mathrm{QCoh}(Y)$ to Y automatically admits a g_Y-lax equivariant structure given by the composite

$$f_*\mathcal{F} \xrightarrow{f_*(t)} f_*(g_X)_*\mathcal{F} \xrightarrow[\sim]{} (g_Y)_*f_*\mathcal{F}.$$

We will further use the notation $f_*(\mathcal{F}, t)$ for $f_*\mathcal{F} \in \mathrm{QCoh}(Y)$ together with the above lax equivariant structure.

The definition above motivates the following

Definition 6.1.3. Let (X, g) be a smooth scheme with an endomorphism. We will denote by $K_0^g(X)$ the usual K_0-group of the category of lax g-equivariant perfect sheaves on X.

Remark 6.1.4. Note that for a proper equivariant morphism $(X, g_X) \xrightarrow{f} (Y, g_Y)$ between smooth schemes[1] the induced pushforward functor f_* is exact and preserves perfect sheaves (as $\mathrm{Coh}(Y) \simeq \mathrm{QCoh}(Y)^{\mathrm{perf}}$ and proper morphism preserves coherent sheaves) and therefore there is an induced morphism

$$K_0^{g_X}(X) \longrightarrow K_0^{g_Y}(Y)$$

which we will also denote by f_*.

Motivated by Corollary 4.4.4 we also introduce the following

Notation 6.1.5. Given a smooth proper scheme together with an endomorphism $X \xrightarrow{g} X$ we will further denote the canonical orientation $\mathcal{O}_{X^g} \simeq \omega_{X^g}$ (see Construction 3.2.1) on X^g by td_g and call it an *equivariant Todd distribution on* (X, g).

Now the abstract formalism of traces readily gives us the

Proposition 6.1.6 (Abstract Grothendieck-Riemann-Roch)**.** Let

$$(X, g_X) \xrightarrow{f} (Y, g_Y)$$

be an equivariant morphism between smooth proper schemes. Then the diagram

$$\begin{tikzcd}
K_0^{g_X}(X) \arrow[r, "\mathrm{ch}(-,-)\,\mathrm{td}_{g_X}"] \arrow[d, "f_*"'] & \pi_0\Gamma(X^{g_X}, \omega_{X^{g_X}}) \arrow[d, "(f^g)_*"] \\
K_0^{g_Y}(Y) \arrow[r, "\mathrm{ch}(-,-)\,\mathrm{td}_{g_Y}"'] & \pi_0\Gamma(Y^{g_Y}, \omega_{X^{g_Y}})
\end{tikzcd}$$

is commutative, i.e. for any perfect g_X-lax equivariant sheaf (E, t) on X there is an equality

$$(f^g)_*(\mathrm{ch}(E, t)\,\mathrm{td}_{g_X}) = \mathrm{ch}(f_*(E, t))\,\mathrm{td}_{g_Y}$$

in $\Gamma(Y^{g_Y}, \omega_{Y^{g_Y}})$, where $\mathrm{ch}(E, t) \in \Gamma(X^{g_X}, \mathcal{O}_{X^{g_X}})$ here is the categorical Chern character 1.1.4.

[1]In fact it is enough to assume that Y is smooth.

Proof. By passing to the induced morphism of traces 0.0.3 in the commutative diagram

$$\begin{tikzcd} & \mathrm{QCoh}(X) \arrow[loop above, "g_{X*}"] \arrow[r, "f_*"] \arrow[d, "\sim"', "\otimes\mathcal{O}_X"] & \mathrm{QCoh}(Y) \arrow[loop above, "g_{Y*}"] \arrow[d, "\sim"', "\otimes\mathcal{O}_Y"] \\ \mathrm{Vect}_k \arrow[ur, phantom] & \mathrm{ICoh}(X) \arrow[loop below, "g_{X*}"] \arrow[r, "f_*"'] & \mathrm{ICoh}(Y) \arrow[loop below, "g_{Y*}"] \end{tikzcd}$$

$$\mathrm{Vect}_k \xrightarrow{E} \mathrm{QCoh}(X)$$

by functoriality we obtain a commutative diagram

$$\begin{tikzcd} k \arrow[r, "\mathrm{ch}(E,t)"'] \arrow[rr, bend left, "\mathrm{ch}(f_*(E,t))"] & \Gamma(X^{g_X}, \mathcal{O}_{X^{g_X}}) \arrow[r] \arrow[d, "\sim"', "\cdot\,\mathrm{td}_{g_X}"] & \Gamma(Y^{g_Y}, \mathcal{O}_{Y^{g_Y}}) \arrow[d, "\sim"', "\cdot\,\mathrm{td}_{g_Y}"] \\ & \Gamma(X^{g_X}, \omega_{X^{g_X}}) \arrow[r, "(f^g)_*"'] & \Gamma(Y^{g_Y}, \omega_{Y^{g_Y}}) \end{tikzcd}$$

where we use Corollaries 1.1.2, 2.2.2, B.1.7 and Proposition B.2.7 to identify morphisms of traces in the above diagram. □

6.2. Equivariant GRR theorem

Now it may be hard to apply Proposition 6.1.6 in practice as in general we don't have a good description of $\Gamma(X^{g_X}, \mathcal{O}_{X^{g_X}})$ and of the Todd distribution td_{g_X}. Fortunately, under some reasonable assumptions one can use ideas of localization to express it in a more computable form.

Assumption 6.2.1. We will further assume that (X, g) is a smooth scheme with an endomorphism $X \xrightarrow{g} X$ such that the reduced classical scheme $\overline{X^g} := \mathcal{H}^0(X^g)^{\mathsf{red}}$ is smooth (but not necessarily connected). We will denote by $\overline{X^g} \xrightarrow{j} X$ the canonical embedding and by $\mathcal{N}_g^\vee$ its conormal bundle. Note that the action of g on Ω^1_X in particular restricts to an endomorphism $\mathcal{N}_g^\vee \xrightarrow{g^*_{|\mathcal{N}_g}} \mathcal{N}_g^\vee$. We will sometimes call $\overline{X^g}$ the *classical fixed locus of* g.

Note that the embedding $\overline{X^g} \overset{j}{\hookrightarrow} X$ is equivariant with respect to the trivial equivariant structure on $\overline{X^g}$ and the given one on X thus induces a morphism $j^g\colon \mathcal{L}\overline{X^g} \longrightarrow X^g$. The theorem below gives a criterion when j^g is an equivalence:

Theorem 6.2.2 (Localization theorem)**.** Let (X,g) be as in the previous notation. Then the pullback map j^g is an equivalence if and only if the determinant $\det(1-g^*_{|\mathcal{N}_g^\vee}) \in \Gamma(\overline{X^g}, \mathcal{O}_{\overline{X^g}})$ is an invertible function.

Proof. Since the map $j^g\colon \mathcal{L}\overline{X^g} \longrightarrow X^g$ is a nil-isomorphism, by [11, Chapter 1, Proposition 8.3.2] the morphism j^g is an equivalence if and only if the induced map on cotangent spaces $\alpha\colon (j^g)^*\mathbb{L}_{X^g} \longrightarrow \mathbb{L}_{\mathcal{L}\overline{X^g}}$ is an equivalence. Moreover, since the inclusion $\overline{X^g} \longrightarrow \mathcal{L}\overline{X^g}$ is a nil-isomorphism, it is enough to prove that $\alpha_{|\overline{X^g}}$ is an equivalence.

Consider now the following commutative diagram of derived schemes

$$\begin{tikzcd} \overline{X^g} \arrow[r, "\Delta"] \arrow[d, "j"'] & \overline{X^g} \times \overline{X^g} \arrow[d, "j\times j"] & \overline{X^g} \arrow[l, "\Delta"'] \arrow[d, "j"] \\ X \arrow[r, "\Delta"'] & X \times X & X. \arrow[l, "(\mathrm{Id}_X, g)"] \end{tikzcd}$$

By definition the limit of the top row is $\mathcal{L}\overline{X^g}$, the limit of the bottom row is X^g and j^g is precisely the induced map on the limits. By applying the absolute cotangent complex functor and pulling everything back to $\mathcal{L}\overline{X^g}$ and then further pulling back along $\overline{X^g} \longrightarrow \mathcal{L}\overline{X^g}$ we then obtain a commutative diagram of sheaves

$$\begin{tikzcd} \Omega^1_{\overline{X^g}} & \Omega^1_{\overline{X^g}} \oplus \Omega^1_{\overline{X^g}} \arrow[l, "\nabla"'] \arrow[r, "\nabla"] & \Omega^1_{\overline{X^g}} \\ j^*\Omega^1_X \arrow[u, "j^*"] & j^*\Omega^1_X \oplus j^*\Omega^1_X \arrow[l, "\nabla"] \arrow[r, "(1,g^*)"'] \arrow[u, "j^*\oplus j^*"'] & j^*\Omega^1_X \arrow[u, "j^*"'] \end{tikzcd}$$

(where by ∇ we denote the codiagonal map) in $\mathrm{QCoh}(\overline{X^g})$. By the Lemma 6.2.3 below, the pushout of the top row is $(\mathbb{L}_{\mathcal{L}\overline{X^g}})_{|\overline{X^g}}$, the pushout of the bottom row is $(\mathbb{L}_{X^g})_{|\overline{X^g}}$ and the induced map between pushouts is $\alpha_{|\overline{X^g}}$. It follows $\alpha_{|\overline{X^g}}$ is an equivalence if and only if the pushout of fibers of the vertical maps

$$\mathcal{N}_g^\vee \xleftarrow{\ \nabla\ } \mathcal{N}_g^\vee \oplus \mathcal{N}_g^\vee \xrightarrow{(1, g^*_{|\mathcal{N}_g^\vee})} \mathcal{N}_g^\vee$$

is nullhomotopic. The above pushout may be computed as the cofiber of the map $\mathcal{N}_g^\vee \oplus \mathcal{N}_g^\vee \longrightarrow \mathcal{N}_g^\vee \oplus \mathcal{N}_g^\vee$ given in block-matrix form by

$$\begin{pmatrix} 1 & 1 \\ 1 & g^*_{|\mathcal{N}_g^\vee} \end{pmatrix}. \tag{7}$$

Finally, the matrix (7) is invertible if and only if $g^*_{|\mathcal{N}_g^\vee} - 1$ is invertible if and only if the determinant $\det(1 - g^*_{|\mathcal{N}_g^\vee})$ is invertible. □

Lemma 6.2.3. Let

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, "r"] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

be a fibered square of derived schemes. Then the induced square

$$\begin{tikzcd} r^*\mathbb{L}_W \arrow[r] \arrow[d] & p^*\mathbb{L}_Y \arrow[d] \\ q^*\mathbb{L}_Z \arrow[r] & \mathbb{L}_X \end{tikzcd}$$

is a pushout in $\mathrm{QCoh}(X)$.

