# Irreducible components of affine Deligne-Lusztig varieties

SIAN NIE[*]

We determine the top-dimensional irreducible components (and their stabilizers in the Frobenius twisted centralizer group) of affine Deligne-Lusztig varieties in the affine Grassmannian of a reductive group, by constructing a natural map from the set of irreducible components to the set of Mirković-Vilonen cycles. This in particular verifies a conjecture by Miaofen Chen and Xinwen Zhu.

# Introduction

## 0.1.  Background

The notion of affine Deligne-Lusztig variety was first introduced by Rapoport in [44], which plays an important role in understanding geometric and arithmetic properties of Shimura varieties. Thanks to the uniformization theorem by Rapoport and Zink [46], the Newton strata of Shimura varieties can be described explicitly in terms of so-called Rapoport-Zink spaces, whose underlying spaces are special cases of affine Deligne-Lusztig varieties.

In [34] and [44], Kottwitz and Rapoport made several conjectures on basic properties of affine Deligne-Lusztig varieties. Most of them have been verified by a number of authors. We mention the works by Rapoport-Richartz [45], Kottwitz [33], Gashi [6], and He [21], [23] on the "Mazur inequality" criterion of non-emptiness; the works by Görtz-Haines-Kottwitz-Reuman [8], He [24], He-Yu [26], Viehmann [48], Hamacher [10], and Zhu [55] on the dimension formula; the works by Hartl-Viehmann [18], [19], Milićević-Viehmann [40], and Hamacher [11] on the irreducible components; and the works by Viehmann [51], Chen-Kisin-Viehmann [3], and the author [41] on the connected components in the hyperspecial case (see also [27], [4] for some partial results in arbitrary parahoric case). For a thorough survey we refer to the report [22]. These advances on affine Deligne-Lusztig varieties have found several interesting applications in arithmetic geometry. For example, the dimension formula leads to a proof by Hamacher [11] for the Grothendieck conjecture on the closure relations of Newton strata of Shimura varieties, and the description of connected components in [3] plays an essential role in the proof by Kisin [30] for the Langlands-Rapoport conjecture on mod $p$ points of Shimura varieties (see [56], [25] for recent progresses).

## 0.2. Main results

This paper is concerned with the parametrization problem of top-dimensional irreducible components of affine Deligne-Lusztig varieties. The problem was first considered by Xiao and Zhu in [53], where they solved the unramified case in order to prove certain cases of the Tate conjecture for Shimura varieties. We will provide a complete parametrization in the general case.

To state the results, we introduce some notations. Let $F$ be a non-archimedean local field with residue field $\mathbb{F}_q$. Let $\breve{F}$ be the completion of the maximal unramified extension of $F$. Denote by $\mathcal{O}_F$ and $\mathcal{O}_{\breve{F}}$ the valuation rings of $F$ and $\breve{F}$ respectively. Let $\sigma$ be the Frobenius automorphism of $\breve{F}/F$.

Let $G$ be a connected reductive group over $\mathcal{O}_F$. Fix $T \subseteq B \subseteq G$, where $T$ is a maximal torus and $B = TU$ is a Borel subgroup with unipotent radical $U$. Denote by $Y$ the cocharacter group of $T$, and by $Y^+$ the set of dominant cocharacters determined by $B$. Let $K = G(\mathcal{O}_{\breve{F}})$. Fix a uniformizer $t \in \mathcal{O}_F$ and set $t^\lambda = \lambda(t) \in G(\breve{F})$ for $\lambda \in Y$. Then we have the Cartan decomposition for the affine Grassmannian

$$\mathrm{Gr} = \mathrm{Gr}_G = G(\breve{F})/K = \sqcup_{\mu \in Y^+} \mathrm{Gr}_\mu^\circ,$$

where $\mathrm{Gr}_\mu^\circ = Kt^\mu K/K$. For $b \in G(\breve{F})$ and $\mu \in Y^+$, the attached affine Deligne-Lusztig set is defined by

$$X_\mu(b) = X_\mu^G(b) = \{g \in G(\breve{F}); g^{-1}b\sigma(g) \in Kt^\mu K\}/K,$$

which is a subscheme locally of finite type in the usual sense if $\mathrm{char}(F) > 0$, and in the sense of Bhatt-Scholze [2] and Zhu [55] if $\mathrm{char}(F) = 0$. By left multiplication it carries an action of the group

$$\mathbb{J}_b = \mathbb{J}_b^G = \{g \in G(\breve{F}); g^{-1}b\sigma(g) = b\}.$$

Up to isomorphism, $X_\mu(b)$ only depends on the $\sigma$-conjugacy class $[b] = [b]_G$ of $b$. Thanks to Kottwitz [31], $[b]$ is uniquely determined by two invariants: the Kottwitz point $\kappa_G(b) \in \pi_1(G)_\sigma = \pi_1(G)/(1-\sigma)(\pi_1(G))$ and the Newton point $\nu_G(b) \in Y_\mathbb{R} = Y \otimes \mathbb{R}$, see [17, §2.1]. Then $X_\mu(b) \neq \emptyset$ if and only if $\kappa_G(t^\mu) = \kappa_G(b)$ and $\nu_G(b) \leqslant \mu^\diamond$, where $\mu^\diamond$ denotes the $\sigma$-average of $\mu$, and $\leqslant$ denotes the partial order on $Y_\mathbb{R}$ such that $v \leqslant v' \in Y_\mathbb{R}$ if $v' - v$ is a non-negative linear combination of coroots in $B$. Moreover, in this case, its dimension is given by

$$\dim X_\mu(b) = \langle \rho_G, \mu - \nu_G(b) \rangle - \frac{1}{2}\mathrm{def}_G(b),$$

where $\rho_G$ is the half-sum of roots of $B$ and $\mathrm{def}_G(b)$ is the *defect* of $b$, see [32, §1.9.1]. Let $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ denote the set of top-dimensional irreducible components of $X_\mu(b)$.

The first goal of this paper is to give an explicit description of the set $\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ of $\mathbb{J}_b$-orbits of $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$. We invoke a conjecture by Miaofen Chen and Xinwen Zhu which suggests a parametrization of $\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ by certain Mirković-Vilonen cycles.

Recall that Mirković-Vilonen cycles are irreducible components of $S^\lambda \cap \mathrm{Gr}_\mu$ for $\mu \in Y^+$ and $\lambda \in Y$, where $S^\lambda = U(\breve{F})t^\lambda K/K$ and $\mathrm{Gr}_\mu = \overline{\mathrm{Gr}_\mu^\circ}$. We write $\mathrm{MV}_\mu = \sqcup_\lambda \mathrm{MV}_\mu(\lambda)$ with $\mathrm{MV}_\mu(\lambda) = \mathrm{Irr}(S^\lambda \cap \mathrm{Gr}_\mu)$ the set of irreducible components.

Let $\widehat{G}$ be the Langlands dual of $G$ defined over $\overline{\mathbb{Q}}_l$ with $l \neq \mathrm{char}(\mathbf{k})$. Denote by $V_\mu = V_\mu^{\widehat{G}}$ the irreducible $\widehat{G}$-module of highest weight $\mu$. The crystal basis (or the canonical basis) $\mathbb{B}_\mu = \mathbb{B}_\mu^{\widehat{G}}$ of $V_\mu$ was first constructed by Lusztig [38] and Kashiwara [29]. In [1, Theorem 3.1], Braverman and Gaitsgory proved that the set $\mathrm{MV}_\mu$ of Mirković-Vilonen cycles admits a $\widehat{G}$-crystal structure and gives rise to a crystal basis of $V_\mu$ via the geometric Satake

isomorphism [39]. In [53, §3.3], Xiao and Zhu constructed a canonical isomorphism $\mathbb{B}_\mu \cong \mathrm{MV}_\mu$ using Littelmann's path model [37], which we denote by $\delta \mapsto S_\delta$. The advantage of using $\mathbb{B}_\mu$ is that its $\widehat{G}$-crystal structure is given in a combinatorial way.

In [17, §2.1], Hamacher and Viehmann proved that, under the partial order $\leqslant$, there is a unique maximal element $\underline{\lambda}_G(b)$ in the set

$$\{\underline{\lambda} \in Y_\sigma = Y/(1-\sigma)Y; \underline{\lambda} = \kappa_G(b), \underline{\lambda}^\diamond \leqslant \nu_G(b)\},$$

which is called "the best integral approximation" of $\nu_G(b)$. Let $V_\mu(\underline{\lambda}_G(b))$ be the sum of $\lambda$-weight spaces $V_\mu(\lambda)$ with $\lambda = \underline{\lambda}_G(b) \in Y_\sigma$, whose basis in $\mathbb{B}_\mu$ and $\mathrm{MV}_\mu$ is denoted by $\mathbb{B}_\mu(\underline{\lambda}_G(b))$ and $\mathrm{MV}_\mu(\underline{\lambda}_G(b))$ respectively.

**Conjecture 0.1** (Chen-Zhu). *There exist natural bijections*

$$\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b) \cong \mathrm{MV}_\mu(\underline{\lambda}_G(b)) \cong \mathbb{B}_\mu(\underline{\lambda}_G(b)).$$

*In particular,* $|\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)| = \dim V_\mu(\underline{\lambda}_G(b))$.

**Remark 0.1.** If $\mathrm{char}(F) > 0$, $X_\mu(b)$ is equi-dimensional by [19] and $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ coincides with the set of irreducible components of $X_\mu(b)$. If $\mathrm{char}(F) = 0$, the equi-dimensionality of $X_\mu(b)$ is not fully established, see [17, Theorem 3.4]. However, $X_\mu(b)$ is always equi-dimensional if $\mu$ is minuscule.

**Remark 0.2.** If $\mu$ is minuscule and either $G$ is split or $b$ is superbasic, Conjecture 0.1 is proved by Hamacher and Viehmann [17] using the method of semi-modules, which originates in the work [5] by de Jong and Oort. If $b$ is unramified, that is, $\mathrm{def}_G(b) = 0$, it is proved by Xiao and Zhu [53] using the geometric Satake. In both cases, the authors obtained complete descriptions of $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$.

**Remark 0.3.** A complete description of $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ was also known for the case where $G$ is $\mathrm{GL}_n$ or $\mathrm{GSp}_{2n}$ and $\mu$ is minuscule, see [49] and [50].

**Remark 0.4.** If the pair $(G, \mu)$ is *fully Hodge-Newton decomposable* (see [9]), $X_\mu(b)$ admits a nice stratification by classical Deligne-Lusztig varieties, whose index set and closure relations are encoded in the Bruhat-Tits building of $\mathbb{J}_b$. Such a stratification has important applications in arithmetic geometry, including the Kudla-Rapoport program [35], [36] and Zhang's Arithmetic Fundamental Lemma [54]. We mention the works by Vollaard-Wedhorn [52], Rapoport-Terstiege-Wilson [47], Howard-Pappas [15], [16], and Görtz-He [7] for some of the typical examples.

Let $I \subseteq G(\breve{F})$ be the standard Iwahori subgroup associated to the triple $T \subseteq B \subseteq G$, see §1.1. By Proposition 1.2, for $b \in G(\breve{F})$, there is a unique standard Levi subgroup $T \subseteq M \subseteq G$ and a *superbasic* element $b_M$ of $M(\breve{F})$, unique up to $M(\breve{F})$-$\sigma$-conjugation, such that $[b_M] = [b]$ and $\nu_M(b_M) = \nu_G(b)$. Moreover, we may and do choose $b_M$ such that $b_M T(\breve{F}) b_M^{-1} = T(\breve{F})$ and $b_M I_M b_M^{-1} = I_M$, where $I_M = M(\breve{F}) \cap I$ is the standard Iwahori subgroup of $M(\breve{F})$. Take $b = b_M$. Let $P = MN$ be the standard parabolic subgroup with $N \subseteq U$ its unipotent radical. Using the Iwasawa decomposition $G(\breve{F}) = N(\breve{F})M(\breve{F})K$ we have

$$\mathrm{Gr} = N(\breve{F})M(\breve{F})K/K = \sqcup_{\lambda \in Y} N(\breve{F})I_M t^\lambda K/K.$$

For $\lambda \in Y$ let $\theta_\lambda^P : N(\breve{F})I_M \to \mathrm{Gr}$ be the map given by $h \mapsto h t^\lambda K$.

Our first goal is to prove Conjecture 0.1.

**Theorem 0.5.** *Let $b$ and $M$ be as above. Then there exists a map*

$$\gamma = \gamma^G : \mathrm{Irr}^{\mathrm{top}} X_\mu(b) \to \mathbb{B}_\mu(\underline{\lambda}_G(b))$$

*such that for $C \in \mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ we have*

$$\overline{\{(ht^\lambda)^{-1} b\sigma(ht^\lambda)K; h \in (\theta_\lambda^P)^{-1}(C)\}} = \epsilon_\lambda^M \overline{S_{\gamma(C)}},$$

*where $\lambda$ is the unique cocharacter such that $N(\breve{F})I_M t^\lambda K/K \cap C$ is open dense in $C$ and $\epsilon_\lambda^M$ is certain Weyl group element for $M$ associated to $\lambda$ (see §1.5). Moreover, $\gamma$ factors through a bijection*

$$\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b) \cong \mathbb{B}_\mu(\underline{\lambda}_G(b)).$$

**Remark 0.6.** The equality $|\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)| = \dim V_\mu(\underline{\lambda}_G(b))$, which is the numerical version of Conjecture 0.1, is proved by Rong Zhou and Yihang Zhu in [57] (even for the quasi-split case), and by the author in an earlier version of this paper, using different approaches.

It is a remarkable feature that the tensor product of two crystals bases is again a crystal basis. So there is a natural map

$$\otimes : \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d} = \mathbb{B}_{\mu_1}^{\widehat{G}} \times \cdots \times \mathbb{B}_{\mu_d}^{\widehat{G}} \longrightarrow \mathbb{B}_{\mu_1}^{\widehat{G}} \otimes \cdots \otimes \mathbb{B}_{\mu_d}^{\widehat{G}} \longrightarrow \sqcup_\mu \mathbb{B}_\mu^{\widehat{G}}(\underline{\lambda}_G(b)),$$

where the first map is given by taking the tensor product, and the second one is the canonical projection to highest weight $\widehat{G}$-crystals.

On the other hand, there is also a "tensor structure" among affine Deligne-Lusztig varieties coming from the geometric Satake. Consider the product $G^d$ of $d$ copies of $G$ together with a Frobenius automorphism given by

$$(g_1, g_2, \ldots, g_d) \mapsto (g_2, \ldots, g_d, \sigma(g_1)).$$

For $\mu_\bullet = (\mu_1, \ldots, \mu_d) \in Y^d$ and $b_\bullet = (1, \ldots, 1, b) \in G^d(\breve{F})$ with $b \in G(\breve{F})$, we can define the affine Deligne-Lusztig variety $X_{\mu_\bullet}(b_\bullet)$ in a similar way. We know (see Corollary 1.6) that the projection $\mathrm{Gr}^d \to \mathrm{Gr}$ to the first factor induces a map

$$\mathrm{pr} : \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet) \to \sqcup_\mu \mathrm{Irr}^{\mathrm{top}} X_\mu(b),$$

which serves as the functor of taking tensor products.

Our second main result shows that the map $\gamma$ (for various $G$) preserves the tensor structures on both sides.

**Theorem 0.7.** *There is a Cartesian square*

$$
\begin{array}{ccc}
\mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet) & \xrightarrow{\;\gamma^{G^d}\;} & \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d} \\
{\scriptstyle \mathrm{pr}} \downarrow & & \downarrow {\scriptstyle \otimes} \\
\sqcup_\mu \mathrm{Irr}^{\mathrm{top}} X_\mu(b) & \xrightarrow{\;\gamma^G\;} & \sqcup_\mu \mathbb{B}_\mu^{\widehat{G}}.
\end{array}
$$

*As a consequence, if $\mathbb{B}_\mu^{\widehat{G}}$ appears in the tensor product $\mathbb{B}_{\mu_\bullet}^{\widehat{G}} = \mathbb{B}_{\mu_1}^{\widehat{G}} \otimes \cdots \otimes \mathbb{B}_{\mu_d}^{\widehat{G}}$, then $\gamma^G$ is determined by $\gamma^{G^d}$ and $\mathrm{Irr}^{\mathrm{top}} X_\mu(b) = \mathrm{pr}((\otimes \circ \gamma^{G^d})^{-1}(\mathbb{B}_\mu^{\widehat{G}}))$.*

**Remark 0.8.** The map $\gamma^G$ coincides with the natural constructions of [53] and [17] for *quasi-minuscule* cocharacters, see [43, Lemme 1.1]. On the other hand, we know that each highest weight module appears in some tensor product of quasi-minuscule highest weight modules. Thus Theorem 0.7 gives a characterization of the map $\gamma^G$ by the tensor structure of $\widehat{G}$-crystals.

**Remark 0.9.** As an essential application, Theorem 0.7, combined with the construction of [17], provides a representation-theoretic construction of $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ up to taking closures. Indeed, by the reduction method in [8, §5], it suffices to consider the case where $b$ is superbasic and $G = \mathrm{Res}_{E/F} \mathrm{GL}_n$ with $E/F$ a finite unramified extension. In this case, we can choose a minuscule cocharacter $\mu_\bullet \in Y^d$ for some $d$ such that $\mathbb{B}_\mu^{\widehat{G}}$ appears in $\mathbb{B}_{\mu_\bullet}^{\widehat{G}}$. As $\mu_\bullet$ is minuscule, both $\mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)$ and $\gamma^{G^d}$ are explicitly constructed in [17]. Then Theorem 0.7 shows how to obtain $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ from $\mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)$ by

taking the projection pr. The key is to decompose the tensor product into simple objects

$$\mathbb{B}_{\mu_\bullet}^{\widehat{G}} = \mathbb{B}_{\mu_1}^{\widehat{G}} \otimes \cdots \otimes \mathbb{B}_{\mu_d}^{\widehat{G}} = \sqcup_\mu (\mathbb{B}_\mu^{\widehat{G}})^{m_{\mu_\bullet}^\mu},$$

which can be solved combinatorially using the "Littlewood-Richardson" rule for $\widehat{G}$-crystals (see [37, §10]). Here $m_{\mu_\bullet}^\mu$ denotes the multiplicity with which $\mathbb{B}_\mu^{\widehat{G}}$ appears in $\mathbb{B}_{\mu_\bullet}^{\widehat{G}}$.

**Remark 0.10.** In the case mentioned above where $G = \mathrm{Res}_{E/F}\mathrm{GL}_n$ and $b$ is superbasic, Viehmann [48] and Hamacher [10] defined a decomposition of $X_\mu(b)$ using extended semi-modules (or extended EL-charts). In particular, $\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ is parameterized by the set of equivalence classes of *top* extended semi-modules, that is, the semi-modules whose corresponding strata are top-dimensional. However, it unclear how to construct all the top extended semi-modules if $\mu$ is non-minuscule. It would be interesting to give an explicit correspondence between the top extended semi-modules and the crystal elements in $\mathbb{B}_\mu(\underline{\lambda}_G(b))$.

The third goal is to give an explicit construction of an irreducible component from each $\mathbb{J}_b$-orbit of $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ and compute its stabilizer. Combined with Theorem 0.5, this will provide a complete parametrization of $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ in theory. If $b$ is unramified, this task has been done by Xiao-Zhu [53]. Otherwise, using Theorem 0.7, it suffices to consider the case where $G$ is adjoint, $\mu$ is minuscule, and $b$ is basic. To handle this case, we consider the decomposition

$$X_\mu(b) = \sqcup_{\lambda \in Y} X_\mu^\lambda(b),$$

where each piece $X_\mu^\lambda(b) = It^\lambda K/K \cap X_\mu(b)$ is a locally closed subset of $X_\mu(b)$.

**Theorem 0.11.** *Keep the assumptions on $G, b, \mu$ as above.*

*(1) $\overline{X_\mu^\lambda(b)} \in \mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ if and only if $\lambda \in Y$ is small;*

*(2) each $\mathbb{J}_b$-orbit of $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ has a representative of the form $\overline{X_\mu^\lambda(b)}$ with $\lambda$ small;*

*(3) if $\lambda \in Y$ is small, then the stabilizer of $\overline{X_\mu^\lambda(b)}$ in $\mathbb{J}_b$ is the standard parahoric subgroup of type $\Pi(\lambda)$, which is of maximal volume among all parahoric subgroups of $\mathbb{J}_b$.*

We refer to §6.4 for the meanings of the smallness of $\lambda$ and the associated type $\Pi(\lambda)$. As a consequence, we obtain the following result without restrictions on $G$, $b$, and $\mu$.

**Theorem 0.12** (He-Zhou-Zhu). *The stabilizer of each top-dimensional irre-ducible component of $X_\mu(b)$ in $\mathbb{J}_b$ is a parahoric subgroup of maximal volume.*

**Remark 0.13.** Theorem 0.12 is first proved by He-Zhou-Zhu [28]. It is also verified in [28] that a parahoric subgroup has maximal volume if and only if its Weyl group has *maximal length* (see Theorem 6.4). This gives an explicit characterization of parahoric subgroups of maximal volume by their types.

The original proof in [28] is based on the twisted orbital integral method (see [57]) and the Deligne-Lusztig reduction method (see [23]). Our proof is based on the combinatorial properties of small cocharacters, which shows that the stabilizers are parahoric subgroups of maximal length.

### 0.3. Strategy

Now we briefly discuss the strategy. First we reduced the problem to the case where $b$ is basic and $G$ is simple and adjoint. If $G$ has no non-zero minuscule coweights, then $b$ is unramified and the problem has been solved by Xiao-Zhu [53]. Thus, it remains to consider the case where $G$ has some non-zero minuscule cocharacter. In particular, any irreducible $\hat{G}$-module appears in some tensor product of irreducible $\hat{G}$-modules with minuscule highest weights (see Lemma 4.6). Combined with the geometric Satake, this observation enables us to decompose the problem into three ingredients: (1) the construction of $\gamma$ in the case where $b$ is superbasic; (2) the equality

$$|\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)| = \dim V_\mu(\underline{\lambda}_G(b))$$

in the case where $\mu$ is minuscule and $b$ is basic; and (3) the construction of irreducible components and the computation of their stabilizers in the situation of (2).

The first ingredient is solved in §3 by combining the semi-module method and Littelmann's path model. For the second ingredients, we consider in §5 the following decomposition

$$X_\mu(b) = \sqcup_{\lambda \in Y} X_\mu^\lambda(b).$$

In Proposition 2.9, we show that $I \cap \mathbb{J}_b$ acts on $\mathrm{Irr} X_\mu^\lambda(b)$ transitively and

$$\mathrm{Irr}^{\mathrm{top}} X_\mu(b) = \sqcup_{\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}} \mathrm{Irr} \overline{X_\mu^\lambda(b)},$$

where $\mathcal{A}_{\mu,b}^{\mathrm{top}}$ is the set of coweights $\lambda$ such that $\dim X_\mu^\lambda(b) = \dim X_\mu(b)$. In particular, the action of $\mathbb{J}_b$ on $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ induces an equivalence rela-tion on $\mathcal{A}_{\mu,b}^{\mathrm{top}}$, and the $\mathbb{J}_b$-orbits of $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ are naturally parameterized

by the corresponding equivalence classes of $\mathcal{A}^{\mathrm{top}}_{\mu,b}$. Therefore, it remains to show the number of these equivalence classes is equal to $\dim V_\mu(\underline{\lambda}_G(b))$. To this end, we give an explicit description of $\mathcal{A}^{\mathrm{top}}_{\mu,b}$ (see Proposition 5.12) and reduce the question to the superbasic case, which has been solved by Hamacher-Viehmann [17, Theorem 1.5]. Finally, to solve the last ingredient, we introduce the notion of small cocharacters in §6. We prove that $X^\lambda_\mu(b)$ is irreducible if and only if $\lambda$ is small, and show its stabilizer is of maximal length in this case. Here we will use a general result [57, Theorem 3.3.1] by Zhou-Zhu showing that the stabilizers are parahoric subgroups, which simplifies the original proof following [53].

**Remark 0.14.** Even if the simple adjoint group $G$ has no non-zero minuscule cocharacters, the above approach still works but is more technically involved, by using quasi-minuscule cocharacters instead.

### 0.4. Comparison with the work [57] by Zhou-Zhu

As mentioned before, this paper aims to give a complete parametrization of $\mathrm{Irr}^{\mathrm{top}}$
$X_\mu(b)$, which consists of three parts: the parametrization of $\mathbb{J}_b\backslash\mathrm{Irr}^{\mathrm{top}}X_\mu(b)$; the construction of representative irreducible components; and the computation of their stabilizers. The major overlap with [57] lies in the first part, see Remark 0.6. A key new feature of this paper is that the $\widehat{G}$-crystal structure plays an essential role in the construction, see Remark 0.8 & 0.9. This in particular enables us to handle the type $A$ case, which is not covered in [57]. There is a minor overlap in the third part, where the difference is that this paper gives an algorithm for computing the stabilizers (see Theorem 0.11); while the work by Zhou-Zhu produces extra interesting information on the volumes of stabilizers, see [57, Theorem C & Remark 1.4.3].

### Acknowledgement

# 1. Preliminaries

We keep the notations in the introduction. Set $K_H = H(\mathcal{O}_{\breve{F}})$ for any subgroup $H \subseteq G$ over $\mathcal{O}_{\breve{F}}$.

## 1.1. Root system

Let $\mathcal{R} = (Y, \Phi^{\vee}, X, \Phi, \mathbb{S}_0)$ be the based root datum of $G$ associated to the triple $T \subseteq B \subseteq G$, where $X$ and $Y$ denote the (absolute) character and cocharacter groups of $T$ respectively; $\Phi$ (resp. $\Phi^{\vee}$) is the roots system (resp. coroot system); $\mathbb{S}_0$ is the set of simple reflections. There is a natural perfect pairing $\langle , \rangle : X \times Y \to \mathbb{Z}$.

Denote by $\Phi^+$ the set of (positive) roots appearing in $B$. Then $\Phi = \Phi^+ \sqcup \Phi^-$ with $\Phi^- = -\Phi^+$. For $\alpha \in \Phi$, we denote by $s_{\alpha}$ the reflection which sends $\lambda \in Y$ to $\lambda - \langle \alpha, \lambda \rangle \alpha^{\vee}$ with $\alpha^{\vee} \in \Phi^{\vee}$ the corresponding coroot of $\alpha$. The Frobenius map of $G$ induces an automorphism of $\mathcal{R}$ of finite order, which is still denoted by $\sigma$. In particular, $\sigma$ acts on $Y_{\mathbb{R}}$ as a linear transformation of finite order.

Let $W_0 = W_G$ be the Weyl group of $T$ in $G$, which is a reflection subgroup of $\mathrm{GL}(Y_{\mathbb{R}})$ generated by $\mathbb{S}_0$. The Iwahori-Weyl group of $T$ in $G$ is given by

$$\tilde{W} = \tilde{W}_G = N_T(\breve{F})/K_T \cong Y \rtimes W_0 = \{t^{\lambda}w; \lambda \in Y, w \in W_0\},$$

where $N_T$ denotes the normalizer of $T$ in $G$. We can embed $\tilde{W}$ into the group of affine transformations of $Y_{\mathbb{R}}$ so that the action of $\tilde{w} = t^{\mu}w$ is given by $v \mapsto \mu + w(v)$. Let $\Phi^+$ be the set of (positive) roots appearing in Borel subgroup $B \supseteq T$ and let

$$\Delta = \Delta_G = \{v \in Y_{\mathbb{R}}; 0 < \langle \alpha, v \rangle < 1, \alpha \in \Phi^+\}$$

be the base alcove. Then we have $\tilde{W} = W^a \rtimes \Omega$, where $W^a = \mathbb{Z}\Phi^{\vee} \rtimes W_0$ is the affine Weyl group and $\Omega$ is the stabilizer of $\Delta$. Let $Y^+$ be the set of dominant cocharacters. For $\chi, \eta \in Y$ we write $\chi \leqslant \eta$ if $\eta - \chi$ is a sum of positive roots. Write $\chi \leq \eta$ if $\bar{\chi} \leqslant \bar{\eta}$. Here $\bar{\eta}, \bar{\chi}$ are the dominant $W_0$-conjugate of $\eta, \chi$ respectively.

For $\alpha \in \Phi$, let $U_{\alpha} \subseteq G$ denote the corresponding root subgroup. We set

$$I = K_T \prod_{\alpha \in \Phi^+} U_{\alpha}(t\mathcal{O}_{\breve{F}}) \prod_{\beta \in \Phi^+} U_{-\beta}(\mathcal{O}_{\breve{F}}) \subseteq G(\breve{F}),$$

which is called the standard Iwahori subgroup associated to $T \subseteq B \subseteq G$. We have the Bruhat decomposition $G(\breve{F}) = \sqcup_{\tilde{w} \in \tilde{W}} I \tilde{w} I$.

## 1.2.  Affine roots

Let $\tilde{\Phi} = \tilde{\Phi}_G = \Phi \times \mathbb{Z}$ be the set of (real) affine roots. Let $a = \alpha + k :=$ $(\alpha, k) \in \tilde{\Phi}$. Denote by $U_a : \mathcal{O}_{\breve{F}} \to G(\breve{F})$, $z \mapsto U_\alpha(zt^k)$ the corresponding one-parameter affine root subgroup. We can view $a$ as an affine function such that $a(v) = -\langle \alpha, v \rangle + k$ for $v \in Y_\mathbb{R}$, whose zero locus $H_a = \{v \in Y_\mathbb{R}; a(v) = 0\}$ is called an affine root hyperplane. Let $s_a = s_{H_a} = t^{k\alpha^\vee} s_\alpha \in \tilde{W}$ denote the corresponding affine reflection. Set $\tilde{\Phi}^+ = \{a \in \tilde{\Phi}; a(\Delta) > 0\}$. Then $\tilde{\Phi} = \tilde{\Phi}^+ \sqcup \tilde{\Phi}^-$ with $\tilde{\Phi}^- = -\tilde{\Phi}^+$. The associated length function $\ell : \tilde{W} \to \mathbb{N}$ is defined by $\ell(\tilde{w}) = |\tilde{\Phi}^- \cap \tilde{w}(\tilde{\Phi}^+)|$. Let $\mathbb{S}^a = \{s_a; a \in \tilde{\Phi}, \ell(s_a) = 1\}$. Then $W^a$ is generated by $\mathbb{S}^a$ and $(W^a, \mathbb{S}^a)$ is a Coxeter system.

Let $\alpha \in \Phi$. Define $\tilde{\alpha} = (\alpha, 0) \in \tilde{\Phi}^+$ if $\alpha < 0$ and $\tilde{\alpha} = (\alpha, 1) \in \tilde{\Phi}^+$ otherwise. Then the map $\alpha \mapsto \tilde{\alpha}$ gives an embedding of $\Phi$ into $\tilde{\Phi}^+$, whose image is $\{a \in \tilde{\Phi}; 0 < a(\Delta) < 1\}$. Let $\Pi$ be the set of roots $\alpha \in \Phi$ such that $\tilde{\alpha}$ is a simple affine root, namely, $\Pi$ consists of minus simple roots and highest positive roots.

**Lemma 1.1.** *Let $\tilde{w}, \tilde{w}' \in \tilde{W}$. Then $I\tilde{w}I\tilde{w}'I \subseteq \cup_{x \leq \tilde{w}} Ix\tilde{w}'I$ and $I\tilde{w}I\tilde{w}'I \subseteq \cup_{x' \leq \tilde{w}'} I\tilde{w}x'I$. Consequently, $\tilde{w}It^\lambda K \subseteq \cup_{x \leq \tilde{w}} It^{x(\lambda)}K$ for $\lambda \in Y$. Here $\leq$ is the usual Bruhat order on $\tilde{W}$ associated to $\ell$.*

## 1.3.  Levi subgroup

Let $M \supseteq T$ be a (semistandard) Levi subgroup of $G$. By replacing the triple $T \subseteq B \subseteq G$ with $T \subseteq B \cap M \subseteq M$, we can define $\Phi_M^\pm$, $\tilde{W}_M$, $W_M^a$, $W_M$, $I_M$, $\tilde{\Phi}_M^\pm$, $\Delta_M$, $\Omega_M$ and so on as above. For $v \in Y_\mathbb{R}$, we denote by $M_v$ the Levi subgroup generated by $T$ and $U_\alpha$ for $\alpha \in \Phi$ such that $\langle \alpha, v \rangle = 0$, and denote by $N_v$ the unipotent subgroup generated by $U_\beta$ for $\beta \in \Phi$ such that $\langle \beta, v \rangle > 0$. We say $M$ is standard if $M = M_v$ for some dominant vector $v \in Y^+$.

## 1.4.  Superbasic element

We say $b \in G(\breve{F})$ is superbasic if none of its $\sigma$-conjugates is contained in a proper Levi subgroup of $G$. In particular, $b$ is basic in $G(\breve{F})$, that is, the Newton point $\nu_G(b)$ is central for $\Phi$.

**Proposition 1.2.** *If $b \in G(\breve{F})$ is basic, then there exists a unique standard Levi subgroup $M \subseteq G$ such that $M(\breve{F}) \cap [b]$ is a (single) superbasic $\sigma$-conjugacy class of $M(\breve{F})$.*

The existence is known. The uniqueness is proved in Appendix B, which is only used in the formulation of Theorem 0.5.

## 1.5. The element $\epsilon_\lambda^G$

Let $\lambda \in Y$ and $\gamma \in \Phi$. We set $\lambda_\gamma = -\tilde\gamma(\lambda)$, that is, $\lambda_\gamma = \langle \gamma, \lambda \rangle$ if $\gamma < 0$ and $\lambda_\gamma = \langle \gamma, \lambda \rangle - 1$ otherwise. Let $U_\lambda$ (resp. $U_\lambda^-$) be the subgroup of $G$ generated by $U_\alpha$ such that $\lambda_\alpha \geqslant 0$ (resp. $\lambda_\alpha < 0$). Notice that $\lambda_\alpha \geqslant 0$ if and only if $\langle \alpha, \lambda - v \rangle > 0$ for any $v \in \Delta$. So $U_\lambda$ and $U_\lambda^-$ are opposite maximal unipotent subgroups normalized by $T$. Thus there exists a unique element $\epsilon_\lambda = \epsilon_\lambda^G \in W_0$ such that $U_\lambda = {}^{\epsilon_\lambda}U := \epsilon_\lambda U \epsilon_\lambda^{-1}$. Here $U$ denotes the unipotent radical of $B$. Set $I_\lambda = I \cap t^\lambda K t^{-\lambda}$ and

$$I_\lambda^- = K_T(I_\lambda \cap U_\lambda^-) = K_T(I \cap U_\lambda^-);$$
$$I_\lambda^+ = K_T(I_\lambda \cap U_\lambda) = K_T t^\lambda K_{U_\lambda} t^{-\lambda}.$$

It follows from the Iwahori decomposition that $I_\lambda = I_\lambda^- I_\lambda^+ = I_\lambda^+ I_\lambda^-$.

Let $p : \tilde W \rtimes \langle \sigma \rangle \to W_0 \rtimes \langle \sigma \rangle$ denote the natural projection, where $\langle \sigma \rangle$ is the finite cyclic subgroup of $\mathrm{GL}(Y_\mathbb{R})$ generated by $\sigma$.

**Lemma 1.3.** *Let $\lambda \in Y$ and $\alpha \in \Phi$. Then*
  *(1) $\lambda_\alpha + \lambda_{-\alpha} = -1$;*
  *(2) $s_{\tilde\alpha}(\lambda) = \lambda - \lambda_\alpha \alpha^\vee$;*
  *(3) $\lambda_\alpha = \omega(\lambda)_{p(\omega)(\alpha)}$ and $\epsilon_{\omega(\lambda)} = p(\omega)\epsilon_\lambda$ for $\omega \in \Omega$.*

*Proof.* The first two statements follow directly by definition. We show the last one. Write $\omega = t^\eta p(\omega)$ for some $\eta \in Y$. Then

$$\langle p(\omega)(\alpha), \omega(\lambda) \rangle = \langle \alpha, \lambda \rangle + \langle p(\omega)(\alpha), \eta \rangle.$$

By the statement (1) we may assume $\alpha > 0$. Since $\omega \in \Omega$, $\langle p(\omega)(\alpha), \eta \rangle = 0$ if $p(\omega)(\alpha) > 0$ and $\langle p(\omega)(\alpha), \eta \rangle = -1$ otherwise. It follows that $\omega(\lambda)_{p(\omega)(\alpha)} = \lambda_\alpha$. In particular, $U_{\omega(\lambda)} = {}^{p(\omega)}U_\lambda$ and hence $\epsilon_{\omega(\lambda)} = p(\omega)\epsilon_\lambda$. $\square$

**Lemma 1.4.** *Let $\lambda, \eta \in Y$ such that $\eta - \lambda$ is minuscule. Then $I_\eta^- \subseteq I_\lambda$.*

*Proof.* It suffices to show $U_\alpha(t^{\epsilon_\alpha} \mathcal{O}_{\breve F}) \subseteq I_\lambda$ for $\lambda_\alpha < 0$, where $\epsilon_\alpha = 0$ if $\alpha < 0$ and $\epsilon_\alpha = 1$ otherwise. If $\lambda_\alpha < 0$, there is nothing to prove. Suppose $\lambda_\alpha > 0$. Then $\langle \alpha, \lambda \rangle > \langle \alpha, \eta \rangle$ and hence $\langle \alpha, \lambda \rangle = \langle \alpha, \eta \rangle + 1$ as $\eta - \lambda$ is minuscule. This means $-1 \leqslant \langle \alpha, \eta \rangle \leqslant 0$. If $\langle \alpha, \eta \rangle = 0$, then $\alpha > 0$ (since $\lambda_\alpha < 0$) and $U_\alpha(t^{\epsilon_\alpha} \mathcal{O}_{\breve F}) = U_\alpha(t \mathcal{O}_{\breve F}) = U_\alpha(t^{\langle \alpha, \lambda \rangle} \mathcal{O}_{\breve F}) \subseteq I_\lambda^+$. If $\langle \alpha, \eta \rangle = -1$, then $\alpha < 0$ (since $\lambda_\alpha > 0$) and $U_\alpha(t^{\epsilon_\alpha} \mathcal{O}_{\breve F}) = U_\alpha(\mathcal{O}_{\breve F}) = U_\alpha(t^{\langle \alpha, \lambda \rangle} \mathcal{O}_{\breve F}) \subseteq I_\lambda^+$. $\square$

## 1.6. The convolution map

Let $d \in \mathbb{Z}_{\geqslant 1}$ and let $G^d$ be the product of $d$ copies of $G$. Let $\sigma_\bullet$ be the Frobenius-type automorphism on $G^d$ given by

$$(g_1, g_2, \ldots, g_d) \mapsto (g_2, \ldots, g_d, \sigma(g_1)).$$

We set $b_\bullet = (1, \ldots, 1, b) \in G(\breve{F})^d$. Let $\mu_\bullet = (\mu_1, \ldots, \mu_d) \in Y^d$ be a dominant cocharacter of $G^d$. Let $X_{\mu_\bullet}(b_\bullet)$ be the corresponding affine Deligne-Lusztig variety in $\mathrm{Gr}^d$ using automorphism $\sigma_\bullet$. Consider the twisted product

$$\mathrm{Gr}^\circ_{\mu_\bullet} := K t^{\mu_1} K \times_K \cdots \times_K K t^{\mu_d} K / K$$

together with the convolution map

$$m_{\mu_\bullet} : \mathrm{Gr}_{\mu_\bullet} := \overline{\mathrm{Gr}^\circ_{\mu_\bullet}} \to \mathrm{Gr}_{|\mu_\bullet|} = \cup_{\mu \leq |\mu_\bullet|} \mathrm{Gr}^\circ_\mu$$

given by $(g_1, \ldots, g_{d-1}, g_d K) \mapsto g_1 \cdots g_d K$, where $|\mu_\bullet| = \mu_1 + \cdots + \mu_d \in Y^+$.

