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Metrics with Positive constant curvature and modular differential equations*

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Let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, where $\mathbb{H}$ is the complex upper half-plane, and $Q(z)$ be a meromorphic modular form of weight 4 on $\mathrm{SL}(2,\mathbb{Z})$ such that the differential equation $\mathcal{L} : y''(z) = Q(z)y(z)$ is Fuchsian on $\mathbb{H}^*$. In this paper, we consider the problem when $\mathcal{L}$ is apparent on $\mathbb{H}$, i.e., the ratio of any two nonzero solutions of $\mathcal{L}$ is single-valued and meromorphic on $\mathbb{H}$. Such a modular differential equation is closely related to the existence of a conformal metric $ds^2 = e^u|dz|^2$ on $\mathbb{H}$ with curvature $1/2$ that is invariant under $z \mapsto \gamma \cdot z$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.

Let $\pm\kappa_\infty$ be the local exponents of $\mathcal{L}$ at ∞. In the case $\kappa_\infty \in \frac{1}{2}\mathbb{Z}$, we obtain the following results:

(a) a complete characterization of $Q(z)$ such that $\mathcal{L}$ is apparent on $\mathbb{H}$ with only one singularity (up to $\mathrm{SL}(2,\mathbb{Z})$-equivalence) at $i = \sqrt{-1}$ or $\rho = (1+\sqrt{3}i)/2$, and
(b) a complete characterization of $Q(z)$ such that $\mathcal{L}$ is apparent on $\mathbb{H}^*$ with singularities only at i and ρ.

We provide two proofs of the results, one using Riemann's existence theorem and the other using Eremenko's theorem on the existence of conformal metric on the sphere.

In the case $\kappa_\infty \notin \frac{1}{2}\mathbb{Z}$, we let $r_\infty \in (0, 1/2)$ be defined by $r_\infty \equiv \pm\kappa_\infty \mod 1$. Assume that $r_\infty \notin \{1/12, 5/12\}$. A special case of an earlier result of Eremenko and Tarasov says that $1/12 < r_\infty < 5/12$ is the necessary and sufficient condition for the existence of the invariant metric. The threshold case $r_\infty \in \{1/12, 5/12\}$ is more delicate. We show that in the threshold case, an invariant metric exists if and only if $\mathcal{L}$ has two linearly independent solutions whose squares are meromorphic modular forms of weight -2 with a pair of conjugate characters on $\mathrm{SL}(2,\mathbb{Z})$. In the non-existence case, our

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example shows that the monodromy data of $\mathcal{L}$ are related to periods of the elliptic curve $y^2 = x^3 - 1728$.

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1. Introduction

A meromorphic function Q on the upper half-plane $\mathbb{H}$ is called a meromorphic modular form of weight $k \in \mathbb{Z}$ with respect to $\mathrm{SL}(2,\mathbb{Z})$ if Q satisfies

$$Q(\gamma \cdot z) = (cz+d)^k Q(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}),$$

and Q is also meromorphic at the cusp ∞. When $k = 0$, a meromorphic modular form is called a modular function. We refer to [1] and [18] for the elementary theory of (holomorphic) modular forms. Given a meromorphic modular form Q of weight 4 on $\mathrm{SL}(2,\mathbb{Z})$, we consider a Fuchsian modular differential equation of second order on $\mathbb{H}$

$$y'' = Q(z)y \quad \text{on } \mathbb{H}, \qquad y' := \frac{dy}{dz}. \tag{1.1}$$

The differential equation (1.1) is called Fuchsian if the order of any pole of Q is less than or equal to 2. At ∞, by using $q = e^{2\pi i z}$, (1.1) can be written as

$$\left(q\frac{d}{dq}\right)^2 y = -\frac{1}{4\pi^2}y'' = -\frac{Q(z)}{4\pi^2}y. \tag{1.2}$$

So (1.1) is Fuchsian at ∞ if and only if Q is holomorphic at ∞.

Suppose that z_0 is a pole of Q. The local exponents of (1.1) are $1/2 \pm \kappa$, $\kappa \geq 0$. If the difference 2κ of the two local exponents is an integer, then the ODE (1.1) might have a solution with a logarithmic singularity at z_0. A singular point z_0 of (1.1) is called *apparent* if the local exponents are $1/2 \pm \kappa$ with $\kappa \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and any solution of (1.1) has no logarithmic singularity near z_0. In such a case, it is necessary that $\kappa > 0$. The ODE (1.1) or Q is called *apparent* if (1.1) is apparent at any pole of Q on $\mathbb{H}$. Clearly, if (1.1) is apparent then the local monodromy matrix at any pole is $\pm I$, where I is the 2×2 identity matrix.

A solution $y(z)$ of (1.1) might be multi-valued. For $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, $y(\gamma\cdot z)$ is understood as the analytic continuation of y along a path connecting z and $\gamma\cdot z$. Even though $y(\gamma\cdot z)$ is not well-defined, the slash operator of weight k ($k\in\mathbb{Z}$) can be defined in the usual way by

$$(1.3) \qquad \left(y\big|_k\gamma\right)(z) := (cz+d)^{-k}y(\gamma\cdot z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}),$$

where $\gamma\cdot z = (az+b)/(cz+d)$. We have the well-known Bol's identity [2]

$$\left(y\big|_{-1}\gamma\right)^{(2)}(z) = \left(y^{(2)}\big|_3\gamma\right)(z).$$

Hence, if $y(z)$ is a solution of (1.1), then $\left(y\big|_{-1}\gamma\right)(z)$ is also a solution of (1.1). Here $f^{(k)}(z)$ is the k-th derivative of $f(z)$.

Suppose that (1.1) is *apparent* and y_i, $i=1,2$, are two independent solutions. Since the local monodromy matrix at any pole of Q is $\pm I$, the ratio $h(z) = y_2(z)/y_1(z)$ is well-defined and meromorphic on $\mathbb{H}$. By Bol's identity, both $\left(y_i\big|_{-1}\gamma\right)(z)$ are solutions of (1.1), where $y_1(\gamma\cdot z)$ and $y_2(\gamma\cdot z)$ are understood as the analytic continuation of $y_1(z)$ and $y_2(z)$ along the same path connecting z and $\gamma\cdot z$. Note that since (1.1) is assumed to be apparent, difference choices of paths from z to $\gamma\cdot z$ only result in sign changes in $y_1(\gamma\cdot z)$ and $y_2(\gamma\cdot z)$. Therefore, there is a matrix $\rho(\gamma)$ in $\mathrm{GL}(2,\mathbb{C})$ such that

$$(1.4) \qquad \begin{pmatrix} \left(y_1\big|_{-1}\gamma\right)(z) \\ \left(y_2\big|_{-1}\gamma\right)(z) \end{pmatrix} = \pm\rho(\gamma)\begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}.$$

Note that $\det\rho(\gamma) = 1$ because the two Wronskians of fundamental solutions $\left(y_1\big|_{-1}\gamma, y_2\big|_{-1}\gamma\right)$ and (y_1,y_2) are equal. Hence ρ is a homomorphism from $\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{PSL}(2,\mathbb{C})$. In this paper, we call the homomorphism $\gamma\mapsto\pm\rho(\gamma)$ *the Bol representation* associated to (1.1).

There is an old problem in conformal geometry related to (1.1). The problem is to find a metric ds^2 with curvature $1/2$ on $\mathbb{H}$ that is locally conformal to the flat metric and invariant under the change $z\mapsto\gamma\cdot z$, $\gamma\in\mathrm{SL}(2,\mathbb{Z})$. Write $ds^2 = e^u|dz|^2$. Below, we collect some basic results concerning the metric which will be proved in Section 2.

(1) The curvature condition is equivalent to saying that u satisfies the curvature equation (2.5). Then

$$Q(z) = -\frac{1}{2}\left(u_{zz} - \frac{1}{2}u_z^2\right) \tag{1.5}$$

is a meromorphic function.

(2) The invariant condition ensures that Q is a meromorphic modular form of weight 4 with respect to $\mathrm{SL}(2,\mathbb{Z})$ and holomorphic at ∞. Moreover, $Q(\infty) \leq 0$.

(3) The metric might have conic singularity at some $p \in \mathbb{H}$ with a conic angle θ_p, and the metric is smooth at p if and only $\theta_p = 1$. Thus Q has a pole at p if and only ds^2 has a conic singularity at p (i.e., $\theta_p \neq 1$), provided that $p \notin \{\gamma \cdot i, \gamma \cdot \rho : \gamma \in \mathrm{SL}(2,\mathbb{Z})\}$, where $i = \sqrt{-1}$ and $\rho = (1+\sqrt{-3})/2$.

(4) Let $1/2 \pm \kappa_p$, $\kappa_p > 0$ be the local exponents at p of (1.1) with this Q. Then $\theta_p = 2\kappa_p/e_p$, where e_p is the elliptic order of p. Moreover, if $\kappa_p \in \frac{1}{2}\mathbb{Z}$ for any p, then (1.1) is automatically apparent.

We say the solution u or the metric $e^u\,|dz|^2$ *realizes* Q or the associated ODE (1.1) is realized by u. We note that for a given Q, finding a metric $e^u\,|dz|^2$ realizing Q is equivalent to solving the curvature equation (2.5) in Section 2 with the RHS being $4\pi \sum n_p\delta_p$, where $n_p = 2\kappa_p - 1$, δ_p is the Dirac measure at $p \in \mathbb{H}$ and the summation runs over all poles of Q on $\mathbb{H}$. In particular, $\kappa_p \in \frac{1}{2}\mathbb{N}$, if and only if the coefficient $n_p \in \mathbb{N}$, the set of positive integers.

In view of this connection, throughout the paper, we assume that the ODE (1.1) satisfy the following conditions ($\mathbf{H}_1$) or ($\mathbf{H}_2$).

($\mathbf{H_1}$) The ODE (1.1) is apparent with the local exponents $1/2 \pm \kappa_p$ at any pole p of Q, $\kappa_p \in \frac{1}{2}\mathbb{N}$, and $Q(\infty) \leq 0$. Denote the local exponents at ∞ by $\pm\kappa_\infty$. Moreover, if $p \notin \{i, \rho\}$, then $\kappa_p > 1/2$.

Note that $Q(z)$ is smooth at p if and only if $\kappa_p = 1/2$, so the requirement $\kappa_p > 1/2$ means that that $Q(z)$ has a pole at p. Note that by (4), the angle θ_ρ at ρ is $2\kappa_\rho/3$ and θ_i at i is κ_i.

($\mathbf{H_2}$) The angles θ_ρ and θ_i are not integers.

Suppose $\kappa_\infty \notin \frac{1}{2}\mathbb{N}$. Then there is $r_\infty \in (0, 1/2)$ such that

$$\text{either } \kappa_\infty \equiv r_\infty \bmod 1 \quad \text{or} \quad \kappa_\infty \equiv -r_\infty \bmod 1. \tag{1.6}$$

Theorem 1.1. *Suppose that* (1.1) *satisfies (*$\mathbf{H}_1$*), (*$\mathbf{H}_2$*), and* $\kappa_\infty \notin \frac{1}{2}\mathbb{N}$. *If* $1/12 < r_\infty < 5/12$*, then there is an invariant metric of curvature* $1/2$ *realizing* Q*. Moreover, the metric is unique. Conversely, if* Q *is realized then* $1/12 \le r_\infty \le 5/12$*.*

Furthermore, assume that $r_\infty = 1/12$ *or* $r_\infty = 5/12$*. Let* χ *be the character of* $\mathrm{SL}(2,\mathbb{Z})$ *determined by*

$$\chi(T) = e^{2\pi i/6}, \qquad \chi(S) = -1,$$

where $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ *and* $S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$*. Then there is an invariant metric of curvature* $1/2$ *realizing* Q *if and only there are two solutions* $y_1(z)$ *and* $y_2(z)$ *of* (1.1) *such that* $y_1(z)^2$ *and* $y_2(z)^2$ *are meromorphic modular forms of weight* -2 *with character* χ *and* $\overline{\chi}$*, respectively, on* $\mathrm{SL}(2,\mathbb{Z})$*.*

Remark. Let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Since $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^*$ is conformally diffeomorphic to the standard sphere S^2, Theorem 1.1 can be formulated in terms of the existence of metrics on S^2 with prescribed singularities at poles of Q and prescribed angle θ_p at each singular point p. In this sense, Theorem 1.1 is a special case of a result of Eremenko and Tarasov [12][1], quoted as Theorem A.1 in the appendix. In the appendix, we give an alternative and self-contained proof of their result in the form of Theorem A.3 as it is elementary and involves only straightforward matrix computation. (In the notation of Theorem A.3, Theorem 1.1 corresponds to the case $\theta_1 = 1/2$, $\theta_2 = 1/3$, and $\theta_3 = 2r_\infty$ or $\theta_3 = 1 - 2r_\infty$, depending on whether $2r_\infty \le 1/2$ or $2r_\infty > 1/2$.)

The threshold case $r_\infty \in \{1/12, 5/12\}$ is more delicate. In Section 6, we provides examples of existence and nonexistence of an invariant metric with $r_\infty = 1/12$. Our examples suggest that to each $Q(z)$ with $r_\infty \in \{1/12, 5/12\}$, one may associate a meromorphic differential 1-form ω of the second kind on a certain elliptic curve E, and whether there exists an invariant metric realizing Q hinges on whether ω is exact, i.e., whether ω is the identity element in the first de Rham cohomology group of E. Also, in the nonexistence example, we find that the entries in the monodromy matrices can be expressed in terms of periods or the central value of the L-function of the elliptic curve $y^2 = x^3 - 1728$. We plan to study the threshold case in more details in the future.

[1]We thank the referee for pointing out this and providing the reference.

Motivated by Theorem 1.1, we consider the datas given below.

$$\begin{aligned}&\text{A set of positive half-integers } \kappa_\rho, \kappa_i, \kappa_j \in \mathbb{N}/2, j = 1, 2, \ldots, m,\\ &\text{such that } 2\kappa_\rho/3 \notin \mathbb{N},\ \kappa_i \notin \mathbb{N};\ \text{a set of inequivalent points } p_j \in \mathbb{H},\\ &j = 1, 2, \ldots, m;\ \text{and a positve number } \kappa_\infty.\end{aligned} \tag{1.7}$$

Definition 1.2. We say Q is *equipped with* (1.7) if

(i) $\{\rho, i, z_j : 1 \le j \le m\}$ are the set of poles of Q;
(ii) The local exponents of Q at ρ, i, z_j are $1/2 \pm \kappa_\rho$, $1/2 \pm \kappa_i$ and $1/2 \pm \kappa_j$, respectively;
(iii) Q is apparent on $\mathbb{H}$; and
(iv) The local exponents at ∞ are $\pm\kappa_\infty$.

Theorem 1.3. *Given* (1.7)*, there are modular forms Q of weight 4 equipped with* (1.7)*. Moreover, the number of such Q is at most $\prod_{j=1}^m (2\kappa_j)$.*

To prove the theorem, we first show that there is a finite set of polynomials such that the set of $Q(z)$ equipped with (1.7) is in a one-to-one correspondence with the set of common zeros of the polynomial. Then the theorem follows immediately from the clasical Bézout theorem. Note that Eremenko and Tarasov [12, Theorem 2.4] has proved a stronger result, which in our setting states that for generic singular points $z_1, \ldots, z_m$, the number of $Q(z)$ is precisely $\prod_{j=1}^m (2\kappa_j)$.

If the local exponents at ∞ are $\pm n/4$, n is odd, then our second result asserts that there is a modular form of weight -4 coming from the equation. In the following, we use $T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$.

Theorem 1.4. *Suppose that* ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$) *hold and $\kappa_\infty = n/4$, n a positive odd integer. Then there is a constant $c \in \mathbb{C}$ such that $F(z) := y_-(z)^2 + c y_+(z)^2$ satisfies*

$$\left(F\big|_{-2} T\right)(z) = \left(F\big|_{-2} S\right)(z) = -F(z),$$

where

$$y_\pm(z) = q^{\pm n/4}\left(1 + \sum_{j\ge1} c_j^\pm q^j\right)$$

are solutions of (1.1)*.*

The constant c is rational if all coefficients of $Q(z)/\pi^2$ in the q-expansion are rational. We conjecture c is positive, but it is not proved yet. Obviously,

$F(z)^2$ is a modular form of weight -4 with respect to $\mathrm{SL}(2,\mathbb{Z})$. Let Γ_2 be the group generated by $T^2 = \left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$ and $ST = \left(\begin{smallmatrix}0&-1\\1&1\end{smallmatrix}\right)$, which is an index 2 subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Then F is a modular form of weight -2 on Γ_2. This fact can help us to compute c and $F(z)$ explicitly. For example, if $Q(z) = -\pi^2 n^2 E_4(z)/4$, then $F(z)$ is holomorphic on $\mathbb{H}$, but with a pole of order n at ∞ (Γ_2 has only one cusp ∞ and two elliptic points of order 3). Thus it is not difficult to prove

Corollary 1.5. *Let $Q(z) = -\pi^2(n/2)^2E_4(z)$, where n is a positive odd integer. Then there is a polynomial $P_{n-1}(x) \in \mathbb{Q}[x]$ of degree $(n-1)/2$ such that*

$$F(z) = \frac{E_4(z)}{\Delta(z)^{1/2}} P_{n-1}(j(z)).$$

Here E_4 and E_6 are the Eisenstein series of weight 4 and 6 on $\mathrm{SL}(2,\mathbb{Z})$ *respectively:*

$$E_4(z) = 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1-q^m} = 1 + 240 \sum_{m=1}^{\infty} \left(\sum_{d|n} d^3\right) q^n, \quad q = e^{2\pi i z},$$

$$E_6(z) = 1 - 504 \sum_{m=1}^{\infty} \frac{m^5 q^m}{1-q^m} = 1 - 504 \sum_{m=1}^{\infty} \left(\sum_{d|n} d^5\right) q^n,$$

$\Delta(z) = (E_4(z)^3 - E_6(z)^2)/1728 = q - 24q^2 + \cdots$, *and* $j(z) = E_4(z)^3/\Delta(z)$.