Proof. Formal from the definition of QCoh-cotangent complex and the universal property of the limit.[2] □

Remark 6.2.4. The theorem above tells us that if the determinant $\det(1 - g^*_{|\mathcal{N}_g})$ is invertible (a condition which is often easy to verify in practice), then the ring $\Gamma(X^g, \mathcal{O}_{X^g})$ (which naturally appears in the abstract GRR-theorem 6.1.6) is equivalent to a ring that we understand much better:

$$-_{|\mathcal{L}\overline{X^g}} \colon \Gamma(X^g, \mathcal{O}_{X^g}) \xrightarrow[\sim]{(j^g)^*} \Gamma(\mathcal{L}\overline{X^g}, \mathcal{O}_{\mathcal{L}\overline{X^g}}) \simeq \bigoplus_{p,q} H^{p,q}(\overline{X^g})[p-q].$$

[2] In fact, one can analogously prove the following more general statement: let $X_\bullet : I \longrightarrow \mathrm{PreStack}$ be a diagram of prestacks admitting cotangent complex. Then $X := \lim_I X_i$ admits cotangent complex and the natural map $\operatorname{colim} p_i^*\mathbb{L}_{X_i} \longrightarrow \mathbb{L}_X$ is an equivalence where $p_i : X \longrightarrow X_i$ are the natural projections.

Moreover, for a perfect g-equivariant sheaf (E,t) on X it is convenient to describe the categorical Chern character $\mathrm{ch}(E,t) \in \Gamma(X^g, \mathcal{O}_{X^g})$ in these terms: by Corollary 1.3.3 we have an equality

$$\mathrm{ch}(E,t)_{|\mathcal{L}(\overline{X^g})} = \mathsf{Tr}_{\mathrm{QCoh}(\overline{X^g})}\left(\exp\left(\mathrm{At}(E_{|\overline{X^g}})\right) \circ t_{|\overline{X^g}}\right)$$

in $\pi_0\Gamma(\overline{X^g}, \mathcal{O}_{\overline{X^g}}) \simeq \bigoplus_p H^{p,p}(\overline{X^g})$. Since the rings $\Gamma(X^g, \mathcal{O}_{X^g})$ and $\Gamma(\mathcal{L}\overline{X^g}, \mathcal{O}_{\mathcal{L}\overline{X^g}})$ under assumptions of the localization theorem are canonically equivalent, by abuse of notations we will sometimes identify $\mathrm{ch}(E,t)$ with its image in $\pi_0\Gamma(\mathcal{L}\overline{X^g}, \mathcal{O}_{\mathcal{L}\overline{X^g}})$.

The conditions of Theorem 6.2.2 are sometimes automatically satisfied. To see this recall the following well-known lemma:

Lemma 6.2.5. Let X be a smooth variety over an algebraically closed field k of characteristic zero and let G be a reductive group acting on X. Then for each G-fixed point $x \in X$ one can choose a set of local coordinates $\{x_1, \dots, x_n\}$ of $\mathcal{O}_{X,x} \simeq k[[x_1, \dots, x_n]]$, such that G acts linearly with respect to them. In particular, the fixed locus X^G is smooth.

Proof. Let $\mathfrak{m}_{\mathcal{O}_{X,x}}$ be the maximal ideal of $\mathcal{O}_{X,x}$. Since the category of G-representations is semisimple, for each $n \geq 2$ the natural surjection $\mathfrak{m}_{\mathcal{O}_{X,x}}/\mathfrak{m}^n_{\mathcal{O}_{X,x}} \twoheadrightarrow \mathfrak{m}_{\mathcal{O}_{X,x}}/\mathfrak{m}^2_{\mathcal{O}_{X,x}}$ admits a G-equivariant section. Passing to the limit $n \to \infty$ we obtain a G-equivariant section $s\colon \mathfrak{m}_{\mathcal{O}_{X,x}}/\mathfrak{m}^2_{\mathcal{O}_{X,x}} \longrightarrow \mathfrak{m}_{\mathcal{O}_{X,x}}$. Let $\{\overline{x}_1, \dots \overline{x}_n\}$ be a basis of $\mathbb{T}_{X,x} \simeq \mathfrak{m}_{X,x}/\mathfrak{m}^2_{\mathcal{O}_{X,x}}$ and put $x_i := s(\overline{x}_i)$. By Nakayama's lemma $\{x_1, \dots, x_n\}$ are local coordinates at the point x and G by construction acts linearly withe respect to them. $\square$

Corollary 6.2.6. Let X be a smooth scheme equipped with an automorphism g of finite order. Then the fixed locus X^g is smooth and the natural map $\mathcal{L}X^g \longrightarrow X^g$ is an equivalence, i.e. the derived fixed locus X^g is formal.

Remark 6.2.7. This result for finite order automorphism was also obtained in [1, Corollary 1.12] by different methods.

In order to proceed further we introduce the following

Definition 6.2.8 (Euler classes)**.** Let (X, g) be as above. Define an *Euler class of* g as

$$e_g := (j^g)^* \,\mathrm{ch}\left(j_*(\mathcal{O}_{\overline{X^g}}, \mathrm{Id}_{\mathcal{O}_{\overline{X^g}}})\right) \quad \in \quad \pi_0\Gamma\big(\mathcal{L}\overline{X^g}, \mathcal{O}_{\mathcal{L}\overline{X^g}}\big),$$

where $\mathcal{L}\overline{X^g} \xrightarrow{j^g} X^g$ is the induced map on fixed loci (see Remark 6.1.2 for the notation $j_*(-,-)$).

To describe the Euler class more explicitly, we recall the following standard result, the proof of which we include here for reader's convenience:

Lemma 6.2.9. Let $Z \overset{i}{\hookrightarrow} X$ be a closed embedding of smooth schemes. Then there exists a canonical isomorphism

$$\mathcal{H}^{-k}(i^* i_* \mathcal{O}_Z) \simeq \Lambda^k(\mathcal{N}^\vee_{Z/X})$$

of quasi-coherent sheaves on Z.

Proof. The case $k = 0$ is obvious, since i is a closed embedding. For $k = 1$ note that by applying the pullback functor i^* to the exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0$$

we obtain an isomorphism $\mathcal{H}^{-1}(i^* i_* \mathcal{O}_Z) \simeq \mathcal{H}^0(i^*\mathcal{I}_Z) \simeq \mathcal{I}_Z/\mathcal{I}_Z^2$. But by smoothness assumption $\mathcal{I}_Z/\mathcal{I}_Z^2$ is isomorphic to the conormal bundle $\mathcal{N}^\vee_{Z/X}$ of Z in X.

Finally, the isomorphism $\mathcal{N}^\vee_{Z/X} \simeq \mathcal{H}^{-1}(i^* i_* \mathcal{O}_Z)$ and multiplication induce a map of algebras in $\mathrm{QCoh}(Z)$

$$\alpha^* \colon \Lambda^*(\mathcal{N}_{Z/X}) \longrightarrow \mathcal{H}^{-*}(i^* i_* \mathcal{O}_Z).$$

By smoothness assumption, Z is a locally complete intersection in X, hence locally both parts are exterior algebras and thus since $\mathcal{H}^{-1}(\alpha)$ is an isomorphism so is α^*. □

Corollary 6.2.10. We have

$$e_g = \mathrm{ch}\left(\mathrm{Sym}(\mathcal{N}_g^\vee[1]), \mathrm{Sym}(g^*_{|\mathcal{N}_g^\vee[1]})\right) = \sum_k (-1)^k \,\mathrm{ch}\left(\Lambda^k(\mathcal{N}_g^\vee), \Lambda^k(g^*_{|\mathcal{N}_g^\vee})\right).$$

Proof. By definition

$$e_g = (j^g)^* \,\mathrm{ch}\left(j_*(\mathcal{O}_{\overline{X^g}}, \mathrm{Id}_{\mathcal{O}_{\overline{X^g}}})\right) = \mathrm{ch}\left(j^* j_*(\mathcal{O}_{\overline{X^g}}, \mathrm{Id}_{\mathcal{O}_{\overline{X^g}}})\right).$$

Next, by the lemma above there is a (Postnikov) filtration on the complex $j^* j_* \mathcal{O}_{\overline{X^g}}$ with associated graded $\mathrm{Sym}(\mathcal{N}_g^\vee[1])$. The statement then follows from the fact that $\mathrm{ch}(-,-)$ is additive in fiber sequences. □

Corollary 6.2.11. Let (X, g) be as in Assumption 6.2.1. Then the following conditions are equivalent:

1. The morphism $1 - (g^*)_{|\mathcal{N}_g^\vee}$ is invertible.
2. The induced map $j^g : \mathcal{L}\overline{X^g} \longrightarrow X^g$ is an equivalence.
3. The Euler class e_g is invertible.

