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A Paley-Wiener theorem for Harish-Chandra modules

Heiko Gimperlein, Bernhard Krötz,
Job Kuit, and Henrik Schlichtkrull

We formulate and prove a Paley-Wiener theorem for Harish-Chandra modules for a real reductive group. As a corollary we obtain a new and elementary proof of the Helgason conjecture.

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1. Introduction

Let G be a real reductive algebraic group and $K \subset G$ a maximal compact subgroup. Let V be a Harish-Chandra module for $(\mathfrak{g}, K)$ where $\mathfrak{g} = \mathrm{Lie}(G)$.

Every Harish-Chandra module admits a completion (globalization) to a representation of G. Such a completion is in general not unique. First and foremost is the smooth completion V^∞ of moderate growth, due to Casselman-Wallach, which is unique up to isomorphism, see [5], [23, Sect. 11] and [2]. Another completion is the G-module V^ω of analytic vectors in V^∞ with its natural compact-open topology.

Following Schmid [21, p. 316] we define the minimal completion of V by the convolution product

$$V_{\min} := C_c^\infty(G) * V \subset V^\infty$$

and endow it with a topology as follows: take a finite dimensional subspace $V_f \subset V$ which generates V, and consider the surjective map

$$C_c^\infty(G) \otimes V_f \twoheadrightarrow V_{\min}.$$

The quotient topology on $V_{\min}$ does not depend on the choice of the finite dimensional generating subspace V_f and thus induces a natural quotient Hausdorff locally convex topology on $V_{\min}$. It is inherent in the construction that $V_{\min}$ embeds equivariantly and continuously into every completion of V, hence the terminology.

Next we review Schmid's interpretation [21] of the Helgason conjecture. The conjecture was stated in [9] and first proven in [12]. Let χ be a character of the algebra $\mathbb{D}(G/K)$ of G-invariant differential operators on G/K. The Helgason conjecture states that the Poisson transform for G/K is an isomorphism between the space of hyperfunction sections of a line bundle over the minimal boundary of G/K and the space $C^\infty(G/K)_\chi$ of joint eigenfunctions of $\mathbb{D}(G/K)$ with eigencharacter χ. Then Schmid's interpretation and extension of the Helgason conjecture is

$$V_{\min} = V^\omega \tag{1.1}$$

as topological vector spaces, for all Harish-Chandra modules V. The equality (1.1) was stated in [21, Theorem on p. 317] and proved in [13, Theorem 2.12].

The objective of this work is to understand the equality (1.1) quantitatively. For that let $G = KAN$ be an Iwasawa decomposition of G and $G = KAK$ the associated Cartan decomposition. Let $\|\cdot\|$ be a Cartan-Killing norm on $\mathfrak{g}$, and define balls $A_R \subset A$ for any $R > 0$ by $A_R = \exp(\mathfrak{a}_R)$ and $\mathfrak{a}_R := \{X \in \mathfrak{a} \mid \|X\| \leq R\}$. This gives us a family of balls $B_R := KA_RK \subset G$, and we write $C_R^\infty(G) \subset C_c^\infty(G)$ for the subspace of functions with support in B_R. We define

$$V_R^{\min} := C_R^\infty(G) * V$$

and endow it with the quotient topology. Note that each of the spaces $V_R^{\min}$ is Fréchet and that

$$V_{\min} = \varinjlim_{R\to\infty} V_R^{\min} \tag{1.2}$$

where the inductive limit is taken in the category of locally convex spaces.