**Theorem 1.5** ([39], [43], [20, Theorem 1.3])**.** *Let notations be as above. Let $\mu \in Y^+$ with $\mu \leq |\mu_\bullet|$ and $y \in \mathrm{Gr}^\circ_\mu$. Then*

*(1) $\dim m_{\mu_\bullet}^{-1}(y) \leqslant \langle \rho, |\mu_\bullet| - \mu \rangle$, and moreover, the number of irreducible components of $m_{\mu_\bullet}^{-1}(y)$ having dimension $\langle \rho, |\mu_\bullet| - \mu \rangle$ equals the multiplicity $m_{\mu_\bullet}^\mu$ with which $\mathbb{B}_\mu^{\widehat{G}}$ occurs in $\mathbb{B}_{\mu_\bullet}^{\widehat{G}} := \mathbb{B}_{\mu_1}^{\widehat{G}} \otimes \cdots \otimes \mathbb{B}_{\mu_d}^{\widehat{G}}$.*

*(2) $m_{\mu_\bullet}^{-1}(y)$ is equi-dimensional of dimension $\langle \rho, |\mu_\bullet| - \mu \rangle$ if $\mu_\bullet$ is minuscule.*

*Here $\rho = \rho_G$ is the half sum of roots in $\Phi^+$.*

Thanks to Zhu [55, §3.1.3], there is a Cartesian square

$$
\begin{array}{ccc}
X_{\mu_\bullet}(b_\bullet) & \longrightarrow & G(\breve{F}) \times_K \mathrm{Gr}^\circ_{\mu_\bullet} \\
\downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{id} \times_K m_{\mu_\bullet}} \\
\cup_{\mu \leq |\mu_\bullet|} X_\mu(b) & \longrightarrow & G(\breve{F}) \times_K \mathrm{Gr}_{|\mu_\bullet|},
\end{array}
$$

where pr is the projection to the first factor; the lower horizontal map is given by $g_1 K \mapsto (g_1, g_1^{-1} b \sigma(g_1) K)$; the upper horizontal map is given by

$$(g_1 K, \ldots, g_d K) \mapsto (g_1, g_1^{-1} g_2, \ldots, g_{d-1}^{-1} g_d, g_d^{-1} b \sigma(g_1) K).$$

Moreover, via the identification

$$\mathbb{J}_b \cong \mathbb{J}_{b_\bullet}, \quad g \mapsto (g, \ldots, g),$$

the above Cartesian square is $\mathbb{J}_b$-equivariant by left multiplication.

**Corollary 1.6.** *Let the notation be as above. Then*

$$\mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet) = \sqcup_{\mu \in Y^+, m_{\mu_\bullet}^\mu \neq 0} \sqcup_{C \in \mathrm{Irr}^{\mathrm{top}} X_\mu(b)} \mathrm{Irr}^{\mathrm{top}} \overline{\mathrm{pr}^{-1}(C)}.$$

*In particular,*

$$\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet) = \sqcup_{\mu \in Y^+, m_{\mu_\bullet}^\mu \neq 0} \sqcup_{C \in \mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)} \mathrm{Irr}^{\mathrm{top}} \overline{\mathrm{pr}^{-1}(C)}$$

*and hence*

$$|\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)| = \sum_{\mu \in Y^+} m_{\mu_\bullet}^\mu |\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)|.$$

*As a consequence, if Theorem 0.5 is true, then the diagram of Theorem 0.7 is Cartesian if it is commutative.*

*Proof.* Using the same strategies of [8] and [10] we have

$$\dim X_{\mu_\bullet}(b_\bullet) = \langle \rho, |\mu_\bullet| - \nu_b \rangle - \frac{1}{2}\mathrm{def}(b) = \dim X_\mu(b) + \langle \rho, |\mu_\bullet| - \mu \rangle.$$

Let $\mu \in Y^+$ and $C \in \mathrm{Irr}^{\mathrm{top}} X_\mu(b)$. By Theorem 1.5 (1),

$$\dim \mathrm{pr}^{-1}(C) = \dim C + \langle \rho, |\mu_\bullet| - \mu \rangle = \dim X_\mu(b) + \langle \rho, |\mu_\bullet| - \mu \rangle \leqslant \dim X_{\mu_\bullet}(b_\bullet),$$

and moreover, the number of irreducible components of $\mathrm{pr}^{-1}(C)$ having dimension $\dim X_{\mu_\bullet}(b_\bullet)$ is equal to $m_{\mu_\bullet}^\mu$ as desired. $\square$

## 1.7. Tensor structure

Let $\mu \in Y^+$. Recall that $V_\mu = V_\mu^{\widehat{G}}$ denotes the simple $\widehat{G}$-module of highest weight $\mu$, and $\mathbb{B}_\mu = \mathbb{B}_\mu^{\widehat{G}}$ denotes the crystal basis of $V_\mu$, which is a highest weight $\widehat{G}$-crystal. We refer to [29], [37] and [53, §3.3] for the definition of $\widehat{G}$-crystals and a realization of $\mathbb{B}_\mu$ using Littelmann's path model.

For $\lambda \in Y$, let $\mathbb{B}_\mu(\lambda)$ be the set of basis elements of weight $\lambda$. Then

$$|\mathbb{B}_\mu(\lambda)| = \dim V_\mu(\lambda),$$

where $V_\mu(\lambda)$ denotes the $\lambda$-weight space of $V_\mu$.

Recall that $\mathrm{MV}_\mu = \mathrm{MV}_\mu^{\widehat{G}}$ denotes the set of Mirković-Vilonen cycles in $\mathrm{Gr}_\mu$. By [1, Theorem 3.1], $\mathrm{MV}_\mu$ admits a $\widehat{G}$-crystal structure, which is

isomorphic to $\mathbb{B}_\mu$. Let $S_1 \in \mathrm{MV}_\mu(\lambda_1)$ and $S_2 \in \mathrm{MV}_\mu(\lambda_2)$ be two Mirković-Vilonen cycles. The twisted product of $S_1$ and $S_2$ is

$$S_1 \tilde{\times} S_2 = (\theta^U_{\lambda_1})^{-1}(S_1)t^{\lambda_1} \times_{K_U} S_2 \subseteq G(\breve{F}) \times_K \mathrm{Gr},$$

where $\theta^U_{\lambda_1} : U(\breve{F}) \to \mathrm{Gr}$ is given by $u \mapsto ut^{\lambda_1}K$. The convolution of $S_1$ and $S_2$ is defined by

$$S_1 \star S_2 = \overline{m(S_1 \tilde{\times} S_2)} \cap S^{\lambda_1 + \lambda_2},$$

where $m : G(\breve{F}) \times_K \mathrm{Gr} \to \mathrm{Gr}$ denotes the usual convolution map. Note that $\overline{S_1 \star S_2} = \overline{m(S_1 \tilde{\times} S_2)}$.

Following [53, Proposition 3.3.15], we fix from now on a bijection $\delta \mapsto S_\delta$ from $\mathbb{B}_\mu$ to $\mathrm{MV}_\mu$ for $\mu \in Y^+$, which is compatible with the tensor product for $\widehat{G}$-crystals, that is, $S_{\delta_1 \otimes \delta_2} = S_{\delta_1} \star S_{\delta_2}$.

### 1.8. Admissible set

Let $P = MN$ be a standard parabolic subgroup with standard Levi subgroup $M = \sigma(M) \supseteq T$ and unipotent radical $N = \sigma(N)$. Let $\mathcal{E}$ be one of groups $I$, $M(\breve{F})$, $N(\breve{F})$ and $P(\breve{F})$. For $n \in \mathbb{Z}_{\geq 0}$ set $\mathcal{E}_n = \mathcal{E} \cap K_n$, where $K_n = \{g \in K = G(\mathcal{O}_{\breve{F}}); g \equiv 1 \mod t^n\}$. Following [8], we say a subset $\mathcal{D} \subseteq \mathcal{E}$ is admissible if there exists some integer $r > 0$ such that $\mathcal{D}\mathcal{E}_r = \mathcal{D}$ and $\mathcal{D}/\mathcal{E}_r \subseteq \mathcal{E}/\mathcal{E}_r$ is a (bounded) locally closed subset. In this case, define

$$\dim \mathcal{D} = \dim \mathcal{D}/\mathcal{E}_r - \dim \mathcal{E}_0/\mathcal{E}_r,$$

and moreover, we can define topological notions for $\mathcal{E}$, such as open/closed subsets, irreducible/connected components and so on, by passing to the quotient $\mathcal{D}/\mathcal{D}_r$. These definitions are independent of the choice of $r$ since the natural quotient map $\mathcal{E}/\mathcal{E}_n \to \mathcal{E}/\mathcal{E}_{n+1}$ is an affine space fiber bundle. For instance, the irreducible components of $\mathcal{D}$ is defined to be the inverse images of the irreducible components of $\mathcal{D}/\mathcal{E}_r \subseteq \mathcal{E}/\mathcal{E}_r$ under the natural projection $\mathcal{E} \to \mathcal{E}/\mathcal{E}_r$. We denote by $\mathrm{Irr}\mathcal{D}$ the set of irreducible components of $\mathcal{D}$ in this sense.

## 2. The set $X_\mu^{P,\lambda}(b)$

Keep the notations in the introduction and §1. In this section, we introduce a decomposition $X_\mu(b) = \sqcup_{\lambda \in Y} X_\mu^{P,\lambda}(b)$ with respect to certain parabolic subgroup $P \subseteq G$, and study the irreducible components of $X_\mu^{P,\lambda}(b)$.

## 2.1. The set $H^P(C)$

Let $P = MN$ be a standard parabolic subgroup as in §1.8. Suppose that $b \in M(\breve{F})$ such that $b$ is basic in $M(\breve{F})$ and $\nu_M(b) = \nu_G(b)$. Moreover, we always assume that $b \in N_T(\breve{F})$ is a lift of an element in $\Omega_M$, whose image in $\tilde{W}_M$ is still denote by $b$. Notice that $b$ normalizes $N(\breve{F})I_M$. Let $\phi_b^P : N(\breve{F})I_M \to N(\breve{F})I_M$ be the Lang's map given by $h \mapsto h^{-1}b\sigma(h)b^{-1}$. For $\lambda \in Y$ let $\theta_\lambda^P : N(\breve{F})I_M \to \mathrm{Gr}$ be the map given by $h \mapsto ht^\lambda K$.

Let $C \subseteq \mathrm{Gr}$ be locally closed and irreducible. We define

$$H^P(C) = H^P(C;b) = \phi_b^P((\theta_\lambda^P)^{-1}(C)) \subseteq N(\breve{F})I_M,$$

where $\lambda \in Y$ such that $N(\breve{F})I_M t^\lambda K/K \cap C$ is open dense in $C$. In this case, the map $\gamma^G$ of Theorem 0.5 can be formulated by

$$\{(ht^\lambda)^{-1}b\sigma(ht^\lambda)K; h \in (\theta_\lambda^P)^{-1}(C)\} = t^{-\lambda}H^P(C)t^{b\sigma(\lambda)}K/K,$$

where $b\sigma(\lambda) \in Y$ is defined by the affine action of $\tilde{W} \rtimes \langle\sigma\rangle$ on $Y$, see §1.1.

For $\mu \in Y^+$ and $\lambda \in Y$ we set $X_\mu^{\lambda,P}(b) = N(\breve{F})I_M t^\lambda K/K \cap X_\mu(b)$.

**Lemma 2.1.** *The map $C \mapsto H^P(C)$ for $C \in \mathrm{Irr}X_\mu^{\lambda,P}(b)$ induces a bijection*

$$(N(\breve{F})I_M \cap \mathbb{J}_b)\backslash\mathrm{Irr}X_\mu^{\lambda,P}(b) \cong \mathrm{Irr}(t^\lambda Kt^\mu Kt^{-b\sigma(\lambda)} \cap N(\breve{F})I_M).$$

*In particular, $H^P(C;b)$ is invariant under left/right multiplication by $K_T$.*

*Proof.* Note that $bt^\chi K = t^{b(\chi)}K$ and $Kt^{-\chi}b^{-1} = Kt^{-b(\chi)}$ for $\chi \in Y$. Therefore,

$$\begin{aligned}(\theta_\lambda^P)^{-1}(X_\mu^{\lambda,P}(b)) &= (\phi_b^P)^{-1}(t^\lambda Kt^\mu Kt^{-\sigma(\lambda)}b^{-1} \cap N(\breve{F})I_M) \\ &= (\phi_b^P)^{-1}(t^\lambda Kt^\mu Kt^{-b\sigma(\lambda)} \cap N(\breve{F})I_M),\end{aligned}$$

As $\phi_b^P$ is an etale covering of $N(\breve{F})I_M$ with Galois group $N(\breve{F})I_M \cap \mathbb{J}_b$, the map $C \mapsto H^P(C)$ for $C \in \mathrm{Irr}X_\mu^{\lambda,P}(b)$ induces a bijection

$$(N(\breve{F})I_M \cap \mathbb{J}_b)\backslash\mathrm{Irr}X_\mu^{\lambda,P}(b) \cong \mathrm{Irr}(t^\lambda Kt^\mu Kt^{-b\sigma(\lambda)} \cap N(\breve{F})I_M).$$

The proof is finished. $\qquad\square$

Now we focus on the basic case. Let $I_{\mathrm{der}}$ be the derived subgroup of $I$.

**Corollary 2.2.** *Assume $b$ is basic. Then the map $C \mapsto H_{\mathrm{der}}^G(C)$ for $C \in \mathrm{Irr}X_{\mu}^{\lambda,G}(b)$ induces a bijection*

$$(I_{\mathrm{der}} \cap \mathbb{J}_b)\backslash\mathrm{Irr}X_{\mu}^{\lambda,G}(b) \cong \mathrm{Irr}(t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I_{\mathrm{der}}).$$

*Here $H_{\mathrm{der}}^G(C) = \phi_b^G((\theta_\lambda^G)^{-1}(C) \cap I_{\mathrm{der}})$.*

*Proof.* Note that $It^\lambda K/K = I_{\mathrm{der}}t^\lambda K/K$ and that $\phi_b^G$ restricts to an etale covering of $I_{\mathrm{der}}$ with Galois group $I_{\mathrm{der}} \cap \mathbb{J}_b$. Then the statement follows in the same way of Lemma 2.1. $\qquad\square$

recall that $p : \tilde{W} \rtimes \langle\sigma\rangle \to W_0 \rtimes \langle\sigma\rangle$ is the natural projection.

**Lemma 2.3.** *Assume $b$ is basic. For $\lambda \in Y$ and $C \in \mathrm{Irr}X_{\mu}^{\lambda,G}(b)$ we have*

$$t^{-\lambda'}H^G(C';b')t^{b'\sigma(\lambda')} = p(\omega)t^{-\lambda}H^G(C;b)t^{b\sigma(\lambda)}p(\omega)^{-1},$$

*where $\omega \in \Omega$, $\dot\omega \in N_T(\breve{F})$ is a lift of $\omega$, $\lambda' = \omega(\lambda)$, $C' = \dot\omega C$ and $b' = \dot\omega b\sigma(\dot\omega)^{-1}$. Note that the right hand side is independent of the choice of $\dot\omega$ by Lemma 2.1.*

*Proof.* As $\omega \in \Omega$, we have $\dot\omega I \dot\omega^{-1} = I$ and hence

$$C' = \dot\omega C \subseteq \dot\omega It^\lambda K/K = It^{\omega(\lambda)}K/K.$$

So $(\theta_{\lambda'}^G)^{-1}(C') = \dot\omega(\theta_\lambda^G)^{-1}(C)\dot\omega^{-1}$, and the statement follows by definition. $\qquad\square$

**Corollary 2.4.** *In the superbasic case, the map $\gamma^G$ in Theorem 0.5 is independent of the choice of $b$.*

*Proof.* Suppose $b$ is superbasic and let $b, b', \lambda, \lambda', \omega, C, C'$ be as in Lemma 2.3. Let $\gamma_b^G$ (resp. $\gamma_{b'}^G$) be the map $\gamma^G$ in Theorem 0.5 defined with respect to $b$ (resp. $b'$). We need to show that $\gamma_b^G(C) = \gamma_{b'}^G(C')$. By definition (for the superbasic case), it suffices to show that

$$(\epsilon_\lambda^G)^{-1}t^{-\lambda}H^G(C;b)t^{b\sigma(\lambda)}K/K = (\epsilon_{\lambda'}^G)^{-1}t^{-\lambda'}H^G(C';b')t^{b'\sigma(\lambda')}K/K,$$

which follows from Lemma 1.3 (3) and Lemma 2.3. $\qquad\square$

**Corollary 2.5.** *Assume $b$ is basic. For $\lambda \in Y$ there are natural bijections*

$$(I \cap \mathbb{J}_b)\backslash\mathrm{Irr}X_{\mu}^{\lambda,G}(b) \cong \mathrm{Irr}(t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I_{U_\lambda}) \cong (I_{\mathrm{der}} \cap \mathbb{J}_b)\backslash\mathrm{Irr}X_{\mu}^{\lambda,G}(b),$$

where $I_{U_\lambda} = I \cap U_\lambda$ and $U_\lambda$ is as in §1.5. Moreover, for $C \in \mathrm{Irr}X_\mu^{\lambda,G}(b)$ we have

$$H^G(C) = K_T H_{\mathrm{der}}^G(C) = H_{\mathrm{der}}^G(C) K_T.$$

*Proof.* By Lemma 2.1 the map $C \mapsto H^G(C)$ for $C \in \mathrm{Irr}X_\mu^{\lambda,G}(b)$ induces a bijection

(a)
$$(I \cap \mathbb{J}_b)\backslash\mathrm{Irr}X_\mu^{\lambda,G}(b) \cong \mathrm{Irr}(t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I) \cong \mathrm{Irr}(t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I_{U_\lambda}),$$

where the second bijection follows from that

$$t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I = I_\lambda^-(t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I_{U_\lambda}).$$

Similarly, by Corollary 2.2 we have

(b) $$\begin{aligned}(I_{\mathrm{der}} \cap \mathbb{J}_b)\backslash\mathrm{Irr}X_\mu^{\lambda,G}(b) &\cong \mathrm{Irr}(t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I_{\mathrm{der}}) \\ &\cong \mathrm{Irr}(t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I_{U_\lambda}).\end{aligned}$$

So the first statement follows.

By (a) and (b), there exist $\mathcal{Z}, \mathcal{Z}' \in \mathrm{Irr}(t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I_{U_\lambda})$ such that

$$(I_\lambda^- \cap I_{\mathrm{der}})\mathcal{Z} = H_{\mathrm{der}}^G(C) \subseteq H^G(C) = I_\lambda^- \mathcal{Z}'.$$

In particular, $\mathcal{Z} = \mathcal{Z}'$. Moreover, $K_T$ normalises $t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I_{U_\lambda}$, and hence normalises each of its irreducible components. So we have

$$H^G(C) = I_\lambda^- \mathcal{Z} = (I_\lambda^- \cap I_{\mathrm{der}})K_T\mathcal{Z} = H_{\mathrm{der}}^G(C)K_T = K_T H_{\mathrm{der}}^G(C).$$

The second statement is proved. $\qquad\square$

**Lemma 2.6.** *For $\lambda, \chi \in Y$ there is a natural bijection*

$$\mathrm{Irr}(t^\lambda K t^\mu K t^{-\chi} \cap I_{U_\lambda}) \cong \mathrm{Irr}(K t^{-\mu} K/K \cap t^{-\chi} I_{U_\lambda} t^\lambda K/K).$$

*Proof.* The map $g \mapsto t^{-\chi}g^{-1}t^\lambda$ gives a bijection

$$t^\lambda K t^\mu K t^{-\chi} \cap I_{U_\lambda} \cong K t^{-\mu} K \cap t^{-\chi} I_{U_\lambda} t^\lambda.$$

By the definition of $U_\lambda$ we have $K_{U_\lambda} \subseteq t^{-\lambda}I_{U_\lambda}t^\lambda$. Therefore,

$$\mathrm{Irr}(t^\lambda K t^\mu K t^{-\chi} \cap I_{U_\lambda}) \cong \mathrm{Irr}((K t^{-\mu} K \cap t^{-\chi} I_{U_\lambda} t^\lambda)/K_{U_\lambda})$$

$$= \operatorname{Irr}(Kt^{-\mu}K/K \cap t^{-\chi}I_{U_\lambda}t^\lambda K/K),$$

where the identity follows from that $t^{-\chi}I_{U_\lambda}t^\lambda/K_{U_\lambda} \cong t^{-\chi}I_{U_\lambda}t^\lambda K/K$. $\qquad\square$

## 2.2. The minuscule and basic case

Suppose $\mu \in Y^+$ is minuscule and $b$ is basic. For $D \subseteq \tilde{W}$ we set $D \cap \mathbb{J}_b = \{\tilde{w} \in D; b\sigma(\tilde{w})b^{-1} = \tilde{w}\}$.

For $\lambda \in Y$ we write $X_\mu^\lambda(b) = X_\mu^{\lambda,G}(b) = It^\lambda K/K \cap X_\mu(b)$. Let $\mathcal{A}_{\mu,b}^G$ and $\mathcal{A}_{\mu,b}^{G,\mathrm{top}}$ be the sets of $\lambda \in Y$ such that $X_\mu^\lambda(b) \neq \emptyset$ and $\dim X_\mu^\lambda(b) = \dim X_\mu(b)$ respectively.

For $\alpha \in \Phi$ define $\alpha^i = p(b\sigma)^i(\alpha) \in \Phi$ and $\tilde{\alpha}^i = (b\sigma)^i(\tilde{\alpha}) \in \tilde{\Phi}$ for $i \in \mathbb{Z}$, where $\tilde{\alpha}$ is as in §1.2 and $p : \tilde{W} \rtimes \langle\sigma\rangle \to W_0 \rtimes \langle\sigma\rangle$ is the natural projection.

For $\lambda \in \mathcal{A}_{\mu,b}^G$ define $\lambda^\natural = -\lambda + b\sigma(\lambda)$, and denote by $R_{\mu,b}^G(\lambda)$ the set of roots $\alpha \in \Phi$ such that $\langle\alpha, \lambda^\natural\rangle = -1$ and $\lambda_\alpha \geqslant 1$. By Lemma 2.7 (1) below, this condition is equivalent to that $\langle\alpha, \lambda^\natural\rangle = -1$ and $\lambda_{\alpha^{-1}} \geqslant 0$.

**Lemma 2.7.** *Let $\lambda \in Y$. Then we have (1) $\langle\alpha, \lambda^\natural\rangle = \lambda_{\alpha^{-1}} - \lambda_\alpha$ for $\alpha \in \Phi$ and (2) $\tilde{w}(\lambda)^\natural = p(\tilde{w})(\lambda^\natural)$ for $\tilde{w} \in \tilde{W} \cap \mathbb{J}_b$.*

*Proof.* Suppose $b \in t^\tau W_0$ for some $\tau \in Y$. Then

$$\langle\alpha, \lambda^\natural\rangle = -\langle\alpha, \lambda\rangle + \langle\alpha, \tau\rangle + \langle\alpha^{-1}, \lambda\rangle.$$

As $b \in \Omega$, $b\sigma$ preserves the fundamental alcove $\Delta$ and hence preserves the set $\{\tilde{\beta}; \beta \in \Phi\}$, see §1.2. Thus $\alpha, \alpha^{-1}$ are both positive or negative if $\langle\alpha, \tau\rangle = 0$; $\alpha < 0$ and $\alpha^{-1} > 0$ if $\langle\alpha, \tau\rangle = -1$; $\alpha > 0$ and $\alpha^{-1} < 0$ if $\langle\alpha, \tau\rangle = 1$. In all cases we have $\langle\alpha, \lambda^\natural\rangle = \lambda_{\alpha^{-1}} - \lambda_\alpha$ as desired. For $\tilde{w} \in \tilde{W} \cap \mathbb{J}_b$, it follows that

$$\tilde{w}(\lambda)^\natural = -\tilde{w}(\lambda) + b\sigma\tilde{w}(\lambda) = -\tilde{w}(\lambda) + \tilde{w}b\sigma(\lambda) = p(\tilde{w})(-\lambda + b\sigma(\lambda)) = p(\tilde{w})(\lambda^\natural).$$

The proof is finished. $\qquad\square$

**Lemma 2.8.** *We have $R_{\mu,b}^G(\omega(\lambda)) = p(\omega)R_{\mu,b}^G(\lambda)$ for $\omega \in \Omega \cap \mathbb{J}_b$, $\lambda \in \mathcal{A}_{\mu,b}^G$.*

*Proof.* Notice that $p(\tilde{w})(\gamma)^i = p(\tilde{w})(\gamma^i)$ for $\gamma \in \Phi$, $\tilde{w} \in \tilde{W} \cap \mathbb{J}_b$ and $i \in \mathbb{Z}$. The statement now follows from Lemma 1.3 (3) and Lemma 2.7 (1). $\qquad\square$

**Proposition 2.9.** *Suppose $\mu$ is minuscule and $b$ is basic. Then $\lambda \in \mathcal{A}_{\mu,b}^G$, that is $X_\mu^\lambda(b) \neq \emptyset$, if and only if $\lambda^\natural$ is conjugate to $\mu$ by $W_0$. Moreover, in this case,*

*(1) $t^\lambda Kt^\mu Kt^{-b\sigma(\lambda)} \cap I = I_\lambda \prod_{\delta \in R_{\mu,b}^G(\lambda)} U_\delta(t^{\langle\delta,\lambda\rangle-1}\mathcal{O}_{\breve{F}})$;*

(2) $X_\mu^\lambda(b)$ *is smooth and* $I \cap \mathbb{J}_b$ *acts on* $\mathrm{Irr}X_\mu^\lambda(b)$ *transitively;*

(3) $\dim X_\mu^\lambda(b) = |R_{\mu,b}^G(\lambda)|$.

*Proof.* By Corollary 2.5 we have

$$(I \cap \mathbb{J}_b)\backslash\mathrm{Irr}X_\mu^\lambda(b) \cong \mathrm{Irr}(t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I) \cong \mathrm{Irr}(K t^\mu K \cap t^{-\lambda}I_{U_\lambda}t^\lambda t^{\lambda^\natural}).$$

As $\mu$ is minuscule, we see that $\emptyset \neq K t^\mu K \cap t^{-\lambda}I_{U_\lambda}t^\lambda t^{\lambda^\natural} \subseteq U_\lambda(\check{F})t^{\lambda^\natural}$, that is, $\lambda \in \mathcal{A}_{\mu,b}^G$, if and only if $\lambda^\natural$ is conjugate to $\mu$. Moreover, in this case,

$$K t^\mu K \cap U_\lambda(\check{F})t^{\lambda^\natural} = K_{U_\lambda}t^{\lambda^\natural}K_{U_\lambda} = (\prod_\alpha U_\alpha(\mathcal{O}_{\check{F}})\prod_\beta U_\beta(t^{-1}\mathcal{O}_{\check{F}}))t^{\lambda^\natural},$$

where $\alpha, \beta$ range over the roots of $U_\lambda$ such that $\langle\alpha, \lambda^\natural\rangle \geqslant 0$ and $\langle\beta, \lambda^\natural\rangle = -1$ in any fixed orders. Here the root subgroups $U_\beta$ commute with each other since $\lambda^\natural$ is minuscule. On the other hand, $t^{-\lambda}I_{U_\lambda}t^\lambda t^{\lambda^\natural} = (\prod_\gamma U_\gamma(t^{-\lambda_\gamma}\mathcal{O}_{\check{F}}))t^{\lambda^\natural}$, where $\gamma$ ranges over the roots of $U_\lambda$ (or $\lambda_\gamma \geqslant 0$) in the above fixed order. Thus

$$K_{U_\lambda}t^{\lambda^\natural}K_{U_\lambda} \cap t^{-\lambda}I_{U_\lambda}t^\lambda t^{\lambda^\natural} = (K_{U_\lambda}\prod_\delta U_\delta(t^{-1}\mathcal{O}_{\check{F}}))t^{\lambda^\natural},$$

where $\delta$ ranges over $R_{\mu,b}^G(\lambda) = \{\gamma \in \Phi; \langle\gamma, \lambda^\natural\rangle = -1, \lambda_\gamma \geqslant 1\}$. Therefore,

$$\begin{aligned}
t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I &= I_\lambda t^\lambda(K t^\mu K \cap t^{-\lambda}I_{U_\lambda}t^\lambda t^{\lambda^\natural})t^{-b\sigma(\lambda)}\\
&= I_\lambda t^\lambda(K_{U_\lambda}t^{\lambda^\natural}K_{U_\lambda} \cap t^{-\lambda}I_{U_\lambda}t^\lambda t^{\lambda^\natural})t^{-b\sigma(\lambda)}\\
&= I_\lambda t^\lambda(K_{U_\lambda}\prod_{\delta\in R_{\mu,b}^G(\lambda)}U_\delta(t^{-1}\mathcal{O}_{\check{F}}))t^{-\lambda}\\
&= I_\lambda \prod_{\delta\in R_{\mu,b}^G(\lambda)}U_\delta(t^{\langle\delta,\lambda\rangle-1}\mathcal{O}_{\check{F}}).
\end{aligned}$$

So the statement (1) follows. The statement (2) follows from Corollary 2.5 by noticing that $t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I$ is smooth and irreducible.

As $(\theta_\lambda^G)^{-1}(X_\mu^\lambda(b)) = (\phi_b^G)^{-1}(t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I)$, we deduce by (1) that

$$\begin{aligned}
\dim X_\mu^\lambda(b) &= \dim((\theta_\lambda^G)^{-1}(X_\mu^\lambda(b))/I_\lambda)\\
&= \dim((\theta_\lambda^G)^{-1}(X_\mu^\lambda(b))) - \dim I_\lambda\\
&= \dim(t^\lambda K t^\mu K t^{-b\sigma(\lambda)} \cap I) - \dim I_\lambda\\
&= |R_{\mu,b}^G(\lambda)|.
\end{aligned}$$

So the statement (3) follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

## 2.3. The set $H^{P^d}(C)$

Let $P = MN$ and $b \in M(L)$ be as in §2.1. Let $G^d$, $\sigma_\bullet$, $b_\bullet$, $\mu_\bullet$ and pr be as in §1.6. We also denote by pr the projections $G^d(\breve{F}) \to G(\breve{F})$ and $Y^d \to Y$ to the first factors.

Let $C \in \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)$ and $\lambda_\bullet \in Y^d$ such that $(N(\breve{F})I_M)^d t^{\lambda_\bullet} K^d/K^d \cap C$ is open dense in $C$. By Corollary 1.6, $\overline{\mathrm{pr}(C)} \in \mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ for some $\mu \in Y^+$. Let $\lambda = \mathrm{pr}(\lambda_\bullet)$. Then $N(\breve{F})I_M t^\lambda K/K \cap \mathrm{pr}(C)$ is open dense in $\overline{\mathrm{pr}(C)}$. Set $\lambda_\bullet^\dagger = b_\bullet \sigma_\bullet(\lambda_\bullet)$, $\phi_{b_\bullet} = \phi_{b_\bullet}^{P^d}$ and $\theta_{\lambda_\bullet} = \theta_\lambda^{P^d}$. By Lemma 2.1,

$$H^{P^d}(C) = \phi_{b_\bullet}(\theta_{\lambda_\bullet}^{-1}(C)) \in \mathrm{Irr}(t^{\lambda_\bullet} K^d t^{\mu_\bullet} K^d t^{-\lambda_\bullet^\dagger} \cap (N(\breve{F})I_M)^d).$$

So we can write

$$H^{P^d}(C) = H_1(C) \times \cdots \times H_d(C),$$

where $H_\tau(C) \in \mathrm{Irr}(t^{\lambda_\tau} K t^{\mu_\tau} K t^{-\lambda_\tau^\dagger} \cap N(\breve{F})I_M)$ for $1 \leqslant \tau \leqslant d$ with $\mu_\bullet = (\mu_1, \ldots, \mu_d)$, $\lambda_\bullet = (\lambda_1, \ldots, \lambda_d)$ and $\lambda_\bullet^\dagger = (\lambda_1^\dagger, \ldots, \lambda_d^\dagger)$.

**Lemma 2.10.** *Let notations be as above. Then we have*

$$\overline{H^P(\overline{\mathrm{pr}(C)})} = \overline{H_1(C) \cdots H_d(C)} \subseteq N(\breve{F})I_M$$

*As a consequence,*

$$\overline{t^{-\lambda} H^P(\overline{\mathrm{pr}(C)}) t^{b\sigma(\lambda)} K/K} = \overline{t^{-\lambda_1} H_1(C) t^{\lambda_1^\dagger} \cdots t^{-\lambda_d} H_d(C) t^{\lambda_d^\dagger} K/K}.$$

*Proof.* As $\overline{\mathrm{pr}((N(\breve{F})I_M)^d t^{\lambda_\bullet} K^d/K^d \cap C)} = \overline{\mathrm{pr}(C)}$, we see that

$$\overline{\mathrm{pr}(\theta_{\lambda_\bullet}^{-1}(C))} = \overline{(\theta_\lambda^P)^{-1}(\overline{\mathrm{pr}(C)})} \subseteq N(\breve{F})I_M.$$

On the other hand, the equality $H^{P^d}(C) = \phi_{b_\bullet}(\theta_{\lambda_\bullet}^{-1}(C))$ means that

$$H_1(C) \times \cdots \times H_d(C) = \{(h_1^{-1} h_2, \ldots, h_{d-1}^{-1} h_d, h_d^{-1} b\sigma(h_1) b^{-1});$$
$$(h_1, \ldots, h_d) \in \theta_{\lambda_\bullet}^{-1}(C)\}.$$

In particular,

$$H_1(C) \cdots H_d(C) = \phi_b^P(\mathrm{pr}(\theta_{\lambda_\bullet}^{-1}(C))).$$

So $\overline{H^P(\overline{\mathrm{pr}(C)})} = \overline{\phi_b^P((\theta_\lambda^P)^{-1}(\overline{\mathrm{pr}(C)}))} = \overline{\phi_b^P(\mathrm{pr}(\theta_{\lambda_\bullet}^{-1}(C)))} = \overline{H_1(C) \cdots H_d(C)}$ as desired. □

## 3. The superbasic case

In this section, we assume that $b$ is superbasic. For $d \in \mathbb{Z}_{\geqslant 1}$ let $\sigma_\bullet$, $b_\bullet$, $\mu_\bullet$ and $P = G$ be as in §1.6. Let $C \in \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)$ and let $\lambda_\bullet \in Y^d$ such that $(I^d t^{\lambda_\bullet} K^d / K^d) \cap C$ is open dense in $C$. Following §2.3 let

$$H^{G^d}(C) = H_1(C) \times \cdots \times H_d(C) \in \mathrm{Irr}(t^{\lambda_\bullet} K^d t^{\mu_\bullet} K^d t^{-b_\bullet \cdot \sigma_\bullet (\lambda_\bullet)} \cap I^d),$$

where $H_\tau(C) \in \mathrm{Irr}(t^{\lambda_\tau} K t^{\mu_\tau} K t^{-\lambda_\tau^\dagger} \cap I)$ for $1 \leqslant \tau \leqslant d$ with $\lambda_\bullet = (\lambda_1, \ldots, \lambda_d)$ and $\lambda_\bullet^\dagger = b_\bullet \sigma_\bullet (\lambda_\bullet) = (\lambda_1^\dagger, \ldots, \lambda_d^\dagger)$. We set $X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) = X_{\mu_\bullet}^{\lambda_\bullet; G^d}(b_\bullet)$ for simplicity.