Proof. The equivalence of the first two assertions is the content of Theorem 6.2.2. To see that the first and the last conditions are equivalent, note that since the natural inclusion $\overline{X^g} \longrightarrow \mathcal{L}\overline{X^g}$ is a nil-isomorphism, the Euler class e_g is invertible if and only if its zero term $e_{g,0}$ is. But by Corollary 6.2.10 and Example 1.3.4 we get

$$\begin{aligned} e_{g,0} &= \sum_{k=0}(-1)^k \operatorname{ch}_0(\Lambda^k(\mathcal{N}_g^\vee), g^*_{|\Lambda^k(\mathcal{N}_g^\vee)}) = \sum_k (-1)^k \operatorname{\mathsf{Tr}}(g^*_{|\Lambda^k(\mathcal{N}_g^\vee)}) \\ &= \det(1 - g^*_{|\mathcal{N}_g^\vee}). \end{aligned}$$

□

Using the Euler class and corollary above, we can describe the Todd distribution td_g (see Notation 6.1.5) in more concrete terms

Proposition 6.2.12. Let (X, g) be as in 6.2.1 and assume that the Euler class e_g is invertible (and so the natural morphism $j^g : \mathcal{L}\overline{X^g} \longrightarrow X^g$ is an equivalence). Then under the composite equivalence

$$\pi_0\Gamma(X^g, \omega_{X^g}) \xrightarrow[\sim]{(j^g)^*} \pi_0\Gamma(\mathcal{L}\overline{X^g}, \omega_{\mathcal{L}\overline{X^g}}) \simeq \bigoplus_p H^{p,p}(\overline{X^g})$$

the Todd distribution $\operatorname{td}_g \in \Gamma(X^g, \omega_{X^g})$ corresponds to $\frac{\operatorname{td}_{\overline{X^g}}}{e_g}$, where $\operatorname{td}_{\overline{X^g}}$ is the ordinary Todd class.

Proof. By applying Proposition 6.1.6 to the canonical inclusion $\overline{X^g} \xrightarrow{j} X$ we obtain

$$(j^g)_*(\operatorname{td}_{\overline{X^g}}) = (j^g)_*\big(\operatorname{ch}(\mathcal{O}_{\overline{X^g}}, \operatorname{Id}_{\mathcal{O}_{\overline{X^g}}}) \operatorname{td}_{\overline{X^g}}\big) = \operatorname{ch}\big(j_*(\mathcal{O}_{\overline{X^g}}, \operatorname{Id}_{\mathcal{O}_{\overline{X^g}}})\big) \operatorname{td}_g .$$

Consequently, by pulling back along j^g (and using that $(j^g)^*(j^g)_*$ is identity) we obtain

$$\operatorname{td}_{\overline{X^g}} = (j^g)^*(j_g)_*(\operatorname{td}_{\overline{X^g}}) = (j^g)^*\big(\operatorname{ch}(j_*(\mathcal{O}_{\overline{X^g}}, \operatorname{Id}_{\mathcal{O}_{\overline{X^g}}})) \operatorname{td}_g\big) = e_g \cdot (j^g)^*(\operatorname{td}_g).$$

We conclude by dividing both parts by e_g. □

As a corollary we obtain:

Theorem 6.2.13 (Equivariant Grothendieck-Riemann-Roch)**.** Let

$$(X, g_X) \xrightarrow{f} (Y, g_Y)$$

be an equivariant morphism between smooth proper schemes such that

- Reduced fixed loci $\overline{X^{g_X}}$ and $\overline{Y^{g_Y}}$ are smooth.
- The induced morphisms on conormal bundles $1-(g_X^*)_{|\mathcal{N}^\vee_{g_X}}$ and $1-(g_Y^*)_{|\mathcal{N}^\vee_{g_Y}}$ are invertible.

Then for a perfect lax g_X-equivariant sheaf (E,t) on X we have an equality

$$(\overline{f^g})_*\left(\mathrm{ch}(E,t)\frac{\mathrm{td}_{\overline{X^{g_X}}}}{e_{g_X}}\right) = \mathrm{ch}\left(f_*(E,t)\right)\frac{\mathrm{td}_{\overline{Y^{g_Y}}}}{e_{g_Y}}$$

in $\bigoplus_p H^{p,p}(\overline{Y^{g_Y}})$.

Proof. This follows immediately from the abstract Grothendieck-Riemann-Roch 6.1.6 and the identification of td_g above. □

Specializing to the case when $Y = *$, we get

Corollary 6.2.14 (Equivariant Hirzebruch-Riemann-Roch)**.** Let (X,g) be as in the theorem above. Then for any lax g-equivariant perfect sheaf (E,t) on X we have

$$\int_{\overline{X^g}} \mathrm{ch}(E,t)\frac{\mathrm{td}_{\overline{X^g}}}{e_g} = \mathsf{Tr}_{\mathrm{Vect}_k}\,\Gamma(X,t).$$

Specializing even further we recover

Corollary 6.2.15 (Holomorphic Atiyah-Bott fixed point formula)**.** Assume that the graph of g intersects the diagonal in $X \times X$ transversely. Then

$$\sum_{x=g(x)} \frac{\mathsf{Tr}(g^*_{|E_x})}{\det(1-d_x g)} = \mathsf{Tr}_{\mathrm{Vect}_k}\,\Gamma(X,t).$$

Proof. By assumption on g the derived fixed locus X^g is discrete, hence $\mathcal{L}\overline{X^g} \simeq X^g$ and the corollary above reads as

$$\sum_{x=g(x)} \frac{\mathrm{ch}(E,t)_{|X^g}}{e_g} = \mathsf{Tr}_{\mathrm{Vect}_k}\,\Gamma(X,t).$$

Let $x \in X^g$ be a fixed point of g. Since X^g is discrete, for any perfect lax g-equivariant sheaf $\mathrm{At}(E_{|\overline{X^g}}) \simeq 0$, hence by Corollary 1.3.3 we have

$$\mathrm{ch}(E,t)_x = x^* \,\mathsf{Tr}_{\mathrm{QCoh}(X^g)}(g^*_{|E_{|X^g}}) = \mathsf{Tr}(g^*_{|E_x}).$$

Moreover, the conormal bundle $\mathcal{N}_g^\vee$ in this case is just the cotangent space at fixed points, hence by the vanishing of the Atiyah class on X^g we deduce

$$(e_g)_x = (e_{g,0})_x = \det(1 - d_x g).$$

$\square$

Specializing in the other direction, we recover

Corollary 6.2.16 (Grothendieck-Riemann-Roch)**.** Let X, Y be smooth proper schemes and $X \xrightarrow{f} Y$ be a morphism. Then for any perfect sheaf E on X we have

$$f_*(\mathrm{ch}(E)\,\mathrm{td}_X) = \mathrm{ch}(f_*E)\,\mathrm{td}_Y$$

where above $\mathrm{ch}(E)$ and $\mathrm{ch}(f_*E)$ are the classical Chern characters.

Proof. Consider the Theorem 6.2.13 in the case when g_X, g_Y, are morphisms. Note that we have $\overline{X^g} = X$ and the map

$$\mathcal{L}X \simeq \mathcal{L}\overline{X^g} \xrightarrow{j^g} X^g \simeq \mathcal{L}X$$

(and analogously for Y) is tautologically equivalent to the identity. Moreover, since the conormal bundles $\mathcal{N}_{g_X}^\vee$ and $\mathcal{N}_{g_Y}^\vee$ are in this case trivial, we have $e_{g_X} = 1$ and $e_{g_Y} = 1$. Hence by Theorem 6.2.13 we obtain

$$f_*(\mathrm{ch}(E, \mathrm{Id}_E)\,\mathrm{td}_X) = \mathrm{ch}(f_*(E, \mathrm{Id}_E))\,\mathrm{td}_Y\,.$$

It is left to note that by Proposition 1.4.9 the categorical Chern character $\mathrm{ch}(E, \mathrm{Id}_E) = \mathrm{ch}(E)$ coincides with the classical one. $\square$

Appendix A. Reminder on formal deformation theory

In this section we review main results of formal deformation theory developed in [11, Chapters 5-9] relevant to this work (hence we need to work over a field of characteristic zero). We start with the following

Definition A.0.1 ([11, Chapter 5, Definition 1.1.1])**.** Define an $(\infty, 1)$-category of *formal moduli problems over* X denoted by $\widehat{\mathrm{Moduli}}_{/X}$ as the full subcategory of $\mathrm{PreStack}_{\mathsf{laft}\,/X}$ (see ([10, Chapter 2, 1.6]) on objects $Y \xrightarrow{p} X$ such that:

- The morphism p is inf-schematic ([11, Chapter 2, Definition 3.1.5]).
- The morphism p is nil-isomorphism, i.e. the induced morphism

$$^{red}Y \xrightarrow{\;^{red}p\;} {}^{red}X$$

  is an equivalence.

Group objects in the category of formal moduli problems over X are called *formal groups over* X.