For small n, P_{n-1} are shown in the following list.

n	F	P_{n-1}
1	$y_-^2 + 3(2^3 y_+)^2$	1
3	$y_-^2 + 3(2^{12} y_+)^2$	$j - 1536$
5	$y_-^2 + 3(2^{18}7^1 y_+)^2$	$j^2 - 2240j + 1146880$
7	$y_-^2 + 3(2^{28}3^1 y_+)^2$	$j^3 - 3072j^2 + 2752512j - 704643072$
9	$(7y_-)^2 + 3(2^{34}11^1 13^1 y_+)^2$	$49j^4 - 192192j^3 + 253034496j^2 - 125954949120j + 19346680184832$

In practice, it seems not easy to verify the apparentness at a singular point with local exponents $1/2 \pm \kappa$, $\kappa \in \frac{1}{2}\mathbb{N}$. Take a simple example

$$\left(q\frac{d}{dq}\right)^2 y = -\frac{1}{4\pi^2} y'' = \left(\frac{n}{2}\right)^2 E_4(z) y \quad \text{on } \mathbb{H}.$$

The local exponents at ∞ are $\pm n/2$. The standard method to verify the apparentness at ∞ is to show that there is a solution $y_-(z)$ having a q-expansion

$$y_-(z) = q^{-n/2}\left(1 + \sum_{j\geq 1} c_j q^j\right).$$

Suppose $E_4(z) = \sum_{j\geq 0}^{\infty} b_j q^j$. Substituting the q-expansion of y_- and E_4 into the equation, then the coefficient c_j satisfies

$$\left(\left(j - \frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2\right) c_j = \left(\frac{n}{2}\right)^2 \sum_{k+\ell=j,\ \ell<j} b_k c_\ell. \tag{1.8}$$

For $j = 1, 2, \ldots, n-1$, c_j can be determined from $c_0 = 1$. However at $j = n$, the LHS of (1.8) vanishes. Therefore, ∞ is apparent if and only the RHS of (1.8) is 0 at $j = n$. If n is small, then it is easy to check that the RHS of (1.8) is not 0 at $j = n$. For a general n, nevertheless, it seems not easy to see why it does not vanish from the recursive relation (1.8). Thus for a modular ODE, the standard method is not efficient for this purpose. We need other ideas.

We consider

$$y''(z) = \pi^2\left(rE_4(z) + s\frac{E_6(z)^2}{E_4(z)^2} + t\frac{E_4(z)^4}{E_6(z)^2}\right) y(z), \tag{1.9}$$

where r, s and t are constant parameters. For simplicity, we denote the potential of (1.9) by $Q_3(z; r, s, t)$ or $Q_3(z)$ for short. Modulo $\mathrm{SL}(2,\mathbb{Z})$, (1.9) has singularities only at ρ and i (recall that $E_4(z_0) = 0$ if and only if z_0 is equivalent to ρ under $\mathrm{SL}(2,\mathbb{Z})$ and $E_6(z_0) = 0$ if and only if z_0 is equivalent to i). Assume the local exponents of (1.9) are $1/2 \pm \kappa_i$ at $i = \sqrt{-1}$ and $1/2 \pm \kappa_\rho$ at $\rho = (1+\sqrt{-3})/2$. Then it is easy to prove that $s = s_{\kappa_\rho}$, $t = t_{\kappa_i}$, where

$$s_{\kappa_\rho} = \frac{1 - 4\kappa_\rho^2}{9}, \qquad \text{and} \qquad t_{\kappa_i} = \frac{1 - 4\kappa_i^2}{4}. \tag{1.10}$$

See Section 3 for the computation.

At ∞, the local exponents are $\pm\kappa_\infty$ if and only if

$$r + s_{\kappa_\rho} + t_{\kappa_i} = -(2\kappa_\infty)^2.$$

In the following, we set the triple $(n_i,\ n_\rho,\ n_\infty)$ by

$$(n_i, n_\rho, n_\infty) = \left(\kappa_i, \frac{2\kappa_\rho}{3}, 2\kappa_\infty\right).$$

Theorem 1.6. *The modular differential equation (1.9) is apparent throughout $\mathbb{H} \cup \{cusps\}$ if and only if the triple (n_i, n_ρ, n_∞) are positive integers satisfying (i) the sum of these three integers is odd, and (ii) the sum of any two of these integers is greater than the third. Moreover, In such a case, the ratio of any two solutions is a modular function on* $\mathrm{SL}(2,\mathbb{Z})$.

For example, if

$$Q(z) = \pi^2\left(\frac{23}{36}E_4(z) - \frac{9n^2-1}{9}\frac{E_6(z)^2}{E_4(z)^2} - \frac{3}{4}\frac{E_4(z)^4}{E_6(z)^2}\right), \quad n \in \mathbb{N},$$

then we have $(n_i, n_\rho, n_\infty) = (1, n, n)$. By Theorem 1.6, (1.9) is apparent throughout $\mathbb{H}\cup\{\text{cusps}\}$. On the other hand, ∞ is not apparent for the ODE

$$y''(z) = -\pi^2 n^2 E_4(z) y(z).$$

As discussed in (1.8), it seems very difficult to verify $(\mathbf{H}_1)$. So we would like to present some examples to show how to verify the condition $(\mathbf{H}_1)$. The first example is

$$y''(z) = \pi^2\left(rE_4(z) + s\frac{E_6(z)^2}{E_4(z)^2}\right)y(z), \tag{1.11}$$

where r, s are constant parameters. For simplicity, we denote the potential of (1.11) by $Q_1(z; r, s)$ or $Q_1(z)$ for short. The singular points modulo $\mathrm{SL}(2,\mathbb{Z})$ is ρ only. If the local exponents are $1/2 \pm \kappa_\rho$, then a simple calculation in Section 3 shows $s = s_{\kappa_\rho}$, where s_{κ_ρ} is given in (1.10).

Theorem 1.7. *Let $\kappa_\rho \in \frac{1}{2}\mathbb{N}$.*

(a) Assume $3 \nmid 2\kappa_\rho$. Then $Q_1(z; r, s)$ is apparent if $s = s_{\kappa_\rho}$ and any $r \in \mathbb{C}$.
(b) Assume $3 | 2\kappa_\rho$. Then there exists a polynomial $P(x) \in \mathbb{Q}[x]$ of degree $2\kappa_\rho/3$ such that $Q_1(z; r, s)$ with $s = s_{\kappa_\rho}$ is apparent if and only if r is a root of $P(x)$. Moreover, r satisfies

$$r + s_{\kappa_\rho} = -\left(\ell + \frac{1}{2}\right)^2, \quad \text{where } \ell = 0, 1, 2, \ldots, \frac{2\kappa_\rho}{3} - 1. \tag{1.12}$$

Next, we consider the ODE

$$y''(z) = \pi^2 \left(rE_4(z) + t\frac{E_4(z)^4}{E_6(z)^2} \right) y(z) \quad \text{on } \mathbb{H}, \tag{1.13}$$

where r and t are constant parameters. For simplicity, the potential of (1.13) is denoted by $Q_2(z; r, t)$ or $Q_2(z)$ for short. Similar to (1.11), (1.13) has local exponents $1/2 \pm \kappa_i$ at i if and only if $t = t_{\kappa_i}$, where t_{κ_i} is given in (1.10).

Theorem 1.8. *Let $\kappa_i \in \frac{1}{2}\mathbb{N}$.*

(a) Assume $\kappa_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. Then (1.13) is apparent if and only if $t = t_{\kappa_i}$ and any $r \in \mathbb{C}$.

(b) Assume $\kappa_i \in \mathbb{N}$. Then there exists a polynomial $P(x) \in \mathbb{Q}[x]$ of degree κ_i such that (1.13) with $t = t_{\kappa_i}$ is apparent if and only if r is a root of $P(x)$. Moreover, r satisfies

(1.14)
$$r + t_{\kappa_i} = -\left(\ell \pm \frac{1}{3}\right)^2, \quad \begin{cases} \ell = 0, 2, 4, \ldots, \kappa_i - 1, & \text{if } \kappa_i \text{ is odd,} \\ \ell = 1, 3, 5, \ldots, \kappa_i - 1, & \text{if } \kappa_i \text{ is even.} \end{cases}$$

We use the Frobenius method to prove Part (a) of Theorem 1.7 and Theorem 1.8. However, due to the modularity, our expansion of functions are expanded in terms of powers of $w_\rho := (z-\rho)/(z-\bar{\rho})$ and $w_i := (z-i)/(z+i)$, not powers of $z - \rho$ and $z - i$ as the standard method does. This kind of expansion has been used in [19] and [21]. We will see in Section 3 that this type of expansions not only simplifies computations greatly, but also obtains the degree of $P(x)$ in Theorem 1.7(b) and Theorem 1.8(b) precisely.

We will present two proofs of (1.12) in Theorem 1.7(b) and (1.14) in Theorem 1.8(b) in Section 4 and Section 5. One is to apply Riemann's existence theorem on compact Riemann surfaces. The other is to apply the existence theorems of the invariant metrics with curvature 1/2. This geometric theorems are obtained by Eremenko [10, 11]. Hopefully, these methods are useful for treating this kind of problems in modular differential equations.

The paper is organized as follows. In Section 2, we will discuss the connection between the invariant metric $ds^2 = e^u |dz|^2$ of curvature 1/2 and modular ODEs, in particular, the relation among the behavior of u near cusps, angles and the local exponents of the realized modular ODE by u. In Section 3, we will explain the expansion of modular forms in terms of the natural coordinate $w = (z - z_0)/(z - \bar{z}_0)$, and prove Theorem 1.7(a) and Theorem 1.8(a). Both Theorem 1.7(b) and Theorem 1.8(b) are proved in

Section 4, and Theorem 1.6 is proved in Section 5. Finally, we will prove Theorem 1.1 and Theorem 1.4 to complete the paper in Section 6 and Section 7 respectively.

2. Curvature equations and the modular ODEs

2.1.

Let M be a compact Riemann surface, $p \in M$, and z be a complex coordinate in an open neighborhood U of p with $z(p) = 0$. We consider the following curvature equation:

$$4u_{z\bar{z}} + e^u = f \quad \text{on } U, \tag{2.1}$$

where $f = 4\pi \sum \alpha_i \delta_{p_i}$ is a sum of Dirac measures and $0 \neq \alpha_i > -1$. The assumption $\alpha_i > -1$ ensures that e^u is locally integrable in a neighborhood of p_i. The L^1-integrability implies

$$u(z) = 2\alpha_i \log|z - p_i| + O(1) \quad \text{near } p_i. \tag{2.2}$$

This is a general result from the elliptic PDE theory, see [4, 5].

Let $w = w(z)$ be a coordinate change and set

$$\hat{u}(w) = u(z) - 2\log\left|\frac{dw}{dz}\right|. \tag{2.3}$$

Then $\hat{u}(w)$ also satisfies

$$4\hat{u}_{w\bar{w}} + e^{\hat{u}} = \hat{f}, \qquad f = 4\pi \sum \alpha_i \delta_{\hat{p}_i},$$

where $\hat{p}_i = w(p_i)$. In other words, $e^u |dz|^2$ is invariant under a coordinate change. Since u has singularities at p_i, the metric $ds^2 = e^u |dz|^2$ has a conic singularity at p_i. If u is a solution of (2.1), then the metric $ds^2 = e^u |dz|^2$ has curvature $1/2$ at any point $p \notin \{p_i\}$. Suppose that M is covered by $\{U_i\}$ and z_i is a coordinate in U_i. We call the collection $\{u_i\}$ to be a solution of (2.1) on M if u_i is a solution of (2.1) on U_i for each i and satisfy the transformation law $u_j = u_i - 2\log\left|\frac{dz_j}{dz_i}\right|$ on $U_i \cap U_j$.

Let g be a metric of M with the curvature K, and the equation (2.1) on M is equivalent to the curvature equation:

$$\Delta_g \hat{u} + e^{\hat{u}} - K = 4\pi \sum \alpha_i \delta_{p_i} \quad \text{on } M, \tag{2.4}$$

where Δ_g is the Beltrami-Laplace operator of (M, g). We could normalize the metric g such that the area of M is equal to 1. In the case when g has a constant curvature, (2.4) can be written as

$$\Delta_g \hat{u} + \rho \left(\frac{e^{\hat{u}}}{\int e^{\hat{u}}} - 1 \right) = 4\pi \sum \alpha_i (\delta_{p_i} - 1) \quad \text{on } M.$$

This nonlinear PDE is often call *a mean field equation* in analysis. See [3, 5, 4, 8, 7, 6] and [14, 15, 16] for the recent development of mean field equations.

In this paper, we consider the compact Riemann surface that is the quotient of $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ by a finite index subgroup Γ of $\mathrm{SL}(2, \mathbb{Z})$, and the equation (2.1) is defined on the upper half space $\mathbb{H}$:

$$4u_{z\bar{z}} + e^u = 4\pi \sum \alpha_i \delta_{p_i} \quad \text{on } \mathbb{H}, \tag{2.5}$$

where the RHS is invariant under the action of Γ, i.e., the set $\{p_i\}$ is invariant under the action of Γ and $\alpha_i = \alpha_j$ if $p_i = \gamma \cdot p_j$ for some $\gamma \in \Gamma$. The transformation law (2.3) for coordinate change is equivalent to asking u to satisfy

$$u(\gamma z) = u(z) + 4 \log |cz + d| \,, \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{2.6}$$

Let s be a cusp of Γ and $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ be a matrix such that $\gamma \cdot \infty = s$. Then we define u_γ by

$$u_\gamma(z) := u(\gamma \cdot z) - 4 \log |cz + d| \,.$$

Thus, u is required to satisfy the following behavior near s: there is $\alpha_s > 0$ such that

$$e^{u_\gamma(z)} = |q_N|^{4\alpha_s} (c + o(1)), \quad q_N = e^{2\pi i z/N}, \; c > 0, \tag{2.7}$$

where N is the width of the cusp s and $o(1) \to 0$ as $q_N \to 0$. Given the RHS of (2.5) and a positive α_s at the cusp s, we ask for a solution u of (2.5) satisfying (2.6) and (2.7) at any cusp.

The conic angle θ, defined at a singularity p_i or a cusp s, is an important geometric quantity. Suppose that a metric ds^2, conformal to the flat metric $|dz|^2$, has a conic singularity at p, and w is a coordinate near p with $w(p) = 0$. If

$$ds^2 = |w|^{2(\theta - 1)} (c + o(1)) \, |dw|^2 \,, \quad c > 0, \tag{2.8}$$

then we call θ the *angle* at p, and $2\pi\theta$ the *total angle* at p. Since ds^2 is required to have a finite area, the angle θ is always *positive*. Note that ds^2 is smooth (as a metric) at p if and only if $\theta = 1$.

Next, we want to calculate the angles of $ds^2 = e^u |dz|^2$, if u is a solution of (2.5). Note that z is not a coordinate of M if p_i is an elliptic point of order $e_i > 1$. Indeed, $w = (z - p_i)^{e_i}$ is the local coordinate near p_i. For simplicity, we denote $z - p_i$ by z ($z(p_i) = 0$). By (2.2), we have $u(z) = 2\alpha_i \log |z| + O(1)$, i.e., $e^{u(z)} = |z|^{2\alpha_i} (c_0 + o(1))$, $c_0 > 0$. Then

$$e^{u(z)} |dz|^2 = |w|^{(2\alpha_i+2)/e_i - 2} (d + o(1)) |dw|^2, \quad d > 0.$$

By (2.8), we have

$$\theta_i = \frac{\alpha_i + 1}{e_i}. \tag{2.9}$$

At a cusp s, the coordinate is $q_N = e^{2\pi i z/N}$, where N is the width of the cusp s. By (2.7),

$$e^{u_\gamma(z)} |dz|^2 = |q_N|^{4\alpha_s - 2} (c + o(1)) |dq_N|^2, \quad c > 0.$$

So the angle θ_s at s is

$$\theta_s = 2\alpha_s. \tag{2.10}$$

2.2. Integrability and modular differential equations

Equation (2.5) is also known as an integrable system. There are two important features related to the integrability. One is that

$$Q(z) := -\frac{1}{2}\left(u_{zz} - \frac{1}{2}u_z^2\right) \quad \text{is a meromorphic function,} \tag{2.11}$$

because $Q(z)_{\bar z} = -\frac{1}{2}(u_{z\bar z z} - u_{z\bar z} u_z) = 0$ by (2.5).

Lemma 2.3. *Each p_i is a double pole of $Q(z)$ with the expansion $\frac{\alpha_i}{2}\left(\frac{\alpha_i}{2} + 1\right)(z - p_i)^{-2} + O\left((z - p_i)^{-1}\right)$.*

Proof. Since $u(z) = 2\alpha_i \log |z - p_i| + O(1)$ near p_i, we have $u_z(z) = \alpha_i(z - p_i)^{-1} + O(1)$ and $u_{zz}(z) = -\alpha_i(z - p_i)^{-2} + O\left((z - p_i)^{-1}\right)$. Then the lemma follows immediately. □

On the other hand, the Liouville theorem asserts that locally any solution u can be expressed as

$$u(z) = \log \frac{8\,|h'(z)|^2}{\left(1 + |h(z)|^2\right)^2}, \tag{2.12}$$

where $h(z)$ is a meromorphic function. Recall the Schwarz derivative

$$\{h, z\} = \left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2. \tag{2.13}$$

Note that the Schwarz derivative can be used to recover h from u. Indeed, a direct computation from (2.12) yields that

$$\{h, z\} = -2Q(z). \tag{2.14}$$

See [3, 14, 15, 16] for the detail of the proofs (2.12)–(2.14). The meromorphic function h is called a *developing map* for the solution u. Any two developing maps h_i, $i = 1, 2$, of u have the same Schwarz derivative by (2.14), thus they can be connected by a Möbius transformation,

$$h_2(z) = \frac{ah_1(z) + b}{ch_1(z) + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}). \tag{2.15}$$

By (2.12), we obtain

$$\frac{|h_1'(z)|^2}{\left(1 + |h_1(z)|^2\right)^2} = \frac{|h_2'(z)|^2}{\left(1 + |h_2(z)|^2\right)^2}, \tag{2.16}$$

which implies that the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an unitary matrix.

Next, we recall the classical Hermite theorem, see [20].

Theorem A. *Let y_i, $i = 1, 2$, be two independent solutions of*

$$y'' = Q(z)y.$$

Then the ratio $h(z) = y_2(z)/y_1(z)$ satisfies $\{h, z\} = -2Q(z)$.