The main result of this section is

**Theorem 3.1.** *Let* $C, \lambda_\bullet, \lambda_\bullet^\dagger$ *be as above. There is* $\gamma^{G^d}(C) = (\gamma_1, \ldots, \gamma_d) \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}$ *such that*

$$\overline{t^{-\lambda_1} H_1(C) t^{\lambda_1^\dagger} \times_K \cdots \times_K t^{-\lambda_d} H_d(C) t^{\lambda_d^\dagger} K / K} = \epsilon_{\lambda_1} \overline{S_{\gamma_1} \widetilde{\times} \cdots \widetilde{\times} S_{\gamma_d}},$$

*where* $\epsilon_{\lambda_1} = \epsilon_{\lambda_1}^G$ *is as in* §1.5. *Moreover, the map* $C \mapsto \gamma^{G^d}(C)$ *factors through a bijection*

$$\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet) \cong \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\underline{\lambda}_{G^d}(b_\bullet)).$$

*In particular, Theorem 0.5 is true if* $b$ *is superbasic by taking* $d = 1$.

### 3.1. Reduction procedure

First we show how to pass to the case where $G = \mathrm{Res}_{E/F} \mathrm{GL}_n$ with $E/F$ an unramified extension.

**Lemma 3.2.** *Let* $f : G \to G'$ *be a central isogeny. Then Theorem 3.1 is true for* $G$ *if and only if it is true for* $G'$.

*Proof.* We still denote by $f$ the induced maps $G(\breve{F}) \to G'(\breve{F})$, $\mathrm{Gr}_G \to \mathrm{Gr}_{G'}$ and so on. Let $\sigma'$ be the Frobenius automorphism of $G'$. Let $C \in \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)$ and $\lambda_\bullet \in Y^d$ such that $I^d t^{\lambda_\bullet} K^d / K^d \cap C$ is open dense in $C$. Let $\omega_\bullet \in \pi_1(G^d)$ such that the corresponding connected component $\mathrm{Gr}_{G^d}^{\omega_\bullet}$ contains $C$. Let $K' = G'(\mathcal{O}_{\breve{F}})$ and $I' \subseteq K'$ the Iwahori subgroup containing $f(I)$. Denote by $\mu_\bullet', \lambda_\bullet', C', b_\bullet', \omega_\bullet'$ the images of $\mu_\bullet, \lambda_\bullet, C, b_\bullet, \omega_\bullet$ under $f$ respectively. Write $\lambda_\bullet = (\lambda_1, \ldots, \lambda_d)$, $b_\bullet = (b_1, \ldots, b_d)$, $\lambda_\bullet' = (\lambda_1', \ldots, \lambda_d')$ and $b_\bullet' = (b_1', \ldots, b_d')$.

By [3, Corollary 2.4.2] and [17, Proposition 3.1], $f$ induces a homeomorphism

$$X_{\mu_\bullet}(b_\bullet) \cap \mathrm{Gr}_{G^d}^{\omega_\bullet} =: X_{\mu_\bullet}(b_\bullet)^{\omega_\bullet} \xrightarrow{\sim} X_{\mu'_\bullet}(b'_\bullet)^{\omega'_\bullet} := X_{\mu'_\bullet}(b'_\bullet)^{\omega'_\bullet} \cap \mathrm{Gr}_{G'^d}^{\omega'_\bullet}.$$

So $C' \in \mathrm{Irr}^{\mathrm{top}} X_{\mu'_\bullet}(b'_\bullet)$ and $I'^d t^{\lambda'_\bullet} K'^d / K'^d \cap C'$ is open dense in $C'$. Moreover, as $f(I_{\mathrm{der}}) = I'_{\mathrm{der}}$ we have

$$f((\theta_{\lambda_\bullet}^{G^d})^{-1}(C) \cap (I_{\mathrm{der}})^d) = (\theta_{\lambda_\bullet}^{G'^d})^{-1}(C') \cap (I'_{\mathrm{der}})^d$$
$$f(H_{\mathrm{der}}^{G^d}(C)) = H_{\mathrm{der}}^{G'^d}(C'),$$

where $H_{\mathrm{der}}^{G^d}(C)$ and $H_{\mathrm{der}}^{G'^d}(C')$ are defined as in Corollary 2.2. By Corollary 2.3,

$$H^{G^d}(C) = H_{\mathrm{der}}^{G^d}(C) T^d(\mathcal{O}_{\breve{F}}) = T^d(\mathcal{O}_{\breve{F}}) H_{\mathrm{der}}^{G^d}(C);$$
$$H^{G'^d}(C') = H_{\mathrm{der}}^{G'^d}(C') T'^d(\mathcal{O}_{\breve{F}}) = T'^d(\mathcal{O}_{\breve{F}}) H_{\mathrm{der}}^{G'^d}(C').$$

Therefore, $f$ induces a surjection and hence a bijection

$$t^{-\lambda_a} H_a(C) t^{\lambda_a^\dagger} \times_K \cdots \times_K t^{-\lambda_c} H_c(C) t^{\lambda_c^\dagger} K/K$$
$$= t^{-\lambda_a} H_{\mathrm{der},a}(C) t^{\lambda_a^\dagger} \times_K \cdots \times_K t^{-\lambda_c} H_{\mathrm{der},c}(C) t^{\lambda_c^\dagger} K/K$$
$$\cong t^{-\lambda'_a} H_{\mathrm{der},a}(C') t^{\lambda_a'^\dagger} \times_{K'} \cdots \times_{K'} t^{-\lambda'_c} H_{\mathrm{der},c}(C') t^{\lambda_c'^\dagger} K'/K'$$
$$= t^{-\lambda'_a} (H_a(C') t^{\lambda_a'^\dagger} \times_{K'} \cdots \times_{K'} t^{-\lambda'_c} H_c(C') t^{\lambda_c'^\dagger} K'/K',$$

where, as in §2.3 we write

$$H_{\mathrm{der}}^{G^d}(C) = H_{\mathrm{der},1}(C) \times \cdots \times H_{\mathrm{der},d}(C)$$
$$H_{\mathrm{der}}^{G'^d}(C') = H_{\mathrm{der},1}(C') \times \cdots \times H_{\mathrm{der},d}(C').$$

By Corollary 2.3, we have the following commutative diagram

(a) $\quad ((I_{\mathrm{der}})^d \cap \mathbb{J}_{b_\bullet}) \backslash \mathrm{Irr} X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) \xrightarrow{\ \sim\ } \mathrm{Irr}(t^{\lambda_\bullet} K^d t^{\mu_\bullet} K^d t^{-\lambda_\bullet^\dagger} \cap (I^d)_{(U^d)_{\lambda_\bullet}})$

$$f \downarrow \qquad\qquad\qquad\qquad\qquad\qquad f \downarrow \wr$$

$\quad ((I'_{\mathrm{der}})^d \cap \mathbb{J}_{b'_\bullet}) \backslash \mathrm{Irr} X_{\mu'_\bullet}^{\lambda'_\bullet}(b'_\bullet) \xrightarrow{\ \sim\ } \mathrm{Irr}(t^{\lambda'_\bullet} K'^d t^{\mu'_\bullet} K'^d t^{-\lambda_\bullet'^\dagger} \cap (I'^d)_{(U^d)_{\lambda'_\bullet}}),$

where the right vertical bijection follows from Lemma 2.6 and the homeomorphism $f : \mathrm{Gr}_{G^d}^{\omega_\bullet} \cong \mathrm{Gr}_{G'^d}^{\omega'_\bullet}$.

Let $\mathbb{J}_{b_\bullet}^0$ and $\mathbb{J}_{b_\bullet'}^0$ be the kernels of the natural projections $\mathbb{J}_{b_\bullet} \to \pi_1(G^d)$ and $\mathbb{J}_{b_\bullet'} \to \pi_1(G'^d)$ respectively. Then we have a commutative diagram

$$
\begin{array}{ccc}
\mathbb{J}_{b_\bullet}^0 \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)^{\omega_\bullet} & \xrightarrow{\sim} & \mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet) \\
\downarrow & & \downarrow \\
\mathbb{J}_{b_\bullet'}^0 \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet'}(b_\bullet')^{\omega_\bullet'} & \xrightarrow{\sim} & \mathbb{J}_{b_\bullet'} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet'}(b_\bullet').
\end{array}
$$

Thus the bijection $\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet) \cong \mathbb{J}_{b_\bullet'} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet'}(b_\bullet')$ follows from the following commutative diagrams

$$
\begin{array}{ccccc}
((I_{\mathrm{der}})^d \cap \mathbb{J}_{b_\bullet}) \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)^{\omega_\bullet} & \xrightarrow{\sim} & (I^d \cap \mathbb{J}_b) \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)^{\omega_\bullet} & \xrightarrow{\sim} & \mathbb{J}_{b_\bullet}^0 \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)^{\omega_\bullet} \\
\downarrow \wr & & \downarrow & & \downarrow \\
((I_{\mathrm{der}}')^d \cap \mathbb{J}_{b_\bullet'}) \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet'}(b_\bullet')^{\omega_\bullet'} & \xrightarrow{\sim} & (I'^d \cap \mathbb{J}_{b_\bullet'}) \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet'}(b_\bullet')^{\omega_\bullet'} & \xrightarrow{\sim} & \mathbb{J}_{b_\bullet'}^0 \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet'}(b_\bullet')^{\omega_\bullet'},
\end{array}
$$

where the left horizontal bijections follow from Corollary 2.3; the right horizontal bijections follow from that $\mathbb{J}_{b_\bullet}^0 = I^d \cap \mathbb{J}_{b_\bullet}$ and $\mathbb{J}_{b_\bullet'}^0 = I'^d \cap \mathbb{J}_{b_\bullet'}$ as $b_\bullet, b_\bullet'$ are superbasic; the leftmost vertical bijection follows from the natural bijection

$$
((I_{\mathrm{der}})^d \cap \mathbb{J}_{b_\bullet}) \backslash \mathrm{Irr} X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) \cong ((I_{\mathrm{der}}')^d \cap \mathbb{J}_{b_\bullet'}) \backslash \mathrm{Irr} X_{\mu_\bullet'}^{\lambda_\bullet'}(b_\bullet')
$$

in the commutative diagram (a). The proof is finished.

$\square$

Let $G_{\mathrm{ad}}$ denote the adjoint group of $G$. As $b$ is superbasic, by [3, Lemma 3.11], $G_{\mathrm{ad}} \cong \prod_i \mathrm{Res}_{F_i/F} \mathrm{PGL}_{n_i}$ for some unramified extensions $F_i/F$. In view of the following natural central isogenies

$$
G \longrightarrow G_{\mathrm{ad}} \cong \prod_i \mathrm{Res}_{F_{d_i}/F} \mathrm{PGL}_{n_i} \longleftarrow \prod_i \mathrm{Res}_{F_{d_i}/F} \mathrm{GL}_{n_i},
$$

we will assume in the rest of this section that $G = \mathrm{Res}_{E/F} \mathrm{GL}_n$ for some unramified extension $E/F$ by Lemma 3.2.

### 3.2. Reduction procedure in the minuscule case

Now we consider the case where $\mu_\bullet$ is minuscule. Let $\mathcal{A}_{\mu_\bullet, b_\bullet} = \mathcal{A}_{\mu_\bullet, b_\bullet}^{G^d}$ and $\mathcal{A}_{\mu_\bullet, b_\bullet}^{\mathrm{top}} = \mathcal{A}_{\mu_\bullet, b_\bullet}^{G^d, \mathrm{top}}$ be defined in §2.2. For $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet, b_\bullet}$ set $\lambda_\bullet^\dagger = b_\bullet \sigma_\bullet(\lambda_\bullet)$, $\lambda_\bullet^\natural = -\lambda_\bullet + \lambda_\bullet^\dagger$ and $\lambda_\bullet^\flat = \epsilon_{\lambda_\bullet}^{-1}(\lambda_\bullet^\natural)$, where $\epsilon_{\lambda_\bullet} := \epsilon_{\lambda_\bullet}^{G^d}$ is defined in §1.5.

Since $\mu_\bullet$ is minuscule, we identify $\mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}$ canonically with the set of cocharacters in $Y^d$ which are conjugate to $\mu_\bullet$. Moreover, for $\zeta_\bullet \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}$, the corresponding Mirković-Vilonen cycle is $S_{\zeta_\bullet} = (K_U)^d t^{\zeta_\bullet} K^d / K^d$.

**Theorem 3.3** ([17, Proposition 1.6]). *Assume $\mu_\bullet$ is minuscule. For $\lambda \in \mathcal{A}_{\mu_\bullet, b_\bullet}$ we have*

*(1) $X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)$ is an affine space;*

*(2) $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet, b_\bullet}^{\mathrm{top}}$ if and only if $\lambda_\bullet^\flat = \epsilon_{\lambda_\bullet}^{-1}(\lambda_\bullet^\natural) \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\underline{\lambda}_{G^d}(b_\bullet))$.*

*Moreover, the maps $\lambda_\bullet \mapsto \lambda_\bullet^\flat$ and $\lambda_\bullet \mapsto \overline{X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)}$ induce a bijection*

$$\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet) \cong \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\underline{\lambda}_{G^d}(b_\bullet)).$$

As a consequence, for $C \in \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)$ there exists $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet, b_\bullet}^{\mathrm{top}}$ such that $\check{C} = \overline{X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)}$. Define

$$\gamma^{G^d}(C) = \lambda_\bullet^\flat \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\underline{\lambda}_{G^d}(b_\bullet)).$$

Moreover, we write

$$H(\lambda_\bullet) := H^{G^d}(C) = t^{\lambda_\bullet} K^d t^{\mu_\bullet} K^d t^{-\lambda_\bullet^\dagger} \cap I^d = H_1(\lambda_\bullet) \times \cdots \times H_d(\lambda_\bullet),$$

where $H_\tau(\lambda) := H_\tau(C)$ for $1 \leqslant \tau \leqslant d$ as in §2.3.

**Remark 3.4.** In [17], the EL-charts for $X_{\mu_\bullet}(b_\bullet)$ are parameterized by cocharacters $\lambda_\bullet$ in $\mathcal{A}_{\mu_\bullet, b_\bullet} \subseteq Y^d \cong (\mathbb{Z}^n)^{dl}$ with $l = \deg E/F$. By [17, Corollary 4.18], the map, sending $\lambda_\bullet$ to its *cotype*, induces a bijection

$$\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet) \cong \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\underline{\lambda}_{G^d}(b_\bullet)).$$

Following [17, Definition 4.13], the cotype of $\lambda_\bullet$ is equal to $\varepsilon_\bullet^{-1}(\lambda_\bullet^\natural)$, where $\varepsilon_\bullet$ lies in $(W_0)^d \cong (\mathfrak{S}_n)^{dl}$ such that

$$n\lambda_{i,j}(\varepsilon_{i,j}(k)) + \varepsilon_{i,j}(k) < n\lambda_{i,j}(\varepsilon_{i,j}(k')) + \varepsilon_{i,j}(k')$$

for $1 \leqslant i \leqslant d$, $1 \leqslant j \leqslant l$, and $1 \leqslant k' < k \leqslant n$. This means that $(\lambda_{i,j})_{\varepsilon_{i,j}(\alpha)} \geqslant 0$ for all positive roots $\alpha$. So $\varepsilon_\bullet = \epsilon_{\lambda_\bullet}$ and the cotype of $\lambda_\bullet$ coincides with $\lambda_\bullet^\flat$.

By the definition of $\gamma^{G^d}$, the second statement of Theorem 3.1 (for the minuscule case) follows from Theorem 3.3. It remains to show the first statement, which follows from the following result.

**Proposition 3.5.** *Let $\mu_\bullet$ be minuscule and let $\lambda_\bullet \in \mathcal{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$. For $1 \leqslant a \leqslant c \leqslant d$,*

$$\overline{t^{-\lambda_a}(H_a(\lambda_\bullet)t^{\lambda_a^\dagger} \times_K \cdots \times_K t^{-\lambda_c}H_c(\lambda_\bullet)t^{\lambda_c^\dagger}K/K}$$

$$= \epsilon_{\lambda_a}\overline{K_U t^{\lambda_a^\flat} \times_K \cdots \times_K K_U t^{\lambda_c^\flat}K/K}$$

$$= \epsilon_{\lambda_a}\overline{S_{\lambda_a^\flat} \tilde{\times} \cdots \tilde{\times} S_{\lambda_c^\flat}},$$

*where $\lambda_\bullet^\flat = (\lambda_1^\flat, \ldots, \lambda_d^\flat)$ and $\lambda_\bullet^\dagger = (\lambda_1^\dagger, \ldots, \lambda_d^\dagger)$.*

As $G = \mathrm{Res}_{E/F}\mathrm{GL}_n$, we have $G^d(\breve{F}) = \prod_{i=1}^{dl} G_i(\breve{F})$, where $l = \deg E/F$ and each $G_i$ is isomorphic to $\mathrm{GL}_n$ over $E$. Moreover, $\sigma_\bullet$ sends $G_i$ to $G_{i-1}$ for $1 \leqslant i \leqslant dl$ with $G_{dl+1} = G_1$. By Lemma 2.9 (1), we see that $H(\lambda_\bullet)$ only depends on $\lambda_\bullet \in \mathcal{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$, the image of $b_\bullet$ in $(\Omega_G)^d$ and the induced action of $\sigma_\bullet$ on the root system. Moreover, by Corollary 2.4, we can assume, by replacing $b$ with a suitable $\Omega$-$\sigma$-conjugate, that

$$b_\bullet = (1, \ldots, 1, b) \in \prod_{i=1}^{dl} G_i(\breve{F}).$$

Let $G' = \mathrm{GL}_n$ and let $\sigma'_\bullet$ be the Frobenius automorphism of $(G')^{dl}$ defined in §1.6. Via the natural identification (over $E$)

$$(G')^{dl} = \prod_{i=1}^{dl} G_i = G^d,$$

we see that the induced actions of $\sigma'_\bullet$ and $\sigma_\bullet$ on the root system coincide. Thus Proposition 3.5 for the triple $(G = \mathrm{Res}_{E/F}\mathrm{GL}_n, d, b_\bullet)$ is a consequence of its counterpart for the triple $(G' = \mathrm{GL}_n, dl, b_\bullet)$. So we will assume that $G = \mathrm{GL}_n$ when $\mu_\bullet$ is minuscule.

### 3.3. The minuscule case with $G = \mathrm{GL}_n$

Assume $G = \mathrm{GL}_n$. Let $T$ and $B$ be the group of diagonal matrices and the group of upper triangular matrices respectively. Let $V = \oplus_{i=1}^n \breve{F}e_i$ be the natural representation of $G(\breve{F})$. Then there are natural identifications $X = \oplus_{i=1}^n \mathbb{Z}e_i$, $Y = \oplus_{i=1}^n \mathbb{Z}e_i^\vee$ and $W_0 = \mathfrak{S}_n$, where $(e_i^\vee)_{1 \leqslant i \leqslant n}$ is the dual basis to $(e_i)_{1 \leqslant i \leqslant n}$ and $\mathfrak{S}_n$ denotes the symmetric group. Then the natural action of $w \in W_0$ on $X$ is given by $w(e_i) = e_{w(i)}$. Moreover, we have $\Phi = \{\alpha_{i,j} = e_i - e_j; 1 \leqslant i \neq j \leqslant n\}$ and the simple roots are $\alpha_i = e_i - e_{i+1}$ for

$1 \leqslant i \leqslant n-1$. Notice that $\Omega$ is a free abelian group of rank one. We fix a generator $\omega \in \Omega$ which sends $e_i$ to $e_{i+1}$ for $i \in \mathbb{Z}$, where $e_{j+n} = te_j$ for $j \in \mathbb{Z}$. So we can assume $b = \omega^m$ for some $m \in \mathbb{Z}$. As $b$ is superbasic, $m$ is coprime to $n$. The embedding $h \mapsto (1, \ldots, 1, h)$ induces an identification $Y = Y_\sigma \cong (Y^d)_{\sigma_\bullet}$, through which we have $\underline{\lambda}_G(b) = \underline{\lambda}_{G^d}(b_\bullet)$.

Suppose $\mu_\bullet \in Y^d$ is minuscule. Let $\lambda_\bullet = (\lambda_1, \ldots, \lambda_d) \in \mathcal{A}_{\mu_\bullet, b_\bullet}$. Following §3.2 we can define $\lambda_\bullet^\dagger = (\lambda_1^\dagger, \ldots, \lambda_d^\dagger)$, $\lambda_\bullet^\natural = (\lambda_1^\natural, \ldots, \lambda_d^\natural)$, $\lambda_\bullet^\flat = (\lambda_1^\flat, \ldots, \lambda_d^\flat)$ and $H(\lambda_\bullet) = H_1(\lambda_\bullet) \times \cdots \times H_d(\lambda_\bullet)$. Notice that $\lambda_\tau^\dagger = \lambda_{\tau+1}$ for $1 \leqslant \tau \leqslant d-1$.

**Lemma 3.6.** *Let* $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet, b_\bullet}$ *and* $1 \leqslant a \leqslant c \leqslant d$. *Then*

$$t^{-\lambda_a} H_a(\lambda_\bullet) t^{\lambda_a^\dagger} \times_K \cdots \times_K t^{-\lambda_c} H_c(\lambda_\bullet) t^{\lambda_c^\dagger} K = K_{U_{\lambda_a}} t^{\lambda_a^\natural} \times_K \cdots \times_K K_{U_{\lambda_c}} t^{\lambda_c^\natural} K.$$

*Proof.* Let $R_{\mu_\bullet, b_\bullet}(\lambda_\bullet) = \sqcup_{\tau=1}^d R_\tau(\lambda_\bullet) \subseteq \sqcup_{\tau=1}^d \Phi$ be as in §2.2, where

$$R_\tau(\lambda_\bullet) = \{\alpha \in \Phi; (\lambda_\tau)_\alpha \geqslant 1, \langle \alpha, \lambda_\tau^\natural \rangle = \langle \alpha, -\lambda_\tau + \lambda_\tau^\dagger \rangle = -1\}.$$

By the proof of Proposition 2.9, we have $H_\tau(\lambda_\bullet) = I_{\lambda_\tau} \Sigma_\tau$, where

$$\Sigma_\tau = \prod_{\alpha \in R_\tau(\lambda_\bullet)} U_\alpha(t^{\langle \alpha, \lambda_\tau \rangle - 1} \mathcal{O}_{\breve{F}}).$$

Thus $(\lambda_\tau^\dagger)_\alpha = (\lambda_\tau)_\alpha - 1 \geqslant 0$ and $U_\alpha(t^{\langle \alpha, \lambda_\tau \rangle - 1} \mathcal{O}_{\breve{F}}) = U_\alpha(t^{\langle \alpha, \lambda_\tau^\dagger \rangle} \mathcal{O}_{\breve{F}}) \subseteq I_{\lambda_\tau^\dagger}^+$ for $\alpha \in R_\tau(\lambda_\bullet)$, which means $\Sigma_\tau \subseteq I_{\lambda_\tau^\dagger}^+$ for $1 \leqslant \tau \leqslant d$. As $\lambda_\tau^\dagger = \lambda_{\tau+1}$ for $1 \leqslant \tau \leqslant d-1$, we have

$$\begin{aligned}
&t^{-\lambda_a} H_a(\lambda_\bullet) t^{\lambda_a^\dagger} \times_K \cdots \times_K t^{-\lambda_{c-1}} H_{c-1} t^{\lambda_{c-1}^\dagger} \times_K t^{-\lambda_c} H_c(\lambda_\bullet) t^{\lambda_c^\dagger} K \\
&= t^{-\lambda_a} H_a(\lambda_\bullet) t^{\lambda_{c+1}} \times_K \cdots \times_K t^{-\lambda_{c-1}} H_{c-1}(\lambda_\bullet) t^{\lambda_c} \times_K t^{-\lambda_c} H_c(\lambda_\bullet) t^{\lambda_c^\dagger} K \\
&= t^{-\lambda_a} I_{\lambda_a} \Sigma_a t^{\lambda_{a+1}} \times_K \cdots \times_K t^{-\lambda_{c-1}} I_{\lambda_{c-1}} \Sigma_{c-1} t^{\lambda_c} \times_K t^{-\lambda_c} I_{\lambda_c} \Sigma_c t^{\lambda_c^\dagger} K \\
&= t^{-\lambda_a} I_{\lambda_a} t^{\lambda_{a+1}} \times_K \cdots \times_K t^{-\lambda_{c-1}} I_{\lambda_{c-1}} t^{\lambda_c} \times_K t^{-\lambda_c} I_{\lambda_c} t^{\lambda_c^\dagger} K \\
&= t^{-\lambda_a} I_{\lambda_a}^+ I_{\lambda_a}^- t^{\lambda_{a+1}} \times_K \cdots \times_K t^{-\lambda_{c-1}} I_{\lambda_{c-1}}^+ I_{\lambda_{c-1}}^- t^{\lambda_c} \times_K (t^{-\lambda_c} I_{\lambda_c}^+ I_{\lambda_c}^- t^{\lambda_c^\dagger}) K \\
&= t^{-\lambda_a} I_{\lambda_a}^+ t^{\lambda_{a+1}} \times_K \cdots \times_K t^{-\lambda_{c-1}} I_{\lambda_{c-1}}^+ t^{\lambda_c} \times_K t^{-\lambda_c} I_{\lambda_c}^+ t^{\lambda_c^\dagger} K \\
&= K_{U_{\lambda_a}} t^{\lambda_a^\natural} \times_K \cdots \times_K K_{U_{\lambda_c}} t^{\lambda_c^\natural} K,
\end{aligned}$$

where the fifth equality follows from Lemma 1.4 that $I_{\lambda_\tau}^- \subseteq I_{\lambda_\tau^\dagger}^+$ for $1 \leqslant \tau \leqslant d$ since $\lambda_\bullet^\natural = -\lambda_\bullet + \lambda_\bullet^\dagger$ is minuscule. The proof is finished. $\qquad \square$

Write $\epsilon_{\lambda_\bullet} = (\epsilon_1, \ldots, \epsilon_d) \in (\mathfrak{S}_n)^d$ with $\epsilon_\tau := \epsilon_{\lambda_\tau}^G$ for $1 \leqslant \tau \leqslant d$. We define $a_{\tau,i} = \epsilon_\tau(i) + n\lambda_\tau(\epsilon_\tau(i))$ for $1 \leqslant \tau \leqslant d$. By the definition of $\epsilon_\tau = \epsilon_{\lambda_\tau}$ (see also Remark 3.4), $a_{\tau,1} > \cdots > a_{\tau,n}$ is the arrangement of the integers $i + n\lambda_\tau(i)$ for $1 \leqslant i \leqslant n$ in the decreasing order. Define $w_{\lambda_\bullet} = (w_\tau)_{1 \leqslant \tau \leqslant d} \in (\mathfrak{S}_n)^d$ such that

$$(*) \qquad a_{\tau,i} = \begin{cases} a_{\tau+1,w_\tau(i)} - n\lambda_\tau^\flat(i), & \text{if } 1 \leqslant \tau \leqslant d-1; \\ a_{1,w_d(i)} - n\lambda_d^\flat(i) + m, & \text{if } \tau = d. \end{cases}$$

**Lemma 3.7.** *We have $\epsilon_\tau = \epsilon_1 w_1^{-1} \cdots w_{\tau-1}^{-1}$ and hence $\lambda_\tau^\flat = w_{\tau-1} \cdots w_1 \epsilon_1^{-1}(\lambda_\tau^\natural)$ for $1 \leqslant \tau \leqslant d$.*

*Proof.* Suppose $2 \leqslant \tau \leqslant d$. Then $a_{\tau,i} = a_{\tau-1,w_{\tau-1}^{-1}(i)} + n\lambda_{\tau-1}^\flat(w_{\tau-1}^{-1}(i))$, that is,

$$\epsilon_\tau(i) + n\lambda_\tau(\epsilon_\tau(i)) = \epsilon_{\tau-1}(w_{\tau-1}^{-1}(i)) + n\lambda_{\tau-1}(\epsilon_{\tau-1}(w_{\tau-1}^{-1}(i))) + n\lambda_{\tau-1}^\flat(w_{\tau-1}^{-1}(i)),$$

which means $\epsilon_\tau(i) = \epsilon_{\tau-1}(w_{\tau-1}^{-1}(i))$. By induction we have $\epsilon_\tau = \epsilon_1 w_1^{-1} \cdots w_{\tau-1}^{-1}$. $\quad\square$

**Lemma 3.8.** *Let $1 \leqslant \tau \leqslant d$ and $1 \leqslant i < j \leqslant n$. We have*
   *(1) $w_\tau(i) > w_\tau(j)$, that is, $a_{\tau+1,w_\tau(i)} < a_{\tau+1,w_\tau(j)}$ if and only if $a_{\tau,i} - a_{\tau,j} < n$ and $\lambda_\tau^\flat(j) - \lambda_\tau^\flat(i) = 1$, in which case $a_{\tau+1,w_\tau(j)} - a_{\tau+1,w_\tau(i)} < n$.*
   *(2) $\ell(w_\tau) = |\{\alpha \in \Phi; (\lambda_\tau)_\alpha \geqslant 0, (\lambda_\tau^\dagger)_\alpha < 0\}|$.*

*Proof.* By $(*)$ we have

$$(i) \qquad a_{\tau+1,w_\tau(i)} - a_{\tau+1,w_\tau(j)} = a_{\tau,i} - a_{\tau,j} + n(\lambda_\tau^\flat(i) - \lambda_\tau^\flat(j))$$
$$= (\epsilon_\tau(i) + n\lambda_\tau(\epsilon_\tau(i))) - (\epsilon_\tau(j) + n\lambda_\tau(\epsilon_\tau(j))).$$

Then the first statement follows from that $\lambda_\tau^\flat$ is minuscule. For $1 \leqslant k \neq l \leqslant n$ we ave $\lambda_{\alpha_{k,l}} \geqslant 0$ if and only if $k + n\lambda(k) > l + n\lambda(l)$. Then it follows from (i) that the map $\gamma \mapsto \epsilon_\tau(\gamma)$ gives a bijection between $\Phi^+ \cap -w_\tau^{-1}(\Phi^+)$ and the set $\{\alpha \in \Phi; (\lambda_\tau)_\alpha \geqslant 0, (\lambda_\tau^\dagger)_\alpha < 0\}$. The second statement is proved. $\quad\square$

For $w \in \mathfrak{S}_n$ we denote by $\mathrm{supp}(w)$ the set of integers $1 \leqslant i \leqslant n-1$ such that the simple reflection $s_{\alpha_i}$ appears in some/any reduced expression of $w$.

**Lemma 3.9.** *Let $1 \leqslant \tau \leqslant d$ and $1 \leqslant i \leqslant n-1$ such that $i \in \mathrm{supp}(w_\tau)$. Then there are roots $\alpha, \beta \geqslant \alpha_i$ such that $w_\tau^{-1}(\alpha) < 0$ and $w_\tau(\beta) < 0$. As a consequence, $a_{\tau,i} - a_{\tau,i+1} < n$ and $a_{\tau+1,i} - a_{\tau+1,i+1} < n$.*

*Proof.* The first statement follows from that $i \in \mathrm{supp}(w_\tau)$. Let $\alpha = \alpha_{j,j'} \in \Phi^+$ such that $w_\tau^{-1}(\alpha) < 0$ and $\alpha_i \leqslant \alpha$. In other words, $j \leqslant i < i+1 \leqslant j'$ and $w_\tau^{-1}(j) > w_\tau^{-1}(j')$. Then we have $a_{\tau+1,i} - a_{\tau+1,i+1} \leqslant a_{\tau+1,j} - a_{\tau+1,j'} < n$ by Lemma 3.8. The inequality $a_{\tau,i} - a_{\tau,i+1} < n$ follows in a similar way. $\square$

**Lemma 3.10.** *We have* $\dim X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) = \langle \rho_\bullet, \mu_\bullet - \lambda_\bullet^\flat \rangle - \ell(w_{\lambda_\bullet})$. *Here* $\rho_\bullet$ *denotes the half sum of positive roots of* $G^d$.

*Proof.* We set

$$E = \{(\tau, i, j); 1 \leqslant \tau \leqslant d, 1 \leqslant i < j \leqslant n, \lambda_\tau^\flat(j) - \lambda_\tau^\flat(i) = 1\}$$
$$E' = \{(\tau, i, j) \in E; a_{\tau,i} - a_{\tau,j} > n\}$$
$$E'' = \{(\tau, i, j); 1 \leqslant \tau \leqslant d, 1 \leqslant i < j \leqslant n, w_\tau(i) > w_\tau(j)\}.$$

Then $E = E' \sqcup E''$ by Lemma 3.8. Applying Proposition 2.9 (3) we have

$$\dim X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) = |E'| = |E| - |E''| = \langle \rho_\bullet, \overline{\lambda_\bullet^\flat} - \lambda_\bullet^\flat \rangle - \ell(w_{\lambda_\bullet}) = \langle \rho_\bullet, \mu_\bullet - \lambda_\bullet^\flat \rangle - \ell(w_{\lambda_\bullet}),$$

where $\overline{\lambda_\bullet^\flat}$ denotes the dominant conjugate of $\lambda_\bullet^\flat$, which equals $\mu_\bullet$ by Proposition 2.9. $\square$

**Lemma 3.11.** *If* $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet, b_\bullet}^{\mathrm{top}}$, *then* $\ell(w_{\lambda_\bullet}) = \sum_{\tau=1}^d \ell(w_\tau) = n - 1$, *and* $w_d \cdots w_1 \in \mathfrak{S}_n$ *is a product of distinct simple reflections.*

*Proof.* Let $\lambda_{m,n} \in \mathbb{Z}^n$ such that $\lambda_{m,n}(i) = \lfloor \frac{im}{n} \rfloor - \lfloor \frac{(i-1)m}{n} \rfloor$ for $1 \leqslant i \leqslant n$. As $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet, b_\bullet}^{\mathrm{top}}$, It follows from [17, §4.4] that $\sum_{\tau=1}^d \lambda_\tau^\flat = \lambda_{m,n}$ and $\dim X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) = \langle \rho_\bullet, \mu_\bullet \rangle - \frac{n-1}{2}$. By Lemma 3.10,

$$\begin{aligned}
\dim X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) &= \langle \rho_\bullet, \mu_\bullet - \lambda_\bullet^\flat \rangle - \ell(w_{\lambda_\bullet}) \\
&= \langle \rho_\bullet, \mu_\bullet \rangle - \langle \rho, \lambda_{m,n} \rangle - \ell(w_{\lambda_\bullet}) \\
&= \langle \rho_\bullet, \mu_\bullet \rangle + \frac{n-1}{2} - \ell(w_{\lambda_\bullet}) \\
&= \langle \rho_\bullet, \mu_\bullet \rangle - \frac{n-1}{2},
\end{aligned}$$

where $\rho$ is the half sum of positive roots of $\mathrm{GL}_n$. Thus $\ell(w_{\lambda_\bullet}) = n - 1$. Moreover, by $(*)$ we see that $\epsilon_1(i) - \epsilon_1 w_d \cdots w_1(i) \equiv a_{1,i} - a_{1,w_d \cdots w_1(i)} \equiv m$ mod $n$ for $1 \leqslant i \leqslant n$. So $w_d \cdots w_1 \in \mathfrak{S}_n$ acts on $\{1, \ldots, n\}$ transitively as $m$ is coprime to $n$. This means that $n - 1 \leqslant \ell(w_d \cdots w_1) \leqslant \ell(w_{\lambda_\bullet}) = n - 1$ and $w_d \cdots w_1$ is a product of distinct simple reflections as desired. $\square$

**Lemma 3.12.** *Let* $\lambda_\bullet \in \mathcal{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$. *Let* $1 \leqslant \tau \leqslant d-1$ *and* $i \in \mathrm{supp}(w_\tau)$. *Then* $\sum_{k=\tau+1}^\iota \langle \alpha_i, \lambda_k^\flat \rangle \in \mathbb{Z}_{\geqslant 0}$ *for* $\tau+1 \leqslant \iota \leqslant d$.

*Proof.* By Lemma 3.11, $i \notin \mathrm{supp}(w_l)$ for $\tau+1 \leqslant l \leqslant d$, and moreover, there exists at most one integer $\tau+1 \leqslant \tau' \leqslant \iota$ (resp. $\tau+1 \leqslant \tau'' \leqslant \iota$) such that $i-1 \in \mathrm{supp}(w_{\tau'})$ (resp. $i+1 \in \mathrm{supp}(w_{\tau''})$). Without loss of generality, we assume such $\tau', \tau''$ exist. Therefore,

(i) for $\tau+1 \leqslant k \leqslant \iota$ we have: (1) $w_k(i) \neq i$ if and only if $k = \tau'$ and $w_{\tau'}(i) < i$; (2) $w_k(i+1) \neq i+1$ if and only if $k = \tau''$ and $w_{\tau''}(i+1) > i+1$.