Next we consider the filtration of V^ω. We recall that a vector $v \in V^\infty$ is analytic if and only if it is K-analytic, i.e. the restricted orbit map

$$f_v : K \to V^\infty, \quad k \mapsto k \cdot v$$

is analytic (see Lemma 4.1). Now for any $r > 0$ we define a K-bi-invariant domain of $K_\mathbb{C}$ by $K_\mathbb{C}(r) := K\exp(i\mathfrak{k}_r)$, where

$$\mathfrak{k}_r = \{X \in \mathfrak{k} \mid \|X\| < r\}.$$

We define $V_r^\omega \subset V^\omega$ to be the subspace of those v for which f_v extends holomorphically to $K_\mathbb{C}(r)$ and endow it with the Fréchet topology of uniform convergence on compacta in $K_{\mathbb{C}}(r)$. We then obtain the filtration of V^ω as an inductive limit in the category of locally convex topological vector spaces

$$V^\omega = \varinjlim_{r\to 0} V_r^\omega. \tag{1.3}$$

In this article we prove that the two filtrations (1.2) and (1.3) are continuously sandwiched into each other: we prove the following two inclusions.

Analytic inclusion:
For all $r > 0$ there exists $R = R(r) > 0$ with $V_r^\omega \subset V_R^{\min}$.

Geometric inclusion:
For all $R > 0$ there exists $r = r(R) > 0$ with $V_R^{\min} \subset V_r^\omega$.

Observe that the equality (1.1) is a consequence already of the analytic inclusion and the minimality of $V_{\min}$. In turn, the geometric inclusion could be obtained from (1.1) and the Grothendieck factorization theorem [8, Ch. 4, Sect. 5, Th. 1] (see also [19, Corollary 24.35]), but we give a direct proof.

By a *Paley-Wiener type theorem for a Harish-Chandra module* V we understand the existence of the geometric and analytic inclusions together with bounds on the numbers $r(R)$ and $R(r)$. In this article we prove such a theorem.

To explain the terminology, we consider the following algebraic type of Fourier transform

$$\mathcal{F} = \bigoplus_{V\in\mathcal{HC}} \mathcal{F}_V : C_c^\infty(G) \to \bigoplus_{V\in\mathcal{HC}} \operatorname{Hom}_{(\mathfrak{g},K)}(V, V^\omega),$$

that is given by

$$\mathcal{F}_V\phi(v) = \phi * v \qquad (V \in \mathcal{HC}, v \in V).$$

Here $\mathcal{HC}$ is the category of Harish-Chandra modules. A complete Paley-Wiener theorem would be a description of the image under $\mathcal{F}$ of the filtration of $C_c^\infty(G)$. A step towards that is the localized version, i.e. for a fixed $V \in \mathcal{HC}$ a description for the image under $\mathcal{F}_V$ of the filtration of $C_c^\infty(G)$ in terms of the filtration on $\operatorname{Hom}_{(\mathfrak{g},K)}(V, V^\omega)$ induced from V^ω. Optimal estimates of $r(R)$ and $R(r)$ determining the geometric and analytic inclusions are an interesting open problem, even for groups of rank one.

1.1. Geometric inclusion

What we termed geometric inclusion has a straightforward relation to a problem concerning the complex geometry of the G-invariant crown domain $\Xi \subset Z_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ of the attached Riemannian symmetric space $Z = G/K$. The crown domain Ξ was first defined in [1] as in (3.1)-(3.2) below and characterized as the largest G-domain $Z \subset \Xi \subset Z_{\mathbb{C}}$ on which G acts properly. If $z_0 = K_{\mathbb{C}}$ is the standard base point, then the crown domain can alternatively be defined as the connected component of the intersection

$$\bigcap_{g\in G} gN_{\mathbb{C}}A_{\mathbb{C}} \cdot z_0 = \bigcap_{k\in K} kN_{\mathbb{C}}A_{\mathbb{C}} \cdot z_0$$

which contains z_0. The latter can also be rephrased by $\Xi \subset Z_{\mathbb{C}}$ being the maximal G-invariant domain containing Z such that for every K-spherical principal series representation $V = V_\lambda$ with $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and non-zero K-spherical vector $v_K = v_{K,\lambda}$ the orbit map

$$f_\lambda : G/K \to V_\lambda^\infty, \quad gK \mapsto \pi_\lambda(g)v_{K,\lambda}$$

extends as a holomorphic map to $\Xi \to V_\lambda^\infty$. (See [18] and [17] for the fact that every f_λ extends holomorphically to Ξ, and [16, Sect. 4] for the fact that Ξ is maximal with respect to this property.)