Let $Q(z)$ be the meromorphic function (2.11) obtained from the solution u. Consider the ODE

$$y'' = Q(z)y. \tag{2.17}$$

Then (2.11) and the Hermite theorem together imply that $h(z)$ is a ratio of two solutions of (2.16).

Theorem 2.4. *Suppose u is a solution of (2.5). Then (2.17) satisfies ($\mathbf{H}_1$) and the followings hold.*

(a) The function $Q(z)$ is a meromorphic modular form of weight 4 with respect to Γ and holomorphic at any cusp. Moreover, at a cusp s, $Q(s) < 0$.
(b) (2.17) is Fuchsian and the local exponents of (2.17) at p_i are $-\alpha_i/2$, $\alpha_i/2+1$, and $\pm\alpha_s$ at a cusp.
(c) If $\alpha_i \in \mathbb{N}$ for all i, then (2.17) is apparent.

Proof. (a) By the chain rule, we have

$$(u\circ\gamma)_z(z) = u_z(\gamma\cdot z)(cz+d)^{-2},$$
$$(u\circ\gamma)_{zz}(z) = u_{zz}(\gamma\cdot z)(cz+d)^{-4} - u_z(\gamma\cdot z)\frac{2c}{(cz+d)^3}.$$

Thus

$$(u\circ\gamma)_{zz} - \frac{1}{2}(u\circ\gamma)_z^2 = \left(u_{zz}(\gamma\cdot z) - \frac{1}{2}u_z^2(\gamma\cdot z)\right) \times (cz+d)^{-4} - u_z(\gamma\cdot z)\cdot\frac{2c}{(cz+d)^3}.$$

On the other hand, the transformation law (2.6) yields

$$(u\circ\gamma)_z(z) = u_z(z) + \frac{2c}{(cz+d)},$$
$$(u\circ\gamma)_{zz} = u_{zz} - \frac{2c^2}{(cz+d)^2}.$$

Hence, we have

$$\begin{aligned}(u\circ\gamma)_{zz} - \frac{1}{2}(u\circ\gamma)_z^2 &= \left(u_{zz} - \frac{1}{2}u_z^2\right) - u_z(z)\cdot\frac{2c}{(cz+d)} - \frac{4c^2}{(cz+d)^2}\\ &= \left(u_{zz} - \frac{1}{2}u_z^2\right) - \frac{2c}{(cz+d)}(u\circ\gamma)_z(z).\end{aligned}$$

Since

$$\frac{-2c}{(cz+d)}(u\circ\gamma)_z = \frac{-2c}{(cz+d)^3}u_z(\gamma\cdot z),$$

we find that $Q := -\frac{1}{2}\left(u_{zz} - \frac{1}{2}u_z^2\right)$ satisfies

$$Q(\gamma \cdot z) = Q(z) \cdot (cz+d)^4.$$

This proves the modularity of Q.

To prove the holomorphy of Q at cusps, without loss of generality, we may assume that the cusp s is ∞. Then $q_N = e^{2\pi iz/N}$ is the local coordinate near ∞, where N is the width of the cusp ∞. By the transformation law of coordinate changes, the solution $\hat{u}$ in terms of q_N should be expressed by $\hat{u}(q_N) = u(z) - 2\log\left|\frac{dq_N}{dz}\right|$. Thus,

$$e^{\hat{u}(q_N)} = \frac{8\left|h'(z)\right|^2}{\left(1+|h(z)|^2\right)^2}\left|\frac{dq_N}{dz}\right|^2 = 8\left|\frac{d}{dq_N}h(z)\right|^2\left(1+|h(z)|^2\right)^{-2}.$$

Hence the developing map $h(z) = \hat{h}(e^{2\pi iz/N}) = \hat{h}(q_N)$, where $q_N = e^{2\pi iz/N}$. Note that

$$\begin{aligned}\{h,z\} &= \left\{\hat{h}, q_N\right\}\left(\frac{dq_N}{dz}\right)^2 + \{q_N, z\} \\ &= \left\{\hat{h}, q_N\right\} q_N^2\left(\frac{-4\pi^2}{N^2}\right) + \frac{2\pi^2}{N^2}.\end{aligned}$$

Since

$$-\frac{1}{2}\left\{\hat{h}, q_N\right\} = \hat{u}_{q_N q_N} - \frac{1}{2}\hat{u}_{q_N}^2 = \frac{\alpha}{2}\left(\frac{\alpha}{2}+1\right)q_N^{-2} + O\left(q_N^{-1}\right),$$

where $\alpha = \theta - 1$, θ is the angle at ∞, we have

$$\lim_{\operatorname{Im} z\to\infty} Q(z) = -\frac{\pi^2}{N^2}\left(1 + \frac{4\alpha}{2}\left(\frac{\alpha}{2}+1\right)\right) = -\frac{\pi^2}{N^2}(1+\alpha)^2 < 0,$$

because $\alpha > -1$. This proves Part (a).

Part (b) is a consequence of Lemma 2.3.

For Part (c), if $\alpha_i \in \mathbb{N}$ then the local exponents $-\alpha_i/2$ and $\alpha_i/2+1$ can be written as $1/2 \pm \kappa_i$, $\kappa_i = (\alpha_i+1)/2 \in \frac{1}{2}\mathbb{N}$ and by the Liouville theorem (2.12), we see easily that $h(z)$ can not have a logarithmic singularity at p_i. The fact that $h(z)$ is a ratio of two solutions of (2.17) implies any solution of (2.17) has no logarithmic singularity. This proves Part (c). □

Together with the Liouville theorem, we have

Proposition 2.5. *Suppose Q is a meromorphic modular form of weight 4 on* $\mathrm{SL}(2,\mathbb{Z})$*. If there are two independent solutions y_1 and y_2 of (2.17) such that $h(z) = y_2(z)/y_1(z)$ satisfies $h(\gamma z) = \frac{ah(z)+b}{ch(z)+d}$ for some unitary matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ depending on γ, for any $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, then Q can be realized.*

Proof. Let $u(z) = \log \frac{8|h'(z)|^2}{(1+|h(z)^2|)^2}$. Since $h(z)$ is unitary, $u(z)$ is well-defined on $\mathbb{H}$ and satisfies (2.6). Further, the Liouville theorem says that $u(z)$ satisfies (2.5). □

2.6. Examples

In this subsection, we will give some examples to indicate how to determine Q provided that the RHS of (2.5) is known and α_∞ is given at ∞. Here, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$.

Example 1. Assume that the RHS of (2.5) is equal to 0. Then $Q := -\frac{1}{2}\left(u_{zz} - \frac{1}{2}u_z^2\right)$ is a holomorphic modular form of weight 4. Thus,

$$Q(z) = \pi^2 r E_4(z). \tag{2.18}$$

Since $\pm\alpha_\infty$ are the local exponents of (1.1) at ∞, we have $r = -4\alpha_\infty^2$. Thus, Q is uniquely determined. Note that at ∞, the angle θ_∞ is equal to $2\alpha_\infty$.

Example 2. Assume that the RHS of (2.5) is $4\pi n \sum \delta_p$, where the summation is over $\gamma \cdot \rho$ for every $\gamma \in \Gamma$. Then Q is a meromorphic modular form of weight 4 whose poles occur at $\gamma \cdot \rho$ and the order is 2. Thus, $E_4(z)^2 Q(z)$ is holomorphic a modular form of wright 12, and then

$$Q(z) = \pi^2 \left(r E_4(z) + s \frac{E_6(z)^2}{E_4(z)^2} \right),$$

where we recall that the graded ring of modular forms on $\mathrm{SL}(2,\mathbb{Z})$ is generated by $E_4(z)$ and $E_6(z)$. By Theorem 2.4, the local exponents at ρ are $-n/2$ and $n/2+1$, which implies $\kappa_\rho = (n+1)/2$, $s = (1-4\kappa_\rho^2)/9$, and $-(r+s)/4 = \alpha_\infty^2$. Thus Q is uniquely determined. Moreover, the angles θ_j in this example are $\theta_i = 1/2$, $\theta_\rho = (n+1)/3$ and $\theta_\infty = 2\alpha_\infty$.

Example 3. Assume that the RHS of (2.5) is equal to $4\pi n \sum \delta_p$, where the summations is over $\gamma \cdot i$ for any $\gamma \in \Gamma$. Reasoning as Example 2, we have

$$Q(z) = \pi^2 \left(r E_4(z) + t \frac{E_4(z)^4}{E_6(z)^2} \right). \tag{2.19}$$

By Theorem 2.4, we have

$$\kappa_i = \frac{n+1}{2}, \quad t = \frac{1-4\kappa_i^2}{4}, \quad \text{and} \quad r+t = -4\alpha_\infty^2.$$

Thus Q is uniquely determined. Moreover, we have $\theta_i = (n+1)/2$, $\theta_\rho = 1/3$, and $\theta_\infty = 2\alpha_\infty$.

Example 4. Assume the RHS of (2.5) is $4\pi\left(n\sum_{p_1}\delta_{p_1} + m\sum_{p_2}\delta_{p_2}\right)$, where p_1, p_2 run over zeros of $E_4(z)$ and $E_6(z)$, respectively. Then

$$Q(z) = \pi^2\left(rE_4(z) + s\frac{E_6(z)^2}{E_4(z)^2} + t\frac{E_4(z)^4}{E_6(z)^2}\right). \tag{2.20}$$

The conditions on the local exponents at ρ, i and ∞ yield that

$$\begin{aligned} &s = \frac{1-4\kappa_\rho^2}{9}, \quad \kappa_\rho = \frac{n+1}{2}; \quad t = \frac{1-4\kappa_i^2}{4}, \quad \kappa_i = \frac{m+1}{2}; \\ &r+s+t = -4\alpha_\infty^2. \end{aligned}$$

Then Q is uniquely determined. Moreover, $\theta_1 = (m+1)/2$, $\theta_2 = (n+1)/3$ and $\theta_\infty = 2\alpha_\infty$.

2.7. Eremenko's theorem

A. Eremenko [10, 11] gave a necessary and sufficient conditions of the angles θ_i, $1 \le i \le 3$, at the three singular points i, ρ, ∞ for the existence of u of (2.5)-(2.7).

When one of angles is an integer, the following conditions are required.

(A) If only one (say θ_1) of angles is an integer, then either $\theta_2+\theta_3$ or $|\theta_2-\theta_3|$ is an integer m of opposite parity to θ_1 with $m \le \theta_1 - 1$. If all the angles are integers, then (1) $\theta_1+\theta_2+\theta_3$ is odd, and (2) $\theta_i < \theta_j + \theta_k$ for $i \ne j \ne k$.

Eremenko's theorem. If one of θ_j is an integer, then a necessary and sufficient condition for the existence of a conformal metric of positive constant curvature on the sphere with three conic singularities of angles θ_1, θ_2, θ_3 ($\theta_j \ne 1$, $1 \le j \le 3$), is that $\{\theta_1, \theta_2, \theta_3\}$ satisfies (A). Moreover, if (A) holds and there is only one integral angle, then the metric is unique.

3. Expansions of Eisenstein series at ρ and i

The q-expansion of a modular form $f(z)$, i.e., the expansion of $f(z)$ with respect to the local parameter q at the cusp ∞, is frequently studied and is of great significance in many problems in number theory. Here we shall review properties of series expansions of modular forms at a point $z_0 \in \mathbb{H}$, other than the cusp ∞.

Definition 3.1. Let Γ be a Fuchsian subgroup of the first kind of $\mathrm{SL}(2,\mathbb{R})$. Let $f(z)$ be a meromorphic modular form of weight k on Γ. Given $z_0 \in \mathbb{H}$, let

$$w = w_{z_0}(z) = \frac{z - z_0}{z - \overline{z}_0}.$$

The expansion of the form

$$f(z) = (1-w)^k \sum_{n \geq n_0} \frac{b_n}{n!} w^n \tag{3.1}$$

is called the *power series expansion* of f at z_0.

One advantage of this expansion is that its coefficients b_n have a simple expression in terms of the Shimura-Maass derivatives of f. To state the result, we recall that if $f : \mathbb{H} \to \mathbb{C}$ is said to be *nearly holomorphic* if it is of the form

$$f(z) = \sum_{d=0}^{n} \frac{f_d(z)}{(z - \overline{z})^d}$$

for some holomorphic functions f_d. If k is an integer and $f : \mathbb{H} \to \mathbb{C}$ is a nearly holomorphic function such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and each f_d is holomorphic at every cusp, then we say f is a *nearly holomorphic modular form* of weight k on Γ.

For a nearly holomorphic function $f : \mathbb{H} \to \mathbb{C}$, we define its *Shimura-Maass* derivative of weight k by

$$(\partial_k f)(z) := \frac{1}{2\pi i}\left(f'(z) + \frac{kf(z)}{z - \overline{z}}\right).$$

We have the following important properties of Shimura-Maass derivatives.

Lemma 3.2 ([19, Equations (1.5) and (1.8)]). *For any nearly holomorphic functions $f, g : \mathbb{H} \to \mathbb{C}$, any integers k and ℓ, and any $\gamma \in \mathrm{GL}^+(2, \mathbb{R})$, we have*

$$\partial_{k+\ell}(fg) = (\partial_k f)g + f(\partial_\ell g)$$

and

$$\partial_k \left(f\big|_k \gamma \right) = (\partial_k f)\big|_{k+2}\gamma.$$

Remark. The second property in the lemma implies that if f is a nearly holomorphic modular form of weight k on Γ, then $\partial_k f$ is a nearly holomorphic form of weight $k+2$ on Γ.

Set also

$$\partial_k^n f = \partial_{k+2n-2} \dots \partial_k f.$$

Then the coefficients b_n in (3.1) has the following expression.

Proposition 3.3 ([21, Proposition 17]). *If $f(z)$ is a holomorphic modular form of weight k on Γ, then the coefficients b_n in (3.1) are*

$$b_n = (\partial_k^n f)(z_0)(-4\pi \operatorname{Im} z_0)^n$$

for $n \geq 0$. That is, we have

$$f(z) = (1-w)^k \sum_{n=0}^{\infty} \frac{(\partial_k^n f)(z_0)(-4\pi \operatorname{Im} z_0)^n}{n!} w^n.$$

Note that there is a misprint in Proposition 17 [21]. The proof of the proposition shows that $b_n = (\partial^n f)(z_0)(-4\pi \operatorname{Im} z_0)^n$, but the statement misses the minus sign.

We will use these properties of power series expansions of modular forms to show that the apparentness of (1.1) at a point z_0 will imply the apparentness at γz_0 for all $\gamma \in \mathrm{SL}(2, \mathbb{Z})$. We first prove two lemmas. The first lemma relates the power series expansion of a meromorphic modular form at z_0 to that at γz_0.

Lemma 3.4. *Assume that f is a meromorphic modular form of weight k on $\mathrm{SL}(2, \mathbb{Z})$. Assume that the power series expansion of f at $z_0 \in \mathbb{H}$ is*

$$f(z) = (1-w)^k \sum_{n=n_0}^{\infty} a_n w^n, \qquad w = w_{z_0}(z) = \frac{z - z_0}{z - \overline{z}_0}.$$

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$*, let* $\widetilde{w} = w_{\gamma z_0}(z) = (z - \gamma z_0)/(z - \gamma \overline{z}_0)$*. Then the power series expansion of* f *at* $\widetilde{z}_0$ *is*

$$(cz_0 + d)^k (1 - \widetilde{w})^k \sum_{n=n_0}^{\infty} a_n \left(\frac{cz_0 + d}{c\overline{z}_0 + d} \right)^n \widetilde{w}^n.$$

Proof. Since every meromorphic modular form on $\mathrm{SL}(2,\mathbb{Z})$ can be written as the quotient of two holomorphic modular forms on $\mathrm{SL}(2,\mathbb{Z})$, it suffices to prove the lemma under the assumption that f is a holomorphic modular form.

According to Proposition 3.3, the power series expansions of f at z_0 and at γz_0 are

$$(1 - w)^k \sum_{n=0}^{\infty} \frac{(\partial_k^n f)(z_0)(-4\pi \operatorname{Im} z_0)^n}{n!} w^n$$

are

$$(1 - \widetilde{w})^k \sum_{n=0}^{\infty} \frac{(\partial_k^n f)(\gamma z_0)(-4\pi \operatorname{Im} \gamma z_0)^n}{n!} \widetilde{w}^n,$$

respectively. Since $\partial^n f(z)$ is modular of weight $k + 2n$ (see the remark following Lemma 3.2), we have

$$(\partial^n f)(\gamma z_0) = (cz_0 + d)^{k+2n} (\partial^n f)(z_0).$$

Also,

$$\operatorname{Im} \gamma z_0 = \frac{\operatorname{Im} z_0}{|cz_0 + d|^2}. \tag{3.2}$$

Thus, if the power series expansion of f at z_0 is

$$(1 - w)^k \sum_{n=0}^{\infty} \frac{b_n}{n!} w^n,$$

then that of f at γz_0 is

$$\begin{aligned} (1 - \widetilde{w})^k &\sum_{n=0}^{\infty} \frac{b_n}{n!} \frac{(cz_0 + d)^{k+2n}}{|cz_0 + d|^{2n}} \widetilde{w}^n \\ &= (cz_0 + d)^k (1 - \widetilde{w})^k \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{cz_0 + d}{c\overline{z}_0 + d} \right)^n \widetilde{w}^n. \end{aligned}$$

This proves the lemma. □

The next lemma expresses $y''(z)$ in terms of w.