Using (i) and the equality from $(*)$

$$\langle \alpha_i, \lambda_k^\flat \rangle = (a_{k+1, w_k(i)} - a_{k+1, w_k(i+1)}) - (a_{k,i} - a_{k,i+1})$$

we deduce that

$$\sum_{k=\tau+1}^\iota \langle \alpha_i, \lambda_k^\flat \rangle$$
$$= \frac{a_{\tau'+1, w_{\tau'}(i)} - a_{\tau'+1, i}}{n}$$
$$+ \frac{a_{\iota+1, i} - a_{\iota+1, i+1}}{n} + \frac{a_{\tau''+1, i+1} - a_{\tau''+1, w_{\tau''}(i+1)}}{n} - \frac{a_{\tau+1, i} - a_{\tau+1, i+1}}{n}$$
$$\geqslant 0,$$

where the inequality follows from that $a_{\tau+1,i} - a_{\tau+1,i+1} < n$ (by Lemma 3.9) and that $a_{\tau'+1, w_{\tau'}(i)} - a_{\tau'+1,i}, a_{\iota+1,i} - a_{\iota+1,i+1}, a_{\tau''+1,i+1} - a_{\tau''+1, w_{\tau''}(i+1)} > 0$. $\square$

*Proof of Proposition 3.5.* By Lemma 3.7 we have $\lambda_k^\natural = \epsilon_{\lambda_a}(w_{k-1} \cdots w_a)^{-1}(\lambda_k^\flat)$ and $U_{\lambda_k} = \epsilon_{\lambda_a}(w_{k-1} \cdots w_a)^{-1} U$ for $a \leqslant k \leqslant d$. By Lemma 3.6,

$$t^{-\lambda_a} H_a(\lambda_\bullet) t^{\lambda_a^\dagger} \times_K \cdots \times_K t^{-\lambda_c} H_c(\lambda_\bullet) t^{\lambda_c^\dagger} K/K$$
$$= K_{U_{\lambda_a}} t^{\lambda_a^\natural} \times_K \cdots \times_K K_{U_{\lambda_c}} t^{\lambda_c^\natural} K/K$$
$$= \epsilon_{\lambda_a} K_U t^{\lambda_a^\flat} \times_K w_a^{-1} K_U t^{\lambda_{a+1}^\flat} w_a$$
$$\times_K \cdots \times_K (w_{c-1} \cdots w_a)^{-1} K_U t^{\lambda_c^\flat} w_{c-1} \cdots w_a K/K$$
$$= \epsilon_{\lambda_a} K_U t^{\lambda_a^\flat} \times_K w_a^{-1} K_U t^{\lambda_{a+1}^\flat} \times_K \cdots \times_K w_{c-1}^{-1} K_U t^{\lambda_c^\flat} K/K$$

Therefore, it suffices to show that for $a \leqslant \tau \leqslant c-1$ and $i \in \mathrm{supp}(w_\tau)$ we have

$$s_i \overline{K_U t^{\lambda_{\tau+1}^\flat} \times_K \cdots \times_K K_U t^{\lambda_c^\flat} K/K} = \overline{K_U t^{\lambda_{\tau+1}^\flat} \times_K \cdots \times_K K_U t^{\lambda_c^\flat} K/K}.$$

Set $U_i = U_{\alpha_i}$, $U_{-i} = U_{-\alpha_i}$ and $U^i = \prod_{0 < \alpha \neq \alpha_i} U_\alpha$ for $1 \leqslant i \leqslant n-1$. Then $U = U_i U^i = U^i U_i$ and $U^i$ is normalized by $U_i$ and $U_{-i}$. As we can take $s_i = U_i(-1)U_{-i}(1)U_i(-1)$, the displayed equality above is equivalent to

$$U_{-i}(1)\overline{K_U t^{\lambda^\flat_{\tau+1}} \times_K \cdots \times_K K_U t^{\lambda^\flat_c} K/K} = \overline{K_U t^{\lambda^\flat_{\tau+1}} \times_K \cdots \times_K K_U t^{\lambda^\flat_c} K/K}.$$

Define $f_\iota$ for $\tau \leqslant \iota \leqslant d$ such that $f_\tau = 1$ and $f_\iota = t^{e_\iota} f_{\iota-1}/(1 + z_\iota f_{\iota-1})$ with $e_\iota = \langle \alpha_i, \lambda^\flat_\iota \rangle$ for $\tau + 1 \leqslant \iota \leqslant c$. We claim that

(i) for generic points $(z_{\tau+1}, \ldots, z_c) \in (\mathcal{O}_{\breve{F}})^{c-\tau}$ we have $1 + z_\iota f_{\iota-1} \in \mathcal{O}_{\breve{F}}^\times$ and $f_\iota \in t^{\sum_{k=\tau+1}^\iota e_k} \mathcal{O}_{\breve{F}} \subseteq \mathcal{O}_{\breve{F}}$ for $\tau + 1 \leqslant \iota \leqslant c$.

if $\iota = \tau + 1$, the claim is true by taking $z_{\tau+1} \in \mathcal{O}_{\breve{F}} \setminus \{-1\}$. Suppose it is true for $\iota - 1$. Then $f_{\iota-1} \in t^{\sum_{k=\tau+1}^{\iota-1} e_k} \mathcal{O}_{\breve{F}} \subseteq \mathcal{O}_{\breve{F}}$ by Lemma 3.12. So there exist generic points $z_\iota \in \mathcal{O}_{\breve{F}}$ such that $1 + z_\iota f_{\iota-1} \in \mathcal{O}_{\breve{F}}^\times$ and hence $f_\iota = t^{e_\iota} f_{\iota-1}/(1 + z_\iota f_{\iota-1}) \in t^{\sum_{k=\tau+1}^\iota e_k} \mathcal{O}_{\breve{F}}$ as desired. The claim (i) is proved.

Let $(z_{\tau+1}, \ldots, z_c) \in (\mathcal{O}_{\breve{F}})^{c-\tau}$ be a generic point as in (i). Using (i) and the commutator relation

$$U_{-\alpha}(f)U_\alpha(z) = U_\alpha(\frac{z}{1+zf})(1+zf)^{-\alpha^\vee} U_{-\alpha}(\frac{f}{1+zf}) \text{ for } 1 + zf \neq 0,$$

we deduce that

$$U_{-i}(1)K_{U^i}U_i(z_{\tau+1})t^{\lambda^\flat_{\tau+1}} \times_K \cdots \times_K K_{U^i}U_i(z_c)t^{\lambda^\flat_c}K/K$$
$$\subseteq K_B t^{\lambda^\flat_{\tau+1}} \times_K U_{-i}(f_{\tau+1})K_{U^i}U_i(z_{\tau+2})t^{\lambda^\flat_{\tau+2}} \times_K \cdots \times_K K_{U^i}U_i(z_c)t^{\lambda^\flat_c}K/K$$
$$\vdots$$
$$\subseteq K_B t^{\lambda^\flat_{\tau+1}} \times_K \cdots \times_K K_B t^{\lambda^\flat_c}K/K$$
$$= K_U t^{\lambda^\flat_{\tau+1}} \times_K \cdots \times_K K_U t^{\lambda^\flat_c}K/K.$$

Therefore, $K_U t^{\lambda^\flat_{\tau+1}} \times_K \cdots \times_K K_U t^{\lambda^\flat_c}K/K$ contains an open dense subset of $U_{-i}(1)K_U t^{\lambda^\flat_{\tau+1}} \times_K \cdots \times_K K_U t^{\lambda^\flat_c}K/K$ as desired. $\qquad\square$

### 3.4. The general case

Finally we consider the general case where $\mu_\bullet \in Y^d$ is an arbitrary dominant cocharacter. The strategy is to reduce it to the minuscule case considered in the previous subsection.

As $G = \mathrm{Res}_{E/F}\mathrm{GL}_n$, there exist $e \in \mathbb{Z}_{\geqslant d}$, a minuscule dominant cocharacter $\upsilon_\bullet \in Y^e$ and a sequence $\Sigma$ of integers $1 = k_1 < \cdots < k_d < k_{d+1} = e+1$

such that $\mu_\tau = \upsilon_{k_\tau} + \cdots + \upsilon_{k_{\tau+1}-1}$ for $1 \leqslant \tau \leqslant d$. Let $\mathrm{pr}_\Sigma : G^e \to G^d$ be the projection given by $(g_1, \ldots, g_e) \mapsto (g_{k_1}, \ldots, g_{k_d})$. By abuse of notation, we still denote by $b_\bullet$ the element $(1, \ldots, 1, b)$ in $G^e(\breve{F})$ and by $\sigma_\bullet$ the Frobenius of $G^e$ given by $(g_1, \ldots, g_e) \mapsto (g_2, \ldots, g_e, \sigma(g_1))$. Then there is a Cartesian square

$$
\begin{array}{ccc}
X_{\upsilon_\bullet}(b_\bullet) & \longrightarrow & G(\breve{F}) \times_K \mathrm{Gr}_{\upsilon_\bullet} \\
\downarrow{\scriptstyle \mathrm{pr}_\Sigma} & & \downarrow{\scriptstyle \mathrm{id} \times_K m_{\upsilon_\bullet}^\Sigma} \\
\cup_{\eta_\bullet \leq \mu_\bullet} X_{\eta_\bullet}(b_\bullet) & \longrightarrow & G(\breve{F}) \times_K \mathrm{Gr}_{\mu_\bullet},
\end{array}
$$

where $m_{\upsilon_\bullet}^\Sigma : \mathrm{Gr}_{\upsilon_\bullet} \to \mathrm{Gr}_{\mu_\bullet}$ is the partial convolution map given by

$$
(g_1, \ldots, g_{e-1}, g_e K) \mapsto (g_{k_1} \cdots g_{k_2-1}, \ldots, g_{k_{d-1}} \cdots g_{k_d-1}, g_{k_d} \cdots g_e K);
$$

the top horizontal map is given by

$$
(g_1 K, \ldots, g_e K) \mapsto (g_1, g_1^{-1} g_2, \ldots, g_{e-1}^{-1} g_e, g_e^{-1} b \sigma(g_1) K);
$$

the bottom horizontal map is given by

$$
(h_1 K, \ldots, h_d K) \mapsto (h_1, h_1^{-1} h_2, \ldots, h_{d-1}^{-1} h_d, h_d^{-1} b \sigma(h_1) K).
$$

For a dominant cocharacter $\eta_\bullet \in Y^d$ we denote by $m_{\upsilon_\bullet}^{\eta_\bullet}$ the multiplicity with which $V_{\eta_\bullet}^{\widehat{G}^d}$ appears in $V_{\upsilon_\bullet}^{\widehat{G}^e}$. Here we view each $\widehat{G}^e$-crystal as a $\widehat{G}^d$-crystal via the embedding $\widehat{G}^d \hookrightarrow \widehat{G}^e$ given by $(h_1, \ldots, h_d) \mapsto (h_1^{(k_2-k_1)}, \ldots, h_d^{(k_{d+1}-k_d)})$.

**Proposition 3.13.** *We have*

$$
|\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\upsilon_\bullet}(b_\bullet)| = \sum_{\eta_\bullet \leq \mu_\bullet} m_{\upsilon_\bullet}^{\eta_\bullet} |\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\eta_\bullet}(b_\bullet)|.
$$

*As a consequence,* $|\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)| = \dim V_{\mu_\bullet}^{\widehat{G}^d}(\lambda_G(b))$.

*Proof.* The first statement follows similarly as Corollary 1.6. To show the second one, we argue by induction on $|\mu_\bullet|$. If $\mu_\bullet$ minuscule, it is proved in Theorem 3.3. Suppose it is true for $|\eta_\bullet| < |\mu_\bullet|$. By the choice of $\upsilon_\bullet$ we have $m_{\upsilon_\bullet}^{\mu_\bullet} = 1$. Therefore,

$$
|\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\upsilon_\bullet}(b_\bullet)| = \sum_{\eta_\bullet \leq \mu_\bullet} m_{\upsilon_\bullet}^{\eta_\bullet} |\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\eta_\bullet}(b_\bullet)|
$$

$$= |\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)| + \sum_{\eta_\bullet < \mu_\bullet} m_{\upsilon_\bullet}^{\eta_\bullet} \dim V_{\eta_\bullet}(\underline{\lambda}_G(b))$$

$$= \dim V_{\upsilon_\bullet}(\underline{\lambda}_G(b))$$

$$= \sum_{\eta_\bullet \leq \mu_\bullet} m_{\upsilon_\bullet}^{\eta_\bullet} \dim V_{\eta_\bullet}(\underline{\lambda}_G(b)),$$

where the second equality follows from the induction hypothesis, and the last equality follows again from Theorem 3.3 as $\upsilon_\bullet$ is minuscule. Therefore, we have $|\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)| = \dim V_{\mu_\bullet}(\underline{\lambda}_G(b))$ as desired. $\qquad \square$

Similar to the definition of $\otimes$ in Theorem 0.7, let $\otimes_\Sigma : \mathbb{B}_{\upsilon_\bullet}^{\widehat{G}^e} \to \sqcup_{\eta_\bullet} \mathbb{B}_{\eta_\bullet}^{\widehat{G}^d}$ denote the map given by

$$(\delta_1, \ldots, \delta_e) \mapsto (\delta_{k_1} \otimes \cdots \otimes \delta_{k_2 - 1}, \ldots, \delta_{k_d} \otimes \cdots \otimes \delta_{k_{d+1} - 1}).$$

*Proof of Theorem 3.1.* Let $C \in \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)$. By Theorem 3.3 and Proposition 3.13, there exists $\xi_\bullet \in \mathcal{A}_{\upsilon_\bullet, b_\bullet}^{\mathrm{top}}$ such that $\xi_\bullet^\flat \in \mathbb{B}_{\upsilon_\bullet}^{\widehat{G}^e}(\underline{\lambda}_G(b))$ and $C = \overline{\mathrm{pr}_\Sigma(X_{\upsilon_\bullet}^{\xi_\bullet}(b_\bullet))}$. Write $\xi_\bullet = (\xi_1, \ldots, \xi_e)$, $\xi_\bullet^\dagger = b_\bullet \sigma_\bullet(\xi_\bullet) = (\xi_1^\dagger, \ldots, \xi_e^\dagger)$ and $\xi_\bullet^\flat = (\xi_1^\flat, \ldots, \xi_e^\flat)$. Define

$$\gamma^{G^d}(C) = \otimes_\Sigma(\xi_\bullet^\flat) = (\gamma_1, \ldots, \gamma_d) \in \sqcup_{\eta_\bullet} \mathbb{B}_{\eta_\bullet}^{\widehat{G}^d}(\underline{\lambda}_G(b)),$$

where $\gamma_\tau = \xi_{k_\tau}^\flat \otimes \cdots \otimes \xi_{k_{\tau+1} - 1}^\flat \in \mathbb{B}^{\widehat{G}} := \sqcup_\eta \mathbb{B}_\eta^{\widehat{G}}$ for $1 \leqslant \tau \leqslant d$.

Let $\lambda_\bullet = \mathrm{pr}_\Sigma(\xi_\bullet) \in Y^d$. Then $(I^d t^{\lambda_\bullet} K^d / K^d) \cap C$ is open dense in $C$. So

$$\overline{(\theta_{\lambda_\bullet}^{G^d})^{-1}(C)} = \overline{\mathrm{pr}_\Sigma((\theta_{\xi_\bullet}^{G^e})^{-1}(X_{\upsilon_\bullet}^{\xi_\bullet}(b_\bullet)))} \subseteq I^d,$$

which means (by the proof of Lemma 2.10) that, for each $1 \leqslant \tau \leqslant d$,

$$\overline{H_\tau(C)} = \overline{H_{k_\tau}(\xi_\bullet) \cdots H_{k_{\tau+1} - 1}(\xi_\bullet)},$$

where $H_\tau(C)$ and $H(\xi_\bullet)$ are defined in §2.3 and §3.2 respectively. Thus for $1 \leqslant a \leqslant c \leqslant d$,

$$\overline{t^{-\lambda_a} H_a(C) t^{\lambda_a^\dagger} \times_K \cdots \times_K t^{-\lambda_c} H_c(C) t^{\lambda_c^\dagger} K / K}$$

$$= m_{\upsilon_\bullet}^\Sigma (t^{-\xi_{k_a}} H_{k_a}(\xi_\bullet) t^{\xi_{k_a}^\dagger} \times_K \cdots \times_K t^{-\xi_{k_{c+1} - 1}} H_{k_{c+1} - 1}(\xi_\bullet) t^{\xi_{k_{c+1} - 1}^\dagger} K / K)$$

$$= m_{\upsilon_\bullet}^\Sigma (\epsilon_{\lambda_a} K_U t^{\xi_{k_a}^\flat} \times_K \cdots \times_K K_U t^{\xi_{k_{c+1} - 1}^\flat} K / K)$$

$$= \epsilon_{\lambda_a} \overline{S_{\gamma_a} \widetilde{\times} \cdots \widetilde{\times} S_{\gamma_c}},$$

where the first equality follows from that $\lambda_\tau = \xi_{k_\tau}$ and $\lambda_\tau^\dagger = \xi_{k_{\tau+1}-1}^\dagger$ for $1 \leqslant \tau \leqslant d$; the second equality follows from Proposition 3.5. In particular, we have

$$\overline{t^{-\lambda_\tau} H_\tau(C) t^{\lambda_\tau^\dagger} K/K} = \epsilon_{\lambda_\tau} \overline{S_{\gamma_\tau}}$$

by taking $a = c = \tau$. On the other hand, as $C \subseteq X_{\mu_\bullet}(b_\bullet)$ it follows that

$$t^{-\lambda_\tau} H_\tau(C) t^{\lambda_\tau^\dagger} K/K \subseteq \mathrm{Gr}_{\mu_\tau}^\circ.$$

Thus, $\gamma_\tau \in \mathbb{B}_{\mu_\tau}^{\widehat{G}}$ and hence $\gamma^{G^d}(C) \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\underline{\lambda}_G(b))$. Now the first statement of Theorem 3.1 follows by taking $a = 1$ and $c = d$.

As $b_\bullet$ is superbasic, $\mathbb{J}_{b_\bullet} = (\Omega^d \cap \mathbb{J}_{b_\bullet})(I^d \cap \mathbb{J}_{b_\bullet})$. By Lemma 2.3 and Lemma 2.5, the map $C \mapsto \gamma^{G^d}(C)$ defined in the previous paragraph induces a map

$$\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet) \to \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\underline{\lambda}_G(b)).$$

Then we have the following commutative diagram

$$
\begin{array}{ccc}
\mathrm{Irr} X_{\upsilon_\bullet}(b_\bullet) & \xrightarrow{\gamma^{G^e}} & \mathbb{B}_{\mu_\bullet}^{\widehat{G}^e}(\underline{\lambda}_G(b)) \\
{\scriptstyle \mathrm{pr}_\Sigma} \downarrow & & \downarrow {\scriptstyle \otimes_\Sigma} \\
\sqcup_{\eta_\bullet} \mathrm{Irr} X_{\eta_\bullet}(b) & \xrightarrow{\gamma^{G^d}} & \sqcup_{\eta_\bullet} \mathbb{B}_{\eta_\bullet}^{\widehat{G}^d}(\underline{\lambda}_G(b)).
\end{array}
$$

As $\gamma^{G^e}$ is bijective and $m_{\upsilon_\bullet}^{\mu_\bullet} = 1$, the map $\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet) \xrightarrow{\gamma^{G^d}} \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\underline{\lambda}_G(b))$ is surjective and hence bijective by Proposition 3.13. $\qquad\square$

## 4. Proof of Theorem 0.5 and 0.7

### 4.1. Irreducible components of $S_{\mu,\eta}^N$

Let $P = MN$ and $b \in M(\breve{F})$ be as in §2.1. For $\mu \in Y^+$ we denote by $I_{\mu,M}$ the set of $M$-dominant cocharacters $\eta$ such that

$$S_{\mu,\eta}^N := N(\breve{F}) t^\eta K/K \cap \mathrm{Gr}_\mu^\circ \neq \emptyset.$$

Define $I_{\mu,b,M} = \{\eta \in I_{\mu,M}; \eta = \kappa_M(b) \in \pi_1(M)_\sigma\}$.

**Proposition 4.1** ([8, Proposition 5.4.2]). *Let* $\eta \in I_{\mu,M}$, *then*

$$\dim S_{\mu,\eta}^N \leqslant \langle \rho, \mu + \eta \rangle - 2\langle \rho_M, \eta \rangle.$$

*Moreover, let $\Sigma_{\mu,\eta}^N$ be the set of irreducible components of $S_{\mu,\eta}^N$ with the maximal possible dimension $\langle \rho, \mu + \eta \rangle - 2\langle \rho_M, \eta \rangle$. Then $|\Sigma_{\mu,\eta}^N|$ equals the multiplicity with which $\mathbb{B}_\eta^{\widehat{M}}$ appears in $\mathbb{B}_\mu^{\widehat{G}}$.*

For $\eta \in I_{\mu,M}$ recall that $\theta_\eta^N : N(\breve{F}) \to \mathrm{Gr}_P$ is the map given by $n \mapsto nt^\eta K_P$. Let $Z^N \in \mathrm{Irr}\, S_{\mu,\eta}^N$ and $g = ht^\eta h' \in K_M t^\eta K_M$ with $h, h' \in K_M$. We define

$$(\theta_\eta^N)^{-1}(Z^N) * g = h(\theta_\eta^N)^{-1}(Z^N)h^{-1}g \subseteq P(\breve{F});$$
$$Z^N * (gK_M) = (\theta_\eta^{-1}(Z^N) * g)K_P/K_P \subseteq \mathrm{Gr}_P,$$

which do not depend on the choices of $h, h' \in K_M$ since the connected group $K_M \cap t^\eta K_M t^{-\eta}$ fixes $S_{\mu,\eta}^N$ and hence fixes each of its irreducible components, by left multiplication. For $\mathcal{D}^M \subseteq K_M t^\eta K_M$ we set

$$(\theta_\eta^N)^{-1}(Z^N) * \mathcal{D}^M = \cup_{g \in \mathcal{D}^M} (\theta_\eta^N)^{-1}(Z^N) * g;$$
$$Z^N * (\mathcal{D}^M K_M/K_M) = \cup_{g \in \mathcal{D}^M} Z^N * (gK_M).$$

Notice that $Z^N * (\mathcal{D}^M K_M/K_M) = ((\theta_\eta^N)^{-1}(Z^N) * \mathcal{D}^M)K_P/K_P$.

**Lemma 4.2.** *Let $Z^N \in \mathrm{Irr}\, S_{\mu,\eta}^N$ and $g \in K_M t^\eta K_M$. Then we have*
*(1) $h((\theta_\eta^N)^{-1}(Z^N) * g) = (\theta_\eta^N)^{-1}(Z^N) * (hg)$ for $h \in K_M$;*
*(2) $u((\theta_\eta^N)^{-1}(Z^N) * g) = (\theta_\eta^N)^{-1}(Z^N) * g$ for $u \in K_N$.*

*Proof.* The first statement follows by definition. The second one follows from that $K_N Z^N = Z^N$ since $K_N$ fixes $S_{\mu,\eta}^N$ and hence fixes each of its irreducible components by left multiplication. $\square$

### 4.2. Iwasawa decomposition of $X_\mu(b)$

Notice that the natural projection $P = MN \to M$ induces a map

$$\beta : X_\mu(b) \hookrightarrow \mathrm{Gr}_G = \mathrm{Gr}_P \to \mathrm{Gr}_M.$$

Let $\eta \in I_{\mu,b,M}$ and let $X_\eta^M(b)$ be the affine Deligne-Lusztig variety defined for $M$. For $Z^N \in \mathrm{Irr}\, S_{\mu,\eta}^N$ and $C^M \subseteq X_\eta^M(b)$ we define

$$X_\mu^{Z^N,C^M}(b) = \{gK_P \in \beta^{-1}(C^M); g^{-1}b\sigma(g)K_P \in Z^N * \mathrm{Gr}_{\eta,M}^\circ\} \subseteq \mathrm{Gr}_P,$$

where $\mathrm{Gr}_{\eta,M}^\circ = K_M t^\eta K_M/K_M$. Notice that the natural projection

$$Z^N * \mathrm{Gr}_{\eta,M}^\circ \to \mathrm{Gr}_{\eta,M}^\circ$$

is a fiber bundle with fibers isomorphic to $Z^N$.

**Proposition 4.3.** *Let $C^M \subseteq X_\eta^M(b)$ be locally closed and irreducible. Then*

    *(1) $\beta^{-1}(C^M) = \cup_{Z^N \in \mathrm{Irr}S_{\mu,\eta}^N} X_\mu^{Z^N,C^M}(b)$;*

    *(2) $\dim X_\mu^{Z^N,C^M}(b) \leqslant \dim X_\mu(b)$, where the equality holds if and only if* $\dim C^M = \dim X_\eta^M(b)$ *and* $Z^N \in \Sigma_{\mu,\eta}^N$;

    *(3) $N(\breve{F}) \cap \mathbb{J}_b$ acts transitively on $\mathrm{Irr}X_\mu^{Z^N,C^M}(b)$;*

    *(4) $X_\mu^{Z^N,C^M}(b)$ is irreducible if $M$ is the centralizer of $\nu_b$;*

    *(5) $t^{-\lambda}H^P(C)t^{b\sigma(\lambda)} = (\theta_\eta^N)^{-1}(Z^N) * (t^{-\lambda}H^M(C^M)t^{b\sigma(\lambda)})$*

*if $C \in \mathrm{Irr}X_\mu^{Z^N,C^M}(b)$ and $C^M \in \mathrm{Irr}X_\eta^{\lambda,M}$ for some $\lambda \in Y$.*

    *Here $X_\eta^{\lambda,M}(b) = I_M t^\lambda K_M / K_M \cap X_\eta^M(b)$, and $H^M(C^M)$ is define in §2.1* *for $G = M$.*

*Proof.* Let $mK_M \in X_\eta^M(b)$. By definition,

$$\beta^{-1}(mK_M) = \{mnK_P; n^{-1}m^{-1}b\sigma(m)\sigma(n)K_P \in S_{\mu,\eta}^N * (b_m K_M)\}.$$

So (1) follows. Moreover, as $\nu_M(b) = \nu_G(b)$ is dominant, it follows from [8, Proposition 5.3.2] that

$$\dim(X_\mu^{Z^N,C^M}(b) \cap \beta^{-1}(mK_M)) = \dim Z^N - \langle 2\rho_N, \eta \rangle.$$

Therefore, by Proposition 4.1 we have

$$\begin{aligned}
\dim X_\mu^{Z^N,C^M}(b) &= \dim C^M + \dim Z^N - 2\langle \rho, \eta \rangle \\
&\leqslant \langle \rho_M, \eta \rangle - \frac{1}{2}\mathrm{def}_M(b) + \langle \rho, \mu + \eta \rangle - 2\langle \rho_M, \eta \rangle - 2\langle \rho_N, \eta \rangle \\
&= \langle \rho, \mu \rangle - \langle \rho_N, \eta \rangle - \frac{1}{2}\mathrm{def}_G(b) \\
&= \langle \rho, \mu - \nu_M(b) \rangle - \frac{1}{2}\mathrm{def}_G(b) \\
&= \dim X_\mu(b),
\end{aligned}$$

where the equality holds if and only if $\dim C^M = \dim X_\eta^M(b)$ and $Z^N \in \Sigma_{\mu,\eta}^N$. So (2) follows.

    The statement (3) follows similarly as [17, Proposition 5.6] which deals with the minuscule case. Notice that for minuscule $\mu$ the sets $S_{\mu,\eta}^N = K_N t^\eta K/K$ and $N(\breve{F}) \cap t^{-\eta}Kt^\mu K = t^{-\eta}(\theta_\eta^N)^{-1}(S_{\mu,\eta}^N)t^\eta$ are irreducible. For general case we only needs to replace $N(\breve{F}) \cap t^{-\eta}Kt^\mu K$ in [17, Claim 1 on page 1630] with the irreducible set $t^{-\eta}(\theta_\eta^N)^{-1}(Z^N)t^\eta$ for $Z^N \in \mathrm{Irr}S_{\mu,\eta}^N$.

The statement (4) follows from (3) and that $N(\breve{F}) \cap \mathbb{J}_b = \{1\}$.
It follows by definition that

$$t^{-\lambda} H^P(X^{Z^N, C^M}(b)) t^{b\sigma(\lambda)} = (\theta_\eta^N)^{-1}(Z^N) * (t^{-\lambda} H^M(C^M) t^{b\sigma(\lambda)}).$$

By (3), all the irreducible components of $X^{Z^N, C^M}(b)$ are conjugate under $N(\breve{F}) \cap \mathbb{J}_b$. Thus $H^P(C) = H^P(X^{Z^N, C^M}(b))$ for $C \in \mathrm{Irr} X^{Z^N, C^M}(b)$, and (5) follows. □

**Corollary 4.4.** *The map*

$$(Z^N, C^M) \mapsto \mathbb{J}_b^M \mathrm{Irr} \overline{X_\mu^{Z^N, C^M}}(b) = (P(\breve{F}) \cap \mathbb{J}_b) \mathrm{Irr} \overline{X_\mu^{Z^N, C^M}}(b)$$

*induces a bijection*

$$(P(\breve{F}) \cap \mathbb{J}_b) \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b) \cong \sqcup_{\eta \in I_{\mu,b,M}} \Sigma_{\mu,\eta}^N \times (\mathbb{J}_b^M \backslash \mathrm{Irr}^{\mathrm{top}} X_\eta^M(b)).$$

*As a consequence, $|\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)| \leqslant \dim V_\mu(\underline{\lambda}_G(b))$.*

*Proof.* The bijection follows from Proposition 4.3 (1), (2), (3). Choose $P = MN$ such that $b$ is superbasic in $M(\breve{F})$. Then

$$\begin{aligned}
|\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)| &\leqslant |(P(\breve{F}) \cap \mathbb{J}_b) \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)| \\
&= \sum_{\eta \in I_{\mu,b,M}} |\Sigma_{\mu,\eta}^N| |\mathbb{J}_b^M \backslash \mathrm{Irr}^{\mathrm{top}} X_\eta^M(b)| \\
&= \sum_{\eta \in I_{\mu,b,M}} |\Sigma_{\mu,\eta}^N| \dim V_\eta^{\widehat{M}}(\underline{\lambda}_M(b)) \\
&= \dim V_\mu(\underline{\lambda}_M(b)) \\
&= \dim V_\mu(\underline{\lambda}_G(b)),
\end{aligned}$$

where the second equality follows from Proposition 3.13 dealing with the subperbasic case, and the last one follows from that $\underline{\lambda}_M(b) = \underline{\lambda}_G(b)$. □

### 4.3. The numerical identity

In this subsection we prove the numerical version of Theorem 0.5.

**Proposition 4.5.** *We have $|\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)| = \dim V_\mu(\underline{\lambda}_G(b))$ if $\mu$ is minuscule and $b$ is basic.*

The proof is given in §5.

**Lemma 4.6.** *If $G$ is simple, adjoint and has some nonzero minuscule cocharacter, then each irreducible $\widehat{G}$-module appears in some tensor product of irreducible $\widehat{G}$-modules with minuscule highest weights.*

*Proof.* Let $\mu \in Y^+$. By the assumption on $G$, there exists a dominant and minuscule cocharacter $\mu_\bullet \in Y^d$ for some $d \in \mathbb{Z}_{\geqslant 1}$ such that $\mu \leq |\mu_\bullet|$ and hence $\mathrm{Gr}_\mu \subseteq m_{\mu_\bullet}(\mathrm{Gr}_{\mu_\bullet})$. By Theorem 1.5 (2), $V_\mu^{\widehat{G}}$ appears in $V_{\mu_\bullet}^{\widehat{G}}$ as desired. $\square$

**Remark 4.7.** The condition in Lemma 4.6 is equivalent to that $G$ is simple, adjoint, and any/some of its absolute factors is of classical type or $E_6$ type or $E_7$ type.

**Proposition 4.8.** *We have $|\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)| = \dim V_\mu(\underline{\lambda}_G(b))$ if $b$ is basic.*

First we reduce Proposition 4.8 to the adjoint case.

**Lemma 4.9.** *Proposition 4.8 is true for $G$ if it is true for $G = G_{\mathrm{ad}}$.*

*Proof.* Choose $\omega \in \pi_1(G)$ such that $X_\mu(b)^\omega := X_\mu(b) \cap \mathrm{Gr}^\omega \neq \emptyset$, where $\mathrm{Gr}^\omega$ is the corresponding connected component of $\mathrm{Gr}$. By [3, Corollary 2.4.2] and [17, Proposition 3.1], the natural projection $G \to G_{\mathrm{ad}}$ induces a universal homeomorphism $X_\mu(b)^\omega \overset{\sim}{\to} X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})^{\omega_{\mathrm{ad}}}$, where $\mu_{\mathrm{ad}}$, $b_{\mathrm{ad}}$ and $\omega_{\mathrm{ad}}$ denote the images of $\mu$, $b$ and $\omega$ respectively under the natural projection $G \to G_{\mathrm{ad}}$. Let $\mathbb{J}_b^0$, $\mathbb{J}_{b_{\mathrm{ad}}}^0$ be the kernels of the natural projections $\mathbb{J}_b \to \pi_1(G)$, $\mathbb{J}_{b_{\mathrm{ad}}} \to \pi_1(G_{\mathrm{ad}})$ respectively. Notice that $X_\mu(b) = (\Omega \cap \mathbb{J}_b) X_\mu(b)^\omega$ as $b \in \Omega$ is basic. By Corollary 4.4,

$$\begin{aligned}
\dim V_\mu(\underline{\lambda}_G(b)) &\geqslant |\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)| = |\mathbb{J}_b^0 \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)^\omega| \\
&\geqslant |\mathbb{J}_{b_{\mathrm{ad}}}^0 \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})^{\omega_{\mathrm{ad}}}| = |\mathbb{J}_{b_{\mathrm{ad}}} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})| \\
&= \dim V_{\mu_{\mathrm{ad}}}(\underline{\lambda}_{G_{\mathrm{ad}}}(b_{\mathrm{ad}})) = \dim V_\mu(\underline{\lambda}_G(b)),
\end{aligned}$$

where the second last equality follows by assumption. $\square$

*Proof of Proposition 4.8 by assuming Proposition 4.5.* By Lemma 4.9, we can assume $G$ is adjoint and simple. If the coweight lattice equals the coroot lattice, then $b$ is unramified and the statement is proved in [53, Theorem 4.4.14]. So we will assume $G$ has a nonzero minuscule coweight. By Lemma 4.6, there exists a minuscule and dominant cocharacter $\mu_\bullet \in Y^d$ for some $d \in \mathbb{Z}_{\geqslant 1}$ such that $\mathbb{B}_\mu^{\widehat{G}}$ appears in $\mathbb{B}_{\mu_\bullet}^{\widehat{G}}$, that is, $m_{\mu_\bullet}^\mu \neq 0$. By Proposition 4.5,

$$\dim V_{\mu_\bullet}^{\widehat{G}}(\underline{\lambda}_G(b)) = |\mathbb{J}_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)|$$

$$= \sum_{\upsilon \leq |\mu_\bullet|} m_{\mu_\bullet}^\upsilon |\mathbb{J}_b \backslash \mathrm{Irr} X_\upsilon(b)|$$

$$\leqslant \sum_{\upsilon \leq |\mu_\bullet|} m_{\mu_\bullet}^\upsilon \dim V_\upsilon^{\widehat{G}}(\underline{\lambda}_G(b))$$

$$= \dim V_{\mu_\bullet}^{\widehat{G}}(\underline{\lambda}_G(b)).$$

where the second equality follows from Corollary 1.6, and the inequality follows from Corollary 4.4. Thus $|\mathbb{J}_b \backslash \mathrm{Irr} X_\upsilon(b)| = \dim V_\upsilon(\underline{\lambda}(b))$ if $m_{\mu_\bullet}^\upsilon \neq 0$. $\qquad \square$

**Theorem 4.10.** *We have* $|\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)| = \dim V_\mu(\underline{\lambda}_G(b))$. *In particular,*

$$(P(\breve{F}) \cap \mathbb{J}_b) \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b) \cong \mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b).$$

*Proof.* By Corollary 4.4 we have

$$|(P(\breve{F}) \cap \mathbb{J}_b) \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)| = \sum_{\eta \in I_{\mu,b,M}} |\Sigma_{\mu,\eta}^N| |\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\eta^M(b)|$$

$$= \sum_{\eta \in I_{\mu,b,M}} |\Sigma_{\mu,\eta}^N| \dim V_\eta^{\widehat{M}}(\underline{\lambda}_M(b))$$

$$= \dim V_\mu(\underline{\lambda}_G(b)),$$

where the second equality follows from Proposition 4.8 since $b$ is basic in $M(\breve{F})$. Now the first statement follows by taking $M$ to be the centralizer of $\nu_G(b)$, in which case $P(\breve{F}) \cap \mathbb{J}_b = \mathbb{J}_b^M = \mathbb{J}_b$. The second statement follows from the equality $|(P(\breve{F}) \cap \mathbb{J}_b) \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)| = \dim V_\mu(\underline{\lambda}_G(b)) = |\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b)|$. $\qquad \square$

## 4.4. Decomposition of MV-cycles

Notice that each $\widehat{G}$-crystal restricts to an $\widehat{M}$-crystal. For $\delta \in \mathbb{B}_\mu^{\widehat{G}}$ we denote by $S_\delta^M$ the corresponding Mirković-Vilonen cycle in $\mathrm{Gr}_M$.

**Lemma 4.11.** *Let* $\delta \in \mathbb{B}_\mu^{\widehat{G}}$ *and let* $\eta \in I_{\mu,M}$ *such that* $\delta$ *lies in a highest weight* $\widehat{M}$-*crystal isomorphic to* $\mathbb{B}_\eta^{\widehat{M}}$. *Then there exists a unique irreducible component* $Z_\delta^N \in \Sigma_{\mu,\eta}^N$ *such that*

$$\overline{S_\delta} = \overline{Z_\delta^N * S_\delta^M}.$$

*Here we view* $S_\delta^M$ *as its open dense subset lying in* $\mathrm{Gr}_{\eta,M}^\circ$.