Given $R > 0$ we define an $\mathrm{Ad}(K)$-invariant open subset in $\mathfrak{k}$ by

$$\mathfrak{k}(R) := \{X \in \mathfrak{k} \mid \exp(iX)B_R \cdot z_0 \subset \Xi\}_0,$$

with the subscript indicating the connected component which contains $0 \in \mathfrak{k}$.

Proposition 1.1. *The following assertions hold.*

(i) For any $r > 0$ with $\mathfrak{k}_r \subset \mathfrak{k}(R)$ we have a continuous embedding

$$V_R^{\min} \subset V_r^\omega.$$

(ii) There exist constants $c, C > 0$ so that

$$\mathfrak{k}_r \subset \mathfrak{k}(R) \quad \textit{if} \quad r < Ce^{-cR}. \tag{1.4}$$

Assertion (i) is Proposition 5.1; assertion (ii) is Proposition 3.1.

It is an interesting problem to determine $\mathfrak{k}(R)$ explicitly, and we do so for two examples in Appendix A. The results in the appendix suggest that the bound (1.4) is sharp modulo the constants $c, C > 0$.

1.2. Analytic inclusion

We now address the more interesting and much more difficult part, namely the analytic inclusion, i.e. to find for given $r > 0$ an $R = R(r) > 0$ such that $V_r^\omega \subset V_{R(r)}^{\min}$. The main theorem of this paper is (see Theorem 10.1 with Remark 10.2):

Theorem 1.2. *Let G be a real reductive algebraic group and V be a Harish-Chandra module. Then there exist constants $c > 0$ and $R_0 > 0$, only depending on G, with the following property: Given $r > 0$, then for all $R > R_0$ satisfying*

$$\frac{(\log R)^2}{R^2} < cr$$

one has a continuous embedding

$$V_r^\omega \subset V_R^{\min}.$$

As a corollary of this theorem we obtain Schmid's identity (1.1), and we can view Theorem 1.2 as a new quantitative version of it. In Appendix B we give a short derivation of the Helgason conjecture from (1.1). Finally, in Appendix C we give an application of our quantitative version to the factorization of analytic eigenfunctions in terms of the Harish-Chandra spherical function.

Let us now explain the idea of the proof. Standard techniques reduce matters quickly to the case when $V = V_\lambda$ is a principal series for which the K-spherical vector is cyclic (see Lemma 4.4). Our approach is based on the Paley-Wiener theorem of Helgason for the Fourier transform on G/K. Let us briefly recall the statement. Let $\mathrm{PW}(\mathfrak{a}_\mathbb{C}^*, C^\infty(K/M))_R$ be the $C^\infty(K/M)$-valued Paley-Wiener space of holomorphic functions on the complexification $\mathfrak{a}_\mathbb{C}^*$ of the Euclidean space $\mathfrak{a}$ with growth rate R, see (6.1) for the formal definition. We realize V_λ in the compact picture, where $V_\lambda^\infty = C^\infty(K/M)$ as K-modules, and denote by $v_{K,\lambda} = \mathbf{1}_{K/M}$ the constant indicator function of K/M. It is then easy to see that the spherical Fourier transform

$$\mathcal{F} : C_c^\infty(G) \to \mathcal{O}(\mathfrak{a}_\mathbb{C}^*, C^\infty(K/M)), \quad f \mapsto (\lambda \mapsto \pi_\lambda(f) v_{K,\lambda})$$

satisfies

$$\mathcal{F}(C_R^\infty(G)) \subset \mathrm{PW}(\mathfrak{a}_\mathbb{C}^*, C^\infty(K/M))_R.$$