Lemma 3.5. *Let $z_0 \in \mathbb{H}$ and set $w = w_{z_0}(z) = (z - z_0)/(z - \overline{z}_0)$. If*

$$y(z) = \frac{1}{1-w} \sum_{n=0}^{\infty} a_n w^{\alpha+n}$$

for some real number α, then

$$\frac{d^2}{dz^2} y(z) = \frac{(1-w)^3}{(z_0 - \overline{z}_0)^2} \sum_{n=0}^{\infty} a_n (\alpha+n)(\alpha+n-1) w^{\alpha+n-2}.$$

Proof. We first note that

$$1 - w = \frac{z_0 - \overline{z}_0}{z - z_0}$$

and hence

$$\frac{dw}{dz} = \frac{z_0 - \overline{z}_0}{(z-z_0)^2} = \frac{(1-w)^2}{z_0 - \overline{z}_0}, \qquad \frac{d^2 w}{dz^2} = -2\frac{z_0 - \overline{z}_0}{(z - z_0)^3} = -\frac{2(1-w)^3}{(z_0 - \overline{z}_0)^2}. \tag{3.3}$$

Let $g(w) = \sum a_n w^{\alpha+n}$. We compute that

$$\frac{dy}{dz} = \left(\frac{1}{(1-w)^2} g(w) + \frac{1}{1-w} \frac{dg(w)}{dw} \right) \frac{dw}{dz}$$

and

$$\begin{aligned} \frac{d^2 y}{dz^2} = &\left(\frac{2}{(1-w)^3} g(w) + \frac{2}{(1-w)^2} \frac{dg(w)}{dw} + \frac{1}{1-w} \frac{d^2 g(w)}{dw^2} \right) \left(\frac{dw}{dz} \right)^2 \\ &+ \left(\frac{1}{(1-w)^2} g(w) + \frac{1}{1-w} \frac{dg(w)}{dw} \right) \frac{d^2 w}{dz^2}. \end{aligned}$$

Using (3.3), we reduce this to

$$\frac{d^2 y}{dz^2} = \frac{(1-w)^3}{(z_0 - \overline{z}_0)^2} \frac{d^2 g(w)}{dw^2}.$$

This proves the lemma. ∎

Proposition 3.6. *Suppose that Q is a meromorphic modular form of weight 4 with respect to $\mathrm{SL}(2, \mathbb{Z})$ such that (1.1) is Fuchsian. Let z_0 be a pole of Q.*

Then the local exponents of (1.1) *at* γz_0 *are the same for all* $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. *Also, if* (1.1) *is apparent at* z_0, *then it is apparent at* γz_0 *for all* $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, $w = (z - z_0)/(z - \overline{z}_0)$, and $\widetilde{w} = (z - \gamma z_0)/(z - \gamma\overline{z}_0)$. It suffices to prove that if

$$y(z) = \frac{1}{1-w} w^\alpha \sum_{n=0}^\infty c_n w^n$$

is a solution of (1.1) near z_0, then

$$\widetilde{y}(z) = \frac{1}{1-\widetilde{w}} \widetilde{w}^\alpha \sum_{n=0}^\infty c_n (C\widetilde{w})^n, \qquad C = \frac{cz_0 + d}{c\overline{z}_0 + d},$$

is a solution of (1.1) near γz_0.

Since (1.1) is assumed to be Fuchsian, the order of poles of $Q(z)$ at z_0 is at most 2. We have

$$Q(z) = (1-w)^4 \sum_{n=-2}^\infty a_n w^n$$

for some complex numbers a_n. Then by Lemma 3.5, $y(z)$ being a solution of (1.1) near z_0 means that

$$\begin{aligned} &\frac{1}{(2i \operatorname{Im} z_0)^2} \sum_{n=0}^\infty c_n (\alpha+n)(\alpha+n-1) w^{\alpha+n-2} \\ &\quad = \left(\sum_{n=-2}^\infty a_n w^n\right)\left(\sum_{n=0}^\infty c_n w^{\alpha+n}\right). \end{aligned} \tag{3.4}$$

On the other hand, by Lemmas 3.5 and 3.4, we have

$$Q(z) = (cz_0 + d)^4 (1-\widetilde{w})^4 \sum_{n=-2}^\infty a_n (C\widetilde{w})^n$$

near γz_0 and

$$\begin{aligned} \widetilde{y}''(z) &= \frac{C^2 (1-\widetilde{w})^3}{(2i \operatorname{Im} \gamma z_0)^2} \sum_{n=0}^\infty c_n (\alpha+n)(\alpha+n-1) C^n \widetilde{w}^{\alpha+n-2} \\ &= (cz_0+d)^4 \frac{(1-\widetilde{w})^3}{(2i \operatorname{Im} z_0)^2} \sum_{n=0}^\infty c_n (\alpha+n)(\alpha+n-1) C^n \widetilde{w}^{\alpha+n-2}, \end{aligned}$$

where in the last step we have used (3.2) and $C = (cz_0 + d)/(c\overline{z}_0 + d)$. From these two expressions and (3.4), we see that if $y(z)$ is a solution of (1.1) near z_0, then $\widetilde{y}(z)$ is a solution of (1.9) near γz_0, and the proof is completed. □

For our purpose, we need the following properties of power series expansions of modular forms on $\mathrm{SL}(2,\mathbb{Z})$. These properties are well-known to experts (see [13], for example). For convenience of the reader, we reproduce the proofs here.

Lemma 3.7. *Let*

$$w_i(z) = \frac{z-i}{z+i}.$$

Then

$$w_i(-1/z) = -w_i(z), \qquad 1 - w_i(-1/z) = -iz(1 - w_i(z)).$$

Also, let $\rho = (1+\sqrt{-3})/2$,

$$w_\rho(z) = \frac{z-\rho}{z-\overline{\rho}}$$

and $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. *Then*

$$w_\rho(\gamma z) = e^{2\pi i/3} w_\rho(z), \qquad 1 - w_\rho(\gamma z) = e^{4\pi i/3}(z-1)(1 - w_\rho(z)).$$

Proof. The proof is straightforward. Here we will only provide details for the case of $w_\rho(z)$.

We have

$$w_\rho(z) = \begin{pmatrix} 1 & -\rho \\ 1 & -\overline{\rho} \end{pmatrix} z.$$

Hence,

$$w_\rho(\gamma z) = \begin{pmatrix} 1 & -\rho \\ 1 & -\overline{\rho} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} z.$$

We then compute that

$$\begin{pmatrix} 1 & -\rho \\ 1 & -\overline{\rho} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -\rho \\ 1 & -\overline{\rho} \end{pmatrix}^{-1} = \begin{pmatrix} (-1-\sqrt{-3})/2 & 0 \\ 0 & (-1+\sqrt{-3})/2 \end{pmatrix}.$$

It follows that

$$w_\rho(\gamma z) = e^{2\pi i/3} w_\rho(z).$$

Then we have

$$1 - w_\rho(\gamma z) = 1 - \rho^2 w_\rho(z) = 1 - \frac{\rho^2 z + 1}{z - \overline{\rho}} = \frac{(1-\rho^2)(z-1)}{z - \overline{\rho}},$$

while

$$1 - w_\rho(z) = \frac{\rho - \rho^{-1}}{z - \overline{\rho}}.$$

Hence,

$$1 - w_\rho(\gamma z) = -\rho(z-1)(1 - w_\rho(z)) = e^{4\pi i/3}(z-1)(1 - w_\rho(z)).$$

This proves the lemma. □

From the lemma, we deduce the following properties of expansions of modular forms at i and ρ. These properties will be crucial in the proofs of Theorem 1.7(a) and Theorem 1.8(a).

Corollary 3.8. *Let $f(z)$ be a meromorphic modular form of even weight k on $\mathrm{SL}(2,\mathbb{Z})$. Suppose that the power series expansion of f at i is*

$$f(z) = (1 - w_i(z))^k \sum_{n=n_0}^{\infty} a_n w_i(z)^n, \qquad w_i(z) = \frac{z-i}{z+i}.$$

Then $a_n = 0$ whenever $n + k/2 \not\equiv 0 \bmod 2$. Also, if the power series expansion of f at $\rho = (1+\sqrt{-3})/2$ is

$$f(z) = (1 - w_\rho(z))^k \sum_{n=n_0}^{\infty} b_n w_\rho(z)^n, \qquad w_\rho(z) = \frac{z-\rho}{z-\overline{\rho}},$$

then $b_n = 0$ whenever $n + k/2 \not\equiv 0 \bmod 3$.

Proof. Here we will only prove the case of ρ. Let $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. Since $f(z)$ is a meromorphic modular form of weight k on $\mathrm{SL}(2,\mathbb{Z})$, we have

$$f(\gamma z) = (z-1)^k f(z) = (z-1)^k (1 - w_\rho(z))^k \sum_{n=n_0}^{\infty} b_n w_\rho(z)^n$$

On the other hand, by the lemma above, we have

$$f(\gamma z) = e^{4\pi i k/3}(z-1)^k (1 - w_\rho(z))^k \sum_{n=n_0}^{\infty} b_n e^{2\pi i n/3} w_\rho(z)^n.$$

Comparing the two expressions, we conclude that $b_n = 0$ whenever $n+k/2 \not\equiv 0 \bmod 3$. $\square$

To determine local exponents of modular differential equations at ρ and i, we need to know the leading terms of the expansions of $E_6(z)^2/E_4(z)^2$ and $E_4(z)^4/E_6(z)^2$.

Lemma 3.9. *(a) Let*

$$w_\rho = w_\rho(z) = \frac{z-\rho}{z-\overline{\rho}}.$$

Then we have

$$\pi^2 \frac{E_6(z)^2}{E_4(z)^2} = (1-w_\rho^4)\left(\frac{3}{4}w_\rho^{-2} + \sum_{n=1}^\infty a_n w_\rho^n\right)$$

for some complex numbers a_n such that $a_n = 0$ whenever $n \not\equiv 1 \bmod 3$.

(b) Let

$$w_i = w_i(z) = \frac{z-i}{z+i}.$$

Then

$$\pi^2 \frac{E_4(z)^4}{E_6(z)^2} = (1-w_i)^4\left(\frac{1}{4}w_i^{-2} + \sum_{n=0}^\infty b_n w_i^n\right)$$

for some complex numbers b_n such that $a_n = 0$ whenever $n \not\equiv 0 \bmod 2$.

Proof. It is known that, as an analytic function on $\mathbb{H}$, $E_4(z)$ has a simple zero at ρ. Also, $E_6(\rho) \neq 0$. Thus, by Corollary 3.8,

$$\pi^2 \frac{E_6(z)^2}{E_4(z)^2} = (1-w_\rho)^4\left(a_{-2}w_\rho^{-2} + \sum_{n=1}^\infty a_n w_\rho^n\right)$$

for some complex numbers a_n such that $a_n = 0$ whenever $n \not\equiv 1 \bmod 3$. To determine the leading coefficient a_{-2}, we use the well-known Ramanujan's identity

$$\frac{1}{2\pi i}E_4'(z) = \frac{E_2(z)E_4(z) - E_6(z)}{3},$$

where $E_2(z)$ is the Eisenstein series of weight 2 on $\mathrm{SL}(2,\mathbb{Z})$ (see [21, Proposition 15]). Hence,

$$\lim_{z\to\rho} w_\rho(z)\frac{E_6(z)}{E_4(z)} = \frac{E_6(\rho)}{\rho-\overline{\rho}}\lim_{z\to\rho}\frac{z-\rho}{E_4(z)} = \frac{E_6(\rho)}{\sqrt{3}i}\frac{1}{E_4'(\rho)}$$
$$= -\frac{E_6(\rho)}{2\pi\sqrt{3}}\frac{3}{E_2(\rho)E_4(\rho)-E_6(\rho)} = \frac{\sqrt{3}}{2\pi},$$

which implies that $a_{-2}=3/4$. This proves Part (a).

The proof of Part (b) is similar. We use another identity

$$\frac{1}{2\pi i}E_6'(z) = \frac{E_2(z)E_6(z)-E_4(z)^2}{2}$$

of Ramanujan's to conclude that the leading term of $\pi^2 E_4(z)^4/E_6(z)^2$ is $w_i^{-2}/4$. We omit the details. □

Corollary 3.10. *The local exponents of the modular differential equation* (1.9) *at ρ and at i are roots of*

$$x^2 - x + \frac{9}{4}s = 0$$

and

$$x^2 - x + t = 0,$$

respectively.

Proof. Here we prove only the case of ρ; the proof of the case of i is similar.

Let $w = w_\rho(z) = (z-\rho)/(z-\overline{\rho})$. Assume that

$$y(z) = \frac{1}{1-w}\sum_{n=0}^{\infty} a_n w^{\alpha+n}, \quad a_0 \neq 0,$$

is a solution of (1.9). By Lemmas 3.9 and 3.5, we have

$$y''(z) = -\frac{(1-w)^3}{3}\left(\alpha(\alpha-1)a_0 w^{\alpha-2}+\cdots\right)$$

while

$$\pi^2\left(rE_4(z) + s\frac{E_6(z)^2}{E_4(z)^2} + t\frac{E_4(z)^4}{E_6(z)^2}\right)y(z)$$
$$= (1-w)^3\left(\frac{3}{4}sa_0w^{\alpha-2}+\cdots\right).$$

Comparing the leading terms, we see that the exponent α satisfies $\alpha^2-\alpha+9s/4=0$. ☐

We are now ready to prove Part (a) of Theorem 1.7.

Proof of Theorem 1.7(a). By Proposition 3.6, we only need to determine when (1.11) is apparent at ρ.

Let $\kappa_\rho \in \frac{1}{2}\mathbb{N}$ and set $s = s_{\kappa_\rho} = (1-4\kappa_\rho)/9$ so that the local exponents of the modular differential equation (1.11) with $s = s_{\kappa_\rho}$, i.e.,

$$y''(z) = \pi^2\left(rE_4(z) + s_{\kappa_\rho}\frac{E_6(z)^2}{E_4(z)^2}\right)y(z) \tag{3.5}$$

at ρ are $1/2 \pm \kappa_\rho$, by Corollary 3.10.

Let $w = w_\rho(z) = (z-\rho)/(z-\overline{\rho})$. According to Corollary 3.8 and Lemma 3.9, we have

$$\pi^2 E_4(z) = (1-w)^4\sum_{n=1}^{\infty} a_n w^n, \tag{3.6}$$

and

$$\pi^2\frac{E_6(z)^2}{E_4(z)^2} = (1-w)^4\left(\frac{3}{4}w^{-2} + \sum_{n=1}^{\infty} b_n w^n\right), \tag{3.7}$$

where a_n and b_n are complex numbers satisfying

$$a_n = b_n = 0 \quad \text{if} \quad n \not\equiv 1 \bmod 3. \tag{3.8}$$

We also remark that $a_1 \neq 0$ since the zero ρ of $E_4(z)$, as a holomorphic function on $\mathbb{H}$, is simple.

Now the differential equation (3.5) is apparent at ρ if and only if it has a solution of the form

$$y(z) = \frac{1}{1-w}w^{1/2-\kappa_\rho}\sum_{n=0}^{\infty} c_n w^n \quad \text{with } c_0 = 1.$$

Plugging this series into (3.5) and using Lemma 3.5, (3.6), and (3.7), we find that the coefficients c_n need to satisfy

$$n\,(n-2\kappa_\rho)\,c_n = -3\sum_{j=0}^{n-2} c_j(ra_{n-j-2} + s_{\kappa_\rho}b_{n-j-2}). \tag{3.9}$$

Due to (3.8) and (3.9), we can inductively prove that

$$c_n = 0 \quad \text{if } n \not\equiv 0 \bmod 3. \tag{3.10}$$

Since the left-hand side of (3.9) vanishes when $n = 2\kappa_\rho$, (3.5) is apparent at ρ if and only if

$$\sum_{j=0}^{2\kappa_\rho-2} c_j(ra_{2\kappa_\rho-j-2} + s_{\kappa_\rho} b_{2\kappa_\rho-j-2}) = 0. \tag{3.11}$$

Suppose $3 \nmid 2\kappa_\rho$. Then, $j \equiv 0 \bmod 3$ and $2\kappa_\rho - j - 2 \equiv 1 \bmod 3$ cannot hold simultaneously. Hence, by (3.8) and (3.10), the condition (3.11) always holds for any r, i.e., (3.5) is apparent at ρ for any r. This proves (a).

For the case $3|2\kappa_\rho$, considering r as an indeterminate and using (3.9) to recursively express c_n as polynomials in r, we find that c_n is a polynomial of degree exactly $n/3$ in r when $3|n$ and $n < 2\kappa_\rho$. (Note that we use the fact that $a_1 \neq 0$ to conclude that the degree is $n/3$.) Thus, the left-hand side of (3.11) is a polynomial $P(r)$ of degree $2\kappa_\rho/3$ in r and (3.5) is apparent at ρ if and only if r is a root of this polynomial $P(x)$. This proves Part (b) except the identity (1.12). □

The proof of Theorem 1.8(a) except (1.14) is very similar to that of Theorem 1.7 and will be omitted.

4. Riemann's existence theorem and its application

In this section, we will use Riemann's existence theorem to prove Theorems 1.6, 1.7(b), and 1.8(b). The basic idea is as follows.

Let $h(z)$ be a modular function on some subgroup Γ of finite index of $\mathrm{SL}(2,\mathbb{Z})$. A simple computation shows that both $y_1(z) = 1/\sqrt{h'(z)}$ and $y_2(z) = h(z)/\sqrt{h'(z)}$ are solutions of

$$y''(z) = Q(z)y(z), \qquad Q(z) = -\frac{1}{2}\{h(z), z\},$$

where $\{h(z), z\}$ is the Schwarz derivative. Using either properties of Schwarz derivatives or direct computation, we can verify that $\{h(z), z\}$ is a meromorphic modular form of weight 4 on Γ. When $h(z)$ has additional symmetry, $\{h(z), z\}$ can be modular on a larger group. Note that, by construction, this differential equation $y''(z) = Q(z)y(z)$ is apparent on $\mathbb{H}$. Thus, one way to prove the theorems is simply to prove the existence of a modular function $h(z)$ such that $-\{h(z), z\}/2 = Q(z)$ for each $Q(z)$ appearing in the theorems. To achieve this, we will use Riemann's existence theorem.

Since some of the readers may not be familiar with Riemann's existence theorem, here we give a quick overview of this important result in the theory of Riemann surfaces. The exposition follows [17, Chapter III].

Let $F : X \to Y$ be a (branched) covering of compact Riemann surfaces of degree d. A point y of Y is a *branch point* if the cardinality of $F^{-1}(y)$ is not d and a point x of X is a *ramification point* if F is not locally one-to-one near x. (In particular, $F(x)$ is a branch point.) Let B be the (finite) set of branch points on Y under F. Pick a point $y_0 \in Y - B$ so that $F^{-1}(y_0)$ has d points, say $x_1, \ldots, x_d$. Every loop γ in $Y - B$ based at y_0 can be lifted to d paths $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_d$ with $\widetilde{\gamma}_j(0) = x_j$ and $\widetilde{\gamma}_j(1) = x_{j'}$ for some $x_{j'}$. The map $j \mapsto j'$ is then a permutation in S_d. The permutation depends only on the homotopy class of γ. In this way, we get a monodromy representation

$$\rho : \pi_1(Y - B, y_0) \to S_d.$$

Note that since $F^{-1}(Y - B)$ is connected, the image of ρ is a transitive subgroup of S_d. Also, let $b \in B$ and $a_1, \ldots, a_k$ be the points in $F^{-1}(b)$ with ramification indices $m_1, \ldots, m_k$, respectively. We can show that if γ is a small loop in $Y - B$ around b based at y_0, then $\rho(\gamma)$ is a product of disjoint cycles of lengths $m_1, \ldots, m_k$.