Let now $\mathcal{Y} \xrightarrow{p} X$ be a formal moduli problem over X. The functor p_* being left adjoint to a symmetric monoidal functor $f^!$ is left lax-monoidal. Hence $p_*\omega_{\mathcal{Y}}$ is naturally a cocommutative coalgebra object of $\mathrm{ICoh}(X)$, that is, an object of $\mathrm{coCAlg}(\mathrm{ICoh}(X))$. Moreover since the functor

$$\widehat{\mathrm{Moduli}}_{/X} \longrightarrow \mathrm{coCAlg}(\mathrm{ICoh}(X))$$

$$\mathcal{Y} \longmapsto p_*\omega_{\mathcal{Y}}$$

is symmetric monoidal (e.g. $p_*\omega_{\mathcal{Y}\times_X\mathcal{Z}} \simeq p_*\omega_{\mathcal{Y}} \otimes p_*\omega_{\mathcal{Z}}$) for a formal group $\widehat{G} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$ the sheaf $p_*\omega_{\widehat{G}}$ is a group object in the category of cocommutative coalgebras, i.e. a cocommutative Hopf algebra. In particular, we can define a functor

$$\mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X}) \xrightarrow{\;\mathrm{Lie}_X\;} \mathrm{LAlg}(\mathrm{ICoh}(X))$$

by setting

$$\mathrm{Lie}_X(\widehat{G}) := \mathrm{Prim}(p_*\omega_{\widehat{G}}) \in \mathrm{LAlg}(\mathrm{ICoh}(X))$$

for a formal group $\widehat{G} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$, where

$$\mathrm{HopfAlg}(\mathrm{ICoh}(X)) \xrightarrow{\mathrm{Prim}} \mathrm{LAlg}(\mathrm{ICoh}(X))$$

is the functor of primitive elements.

Now the following crucial theorem relates groups in the category of formal moduli problems over X and Lie algebras in the category of quasi-coherent sheaves on X:

Theorem A.0.2 ([11, Chapter 7, Theorem 3.6.2 and Proposition 5.1.2])**.** We have:

1. There is an equivalence of $(\infty,1)$-categories

$$\mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X}) \xrightarrow[\sim]{\mathrm{Lie}_X} \mathrm{LAlg}(\mathrm{ICoh}(X)),$$

   where $\mathrm{LAlg}(\mathrm{ICoh}(X))$ is the $(\infty,1)$-category of algebras in $\mathrm{ICoh}(X)$ over the Lie operad. Moreover, for a formal group $\widehat{G} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$ the underlying ind-coherent sheaf of $\mathrm{Lie}_X(\widehat{G}) \in \mathrm{LAlg}(\mathrm{ICoh}(X))$ is equivalent to $\mathbb{T}_{\widehat{G}/X,e} := e^!\mathbb{T}_{\widehat{G}/X}$, where $X \xrightarrow{e} \widehat{G}$ is the identity section and $\mathbb{T}$ denotes tangent sheaf.
2. For $\widehat{G} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$ there is an equivalence of $(\infty,1)$-categories

$$\mathrm{Rep}_{\widehat{G}}(\mathrm{ICoh}(X)) \xrightarrow[\sim]{} \mathrm{Mod}_{\mathrm{Lie}_X(\widehat{G})}(\mathrm{ICoh}(X)).$$

Now in classical theory of Lie groups for a (real) Lie group G with Lie algebra $\mathfrak{g}$ there is an exponential map $\mathfrak{g} \xrightarrow{\exp_G} G$, which is a diffeomorphism in a small enough neighborhoods of $0 \in \mathfrak{g}$ and $1_G \in G$. The same story works even better in the formal world since one does not need to consider neighborhoods. In order to formulate this statement explicitly in our setting, we first need the following

Definition A.0.3. For $E \in \mathrm{ICoh}(X)$ define a *vector prestack* $\mathbb{V}(E) \in \widehat{\mathrm{Moduli}}_{X//X}$ *of* E by the property that for any $\mathcal{Y} \in \widehat{\mathrm{Moduli}}_{X//X}$ there is a natural equivalence

$$\mathrm{Hom}_{\widehat{\mathrm{Moduli}}_{X//X}}(\mathcal{Y}, \mathbb{V}(E)) := \mathrm{Hom}_{\mathrm{QCoh}(X)}(I(p_*\omega_{\mathcal{Y}}), E),$$

where for a coaugmented coalgebra $C \in \mathrm{coCAlg}^{\mathsf{coaug}}(\mathrm{QCoh}(X))$ we let $I(C) := \mathrm{cofib}(\mathcal{O}_X \longrightarrow C)$ to be the coaugmentation ideal of C.

Remark A.0.4. In fact vector prestack can be seen as a part of a more general construction. Namely, define a functor

$$\mathrm{coCAlg}(\mathrm{ICoh}(X)) \xrightarrow{\mathrm{Spec}_X^{\mathsf{inf}}} \widehat{\mathrm{Moduli}}_{/X}$$

as the right adjoint to the functor $\mathcal{Y} \longmapsto p_*\omega_{\mathcal{Y}}$ so that for any formal moduli problem $\mathcal{Y} \in \widehat{\mathrm{Moduli}}_{/X}$ we have an equivalence

$$\mathrm{Hom}_{\widehat{\mathrm{Moduli}}_{/X}}(\mathcal{Y}, \mathrm{Spec}_X^{\mathsf{inf}}\, C) \simeq \mathrm{Hom}_{\mathrm{coCAlg}(\mathrm{ICoh}(X))}(p_*\omega_{\mathcal{Y}}, C).$$

Then it is straightforward to see that for a sheaf $E \in \mathrm{ICoh}(X)$ there is a natural equivalence $\mathbb{V}(E) \simeq \mathrm{Spec}_X^{\mathsf{inf}}\,(\mathrm{Sym}(E))$, where

$$\mathrm{ICoh}(X) \xrightarrow{\mathrm{Sym}} \mathrm{coCAlg}^{\mathsf{nil}}(\mathrm{ICoh}(X)) \qquad E \mapsto \bigoplus_{n=0}^{\infty}(E^{\otimes n})_{\Sigma_n}$$

is the symmetric algebra functor, endowed with its canonical cofree ind-nilpotent co commutative coalgebra structure. We refer interested reader to [11, Chapter 7, 1.3] for through discussion of the inf-spectrum functor $\mathrm{Spec}^{\mathsf{inf}}$.

Example A.0.5. Unwinding the definitions one finds that for $E \in \mathrm{QCoh}(X)$ such that $E^\vee \in \mathrm{Coh}^{<0}(X)$ the vector prestack $\mathbb{V}(\Upsilon_X(E))$ is equivalent to the "vector bundle associated to E", i.e.

$$\mathbb{V}(\Upsilon_X(E)) \simeq \mathrm{Spec}_{/X}(\mathrm{Sym}_{\mathrm{QCoh}(X)}(E^\vee)).$$

In the case when $E^\vee \in \mathrm{Coh}^{\leq 0}(X)$, there is a similar equivalence if we take the formal completion at the zero section of the right-hand side.

In these notations we finally have

Theorem A.0.6 ([11, Chapter 7, Corollary 3.2.2.])**.** Let $\widehat{G} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$ be a formal group over X. Then there is a functorial equivalence

$$\mathbb{V}(\mathrm{Lie}_X(\widehat{G})) \xrightarrow[\sim]{\exp_{\widehat{G}}} \widehat{G}$$

of formal moduli problems over X.

Idea of the proof. Given a Lie algebra $\mathfrak{g} \in \mathrm{ICoh}(X)$ its universal enveloping algebra is naturally a cocommutative Hopf algebra, i.e. a group object in the category $\mathrm{coCAlg}(\mathrm{ICoh}(X))$. Since the functor $\mathrm{Spec}^{\mathsf{inf}}$ is monoidal, we see that $\exp_X(\mathfrak{g}) := \mathrm{Spec}^{\mathsf{inf}}\,(U(\mathfrak{g}))$ is a group object in $\widehat{\mathrm{Moduli}}_{/X}$. In fact, one can show that the construction $\mathfrak{g} \longmapsto \exp_X(\mathfrak{g})$ is the inverse to the Lie_X functor from Theorem A.0.2. But by [11, Chapter 6, Corollary 1.7.3] there is canonical equivalence of cocommutative coalgebras $U(\mathfrak{g}) \simeq \mathrm{Sym}_{\mathrm{ICoh}(X)}(\mathfrak{g})$ (aka Milnor-Moore theorem), hence for $\mathfrak{g} = \mathrm{Lie}_X(\widehat{G})$ we have

$$\widehat{G} \simeq \exp_X(\mathfrak{g}) = \mathrm{Spec}^{\mathsf{inf}}\left(U(\mathfrak{g})\right) \simeq \mathrm{Spec}^{\mathsf{inf}}\left(\mathrm{Sym}_{\mathrm{ICoh}(X)}(\mathfrak{g})\right) = \mathbb{V}(\mathfrak{g}).$$

□

Remark A.0.7. Notice that for a formal group $\widehat{A} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$ with abelian Lie algebra the map $\exp_{\widehat{A}}$ above is not only an equivalence of formal moduli problems, but moreover an equivalence of formal groups. For example, in the case $\widehat{G} := \widehat{\mathbb{G}_m}$ the map

$$\widehat{\mathbb{G}_a} \simeq \mathbb{V}(\mathrm{Lie}_X(\widehat{\mathbb{G}_m})) \xrightarrow[\sim]{\exp_{\widehat{\mathbb{G}_m}}} \widehat{\mathbb{G}_m}$$

is the usual formal exponent.

Appendix B. Correspondences and traces

B.1. Ind-coherent sheaves and morphism of traces

In this section we discuss how one can calculate the morphism of traces in the setting of ind-coherent sheaves using the category of correspondences. We start with the following

Theorem B.1.1 ([10, Chapter 5, Theorem 2.1.4., Theorem 4.1.2])**.** The ind-coherent sheaves functor can be lifted to a symmetric monoidal functor

$$\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}} \xrightarrow{\mathrm{ICoh}} 2\,\mathrm{Cat}_k\,.$$

Where $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$ is the $(\infty, 2)$-category which can be informally described as follows:

1. Its objects are those of $\mathrm{Sch}_{\mathsf{aft}}$.
2. Given $X, Y \in \mathrm{Sch}_{\mathsf{aft}}$ a morphism from X to Y in $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$ is given by a span

   $$X \xleftarrow{g} W \xrightarrow{f} Y$$

   and the composition of morphisms is given by taking pullbacks.