*Proof.* Let $\chi \in I_{\mu,M}$. Let $Z^N \in \Sigma^N_{\mu,\chi}$ and $\xi \in \mathbb{B}^{\widehat{M}}_\chi(\lambda)$ for some $\lambda \in Y$. Then $Z^N * S^M_\xi \subseteq S^\lambda$ is irreducible as the natural projection $Z^N * S^M_\xi \to S^M_\xi$ is a fiber bundle with fibers isomorphic to $Z^N$. Moreover,

$$\begin{aligned}
\dim Z^N * S^M_\xi &= \dim Z^N + \dim S^M_\xi \\
&= \langle \rho, \mu + \chi \rangle - 2\langle \rho_M, \chi \rangle + \langle \rho_M, \chi + \lambda \rangle \\
&= \langle \rho, \mu + \lambda \rangle + \langle \rho_N, \chi - \lambda \rangle \\
&= \langle \rho, \mu + \lambda \rangle \\
&= \dim(S^\lambda \cap \mathrm{Gr}_\mu),
\end{aligned}$$

where the fourth equality follows from that $\chi - \lambda \in \mathbb{Z}\Phi^\vee_M$. Therefore,

$$\overline{Z * S^M_\xi} \in \mathrm{Irr}(\overline{S^\lambda \cap \mathrm{Gr}_\mu}) \cong \mathrm{MV}_\mu(\lambda) = \mathbb{B}^{\widehat{G}}_\mu(\lambda).$$

Hence the map $(Z^N, \xi) \mapsto \overline{Z^N * S^M_\xi}$ gives an embedding

$$\sqcup_\chi (\Sigma^N_{\mu,\chi} \times \mathbb{B}^{\widehat{M}}_\chi) \hookrightarrow \mathrm{MV}_\mu \cong \mathbb{B}^{\widehat{G}}_\mu,$$

which is bijective since $\sum_\chi |\Sigma^N_{\mu,\chi}||\mathbb{B}^{\widehat{M}}_\chi| = |\mathbb{B}^{\widehat{G}}_\mu|$ by Proposition 4.1. Thus there exist unique $\kappa \in I_{\mu,M}$, $\zeta \in \mathbb{B}^{\widehat{M}}_\kappa$ and $Z^N_\delta \in \Sigma^N_{\mu,\kappa}$ such that $\overline{S_\delta} = \overline{Z^N_\delta * S^M_\zeta}$. It remains to show $\zeta = \delta \in \mathbb{B}^{\widehat{M}}_\eta$, that is, $\overline{\pi_P(S_\delta)} = \overline{S^M_\delta}$ with $\pi_P : \mathrm{Gr}_P \to \mathrm{Gr}_M$ the natural projection. In view of the construction of MV cycles using Littelmann's path model [53, Proposition 3.3.12 & 3.3.15], it suffices to consider the case where $\mu$ is a quasi-minuscule cocharacter of $G$. Then the statement follows from the explicit construction in [53, §3.2.5 & Definition 3.3.6]. □

## 4.5. Proof of Theorem 0.5

Take $P = MN$ such that $b$ is superbasic in $M(\breve{F})$. Let $C \in \mathrm{Irr}^{\mathrm{top}} X_\mu(b)$. By Corollary 4.4, there exist $\eta \in I_{\mu,b,M}$ and $\lambda \in Y$ such that $C \subseteq \overline{X^{Z^N, C^{\lambda,M}}_\mu}(b)$ for some $(Z^N, C^{\lambda,M}) \in \Sigma^N_{\mu,\eta} \times \mathrm{Irr} X^{\lambda,M}_\eta$ such that $\overline{C^{\lambda,M}} \in \mathrm{Irr} X^M_\eta(b)$. In particular, $(N(\breve{F})I_M t^\lambda K/K) \cap C$ is open dense in $C$. Let $\gamma^M(\overline{C^{\lambda,M}}) \in \mathbb{B}^{\widehat{M}}_\eta(\underline{\lambda}_M(b))$ be as in Theorem 3.1 such that

$$\overline{t^{-\lambda} H^M(C^{\lambda,M}) t^{b\sigma(\lambda)} K_M/K_M} = \epsilon^M_\lambda \overline{S^M_{\gamma^M(\overline{C^{\lambda,M}})}} \subseteq \mathrm{Gr}_M.$$

By Lemma 4.11, there exists $\gamma(C) \in \mathbb{B}_\mu(\underline{\lambda}(b))$ such that

$$\overline{Z^N * S^M_{\gamma^M(\overline{C^{\lambda,M}})}} = \overline{S_{\gamma(C)}}.$$

By Proposition 4.3 (5),

$$\overline{t^{-\lambda} H^P(C) t^{b\sigma(\lambda)} K/K} = \overline{Z^N * \epsilon^M_\lambda S_{\gamma^M(\overline{C^{\lambda,M}})}} = \epsilon^M_\lambda \overline{Z^N * S_{\gamma^M(\overline{C^{\lambda,M}})}} = \epsilon^M_\lambda \overline{S_{\gamma(C)}}.$$

So the first statement follows.

Let $C' \in \mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ be a conjugate of $C$ under $\mathbb{J}_b$. By Theorem 4.10, $C'$ and $C$ are conjugate under $P(\breve{F}) \cap \mathbb{J}_b$, which, combined with Corollary 4.4, implies that $C' \subseteq \overline{X^{Z^N, C^{\lambda',M}}_\mu}(b)$ for some $\lambda' \in Y$ and $C^{\lambda',M} \in \mathrm{Irr} X^{\lambda',M}_\eta(b)$ such that $\overline{C^{\lambda,M}}$ and $\overline{C^{\lambda',M}}$ are conjugate by $\mathbb{J}^M_b$. By Theorem 3.1, we have

$$\gamma^M(\overline{C^{\lambda',M}}) = \gamma^M(\overline{C^{\lambda,M}})$$

and hence $\gamma(C') = \gamma(C)$. So $\gamma$ is invariant on the $\mathbb{J}_b$-orbits of $\mathrm{Irr}^{\mathrm{top}} X_\mu(b)$.

It remains to show $\gamma$ induces a bijection $\mathbb{J}_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b) \cong \mathbb{B}_\mu(\underline{\lambda}_G(b))$. By Theorem 4.10 it suffices to show it is surjective. Let $\delta \in \mathbb{B}_\mu(\underline{\lambda}_G(b))$. Suppose $\delta \in \mathbb{B}^{\widehat{M}}_\eta(\underline{\lambda}_G(b))$ for some $\eta \in I_{\mu,b,M}$. It follows from Theorem 3.1 that there exists $C^M \in \mathrm{Irr}^{\mathrm{top}} X^M_\eta(b)$ such that

$$\gamma^M(C^M) = \delta \in \mathbb{B}^{\widehat{M}}_\eta(\underline{\lambda}_G(b)).$$

Let $\phi \in Y$ and $C^{\phi,M} \in \mathrm{Irr} X^{\phi,M}_\eta(b)$ such that $\overline{C^{\phi,M}} = C^M$. Let $Z^N_\delta \in \Sigma^N_{\mu,\eta}$ be as in Lemma 4.11 such that $\overline{S_\delta} = \overline{Z^N_\delta * S^M_\delta}$. By the construction in the previous paragraph, we have $\gamma(C) = \delta$ for any $C \in \mathrm{Irr} X^{Z^N_\delta, C^{\phi,M}}_\mu(b)$. So $\gamma$ is surjective as desired.

## 4.6. Proof of Theorem 0.7

Let $C \in \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)$ and $C' \in \mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ for some $\mu \in Y^+$ such that $\overline{C'} = \mathrm{pr}(C)$. One should not confuse with the notation in the previous subsection. Assume $\gamma^{G^d}(C) = \gamma_\bullet = (\gamma_1, \ldots, \gamma_d) \in \mathbb{B}^{\widehat{G}^d}_{\mu_\bullet}$. By Corollary 1.6, it suffices to show that

$$\gamma(C') = \gamma_1 \otimes \cdots \otimes \gamma_d.$$

It follows from Corollary 4.4 that there exist $Z_\bullet^{N^d} = (Z_1^N, \cdots, Z_d^N) \in \Sigma_{\mu_\bullet, \eta_\bullet}^{N^d}$ and $C^{\lambda_\bullet, M^d} \in \mathrm{Irr} X_{\eta_\bullet}^{\lambda_\bullet, M^d}(b_\bullet)$ for some $\eta_\bullet \in I_{\mu_\bullet, M^d}$ and $\lambda_\bullet \in Y^d$ such that $\overline{C^{\lambda_\bullet, M^d}} \in \mathrm{Irr}^{\mathrm{top}} X_{\eta_\bullet}^{M^d}(b_\bullet)$ and

(i) $$C \subseteq \overline{X_{\mu_\bullet}^{Z_\bullet^{N^d}, C^{\lambda_\bullet, M^d}}(b_\bullet)}.$$

Let $\mathcal{Z}_\bullet^{N^d} = (\theta_{\eta_\bullet}^{N^d})^{-1}(Z_\bullet^{N^d}) = (\mathcal{Z}_1^N, \cdots, \mathcal{Z}_d^N)$. By (i) and Proposition 4.3 (5),

(ii) $$\overline{t^{-\lambda_\bullet} H^{P^d}(C) t^{b_\bullet \sigma_\bullet(\lambda_\bullet)}} = \overline{\mathcal{Z}_\bullet^{N^d} * (t^{-\lambda_\bullet} H^{M^d}(C^{\lambda_\bullet, M^d}) t^{b_\bullet \sigma_\bullet(\lambda_\bullet)})}.$$

Set $\lambda_\bullet = (\lambda_1, \ldots, \lambda_d)$, $\lambda_\bullet^\dagger = b_\bullet \sigma_\bullet(\lambda_\bullet) = (\lambda_1^\dagger, \ldots, \lambda_d^\dagger)$ and

$$H^{P^d}(C) = H_1(C) \times \cdots \times H_d(C);$$
$$H^{M^d}(C^{\lambda_\bullet, M^d}) = H_1(C^{\lambda_\bullet, M^d}) \times \cdots \times H_d(C^{\lambda_\bullet, M^d}).$$

Applying Theorem 0.5 (for $C$ and $\overline{C^{\lambda_\bullet, M^d}}$ respectively) and Lemma 4.11 we have

$$\epsilon_{\lambda_\bullet}^{M^d} \overline{S_{\gamma_\bullet}^{G^d}} = \overline{t^{-\lambda_\bullet} H^{P^d}(C) t^{\lambda_\bullet^\dagger} K^d / K^d} = \overline{\mathcal{Z}_\bullet^{N^d} * (t^{-\lambda_\bullet} H^{M^d}(C^{\lambda_\bullet, M^d}) t^{\lambda_\bullet^\dagger}) K^d / K^d}$$
$$= \epsilon_{\lambda_\bullet}^{M^d} \overline{Z_\bullet^{N^d} * S_{\gamma_\bullet}^{M^d}}.$$

In particular, for $1 \leqslant \tau \leqslant d$ we have

(iii) $$\overline{S_{\gamma_\tau}} = (\epsilon_{\lambda_\tau}^M)^{-1} \overline{\mathcal{Z}_\tau^N * (t^{-\lambda_\tau} H_\tau(C^{\lambda_\bullet, M^d}) t^{\lambda_\tau^\dagger}) K / K} = \overline{Z_\tau^N * S_{\gamma_\tau}^M}.$$

Let $\lambda = \mathrm{pr}(\lambda_\bullet) = \lambda_1$. As $\overline{C'} = \overline{\mathrm{pr}(C)} \subseteq \mathrm{Gr}$, we see that $N(\breve{F}) I_M t^\lambda K / K \cap C'$ is open dense in $C'$. By Theorem 0.5,

$$\epsilon_\lambda^M \overline{S_{\gamma(C')}}$$
$$= \overline{t^{-\lambda} H^P(C') t^{b\sigma(\lambda)} K / K}$$
$$= \overline{t^{-\lambda_1} H_1(C) t^{\lambda_1^\dagger} \cdots t^{-\lambda_d} H_d(C) t^{\lambda_d^\dagger} K / K}$$
$$= \overline{(\mathcal{Z}_1^N * (t^{-\lambda_1} H_1(C^{\lambda_\bullet, M^d}) t^{\lambda_1^\dagger})) \cdots (\mathcal{Z}_d^N * (t^{-\lambda_d} H_d(C^{\lambda_\bullet, M^d}) t^{\lambda_d^\dagger})) K / K}$$
$$= \overline{m(\mathcal{Z}_1^N * (t^{-\lambda_1} H_1(C^{\lambda_\bullet, M^d}) t^{\lambda_1^\dagger}) \times_K \cdots \times_K (\mathcal{Z}_d^N * (t^{-\lambda_d} H_d(C^{\lambda_\bullet, M^d}) t^{\lambda_d^\dagger})) K / K)}$$
$$= \overline{m(\epsilon_{\lambda_1}^M(Z_1^N * S_{\gamma_1}^M) \tilde{\times} \cdots \tilde{\times} (Z_d^N * S_{\gamma_d}^M))}$$
$$= \epsilon_{\lambda_1}^M \overline{m(S_{\gamma_1} \tilde{\times} \cdots \tilde{\times} S_{\gamma_d})}$$

$$= \epsilon_{\lambda_1}^M \overline{S_{\gamma_1} \star \cdots \star S_{\gamma_d}}$$
$$= \epsilon_\lambda^M \overline{S_{\gamma_1 \otimes \cdots \otimes \gamma_d}},$$

where $m : G(\breve{F}) \times_K \cdots \times_K G(\breve{F}) \times_K \mathrm{Gr} \to \mathrm{Gr}$ is the usual convolution map; the second equality follows from Lemma 2.10; the third one follows from (ii) and that $\lambda_\bullet^\dagger = (\lambda_1^\dagger, \ldots, \lambda_d^\dagger) = (\lambda_2, \ldots, \lambda_d, b\sigma(\lambda_1))$; the fifth one follows from Lemma 4.2, Theorem 3.1 and (iii). So $\gamma(C') = \gamma_1 \otimes \cdots \otimes \gamma_d$ as desired.

## 5. Proof of Proposition 4.5

We keep the notations in §1. Let $\mu \in Y^+$ be minuscule, and let $b \in G(\breve{F})$ be basic which is a lift of an element in $\Omega$. To prove Proposition 4.5, we assume by Lemma 4.9 that $G$ is simple and adjoint. Then $\sigma$ acts transitively on the connected components of (the Dynkin diagram of) $\mathbb{S}_0$. Let $d$ be the number of connected components of $\mathbb{S}_0$.

For simplicity, we also denote by $\tilde{w} \in \tilde{W} \cap \mathbb{J}_b = \{x \in \tilde{W}; b\sigma(x)b^{-1} = x\}$ some lift of $\tilde{w}$ in $N_T(\breve{F})$ that lies in $\mathbb{J}_b$.

### 5.1. Orthogonal subset of roots

We say a subset $D \subseteq \Phi$ is strongly orthogonal if $\beta' \pm \beta \notin \Phi$ for any $\beta', \beta \in D$. In particular, if $D$ is strongly orthogonal, then it is orthogonal, that is, $\langle \beta', \beta^\vee \rangle = 0$ for any $\beta \neq \beta' \in D$.

Let $\alpha \in \Phi$. Set $\mathcal{O}_\alpha = \{\alpha^i; i \in \mathbb{Z}\}$ and $\mathcal{O}_{\tilde{\alpha}} = \{\tilde{\alpha}^i; i \in \mathbb{Z}\}$, where $\alpha^i$ and $\tilde{\alpha}^i$ are as in §2.2. Let $W_{\mathcal{O}_{\tilde{\alpha}}}$ be the parabolic subgroup of $\tilde{W}$ generated by $s_{\tilde{\beta}}$ for $\beta \in \mathcal{O}_\alpha$. Recall that $\Pi$ is the set of minus simple roots and highest roots of $\Phi$.

**Lemma 5.1.** *Let $\alpha \in \Pi$ such that $W_{\mathcal{O}_{\tilde{\alpha}}}$ is finite. Let $\tilde{w}_\alpha$ be the longest element of $W_{\mathcal{O}_{\tilde{\alpha}}}$. Then $W_{\mathcal{O}_{\tilde{\alpha}}} \cap \mathbb{J}_b = \{1, \tilde{w}_\alpha\}$ and one of the following cases occurs:*

*(1) $\langle \alpha^d, \alpha^\vee \rangle = -1$, $|\mathcal{O}_\alpha| = 2d$, $\mathcal{O}_{\alpha + \alpha^d}$ is strongly orthogonal (as $|\mathcal{O}_{\alpha + \alpha^d}| = d$) and $\tilde{w}_\alpha = \prod_{\xi \in \mathcal{O}_{\alpha + \alpha^d}} s_{\tilde{\xi}}$;*

*(2) $\mathcal{O}_\alpha$ is strongly orthogonal and hence $\tilde{w}_\alpha = \prod_{\beta \in \mathcal{O}_\alpha} s_{\tilde{\beta}}$.*

*In particular, any affine reflection of $W^a \cap \mathbb{J}_b$ is equal to $\prod_{c \in \mathcal{O}_a} s_c$ for some $a = (\gamma, k) \in \Phi \times \mathbb{Z} = \tilde{\Phi}$ such that $\mathcal{O}_\gamma$ is strongly orthogonal.*

*Proof.* The first statement follows from a case-by-case analysis. The "In particular" part follows by noticing that each reflection of $W^a \cap \mathbb{J}_b$ is conjugate to some $\tilde{w}_\alpha$ as in the first statement. $\qquad \square$

## 5.2. Characterization of $\mathcal{A}_{\mu,b}^{\mathrm{top}}$

Let $\lambda \in Y$. Let $X_\mu^\lambda(b) = It^\lambda K/K \cap X_\mu(b)$, $\mathcal{A}_{\mu,b} = \mathcal{A}_{\mu,b}^G$, $\mathcal{A}_{\mu,b}^{\mathrm{top}} = \mathcal{A}_{\mu,b}^{\mathrm{top},G}$ and $R(\lambda) = R_{\mu,b}^{\mathrm{top}}(\lambda)$ be as in §2.2. By Proposition 2.9, $\lambda \in \mathcal{A}_{\mu,b}$ if and only if $\lambda^\natural = -\lambda + b\sigma(\lambda)$ is conjugate to $\mu$ by $W_0$.

Let $V = Y \otimes_{\mathbb{Z}} \mathbb{R}$ and $V^{p(b\sigma)} = \{v \in V; p(b\sigma)(v) = v\}$. Define

$$V_{\mathrm{gen}}^{p(b\sigma)} = \{v \in V^{p(b\sigma)}; \langle \alpha, v \rangle = 0 \Leftrightarrow \langle \alpha, V^{p(b\sigma)} \rangle = 0, \forall \alpha \in \Phi\},$$

which is open dense in $V^{p(b\sigma)}$. Notice that $V_{\mathrm{gen}}^{p(b\sigma)} \cap Y \neq \emptyset$. Let $M_b \supseteq T$ be the Levi subgroup with root system $\{\alpha \in \Phi; \langle \alpha, V^{p(b\sigma)} \rangle = 0\}$. By definition, for any $v \in V_{\mathrm{gen}}^{p(b\sigma)}$ the centralizer $M_v$ (see §1.3) of $v$ in $G$ coincides with $M_b$.

Fix $v \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$. Denote by $\bar{v}$ the unique dominant $W_0$-conjugate of $v$. Let $z$ be the minimal element of $W_0$ such that $z(v) = \bar{v}$. Let $N_v = \prod_{\alpha \in \Phi; \langle \alpha, v \rangle > 0} U_\alpha$. Set $M = M_{\bar{v}} = {}^z M_v = {}^z M_b$ and $b_M = z b \sigma(z)^{-1}$. By [14, Lemma 3.1], $b_M$ is a lift of some element in $\Omega_M$, and is superbasic in $M(\breve{F})$.

**Lemma 5.2.** *Let $\lambda \in \mathcal{A}_{\mu,b}$ and $\alpha \in \Phi - \Phi_{M_b}$. Then $\mathcal{O}_\alpha \cap \mathcal{O}_{-\alpha} = \emptyset$ and*

$$|R(\lambda) \cap (\mathcal{O}_\alpha \cup \mathcal{O}_{-\alpha})| \leqslant \frac{1}{2} \sum_{\beta \in \mathcal{O}_\alpha} |\langle \beta, \lambda^\natural \rangle|,$$

*where the equality holds if and only if either $\lambda_\beta \geqslant 0$ for $\beta \in \mathcal{O}_\alpha$ or $\lambda_\beta \leqslant -1$ for $\beta \in \mathcal{O}_\alpha$.*

*Proof.* As $\mu$ is minuscule, it follows from Proposition 2.9 and Lemma 2.7 (1) that $\lambda^\natural$ is minuscule and that

(i)                      $\lambda_{\gamma^{-1}} - \lambda_\gamma = \langle \gamma, \lambda^\natural \rangle \in \{0, \pm 1\}$ for $\gamma \in \Phi$.

By assumption, we have $\langle \alpha, v \rangle \neq 0$ and hence $\mathcal{O}_\alpha \cap \mathcal{O}_{-\alpha} = \emptyset$. By symmetry, we may assume $\lambda_\alpha \geqslant 0$ and there exist integers

$$0 = b_0 \leqslant c_1 < b_1 \leqslant \cdots \leqslant c_r < b_r = |\mathcal{O}_\alpha|$$

such that for $1 \leqslant k \leqslant r$ we have

$$\lambda_{\alpha^i} < 0 \text{ for } b_{k-1} + 1 \leqslant i \leqslant c_k \text{ and } \lambda_{\alpha^j} \geqslant 0 \text{ for } c_k + 1 \leqslant j \leqslant b_k.$$

It follows from (i) that

(ii)     if $b_{k-1} < c_k$, then $\lambda_{\alpha^{b_{k-1}+1}} = \lambda_{\alpha^{c_k}} = -1$ and $\lambda_{\alpha^{c_k+1}} = \lambda_{\alpha^{b_{k-1}}} = 0$.

If $b_{k-1} < c_k$ for some $1 \leqslant k \leqslant r$, we have

$$
\begin{aligned}
&|R(\lambda) \cap \{\pm \alpha^i; b_{k-1} + 1 \leqslant i \leqslant c_k\}| \\
&= |R(\lambda) \cap \{-\alpha^i; b_{k-1} + 1 \leqslant i \leqslant c_k\}| \\
&= |\{b_{k-1} + 1 \leqslant i \leqslant c_k; \lambda_{-\alpha^{i-1}} \geqslant 0, \lambda_{-\alpha^i} - \lambda_{-\alpha^{i-1}} = 1\}| \\
&= |\{b_{k-1} + 1 \leqslant i \leqslant c_k; \lambda_{\alpha^{i-1}} \leqslant -1, \lambda_{\alpha^{i-1}} - \lambda_{\alpha^i} = 1\}| \\
&= |\{b_{k-1} + 1 \leqslant i \leqslant c_k; \lambda_{\alpha^{i-1}} - \lambda_{\alpha^i} = 1\}| - 1 \\
&= -\frac{1}{2} + \frac{1}{2} \sum_{i=b_{k-1}+1}^{c_k} |\lambda_{\alpha^{i-1}} - \lambda_{\alpha^i}| \\
&= -\frac{1}{2} + \frac{1}{2} \sum_{i=b_{k-1}+1}^{c_k} |\langle \alpha^i, \lambda^\natural \rangle|,
\end{aligned}
$$

where the first equality follows from that $\lambda_{\alpha^i} < 0$ and hence $\alpha^i \notin R(\lambda)$ for $b_{k-1} + 1 \leqslant i \leqslant c_k$; the third one follows from that $\lambda_{-\gamma} = -1 - \lambda_\gamma$ for $\gamma \in \Phi$; the fourth one follows from that $\lambda_{\alpha^i} \leqslant -1$ for $b_{k-1} + 1 \leqslant i \leqslant c_k$ but $1 + \lambda_{\alpha^{b_{k-1}+1}} = \lambda_{\alpha^{b_{k-1}}} = 0$ by (ii); the fifth one follows from (i) and the equality $\sum_{i=b_{k-1}+1}^{c_k} \lambda_{\alpha^{i-1}} - \lambda_{\alpha^i} = \lambda_{\alpha^{b_{k-1}}} - \lambda_{\alpha^{c_k}} = 1$ by (ii); the last one follows from (i).

Similarly, for $1 \leqslant k \leqslant r$,

$$
\begin{aligned}
&|R(\lambda) \cap \{\pm \alpha^i; c_k + 1 \leqslant i \leqslant b_k\}| \\
&= |\{c_k + 1 \leqslant i \leqslant b_k; \lambda_{\alpha^i} \geqslant 1, \lambda_{\alpha^i} - \lambda_{\alpha^{i-1}} = 1\}| \\
&= \begin{cases} |\{c_k + 1 \leqslant i \leqslant b_k; \lambda_{\alpha^i} - \lambda_{\alpha^{i-1}} = 1\}|, & \text{if } b_{k-1} = c_k; \\ |\{c_k + 1 \leqslant i \leqslant b_k; \lambda_{\alpha^i} - \lambda_{\alpha^{i-1}} = 1\}| - 1, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{2} \sum_{i=c_k+1}^{b_k} |\langle \alpha^i, \lambda^\natural \rangle|, & \text{if } b_{k-1} = c_k \\ -\frac{1}{2} + \frac{1}{2} \sum_{i=c_k+1}^{b_k} |\langle \alpha^i, \lambda^\natural \rangle|, & \text{otherwise.} \end{cases}
\end{aligned}
$$

where the second equality follows from that $\lambda_{\alpha^i} \geqslant 0$ for $c_k + 1 \leqslant i \leqslant b_k$ and that $\lambda_{\alpha^{c_k}} \geqslant 0$ if and only if $b_{k-1} = c_k$.

Therefore,

$$
|R(\lambda) \cap (\mathcal{O}_\alpha \cup \mathcal{O}_{-\alpha})| \leqslant \frac{1}{2} \sum_{\beta \in \mathcal{O}_\alpha} |\langle \beta, \lambda^\natural \rangle|,
$$

where the equality holds if and only if $b_{k-1} = c_k$ for $1 \leqslant k \leqslant r$, that is, $\lambda_\beta \geqslant 0$ for $\beta \in \mathcal{O}_\alpha$. The proof is finished. $\qquad \square$

**Lemma 5.3.** *Let $\alpha \in \Pi$ (see §1.2). Then $W_{\mathcal{O}_{\bar{\alpha}}}$ is infinite if and only if $\mathcal{O}_\alpha = \Pi$. Moreover, in this case, $M_b = G$.*

*Proof.* It follows from a case-by-case analysis on the Dynkin diagram of $\mathbb{S}_0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

For $\eta \in Y$ let $\overline{\eta}^M$ be the unique $W_M$-conjugate of $\eta$ which is $M$-dominant.

**Lemma 5.4.** *For $\lambda \in \mathcal{A}_{\mu,b}$ the map $\alpha \mapsto z(\alpha)$ gives a bijection $R(\lambda) \cap \Phi_{M_b} \cong R^M_{z(\lambda^\natural),b_M}(z(\lambda))$. As a consequence, $|R(\lambda) \cap \Phi_{M_b}| \leqslant \dim X^M_{z(\lambda^\natural)}(b_M)$. Here the subset $R^M_{z(\lambda^\natural),b_M}(z(\lambda)) \subseteq \Phi_M$ is defined in §2.2 for $G = M$, $b = b_M$, and $\mu = \overline{z(\lambda^\natural)}$.*

*Proof.* Since $z(\Phi^+_{M_b}) = \Phi^+_M$, we have $\lambda_\alpha = z(\lambda)_{z(\alpha)}$ for $\alpha \in \Phi_{M_v}$. Hence the first statement follows. The second statement follows from Proposition 2.9 that $|R^M_{z(\lambda^\natural),b_M}(z(\lambda))| = \dim X^{z(\lambda),M}_{z(\lambda^\natural)}(b_M)$. $\qquad\qquad\qquad$ $\square$

**Corollary 5.5.** *Let $\lambda \in \mathcal{A}_{\mu,b}$. Then $\lambda \in \mathcal{A}^{\mathrm{top}}_{\mu,b}$ if and only if (1) $z(\lambda) \in \mathcal{A}^{M,\mathrm{top}}_{z(\lambda^\natural),b_M}$ and (2) for each $\alpha \in \Phi - \Phi_{M_b}$, either $\lambda_\beta \geqslant 0$ for $\beta \in \mathcal{O}_\alpha$ or $\lambda_\beta \leqslant -1$ for $\beta \in \mathcal{O}_\alpha$. Here $\mathcal{A}^{M,\mathrm{top}}_{z(\lambda^\natural),b_M}$ is defined in §5.2 for $G = M$, $b = b_M$, and $\mu = \overline{z(\lambda^\natural)}^M$.*

*Proof.* As $\lambda^\natural$ is conjugate to $\mu$, we have

$$|\langle \rho, \mu \rangle| = \frac{1}{2} \sum_{\alpha \in \Phi^+_{M_v} \cup \Phi_{N_v}} |\langle \alpha, \lambda^\natural \rangle|.$$

Therefore,

$$
\begin{aligned}
\dim X^\lambda_\mu(b) &= |R(\lambda)| \\
&= |R(\lambda) \cap \Phi_{M_b}| + |R(\lambda) \cap (\Phi - \Phi_{M_b})| \\
&= |R(\lambda) \cap \Phi_{M_b}| + \sum_{\mathcal{O}} |R(\lambda) \cap \pm\mathcal{O}| \\
&\leqslant |R(\lambda) \cap \Phi_{M_b}| + \frac{1}{2} \sum_{\alpha \in \Phi_{N_v}} |\langle \alpha, \lambda^\natural \rangle| \\
&= |R(\lambda) \cap \Phi_{M_b}| + \langle \rho, \overline{\lambda^\natural} \rangle - \langle \rho_M, \overline{z(\lambda^\natural)}^M \rangle \\
&\leqslant \dim X^M_{z(\lambda^\natural)}(b_M) + \langle \rho, \mu \rangle - \langle \rho_M, \overline{z(\lambda^\natural)}^M \rangle \\
&= \dim X_\mu(b),
\end{aligned}
$$

where $\mathcal{O}$ ranges over $p(b\sigma)$-orbits of $\Phi_{N_v}$, and moreover, by Lemma 5.2 and Lemma 5.4 the equality holds if and only if the conditions (1) and (2) hold. The proof is finished. $\qquad\square$

## 5.3.  The action of $W_{\mathcal{O}_{\tilde{\alpha}}} \cap \mathbb{J}_b$ on $\mathrm{Irr}X_\mu(b)$

Notice that $\mathbb{J}_b$ is generated by $I \cap \mathbb{J}_b$, $\Omega \cap \mathbb{J}_b$ and $W_{\mathcal{O}_{\tilde{\alpha}}} \cap \mathbb{J}_b$ for $\alpha \in \Pi$. In this subsection we study the action of $W_{\mathcal{O}_{\tilde{\alpha}}} \cap \mathbb{J}_b$ on $\mathrm{Irr}X_\mu(b)$. Assume that $W_{\mathcal{O}_{\tilde{\alpha}}}$ is finite and let $\tilde{w}_\alpha$ be the longest element of $W_{\mathcal{O}_{\tilde{\alpha}}}$.

**Lemma 5.6.** *Let $\alpha, \tilde{w}_\alpha$ be as in §5.3. Then $\{\lambda_\beta; \beta \in \mathcal{O}_\alpha\} = \{-\tilde{w}_\alpha(\lambda)_\beta; \beta \in \mathcal{O}_\alpha\}$ for $\lambda \in Y$.*

*Proof.* Recall that $\lambda_\beta = -\tilde{\beta}(\lambda)$ for $\beta \in \Phi$. The statement follows by noticing that $\tilde{w}_\alpha$ sends $\mathcal{O}_{\tilde{\alpha}}$ to $-\mathcal{O}_{\tilde{\alpha}}$. $\qquad\square$

**Lemma 5.7.** *Let $\alpha, \tilde{w}_\alpha$ be as in §5.3. Let $\lambda \in Y$ such that either $\lambda_\beta \geqslant 1$ for $\beta \in \mathcal{O}_\alpha$ or $\lambda_\beta \leqslant -1$ for $\beta \in \mathcal{O}_\alpha$. For $\gamma \in \Phi$ with $\lambda_\gamma \geqslant 0$ we have $\tilde{w}_\alpha(\lambda)_{p(\tilde{w}_\alpha)(\gamma)} \geqslant 0$. If, moreover, $\lambda \in \mathcal{A}_{\mu,b}$, then $p(\tilde{w}_\alpha)R(\lambda) = R(\tilde{w}_\alpha(\lambda))$.*

*Proof.* We argue by contradiction. Set $\tilde{w} = \tilde{w}_\alpha$, $\lambda' = \tilde{w}(\lambda)$ and $\gamma' = p(\tilde{w})(\gamma)$. Suppose $\lambda_\gamma \geqslant 0$ but $\lambda'_{\gamma'} < 0$, that is,

(i) $\langle \gamma, \lambda \rangle \geqslant 0$, and $\gamma < 0$ if $\langle \gamma, \lambda \rangle = 0$;

(ii) $\langle \gamma', \lambda' \rangle \leqslant 0$, and $\gamma' > 0$ if $\langle \gamma', \lambda' \rangle = 0$.

By assumption and Lemma 5.1, we have

(iii) $\lambda_\beta \geqslant 1$ or $\lambda_\beta \leqslant -1$ if $\beta \in \Phi$ is a sum of roots in $\mathcal{O}_\alpha$.

Case(1): $\langle \alpha^d, \alpha^\vee \rangle \neq -1$. Then $\mathcal{O}_\alpha$ is orthogonal and $\tilde{w} = \tilde{w}_\alpha = \prod_{\beta \in \mathcal{O}_\alpha} s_{\tilde{\beta}}$. So $\lambda' = \tilde{w}(\lambda) = p(\tilde{w})(\lambda - \sum_{\beta \in \mathcal{O}_\alpha \cap \Phi^+} \beta^\vee)$ and hence $\langle \gamma', \lambda' \rangle = \langle \gamma, \lambda - \sum_{\beta \in E \cap \Phi^+} \beta^\vee \rangle$, where $E = \{\beta \in \mathcal{O}_\alpha; \langle \gamma, \beta^\vee \rangle \neq 0\}$. If $E \subseteq \Phi^-$, then $\langle \gamma', \lambda' \rangle = \langle \gamma, \lambda \rangle$. By (i) and (ii) this implies that $\gamma' > 0$, $\gamma < 0$ and $\langle \gamma, \lambda \rangle = 0$. As $E$ consists of minus simple roots and $\gamma' = p(\tilde{w})(\gamma) = (\prod_{\beta \in E} s_\beta)(\gamma)$, we deduce that $\gamma$ is a sum of roots in $E$, contradicting (iii) since $\lambda_\gamma = 0$. Thus $E$ contains a unique highest root $\theta$ of $\Phi^+$ and $\langle \gamma', \lambda' \rangle = \langle \gamma, \lambda - \theta^\vee \rangle$. By (i), (ii) and that $\langle \gamma, \theta^\vee \rangle \neq 0$, we have $\langle \gamma, \theta^\vee \rangle \geqslant 1$. If $\gamma = \theta \in \mathcal{O}_\alpha$, then $\gamma' = -\theta < 0$ (since $\mathcal{O}_\alpha$ is orthogonal). As $\lambda_\gamma \geqslant 0$ and $\gamma \in \mathcal{O}_\alpha$, by (iii) we have $\langle \lambda, \gamma \rangle = \lambda_\gamma + 1 \geqslant 2$. So $\lambda'_{\gamma'} = \langle \gamma, \lambda \rangle - 2 \geqslant 2 - 2 = 0$, which is a contradiction. So $\gamma \neq \pm\theta$ and hence $\langle \gamma, \theta^\vee \rangle = 1$ (since $\theta$ is a long root). By (i) and (ii) we have $0 \leqslant \langle \gamma, \lambda \rangle \leqslant 1$. If $\langle \gamma, \lambda \rangle = 1$, then $\gamma' = (\prod_{\beta \in E - \{\theta\}} s_\beta)(\gamma - \theta) > 0$ by (ii). As $\gamma - \theta \in \Phi^-$, $\gamma - \theta$ is a sum of roots in $E - \{\theta\}$, contradicting that $\mathcal{O}$ is strongly orthogonal by Lemma 5.1 (2). So $\langle \gamma, \lambda \rangle = 0$ and hence $\gamma < 0$ by (i). In particular, $\langle \gamma, \theta^\vee \rangle \leqslant 0$ as $\theta^\vee$ is dominant, which contradicts that $\langle \gamma, \theta^\vee \rangle = 1$.