To state the version of Riemann's existence theorem used in the paper, let us consider the case $Y = \mathbb{P}^1(\mathbb{C})$. Let $B = \{b_1, \ldots, b_n\}$ be the set of branch points of $F : X \to \mathbb{P}^1(\mathbb{C})$. Let γ_j, $j = 1, \ldots, n$, be loops that circles b_j once but no other branch points. Then $\pi_1(\mathbb{P}^1(\mathbb{C}) - B, y_0)$ is generated by the homotopy classes $[\gamma_j]$, subject to a single relation $[\gamma_1] \ldots [\gamma_n] = 1$ (with a suitable ordering of the points b_j). Thus, the image of ρ is generated by $\sigma_j = \rho(\gamma_j)$ satisfying the relation $\sigma_1 \cdots \sigma_n = 1$. Then Riemann's existence theorem states as follows (see [17, Corollary 4.10]).

Theorem B (Riemann's existence theorem)**.** *Let $B = \{b_1, \ldots, b_n\}$ be a finite subset of $\mathbb{P}^1(\mathbb{C})$. Then there exists a one-to-one correspondence between the set of isomorphism classes of coverings $F : X \to \mathbb{P}^1(\mathbb{C})$ of compact Riemann surfaces of degree d whose branch points lie in B and the set of (simultaneous) conjugacy classes of n-tuples $(\sigma_1, \ldots, \sigma_n)$ of permutations in S_d such that $\sigma_1 \ldots \sigma_n = 1$ and the group generated by the σ_j's is transitive.*

Moreover, if the disjoint cycle decomposition of σ_j is a product of k cycles of lengths $m_1, \ldots, m_k$, then $F^{-1}(b_j)$ has k points with ramification indices $m_1, \ldots, m_k$, respectively.

We now use this result to prove Theorems 1.6, 1.7(b), and 1.8(b). Since the proofs are similar, we will provide details only for Theorem 1.7(b).

Proof of Theorem 1.7(b). Assume that $3|2\kappa_\rho$. Let Γ_2 be the subgroup of index of 2 of $\mathrm{SL}(2,\mathbb{Z})$ generated by

$$\gamma_1 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Note that

$$\gamma_1\gamma_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

The group Γ_2 has a cusp ∞ and two elliptic points $\rho_1 = (1+\sqrt{-3})/2$ and $\rho_2 = (-1+\sqrt{-3})/2$ of order 3, fixed by γ_1 and γ_2, respectively. Let

$$j_2(z) = \frac{E_6(z)}{\eta(z)^{12}},$$

which is a Hauptmodul for Γ_2, and set

$$J_2(z) = \frac{24}{j_2(z)}.$$

We have $J_2(\infty) = 0$, $J_2(\rho_1) = 1/\sqrt{-3}$, and $J_2(\rho_2) = -1/\sqrt{-3}$.

Set $\ell_0 = 2\kappa_\rho/3$. We first show that for each $\ell \in \{0,\dots,\ell_0-1\}$, there exists a modular function $h(z)$ on Γ_2 such that the covering $h : X(\Gamma_2) \to \mathbb{P}^1(\mathbb{C})$ of compact Riemann surfaces is ramified precisely at ∞, ρ_1, and ρ_2 with ramification index $2\ell+1$, ℓ_0, and ℓ_0, respectively. Note that by the Riemann-Hurwitz formula, such a covering has degree $\ell_0+\ell$, i.e., such a modular function $h(z)$ will be a rational function of degree $\ell_0+\ell$ in $J_2(z)$.

Consider the two ℓ_0-cycles

$$\sigma_1 = (1,\dots,\ell_0), \qquad \sigma_2 = (\ell_0+\ell, \ell_0+\ell-1, \dots, \ell+1)$$

in the symmetric group $S_{\ell_0+\ell}$. Since $\ell < \ell_0$, we have

$$\sigma_2\sigma_1 = (1,\dots,\ell,\ell_0+\ell,\ell_0+\ell-1,\dots,\ell_0),$$

which is a $(2\ell+1)$-cycle. (Notice that if $\ell \geq \ell_0$, then σ_1 and σ_2 are disjoint.) It is clear that when $\ell < \ell_0$, the subgroup generated by σ_1 and σ_2 is a transitive subgroup of $S_{\ell_0+\ell}$. Thus, by Riemann's existence theorem, there exists a covering of compact Riemann surfaces $H : X \to \mathbb{P}^1(\mathbb{C})$ of degree $\ell_0 + \ell$ ramified at three points ζ_1, ζ_2, and ζ_3 of $\mathbb{P}^1(\mathbb{C})$ with corresponding monodromy σ_1, σ_2, and $\sigma_1^{-1}\sigma_2^{-1}$, respectively. By the Riemann-Hurwitz formula, the genus of X is 0, and H is a rational function from $\mathbb{P}^1(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{C})$. Furthermore, by applying a suitable linear fractional transformation on the variable of H, we may assume that the three ramified points in $H^{-1}(z_j)$ are $0 = J_2(\infty)$, $1/\sqrt{-3} = J_2(\rho_1)$, and $-1/\sqrt{-3} = J_2(\rho_2)$, respectively. Set $h(z) = H(J_2(z))$. Then $h(z)$ has the required properties that the only points of $X(\Gamma_2)$ ramified under $h : X(\Gamma_2) \to \mathbb{P}^1(\mathbb{C})$ are ρ_1, ρ_2, and the cusp ∞ with ramified indices ℓ_0, ℓ_0, and $2\ell + 1$, respectively.

Now consider the Schwarz derivative $\{h(z), z\}$, which is a meromorphic modular form of weight 4 on Γ_2. We claim that it is in fact modular on the bigger group $\mathrm{SL}(2, \mathbb{Z})$.

Indeed, to show $\{h(z), z\}$ is modular on $\mathrm{SL}(2, \mathbb{Z})$, it suffices to prove that $\{h(z), z\}\big|T = \{h(z), z\}$, where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Let $\widetilde{h}(z) = h(z + 1)$. Now the automorphism on $X(\Gamma_2)$ induced by T interchanges ρ_1 and ρ_2. Thus, the ramification data of the covering $\widetilde{h} : X(\Gamma_2) \to \mathbb{P}^1(\mathbb{C})$ is the same as that of h. By the Riemann's existence theorem, h and $\widetilde{h}$ are related by a linear fractional transformation, i.e., $\widetilde{h} = (ah + b)/(ch + d)$ for some $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. It follows that $\{h(z), z\}\big|T = \{h(z), z\}$ by the well-known property $\{(af(z) + b)/(cf(z) + d), z\} = \{f(z), z\}$ of the Schwarz derivative. This proves that $\{h(z), z\}$ is a meromorphic modular form of weight 4 on the larger group $\mathrm{SL}(2, \mathbb{Z})$.

Furthermore, since ρ_1 is an elliptic point of order 3, a local parameter for ρ_1 as a point on the compact Riemann surface $X(\Gamma_2)$ is w^3, where $w = (z - \rho)/(z - \overline{\rho})$. Therefore, we have

$$h(z) = d_0 + \sum_{n=3\ell_0}^{\infty} d_n w^n,$$

for some complex numbers d_n with $d_{3\ell_0} \neq 0$ and $d_n = 0$ whenever $3 \nmid n$. For convenience, set

$$\begin{aligned} A &= \sum_{n=3\ell_0}^{\infty} n d_n w^{n-1}, \\ B &= \sum_{n=3\ell_0}^{\infty} n(n-1) d_n w^{n-2}, \end{aligned}$$

$$C = \sum_{n=3\ell_0}^{\infty} n(n-1)(n-2)d_n w^{n-3}.$$

Using (3.3), we compute that

$$h'(z) = \frac{(1-w)^2}{\rho-\overline{\rho}}A,$$

$$h''(z) = \frac{(1-w)^4}{(\rho-\overline{\rho})^2}B - 2\frac{(1-w)^3}{(\rho-\overline{\rho})^2}A,$$

$$h'''(z) = \frac{(1-w)^6}{(\rho-\overline{\rho})^3}C - 6\frac{(1-w)^5}{(\rho-\overline{\rho})^3}B + 6\frac{(1-w)^4}{(\rho-\overline{\rho})^3}A,$$

and hence

$$\{h(z),z\} = \frac{(1-w)^4}{(\rho-\overline{\rho})^2}\left(\frac{C}{A} - \frac{3}{2}\frac{B^2}{A^2}\right) = -\frac{(1-w)^4}{3}\left(\frac{1-9\ell_0^2}{2w^2} + cw + \cdots\right)$$

for some c. It follows that, by (3.7),

$$\{h(z),z\} + 2\pi^2 s_{\kappa_\rho}\frac{E_6(z)^2}{E_4(z)^2}, \qquad s_{\kappa_\rho} = \frac{1-4\kappa_\rho^2}{9} = \frac{1}{9} - \ell_0^2,$$

is a holomorphic modular form of weight 4 on $\mathrm{SL}(2,\mathbb{Z})$. By comparing the leading coefficients of the Fourier expansions at the cusp ∞, we conclude that,

$$\{h(z),z\} = -2\pi^2\left(rE_4(z) + s_{\kappa_\rho}\frac{E_6(z)^2}{E_4(z)^2}\right),$$

where $r = -(2\ell+1)^2/4 - s_{\kappa_\rho} = \ell_0^2 - (2\ell+1)^2/4 - 1/9$. Equivalently, $1/\sqrt{h'(z)}$ and $h(z)/\sqrt{h'(z)}$ are solutions of (1.11) which also implies that the singularity of (1.11) at ρ is apparent.

Finally, since we have found ℓ_0 different r such that (1.11) has an apparent singularity at ρ for the given s_{κ_ρ}, by Part (a), this proves the theorem. $\square$

Example. For small κ_ρ, the modular functions $h(z)$ appearing in the proof are given by

κ_ρ	ℓ	(r,s)	$h(z)$
$\frac{3}{2}$	0	$\left(\frac{23}{36}, -\frac{8}{9}\right)$	J_2
3	0	$\left(\frac{131}{36}, -\frac{35}{9}\right)$	$\frac{J_2}{1-3J_2^2}$
3	1	$\left(\frac{59}{36}, -\frac{35}{9}\right)$	$\frac{J_2^3}{1+9J_2^2}$

Proof of Theorem 1.8(b). Assume that $\kappa_i \in \mathbb{N}$. Let Γ_3 be the subgroup of index 3 of $\mathrm{SL}(2,\mathbb{Z})$ generated by

$$\gamma_1 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We note that

$$\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

The group Γ_3 has one cusp and three elliptic points $z_1 = 1+i$, $z_2 = (1+i)/2$, and $z_3 = i$ of order 2, fixed by γ_j, $j = 1,2,3$, respectively. Let

$$j_3(z) = \frac{E_4(z)}{\eta(z)^8}$$

be a Hauptmodul for Γ_3 and set

$$J_3(z) = 12 j_3(z)^{-1}.$$

Note that $j_3(z)^3$ is equal to the elliptic j-function $j(z)$. Since $j(i) = 1728$ and $j(\rho) = 0$, we have $\{J_3(z_1), J_3(z_2), J_3(z_3)\} = \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$, $J_3(\rho) = \infty$, and $J_3(\infty) = 0$.

Consider the case $r + t_{\kappa_i} = -(\ell + 1/3)^2$ first. Our goal here is to construct a modular function $h(z)$ on Γ_3, for each ℓ in the range, such that the covering $h : X(\Gamma_3) \to \mathbb{P}^1(\mathbb{C})$ has degree

$$d = \frac{1}{2}(3\kappa_i + 3\ell - 1)$$

and is ramified at precisely the cusp ∞ and the three elliptic points z_1, z_2, and z_3 with ramification indices $3\ell+1$, κ_i, κ_i, and κ_i, respectively. (Notice that κ_i and ℓ have opposite parities, so d is an integer.) Since the covering has four branch points, it is not easy to apply Riemann's existence theorem directly to get $h(z)$. Instead, we shall use the following idea.

For convenience, set

$$m=\frac{1}{2}(\kappa_i+\ell-1),\qquad m'=\frac{1}{2}(\kappa_i-\ell-1). \tag{4.1}$$

We claim that there exists a rational function $H(x)$ of degree d in x of the form

$$H(x)=\frac{x^{3\ell+1}G(x)^3}{F(x)^3},\quad \deg F(x)=m,\quad \deg G(x)=m',$$

such that $xF(x)G(x)$ is squarefree and

$$H(x)-1=\frac{(x-1)^{\kappa_i}L(x)}{F(x)^3}$$

for some polynomial L of degree $d-\kappa_i$ with no repeated roots. That is, $H(x)$ is a rational function such that

(i) the covering $H:\mathbb{P}^1(\mathbb{C})\to\mathbb{P}^1(\mathbb{C})$ branches at precisely ∞, 0, and 1 (note that by the Riemann-Hurwitz formula, H cannot have other branch points),

(ii) the monodromy σ_∞ around ∞ is a product of m disjoint 3-cycles, the monodromy σ_0 around 0 is a disjoint product of a $(3\ell+1)$-cycle and m' 3-cycles, and the monodromy σ_1 around 1 is a κ_i-cycle,

(iii) the unique unramified point in $H^{-1}(\infty)$ is ∞, the unique point of ramification index $3\ell+1$ in $H^{-1}(0)$ is 0, and the unique ramified point in $H^{-1}(1)$ is 1.

Suppose that such a rational function $H(x)$ exists. We define $h:X(\Gamma_3)\to\mathbb{P}^1(\mathbb{C})$ by

$$h(z)=H(J_3(z)^3)^{1/3}=\frac{J_3(z)^{3\ell+1}G(J_3(z)^3)}{F(J_3(z)^3)}.$$

From the construction, we see that h ramifies only at $z_1=1+i$, $z_2=(1+i)/2$, $z_3=i$, and ∞ with ramification indices κ_i, κ_i, κ_i, and $3\ell+1$, respectively. Then following the proof of Theorem 1.7(b), we can prove that the Schwarz derivative $\{h(z),z\}$ is a meromorphic modular form on the larger group $\mathrm{SL}(2,\mathbb{Z})$ and that

$$\{h(z),z\}=-2\pi^2\left(rE_4(z)+t_{\kappa_i}\frac{E_4(z)^4}{E_6(z)^2}\right),\quad r=-\left(\ell+\frac{1}{3}\right)^2-t_{\kappa_i},$$

which is equivalent to the assertion that $1/\sqrt{h'(z)}$ and $h(z)/\sqrt{h'(z)}$ are solutions of (1.13) with $t = t_{\kappa_i}$ and $r = -(\ell + 1/3)^2 - t_{\kappa_i}$ and hence implies that (1.13) is apparent with these parameters.

It remains to prove that a rational function $H(x)$ with properties described above exists. According to Riemann's existence theorem, it suffices to find σ_∞ that is a product of m disjoint 3-cycles and σ_1 that is a κ_i-cycle in S_d such that $\sigma_1\sigma_\infty$ is a disjoint product of a cycle of length $3\ell + 1$ and m' cycles of length 3. Indeed, we find that we may choose

$$\sigma_\infty = (2, 3, 4)(5, 6, 7) \dots (3m - 1, 3m, 3m + 1)$$

and

$$\sigma_1 = (1, 2, 5, 8, \dots, 3m - 1, 3m' + 1, 3m' - 2, \dots, 7, 4).$$

Then

$$\sigma_1\sigma_\infty = (1, 2, 3)(4, 5, 6) \dots (3m' - 2, 3m' - 1, 3m')(3m' + 1, 3m' + 2, \dots, d).$$

This settles the case $r + t_{\kappa_i} = -(\ell + 1/3)^2$.