3. Given two morphisms $W_1, W_2 \in \mathrm{Hom}_{\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}}(X, Y)$ a 2-morphism from W_1 to W_2 is given by a commutative diagram

$$\begin{tikzcd} & W_1 \arrow[dl] \arrow[dr] \arrow[dd, "h"] & \\ X & & Y \\ & W_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in $\mathrm{Sch}_{\mathsf{aft}}$ where h is proper.

The symmetric monoidal structure on $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$ is given by the cartesian product of underlying objects of $\mathrm{Sch}_{\mathsf{aft}}$. Once again, we refer to [10, Part III] for a discussion of the category of correspondences.

Remark B.1.2. In [10, Chapter 7] the category $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$ was denoted by $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}_{\mathsf{all,all}}$.

Informally speaking, the functor above maps $X \in \mathrm{Sch}_{\mathsf{aft}}$ to the category $\mathrm{ICoh}(X) \in \mathrm{Cat}_k$ and a morphism $X \xleftarrow{g} W \xrightarrow{f} Y$ in $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$ to the composite $\mathrm{ICoh}(X) \xrightarrow{g^!} \mathrm{ICoh}(W) \xrightarrow{f_*} \mathrm{ICoh}(Y)$ in $2\,\mathrm{Cat}_k$.

Remark B.1.3. Since by ([10, Chapter 9, Proposition 2.3.4.]) every object $X \in \mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$ is self-dual via the morphisms

$$* \xleftarrow{\quad p \quad} X \xrightarrow{\quad \Delta \quad} X \times X$$

and

$$X \times X \xleftarrow{\quad \Delta \quad} X \xrightarrow{\quad p \quad} *$$

we see that the category $\mathrm{ICoh}(X) \in 2\,\mathrm{Cat}_k$ is also self-dual. Moreover, note that by ([10, Chapter 9, Proposition 2.3.4.]) every morphism in $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$ of the form

$$X \xleftarrow{\quad \mathrm{Id}_X \quad} X \xrightarrow{\quad f \quad} Y$$

where f is proper admits a right adjoint given by

$$Y \xleftarrow{\quad f \quad} X \xrightarrow{\quad \mathrm{Id}_X \quad} X$$

Our goal now is to understand morphism of traces in the setting of correspondences. We start by calculating classical traces in the category of correspondences

Proposition B.1.4. The trace of the endomorphism $X \xleftarrow{g} Y \xrightarrow{f} X$ in $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$ is given by

$$* \longleftarrow X^{f=g} \longrightarrow * ,$$

where $X^{f=g}$ is defined as the pullback

$$\begin{tikzcd} X^{f=g} \arrow[r, "i"] \arrow[d, "j"'] & Y \arrow[d, "(f,g)"] \\ X \arrow[r, "\Delta"'] & X \times X. \end{tikzcd}$$

Proof. By definition, the trace is given by the composite

$$\begin{tikzcd}[column sep=small] & X \arrow[dl, "p"'] \arrow[dr, "\Delta"] & & Y \times X \arrow[dl, "g\times\mathrm{Id}_X"'] \arrow[dr, "f\times\mathrm{Id}_X"] & & X \arrow[dl, "\Delta"'] \arrow[dr, "p"] & \\ * & & X \times X & & X \times X & & *. \end{tikzcd}$$

Since the composition in $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$ is given by taking pullback, the result follows. ☐

Corollary B.1.5. Applying the functor $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}} \xrightarrow{\mathrm{ICoh}} 2\,\mathrm{Cat}_k$ we see that the trace of the endomorphism $\mathrm{ICoh}(X) \xrightarrow{f_* g^!} \mathrm{ICoh}(X)$ in $2\,\mathrm{Cat}_k$ is given by $\Gamma(X^{f=g}, \omega^{\mathrm{ICoh}}_{X^{f=g}})$.

We are now going to understand morphism of traces in the setting of correspondences (and therefore in the setting of ind-coherent sheaves). In order to simplify notation we will denote a morphism

$$X \xleftarrow{\quad g \quad} W \xrightarrow{\quad f \quad} Y$$

by $\langle\, {}^{g}_{X}W^{f}_{Y} \,\rangle$. Here is the main

Proposition B.1.6. Given a (not necessary commutative) diagram

$$\begin{tikzcd}
X \arrow[r, "\langle {}^{g}_{X}Y^{f}_{X} \rangle"] \arrow[d, "\langle {}^{\mathrm{Id}_X}_{X}X^{s}_{U} \rangle"'] & X \arrow[d, "\langle {}^{\mathrm{Id}_X}_{X}X^{s}_{U} \rangle"] \arrow[dl, Rightarrow, "T"'] \\
U \arrow[r, "\langle {}^{b}_{U}V^{a}_{U} \rangle"'] & U
\end{tikzcd}$$

in $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$ where the 2-morphism T is given by a choice of some commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "g"'] \arrow[dr, "s\circ f"] \arrow[dd, "t"] & \\
X & & U \\
 & X\ {}^{s}\times^{b}_{U} V \arrow[ul, "\mathsf{pr}_X"] \arrow[ur, "a\circ\mathsf{pr}_V"'] &
\end{tikzcd}$$

$$\alpha_1 : b \circ \mathsf{pr}_V \circ t \simeq s \circ \mathsf{pr}_X \circ t$$
$$\alpha_2 : \mathsf{pr}_X \circ t \simeq g$$
$$\alpha_3 : a \circ \mathsf{pr}_V \circ t \simeq s \circ f$$

where $t \in \mathrm{Hom}_{\mathrm{Sch}_{\mathsf{aft}}}(Y, X\ {}^{s}\times^{b}_{U} V)$ is proper, the induced morphism of traces

$$X^{f=g} \simeq \mathsf{Tr}_{\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}}\left(\langle {}^{g}_{X}Y^{f}_{X} \rangle\right) \xrightarrow{\mathsf{Tr}\left(\langle {}^{\mathrm{Id}_X}_{X}X^{s}_{U} \rangle, T\right)} \mathsf{Tr}_{\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}}\left(\langle {}^{b}_{U}V^{a}_{U} \rangle\right) \simeq U^{a=b}$$

is obtained as the map of pullbacks from the commutative diagram

$$\begin{tikzcd}
X^{f=g} \arrow[rr] \arrow[dr] \arrow[dd] & & X^{f=g} \arrow[dr] \arrow[dd, "s\circ i_X" near start] & \\
 & X^{f=g} \arrow[rr] \arrow[dd, "\mathsf{pr}_V \circ t \circ j_X" near end] & & X^{f=g} \arrow[dd, "(s\circ i_X, s\circ i_X)"] \\
U^{a=b} \arrow[rr, "i_U" near end] \arrow[dr, "j_U"'] & & U \arrow[dr, "\Delta"] & \\
 & V \arrow[rr, "(b,a)"] & & U \times U,
\end{tikzcd}$$

where the commutativity of the front square is given by the equivalences

$$b \circ \mathsf{pr}_V \circ t \circ j_X \overset{\alpha_1 \circ j_X}{\simeq} s \circ \mathsf{pr}_X \circ t \circ j_X \overset{s \circ \alpha_2 \circ j_X}{\simeq} s \circ g \circ j_X \simeq s \circ i_X$$

and

$$a \circ \mathsf{pr}_V \circ t \circ j_X \overset{\alpha_3 \circ j_X}{\simeq} s \circ f \circ j_X \simeq s \circ i_X.$$

Proof. Unwinding the definitions, we see that the morphism of traces is induced by the diagram

$$\begin{tikzcd}
* \arrow[r, "\langle {}_{*}X^{\Delta}_{X\times X} \rangle"] \arrow[d, "\mathrm{Id}_*"'] & X \times X \arrow[r, "\langle {}^{g\times \mathrm{Id}_X}_{X\times X} Y\times X^{f\times \mathrm{Id}_X}_{X\times X} \rangle"] \arrow[d] \arrow[dl, Rightarrow] & X \times X \arrow[r, "\langle {}_{X\times X}^{\Delta} X_{*} \rangle"] \arrow[d] \arrow[dl, Rightarrow] & * \arrow[d, "\mathrm{Id}_*"] \arrow[dl, Rightarrow] \\
* \arrow[r, "\langle {}_{*}U^{\Delta}_{U\times U} \rangle"'] & U \times U \arrow[r, "\langle {}^{b\times \mathrm{Id}_U}_{U\times U} V\times U^{a\times \mathrm{Id}_U}_{U\times U} \rangle"'] & U \times U \arrow[r, "\langle {}_{U\times U}^{\Delta} U_{*} \rangle"'] & *
\end{tikzcd}$$

where

1. The two middle vertical morphisms are given by $\langle {}^{\mathrm{Id}_X \times \mathrm{Id}_X}_{X\times X} X \times X^{s\times s}_{U\times U} \rangle$ (see remark B.1.3).
2. The left square is induced by the 2-morphism

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dd, "s"] \arrow[dr, "(s{,}s)"] & \\
* & & U \times U \\
 & U \arrow[ul] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

in $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$.

3. The middle square is induced by the 2-morphism

$$\begin{tikzcd}
 & Y \times X \arrow[dl, "g\times \mathrm{Id}_X"'] \arrow[dd, "t\times \mathrm{Id}_X"] \arrow[dr, "(s\circ f)\times s"] & \\
X \times X & & U \times U \\
 & (X\ {}^{s}\!\times^{b}_{U} V) \times X \arrow[ul, "\mathsf{pr}_X \times \mathrm{Id}_X"] \arrow[ur, "(a\circ \mathsf{pr}_V)\times s"'] &
\end{tikzcd}$$

in $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$.