Case(2): $\langle \alpha^d, \alpha^\vee \rangle = -1$. Let $\xi = \alpha + \alpha^d$. Then $|\mathcal{O}_\xi| = d$ and $\tilde{w} = \tilde{w}_\alpha = \prod_{\beta \in \mathcal{O}_\xi} s_{\tilde{\beta}}$ by Lemma 5.1. So $\lambda' = \tilde{w}(\lambda) = p(\tilde{w})(\lambda - \sum_{\beta \in \mathcal{O}_\xi \cap \Phi^+} \beta^\vee)$ and hence $\langle \gamma', \lambda' \rangle = \langle \gamma, \lambda - \sum_{\beta \in E \cap \Phi^+} \beta^\vee \rangle$, where $E = \{\beta \in \mathcal{O}_\xi; \langle \gamma, \beta^\vee \rangle \neq 0\}$. Notice that $E$ consists of at most one element. If $E = \emptyset$, then $\gamma = \gamma'$ and $\langle \gamma', \lambda' \rangle = \langle \gamma, \lambda \rangle$, contradicting (i) and (ii). So $E = \{\xi^{i_0}\}$ for some $1 \leqslant i_0 \leqslant d$. If $\xi^{i_0} < 0$, then $\alpha^{i_0}, \alpha^{i_0+d}$ are both minus simple roots and $\langle \gamma', \lambda' \rangle = \langle \gamma, \lambda \rangle$. By (i) and (ii) we have $\gamma' > 0$, $\gamma < 0$ and $\langle \gamma, \lambda \rangle = 0$. As $\gamma' = s_{\xi^{i_0}}(\gamma) = s_{\alpha^{i_0}} s_{\alpha^{i_0+d}} s_{\alpha^{i_0}}(\gamma)$, we deduce that $\gamma$ is a sum of roots in $\{\alpha^{i_0}, \alpha^{i_0+d}\}$, contradicting (iii). So $\xi^{i_0} > 0$ and $\langle \gamma', \lambda' \rangle = \langle \gamma, \lambda - (\xi^{i_0})^\vee \rangle$. Moreover, as $\xi^{i_0} = \alpha^{i_0} + \alpha^{i_0+d}$, exactly one of $\{\alpha^{i_0}, \alpha^{i_0+d}\}$ is a positive highest root. By symmetry, we can assume $\alpha^{i_0} < 0$ and $\alpha^{i_0+d} > 0$. By (ii) and that $\langle \gamma, (\xi^{i_0})^\vee \rangle \neq 0$ we have $\langle \gamma, (\xi^{i_0})^\vee \rangle \geqslant 1$. If $\gamma = \xi^{i_0}$, then $\gamma' = -\xi^{i_0} < 0$. By (iii) we have $\langle \lambda, \gamma \rangle = \lambda_\gamma + 1 \geqslant 2$ and hence $\lambda'_{\gamma'} = \langle \gamma, \lambda \rangle - 2 \geqslant 2 - 2 = 0$, which is a contradiction. So $\gamma \neq \pm \xi^{i_0}$ and $\langle \gamma, (\xi^{i_0})^\vee \rangle = 1$ (since $\xi^{i_0}$ is a long root), which means $\gamma' = s_{\xi^{i_0}}(\gamma) = \gamma - \xi^{i_0}$. By (i) and (ii) we have $0 \leqslant \langle \gamma, \lambda \rangle \leqslant 1$. If $\langle \gamma, \lambda \rangle = 1$, then $\langle \gamma', \lambda' \rangle = 0$ and hence $0 < \gamma' = \gamma - \xi^{i_0} = (\gamma - \alpha^{i_0+d}) - \alpha^{i_0} \leqslant -\alpha^{i_0}$ by (ii), where the last inequality follows from that $\alpha^{i_0+d}$ is a positive highest root. As $-\alpha_{i_0}$ is a simple root, we deduce that $\gamma' = -\alpha^{i_0}$ and hence $\gamma = \alpha^{i_0+d} \in \mathcal{O}_\alpha$, contradicting (iii) since $\lambda_\gamma = 0$. So $\langle \gamma, \lambda \rangle = 0$ and hence $\gamma < 0$ by (i), which together with the equality $\gamma' = \gamma - \xi^{i_0} \in \Phi$ implies that $0 \leqslant \gamma' + \alpha^{i_0+d} = \gamma - \alpha^{i_0} < -\alpha^{i_0}$. So $\gamma - \alpha^{i_0} = 0$, that is, $\gamma = \alpha^{i_0} \in \mathcal{O}_\alpha$, which contradicts (iii) since $\lambda_\gamma = 0$. The first statement is proved.

Let $\gamma \in R(\lambda)$, that is, $\langle \gamma, \lambda^\natural \rangle = -1$ and $\lambda_{\gamma^{-1}} \geqslant 0$. By Lemma 2.7 and the first statement of the lemma we have

$$\langle p(\tilde{w})(\gamma), \tilde{w}(\lambda)^\natural \rangle = \langle p(\tilde{w})(\gamma), p(\tilde{w})(\lambda^\natural) \rangle = \langle \gamma, \lambda^\natural \rangle = -1$$

and $\tilde{w}(\lambda)_{p(\tilde{w})(\gamma)^{-1}} = \tilde{w}(\lambda)_{p(\tilde{w})(\gamma^{-1})} \geqslant 0$, that is, $p(\tilde{w})(\gamma) \in R(\tilde{w}(\lambda))$ and hence $p(\tilde{w})R(\lambda) \subseteq R(\tilde{w}(\lambda))$. By symmetry (see Lemma 5.6), we have $p(\tilde{w})R(\lambda) = R(\tilde{w}(\lambda))$. The second statement follows. $\qquad\square$

**Lemma 5.8.** *Let $\alpha, \tilde{w}_\alpha$ be as in §5.3. For $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ we have*

*(1) either $\lambda_\beta \geqslant 0$ for $\beta \in \mathcal{O}_\alpha$ or $\lambda_\beta \leqslant -1$ for $\beta \in \mathcal{O}_\alpha$;*

*(2) if $\lambda \neq \tilde{w}_\alpha(\lambda) \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$, then either $\lambda_\beta \geqslant 1$ for $\beta \in \mathcal{O}_\alpha$, or $\lambda_\beta \leqslant -1$ for $\beta \in \mathcal{O}_\alpha$;*

*(3) if $\lambda' \in W_{\mathcal{O}_{\tilde{\alpha}}}(\lambda) \cap \mathcal{A}_{\mu,b}^{\mathrm{top}}$, then $\lambda' = \lambda$ or $\lambda' = \tilde{w}_\alpha(\lambda)$.*

*Proof.* The first statement follows from Lemma 5.3 and Corollary 5.5 (2).

Suppose $\lambda \neq \tilde{w}_\alpha(\lambda) \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$. By Lemma 5.6 we have

$$\{\lambda_\beta; \beta \in \mathcal{O}_\alpha\} = \{-\tilde{w}_\alpha(\lambda)_\beta; \beta \in \mathcal{O}_\alpha\},$$

which, together with (1), implies that either $\lambda_\beta = 0$ for $\beta \in \mathcal{O}_\alpha$ or $\lambda_\beta \geqslant 1$ for $\beta \in \mathcal{O}_\alpha$ or $\lambda_\beta \leqslant -1$ for $\beta \in \mathcal{O}_\alpha$. By Lemma 1.3 (2), the first case implies that $\lambda = \tilde{w}_\alpha(\lambda)$, contradicting our assumption. So the statement (2) follows.

Suppose $\lambda' \in W_{\mathcal{O}_{\tilde{\alpha}}}(\lambda) \cap \mathcal{A}_{\mu,b}^{\mathrm{top}}$. By (1) and the equality $\chi_\beta = -\tilde{\beta}(\chi)$ for $\beta \in \Phi$ and $\chi \in Y$, we see that $\lambda$ and $\lambda'$ are contained in the union of

$$\{y \in Y_\mathbb{R}; \tilde{\beta}(y) \leqslant 0, \beta \in \mathcal{O}_\alpha\} \text{ and } \{y \in Y_\mathbb{R}; \tilde{\beta}(y) > 0, \beta \in \mathcal{O}_\alpha\},$$

which are the closed anti-dominant Weyl chamber and the open dominant Weyl chamber for $W_{\mathcal{O}_{\tilde{\alpha}}}$ respectively. Therefore, as $\lambda' \in W_{\mathcal{O}_{\tilde{\alpha}}}(\lambda)$, we see that $\lambda = \lambda'$ if they are both dominant or both anti-dominant for $W_{\mathcal{O}_{\tilde{\alpha}}}$, and $\lambda' = \tilde{w}_\alpha(\lambda)$ otherwise. The statement (3) is proved. $\qquad\square$

**Corollary 5.9.** *Let $\alpha, \tilde{w}_\alpha$ be as in §5.3. For $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ we have*

*(1) $\tilde{w}_\alpha \overline{X_\mu^\lambda(b)} = \overline{X_\mu^\lambda(b)}$ if $\lambda_\beta = 0$ for some $\beta \in \mathcal{O}_\alpha$;*

*(2) $\tilde{w}_\alpha X_\mu^\lambda(b) \subseteq X_\mu^{\tilde{w}_\alpha(\lambda)}(b)$ if $\lambda_\beta \leqslant -1$ for $\beta \in \mathcal{O}_\alpha$.*

*Proof.* Let $\lambda' \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ such that $\mathrm{Irr}\overline{X_\mu^{\lambda'}}$ intersects $\mathrm{Irr}(\tilde{w}_\alpha \overline{X_\mu^\lambda(b)})$. By Lemma 1.1, $\lambda' \in W_{\mathcal{O}_{\tilde{\alpha}}}(\lambda)$. Thus $\lambda' = \lambda$ or $\lambda' = \tilde{w}_\alpha(\lambda)$ by Lemma 5.8 (3). If $\lambda_\beta = 0$ for some $\beta \in \mathcal{O}_\alpha$, then $\lambda' = \lambda$ by Lemma 5.8 (2). So the statement (1) follows.

Suppose $\lambda_\beta \leqslant -1$ for $\beta \in \mathcal{O}_\alpha$. Then $s_{\tilde{\beta}}t^\lambda > t^\lambda$ for $\beta \in \mathcal{O}_\alpha$. Thus $\ell(\tilde{w}_\alpha t^\lambda) = \ell(\tilde{w}_\alpha) + \ell(t^\lambda)$ and

$$\tilde{w}_\alpha I t^\lambda K \subseteq I \tilde{w}_\alpha t^\lambda K = I t^{\tilde{w}_\alpha(\lambda)} K.$$

So $\tilde{w}_\alpha X_\mu^\lambda(b) \subseteq X_\mu^{\tilde{w}_\alpha(\lambda)}(b)$ and the statement (2) follows. $\qquad\square$

## 5.4. Equivalence relations on $\mathcal{A}_{\mu,b}^{\mathrm{top}}$ and $\mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$

For $\lambda, \lambda' \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$, we write $\lambda \sim \lambda'$ if $\mathbb{J}_b \mathrm{Irr}\overline{X_\mu^\lambda(b)} = \mathbb{J}_b \mathrm{Irr}\overline{X_\mu^{\lambda'}(b)}$. Notice that $\mathbb{J}_b \mathrm{Irr}X_\mu^\lambda(b)$ -is a single $\mathbb{J}_b$-orbit of $\mathrm{Irr}X_\mu(b)$ by Proposition 2.9.

Let $v \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$. Let $\mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$ (resp. $\mathcal{A}_{\mu,b}(v)$) denote the set of $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ (resp. $\lambda \in \mathcal{A}_{\mu,b}$) such that $\lambda_\alpha \geqslant 0$ for $\alpha \in \Phi_{N_v}$. Here $\Phi_{N_v} = \{\alpha \in \Phi; \langle \alpha, v \rangle > 0\}$ is the set of roots in $N_v$.

**Lemma 5.10.** *Let $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$. Then $\lambda \sim \chi$ for some $\chi \in \mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$.*

*Proof.* Let $n \in \mathbb{Z}$. By Lemma 1.1, $t^{nv} I t^\lambda K \subseteq \cup_{x \leq t^\lambda} I t^{nv} x K$. Let $\chi_{x,n} \in Y$ such that $I t^{nv} x K = I t^{\chi_{x,n}} K$ for $x \leq t^\lambda$. Then $t^{nv} \mathrm{Irr}X_\mu^\lambda(b) \subseteq \cup_{x \leq t^\lambda} \mathrm{Irr}\overline{X_\mu^{\chi_{x,n}}(b)}$.

Thus for sufficiently large $n$ we have $(\chi_{x,n})_\alpha \geqslant 0$ for $x \leq t^\lambda$ and $\alpha \in \Phi_{N_v}$. So the statement follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

**Lemma 5.11.** *Let* $\lambda, \lambda' \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ *such that* $\lambda \sim \lambda'$*. Then there exists a sequence* $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_r = \lambda' \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ *such that* $\lambda_i = \tilde{w}_{\alpha_i}(\lambda_{i-1})$ *and* $R(\lambda_i) = p(\tilde{w}_{\alpha_i})R(\lambda_{i-1})$ *for* $1 \leqslant i \leqslant r$*, where* $\tilde{w}_{\alpha_i} \in \Omega \cap \mathbb{J}_b$ *or* $\tilde{w}_{\alpha_i}$ *is the longest element of* $W_{\mathcal{O}_{\tilde{\alpha}_i}}$ *for some* $\alpha_i \in \Pi$*.*

*Proof.* By assumption, there exist $C \in \mathrm{Irr}X_\mu^\lambda(b)$ and $g \in \mathbb{J}_b$ such that $gC \subseteq \overline{X_\mu^{\lambda'}(b)}$. Since $b \in \Omega$, $\mathbb{J}_b$ is generated by $I \cap \mathbb{J}_b$, $\Omega \cap \mathbb{J}_b$ and $W_{\mathcal{O}_{\tilde{\alpha}}} \cap \mathbb{J}_b$ for $\alpha \in \Pi$ such that $W_{\mathcal{O}_{\tilde{\alpha}}}$ is finite. We may assume $g$ lies in one of the sets $I \cap \mathbb{J}_b$, $\Omega \cap \mathbb{J}_b$ and $W_{\mathcal{O}_{\tilde{\alpha}}} \cap \mathbb{J}_b$. If $g \in I \cap \mathbb{J}_b$, then $\lambda = \lambda'$ and there is nothing to prove. If $g = \omega$ for some $\omega \in \Omega \cap \mathbb{J}_b$, then $\omega I t^\lambda K/K = I t^{\omega(\lambda)}K/K$. Hence $\lambda' = \omega(\lambda)$ and $R(\lambda') = p(\omega)R(\lambda)$ by Lemma 2.8. Suppose $g \in W_{\mathcal{O}_{\tilde{\alpha}}} \cap \mathbb{J}_b = \{1, \tilde{w}_\alpha\}$, where $\tilde{w}_\alpha$ is the unique longest element of $W_{\mathcal{O}_{\tilde{\alpha}}}$. Then $\lambda'$ equals $\lambda$ or $\tilde{w}_\alpha(\lambda)$ by Lemma 5.8 (3). So we can assume that $\lambda \neq \lambda' = \tilde{w}_\alpha(\lambda)$ and it remains to show $R(\lambda') = p(\tilde{w}_\alpha)R(\lambda)$. The statement follows from Lemma 5.8 (2) and Lemma 5.7. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

**Proposition 5.12.** *We have* $\mathcal{A}_{\mu,b}^{\mathrm{top}} = \cup_{v' \in p(\tilde{W} \cap \mathbb{J}_b)(v)} \mathcal{A}_{\mu,b}^{\mathrm{top}}(v')$*.*

*Proof.* Let $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$. By Lemma 5.10, there exist $v' \in p(\tilde{W} \cap \mathbb{J}_b)(v)$ and $\chi \in \mathcal{A}_{\mu,b}^{\mathrm{top}}(v')$ such that $\lambda \sim \chi$. By Lemma 5.11, we can assume that $\chi \neq \lambda = \tilde{w}(\chi) \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$, where (1) $\tilde{w} \in \Omega \cap \mathbb{J}_b$ or (2) $\tilde{w} = \tilde{w}_\alpha$ for some $\alpha \in \Pi$ as in §5.3. It suffices to show $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}(p(\tilde{w})(v'))$, that is, $\lambda_{p(\tilde{w})(\beta)} = \tilde{w}(\chi)_{p(\tilde{w})(\beta)} \geqslant 0$ for $\beta \in \Phi_{N_{v'}}$. Notice that $\chi_\beta \geqslant 0$ for $\beta \in \Phi_{N_{v'}}$. Then the case (1) follows from Lemma 1.3 (3), and the case (2) follows from Lemma 5.8 (2) and Lemma 5.7 as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

## 5.5. The action of $\tilde{W} \cap \mathbb{J}_b$ on $V^{b\sigma}$

Notice that $W^a \cap \mathbb{J}_b$ preserves the affine space $V^{b\sigma} = \{v \in V; b\sigma(v) = v\}$. Via the restriction to $V^{b\sigma}$ we can identify $W^a \cap \mathbb{J}_b$ with an affine reflection group of $V^{b\sigma}$, whose affine root hyperplanes are $H_a \cap V^{b\sigma}$ for $a \in \tilde{\Phi}^+$ with $V^{b\sigma} \neq H_a \cap V^{b\sigma} \neq \emptyset$. Moreover, $\Delta \cap V^{b\sigma}$ is an alcove for $W^a \cap \mathbb{J}_b$, with respect to which the simple affine reflections are the longest elements of $W_{\mathcal{O}_{\tilde{\alpha}}}$ for $\alpha \in \Pi$ with $W_{\mathcal{O}_{\tilde{\alpha}}}$ finite. We fix a special point $e'$ in the closure of $\Delta \cap V^{b\sigma}$ for $W^a \cap \mathbb{J}_b$.

We recall a lemma on root systems.

**Lemma 5.13.** *Let $E$ be some euclidean space and let $\Sigma \subseteq E$ be a root system. Let $v_1, v_2 \in E$ be two regular points for $\Sigma$ (that is, not contained in any root hyperplane of $\Sigma$). Then there exists root hyperplanes $H_1, \ldots, H_r$ separating $v_1$ from $v_2$ such that $s_{H_1} \cdots s_{H_r}(v_1)$ and $v_2$ are in the same Weyl chamber.*

**Lemma 5.14.** *Let $e'$ be as in §5.5. Let $v \in V_{\mathrm{gen}}^{p(b\sigma)}$. Then for $\tilde{w} \in \tilde{W} \cap \mathbb{J}_b$ there exist affine root hyperplanes $H_1, \ldots, H_r$ of $V^{b\sigma}$ passing through $e'$ such that*

*(1) $H_i$ separates $e' + v$ from $e' + p(\tilde{w})^{-1}(v)$ for $1 \leqslant i \leqslant r$;*

*(2) $e' + v = s_{H_1} \cdots s_{H_r}(e' + p(\tilde{w})^{-1}(v))$.*

*Moreover, $s_{H_i} = \prod_{\beta \in \mathcal{O}_{\alpha_i}} s_{\tilde{\beta}}$ for some $\alpha_i \in \Phi$ such that $\mathcal{O}_{\alpha_i}$ is strongly orthogonal.*

*Proof.* First note that $V^{p(b\sigma)}$ is the underlining vector space of the affine space $V^{b\sigma}$. As $p(\tilde{w})$ preserves $V^{p(b\sigma)}$, we see that $v, p(\tilde{w})^{-1}(v) \in V_{\mathrm{gen}}^{p(b\sigma)}$. Hence $e' + v, e' + p(\tilde{w})(v)$ are regular points for the root system associated to $W^a \cap \mathbb{J}_b$ with origin $e'$. By Lemma 5.13, there are affine root hyperplanes $H_1, \ldots, H_r$ of $V^{b\sigma}$ (passing through $e'$) separating $e' + v$ from $e' + p(\tilde{w})^{-1}(v)$ such that $s_{H_1} \cdots s_{H_r}(e' + p(\tilde{w})^{-1}(v))$ and $e' + v$ are contained in the same Weyl chamber of $V^{b\sigma}$ with origin $e'$. Suppose $e' + v \neq s_{H_1} \cdots s_{H_r}(e' + p(\tilde{w})^{-1}(v))$, that is, $v \neq p(s_{H_1} \cdots s_{H_r} \tilde{w}^{-1})(v) \in W_0(v)$. Then there exists $\alpha \in \Phi$ whose root hyperplane $H_\alpha$ separates $v$ from $p(s_{H_1} \cdots s_{H_r} \tilde{w}^{-1})(v)$. In particular, $H_\alpha \cap V^{b\sigma}$ is an affine root hyperplane for $W^a \cap \mathbb{J}_b$. As $e'$ is a special point for $W^a \cap \mathbb{J}_b$, there exists some affine root hyperplane $H$ of $V^{b\sigma}$ passing through $e'$ which is parallel to $H_\alpha \cap V^{b\sigma}$. Then $H$ separates $e' + v$ from $s_{H_1} \cdots s_{H_r}(e' + p(\tilde{w})^{-1}(v))$, contradicting our assumption. So we have $e' + v = s_{H_1} \cdots s_{H_r}(e' + p(\tilde{w})^{-1}(v))$ as desired.

Now we show the "Moreover" part. By Lemma 5.1, there exists $a_i = (\alpha_i, k_i) \in \tilde{\Phi}^+$ such that $\mathcal{O}_{\alpha_i}$ is strongly orthogonal and $s_{H_i} = \prod_{a \in \mathcal{O}_{a_i}} s_a$ by viewing $s_{H_i}$ as an element of $\tilde{W} \cap \mathbb{J}_b$. Notice that $e' \in H_i = H_{a_i} \cap V^{b\sigma}$, that is, $a_i(e') = -\langle \alpha_i, e' \rangle + k_i = 0$. As $e'$ lies in the closure of $\Delta$, we have $|\langle \alpha_i, e' \rangle| \leqslant 1$, which together with the inclusion $a_i \in \tilde{\Phi}^+$ implies that either $\alpha_i > 0$ and $k_i = 1$ or $\alpha_i < 0$ and $k_i = 0$. In either case, $a_i = \tilde{\alpha}_i$ and the proof is finished. $\qquad\square$

## 5.6. Characterization of the equivalence relation

Let $v$, $z$, $M$, $b_M$ be as in §5.2. We give an explicit description of the equivalence relation $\sim$ on $\mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$.

**Lemma 5.15.** *Let* $\lambda, \lambda' \in \mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$ *such that* $\lambda \sim \lambda'$. *Then* $\lambda' = y(\lambda)$ *for some* $y \in \Omega_{M_b} \cap \mathbb{J}_b$.

*Proof.* First note that it suffices to find an element $y \in \tilde{W} \cap \mathbb{J}_b$ such that $p(y)(v) = v$ and $\lambda' = y(\lambda)$. Indeed, the conditions $p(y)(v) = v$ and $y \in \mathbb{J}_b$ imply that $y \in \tilde{W}_{M_b}$ and $y(V^{b\sigma}) = V^{b\sigma}$. Noticing that $V^{b\sigma} \subseteq \Delta_{M_b}$ (since $V^{b\sigma} \cap \Delta \neq \emptyset$), we have $y(\Delta_{M_b}) = \Delta_{M_b}$ and hence $y \in \Omega_{M_b}$.

By Lemma 5.11, there is $\tilde{w} \in \tilde{W} \cap \mathbb{J}_b$ such that $\lambda' = \tilde{w}(\lambda)$ and $R(\lambda') = p(\tilde{w})R(\lambda)$. If $p(\tilde{w})(v) = v$, the statement follows as in the above paragraph. Suppose $p(\tilde{w})(v) \neq v$. Let $e'$, $H_i$ and $\alpha_i$ for $1 \leqslant i \leqslant r$ be as in Lemma 5.14. We construct $x_i \in W^a \cap \mathbb{J}_b$ such that $p(x_i) = p(s_{H_i})$ and $x_i(\lambda) = \lambda$ as follows.

As $H_i$ separates $e' + v$ from $e' + p(\tilde{w})^{-1}(v)$, without loss of generality we may assume that

$$\langle \alpha_i, v \rangle < 0 < \langle \alpha_i, p(\tilde{w})^{-1}(v) \rangle = \langle p(\tilde{w})(\alpha_i), v \rangle.$$

Let $\alpha \in \mathcal{O}_{-\alpha_i}$. As $v \in V^{p(b\sigma)}$ and that $p(\tilde{w})$ commutes with $p(b\sigma)$, we have

$$\langle p(\tilde{w})(\alpha), v \rangle = \langle p(\tilde{w})(-\alpha_i), v \rangle < 0.$$

So $-p(\tilde{w})(\alpha) \in \Phi_{N_v}$. Moreover, as $\lambda' \in \mathcal{A}_{\mu,b}(v)$, we have $\lambda'_{-p(\tilde{w})(\alpha)} \geqslant 0$ and

$$\lambda'_{p(\tilde{w})(\alpha)} = -\lambda'_{-p(\tilde{w})(\alpha)} - 1 \leqslant -1.$$

By definition, $p(\tilde{w})(\alpha) \notin R(\lambda')$ and hence $R(\lambda') \cap p(\tilde{w})\mathcal{O}_{-\alpha_i} = \emptyset$. Therefore, $R(\lambda) \cap \mathcal{O}_{-\alpha_i} = \emptyset$ as $R(\lambda') = p(\tilde{w})R(\lambda)$.

We claim that $\lambda_\beta$ is invariant for $\beta \in \mathcal{O}_{-\alpha_i} \subseteq \Phi_{N_v}$. Otherwise, there exists $\xi \in \mathcal{O}_{-\alpha_i}$ such that $\langle \xi, \lambda^\natural \rangle = \lambda_{\xi^{-1}} - \lambda_\xi \neq 0$ (see Lemma 2.7). Since $\lambda^\natural$ is minuscule and

$$\sum_{\beta \in \mathcal{O}_{-\alpha_i}} \langle \beta, \lambda^\natural \rangle = \sum_{\beta \in \mathcal{O}_{-\alpha_i}} \lambda_{\beta^{-1}} - \lambda_\beta = 0,$$

there exists $\gamma \in \mathcal{O}_{-\alpha_i}$ such that $\langle \gamma, \lambda^\natural \rangle = \lambda_{\gamma^{-1}} - \lambda_\gamma = -1$. On the other hand, we have $\lambda_{\gamma^{-1}} \geqslant 0$ since $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$ and $\gamma^{-1} \in \mathcal{O}_{-\alpha_i} \subseteq \Phi_{N_v}$. So $\gamma \in R(\lambda)$, which contradicts that $R(\lambda) \cap \mathcal{O}_{-\alpha_i} = \emptyset$. The claim is proved.

Let $c_i = \lambda_\beta = -\lambda_{-\beta} - 1 \in \mathbb{Z}$ for $\beta \in \mathcal{O}_{-\alpha_i}$, which is a constant by the above claim. Let $\psi_i = \sum_{\delta \in \mathcal{O}_{\alpha_i}} \lambda_\delta \delta^\vee = (-c_i - 1) \sum_{\delta \in \mathcal{O}_{\alpha_i}} \delta^\vee$ and $x_i = t^{\psi_i} s_{H_i}$.

Then $p(x_i) = p(s_{H_i})$ and $t^{\psi_i} \in W^a \cap \mathbb{J}_b$. Moreover, by Lemma 1.3 (2) and that $\mathcal{O}_{\alpha_i}$ is orthogonal (see Lemma 5.14) we have

$$x_i(\lambda) = \psi_i + (\prod_{\delta \in \mathcal{O}_{\alpha_i}} s_{\tilde{\delta}})(\lambda) = \psi_i + \lambda - \sum_{\delta \in \mathcal{O}_{\alpha_i}} \lambda_\delta \delta^\vee = \lambda.$$

Thus $x_i$ satisfies our requirements.

Let $y = \tilde{w} x_r \cdots x_1 \in \tilde{W} \cap \mathbb{J}_b$. Then it follows that $y(\lambda) = \tilde{w}(\lambda) = \lambda'$ and $p(y)(v) = p(\tilde{w})p(s_{H_r}) \cdots p(s_{H_1})(v) = v$ as desired. $\quad\square$

**Lemma 5.16.** *Let $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$ and $\tilde{w} \in \Omega_{M_b} \cap \mathbb{J}_b$. Then $\tilde{w}(\lambda) \in \mathcal{A}_{\mu,b}$. Moreover, if $\tilde{w}(\lambda) \in \mathcal{A}_{\mu,b}(v)$, then $\tilde{w}(\lambda) \in \mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$ and $\lambda \sim \tilde{w}(\lambda)$.*

*Proof.* Let $\chi = \tilde{w}(\lambda)$. By Lemma 2.7, $\chi^\natural \in W_{M_b}(\lambda^\natural)$ and hence $\chi \in \mathcal{A}_{\mu,b}$ by Proposition 2.9. As $\tilde{w} \in \Omega_{M_b} \cap \mathbb{J}_b$, it follows the same way as Lemma 2.8 that $p(\tilde{w})(R(\lambda) \cap \Phi_{M_b}) = R(\chi) \cap \Phi_{M_b}$. Combining Proposition 2.9 with Lemma 5.4 we have

(a) $\qquad \dim X_{z(\chi^\natural)}^{z(\chi),M}(b_M) = |R(\chi) \cap \Phi_{M_b}| = |R(\lambda) \cap \Phi_{M_b}|$
$$= \dim X_{z(\lambda^\natural)}^M(b_M) = \dim X_{z(\chi^\natural)}^M(b_M),$$

where the third equality follows from Corollary 5.5 (1) by noticing that $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$, and the last one follows from that $z(\chi^\natural), z(\lambda^\natural)$ are conjugate by $W_M$.

Suppose $\chi \in \mathcal{A}_{\mu,b}(v)$. Then $\chi_\beta \geqslant 0$ and $\chi_{-\beta} = -\chi_\beta - 1 \leqslant -1$ for $\beta \in \Phi_{N_v}$, which together with (a) implies that $\chi \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ by Corollary 5.5.

For $\eta \in \mathcal{A}_{\mu,b}(v)$ we have $-\langle \beta, \eta \rangle \geqslant \eta_{-\beta} \geqslant 0$ for $\beta \in \Phi_{N_{-v}} = -\Phi_{N_v}$. Then

$$I t^\eta K / K = I_{M_b} I_{N_v} I_{N_{-v}} t^\eta K / K = I_{M_b} I_{N_v} t^\eta K / K,$$

which implies that $t^{nv} I t^\eta K / K \subseteq I t^{nv+\eta} K / K$ for $n \in \mathbb{Z}_{\geqslant 0}$. In particular, $t^{nv} X_\mu^\chi(b) \subseteq X_\mu^{nv+\chi}(b)$ and hence $\chi \sim nv + \chi$ for $n \in \mathbb{Z}_{\geqslant 0}$. Choose $n$ sufficiently large so that $\tilde{w} t^{nv} I_{N_v} t^{-nv} \subseteq I_{N_v} \tilde{w}$. Then

$$\begin{aligned}
\tilde{w} t^{nv} X_\mu^\lambda(b) &\subseteq \tilde{w} t^{nv} I t^\lambda K / K \\
&= \tilde{w} t^{nv} I_{M_b} I_{N_v} t^\lambda K / K \\
&= I_{M_b} \tilde{w} (t^{nv} I_{N_v} t^{-nv}) t^{nv+\lambda} K / K \\
&\subseteq I_{M_b} I_{N_v} \tilde{w} t^{nv+\lambda} K / K \\
&\subseteq I t^{nv+\chi} K / K.
\end{aligned}$$

Therefore, $\tilde{w} t^{nv} X_\mu^\lambda(b) \subseteq X_\mu^{nv+\chi}(b)$ and hence $\lambda \sim nv + \chi \sim \chi$ as desired. $\quad\square$

Combining Lemma 5.15 with Lemma 5.16 we have

**Corollary 5.17.** *Let $\lambda, \lambda' \in \mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$. Then $\lambda \sim \lambda'$ if and only if $\lambda' = \tilde{w}(\lambda)$ for some $\tilde{w} \in \Omega_{M_b} \cap \mathbb{J}_b$.*

## 5.7. End of the proof

Let $v$, $z$, $M$, $b_M$ be as in §5.2. Recall that $I_{\mu,M}$ is the set of $W_M$-orbits of $W_0(\mu)$, and $I_{\mu,b_M,M} = \{\eta \in I_{\mu,M}; \kappa_M(t^\eta) = \kappa_M(b_M)\}$. Let $\tilde{\mathcal{A}}_{\mu,b}^{\mathrm{top}}(v)$ denote the set of equivalence classes of $\mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$ with respect to $\sim$ defined in §5.4. For $\eta \in I_{\mu,b_M,M}$ let $\mathcal{A}_{\eta,b_M}^{M,\mathrm{top}} := \mathcal{A}_{\overline{\eta}^M,b_M}^{M,\mathrm{top}}$ be defined in §5.2. Similarly we can define an equivalence relation $\sim_M$ on $\mathcal{A}_{\eta,b_M}^{M,\mathrm{top}}$, and denote by $\tilde{\mathcal{A}}_{\eta,b_M}^{M,\mathrm{top}}$ the set of corresponding equivalence classes. As $b_M$ is superbasic in $M(\breve{F})$, we have $\chi \sim_M \chi' \in \mathcal{A}_{\eta,b_M}^{M,\mathrm{top}}$ if and only if $\chi' = \tilde{w}(\chi)$ for some $\tilde{w} \in \Omega_M \cap \mathbb{J}_{b_M}^M$.

*Proof of Proposition 4.5.* We show that there are bijections

$$\mathbb{J}_b\backslash\mathrm{Irr}^{\mathrm{top}}X_\mu(b) \xleftarrow{\Psi_1} \tilde{\mathcal{A}}_{\mu,b}^{\mathrm{top}}(v) \xrightarrow{\Psi_2} \sqcup_{\eta\in I_{\mu,b_M,M}} \tilde{\mathcal{A}}_{\eta,b_M}^{M,\mathrm{top}},$$

where $\Psi_1$ and $\Psi_2$ are given by $\lambda \mapsto \mathbb{J}_b\overline{\mathrm{Irr}X_\mu^\lambda(b)}$ and $\lambda \mapsto z(\lambda)$ respectively. Indeed, by Proposition 2.9 and Lemma 5.10 we see that $\Psi_1$ is bijective.

Let $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$. By Corollary 5.5, $z(\lambda) \in \mathcal{A}_{z(\lambda^\natural),b_M}^{M,\mathrm{top}}$ and $z(\lambda^\natural) \in I_{\mu,b_M,M}$. Moreover, by Lemma 2.7 and Corollary 5.17 we deduce that

$$\lambda \sim \lambda' \in \mathcal{A}_{\mu,b}^{\mathrm{top}}(v) \Leftrightarrow z(\lambda) \sim_M z(\lambda') \in \mathcal{A}_{z(\lambda^\natural),b_M}^{M,\mathrm{top}}.$$

So $\Psi_2$ is well defined. On the other hand, let $\chi \in \mathcal{A}_{\eta,b_M}^{M,\mathrm{top}}$ with $\eta \in I_{\mu,b_M,M}$. By Corollary 5.5 and 5.17, the map $\chi \to nv + z^{-1}(\chi)$ with $n \gg 0$ induces the inverse map of $\Psi_2$. So $\Psi_2$ is also bijective.

Therefore,

$$\begin{aligned}
|\mathbb{J}_b\backslash\mathrm{Irr}^{\mathrm{top}}X_\mu(b)| &= \sum_{\eta\in I_{\mu,b_M,M}} |\tilde{\mathcal{A}}_{\eta,b_M}^{M,\mathrm{top}}|; \\
&= \sum_{\eta\in I_{\mu,b_M,M}} \dim V_\eta^{\widehat{M}}(\underline{\lambda}_M(b_M)) \\
&= \sum_{\eta\in I_{\mu,M}} \dim V_\eta^{\widehat{M}}(\underline{\lambda}_M(b_M)) \\
&= \dim V_\mu(\underline{\lambda}_M(b_M)) \\
&= \dim V_\mu(\underline{\lambda}_G(b)),
\end{aligned}$$

where the second equality follows from [17, Theorem 1.5]; the fourth one follows from that $V_\mu = \oplus_{\eta \in I_{\mu,M}} V_\eta^{\widehat{M}}$ as $\mu$ is minuscule. The proof is finished.

$\square$

## 6. The stabilizer in $\mathbb{J}_b$

In this section, we give an algorithm to compute the stabilizer $N_{\mathbb{J}_b}(C)$ of $C \in \mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ in $\mathbb{J}_b$.

### 6.1. Reduction to the adjoint case

Let $G_{\mathrm{ad}}$ be the adjoint quotient of $G$. By [3, §2], the natural projection $f : G \to G_{\mathrm{ad}}$ induces a Cartesian square

$$
\begin{array}{ccc}
X_\mu(b) & \xrightarrow{\ f\ } & X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}}) \\
\downarrow & & \downarrow \\
\pi_1(G) & \xrightarrow{\ f\ } & \pi_1(G_{\mathrm{ad}}),
\end{array}
$$

where the vertical maps are the natural projections; $\mu_{\mathrm{ad}}$ and $b_{\mathrm{ad}}$ are the images of $\mu$ and $b$ under $f$ respectively. In particular, the stabilizer $N_{\mathbb{J}_b}(C)$ can be computed from the stabilizer $N_{\mathbb{J}_{b_{\mathrm{ad}}}}(C_{\mathrm{ad}})$ of $C_{\mathrm{ad}} = f(C)$ in $\mathbb{J}_{b_{\mathrm{ad}}}$ via the following natural Cartesian square

$$
\begin{array}{ccc}
N_{\mathbb{J}_b}(C) & \xrightarrow{\ f\ } & N_{\mathbb{J}_{b_{\mathrm{ad}}}}(C_{\mathrm{ad}}) \\
\downarrow & & \downarrow \\
\mathbb{J}_b^0 & \xrightarrow{\ f\ } & \mathbb{J}_{b_{\mathrm{ad}}},
\end{array}
$$

where the vertical maps are the natural inclusions, and $\mathbb{J}_b^0$ is the kernel of the natural projection $\mathbb{J}_b \to \pi_1(G)$. Thus we can assume $G$ is adjoint and simple.