The case $r + t_{\kappa_i} = -(\ell - 1/3)^2$ can be dealt with in the same way. The difference is that the rational function $H(x)$ in this case has degree

$$d = \frac{3}{2}(\kappa_i + \ell - 1)$$

and is of the form

$$H(x) = \frac{x^{3\ell-1}G(x)^3}{F(x)^3}, \quad \deg F(x) = m, \quad \deg G(x) = m',$$

where m and m' are the same as those in (4.1), such that $xF(x)G(x)$ is squarefree and

$$H(x) - 1 = \frac{(x - 1)^{\kappa_i} L(x)}{F(x)^3}$$

for some polynomial $L(x)$ of degree $d - \kappa_i$ with no repeated roots. I.e., σ_∞ in this case is a disjoint product of m 3-cycles, σ_0 is a a disjoint product of $(3\ell - 1)$-cycle and m' 3-cycles, and σ_1 is a κ_i-cycle. We choose

$$\sigma_\infty = (1, 2, 3)(4, 5, 6) \dots (3m - 2, 3m - 1, 3m)$$

and

$$\sigma_1 = (1, 4, 7, \dots, 3m - 2, 3m, 3m - 3, \dots, 3\ell)$$

with

$$\sigma_1\sigma_\infty = (1, 2, 3, 4, \ldots, 3\ell-1)(3\ell, 3\ell+1, 3\ell+2)\ldots(3m-3, 3m-2, 3m-1).$$

The rest of proof is the same as the case of $r + t_{\kappa_i} = -(\ell+1/3)^2$. This completes the proof that (1.14) is the complete list of parameters r such that (1.13) with $t = t_{\kappa_i}$ is apparent. ☐

Example. For small κ_i, the modular functions $h(z)$ in the proof are given by

κ_i	$\ell \pm 1/3$	(r,t)	$h(z)$
1	$\frac{1}{3}$	$\left(\frac{23}{36}, -\frac{3}{4}\right)$	J_3
2	$\frac{2}{3}$	$\left(\frac{119}{36}, -\frac{15}{4}\right)$	$\frac{J_3^2}{1+2J_3^3}$
2	$\frac{4}{3}$	$\left(\frac{71}{36}, -\frac{15}{4}\right)$	$\frac{J_3^4}{1-4J_3^3}$

Proof of Theorem 1.6. Assume that n_i, n_ρ, and n_∞ are positive integers satisfying the two conditions. We note that the parameters r, s, and t in (1.9) are

$$r = -n_\infty^2 + n_\rho^2 + n_i^2 - \frac{13}{36}, \qquad s = \frac{1}{9} - n_\rho^2, \qquad t = \frac{1}{4} - n_i^2. \tag{4.2}$$

Let

$$d = \frac{1}{2}(n_i + n_\rho + n_\infty - 1).$$

By the second condition, we have

$$d - n_i = \frac{1}{2}(n_\rho + n_\infty - n_i - 1) \geq 0$$

and similarly, $d - n_\rho \geq 0$. Thus, there are cycles of lengths n_i and n_ρ in the symmetric group S_d. Choose

$$\sigma_1 = (1, \ldots, n_i), \qquad \sigma_2 = (d, d-1, \ldots, d-n_\rho+1)$$

By the second condition again, we have

$$n_i - (d - n_\rho + 1) = \frac{1}{2}(n_i + n_\rho - n_\infty - 1) \geq 0.$$

In other words, the two cycles are not disjoint. We then compute that

$$\sigma_2\sigma_1 = (1, \ldots, d - n_\rho, d, d-1, \ldots, n_i).$$

This is a cycle of length

$$d - n_\rho + (d - n_i + 1) = 2d - n_\rho - n_i + 1 = n_\infty.$$

It is clear that the subgroup of S_d generated by σ_1 and σ_2 is transitive. Thus, by Riemann's existence theorem, given three distinct points ζ_1, ζ_2, and ζ_3 on $\mathbb{P}^1(\mathbb{C})$, there is a covering $H : X \to \mathbb{P}^1(\mathbb{C})$ of compact Riemann surfaces of degree d branched at ζ_1, ζ_2, and ζ_3 with monodromy σ_1, σ_2, and $\sigma_3 = \sigma_1^{-1}\sigma_2^{-1}$, respectively. By the Riemann-Hurwitz formula, the genus of X is 0 and we may assume that $X = \mathbb{P}^1(\mathbb{C})$. Applying a suitable linear fractional transformation (i.e., an automorphism of X) if necessary, we may assume that the ramification points on X are $1728 = j(i)$, $0 = j(\rho)$, and $\infty = j(\infty)$ with ramification indices n_i, n_ρ, and n_∞, respectively. Let $h : X_0(1) \to \mathbb{P}^1(\mathbb{C})$ be defined by $h(z) = H(j(z))$. Following the same computation as in the proof of Theorem 1.7(b), we can show that

$$\{h(z), z\} = -2\pi^2 \left(rE_4(z) + s\frac{E_6(z)^2}{E_4(z)^2} + t\frac{E_4(z)^4}{E_6(z)^2} \right)$$

with r, s, and t given as (4.2) (details omitted). This implies that the singularities of (1.9) are all apparent.

Conversely, assume that the differential equation (1.9) is apparent throughout $\mathbb{H} \cup \{\text{cusps}\}$. Let $\pm n_\infty/2$ be the local exponents at ∞. Then a fundamental pair of solutions near ∞ is

$$y_\pm(z) = q^{\pm n_\infty/2} \left(1 + \sum_{n=1}^{\infty} c_n^\pm q^n \right).$$

Let $h(z) = y_+(z)/y_-(z)$. Since (1.9) is apparent throughout $\mathbb{H}$, $h(z)$ is a single-valued function on $\mathbb{H}$. Arguing as in the second proof of Theorem 1.6, we see that $h(z)$ is a modular function on $\mathrm{SL}(2,\mathbb{Z})$. Now since

$$\{h(z), z\} = -2\pi^2 \left(rE_4(z) + s\frac{E_6(z)^2}{E_4(z)^2} + t\frac{E_4(z)^4}{E_6(z)^2} \right)$$

have poles only at points equivalent to ρ or i under $\mathrm{SL}(2,\mathbb{Z})$, the covering $X_0(1) \to \mathbb{P}^1(\mathbb{C})$ defined by $z \mapsto h(z)$ can only ramify at ρ, i, or ∞. From the computation above, we see that their ramification indices must be n_ρ, n_i, and n_∞, respectively. Then by the Riemann-Hurwitz formula, $n_\rho+n_i+n_\infty$ must be odd and the degree of the covering is $(n_\rho+n_i+n_\infty-1)/2$. Since the ramification indices n_ρ, n_i, and n_∞ cannot exceed the degree of the covering, we conclude that the sum of any two of n_ρ, n_i, and n_∞ must be greater than the remaining one. This completes the proof of the theorem. □

5. Eremenko's Theorem and its applications

Second proof of (1.12). In Section 2.3, Example 2 shows that the angle of Q_1 at i, ρ and ∞ are

$$\theta_1 = \frac{1}{2}, \quad \theta_2 = \frac{2\kappa_\rho}{3}, \quad \text{and} \quad \theta_\infty = \sqrt{-(r+s_{\kappa_\rho})}. \tag{5.1}$$

First, we consider θ_2 is even, say $\theta_2 = 2\ell_0$. By Eremanko's Theorem in Section 2, the curvature equation (2.5) has a solution if and only if either $|\theta_\infty - \theta_1| = 2\ell+1$ or $\theta_\infty + \theta_1 = 2\ell + 1$ for some $\ell \in \mathbb{Z}_{\geq 0}$ and $\ell \leq \ell_0 - 1$. Since $\theta_\infty > 0$, the condition $|\theta_\infty - \theta_1| = 2\ell + 1 \geq 1$ implies $\theta_\infty - \theta_1 > 0$ and then $\theta_\infty - \theta_1 = 2\ell + 1$. This is equivalent to $-(r + s_{\kappa_\rho}) = \theta_\infty^2 = (2\ell+1+1/2)^2$, $\ell = 0, \ldots, \ell_0 - 1$. The second condition $\theta_\infty + \theta_1 = 2\ell+1$ is equivalent to $-(r+s_{\kappa_\rho}) = \theta_\infty^2 = (2\ell+1/2)^2$, $\ell = 0, \ldots, \ell_0-1$. Therefore, there are exactly $2\ell_0$ different θ_∞ such that the curvature equation (2.5) has a solution and each of such a curvature equation is associated with the modular form $Q_1(z; r, s_{\kappa_\rho})$ with (r, s_{κ_ρ}) where $r+s_{\kappa_\rho} = -(\ell+1/2)^2$ for some $\ell \in \{0, \ldots, 2\ell_0-1\}$. By Theorem 2.4, for each (r, s_{κ_ρ}), the ODE (1.11) is apparent. However, the first part of Theorem 1.7(b) says that there exists a polynomial $P(x)$ of degree $2\kappa_\rho/3$ such that (2.5) with (r, s_{κ_ρ}) is apparent if and only if $P(r) = 0$. Therefore, $P(r)$ has distinct roots and each root satisfies $r + s_{\kappa_\rho} = -(\ell+1/2)^2$ for some integer ℓ, $0 \leq \ell \leq 2\ell_0 - 1 = \theta_2 - 1$. The proves (1.12) when θ_2 is even.

For the case θ_2 is odd, the idea of the proof is basically the same. By noting $\theta_1 = 1/2$, the Eremenko theorem in Section 2 implies either $|\theta_\infty - 1/2| = \ell$ or $\theta_\infty + 1/2 = \ell$, where ℓ is even because θ_2 is odd. The first condition can be replaced by $\theta_\infty - 1/2 = \ell$. Thus we have $\theta_\infty = \ell + 1/2$ or $\theta_\infty = \ell - 1/2 = (\ell-1)+1/2$, that is $r+s = -(\ell+1/2)^2$, $\ell = 0, 1, 2, \ldots, \theta_2 - 1$. The proof of (1.12) is complete. □

Second proof of (1.14). The angles for $Q_2(z)$ are $\theta_1 = \kappa_i$, $\theta_2 = 1/3$, and $\theta_\infty = \sqrt{-(r+t_i)}$, where $\frac{1}{2} \pm \kappa_i$ are the local exponents of (1.13). Hence

$$\kappa_i - \frac{1}{2} + 1 = m + \frac{1}{2}$$

i.e., $\theta_1 = \kappa_i$ is an integer. Hence, there is a solution u of (2.5)-(2.7) with the RHS equals to $4\pi n \sum \delta_p$, where the summation runs over $\gamma \cdot i$, $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, if and only if either $\theta_\infty - \theta_2 = |\theta_\infty - \theta_2| = \ell$ or $\theta_\infty + \theta_2 = \ell$ where $\ell \leq \kappa_i - 1$ and ℓ has the opposite parity of κ_i. Hence, $\theta_\infty = \ell \pm 1/3$ and $r + t_i = -(\ell \pm 1/3)^2$. This proves (1.14). □

Second proof of Theorem 1.6. Suppose that the ODE (1.9) has local exponents $\pm n_\infty$ at ∞, $n_\infty \in \frac{1}{2}\mathbb{N}$. We claim that (1.9) *is apparent throughout* $\mathbb{H}^*$ *if and only if* $Q_3(z) = Q_3(z; r, s, t)$ *is realized by a metric with curvature* 1/2. It is clear that the second statement implies the first statement. So it suffices to prove the other direction.

Suppose that (1.9) is apparent throughout $\mathbb{H}^*$. Let $y_\pm(z) = q^{\pm n_\infty/2}(1 + O(q))$ be two solutions of (1.9) and set $h(z) = y_+(z)/y_-(z)$. Since (1.9) is apparent on $\mathbb{H}$, $h(z)$ is a meromorphic single-valued function on $\mathbb{H}$ and its Schwarz derivative is $-2Q_3(z)$. Recall Bol's theorem that there is a homomorphism $\rho : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{C})$ such that

$$\begin{pmatrix} \left(y_1\big|_{-1}\gamma\right)(z) \\ \left(y_2\big|_{-1}\gamma\right)(z) \end{pmatrix} = \pm\rho(\gamma) \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}, \quad \gamma \in \mathrm{SL}(2,\mathbb{C}).$$

Clearly, $\rho(T) = \pm I$ because ∞ is apparent. Note that $\ker\rho$ is a normal subgroup of $\mathrm{SL}(2,\mathbb{Z})$ and contains $\gamma T \gamma^{-1}$ for any $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. In particular, $\ker\rho$ contains both $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $STS^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$, where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ generate $\mathrm{SL}(2,\mathbb{Z})$, we conclude that $\ker\rho = \mathrm{SL}(2,\mathbb{Z})$. In other words, $\rho(\gamma) = \pm I$ and $h(z)$ is a modular function on $\mathrm{SL}(2,\mathbb{Z})$. Thus we have a solution $u := \log \frac{8|h'(z)|^2}{\left(1+|h(z)|^2\right)^2}$ which realizes Q_3. This proves the claim.

Now, we apply the Eremenko theorem with the angles given by $\theta_1 = \kappa_i$, $\theta_2 = 2\kappa_\rho/3$ and $\theta_3 = n_\infty$. Our necessary and sufficient condition in Theorem 1.6 is identically the same as the condition of Eremenko's theorem for the existence of u with three integral angles. This proves Theorem 1.6. □

Theorem 5.1. *Suppose* $\kappa_i \in \mathbb{N}$ *and* $\kappa_\rho, \kappa_\infty \in \frac{1}{2}\mathbb{N}$ *such that* $2\kappa_\rho/3 \in \mathbb{N}$. *If* $Q_3(z; r, s, t)$ *is apparent at* ρ *and* i, *then* Q *can be realized.*

Proof. By the assumption, we have that θ_i, $1 \leq i \leq 3$, are all integers. Now, given κ_i and κ_ρ, s and t are determined by the same formula in our paper. Further, there are polynomials P_1 and P_2:

- $Q_3(z; r, s, t)$ is apparent at i if and only if $P_1(r) = 0$, and $\deg P_1(r) = \kappa_i$.

- $Q_3(z;r,s,t)$ is apparent at ρ if and only if $P_2(r)=0$, and $\deg P_2 = 2\kappa_\rho/3$.

Therefore, $Q_3(z;r,s,t)$ is apparent at i and ρ if and only if

$$r \in \{r : P_1(r) = P_2(r) = 0\}.$$

Now, we claim that under the assumption $\theta_1 \in \mathbb{N}$, $Q_3(z;r,s,t)$ is apparent if and only if the local exponents at ∞ are $\pm\kappa_\infty/2$, $\kappa_\infty \in \mathbb{N}$ and the curvature equation has a solution.

By Eremenko's Theorem (Section 2.4), (recall $\theta_1 = \kappa_i$, $\theta_2 = 2\kappa_\rho/3$, $\theta_3 = 2\kappa_\infty$) the curvature equation has a solution if and only if $\theta_1+\theta_2+\theta_3$ is odd and $\theta_i < \theta_j + \theta_k$, $i \neq j \neq k$. This condition is equivalent to

(a)

$$\theta_2 - \theta_1 < \theta_3 < \theta_2 + \theta_1, \quad \text{and}$$

(b)

$$\theta_1 - \theta_2 < \theta_3 < \theta_1 + \theta_2.$$

Since $\theta_1+\theta_2+\theta_3$ is odd, we have θ_2 solutions of the curvature equation if $\theta_1 > \theta_2$, θ_1 solutions if $\theta_2 > \theta_1$.

Now, $\deg P_1 = \kappa_i = \theta_1$ and $\deg P_2 = 2\kappa_\rho/3 = \theta_2$. Then

$$\begin{aligned}\min\{\theta_1,\theta_2\} &\geq |\{r : P_1(r) = P_2(r) = 0\}| \\ &=\geq \#\text{ of curvature equations} \geq \min\{\theta_1,\theta_2\}.\end{aligned}$$

Thus

$$|\{r : P_1(r) = P_2(r) = 0\}| = \#\text{ of curvature equations.}$$

This proves the theorem. □

Remark. In fact, the proof shows that if $\deg P_i \leq \deg P_j$, then P_i is a factor of P_j.

6. proof of Theorem 1.1 and Theorem 1.4

Proof of Theorem 1.1. Let ρ be the Bol representation associated to (1.1), and set $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, and $R = TS = \left(\begin{smallmatrix}1&-1\\1&0\end{smallmatrix}\right)$. They satisfy

$$S^2 = -I, \quad \text{and} \quad R^3 = -I. \tag{6.1}$$

Assume that $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ hold. It follows from either [12, Theorem 2.5], quoted as Theorem A.1 in the appendix, or Theorem A.3 (with $\theta_1 = 1/2$, $\theta_2 = 1/3$, and $\theta_3 = 2r_\infty$ or $\theta_3 = 1 - 2r_\infty$, depending on whether $2r_\infty \le 1/2$ or $2r_\infty > 1/2$) in the appendix that if $1/12 < r_\infty < 5/12$, then an invariant metric realizing $Q(z)$ exists, and if $0 < r_\infty < 1/12$ or $5/12 < r_\infty \le 1/2$, then there does not exist an invariant metric realizing $Q(z)$. So here we are concerned with the case $r_\infty = 1/12$ or $r_\infty = 5/12$.

Assume that $r_\infty = 1/12$. Then there exists a basis $\{y_1(z), y_2(z)\}$ for the solution space of (1.1) such that

$$\rho(T) = \pm \begin{pmatrix} \epsilon & 0 \\ 0 & \bar{\epsilon} \end{pmatrix}, \qquad \epsilon = e^{2\pi i/12}. \tag{6.2}$$

Since $S^2 = -I$, we have $\rho(S)^2 = \pm I$. The matrix $\rho(S)$ cannot be equal to $\pm I$ as the relation $R = TS$ will imply that the eigenvalues of $\rho(R)$ are $\pm e^{2\pi i/12}$ or $\pm e^{-2\pi i/12}$, which is absurd. It follows that $\operatorname{tr} \rho(S) = 0$ and we have

$$\rho(S) = \pm \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \qquad \rho(R) = \pm\rho(T)\rho(S) = \pm \begin{pmatrix} \epsilon a & \epsilon b \\ \bar{\epsilon} c & -a\bar{\epsilon} \end{pmatrix} \tag{6.3}$$

for some $a, b, c \in \mathbb{C}$. Since $\rho(R)^3 = \pm I$, $\det \rho(R) = 1$, and $\rho(R) \ne \pm I$ by a similar reason as above, the characteristic polynomial of $\rho(R)$ has to be $x^2 - x + 1$ or $x^2 + x + 1$. In particular, we have $\operatorname{tr} \rho(R) = \pm 1$, i.e., $a(\epsilon - \bar{\epsilon}) = \pm 1$ and hence $a = \pm i$ and $bc = 0$. Under the assumption that there is an invariant metric realizing $Q(z)$, the matrices $\rho(S)$, $\rho(T)$, and $\rho(R)$ must be unitary, after a simultaneous conjugation. (See the discussion in Section 2.2.) If one of b and c is not 0, this cannot happen. Therefore, we have $b = c = 0$. This implies that the function $y_1(z)^2$, which is meromorphic throughout $\mathbb{H}$ since the local exponents at every singularity are in $\frac{1}{2}\mathbb{Z}$, satisfies

$$y_1(Tz)^2 = e^{2\pi i/6} y_1(z)^2, \qquad y_1(Sz)^2 = -z^{-2} y_1(z)^2.$$

It follows that $y_1(z)^2$ is a meromorphic modular form of weight -2 with character χ on $\mathrm{SL}(2, \mathbb{Z})$. Likewise, we can show that $y_2(z)^2$ is a meromorphic modular form of weight -2 with character $\overline{\chi}$. This proves that if there is an invariant metric realizing $Q(z)$, then there are solutions $y_1(z)$ and $y_2(z)$ with the stated properties. The proof of the case $r_\infty = 5/12$ is similar and is omitted.