4. The right square is induced by the 2-morphism

$$
\begin{tikzcd}
 & X \arrow[dl, "\Delta"'] \arrow[dr] \arrow[dd] & \\
X \times X & & * \\
 & X \times_U X \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

in $\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}}$.

Consequently, we see that the whole morphism of traces is given by the composite of the morphisms

$$Y\ {}^{(f,g)}\!\times^{\Delta}_{(X\times X)} X \longrightarrow Y\ {}^{(f,g)}\!\times_{(X\times X)} (X\times_U X) \simeq X\ {}^{\Delta}\!\times^{g\times \mathrm{Id}_X}_{(X\times X)} (Y\times X)\ {}^{(s\circ f)\times s}\!\times^{\Delta}_{U\times U} U$$

$$X\ {}^{\Delta}\!\times^{g\times \mathrm{Id}_X}_{(X\times X)} (Y\times X)\ {}^{(s\circ f)\times s}\!\times^{\Delta}_{U\times U} U \longrightarrow X\ {}^{\Delta}\!\times^{\mathsf{pr}_X\times \mathrm{Id}_X}_{(X\times X)} \big((X\times_V Y)\times X\big)\ {}^{(a\circ \mathsf{pr}_V)\times s}\!\times^{\Delta}_{U\times U} U$$

and

$$X\ {}^{\Delta}\!\times^{\mathsf{pr}_X\times \mathrm{Id}_X}_{(X\times X)} \big((X\times_V Y)\times X\big)\ {}^{(a\circ \mathsf{pr}_V)\times s}\!\times^{\Delta}_{U\times U} U \simeq X\ {}^{(s,s)}\!\times^{(b,a)}_{(U\times U)} V \longrightarrow U\ {}^{\Delta}\!\times^{(b,a)}_{(U\times U)} V.$$

In particular, we can rewrite the whole composition as the left vertical morphism from $X^{f=g}$ to $U^{a=b}$ in the commutative diagram

$$
\begin{tikzcd}
X^{f=g} \arrow[rr] \arrow[dr] \arrow[dd] & & Y \arrow[rr, "f"] \arrow[dr] & & X \arrow[dr] \arrow[dd] & \\
 & X^{f=g} \arrow[rr, "j_X"] \arrow[dd, "j_X"] & & Y \arrow[rr, "f"] & & X \arrow[dd, "\Delta"] \\
Y\ {}^{s\circ f}\!\times^{s\circ g}_U Y \arrow[rr] \arrow[dr] \arrow[dd] & & Y\ {}^{s\circ f}\!\times^{s}_U X \arrow[rr] \arrow[dr] \arrow[dd] & & X\ {}^{s}\!\times^{s}_U X \arrow[dr] & \\
 & Y \arrow[rr, "(\mathrm{Id}_Y,g)"] \arrow[dd, "(g,\mathsf{pr}_V\circ t)"] & & Y\times X \arrow[rr, "f\times \mathrm{Id}_X"] \arrow[dd, "t\times \mathrm{Id}_X"] & & X\times X \\
A \arrow[rr] \arrow[dr] \arrow[dd] & & (X\ {}^{s}\!\times^{b}_U V)\ {}^{a\circ \mathsf{pr}_V}\!\times^{s}_U X \arrow[rr] \arrow[dr] & & V \arrow[dr, "(\mathrm{Id}_V,a)"] \arrow[dd] & \\
 & X\ {}^{s}\!\times^{b}_U V \arrow[rr, "\Delta_X\times_U V"] \arrow[dd, "\mathsf{pr}_V"] & & (X\ {}^{s}\!\times^{b}_U V)\times X \arrow[rr, "\mathsf{pr}_V\times s"] & & V\times U \arrow[dd] \\
U^{a=b} \arrow[rr] \arrow[dr] & & U^{a=b} \arrow[rr, "j_U"] \arrow[dr, "j_U"] & & V \arrow[dr, "(\mathrm{Id}_V,a)"] & \\
 & V \arrow[rr] & & V \arrow[rr, "(\mathrm{Id}_V,b)"] & & V\times U
\end{tikzcd}
$$

where all the horizontal squares are pullback squares. Consequently, we see that we can rewrite the morphism of traces as the left vertical morphism from $X^{f=g}$ to $U^{a=b}$ in the commutative diagram

$$
\begin{tikzcd}
X^{f=g} \arrow[rr] \arrow[dr] \arrow[dd] & & Y \arrow[dr] \arrow[dd] & \\
& X^{f=g} \arrow[rr, "j_X"] \arrow[dd, "j_X"] & & Y \arrow[dd, "(\mathrm{Id}_Y{,}f)"] \\
Y \; {}^{s\circ f}\times^{s\circ g}_U Y \arrow[rr] \arrow[dr] \arrow[dd] & & Y \; {}^{s\circ f}\times^{s}_U X \arrow[dr] \arrow[dd] & \\
& Y \arrow[rr, "(\mathrm{Id}_Y{,}g)"] \arrow[dd, "(g{,}\mathsf{pr}_V \circ t)"] & & Y \times X \arrow[dd, "(\mathsf{pr}_V \circ t)\times s"] \\
A \arrow[rr] \arrow[dr] \arrow[dd] & & V \arrow[dr, "(\mathrm{Id}_V{,}a)"] \arrow[dd] & \\
& X \; {}^{s}\times^{b}_U V \arrow[rr, "s\times \mathrm{Id}_V"] \arrow[dd, "\mathsf{pr}_V"] & & V \times U \arrow[dd] \\
U^{a=b} \arrow[rr] \arrow[dr] & & V \arrow[dr, "(\mathrm{Id}_V{,}a)"] & \\
& V \arrow[rr, "(\mathrm{Id}_V{,}b)"] & & V \times U.
\end{tikzcd}
$$

It is only left to note that the above morphism can be rewritten precisely as in the statement of the proposition. ☐

Corollary B.1.7. Applying in the setting of the above proposition the functor

$$\mathrm{Corr}(\mathrm{Sch}_{\mathsf{aft}})^{\mathsf{proper}} \xrightarrow{\mathrm{ICoh}} 2\,\mathrm{Cat}_k$$

we see that given a commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "g_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\
Y \arrow[r, "g_Y"'] & Y
\end{tikzcd}
$$

the morphism of traces

$$\Gamma(X^{g_X}, \omega^{\mathrm{ICoh}}_{X^{g_X}}) \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}\big((g_X)_*\big) \longrightarrow \mathsf{Tr}_{2\,\mathrm{Cat}_k}\big((g_Y)_*\big) \simeq \Gamma(Y^{g_Y}, \omega^{\mathrm{ICoh}}_{Y^{g_Y}})$$

induced by the diagram

$$\begin{tikzcd} \mathrm{ICoh}(X) \arrow[r, "(g_X)_*"] \arrow[d, bend right, "f_*"'] & \mathrm{ICoh}(X) \arrow[d, bend right, "f_*"'] \arrow[dl, Rightarrow] \\ \mathrm{ICoh}(Y) \arrow[r, "(g_Y)_*"'] \arrow[u, bend right, "f^!"'] & \mathrm{ICoh}(Y) \arrow[u, bend right, "f^!"'] \end{tikzcd}$$

is given by the counit of the adjunction $(f^g)_* \dashv (f^g)^!$ where $X^{g_X} \xrightarrow{f^g} Y^{g_Y}$ is the induced morphism between fixed points.

B.2. Decorated correspondences and orientations

Our goal in this section is to show the morphism of traces

$$\Gamma(X^g, \mathcal{O}_{X^g}) \simeq \mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*^{\mathrm{QCoh}}) \xrightarrow[\sim]{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes \mathcal{O}_X)} \mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*^{\mathrm{ICoh}}) \simeq \Gamma(X^g, \omega_{X^g})$$

induced by the diagram

$$\begin{tikzcd} \mathrm{QCoh}(X) \arrow[r, "g_*"] \arrow[d, "-\otimes\mathcal{O}_X"'] & \mathrm{QCoh}(X) \arrow[d, "-\otimes\mathcal{O}_X"] \\ \mathrm{ICoh}(X) \arrow[r, "g_*"'] & \mathrm{ICoh}(X) \end{tikzcd}$$

is induced by the canonical orientation (Construction 3.2.1). Using the fact that for an eventually coconnective morphism $X \xrightarrow{f} Y$ almost of finite type between Noetherian schemes one has a Grothendieck formula $f^!- \simeq \omega_f \otimes f^*-$ ([9, Corollary 7.2.5.]) we will reduce the calculation of the morphism of traces to a simple calculation in a version of the category of correspondences.

We start with the following

Definition B.2.1. We define an $(\infty, 2)$-*category of correspondences decorated by* QCoh denoted by $\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}$ as follows:

1. Its objects are those of Sch.
2. Given $X, Y \in \mathrm{Sch}$ a morphism from X to Y in $\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}$ is given by a span
$$X \xleftarrow{g} W \xrightarrow{f} Y$$
together with a quasi-coherent sheaf $\mathcal{F}_W \in \mathrm{QCoh}(Y)$ on W and the composition of morphisms is given by taking pullbacks of schemes and box products of sheaves.
3. Given two morphisms $(W_1, \mathcal{F}_{W_1}), (W_2, \mathcal{F}_{W_2}) \in \mathrm{Hom}_{\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}}(X, Y)$ a 2-morphism from $(W_1, \mathcal{F}_{W_1})$ to $(W_2, \mathcal{F}_{W_2})$ is given by a commutative diagram
$$\begin{tikzcd} & W_1 \arrow[dl] \arrow[dd, "h"] \arrow[dr] & \\ X & & Y \\ & W_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$
in Sch and a morphism $h^*\mathcal{F}_{W_2} \longrightarrow \mathcal{F}_{W_1}$ in $\mathrm{QCoh}(\mathcal{F}_{W_1})$.