### 6.2. Reduction to the basic case

Now we show how to pass to the case where $b$ is basic. Let $P = MN$ and $\beta : X_\mu(b) \to \mathrm{Gr}_M$ be as in §2.1 such that $M$ is the centralizer of $\nu_G(b)$. In

particular, $\mathbb{J}_b = \mathbb{J}_b^M$. Let $\eta \in I_{\mu,b,M}$ such that $X_\eta^M(b)$ contains an open dense subset of $\beta(C)$. Let

$$C^M = \overline{\beta(C) \cap X_\eta^M(b)} \subseteq X_\eta^M(b).$$

By Proposition 4.3 (1) and (4), $C = \overline{X_\mu^{Z^N, C^M}(b)}$ for some $Z^N \in \Sigma_{\mu,\eta}^N$. Note that $jX_\mu^{Z^N, C^M}(b) = X_\mu^{Z^N, jC^M}(b)$ for $j \in \mathbb{J}_b^M = \mathbb{J}_b$. So we have $N_{\mathbb{J}_b}(C) = N_{\mathbb{J}_b^M}(C^M)$.

## 6.3. Reduction to the minuscule case

Assume $b$ is basic. If $G$ has no nonzero minuscule cocharacters, then $b$ is unramified and $N_{\mathbb{J}_b}(C)$ is determined in [53, Theorem 4.4.14]. Otherwise, by Lemma 4.6, there exists a dominant minuscule cocharacter $\mu_\bullet \in Y^d$ for some $d \in \mathbb{Z}_{\geqslant 1}$ such that $\mathbb{B}_\mu^{\widehat{G}}$ occurs in

$$\mathbb{B}_{\mu_\bullet}^{\widehat{G}} = \mathbb{B}_{\mu_1}^{\widehat{G}} \otimes \cdots \otimes \mathbb{B}_{\mu_d}^{\widehat{G}}.$$

Let $X_{\mu_\bullet}(b_\bullet)$ be as in §1.6. By Theorem 0.7, there exists $C' \in \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)$ such that

$$\overline{\mathrm{pr}(C')} = \overline{C} \subseteq \mathrm{Gr},$$

and moreover, the map $g \mapsto (g, \ldots, g)$ gives an isomorphism $N_{\mathbb{J}_b}(C) \cong N_{\mathbb{J}_{b_\bullet}}(C')$.

## 6.4. Small cocharacters

In the rest of the section we assume that $G = G_{\mathrm{ad}}$ is simple, $\mu_\bullet$ is minuscule and $b$ is basic. By abuse of notation, we write $X_\mu(b)$ for $X_{\mu_\bullet}(b_\bullet)$ by assuming that $\mu$ is minuscule in the rest of this section. Then we can adopt the notation in §5.

Let $v \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$. For $D \subseteq \Phi$ we set $D(v, +) = \{\alpha \in D; \langle \alpha, v \rangle > 0\}$. We say $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ is $v$-small if $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$ (see §5.4) and for each $\alpha \in \Pi(v, +)$ (see §1.2) there exists $\beta \in \mathcal{O}_\alpha$ such that $\lambda_\beta = 0$. We say $v$ is permissible if $v$-small cocharacters exist.

We say $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ is small if it is $v$-small for some $v \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$, and we define $\Pi(\lambda)$ to be the set of roots $\alpha \in \Pi - \Phi_{M_b}$ such that $\lambda_\beta \geqslant 0$ for some/any $\beta \in \mathcal{O}_\alpha$ (see Corollary 5.5). By definition, $\Pi(\lambda) = \Pi(v, +)$ if $\lambda$ is $v$-small.

**Lemma 6.1.** *If $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ is not small, then there exists $\alpha \in \Pi$ such that $W_{\mathcal{O}_{\tilde{\alpha}}}$ is finite and $\lambda_\beta \geqslant 1$ for $\beta \in \mathcal{O}_\alpha$*

*Proof.* By Proposition 5.12, there exists $v \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$ such that $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}(v)$. As $\lambda$ is not $v$-small, there exists $\alpha \in \Pi(v,+) - \Phi_{M_b}$ such that $\lambda_\beta \geqslant 1$ for $\beta \in \mathcal{O}_\alpha$. Moreover, $W_{\mathcal{O}_{\tilde{\alpha}}}$ is finite by Lemma 5.3. $\qquad\square$

**Proposition 6.2.** *For each $C \in \mathrm{Irr}^{\mathrm{top}} X_\mu(b)$ there exists small $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ such that $C \in \mathbb{J}_b \mathrm{Irr} \overline{X_\mu^\lambda(b)}$.*

*Proof.* Recall the dominance order $\leq$ on $Y$ defined in §1.1. For $\eta, \chi \in Y$ write $\eta \trianglelefteq \chi$ if either $\eta \lneq \chi$ (see §1.1 for the definition of $\leq$) or $\eta \in W_0(\chi)$ and $\chi \leqslant \eta$. Let $\lambda$ be a minimal cocharacter in the set

$$\{\chi \in \mathcal{A}_{\mu,b}^{\mathrm{top}}; C \in \mathbb{J}_b \mathrm{Irr} \overline{X_\mu^\chi(b)}\}$$

under the partial order $\trianglelefteq$. We show that $\lambda$ is small.

Suppose $\lambda$ is not small. Let $\alpha \in \Pi$ as in Lemma 6.1, and let $\tilde{w} = \tilde{w}_\alpha \in \mathbb{J}_b$ be the maximal element of $W_{\mathcal{O}_{\tilde{\alpha}}}$. By Lemma 5.1,

$$\tilde{w} = \tilde{w}_\alpha = \prod_{\beta \in \mathcal{O}_\gamma} s_{\tilde{\beta}},$$

where $\mathcal{O}_\gamma$ is orthogonal with $\gamma = \alpha$ if $\langle \alpha^d, \alpha^\vee \rangle \neq -1$, and $\gamma = \alpha^d + \alpha$ otherwise. In particular, $\lambda_\beta \geqslant 1$ for $\beta \in \mathcal{O}_\gamma$. Let $\lambda' = \tilde{w}(\lambda)$. By Corollary 5.9, $\lambda' \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ and $C \in \mathbb{J}_b \mathrm{Irr} \overline{X_\mu^\lambda(b)} = \mathbb{J}_b \mathrm{Irr} \overline{X_\mu^{\lambda'}(b)}$. Moreover, as $\mathcal{O}_\gamma$ is orthogonal,

$$\lambda' = \tilde{w}(\lambda) = p(\tilde{w})(\lambda - \sum_{\beta \in \Phi^+ \cap \mathcal{O}_\gamma} \beta^\vee).$$

If $\Phi^+ \cap \mathcal{O}_\gamma \neq \emptyset$, we have $\lambda' \lneq \lambda$ since $\langle \beta, \lambda \rangle = \lambda_\beta + 1 \geqslant 2$ for $\beta \in \Phi^+ \cap \mathcal{O}_\gamma$. Otherwise,

$$\lambda' = p(\tilde{w})(\lambda) = \lambda - \sum_{\beta \in \mathcal{O}_\gamma} \lambda_\beta \beta^\vee > \lambda.$$

Thus, in either case we have $\lambda' \triangleleft \lambda$, contradicting the choice of $\lambda$. So $\lambda$ is small as desired. $\qquad\square$

We say a root $\alpha \in \Phi(v,+)$ is indecomposable (in $\Phi(v,+)$) if it is not a sum of roots in $\Phi(v,+) \setminus \{\alpha\}$.

**Lemma 6.3.** *Let $v \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$ be permissible. Then each root of $\Pi(v,+)$ is indecomposable in $\Phi(v,+)$.*

*Proof.* For $\alpha \in \Pi(v, +)$ set $Y'(v, \alpha) = \{\lambda \in Y; \lambda_\alpha = 0, \lambda_\beta \geqslant 0 \text{ for } \beta \in \Phi(v, +)\}$. We claim that

(a) $Y'(v, \alpha) \neq \emptyset$.

By assumption, there is a $v$-small cocharacter $\chi$. By definition, $\chi \in Y'(v, \gamma)$ for some $\gamma \in \mathcal{O}_\alpha$. By Lemma 1.3 (3), $Y'(v, \alpha)$ and $Y'(v, \gamma)$ are conjugate by $\langle b\sigma \rangle$. So (a) is proved.

Suppose $\alpha = \sum_{i \in D} \alpha_i$ for some $\alpha \neq \alpha_i \in \Phi(v, +)$. Let $\lambda \in Y'(v, \alpha)$ by (a). Then $\lambda_\alpha = 0$ and $\langle \alpha_i, \lambda \rangle \geqslant \lambda_{\alpha_i} \geqslant 0$ for $i \in D$. If $\alpha < 0$ is a minus simple root, then there exists $i_0 \in D$ such that $\alpha_{i_0} > 0$. Hence $\lambda_\alpha = \langle \alpha, \lambda \rangle \geqslant \langle \alpha_{i_0}, \lambda \rangle = \lambda_{\alpha_{i_0}} + 1 \geqslant 1$, which is a contradiction. If $\alpha > 0$ is the highest root, then there exist $i_1 \neq i_2 \in D$ such that $\alpha_{i_1}, \alpha_{i_2} > 0$. Again we have $\langle \alpha, \lambda \rangle \geqslant \langle \alpha_{i_1}, \lambda \rangle + \langle \alpha_{i_2}, \lambda \rangle \geqslant 2$ and hence $\lambda_\alpha \geqslant 1$, which is also a contradiction. So $\alpha$ is indecomposable as desired. $\square$

Let $J = p(b\sigma)(J) \subseteq \Pi$ such that the corresponding parabolic subgroup $W_J$ (generated by $s_{\tilde{\alpha}}$ for $\alpha \in J$) is finite. By a standard parahoric subgroup of type $J$ we mean a subgroup of $\mathbb{J}_b$ generated by $I \cap \mathbb{J}_b$ and $W_J \cap \mathbb{J}_b$. We say a standard parahoric subgroup of type $J$ is of maximal length if the length, of the maximal element of $W_J$, is maximal among all standard parahoric subgroups of $\mathbb{J}_b$. The following result will be proved in Appendix A.

**Proposition 6.4.** *If $v \in V_{\text{gen}}^{p(b\sigma)} \cap Y$ is permissible, then the parahoric subgroup of type $\Pi(v, +)$ is of maximal length.*

## 6.5. Irreducibility implies smallness

Suppose $\lambda \in \mathcal{A}_{\mu, b}^{\text{top}}$ is not small. Let $\alpha \in \Pi$ be as in Lemma 6.1, and let $\tilde{w} = \tilde{w}_\alpha \in W_{\mathcal{O}_{\tilde{\alpha}}}$ be the longest element. Suppose $X_\mu^\lambda(b)$ is irreducible. By Lemma 5.6, Lemma 5.8 and Corollary 5.9 (2), $\lambda \neq \tilde{w}(\lambda) \in \mathcal{A}_{\mu, b}^{\text{top}}$ and $\tilde{w} X_\mu^{\tilde{w}(\lambda)}(b) \subseteq X_\mu^\lambda(b)$. Hence $\tilde{w} \overline{X_\mu^{\tilde{w}(\lambda)}(b)} = \overline{X_\mu^\lambda(b)}$ is also irreducible. In particular,

$$\tilde{w} N_{\mathbb{J}_b}(\overline{X_\mu^{\tilde{w}(\lambda)}(b)}))\tilde{w}^{-1} = N_{\mathbb{J}_b}(\overline{X_\mu^\lambda(b)}).$$

Notice that $\tilde{w} \in W^a \cap \mathbb{J}_b$ and $N_{\mathbb{J}_b}(\overline{X_\mu^{\tilde{w}(\lambda)}(b)}))$, $N_{\mathbb{J}_b}(\overline{X_\mu^\lambda(b)})$ are both standard parahoric subgroups containing $I \cap \mathbb{J}_b$. Thus $\tilde{w} \in N_{\mathbb{J}_b}(\overline{X_\mu^{\tilde{w}(\lambda)}(b)})) = N_{\mathbb{J}_b}(\overline{X_\mu^\lambda(b)})$, which is a contradiction. So $X_\mu^\lambda(b)$ is not irreducible as desired.

## 6.6. Smallness implies irreducibility

Now we show $X_\mu^\lambda(b)$ is irreducible if $\lambda$ is small. We need some results on permissible vectors introduced in §6.4.

Let $(\,,\,): V \times V \to \mathbb{R}$ be the Killing form such that $\langle \beta, \gamma^\vee \rangle = \frac{2(\beta^\vee, \gamma^\vee)}{(\beta^\vee, \beta^\vee)}$ for $\beta, \gamma \in \Phi$. For any $p(b\sigma)$-orbit $\mathcal{O}$ of $\Pi$ we set $r_\mathcal{O} = \sum_{\xi \in \mathcal{O}} \xi$ and $r_\mathcal{O}^\vee = \sum_{\xi \in \mathcal{O}} \xi^\vee$. Then we have the identification $r_\mathcal{O} = \frac{2}{(\xi^\vee, \xi^\vee)} r_\mathcal{O}^\vee$ via the bilinear form $(\,,\,)$, where $\xi$ is any/some root in $\mathcal{O}$.

**Lemma 6.5.** *If $v \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$ is permissible, then $\{r_\mathcal{O}^\vee; \mathcal{O} \in \Pi(v, +)/\langle p(b\sigma) \rangle\}$ is a basis of $V^{p(b\sigma)}$.*

*Proof.* By Proposition 6.4, $\Pi(v, +)$ is a maximal proper $p(b\sigma)$-stable subset of $\Pi$. Hence $\{r_\mathcal{O}^\vee; \mathcal{O} \in \Pi(v, +)/\langle p(b\sigma) \rangle\}$ is linearly independent, and moreover,

$$|\Pi(v, +)/\langle p(b\sigma) \rangle| = |\Pi/\langle p(b\sigma) \rangle| - 1 = \dim V^{p(b\sigma)}.$$

So the statement follows. $\qquad\square$

**Lemma 6.6.** *Let $v \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$ be permissible. Let $\gamma$ be an indecomposable root in $\Phi(v, +)$. Then there exists $\alpha \in \Pi(v, +)$ such that $\gamma - \alpha \in \Phi_{M_b} \sqcup \{0\}$.*

*Proof.* Suppose $\langle \gamma, \beta^\vee \rangle \leqslant 0$ for $\beta \in \Pi(v, +)$. Then $(r_\mathcal{O}^\vee, r_{\mathcal{O}'}^\vee) \leqslant 0$ for $\mathcal{O}, \mathcal{O}' \in (\Pi(v, +)/\langle p(b\sigma) \rangle) \cup \{\mathcal{O}_\gamma\}$. Thus the set

$$\{r_\mathcal{O}^\vee; \mathcal{O} \in \Pi(v, +)/\langle p(b\sigma) \rangle\} \sqcup \{r_{\mathcal{O}_\gamma}^\vee\} \subseteq \Phi(v, +)$$

is linearly independent, which contradicts Lemma 6.5.

Thus $\langle \gamma, \alpha^\vee \rangle > 0$ for some $\alpha \in \Pi(v, +)$. Notice that $\alpha$ is indecomposable in $\Phi(v, +)$ by Lemma 6.3. If $\gamma = \alpha$, the statement follows. Otherwise, $\delta := \alpha - \gamma$ is also a root. Suppose $\langle \delta, v \rangle \neq 0$. Then $\alpha = \gamma + \delta$ (resp. $\gamma = \alpha + (-\delta)$) is indecomposable if $\langle \delta, v \rangle > 0$ (resp. $\langle \delta, v \rangle < 0$), which contradicts that $\alpha$ and $\gamma$ are indecomposable in $\Phi(v, +)$. So we have $\langle \delta, v \rangle = 0$, that is, $\delta \in \Phi_{M_b}$ as desired. $\qquad\square$

**Corollary 6.7.** *Let $v, v' \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$ be permissible. Then there exists $\varepsilon \in \Omega \cap \mathbb{J}_b$ such that $\Pi(p(\varepsilon)(v), +) = \Pi(v', +)$ and hence $\Phi(p(\varepsilon)(v), +) = \Phi(v', +)$.*

*Proof.* By Proposition 6.4, one checks (using that $G$ is adjoint) that there exists $\varepsilon \in \Omega \cap \mathbb{J}_b$ such that

$$\Pi(p(\varepsilon)(v), +) = p(\varepsilon)(\Pi(v, +)) = \Pi(v', +).$$

By replacing $v$ with $p(\varepsilon)(v)$, we may assume further $\Pi(v,+) = \Pi(v',+)$, and it remains to show $\Phi(v,+) = \Phi(v',+)$. Otherwise, there exists an indecomposable root $\gamma$ in $\Phi(v,+)$ such that $\langle \gamma, v' \rangle < 0$. By Lemma 6.6, there exists $\alpha \in \Pi(v,+) = \Pi(v',+)$ such that $\gamma - \alpha \in \Phi_{M_b} \sqcup \{0\}$. Hence $\langle \gamma, v' \rangle = \langle \alpha, v' \rangle > 0$, which contradicts our assumption. So $\Phi(v,+) = \Phi(v',+)$ as desired. $\square$

**Proposition 6.8.** *Let* $\lambda, \lambda' \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ *be small cocharacters such that* $\lambda \sim \lambda'$. *Then* $\lambda, \lambda'$ *are conjugate under* $\Omega \cap \mathbb{J}_b$. *In particular,* $X_\mu^\lambda(b)$ *and* $X_\mu^{\lambda'}(b)$ *are conjugate by* $\Omega \cap \mathbb{J}_b$.

*Proof.* By Proposition 5.12, there exist permissible vectors $v, v' \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$ such that $\lambda, \lambda'$ are $v$-small and $v'$-small respectively. In particular, $v, v'$ are both permissible. By Corollary 6.7, there exists $\varepsilon \in \Omega \cap \mathbb{J}_b$ such that

$$p(\varepsilon)(\Phi(v',+)) = \Phi(p(\varepsilon)(v'),+) = \Phi(v,+).$$

Thus $\varepsilon(\lambda')$ is also $v$-small. By replacing $\lambda'$ with $\varepsilon(\lambda')$, we may assume $\lambda, \lambda'$ are both $v$-small. By Lemma 5.15, there exists $x \in \Omega_{M_b} \cap \mathbb{J}_b$ such that $x(\lambda) = \lambda'$. It suffices to show $x \in \Omega$.

First we claim that
(a) $x(\tilde{\alpha})$ is a simple affine root for each $\alpha \in \Pi(v,+)$.

Indeed, let $\gamma = p(x)(\alpha) \in \Phi(v,+)$. As $\lambda$ is $v$-small, we may assume $\lambda_\alpha = 0$ (by replacing $\alpha$ by a suitable $\langle p(b\sigma) \rangle$-conjugate). Then we have

$$U_{x(\tilde{\alpha})} = x U_{\tilde{\alpha}} x^{-1} = x t^\lambda U_\alpha(\mathcal{O}_{\breve{F}}) t^{-\lambda} x^{-1} = t^{x(\lambda)} U_\gamma(\mathcal{O}_{\breve{F}}) t^{-x(\lambda)} \subseteq I,$$

where the last inclusion follows from that $x(\lambda) = \lambda'$ is $v$-small. So $x(\tilde{\alpha}) \in \tilde{\Phi}^+$. By Lemma 6.3, $\alpha$ is indecomposable in $\Phi(v,+)$. Hence $\gamma$ is also indecomposable in $\Phi(v,+)$. Applying Lemma 6.6 we deduce that there exists $\beta \in \Pi(v,+)$ such that either $\gamma = \beta$ or $\gamma = \beta + \delta$ for some $\delta \in \Phi_{M_b}$. By symmetry, $x^{-1}(\tilde{\beta}) \in \tilde{\Phi}^+$. In the former case, $x(\tilde{\alpha}) = \tilde{\beta} + m$ for some $m \in \mathbb{Z}_{\geqslant 0}$. Then $\tilde{\alpha} = x^{-1}(\tilde{\beta}) + m$, which means $m = 0$ since $\tilde{\alpha}$ is simple and $x^{-1}(\tilde{\beta}) \in \tilde{\Phi}^+$. So we have $x(\tilde{\alpha}) = \tilde{\beta}$ as desired. In the latter case, we have $x(\tilde{\alpha}) = \tilde{\beta} + \tilde{\delta} + m$ for some $m \in \mathbb{Z}_{\geqslant 0}$. As $\delta \in \Phi_{M_b}$ and $x \in \Omega_{M_b}$, we have $x^{-1}(\tilde{\delta}) \in \tilde{\Phi}_{M_b}^+$. Then

$$\tilde{\alpha} = x^{-1}(\tilde{\beta}) + x^{-1}(\tilde{\delta}) + m,$$

which is a contradiction since $\tilde{\alpha}$ is simple but $x^{-1}(\tilde{\beta}), x^{-1}(\tilde{\delta}) \in \tilde{\Phi}^+$. Thus (a) is proved.

By (a) we see that $x$ permutes the hyperplanes $H_\mathcal{O}$ for $\mathcal{O} \in \Pi(v,+)/\langle p(b\sigma) \rangle$, where

$$H_\mathcal{O} = \{ h \in V^{b\sigma}; \tilde{\alpha}(h) = 0 \text{ for any/some } \alpha \in \mathcal{O} \}$$

whose underlining vector space is

$$V_{\mathcal{O}} = \{r \in V^{p(b\sigma)}; \langle \alpha, r \rangle = 0 \text{ for any/some } \alpha \in \mathcal{O}\}$$
$$= \{r \in V^{p(b\sigma)}; \langle r_{\mathcal{O}}, r \rangle = 0 = (r_{\mathcal{O}}^{\vee}, r)\}.$$

By Lemma 6.5, the subspaces $V_{\mathcal{O}}$ are distinct and their intersection $\cap_{\mathcal{O}} V_{\mathcal{O}}$ is trivial. On the other hand, as $x \in \Omega_{M_b}$, $p(x)$ is product of reflections $s_{\alpha}$ such that $\langle \alpha, v \rangle = 0 = \langle \alpha, V^{p(b\sigma)} \rangle$. In particular, $p(x)$ acts on $V^{p(b\sigma)}$ trivially, which means $x$ acts on $V^{b\sigma}$ as a translation by some vector $\iota \in V^{p(b\sigma)}$. Thus $x$ fixes each $H_{\mathcal{O}}$ and hence $\iota \in \cap_{\mathcal{O}} V_{\mathcal{O}} = \{0\}$, that is, $x$ acts trivially on $V^{b\sigma}$. In particular, $x$ fixes the nonempty subset $\Delta \cap V^{b\sigma}$, which means $x \in \Omega \cap \mathbb{J}_b$ as desired. $\qquad\square$

We recall the following result in [57, Theorem 3.1.1].

**Theorem 6.9.** *Let $Z$ be an irreducible component of $X_{\mu}(b)$. Then the stabilizer of $Z$ in $\mathbb{J}_b$ is a parahoric subgroup of $\mathbb{J}_b$.*

**Corollary 6.10.** *Let $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$. Then $X_{\mu}^{\lambda}(b)$ is irreducible if and only if $\lambda$ is small.*

*Proof.* In view of §6.5, it remains to show the "if" part. Let $\lambda \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ be small. Thanks to Theorem 6.9, there exists $C' \in \mathbb{J}_b\mathrm{Irr}\overline{X_{\mu}^{\lambda}(b)}$ whose stabilizer in $\mathbb{J}_b$ contains $I \cap \mathbb{J}_b$. Let $\lambda' \in \mathcal{A}_{\mu,b}^{\mathrm{top}}$ such that $C' \in \mathrm{Irr}\overline{X_{\mu}^{\lambda'}(b)}$. As $I \cap \mathbb{J}_b$ fixes $C'$, and acts transitively on $\mathrm{Irr}X_{\mu}^{\lambda'}(b)$ (by Lemma 2.9), we see that $\overline{X_{\mu}^{\lambda'}(b)} = C'$ is irreducible, and hence $\lambda'$ is small. Noticing that $\lambda \sim \lambda'$, we deduce by Proposition 6.8 that $X_{\mu}^{\lambda}(b)$ is also irreducible as desired. $\qquad\square$

## 6.7. Computation of the stabilizer

Suppose $C = \overline{X_{\mu}^{\lambda}(b)} \in \mathrm{Irr}^{\mathrm{top}}X_{\mu}(b)$ with $\lambda$ small. Notice that $\mathbb{J}_b$ is generated by $I \cap \mathbb{J}_b$, $\Omega \cap \mathbb{J}_b$, and the longest element $\tilde{w}_{\alpha}$ of $W_{\mathcal{O}_{\tilde{\alpha}}}$ for $\alpha \in \Pi$ such that $W_{\mathcal{O}_{\tilde{\alpha}}}$ is finite. by definition, $I \cap \mathbb{J}_b \subseteq N_{\mathbb{J}_b}(C)$. So $N_{\mathbb{J}_b}(C)$ is a standard parahoric subgroup of $\mathbb{J}_b$, and it remains to determine which $\tilde{w}_{\alpha}$ fixes $C$. By Lemma 5.8, either $\lambda_{\beta} \leqslant -1$ for $\beta \in \mathcal{O}_{\alpha}$ or $\lambda_{\beta} \geqslant 0$ for $\beta \in \mathcal{O}_{\alpha}$. In the former case, we have $\alpha \notin \Pi(\lambda)$ and $\tilde{w}_{\alpha}C \neq C$ by Corollary 5.9 (2) and Lemma 5.6. Suppose the latter case occurs. Then $\alpha \in \Pi(\lambda)$ and $\lambda_{\beta} = 0$ for some $\beta \in \mathcal{O}_{\alpha}$ since $\lambda$ small. So $\tilde{w}C = C$ by Corollary 5.9 (1). Therefore, $N_{\mathbb{J}_b}(C)$ is the parahoric subgroup of $\mathbb{J}_b$ generated by $I \cap \mathbb{J}_b$ and the longest element $\tilde{w}_{\alpha}$ of $W_{\mathcal{O}_{\tilde{\alpha}}}$ for $\alpha \in \Pi(\lambda)$. Moreover, $N_{\mathbb{J}_b}(C)$ is of maximal length by Proposition 6.4.

## Appendix A.  Proof of Proposition 6.4

We assume that $b$ is basic and $G$ is simple and adjoint.

### A.1.  Reduction to absolutely simple case

For practical computation, we need to pass to the case where $G$ is absolutely simple, that is, the root system $\Phi$ of $G$ is irreducible. By assumption,

$$G_{\mathcal{O}_{\breve{F}}} = G_1 \times \cdots \times G_h,$$

where each $G_i$ is an absolutely simple factor of $G$ and the Frobenius automorphism $\sigma$ sends $G_i$ to $G_{i-1}$ for $i \in \mathbb{Z}/h\mathbb{Z}$. Let

$$\pi : G_{\mathcal{O}_{\breve{F}}} \to G_1$$

be the projection to the first factor, which induces an identification

$$\mathbb{J}_b = \mathbb{J}_b^G \cong \mathbb{J}_{\underline{b}_1}^{G_1} = \mathbb{J}_{\underline{b}_1},$$

where $\underline{b}_1 = \pi(b\sigma(b)\cdots\sigma^{h-1}(b)) \in G_1(\breve{F})$ and the Frobenius automorphism of $G_1$ is given by $\sigma^h$.

The following lemma follows similarly as Corollary 1.6.

**Lemma A.1.** *The projection $\pi$ induces a $\mathbb{J}_b$-equivariant map*

$$\pi : \mathrm{Irr}^{\mathrm{top}} X_\mu(b) \to \sqcup_{\mu_1} \mathrm{Irr}^{\mathrm{top}} X_{\mu_1}^{G_1}(\underline{b}_1).$$

*Moreover, $N_{\mathbb{J}_b}(C) = N_{\mathbb{J}_{\underline{b}_1}}(\pi(C))$ for $C \in \mathrm{Irr} X_\mu(b)$.*

Now we can assume $G$ is absolutely simple by Lemma A.1. Moreover, we adopt the notation in §6.3. Notice that $V_{\mathrm{gen}}^{p(b\sigma)}$ is an open dense subset of $V^{p(b\sigma)}$. Notice that the diagonal map gives an isomorphism $V^{p(b\sigma)} \cong (V^d)^{p(b_\bullet \sigma_\bullet)}$.

Fix $v \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$ and let $z, M_b, M, b_M$ be as in §5.2. Notice that $b_M$ is a superbasic element of $M(\breve{F})$. We define $Y(v) = \{\lambda \in Y ; \lambda_\alpha \geqslant 0, \forall \alpha \in \Phi(v, +)\}$.

The following lemma is a reformulation of Corollary 5.5 and small cocharacters in §6.4.

**Lemma A.2.** *Let* $\lambda_\bullet = (\lambda_1, \ldots, \lambda_d) \in Y^d$. *Then we have*

*(1)* $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet, b_\bullet}^{G^d, \mathrm{top}}$ *if and only if (1)* $\lambda_i \in Y(v)$ *for* $1 \leqslant i \leqslant d$ *and*
*(2)* $z(\lambda_\bullet) := (z(\lambda_1), \ldots, z(\lambda_d)) \in \mathcal{A}_{z(\lambda_\bullet), b_{M_\bullet}}^{M^d, \mathrm{top}}$, *where* $b_{M_\bullet} = (1, \ldots, 1, b_M) \in$
$M^d(\breve{F})$.

*(2)* $\lambda_\bullet$ *is* $v$-*small if* $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet, b_\bullet}^{G^d, \mathrm{top}}$ *and for each* $p(b\sigma)$-*orbit* $\mathcal{O}$ *in* $\Pi(v, +)$
*we have* $(\lambda_i)_\alpha = 0$ *for some* $\alpha \in \mathcal{O}$ *and* $1 \leqslant i \leqslant d$.

To prove Proposition 6.4, we need some properties of permissible vectors introduced in §6.4. Let $M' \supseteq T$ be a Levi subgroup and let $\lambda, \eta \in Y$. Define

$$\mathcal{H}_{M'}(\lambda, \eta) = \{\gamma \in \Phi_{M'}; \lambda_\gamma \geqslant 0, \eta_\gamma \leqslant -1\} = -\mathcal{H}_{M'}(\eta, \lambda).$$

**Lemma A.3.** *For* $\lambda, \eta \in Y$ *we have*

*(0)* $b\sigma(\lambda)_{p(b\sigma)(\gamma)} = \lambda_\gamma$ *for* $\gamma \in \Phi$;
*(1)* $\mathcal{H}_{M_b}(\lambda, \eta) \subseteq \mathcal{H}_{M_b}(\lambda, \chi) \cup \mathcal{H}_{M_b}(\chi, \eta)$ *for* $\chi \in Y$;
*(2)* $z(\mathcal{H}_{M_b}(\lambda, \eta)) = \mathcal{H}_M(z(\lambda), z(\eta))$;
*(3)* $p(b\sigma)(\mathcal{H}_{M_b}(\lambda, \eta)) = \mathcal{H}_{M_b}(b\sigma(\lambda), b\sigma(\eta))$.

*Proof.* Note that (1) follows by definition, and (3) follows from (0) which is proved in Lemma 1.3 (3). As $z(\Phi_{M_b}^+) = \Phi_M^+$, we have $z(\lambda)_{z(\alpha)} = \lambda_\alpha$ for $\alpha \in \Phi_{M_b}$, from which (2) follows. □

**Corollary A.4.** *Let* $\lambda_\bullet = (\lambda_1, \ldots, \lambda_d) \in \mathcal{A}_{\mu_\bullet, b_\bullet}^{G^d, \mathrm{top}}$. *For* $1 \leqslant i \leqslant d$ *we have*

$$|\mathcal{H}_{M_b}(\lambda_i, b\sigma(\lambda_i))| \leqslant \mathrm{def}(b),$$

*where* $\mathrm{def}(b)$ *denotes the defect of* $b$.

*Proof.* By Lemma A.2 (2), $z(\lambda_\bullet) \in \mathcal{A}_{z(\lambda_\bullet), b_{M_\bullet}}^{M^d, \mathrm{top}}$. Moreover, $b_M$ is superbasic in $M(\breve{F})$. By Lemma A.3 we have

$$\begin{aligned}
\mathrm{def}(b) &= \mathrm{rk}_F(M) \\
&= |\mathcal{H}_M(z(\lambda_1), z(\lambda_2))| + \cdots + |\mathcal{H}_M(z(\lambda_{d-1}), z(\lambda_d))| \\
&\quad + |\mathcal{H}_M(z(\lambda_d), z(b\sigma(\lambda_1)))| \\
&= |\mathcal{H}_{M_b}(\lambda_1, \lambda_2)| + \cdots + |\mathcal{H}_{M_b}(\lambda_{d-1}, \lambda_d)| + |\mathcal{H}_{M_b}(\lambda_d, b\sigma(\lambda_1))| \\
&\geqslant |\mathcal{H}_{M_b}(\lambda_1, \lambda_i)| + |\mathcal{H}_{M_b}(\lambda_i, b\sigma(\lambda_1))| \\
&= |\mathcal{H}_{M_b}(b\sigma(\lambda_1), b\sigma(\lambda_i))| + |\mathcal{H}_{M_b}(\lambda_i, b\sigma(\lambda_1))| \\
&\geqslant |\mathcal{H}_{M_b}(\lambda_i, b\sigma(\lambda_i))|,
\end{aligned}$$

where $\mathrm{rk}_F(M)$ denote the $F$-semisimple rank of $M$, and the second equality follows from Lemma 3.8 and Lemma 3.10. □

For $\alpha \in \Pi(v, +)$ we define

$$Y(v, \alpha) = \{\lambda \in Y(v); \lambda_\alpha = 0, |\mathcal{H}_{M_b}(\lambda, b\sigma(\lambda))| \leqslant \mathrm{def}(b)\}.$$

**Lemma A.5.** *If $v \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$ is permissible, then we have $Y(v, \alpha) \neq \emptyset$ for $\alpha \in \Pi(v, +)$.*

*Proof.* By assumption, there is a $v$-small cocharacter $\lambda_\bullet = (\lambda_1, \ldots, \lambda_d)$. By definition, there exists a $\langle p(b\sigma) \rangle$-conjugate $\gamma$ of $\alpha$ such that $(\lambda_i)_\gamma = 0$ for some $1 \leqslant i \leqslant d$. Moreover, we have $|\mathcal{H}_{M_b}(\lambda_i, b\sigma(\lambda_i))| \leqslant \mathrm{def}(b)$ by Lemma A.4. So $\lambda_i \in Y(v, \gamma)$. By Lemma A.3, $Y(v, \gamma)$ and $Y(v, \alpha)$ are conjugate under $\langle b\sigma \rangle$. So the statement follows. $\square$

For $\alpha, \beta \in \Phi$ we write $\alpha \to \beta$ if there exists a sequence $\alpha = \gamma_0, \gamma_1, \ldots, \gamma_r = \beta$ of roots in $\Phi - \Phi_{M_b}$ such that $\gamma_i - \gamma_{i-1} \in \Phi^+$ is a simple root for $1 \leqslant i \leqslant r$.

**Lemma A.6.** *Assume $v \in V_{\mathrm{gen}}^{p(b\sigma)} \cap Y$ is permissible. If $\alpha \to \beta$ with $\alpha \in \Phi(v, +)$, then $\beta \in \Phi(v, +)$.*

*Proof.* We can assume $\beta - \alpha$ is a simple root. Notice that $\langle \beta, v \rangle \neq 0$ since $\beta \notin \Phi_{M_b} = \Phi_{M_v}$. If $\beta \notin \Phi(v, +)$, then $-\beta \in \Phi(v, +)$. Thus $-\beta + \alpha \in \Pi(v, +)$ is decomposable in $\Phi(v, +)$, contradicting Lemma 6.3. $\square$

### A.2. The classification

Now we apply Lemma 6.3 and Lemma A.5 to prove Proposition 6.4 when $b$ is ramified, that is, $b \in \Omega$ and the identity 1 are not $\sigma$-conjugate under $\Omega$ (noticing that $G$ is adjoint).

We argue by a case-by-case analysis on the (connected) Dynkin diagram of $\mathbb{S}_0$. The simple roots $\alpha_i$ of $\Phi^+$ are labeled as in [12, §11.4]. If the fundamental coweight $\varpi_i^\vee$ of $\alpha_i$ is minuscule, we denote by $\omega_i \in \Omega \cap t^{\varpi_i^\vee} W_0$ the unique length zero element. Let $\theta > 0$ denote the highest root.

For classical types we fix an ambient vector space $V_0 = \oplus_{i=1}^n \mathbb{R}e_i^\vee$ (of $\Phi^\vee$) and its dual $V_0^* = \oplus_{i=1}^n \mathbb{R}e_i$ together with a pairing $\langle, \rangle$ between $V_0$ and $V_0^*$ such that $\langle e_i, e_j^\vee \rangle = \delta_{i,j}$.

**A.2.1. Type $D_n$.** The simple roots are $\alpha_i = e_i - e_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $\alpha_n = e_{n-1} + e_n$.

Case(4.1.1): $\sigma = \mathrm{id}$ and $b = \omega_1$. Then $V^{p(b\sigma)} = \oplus_{i=2}^{n-1} \mathbb{R}e_i^\vee$ and $\Phi_{M_b}^+ = \{e_1 \pm e_n\}$. Suppose $\alpha_i \in \Phi(v, +)$ for some $2 \leqslant i \leqslant n-2$. Then

$$\alpha_i = e_i - e_{i+1} \to e_i - e_{n-1} \to e_1 - e_{n-1};$$

$$\alpha_i = e_i - e_{i+1} \to e_i - e_n \to e_i + e_{n-1} \to e_2 + e_{n-1}.$$

So $e_2 + e_{n-1}, e_1 - e_{n-1} \in \Phi(v, +)$ by Lemma A.6. Hence

$$\theta = (e_2 + e_{n-1}) + (e_1 - e_{n-1}) \in \Pi(v, +)$$

is decomposable in $\Phi(v, +)$, contradicting Lemma 6.3. Suppose $\alpha_1, \alpha_n, \alpha_{n-1} \in \Phi(v, +)$. Then we have $\alpha_n \to e_2 + e_n$ and $\alpha_{n-1} \to e_2 - e_n$. Hence $e_2 \pm e_n \in \Phi(v, +)$ and $\theta = \alpha_1 + (e_2 + e_n) + (e_2 - e_n)$ is decomposable in $\Phi(v, +)$, a contradiction. Thus $\Pi(v, +)$ equals $\Pi \setminus \{-\alpha_1, \theta\}$ or $\Pi \setminus \{-\alpha_{n-1}, -\alpha_n\}$ as desired.