The proof of the converse statement is easy. If there exist solutions $y_1(z)$ and $y_2(z)$ of (1.1) such that $y_1(z)^2$ and $y_2(z)^2$ are meromorphic modular forms of weight -2 with character χ and $\overline{\chi}$, respectively, on $\mathrm{SL}(2,\mathbb{Z})$, then $y_1(Tz)^2 = e^{2\pi i/6}y_1(z)^2$ and $y_2(Tz)^2 = e^{-2\pi i/6}y_2(z)^2$, which implies that $y_1(z)^2$ and $y_2(z)^2$ are of the form $y_1(z)^2 = q^{1/6}\sum_{j\ge n_0} c_j q^j$ and $y_2(z)^2 = q^{-1/6}\sum_{j\ge n_0} d_j q^j$. It follows that $r_\infty = 1/12$ or $r_\infty = 5/12$. It is clear that with respect to the basis $\{y_1(z), y_2(z)\}$, the Bol representation is given by

$$\rho(T) = \pm\begin{pmatrix} e^{2\pi i/12} & 0 \\ 0 & e^{-2\pi i/12}\end{pmatrix}, \qquad \rho(S) = \pm\begin{pmatrix} \pm i & 0 \\ 0 & -i\end{pmatrix},$$

and hence is unitary. It follows that there is an invariant metric of curvature $1/2$ realizing $Q(z)$. This proves the theorem. □

We now give two examples with $r_\infty = 1/12$, one of which can be realized by some invariant metric of curvature $1/2$, while the other of which can not. Note that Theorem 1 of [10] implies that when (1.1) does not have $\mathrm{SL}(2,\mathbb{Z})$-inequivalent singularities outside $\{i,\rho\}$, $1/12 < r_\infty < 5/12$ is the necessary and sufficient condition for the existence of an invariant metric of curvature $1/2$ realizing Q. The examples we provide below show that when (1.1) has $\mathrm{SL}(2,\mathbb{Z})$-inequivalent singularities other than i and ρ, this condition is no longer a necessary condition.

Example. Let $\eta(z) = q^{1/24}\prod_{n=1}^\infty (1-q^n) = \Delta(z)^{1/24}$,

$$x(z) = \frac{E_4(z)}{\eta(z)^8} = q^{-1/3} + \cdots, \qquad y(z) = \frac{E_6(z)}{\eta(z)^{12}} = q^{-1/2} + \cdots, \tag{6.4}$$

and $h(z) = x(z)/y(z) = q^{1/6} + \cdots$. They are modular functions on the unique normal subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$ of index 6 such that $\mathrm{SL}(2,\mathbb{Z})/\Gamma$ is cyclic. (Another way to describe Γ is that $\Gamma = \ker\chi$, where χ is the character of $\mathrm{SL}(2,\mathbb{Z})$ such that $\chi(S) = -1$ and $\chi(R) = e^{2\pi i/3}$.) Using Ramanujan's identities

$$\begin{aligned} D_q E_2(z) &= \frac{E_2(z)^2 - E_4(z)}{12}, \\ D_q E_4(z) &= \frac{E_2(z)E_4(z) - E_6(z)}{3}, \\ D_q E_6(z) &= \frac{E_2(z)E_6(z) - E_4(z)^2}{2}, \end{aligned}$$

where $D_q = q\,d/dq$ (see [21, Proposition 15]) and the relation $\Delta(z) = (E_4(z)^3 - E_6(z)^2)/1728$, we can compute that

$$\{h(z), z\} = (2\pi i)^2 Q_0(z)$$

where

$$Q_0(z) = E_4(z)\left(-\frac{1}{72} - \frac{9(E_4(z)^3 - E_6(z)^2)^2}{(3E_4(z)^3 - 2E_6(z)^2)^2} + \frac{5}{2}\frac{E_4(z)^3 - E_6(z)^2}{3E_4(z)^3 - 2E_6(z)^2}\right).$$

Thus,

$$y_+(z) = \frac{h(z)}{\sqrt{D_q h(z)}} = q^{1/12} + \cdots, \quad y_-(z) = \frac{1}{\sqrt{D_q h(z)}} = q^{-1/12} + \cdots$$

are solutions of the differential equation $y''(z) = Q(z)y(z)$, where $Q(z) = -(2\pi i)^2 Q_0(z)/2$. The meromorphic modular form $Q(z)$ has only one SL(2,$\mathbb{Z}$)-inequivalent singularity at the point z_1 such that $3E_4(z_1)^3 - 2E_6(z_1)^2 = 0$ and is holomorphic at the elliptic points i and ρ. In the notation of Theorem 1.1, we have $r_\infty = 1/12$. This provides an example of an invariant metric of curvature 1/2 realizing a meromorphic modular form of weight 4 with a threshold r_∞. Note that with respect to the basis $\{y_+, y_-\}$, the Bol representation is given by

$$\rho(T) = \pm\begin{pmatrix} e^{2\pi i/12} & 0 \\ 0 & e^{-2\pi i/12}\end{pmatrix}, \qquad \rho(S) = \pm\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix},$$

both of which are unitary. (The information about $\rho(S)$ follows from the transformation formula $\eta(-1/z) = \sqrt{z/i}\,\eta(z)$ and the fact that $D_q h(z) = C\eta(z)^4(3E_4(z)^3 - 2E_6(z)^2)/E_6(z)^2$ for some constant C.)

Example. Let $x(z)$ and $y(z)$ be defined by (6.4), and Γ be the unique normal subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of index 6 such that $\mathrm{SL}(2,\mathbb{Z})/\Gamma$ is cyclic. The modular curve $X(\Gamma) := \Gamma\backslash\mathbb{H}^*$ has one cusp of width 6, no elliptic points, and is of genus 1. Since the modular functions $x(z)$ and $y(z)$ on Γ have only a pole of order 2 and 3, respectively, at the cusp ∞ and are holomorphic elsewhere, they generate the function field of $X(\Gamma)$. Then from the relation $E_4(z)^3 - E_6(z)^2 = 1728\eta(z)^{24}$, we see that $x(z)$ and $y(z)$ satisfies

$$y^2 = x^3 - 1728,$$

which we may take as the defining equation for $X(\Gamma)$. Let $f(z)$ be a meromorphic modular form of weight 2 on Γ such that all residues on $\mathbb{H}$ are 0. Equivalently, let $\omega = f(z)\,dz$ be a meromorphic differential 1-form of the second kind on $X(\Gamma)$. Consider

$$y_1(z) = \frac{1}{\sqrt{f(z)}}\int_{z_0}^{z} f(u)\,du, \qquad y_2(z) = \frac{1}{\sqrt{f(z)}},$$

where z_0 is a fixed point in $\mathbb{C}$ that is not a pole of $f(z)$. Under the assumption that all residues of $f(z)$ are 0, the integral in the definition of $y_1(z)$ does not depend on the choice of path of integration from z_0 to z. A straightforward computation shows that the Wronskian of y_1 and y_2 is a constant and hence $y_1(z)$ and $y_2(z)$ are solutions of the differential equation $y''(z) = Q(z)y(z)$, where

$$Q(z) = \frac{3f'(z)^2 - 2f(z)f''(z)}{4f(z)^2}$$

can be shown to be a meromorphic modular form of weight 4 on Γ. (The numerator of $Q(z)$ is a constant mulitple of the Rankin-Cohen bracket $[f,f]_2$ and hence a mermomorphic modular form of weight 8. See [9].) By construction, this differential equation is apparent throughout $\mathbb{H}$. Furthermore, if $f(z)$ is chosen in a way such that $f(\gamma z) = \chi(\gamma)(cz+d)^2 f(z)$ holds for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ for some character χ of $\mathrm{SL}(2,\mathbb{Z})$ with $\Gamma \subset \ker\chi$, then $Q(z)$ is modular on $\mathrm{SL}(2,\mathbb{Z})$. We now utilize this construction of modular differential equations to find $Q(z)$ that cannot be realized, i.e., the monodromy group is not unitary.

We let $\omega_1 = dx/y$ and $\omega_2 = d(x/y^3)$. Note that ω_1 is a holomorphic 1-form on the curve $y^2 = x^3 - 1728$, while ω_2 is an exact 1-form and hence a meromorphic 1-form of the second kind. Using Ramanujan's identities, we check that $\omega_1 = f_1(z)\,dz$ and $\omega_2 = f_2(z)\,dz$ with

$$f_1(z) = -\frac{2\pi i}{3}\eta(z)^4, \quad f_2(z) = 2\pi i\frac{\eta(z)^4}{E_6(z)^4}\left(\frac{7}{6}E_4(z)^3\Delta(z) + 576\Delta(z)^2\right).$$

Now we choose, say,

$$\omega = -\frac{3}{2\pi i}\left(\omega_1 + \omega_2\right)$$

and let $f(z) = q^{1/6} + \cdots$ be the meromorphic modular form of weight 2 such that $\omega = f(z)\,dz$. Let $y''(z) = Q(z)y(z)$ be the differential equation obtained from $f(z)$ using the construction described above. Note that $f(z+1) = e^{2\pi i/6}f(z)$ and using $\eta(-1/z) = \sqrt{z/i}\,\eta(z)$, we have $f(-1/z) = -z^2 f(z)$. Thus, $f(\gamma z) = \chi(\gamma)(cz+d)^2 f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, where χ is the character of $\mathrm{SL}(2,\mathbb{Z})$ such that $\chi(T) = e^{2\pi i/6}$ and $\chi(S) = -1$. According the discussion above, the function $Q(z)$ is a meromorphic modular form of weight 4 with trivial character on $\mathrm{SL}(2,\mathbb{Z})$. Note that $f(z)$ has zeros at points where $6E_6(z)^4 - 7E_4(z)^3\Delta(z) - 3456\Delta(z)^2 = 0$. Now let us compute its Bol representation.

We choose $z_0 = i\infty$ and find that

$$y_2(z) = q^{-1/12}\left(1 + \sum_{j=1}^{\infty} c_j q^j\right), \qquad y_1(z) = q^{1/12}\sum_{j=0}^{\infty} d_j q^j$$

for some c_j and d_j with $d_0 \neq 0$. Therefore, the local exponents at ∞ are $\pm 1/12$ and

$$\rho(T) = \pm \begin{pmatrix} e^{2\pi i/12} & 0 \\ 0 & e^{-2\pi i/12} \end{pmatrix}.$$

Also, since $f(-1/z) = -z^2 f(z)$, we have

$$\begin{aligned} \int_{i\infty}^{-1/z} f(u)\,du &= \int_0^z f(-1/u)\,\frac{du}{u^2} = -\int_0^z f(u)\,du \\ &= -\int_0^{i\infty} f(u)\,du - \int_{i\infty}^{z} f(u)\,du. \end{aligned}$$

Thus,

$$\rho(S) = \pm \begin{pmatrix} i & C \\ 0 & -i \end{pmatrix}, \qquad C = i\int_0^{i\infty} f(u)\,du.$$

Now recall that $\omega = f(z)\,dz$ is equal to $-3(\omega_1 + \omega_2)/(2\pi i)$. Since $\omega_2 = d(x/y^3)$ is an exact 1-form on $X(\Gamma)$ and the modular curve $X(\Gamma)$ has only one cusp, which in particular says that ∞ and 0 are mapped to the same point on $X(\Gamma)$ under the natural map $\mathbb{H}^* \to X(\Gamma)$, the integral $\int_0^{i\infty} f_2(u)\,du$ is equal to 0. Therefore, we have

$$C = i\int_0^{i\infty} \eta(u)^4\,du.$$

This constant C can be expressed in terms of the central value of the L-function of the elliptic curve $E : y^2 = x^3 - 1728$, which is known to be nonzero. From this, it is straightforward to check that there is no simultaneous conjugation such that $\rho(T)$ and $\rho(S)$ both become unitary.

Proof of Theorem 1.4. We use the notations in the proof of Theorem 1.1. Since $\kappa_\infty = n/4$ for some odd integer n, with respect to the basis $\{y_+(z), y_-(z)\}$, we have $\rho(T) = \pm\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. If $\rho(S) = \pm I$, then $\rho(R) = \pm\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, which is a contradiction to $\rho(R)^3 = \pm I$. Hence, $\rho(S) \neq \pm I$, and we have $\operatorname{tr}\rho(S) = 0$. Then, by a choosing a suitable scalar r, the matrix of $\rho(S)$ with respect to $\{ry_+(z), y_-(z)\}$ will be of the form

$$\rho(S) = \pm \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

for some $a, b \in \mathbb{C}$ with $a^2 + b^2 = -1$, while $\rho(T)$ is still $\pm\left(\begin{smallmatrix} i & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Set $F(z) = r^2 y_+(z)^2 + y_-(z)^2$. We then compute that $F(Tz) = -F(z)$ and

$$\begin{aligned}(F|_{-2}S)(z) &= (ary_+(z) + by_-(z))^2 + (bry_+(z) - ay_-(z))^2 \\ &= -r^2 y_1(z)^2 - y_2(z)^2 = -F(z).\end{aligned}$$

This proves the theorem. □

7. Existence of the curvature equation

In this section, we will prove Theorem 1.3 equipped with the data (1.7). The main purpose of this section is to prove the existence and the number of such Q equipped with data (1.7). The discussion will be divided into several cases depending on κ_ρ and κ_i.

Lemma 7.1. *Suppose $F(z)$ is a modular form of weight 4 with respect to $\mathrm{SL}(2,\mathbb{Z})$, and is holomorphic except at ρ and i. If the pole order of $F(z)$ at ρ or $i \leq 1$, then $F(z)$ is holomorphic.*

Proof. Let n_1 and n_2 be the orders of poles at i and ρ respectively. The counting zero formula of meromorphic modular form (see [18]) says

$$m - \frac{n_1}{2} - \frac{n_2}{3} = \frac{4}{12}, \qquad m \text{ is a non-negative integer.}$$

By the assumption, $n_i \leq 1$. From the identity, it is easy to see $n_1 \leq 0$ and $n_2 \leq 0$. □

Let $t_j = E_6(z_j)^2/E_4(z_j)^3$ and define $F_j(z) = E_6(z)^2 - t_j E_4(z)^3$. By the theorem of counting zeros of modular forms [18, p. 85, Theorem 3], $F_j(z)$ has a (simple) zero at $z_j \in \mathbb{H}$.

Lemma 7.2. *Suppose that Q satisfies the conditions (i) and (ii) in Definition 1.2. Then*

$$Q = \pi^2 \left(Q_3(z; r, s, t) + \sum_{j=1}^{m} \frac{r_1^{(j)} E_4(z)^4 F_j(z) + r_2^{(j)} E_4(z)^7}{F_j(z)^2} \right), \tag{7.1}$$

where r, $r_1^{(j)}$ are free parameters and s, t, $r_2^{(j)}$ are uniquely determined by

$$\begin{aligned} &s = s_{\kappa_\rho} := (1-4\kappa_\rho^2)/9, \quad t = t_{\kappa_i} = (1-4\kappa_i^2)/4, \quad \textit{and} \\ &r_2^{(j)} = r_{2,\kappa_j}^{(j)} = t_j(t_j-1)^2(1-4\kappa_j^2)/4. \end{aligned} \tag{7.2}$$

Proof. Let $\hat{Q}$ denote the RHS of (7.1). Then it is a straightforward computation to show that (ii) in Definition 1.2 holds at p_j if and only if $s = s_{\kappa_\rho}$ if $p_j = \rho$, $t = t_{\kappa_i}$ if $p_j = i$, and $r_2^{(j)} = r_{2,\kappa_j}^{(j)}$ if $p_j = z_j$. By the choice of s, t and $r_2^{(j)}$, $Q - \hat{Q}$ might contain simple poles only. Further, we can choose $r_1^{(j)}$ to make $Q - \hat{Q}$ holomorphic at z_j. By Lemma 7.1, $Q - \hat{Q}$ is automatically smooth at ρ and i. Therefore, $Q - \hat{Q}$ is a holomorphic modular form of weight 4, and the lemma follows immediately because $E_4(z)$, up to a constant, is the only holomorphic modular form of weight 4. ☐

Now we are in the position to prove Theorem 1.3.

Proof of Theorem 1.3. We first calculate the parameters $r, r_1^{(j)}$, $1 \le j \le m$, such that Q is apparent at z_j. For simplicity, we assume $j = 1$. From (7.1), we do the Taylor expansion at $z = z_1$.

$$\begin{aligned} Q(z) &= a_{-2}(z-z_1)^{-2} + (r_1 b_{-1} + a_{-1})(z-z_1)^{-1} \\ &+ \sum_{j=0}^{\infty} \left(a_j + r_1 b_j + c_j\left(r, r_1^{(2)}, \dots, r_1^{(m)}\right)\right)(z-z_1)^j := \sum_{j=-2}^{\infty} A_j (z-z_1)^j, \end{aligned}$$

where a_j, b_j are independent of r, $r_1^{(j)}$ and $c_j(r, r_1^{(2)}, \dots, r_1^{(m)})$ is linear in all variables, and also

$$y(z) = (z-z_1)^{1/2-\kappa_1}\left(1 + \sum_{j=1}^{\infty} d_j (z-z_1)^j\right).$$

Then we derive the recursive formula by comparing both sides of (1.1) with Q in (7.1),

$$j(j-2\kappa_1)d_j = \sum_{k+\ell=j-2,\ k<j} d_k A_\ell, \quad A_{-1} = a_{-1} + r_1 b_{-1}, \tag{7.3}$$

where $d_0 = 1$ and

$$d_1 = \frac{1}{1-2\kappa_1} d_0 A_{-1} = \frac{b_{-1}}{1-2\kappa_1} r_1 + \text{terms of lower orders.}$$

By induction,

$$j(j-2\kappa_1)d_j = d_{j-1}A_{-1} + d_{j-2}A_0 + d_{j-3}A_1 + \cdots + d_0A_{j-2} \tag{7.4}$$
$$= \frac{b_{-1}^{j-1}}{(1-2\kappa_1)\cdots((j-1)-2\kappa_1)} r_1^{j-1} + \text{ terms of lower orders.}$$

At $j = 2k_1$, the RHS of (7.4) is

$$P_1\left(r, r_1^{(1)}, \ldots, r_1^{(m)}\right) := d_{2\kappa_1-1}A_{-1} + d_{2\kappa_1-2}A_0 + \cdots + d_0A_{\kappa_1-2}.$$

Clearly, $\deg P_1 = 2\kappa_1$ and

$$P_1 = B_0 r_1^{2\kappa_1} + \text{terms of lower orders}, \quad B_0 \neq 0. \tag{7.5}$$

We summarized what are known:

- $\kappa_i \notin \mathbb{N}$, then Q is apparent at i for any tuple $\left(r, r_1^{(j)}\right)$.
- $2\kappa_p/3 \notin \mathbb{N}$, then Q is apparent at ρ for any tuple $\left(r, r_1^{(j)}\right)$,
- $1/2 \pm \kappa_j$, there is a polynomial $P_j\left(r, r_1^{(1)}, \ldots, r_1^{(m)}\right)$ of degree $2\kappa_j$ such that Q is apparent at z_j if and only if $P_j\left(r, r_1^{(1)}, \ldots, r_1^{(m)}\right) = 0$.