Notation B.2.2. We will further denote by $\langle\, {}^{g}_{X}W^{f}_{Y}, \mathcal{F}_W \,\rangle$ the morphism $X \xleftarrow{g} (W, \mathcal{F}_W) \xrightarrow{f} Y$ in $\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}$ with the correspondence given by $X \xleftarrow{g} W \xrightarrow{f} Y$ and a sheaf given by $\mathcal{F}_W \in \mathrm{QCoh}(W)$ and depict it as
$$\begin{tikzcd} & (W, \mathcal{F}_W) \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & Y. \end{tikzcd}$$
In the case we omit $\mathcal{F}_W$ from notation we assume that it is given by $\mathcal{O}_W$.

Now note that the $(\infty, 2)$-category $\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}$ is symmetric monoidal with the monoidal structure given by the cartesian product of underlying objects of Sch (the morphisms are tensored by taking box product of the corresponding sheaves). Moreover, if $X \in \mathrm{Sch}$ is a scheme and $\mathcal{M} \in \mathrm{Pic}(X)$

is a line bundle on X then it is straightforward to see that the morphisms

$$\begin{tikzcd}[column sep=small] & (X,\mathcal{M}) \arrow[dl, "p"'] \arrow[dr, "\Delta"] & & & (X,\mathcal{M}^{-1}) \arrow[dl, "\Delta"'] \arrow[dr, "p"] & \\ * & & X\times X & X\times X & & * \end{tikzcd}$$

in $\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}$ exhibit $X \in \mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}$ as a self-dual object.

Proposition B.2.3. Let $X \in \mathrm{Sch}$ be a scheme with an endomorphism $X \xrightarrow{g} X$ and $\mathcal{M} \in \mathrm{Pic}(X)$ be a line bundle on X. Then there is an equivalence

$$\mathsf{Tr}^{\mathcal{M}}_{\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}}(g) \overset{\eta_{\mathcal{M}}}{\simeq} \langle\, {}_*(X^g)_* \,\rangle$$

in $\mathrm{Hom}_{\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}}(*,*)$, where $\mathsf{Tr}^{\mathcal{M}}_{\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}}(g)$ is the trace of $X \xrightarrow{g} X$ in $\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}$ with respect to the dualization data

$$* \xleftarrow{p} (X,\mathcal{M}) \xrightarrow{\Delta} X\times X$$

and

$$X\times X \xleftarrow{\Delta} (X,\mathcal{M}^{-1}) \xrightarrow{p} *$$

on $X \in \mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}$.

Proof. By definition the trace $\mathsf{Tr}^{\mathcal{M}}_{\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}}(g)$ is given by the composite

$$\begin{tikzcd}[column sep=small] & (X,\mathcal{M}) \arrow[dl, "p"'] \arrow[dr, "\Delta"] & & X\times X \arrow[dl, "\mathrm{Id}_X\times\mathrm{Id}_X"'] \arrow[dr, "g\times\mathrm{Id}_X"] & & (X,\mathcal{M}^{-1}) \arrow[dl, "\Delta"'] \arrow[dr, "p"] & \\ * & & X\times X & & X\times X & & * \end{tikzcd}$$

which composing the first two morphisms can be rewritten as

$$\begin{tikzcd}[column sep=small] & (X,\mathcal{M}) \arrow[dl, "p"'] \arrow[dr, "{(g,\mathrm{Id}_X)}"] & & (X,\mathcal{M}^{-1}) \arrow[dl, "\Delta"'] \arrow[dr, "p"] & \\ * & & X\times X & & *. \end{tikzcd}$$

The equivalence $\eta_{\mathcal{M}}$ is now induced from the pullback diagram

$$\begin{tikzcd} X^g \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "\Delta"] \\ X \arrow[r, "(\mathrm{Id}_X, g)"'] & X \times X \end{tikzcd}$$

and the equivalence $i^*\mathcal{M} \otimes i^*\mathcal{M}^{-1} \simeq \mathcal{O}_{X^g}$. $\square$

We now prove the following

Proposition B.2.4. Let X be a scheme with an endomorphism $X \xrightarrow{g} X$ and $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Pic}(X)$ be two line bundles on X. Then the morphism of traces

$$\langle\, {}_*(X^g)_* \,\rangle \overset{\eta_{\mathcal{M}_1}}{\simeq} \mathsf{Tr}^{\mathcal{M}_1}_{(\,\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}})^{2-\mathsf{op}}}(g) \longrightarrow \mathsf{Tr}^{\mathcal{M}_2}_{(\,\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}})^{2-\mathsf{op}}}(g) \overset{\eta_{\mathcal{M}_2}}{\simeq} \langle\, {}_*(X^g)_* \,\rangle$$

in $\mathrm{Hom}_{(\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}})^{2-\mathsf{op}}}(*, *)$ induced by the commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\mathrm{Id}_X"'] & X \arrow[d, "\mathrm{Id}_X"] \\ X \arrow[r, "g"'] & X \end{tikzcd}$$

is given by the identity 2-morphism

$$\begin{tikzcd} & X^g \arrow[dl] \arrow[dr] \arrow[dd, "\mathrm{Id}_{X^g}"] & \\ * & & * \\ & X^g \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Proof. By [15, Example 1.2.5] the morphism of traces is induced by the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
* \arrow[r, "\langle {}_{*}X^{\Delta}_{X\times X}{,}\mathcal{M}_1 \rangle"] \arrow[d, "\mathrm{Id}_*"'] & X\times X \arrow[r, "\langle {}^{\mathrm{Id}_{X\times X}}_{X\times X}X\times X^{g\times\mathrm{Id}_X}_{X\times X} \rangle"] \arrow[d] \arrow[dl, Rightarrow] & X\times X \arrow[r, "\langle {}^{\Delta}_{X\times X}X_*{,}\mathcal{M}_1^{-1} \rangle"] \arrow[d] \arrow[dl, Rightarrow] & * \arrow[d, "\mathrm{Id}_*"] \arrow[dl, Rightarrow] \\
* \arrow[r, "\langle {}_{*}X^{\Delta}_{X\times X}{,}\mathcal{M}_2 \rangle"'] & X\times X \arrow[r, "\langle {}^{\mathrm{Id}_{X\times X}}_{X\times X}X\times X^{g\times\mathrm{Id}_X}_{X\times X} \rangle"'] & X\times X \arrow[r, "\langle {}^{\Delta}_{X\times X}X_*{,}\mathcal{M}_2^{-1} \rangle"'] & *
\end{tikzcd}$$

where the vertical morphisms are given by

$$X\times X \xleftarrow{\mathrm{Id}_{X\times X}} (X\times X, \mathcal{M}_1^{-1}\boxtimes\mathcal{M}_2) \xrightarrow{\mathrm{Id}_{X\times X}} X\times X.$$

The result now follows from equivalences

$$\langle {}^{\Delta}_{X\times X}X_*, \mathcal{M}_1^{-1} \rangle \circ \langle {}^{\mathrm{Id}_{X\times X}}_{X\times X}X\times X^{g\times\mathrm{Id}_X}_{X\times X} \rangle \circ \langle {}_{*}X^{\Delta}_{X\times X}, \mathcal{M}_1 \rangle \simeq \langle {}_{*}(X^g)_* \rangle\ ,$$

$$\begin{gathered}\langle {}^{\Delta}_{X\times X}X_*, \mathcal{M}_2^{-1} \rangle \circ \langle {}^{\mathrm{Id}_{X\times X}}_{X\times X}X\times X^{\mathrm{Id}_{X\times X}}_{X\times X}, \mathcal{M}_1^{-1}\boxtimes\mathcal{M}_2 \rangle \circ \\ \circ \langle {}^{\mathrm{Id}_{X\times X}}_{X\times X}X\times X^{g\times\mathrm{Id}_X}_{X\times X} \rangle \circ \langle {}_{*}X^{\Delta}_{X\times X}, \mathcal{M}_1 \rangle \simeq \langle {}_{*}(X^g)_* \rangle,\end{gathered}$$

$$\begin{gathered}\langle {}^{\Delta}_{X\times X}X_*, \mathcal{M}_2^{-1} \rangle \circ \langle {}^{\mathrm{Id}_{X\times X}}_{X\times X}X\times X^{g\times\mathrm{Id}_X}_{X\times X} \rangle \circ \\ \circ \langle {}^{\mathrm{Id}_{X\times X}}_{X\times X}X\times X^{\mathrm{Id}_{X\times X}}_{X\times X}, \mathcal{M}_1^{-1}\boxtimes\mathcal{M}_2 \rangle \circ \langle {}_{*}X^{\Delta}_{X\times X}, \mathcal{M}_1 \rangle \simeq \langle {}_{*}(X^g)_* \rangle,\end{gathered}$$

$$\langle {}^{\Delta}_{X\times X}X_*, \mathcal{M}_2^{-1} \rangle \circ \langle {}^{\mathrm{Id}_{X\times X}}_{X\times X}X\times X^{g\times\mathrm{Id}_X}_{X\times X} \rangle \circ \langle {}_{*}X^{\Delta}_{X\times X}, \mathcal{M}_2 \rangle \simeq \langle {}_{*}(X^g)_* \rangle$$

and the fact that the corresponding 2-morphisms are given by identity maps Id_{X^g}. ☐

To see why the proposition above is useful, we have the following generalization of [10, Chapter 5, 5.3.1]

Theorem B.2.5. The quasi-coherent sheaves functor can be lifted to a symmetric monoidal functor

$$\big(\,\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}\big)^{2-\mathsf{op}} \xrightarrow{\widetilde{\mathrm{QCoh}}} 2\,\mathrm{Cat}_k$$

where:

- An object $X \in \mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}$ is sent to the category of quasi-coherent sheaves $\mathrm{QCoh}(X) \in 2\,\mathrm{Cat}_k$ on it.
- A morphism

$$\begin{tikzcd} & (W, \mathcal{F}_W) \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

in $\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}$ is sent to the morphism

$$\mathrm{QCoh}(X) \xrightarrow{f_*(\mathcal{F}_W \otimes g^* -)} \mathrm{QCoh}(Y)$$

in $2\,\mathrm{Cat}_k$.
- A 2-morphism

$$\begin{tikzcd} & (W_1, \mathcal{F}_{W_1}) \arrow[dl, "g"'] \arrow[dr, "f"] \arrow[dd, "h"] & \\ X & & Y \\ & (W_2, \mathcal{F}_{W_2}) \arrow[ul, "t"] \arrow[ur, "s"'] & \end{tikzcd}$$

with $h^*\mathcal{F}_{W_2} \xrightarrow{\eta} \mathcal{F}_{W_1}$ is sent to the 2-morphism

$$s_*(\mathcal{F}_{W_2} \otimes t^* -) \longrightarrow s_*(h_* h^* \mathcal{F}_{W_2} \otimes t^* -) \simeq$$

$$\simeq s_* h_*(h^*\mathcal{F}_{W_2} \otimes h^* t^* -) \simeq f_*(h^*\mathcal{F}_{W_2} \otimes g^* -) \xrightarrow{\eta} f_*(\mathcal{F}_{W_1} \otimes g^* -)$$

in $2\,\mathrm{Cat}_k$.

Example B.2.6. Let X be a Gorenstein Noetherian scheme. Then the functor $\widetilde{\mathrm{QCoh}}$ sends the morphism

$$\begin{tikzcd} & (X, \omega_X) \arrow[dl, "p"'] \arrow[dr, "\Delta"] & \\ * & & X \times X \end{tikzcd}$$

in $\left(\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}\right)^{2-\mathsf{op}}$ to

$$\Delta_* \omega_X \in \mathrm{Hom}_{2\,\mathrm{Cat}_k}(\mathrm{Vect}_k, \mathrm{QCoh}(X \times X)) \simeq \mathrm{QCoh}(X \times X)$$

and the morphism

$$\begin{tikzcd} & (X,\omega_X^{-1}) \arrow[dl, "\Delta"'] \arrow[dr, "p"] & \\ X\times X & & * \end{tikzcd}$$

to

$$\Gamma(X,\omega_X^{-1}\otimes\Delta^*-)\simeq\Gamma(X,\omega_\Delta\otimes\Delta^*-)\simeq$$

$$\simeq\Gamma(X,\Delta^!-)\in\mathrm{Hom}_{2\,\mathrm{Cat}_k}(\mathrm{QCoh}(X\times X),\mathrm{Vect}_k).$$

We can now prove

Proposition B.2.7. For a classical smooth scheme X together with an endomorphism $X\xrightarrow{g}X$ the morphism of traces

$$\Gamma(X^g,\mathcal{O}_{X^g})\overset{\alpha_{\mathrm{QCoh}}}{\simeq}\mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*^{\mathrm{QCoh}})\xrightarrow[\sim]{\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_X)}\mathsf{Tr}_{2\,\mathrm{Cat}_k}(g_*^{\mathrm{ICoh}})\overset{\alpha_{\mathrm{ICoh}}^{-1}}{\simeq}\Gamma(X^g,\omega_{X^g})$$

induced by the diagram

$$\begin{tikzcd} \mathrm{QCoh}(X) \arrow[r, "g_*"] \arrow[d, "-\otimes\mathcal{O}_X"'] & \mathrm{QCoh}(X) \arrow[d, "-\otimes\mathcal{O}_X"] \\ \mathrm{ICoh}(X) \arrow[r, "g_*"'] & \mathrm{ICoh}(X), \end{tikzcd}$$

can be obtained by applying the global sections functor $\Gamma(X^g,-)$ to the canonical orientation $\mathcal{O}_{X^g}\xrightarrow[\sim]{\mathrm{u}_C}\omega_{X^g}$ (see Construction 3.2.1) on X^g, where the equivalences α_{QCoh} and α_{ICoh} above are given by Corollary 1.1.2 and Corollary 2.2.2 respectively.

Proof. Due to the equivalence $\mathrm{ICoh}(X)\simeq\mathrm{QCoh}(X)$ as X is smooth and classical (see Example 2.1.5) and Example B.2.6 above we note that the morphism of traces we are interested in can be obtained by applying the functor $\left(\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}\right)^{2-\mathsf{op}}\xrightarrow{\widetilde{\mathrm{QCoh}}}2\,\mathrm{Cat}_k$ to the morphism of traces

$$\mathsf{Tr}^{\mathcal{O}_X}_{\left(\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}\right)^{2-\mathsf{op}}}(g)\longrightarrow\mathsf{Tr}^{\omega_X}_{\left(\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}\right)^{2-\mathsf{op}}}(g)$$

in $\big(\operatorname{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}\big)^{2-\mathsf{op}}$. Now using equivalences $\eta_{\mathcal{O}_X}$ and η_{ω_X} we can form a commutative diagram

$$\begin{tikzcd}
\Gamma(X^g,\mathcal{O}_{X^g}) \arrow[r, "\alpha_{\mathrm{QCoh}}", "\sim"'] \arrow[dr] & \widetilde{\mathrm{QCoh}}\Big(\mathsf{Tr}^{\mathcal{O}_X}_{\left(\operatorname{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}\right)^{2-\mathsf{op}}}(g)\Big) \arrow[rr, "\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_X)", "\sim"'] \arrow[d, "\widetilde{\mathrm{QCoh}}(\eta_{\omega_X})", "\sim"'] & & \widetilde{\mathrm{QCoh}}\Big(\mathsf{Tr}^{\omega_X}_{\left(\operatorname{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}\right)^{2-\mathsf{op}}}(g)\Big) \arrow[r, "\alpha^{-1}_{\mathrm{ICoh}}", "\sim"'] & \Gamma(X^g,\omega_{X^g}) \\
& \widetilde{\mathrm{QCoh}}(\langle\,{}_*(X^g)_*\,\rangle) \arrow[rr] & & \widetilde{\mathrm{QCoh}}(\langle\,{}_*(X^g)_*\,\rangle) \arrow[u, "\widetilde{\mathrm{QCoh}}(\eta^{-1}_{\omega_X})", "\sim"'] \arrow[ur] &
\end{tikzcd}$$

Since the left diagonal morphism is identity by the construction and the bottom horizontal morphism is identity by Proposition B.2.4, we see that the morphism of traces $\mathsf{Tr}_{2\,\mathrm{Cat}_k}(-\otimes\mathcal{O}_X)$ is given by the right diagonal morphism, that is, unwinding the definition of the morphism η_{ω_X} from Proposition B.2.3 by the composite of

$$\Gamma(X^g,\mathcal{O}_{X^g}) \simeq \Gamma(X^g, i^*\omega_X^{-1}\otimes i^*\omega_X) \simeq$$

$$\simeq \Gamma(X,\omega_X^{-1}\otimes i_*i^*\omega_X) \simeq \Gamma(X,\omega_X^{-1}\otimes \Delta^*(g\times \mathrm{Id}_X)_*\Delta_*\omega_X)$$

and

$$\Gamma(X,\omega_X^{-1}\otimes\Delta^*(g\times\mathrm{Id}_X)_*\Delta_*\omega_X) \simeq \Gamma(X,\Delta^!(\mathrm{Id}_X,g)_*\omega_X) \simeq$$

$$\simeq \Gamma(X, i_*i^!\omega_X) \simeq \Gamma(X, i_*\omega_{X^g}) \simeq \Gamma(X^g,\omega_{X^g}).$$

The result now follows from observation that the morphisms

$$i_*\mathcal{O}_{X^g} \simeq i_*(i^*\omega_X^{-1}\otimes i^*\omega_X) \simeq \omega_X^{-1}\otimes i_*i^*\omega_X \simeq$$

$$\simeq \omega_X^{-1}\otimes\Delta^*(g,\mathrm{Id}_X)_*\omega_X \simeq \Delta^!(\mathrm{Id}_X,g)_*\omega_X \simeq i_*i^!\omega_X \simeq i_*\omega_{X^g}$$

and

$$i_*i^*\mathcal{O}_{X^g} \simeq i_*(i^*\omega_X^{-1}\otimes i^*\omega_X) \simeq i_*(i^*\omega_{X/X\times X}\otimes i^*\omega_X) \simeq$$

$$\simeq i_*(\omega_{X^g/X}\otimes i^*\omega_X) \simeq i_*i^!\omega_X \simeq i_*\omega_{X^g}$$

coincide and Construction 3.2.1. □

Remark B.2.8. Using the fact that the functor

$$\big(\operatorname{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}\big)^{2-\mathsf{op}} \xrightarrow{\mathrm{QCoh}} 2\,\mathrm{Cat}_k$$

sends a morphism

$$\begin{tikzcd}
& (X,E) \arrow[dl, "p"'] \arrow[dr, "\mathrm{Id}_X"] & \\
* & & X
\end{tikzcd}$$

in $\left(\mathrm{Corr}(\mathrm{Sch})^{\mathrm{QCoh}}\right)^{2-\mathsf{op}}$ to the morphism $\mathrm{Vect}_k \xrightarrow{E} \mathrm{QCoh}(X)$ in $2\,\mathrm{Cat}_k$ one can also obtain a proof of [15, Proposition 2.2.3.] by calculating appropriate trace in the category of decorated correspondences and then mapping it to $2\,\mathrm{Cat}_k$.

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