Case(4.1.2): $n$ is odd and $b\sigma$ is of order 4. Let $m = (n-1)/2 \geqslant 2$. Then we have $V^{p(b\sigma)} = \oplus_{i=2}^m \mathbb{R}(e_i^\vee - e_{n+1-i}^\vee)$ and

$$\Phi_{M_b}^+ = \{e_i + e_{n+1-i}; 2 \leqslant i \leqslant m\} \cup \{e_1 \pm e_{m+1}, e_1 \pm e_n, e_{m+1} \pm e_n\}.$$

Denote by $T \subseteq M^1$ (resp. $T \subseteq M^i$ for $2 \leqslant i \leqslant m$) the Levi subgroup of $M_b$ whose set of positive roots is $\{e_1 \pm e_{m+1}, e_1 \pm e_n, e_{m+1} \pm e_n\}$ (resp. $\{e_i + e_{n+1-i}\}$).

Suppose $\alpha_i, \alpha_{n-i} \in \Phi(v, +)$ for some $2 \leqslant i \leqslant m-1$. Then $m \geqslant 3$ and

$$\begin{aligned}
\alpha_{n-i} = e_{n-i} - e_{n-i+1} &\to e_{n-i} - e_n \to e_{n-i} + e_{n-1} \to e_{m+1} \\
&+ e_{n-1} \to e_{m+1} + e_{m+2}; \\
\alpha_i = e_i - e_{i+1} &\to e_i - e_{m+1} \to e_2 - e_{m+1}; \\
\alpha_i = e_i - e_{i+1} &\to e_i - e_{m+2} \to e_1 - e_{m+2}.
\end{aligned}$$

So $e_{m+1} + e_{m+2}, e_2 - e_{m+1}, e_1 - e_{m+2} \in \Phi(v, +)$ and $\theta = (e_{m+1} + e_{m+2}) + (e_2 - e_{m+1}) + (e_1 - e_{m+2})$ is decomposable, a contradiction.

Suppose $\alpha_n \in \Phi(v, +)$. Then $\alpha_n \to e_{m+2} + e_n$ and hence $e_{m+2} + e_n \in \Phi(v, +)$, that is, $v(m+2) > 0$ as $v(n) = v(m+1) = v(1) = 0$. Thus

$$-\alpha_{m+1} = e_{m+2} - e_{m+1}, e_{m+2} \pm e_1, e_{m+2} \pm e_n \in \Pi(v, +).$$

Let $\lambda \in Y(v, -\alpha_{m+1})$. Then $\lambda_{-\alpha_m} = 0$ and $\lambda_{e_{m+2} \pm e_1}, \lambda_{e_{m+2} \pm e_n} \geqslant 0$, which means $\lambda(m+1) = \lambda(m+2)$ and

$$1 - \lambda(m+1) \leqslant \lambda(1) \leqslant \lambda(m+1), \; 1 - \lambda(m+1) \leqslant \lambda(n) \leqslant \lambda(m+1) - 1.$$

It follows that $|\mathcal{H}_{M^1}(\lambda, b\sigma(\lambda))| \geqslant 4$ because

$$e_{m+1} \pm e_1, e_{m+1} \pm e_n \in \mathcal{H}_{M^1}(\lambda, b\sigma(\lambda)).$$

On the other hand, one checks that $|\mathcal{H}_{M^i}(\lambda, b\sigma(\lambda))| = 1$ for $2 \leqslant i \leqslant m$. Thus

$$|\mathcal{H}_{M_b}(\lambda, b\sigma(\lambda))| = \sum_{j=1}^{m} |\mathcal{H}_{M^j}(\lambda, b\sigma(\lambda))| \geqslant m + 3 > m + 2 = \text{def}(b),$$

contradicting Lemma A.4. Thus $\Pi(v, +) = \Pi \setminus \{-\alpha_m, -\alpha_{m+1}\}$ as desired.

Case(4.1.3): $n$ is even and $b\sigma$ is of order 4. Let $m = n/2 \geqslant 2$. Then we have $V^{p(b\sigma)} = \oplus_{i=2}^{m} \mathbb{R}(e_i^{\vee} - e_{n+1-i}^{\vee})$ and

$$\Phi_{M_b}^{+} = \{e_i + e_{n+1-i}; 2 \leqslant i \leqslant m\} \cup \{e_1 \pm e_n\}.$$

Suppose $\alpha_i, \alpha_{n-i} \in \Phi(v, +)$ for some $2 \leqslant i \leqslant m - 1$. Then $m \geqslant 3$ and

$$\alpha_{n-i} = e_{n-i} - e_{n-i+1} \to e_{n-i} + e_n \to e_{n-i} + e_{n-i+2} \to e_m + e_{m+2};$$
$$\alpha_i = e_i - e_{i+1} \to e_2 - e_{m+2};$$
$$\alpha_i = e_i - e_{i+1} \to e_1 - e_m.$$

So $e_m + e_{m+2}, e_2 - e_{m+2}, e_1 - e_m \in \Phi(v, +)$ and $\theta = (e_m + e_{m+2}) + (e_2 - e_{m+2}) + (e_1 - e_m)$ is decomposable in $\Phi(v, +)$, a contradiction. Suppose $\alpha_1, \alpha_{n-1}, \alpha_n \in \Phi(v, +)$. Then $\alpha_{n-1} \to e_2 - e_n$ and $\alpha_n \to e_2 + e_n$. So $e_2 \pm e_n \in \Phi(v, +)$ and $\theta = \alpha_1 + (e_2 + e_n) + (e_2 - e_n)$ is decomposable in $\Phi(v, +)$, a contradiction. Therefore, $\Pi(v, +) = \Pi \setminus \{-\alpha_m\}$ as desired.

Case(4.1.4): $b\sigma$ is of order 2 and $b \in \{\omega_{n-1}, \omega_n\}$. Let $m = \lfloor n/2 \rfloor \geqslant 2$. Then we have $V^{p(b\sigma)} = \oplus_{i=1}^{m} \mathbb{R}(e_i^{\vee} - e_{n+1-i}^{\vee})$ and

$$\Phi_{M_b}^{+} = \{e_i + e_{n+1-i} \in \Phi; 1 \leqslant i \leqslant m\}.$$

Suppose $\alpha_i, \alpha_{n-i} \in \Phi(v, +)$ for some $2 \leqslant i \leqslant m$. Then

$$\alpha_{n-i} = e_{n-i} - e_{n-i+1} \to e_{n-i} + e_n \to e_2 + e_n;$$
$$\alpha_i = e_i - e_{i+1} \to e_1 - e_n.$$

So $e_2 + e_n, e_1 - e_n \in \Phi(v, +)$ and $\theta = (e_2 + e_n) + (e_1 - e_n)$ is decomposable in $\Phi(v, +)$, a contradiction. It is also impossible that $\alpha_1, \alpha_{n-1}, \alpha_n \in \Phi(v, +)$ as in Case(4.1.3). Therefore, we deduce that $\Pi(v, +)$ equals $\Pi \setminus \{-\alpha_{n-1}, -p(b\sigma)(\alpha_{n-1})\}$ or $\Pi \setminus \{-\alpha_n, -p(b\sigma)(\alpha_n)\}$ as desired.

**A.2.2. Type $B_n$.**   The simple roots are $\alpha_i = e_i - e_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $\alpha_n = e_n$. We can assume $\sigma = \mathrm{id}$ and $b = \omega_1$. In this case, $V^{p(b\sigma)} = \oplus_{i=2}^n \mathbb{R} e_i^\vee$ and $\Phi_{M_b}^+ = \{e_1\}$. Suppose $\alpha_i \in \Phi(v,+)$ for some $2 \leqslant i \leqslant n-1$. Then

$$\alpha = e_i - e_{i+1} \to e_2 - e_n \to e_2 \to e_2 + e_n \to e_1 + e_n.$$

So $e_2 - e_n, e_1 + e_n \in \Phi(v,+)$ and $\theta = (e_2 - e_n) + (e_1 + e_n)$ is decomposable in $\Phi(v,+)$, a contradiction.

Suppose $\alpha_1 \in \Phi(v,+)$. Then $\alpha_1 \to e_1 - e_n \in \Phi(v,+)$, which means $v(n) < 0$ as $v(1) = 0$. So we have $-\alpha_n = -e_n, \pm e_1 - e_n \in \Phi(v,+)$. Let $\lambda \in Y(v, -\alpha_n)$. Then $\lambda_{-\alpha_n} = 0$ and $\lambda_{\pm e_1 - e_n} \geqslant 0$, which means $\lambda(n) = 0$, $\lambda(1) - \lambda(n) \geqslant 1$ and $-\lambda(1) - \lambda(n) \geqslant 0$, a contradiction. Thus $\Pi(v,+) = \Pi \setminus \{-\alpha_n\}$ as desired.

**A.2.3. Type $C_n$.**   The simple roots are $\alpha_i = e_i - e_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $\alpha_n = 2e_n$. We can assume $\sigma = \mathrm{id}$ and $b = \omega_n$. Let $m = \lfloor n/2 \rfloor \geqslant 1$. Then $V^{p(b\sigma)} = \oplus_{i=1}^m \mathbb{R}(e_i^\vee - e_{n+1-i}^\vee)$ and

$$\Phi_{M_b}^+ = \{e_i + e_{n+1-i}; 1 \leqslant i \leqslant m+1\}.$$

Case(4.3.1): $n = 2m$. Suppose $\alpha_i, \alpha_{n-i} \in \Phi(v,+)$ for some $1 \leqslant i \leqslant m-1$. Then $m \geqslant 2$ and

$$\alpha_{n-i} = e_{n-i} - e_{n-i+1} \to e_{n-i} + e_n \to e_m + e_n;$$
$$\alpha_i = e_i - e_{i+1} \to e_1 - e_m \to e_1 - e_n$$

So $e_m + e_n, e_1 - e_m, e_1 - e_n \in \Phi(v,+)$ and $\theta = (e_m + e_n) + (e_1 - e_m) + (e_1 - e_n)$ is decomposable in $\Phi(v,+)$, a contradiction.

Suppose $\alpha_n, -\theta \in \Phi(v,+)$. Then $\alpha_n = 2e_n \to 2e_{m+1}$ and $-\theta \to -2e_m$, which means $2e_{m+1}, -2e_m \in \Phi(v,+)$. Let $\lambda \in Y(v, -\alpha_m)$. Then $\lambda_{-\alpha_m} = 0$ and $\lambda_{2e_{m+1}}, \lambda_{-2e_m} \geqslant 0$, which means $\lambda(m+1) = \lambda(m)$, $\lambda(m+1) > 0$, $\lambda(m) \leqslant 0$, a contradiction. Thus $\Pi(v,+) = \Pi \setminus \{-\alpha_m\}$ as desired.

Case(4.3.2): $n = 2m+1$. Suppose $\alpha_i, \alpha_{n-i} \in \Phi(v,+)$ for some $1 \leqslant i \leqslant m$. Then

$$\alpha_{n-i} = e_{n-i} - e_{n-i+1} \to e_{n-i} + e_n \to e_{m+1} + e_{m+2};$$
$$\alpha_i = e_i - e_{i+1} \to e_1 - e_{m+1} \to e_1 - e_{m+2}.$$

So $e_{m+1} + e_{m+2}, e_1 - e_{m+1}, e_1 - e_{m+2} \in \Phi(v,+)$ and $\theta = (e_{m+1} + e_{m+2}) + (e_1 - e_{m+1}) + (e_1 - e_{m+2})$ is decomposable in $\Phi(v,+)$, a contradiction. Thus $\Pi(v,+) = \Pi \setminus \{-\alpha_n, \theta\}$ as desired.

**A.2.4. Type $A_{n-1}$.** The simple roots are $\alpha_i = e_i - e_{i+1}$ for $1 \leqslant i \leqslant n-1$. Let $\varsigma_0$ be the automorphism exchanging $\alpha_i$ and $\alpha_{n-i}$ for $1 \leqslant i \leqslant n-1$.

Case(4.4.1): $\sigma = \mathrm{id}$. Suppose $\langle b \rangle = \langle \omega_h \rangle$ for some $1 \leqslant h \leqslant n-1$ dividing $n$. Then

$$\Phi^+_{M_b} = \{e_i - e_j \in \Phi^+; i - j \in h\mathbb{Z}\}.$$

If $h = 1$, then $v = 0$ and $\Pi(v, +) = \emptyset$ as desired. Suppose $h \geqslant 2$ and we can assume $\alpha_1 \in \Pi(v, +)$. If $\alpha_i \in \Phi(v, +)$ for some $2 \leqslant i \leqslant h$, then

$$\alpha_1 = e_1 - e_2 \rightarrow e_1 - e_i \rightarrow e_1 - e_h$$
$$\alpha_i = e_i - e_{i+1} \rightarrow e_i - e_{h+1}.$$

So $e_1 - e_{h+1}, e_1 - e_h$ and their $\langle p(b\sigma) \rangle$-conjugates are contained in $\Phi(v, +)$. Hence

$$\theta = (e_1 - e_{h+1}) + (e_{h+1} - e_{2h+1}) + \cdots + (e_{n-2h+1} - e_{n-h+1}) + (e_{n-h+1} - e_n)$$

is decomposable in $\Phi(v, +)$, a contradiction. Thus $\Pi(v, +) = \Pi \setminus \mathcal{O}$, where $\mathcal{O}$ is any $p(b\sigma)$-orbit of $\Pi$.

Case(4.4.2): $\sigma = \varsigma_0$, $b = \omega_1$ and $n \geqslant 4$ is even. Let $m = n/2 \geqslant 2$. Then we have

$$\Phi^+_{M_b} = \{e_1 - e_{m+1}\}.$$

Suppose $\alpha_i, \alpha_{n+1-i} \in \Phi(v, +)$ for some $2 \leqslant i \leqslant m-1$, then $m \geqslant 3$ and

$$\alpha_i = e_i - e_{i+1} \rightarrow e_i - e_{n+1-i} \rightarrow e_1 - e_{n+1-i}$$
$$\alpha_{n+1-i} = e_{n+1-i} - e_{n+2-i} \rightarrow e_{n+1-i} - e_n.$$

So $e_1 - e_{n+1-i}, e_{n+1-i} - e_n \in \Phi(v, +)$ and $\theta = (e_1 - e_{n+1-i}) + (e_{n+1-i} - e_n)$ is decomposable in $\Phi(v, +)$, a contradiction. Suppose $\alpha_1, \alpha_m \in \Phi(v, +)$, then $\alpha_1 \rightarrow e_1 - e_m$ and $\alpha_m \rightarrow e_m - e_n$. So $\theta = (e_1 - e_m) + (e_m - e_n)$ is decomposable in $\Phi(v, +)$, a contradiction. Thus $\Pi(v, +)$ equals $\Pi \setminus \{-\alpha_m, -\alpha_{m+1}\}$ or $\Pi \setminus \{-\alpha_1, \theta\}$ as desired.

**A.2.5. Type $E_6$.** The simple roots $\alpha_i$ for $1 \leqslant i \leqslant 6$ are labeled as in [12, §11.4]. We can assume $\sigma = \mathrm{id}$ and $b = \omega_1$. Then we have

$$V^{p(b\sigma)} = \mathbb{R}\alpha_4^\vee \oplus \mathbb{R}(\alpha_2^\vee + \alpha_3^\vee + 2\alpha_4^\vee + \alpha_5^\vee).$$

Suppose $\alpha = \alpha_i \in \Phi(v, +)$ with $i = 2$ or $i = 4$. Let $\beta = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ and $\gamma = \beta + \alpha_1 + \alpha_4 + \alpha_6$. Then

$$\alpha \to \beta \to \beta + \alpha_4 \to \gamma.$$

So $\beta, \gamma \in \Phi(v, +)$ and $\theta = \beta + \gamma$ is decomposable in $\Phi(v, +)$, a contradiction. Thus $\Pi(v, +) = \Pi \setminus \{-\alpha_1, -\alpha_6, \theta\}$ as desired.

**A.2.6. Type $E_7$.** The simple roots $\alpha_i$ for $1 \leqslant i \leqslant 7$ are labeled as in [12, §11.4]. We can assume $\sigma = \mathrm{id}$ and $b = \omega_7$. Then we have

$$\Phi^+_{M_b} = \{\gamma + \alpha_3, \gamma + \alpha_5 + \alpha_6 - \alpha_1, \gamma + \alpha_5\},$$

where $\gamma = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$. Suppose $\alpha = \alpha_i \in \Phi(v, +)$ for some $1 \leqslant i \leqslant 6$. Let $\xi = \gamma + \alpha_3 + \alpha_5 - \alpha_7$. Then $\gamma - \beta, \xi - \beta \notin \mathbb{Z}_{\geqslant 0}\Pi_0$ for $\beta \in \Phi^+_{M_b}$, which implies that

$$\alpha \to \gamma \text{ and } \alpha \to \xi.$$

So $\gamma, \xi \in \Phi(v, +)$ and $\theta = \gamma + \xi$ is decomposable in $\Phi(v, +)$, a contradiction. Thus $\Pi(v, +) = \Pi \setminus \{-\alpha_7, \theta\}$ as desired.

## Appendix B.  Proof of Proposition 1.2

Let $b \in \Omega$ be basic. Let $J \subseteq \mathbb{S}_0$ be a minimal $\sigma$-stable subset such that $[b] \cap M_J(\breve{F}) \neq \emptyset$. Then $[b] \cap M_J(\breve{F})$ is a superbasic $\sigma$-conjugacy class of $M(\breve{F})$. Suppose there is another minimal $\sigma$-stable subset $J' \subseteq \mathbb{S}_0$ such that $[b] \cap M_{J'}(\breve{F}) \neq \emptyset$. To prove Lemma 1.2, we have to show that $J = J'$. Choose $x \in \Omega_J$ and $x' \in \Omega_{J'}$ such that $x, x' \in [b]$. Let ${}^J W_0^J$ be the set of elements $w \in W_0$ which are minimal in its double coset $W_{J'} w W_J$. For $u \in W_0$ we set $\mathrm{supp}_\sigma(u) = \cup_{i \in \mathbb{Z}} \sigma^i(\mathrm{supp}(u)) \subseteq \mathbb{S}_0$, where $\mathrm{supp}(u) \subseteq \mathbb{S}_0$ is the set of simple reflections that appear in some/any reduced expression of $u$.

Following [13], we say $\tilde{w} \in \tilde{W}$ is $\sigma$-straight if

$$\ell(\tilde{w}\sigma(\tilde{w}) \cdots \sigma^{n-1}(\tilde{w})) = n\ell(\tilde{w}) \text{ for } n \in \mathbb{Z}_{\geqslant 1}.$$

Moreover, we say a $\sigma$-conjugacy class of $\tilde{W}$ is straight if it contains some $\sigma$-straight element. By [13, Proposition 3.2], the $\sigma$-conjugacy classes of $x$ and $x'$ are straight. Moreover, as $x, x' \in [b]$, these two straight $\sigma$-conjugacy classes coincide by [13, Theorem 3.3]. Thus there exists $\tilde{w} \in \tilde{W}$ such that

$\tilde{w}x = x'\sigma(\tilde{w})$. Write $p(\tilde{w}) = uzw^{-1}$ with $u \in W_{J'}$, $w \in W_J$ and $z \in {}^{J'}W_0^J$. By taking the projection $p$, we have

$$zw^{-1}p(x)\sigma(w) = u^{-1}p(x')\sigma(u)\sigma(z).$$

Notice that $z, \sigma(z) \in {}^{J'}W_0^J$. Moreover, we have $\operatorname{supp}_\sigma(w^{-1}p(x)\sigma(w)) = J$ and $\operatorname{supp}_\sigma(u^{-1}p(x')\sigma(u)) = J'$ by the minimality of $J$ and $J'$. This means that $z = \sigma(z)$ and $zJz^{-1} = J'$. So Proposition 1.2 follows from the following lemma.

**Lemma B.1.** *Let $J \subseteq \mathbb{S}_0$ be a minimal $\sigma$-stable subset such that $[b] \cap M_J(\check{F}) \neq \emptyset$. If there exists $z = \sigma(z) \in W_0^J$ such that $zJz^{-1} \subseteq \mathbb{S}_0$, then $zJz^{-1} = J$.*

*Proof.* It suffices to consider the case where $G = G_{\mathrm{ad}}$ and $\mathbb{S}_0$ is connected. If $b$ is unramified, that is, $1 \in [b]$, then we can take $J = \emptyset$ and the statement is trivial. So we assume that $b$ is not unramified. By the discussion above, it suffices to show the statement for some fixed $J$, and we can take $J$ as follows. Let $v$ be a generic point of $Y_{\mathbb{R}}^{p(b\sigma)}$, that is, if $\langle \alpha, v \rangle = 0$ for some $\alpha \in \Phi$, then $\langle \alpha, Y_{\mathbb{R}}^{p(b\sigma)} \rangle = 0$. Then we take $J$ to be the set of simple reflections $s$ such that $s(\bar{v}) = \bar{v}$, where $\bar{v}$ is the unique dominant $W_0$-conjugate of $v$. Then $[b] \cap M_J(\check{F})$ is a superbasic $\sigma$-conjugacy class of $M_J(\check{F})$ by [14, Lemma 3.1].

   Case(1): $\mathbb{S}_0$ is of type $A_{n-1}$ for $n \geqslant 2$. Take the simple roots as $\alpha_i = e_i - e_{i+1}$ for $1 \leqslant i \leqslant n - 1$. Let $\omega_1$ be the generator of $\Omega \cong \mathbb{Z}/n\mathbb{Z}$ such that $\omega_1 \in t^{\varpi_1^\vee}W_0$, where $\varpi_1^\vee$ is the fundamental coweight corresponding to the simple root $\alpha_1$. Assume $b = \omega_1^m$ for some $m \in \mathbb{Z}$.

   Case(1.1): $\sigma = \mathrm{id}$. Let $h$ be the greatest common divisor of $m$ and $n$, and $f = n/h$. Then we can take

$$J = \{s_{i+jf}; 1 \leqslant i \leqslant f - 1, 0 \leqslant j \leqslant h - 1\}.$$

Here, and in the sequel, $s_i$ denotes the simple reflection corresponding to the simple root $\alpha_i$. By assumption, $z$ sends each of the subsets

$$D_j = \{1 + jf, 2 + jf, \ldots, (j+1)f\}, \quad 0 \leqslant j \leqslant h - 1$$

to a subset of the form

$$\{k + 1, k + 2, \cdots k + f\} \subseteq \{1, 2, \cdots, n\}.$$

This implies that $z$ permutes the sets $D_j$ for $0 \leqslant j \leqslant h - 1$. In particular, $zJz^{-1} = J$ as desired.

Case(1.2): $\sigma$ is of order 2. In this case, $\sigma$ sends $\alpha_i$ to $\alpha_{n-i}$ for $1 \leqslant i \leqslant n-1$. As $b$ is not unramified, $n$ is even. Moreover, we can take $b = \omega_1$ and

$$J = \{s_{n/2}\}.$$

Noticing that $zJz^{-1} \subseteq \mathbb{S}_0$ is $\sigma$-stable and that $s_{n/2}$ is the unique simple reflection fixed by $\sigma$, we deduce that $zJz^{-1} = J$ as desired.

Case(2): $\mathbb{S}_0$ is of type $B_n$ for $n \geqslant 2$. Then $\sigma = \mathrm{id}$ and $b$ is of order 2 (since it is not unramified). Take the simple roots as $\alpha_n = e_n$ and $\alpha_i = e_i - e_{i+1}$ for $1 \leqslant i \leqslant n-1$. Then we can take

$$J = \{s_n\}.$$

Noticing that $\alpha_n$ is the unique short simple root, we have $z(\alpha_n) = \alpha_n$ and $zJz^{-1} = J$ as desired.

Case(3): $\mathbb{S}_0$ is of type $C_n$ for $n \geqslant 3$. Then $\sigma = \mathrm{id}$ and $b$ is of order 2. Take the simple roots as $\alpha_n = 2e_n$ and $\alpha_i = e_i - e_{i+1}$ for $1 \leqslant i \leqslant n-1$. We can take

$$J = \{s_1, s_3, \ldots, s_{2\lfloor \frac{n-1}{2} \rfloor + 1}\}.$$

If $n$ is odd, $J$ corresponds to the unique orthogonal subset of $(n+1)/2$ simple roots, which means $zJz^{-1} = J$ as desired. If $n$ is even, $J$ corresponds to the unique orthogonal subset of $n/2$ short simple roots, which also means $zJz^{-1} = J$ as desired.

Case(4): $\mathbb{S}_0$ is of type $D_n$ for $n \geqslant 4$. Take the simple roots as $\alpha_n = e_{n-1} + e_n$ and $\alpha_i = e_i - e_{i+1}$ for $1 \leqslant i \leqslant n-1$. As $b$ is not unramified, we have $\sigma^2 = 1$. The Weyl group $W_0$ is the set of permutations $w$ of $\{\pm 1, \ldots, \pm n\}$ such that $z(\pm i) = \pm z(i)$ for $1 \leqslant i \leqslant n$ and $\mathrm{sgn}(w) = 0 \in \mathbb{Z}/2\mathbb{Z}$, where

$$\mathrm{sgn}(w) = |\{1 \leqslant i \leqslant n; iw(i) < 0\}| \mod 2.$$

Case(4.1): $\sigma = \mathrm{id}$. If $b \in t^{\varpi_1^\vee} W_0$, we can take

$$J = \{s_{n-1}, s_n\}.$$

As $zJz^{-1} \subseteq \mathbb{S}_0$, $z$ preserves the set $\{\pm(n-1), \pm n\}$ and hence $zJz^{-1} = J$ as desired. If $b \in t^{\varpi_n^\vee} W_0$, we can take

$$J = \begin{cases} J_1 := \{s_1, s_3 \cdots, s_{n-3}, s_{n-1}\}, & \text{if } n \text{ is even, } \frac{n}{2} \text{ is even;} \\ J_2 := \{s_1, s_3 \cdots, s_{n-3}, s_n\}, & \text{if } n \text{ is even, } \frac{n}{2} \text{ is odd;} \\ J_0 := \{s_1, s_3 \cdots, s_{n-2}, s_{n-1}, s_n\}, & \text{otherwise.} \end{cases}$$

Suppose $n, n/2$ are even and that $J \neq zJz^{-1} \subseteq \mathbb{S}_0$. Then $zJz^{-1} = zJ_1z^{-1} = J_2$ because $J_1, J_2$ correspond to the only two maximal orthogonal subset of simple roots which do not contain $\{s_{n-1}, s_n\}$. By composing $z$ with a suitable element in the symmetric group of $\{1, \ldots, n\}$, we can assume that $z(\alpha_{1+2j}) = \alpha_{1+2j}$ for $0 \leqslant j \leqslant n/2 - 2$ and $z(\alpha_{n-1}) = \alpha_n$. This implies that $\operatorname{sgn}(z) = 1$, which is a contradiction as desired. The case where $n$ is even and $n/2$ is odd follows in a similar way. Suppose $n$ is odd. Then $J = J_0$ is the unique Dynkin subdiagram of $\mathbb{S}_0$ which is of type $(A_1)^{\frac{n-3}{2}} \times A_3$. So $zJz^{-1} = J$ as desired.

Case(4.1): $\sigma$ is of order 2. By symmetry, we can assume $\sigma(\alpha_n) = \alpha_{n-1}$. As $b$ is not unramified, we can assume $b \in t^{\varpi_n^\vee} W_0$. We can take

$$J = \begin{cases} \{s_1, s_3 \cdots, s_{n-3}, s_{n-1}, s_n\}, & \text{if } n \text{ is even}; \\ \{s_1, s_3 \cdots, s_{n-2}\}, & \text{otherwise.} \end{cases}$$

If $n$ is even, then $J$ corresponds to the unique orthogonal subset of $(n+2)/2$ simple roots. So $zJz^{-1} = J$ as desired. If $n$ is odd, then $J$ corresponds to the unique orthogonal $\sigma$-stable subset of $(n-1)/2$ simple roots except $\alpha_{n-1}$ and $\alpha_n$. So $zJz^{-1} = J$ as desired.

Case(5): $\mathbb{S}_0$ is of type $E_6$. As $b$ is not unramified, $\sigma = \operatorname{id}$ and we can assume $b \in t^{\varpi_1^\vee} W_0$. Here, and in the sequel, we using the labeling of $E_6$ and $E_7$ as in [12, §11]. We can take

$$J = \{s_1, s_3, s_5, s_6\}.$$

Then $zJz^{-1} = J$ since $J \subseteq \mathbb{S}_0$ is the unique Dynkin subdiagram of type $A_2 \times A_2$.

Case(6): $\mathbb{S}_0$ is of type $E_7$. Then $\sigma = \operatorname{id}$ and $b \in t^{\varpi_7^\vee} W_0$. We can take

$$J = \{s_2, s_5, s_7\}.$$

Then the statement is verified by computer or by the Lusztig-Spaltenstein algorithm. The proof is finished. $\qquad\square$

## References

[1] A. Braverman, D. Gaitsgory, *Crystals via the affine Grassmannian*, Duke Math. J. 107 (2001), 561–575.

[2] B. Bhatt, P. Scholze, *Projectivity of the Witt vector affine Grassmannian*, Invent. Math. 209, 329–423.

[3] M. Chen, M. Kisin, E. Viehmann, *Connected components of affine Deligne-Lusztig varieties in mixed characteristic*, Compos. Math. 151 (2015), 1697–1762.

[4] L. Chen, S. Nie, *Connected components of closed affine Deligne-Lusztig varieties*, Math. Ann. 375 (2019), 1355–1392.

[5] A. de Jong, F. Oort, *Purity of the stratification by Newtion polygons*, JAMS 13 (2000), 209–241.

[6] Q. Gashi, *On a conjecture of Kottwitz and Rapoport*, Ann. Sci. École Norm. Sup. 43 (2010), 1017–1038.

[7] U. Görtz, X. He, *Basic loci in Shimura varieties of Coxeter type*, Cambridge J. Math. 3 (2015), no. 3, 323–353.

[8] U. Görtz, T. Haines, R. Kottwitz and D. Reuman, *Dimensions of some affine Deligne-Lusztig varieties*, Ann. Sci. École. Norm. Sup. (4) 39 (2006), 467–511.

[9] U. Görtz, X. He, S. Nie, *Fully Hodge-Newton decomposable Shimura varieties*, Peking Math. J. 2 (2019), 99–154.

[10] P. Hamacher, *The dimension of affine Deligne-Lusztig varieties in the affine Grassmannian*, IMRN 23 (2015), 12804–12839.

[11] P. Hamacher, *The geometry of Newton strata in the reduction modulo p of Shimura varieties of PEL type*, Duke Math. J. 164 (2015), 2809–2895.

[12] J. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, New York-Berlin, 1972.

[13] X. He, S. Nie, *Minimal length elements of extended affine Weyl groups*, Compositio Math. 150 (2014), 1903–1927.

[14] X. He, S. Nie, *On the acceptable elements*, IMRN (2018), 907–931.

[15] B. Howard, G. Pappas, *On the supersingular locus of the $GU(2,2)$ Shimura variety*, Algebra Number Theory 8 (2014), 1659–1699.

[16] B. Howard, G. Pappas, *Rapoport-Zink spaces for spinor groups*, arXiv:1509.03914 (2015), to appear in Compos. Math.

[17] P. Hamacher, E. Viehmann, *Irreducible components of minuscule affine Deligne-Lusztig varieties*, Algebra Number Theory 12 (2018), 1611–1634.

[18] U. Hartl, E. Viehmann, *The Newton stratification on deformations of local G-shtukas*, J. Reine Angew. Math. 656 (2011), 87–129.

[19] U. Hartl, E. Viehmann,, *Foliations in deformation sapces of local G-shtukas*, Adv. Math. 299 (2012), 54–78.

[20] T. Haines, *Equidimensionality of convolution morphisms and applications to saturation problems, appendix with M. Kapovich and J. Millson*, Adv. Math., 207 (2006), 297–327.

[21] X. He, *Geometric and homological properties of affine Deligne-Lusztig varieties*, Ann. Math. 179 (2014), 367–404.

[22] X. He, *Kottwitz-Rapoport conjecture on unions of affine Deligne-Lusztig varieties*, Ann. Sci. Éc. Norm. Supér. (4) 49 (2016), 1125–1141.

[23] X. He, *Some results on affine Deligne-Lusztig varieties*, Proc. Int. Cong. of Math. 2018, Rio de Janeiro, Vol. 1 (1341–1362).

[24] X. He, *Cordial elements and dimensions of affine Deligne-Lusztig varieties*, arXiv:2001.03325.

[25] Pol van Hoften, *Mod p points on Shimura varieties of parahoric level (with an appendix by Rong Zhou)*, arXiv:2010.10496.

[26] X. He, Q. Yu, *Dimension formula of the affine Deligne-Lusztig variety $X(\mu, b)$*, Math. Ann. 379 (2021), 1747–1765.

[27] X. He, R. Zhou, *On the connected components of affine Deligne-Lusztig varieties*, Duke Math. J. 169 (2020), 2697–2765.

[28] X. He, R. Zhou, Y. Zhu, *Component stabilizers of affine Deligne-Lusztig varieties in the affine Grassmannian*, arXv:2109.02594.

[29] M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Comm. Math. Phys. 133 (1990), 249–260.

[30] M. Kisin, *Mod p points on Shimura varieties of abelian type*, J. Amer. Math. Soc. 30 (2017), 819–914.

[31] R. Kottwitz, *Isocrystals with additional structure*, Compositio Math. 56 (1985), 201–220.

[32] R. Kottwitz, *Dimensions of Newton strata in the adjoint quotient of reductive groups*, Pure Appl. Math. Q. 2 (2006), 817–836.

[33] R. Kottwitz, *On the Hodge-Newton decomposition for split groups*, IMRN 26 (2003), 1433–1447.

[34] R. Kottwitz, M. Rapoport, *On the existence of F-crystals*, Comment. Math. Helv. 78 (2003), 153–184.

[35] S. Kudla, M. Rapoport, *Arithmetic Hirzebruch-Zagier cycles*, J. Reine Angew. Math. **515** (1999), 155–244.

[36] S. Kudla, M. Rapoport, *Special cycles on unitary Shimura varieties I. Unramified local theory*, Invent. Math. 184 (2011), 629–682.

[37] P. Littelman, *Paths and root operators in representation theory*, Ann. Math. 142 (1995), 499–525.

[38] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. 3 (1990), 447–498.

[39] I. Mirković, K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. 166 (2007), 95–143.

[40] E. Milićević, E. Viehmann, *Generic Newton points and the Newton poset in Iwahori-double cosets*, Forum Math. Sigma 8 (2020), e50, 18 pp.

[41] S. Nie, *Connected components of affine Deligne-Lusztig varieties in affine Grassmannians*, Amer. J. Math. 140 (2018), 1357–1397.

[42] S. Nie, *Semi-modules and irreducible components of affine Deligne-Lusztig varieties*, arXiv:1802.04579.

[43] B-C. Ngô, P. Polo, *Résolutions de Demazure affines et formule de Casselman-Shalika géométrique*, J. Alg. Geom. 10 (2001), 515–547.

[44] M. Rapoport, *A guide to the reduction modulo p of Shimura varieties*, Astérisque (2005), no. 298, 271–318.

[45] M. Rapoport, M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. 103 (1996), 153–181.

[46] M. Rapoport, T. Zink, *Period spaces for p-divisible groups*, Ann. Math. Studies. 141, Princeton University Press, 1996.

[47] M. Rapoport, U. Terstiege, S. Wilson, *The supersingular locus of the Shimura variety for $GU(1, n-1)$ over a ramified prime*, Math. Z. 276 (2014), 1165–1188.

[48] E. Viehmann, *The dimension of some affine Deligne-Lusztig varieties*, Ann. Sci. École. Norm. Sup. (4) 39 (2006), 513–526.

[49] E. Viehmann, *Moduli spaces of p-divisible groups*, J. Alg. Geom. 17 (2008), 341–374.

[50] E. Viehmann, *The global structure of moduli spaces of polarized p-divisible groups*, Doc. Math. 13 (2008), 825–852.

[51] E. Viehmann, *Connected components of closed affine Deligne-Lusztig varieties*, Math. Ann. 340 (2008), 315–333.

[52] I. Vollaard, T. Wedhorn, *The supersingular locus of the Shimura variety of* $\mathrm{GU}(1, n-1)$ *II*, Invent. Math. 184 (2011), 591–627.

[53] L. Xiao and X. Zhu, *Cycles of Shimura varieties via geometric Satake*, arXiv:1707.05700.

[54] W. Zhang, *On arithmetic fundamental lemmas*, Invent. Math. 188 (2012), 197–252.

[55] X. Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, Ann. Math. 185 (2017), 403–492.

[56] R. Zhou, *Mod p isogeny classes on Shimura varieties with parahoric level structure*, Duke Math. J. 169 (2020), 2937–3031.

[57] R. Zhou, Y. Zhu *Twisted orbital integrals and irreducible components of affine Deligne-Lusztig varieties*, Camb. J. Math. 8 (2020), 149–241.

SIAN NIE
ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE
CHINESE ACADEMY OF SCIENCES
BEIJING 100190, CHINA
AND
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF CHINESE ACADEMY OF SCIENCES
CHINESE ACADEMY OF SCIENCES
BEIJING 100049, CHINA
*E-mail address:* niesian@amss.ac.cn