Since κ_∞ is given, we have $\kappa_\infty = \sqrt{-Q(\infty)}/2$, and then

$$r + \sum_{j=1}^{m} (1-t_j)\, r_j^{(1)} + e = 0, \tag{7.6}$$

where e is given. By Bezout's theorem, we have $N = \prod_{j=1}^{m}(2\kappa_j)$ common roots with multiplicity of (7.5) and (7.6) because by (7.5) it is easy to see that there are no solutions at ∞. This proves the theorem. □

Appendix A. Curvature equations on S^2 with multiple singularities

Let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Since $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^* \simeq \mathbb{C} \cup \{\infty\}$, the equation (2.5) in the case $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ can be transformed into the mean field equations on $\mathbb{C}$:

$$\begin{cases} \Delta u + e^u = 4\pi\left(\alpha_1\delta_0 + \alpha_2\delta_1 + \sum_{j=1}^m n_j\delta_{p_j}\right) & \text{on } \mathbb{C}, \\ u(z) = -(4+2\alpha_3)\log|z| + O(1) & \text{as } |z| \to \infty, \end{cases} \tag{A.1}$$

where we assume that the isomorphism maps the points $i = \sqrt{-1}$, $\rho = (1+\sqrt{-3})/2$, and ∞ of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^*$ to 0, 1, and ∞, respectively, δ_p is the Dirac measure at $p \in \mathbb{C}$, $\alpha_k > -1$ for $k = 1, 2, 3$ and $n_j \in \mathbb{N}$. For any solution u of (A.1), the conformal metric $e^u|dz|^2$ has the angles λ_1, λ_2, and σ_j at 0, 1, and p_j, respectively, where

$$\lambda_1 = \alpha_1 + 1, \qquad \lambda_2 = \alpha_2 + 1, \qquad \sigma_j = n_j + 1. \tag{A.2}$$

Throughout the appendix, we assume that α_k are not integers for $k = 1, 2, 3$ and all p_j are distinct. To find a solution for (A.1), we first associate to (A.1) a second-order ODE

$$y''(z) + Q(z)y(z) = 0, \qquad z \in \mathbb{C}, \tag{A.3}$$

where

$$Q(z) = \left(\frac{\frac{\alpha_1}{2}(\frac{\alpha_1}{2}+1)}{z^2} + \frac{r_1}{z}\right) + \left(\frac{\frac{\alpha_2}{2}(\frac{\alpha_2}{2}+1)}{(z-1)^2} + \frac{r_2}{z-1}\right) + \sum_{j=1}^{m} \frac{\frac{n_j}{2}(\frac{n_j}{2}+1)}{(z-p_j)^2} + \frac{s_j}{z-p_j} \tag{A.4}$$

for some free parameters r_0, r_1, s_j. It is known that (A.1) has a solution if and only if the monodromy group of (A.3) is projectively unitary.

Note that the local exponents of (A.3) at 0 and 1 are $\{-\alpha_1/2, 1+\alpha_1/2\}$ and $\{-\alpha_2/2, 1+\alpha_2/2\}$, respectively. Since $\alpha_1, \alpha_2 \notin \mathbb{Z}$, the differences of the local exponents are not integers. At each p_j, there is a polynomial $P_j(r_1, r_2, s_j)$ such that (A.3) is apparent if and only if $P_j(r_1, r_2, s_j) = 0$. The derivation of the polynomials P_j is the same as Lemma 7.2. Moreover, the asymptotic behavior of u at ∞ yields that (A.3) is Fuchsian at ∞ with local exponents $-\alpha_3/2$ and $1+\alpha_3/2$. Thus, we have

$$r_1 + r_2 + \sum_j s_j = 0$$

and

$$\begin{aligned}\frac{\alpha_\infty}{2}(\frac{\alpha_\infty}{2}+1) &= \lim_{z\to\infty} z^2 Q(z) \\ &= r_1 + \sum_{j=1}^{m} s_j p_j + \sum_{k\in\{0,1\}} \frac{\alpha_k}{2}(\frac{\alpha_k}{2}+1) + \sum_{j=1}^{m} \frac{n_j}{2}(\frac{n_j}{2}+1).\end{aligned}$$

Therefore, for given local exponent data for (A.1), the Bézout theorem implies that there are at most $\prod_{j=1}^m(n_j+1)$ distinct Q such that (A.3) realizes the mean field equation (A.1) for given data. Theorem 2.5 of [12] is to give a necessary and sufficient condition to ensure that the projective monodromy group of (A.3) is unitary, i.e., that (A.1) has a solution.

Theorem A.1 ([12, Theorem 2.5])**.** *Suppose that $\alpha_1, \alpha_2, \alpha_3$ are not integers and all combinations*

$$\alpha_1 \pm \alpha_2 \pm \alpha_3 \text{ are not integers} \tag{A.5}$$

for any choice of signs. Then (A.1) has a solution if and only if

$$\cos^2 \pi\alpha_1 + \cos^2 \pi\alpha_2 + \cos^2 \pi\alpha_3 + 2(-1)^{\sigma+1} \cos \pi\alpha_1 \cos \pi\alpha_2 \cos \pi\alpha_3 < 1,$$

where $\sigma = \sum_{j=1}^m n_j$. Moreover, the number of distinct solutions of (A.1) is less than or equal to $\prod_{j=1}^m(n_j+1)$.

We remark that the notations α_j here differ from those used in [12] by 1.

Note that when (A.1) arises from the differential equation (1.1) considered in Theorem 1.1, we have

$$\alpha_1 = \kappa_i - 1, \quad \alpha_2 = 2\kappa_\rho/3 - 1, \quad \alpha_3 = 2\kappa_\infty, \quad n_j = 2\kappa_{p_j} - 1,$$

where $\kappa_i, \kappa_\rho, \kappa_{p_j} \in \frac{1}{2}\mathbb{N}$ are the local exponent data in ($\mathbf{H}_1$) and ($\mathbf{H}_2$). Hence, $\alpha_1 \in \frac{1}{2} + \mathbb{Z}$ and $\alpha_2 = \pm\frac{1}{3} + \mathbb{Z}$ and the condition (A.5) is equivalent to $r_\infty \neq 1/12, 5/12$. Thus, the first half of Theorem 1.1 is a special case of Eremenko and Tarasov's theorem. In the remainder of the appendix, we provide an alternative and self-contained proof of Theorem A.1.

For $k = 1, 2, 3$, let $\theta_k \in (0, 1/2]$ be real numbers such that

$$\alpha_k \equiv \pm\theta_k \mod 1, \quad \text{and} \quad \alpha_k = \ell_k \pm \theta_k. \tag{A.6}$$

Let $S = \{0, 1, \infty, p_1, \dots, p_m\}$ be the set of singular points of (A.3). Choose a base point z_0 near ∞ and consider the monodromy represenation $\rho : \pi_1(\mathbb{C} \setminus S, z_0) \to \mathrm{SL}(2, \mathbb{C})$ of (A.3). Let $\beta_j, \gamma_k \in \pi_1(\mathbb{C}\setminus S, z_0)$ such that β_j, $1 \le j \le m$, (resp. γ_0, γ_1) is a simple loop encircling p_j (resp. 0, 1) counterclockwise, while γ_∞ is a simple loop around ∞ clockwise such that

$$\gamma_0\gamma_1 \prod_{j=1}^m \beta_j = \gamma_\infty, \quad \text{in } \pi_1(\mathbb{C} \setminus S, z_0).$$

Since the local exponents at ∞ are $\{-\alpha_3/2, 1+\alpha_3/2\}$ with $\alpha_3 = \ell_3 \pm \theta_3$ and any solution has no logarithmic singularities, we can choose local solutions $y_{\infty,+}$, $y_{\infty,-}$ near ∞ such that with respect to $(y_{\infty,+}, y_{\infty,-})$, the monodromy matrix $\rho(\gamma_\infty)$ is given by

$$\begin{aligned}\rho(\gamma_\infty) &= \begin{pmatrix} e^{\pi i(\theta_3 \pm \ell_3)} & 0 \\ 0 & e^{-\pi i(\theta_3 \pm \ell_3)} \end{pmatrix} \\ &= (-1)^{\ell_3} \begin{pmatrix} e^{\pi i\theta_3} & 0 \\ 0 & e^{-\pi i\theta_3} \end{pmatrix} =: (-1)^{\ell_3} T.\end{aligned} \tag{A.7}$$

For any $1 \le j \le m$, since the local exponents at p_j are $\{-n_j/2, 1+n_j/2\}$ with $n_j \in \mathbb{N}$, we see that the monodromy matrix $\rho(\beta_j)$ is $(-1)^{n_j} I_2$. Set

$$R := (-1)^{\ell_1} \rho(\gamma_0)^{-1}, \qquad S := (-1)^{\ell_2} \rho(\gamma_1). \tag{A.8}$$

We have

$$(-1)^{\ell_1+\ell_2} R^{-1} S \prod_j^m (-1)^{n_j} I_2 = (-1)^{\ell_3} T,$$

i.e.,

$$S = (-1)^{\sum_j n_j + \sum_k \ell_k} RT. \tag{A.9}$$

Let R, S, and T be three matrices in $\mathrm{SL}(2,\mathbb{C})$ such that

(i) the eigenvalues of R, S, and T are $\delta_1^{\pm 1}$, $\delta_2^{\pm 1}$ and $\delta_3^{\pm 1}$, respectively, where $\delta_j = e^{\pm \pi i \theta_j}$ with $0 < \theta_j < 1$ and $i = \sqrt{-1}$,

(ii) the triple $(\theta_1, \theta_2, \theta_3)$ satisfies
$$0 < \theta_i + \theta_j \le 1, \quad \forall i \ne j,$$
and

(iii) $\theta_3 = \max_{1 \le j \le 3} \theta_j$ and $T = \mathrm{diag}(\delta_3, \bar{\delta}_3) = \left(\begin{smallmatrix} \delta_3 & 0 \\ 0 & \bar{\delta}_3 \end{smallmatrix}\right) \in \mathrm{SU}(2,\mathbb{C})$.

Lemma A.2. *Suppose* $R = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, T, $S = RT \in \mathrm{SL}(2,\mathbb{C})$ *satisfy (i)-(iii). Then the following hold.*

(a) $|a| < 1$ *if and only if* $\theta_1 + \theta_2 > \theta_3$.

(b) $|a| = 1$ *if and only if* $\theta_1 + \theta_2 = \theta_3$.

Proof. Note

$$S = RT = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta_3 & 0 \\ 0 & \bar{\delta}_3 \end{pmatrix} = \begin{pmatrix} \delta_3 a & b\bar{\delta}_3 \\ \delta_3 c & d\bar{\delta}_3 \end{pmatrix}.$$

Using the invariance of $\operatorname{tr} R$ and $\operatorname{tr} S$ under conjugation, we have

$$\begin{cases} a + d = \delta_1 + \bar{\delta}_1 \in \mathbb{R}, \\ \delta_3 a + \bar{\delta}_3 d = \delta_2 + \bar{\delta}_2 \in \mathbb{R}. \end{cases}$$

Since $\delta_3 \neq \pm 1$, we easily obtain

$$d = \bar{a}, \quad a = \frac{\delta_2 + \bar{\delta}_2 - \bar{\delta}_3(\delta_1 + \bar{\delta}_1)}{\delta_3 - \bar{\delta}_3}. \tag{A.10}$$

Consequently,

$$a = \frac{2\cos \pi\theta_2 - 2\bar{\delta}_3 \cos \pi\theta_1}{\pm 2i \sin \pi\theta_3} = \pm \frac{i(\bar{\delta}_3 \cos \pi\theta_1 - \cos \pi\theta_2)}{\sin \pi\theta_3}.$$

Thus

$$\begin{aligned} |a|^2 &= \frac{(\bar{\delta}_3 \cos \pi\theta_1 - \cos \pi\theta_2)(\delta_3 \cos \pi\theta_1 - \cos \pi\theta_2)}{\sin^2 \pi\theta_3} \\ &= \frac{\cos^2 \pi\theta_1 - 2\cos \pi\theta_1 \cos \pi\theta_3 \cos \pi\theta_2 + \cos^2 \pi\theta_2}{\sin^2 \pi\theta_3}. \end{aligned} \tag{A.11}$$

Let

$$\begin{aligned} \Delta &:= \cos^2 \pi\theta_1 - 2\cos \pi\theta_1 \cos \pi\theta_2 \cos \pi\theta_3 + \cos^2 \pi\theta_2 - \sin^2 \pi\theta_3 \\ &= \cos^2 \pi\theta_1 + \cos^2 \pi\theta_2 + \cos^2 \pi\theta_3 - (1 + 2\cos \pi\theta_1 \cos \pi\theta_2 \cos \pi\theta_3). \end{aligned}$$

Then (A.11) implies that $\Delta < 0$ if and only if $|a| < 1$.

Now using the formulas $\cos(x+y) = \cos x \cos y - \sin x \sin y$ and $\cos^2 x = (1 + \cos(2x))/2$, we deduce that

$$\begin{aligned} \Delta &= \cos^2 \pi\theta_3 - \cos \pi\theta_3(\cos \pi(\theta_1 + \theta_2) + \cos \pi(\theta_1 - \theta_2)) \\ &\quad + \frac{1}{2}(\cos(2\pi\theta_1) + \cos(2\pi\theta_2)) \\ &= \cos^2 \pi\theta_3 - \cos \pi\theta_3(\cos \pi(\theta_1 + \theta_2) + \cos \pi(\theta_1 - \theta_2)) \\ &\quad + \cos \pi(\theta_1 + \theta_2) \cos \pi(\theta_1 - \theta_2), \end{aligned}$$

so

$$\Delta = (\cos \pi\theta_3 - \cos \pi(\theta_1 + \theta_2))(\cos \pi\theta_3 - \cos \pi(\theta_1 - \theta_2)).$$

Since the assumptions (i)-(iii) give $1 > \theta_3 > |\theta_1 - \theta_2|$, we have $\cos \pi\theta_3 - \cos \pi(\theta_1 - \theta_2) < 0$, so the desired results follow. The proof is complete. □

We now give an alternative proof of Theorem A.1, which is stated in the following equivalent form.

Theorem A.3. *Assume that* (A.5) *holds.*

(a) Suppose that $\sum_{k=1}^{3} \ell_k + \sum_{j=1}^{m} n_j$ *is an even integer. Then* (A.1) *has a solution if and only if* $\theta_i + \theta_j > \theta_k$ *for any* $i \neq j \neq k$.
(b) Suppose that $\sum_{k=1}^{3} \ell_k + \sum_{j=1}^{m} n_j$ *is an odd integer. Then* (A.1) *has a solution if and only if* $\theta_1 + \theta_2 + \theta_3 > 1$.

Proof. Let R, S, and T be defined by (A.7) and (A.8). We need to determine when they are simultaneously conjugate to unitary matrices, under the assumption that (A.5) holds.

Consider first the case $\sum \ell_k + \sum n_j$ is even. In such a case, we have $S = RT$. Since for any permutation τ of the three points 0, 1, and ∞, there is always a Möbius transformation γ satisfying $\gamma z = \tau(z)$ for all $z \in \{0, 1, \infty\}$, without loss of generality, we may assume that $\theta_3 = \max_k \theta_k$. Then the condition $\theta_i + \theta_j > \theta_k$ for any $i \neq j \neq k$ simply means $\theta_1 + \theta_2 > \theta_3$, which we assume now. Moreover, we may assume that $T = \begin{pmatrix} \delta_3 & 0 \\ 0 & \overline{\delta}_3 \end{pmatrix}$ after a common conjugation, where $\delta_3 = e^{\pi i \theta_3}$.

Write $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By (A.10), we have $d = \overline{a}$. By Lemma A.2, $\theta_1 + \theta_2 > \theta_3$ if and only if $|a| < 1$ and hence $bc = |a|^2 - 1 < 0$. Set $P = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$, where μ is a real number such that

$$\mu^2 = -\frac{bc}{|c|^2} = -\frac{b}{\overline{c}}.$$

We have $P^{-1}TP = T$ and

$$P^{-1}RP = \begin{pmatrix} a & \mu^{-1}b \\ \mu c & d \end{pmatrix},$$

which is unitary since $\mu^{-1}b = -\mu\overline{c} = -\overline{\mu c}$. This proves that if $\theta_1 + \theta_2 > \theta_3$, then (A.1) has a solution.

Conversely, suppose that (A.1) has a solution. Then there exists a matrix P such that $\hat{T} = P^{-1}TP$ and $\hat{R} = P^{-1}RP$ are both unitary. Now it is known that every matrix in $\mathrm{SU}(2, \mathbb{C})$ is conjugate to a diagonal matrix and the conjugation can be taken inside $\mathrm{SU}(2, \mathbb{C})$. Hence, there exists a matrix Q in $\mathrm{SU}(2, \mathbb{C})$ such that $Q^{-1}\hat{T}Q = T$. Then $Q^{-1}\hat{R}Q \in \mathrm{SU}(2, \mathbb{C})$. In particular, the $(1, 1)$-entry of $Q^{-1}\hat{R}Q$ has absolute value ≤ 1. Since PQ commutes with T and T is diagonal but not a scalar matrix, PQ must be a diagonal matrix. Therefore, the $(1, 1)$-entry of R also has absolute value ≤ 1. It follows that, by Lemma A.2, $\theta_1 + \theta_2 > \theta_3$ (as the case $\theta_1 + \theta_2 = \theta_3$ is excluded from our consideration by (A.5)). We conclude that under the assumptions that (A.5) holds and that $\sum \ell_k + \sum n_j$ is even, (A.1) has a solution if and only if $\theta_i + \theta_j > \theta_k$ for any $i \neq j \neq k$.

For the case $\sum \ell_k + \sum n_j$ is odd, we simply apply the result in Part (a) to $\theta_1, \theta_2, 1 - \theta_3$ with T replaced by $-T$ and conclude that (A.1) has a solution if and only if $\theta_1 + \theta_2 + \theta_3 > 1$. This completes the proof. □

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