Let W be the Weyl group of $\Sigma(\mathfrak{g}, \mathfrak{a})$. For $w \in W$ we denote by

$$J_{w,\lambda} : V_\lambda^\infty \simeq C^\infty(K/M) \to V_{w\lambda}^\infty \simeq C^\infty(K/M)$$

the normalized (i.e. fixing $\mathbf{1}_{K/M}$) intertwining operator and recall that $\lambda \mapsto J_{w,\lambda}$ is meromorphic. With that we obtain an action of W on the space of $C^\infty(K/M)$-valued meromorphic functions,

$$W \times \mathfrak{M}(\mathfrak{a}_\mathbb{C}^*, C^\infty(K/M)) \to \mathfrak{M}(\mathfrak{a}_\mathbb{C}^*, C^\infty(K/M)), \quad (w, f) \mapsto w \circ f,$$

$$(w \circ f)(\lambda) = J_{w,w^{-1}\lambda} f(w^{-1}\lambda) \qquad (\lambda \in \mathfrak{a}_\mathbb{C}^*).$$

In this framework Helgason's Paley-Wiener theorem [10] asserts that

$$\mathcal{F}(C_R^\infty(G)) = \mathrm{PW}_W(\mathfrak{a}_\mathbb{C}^*, C^\infty(K/M))_R,$$

where the subscript W refers to invariant functions for the action defined above. However, from the geometric inclusion it follows that $\mathcal{F}(C_R^\infty(G))(\lambda) \subset C_R^\infty(G) * V_\lambda \subset V_\lambda^\omega$. Thus we observe that the intertwining relations force analyticity, i.e. we have

$$\mathrm{PW}_W(\mathfrak{a}_\mathbb{C}^*, C^\infty(K/M))_R = \mathrm{PW}_W(\mathfrak{a}_\mathbb{C}^*, C^\omega(K/M))_R,$$

and this observation was the motivation for our approach to the analytic inclusion.

Fix $\lambda_0 \in \mathfrak{a}_\mathbb{C}^*$ such that V_{λ_0} is cyclic for the K-spherical vector. We explicitly construct for any given analytic vector $v \in V_{\lambda_0}^\omega(r)$ a holomorphic function

$$f_v : \mathfrak{a}_\mathbb{C}^* \to C^\infty(K/M)$$

such that its average

$$\mathcal{A}(f_v) := \sum_{w \in W} w \circ f_v$$

lies in the Paley-Wiener space for a certain $R > 0$ and such that $\mathcal{A}(f_v)(\lambda_0) = v$. The Paley-Wiener theorem then yields that $v \in C_R^\infty(G) * V_{\lambda_0}$, proving the theorem.

We point out that our proof is in essence an $\mathrm{SL}(2,\mathbb{R})$-proof. More precisely, in Section 8 we provide a variety of estimates for products of Γ-functions, which lie at the core of the construction for $G = \mathrm{SL}(2,\mathbb{R})$. Given the framework provided by Kostant in [14], the general case of a reductive group G is then a consequence of the one-variable estimates in Section 8.

2. Preliminaries

Let G be the real points of a connected algebraic reductive group defined over $\mathbb{R}$ and let $\mathfrak{g}$ be its Lie algebra. Subgroups of G are denoted by capitals. The corresponding subalgebras are denoted by the corresponding fraktur letter. The unitary dual of a subgroup S of G we denote by $\widehat{S}$.

We denote by $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ the complexification of $\mathfrak{g}$ and by $G_\mathbb{C}$ the group of complex points. We fix a Cartan involution θ and write K for the maximal compact subgroup that is fixed by θ. We also write θ for the derived automorphism of $\mathfrak{g}$. We write $K_\mathbb{C}$ for the complexification of K, i.e. $K_\mathbb{C}$ is the subgroup of $G_\mathbb{C}$ consisting of the fixed points for the analytic extension of θ.

The Cartan involution induces the infinitesimal Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal abelian subspace. Diagonalize $\mathfrak{g}$ under $\operatorname{ad}\mathfrak{a}$ to obtain the familiar root space decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}^\alpha,$$

with $\mathfrak{m} = \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$ as usual. Let A be the connected subgroup of G with Lie algebra $\mathfrak{a}$ and let $M = Z_K(\mathfrak{a})$. We fix an Iwasawa decomposition $G = KAN$ of G. We define the projections $\mathbf{k} : G \to K$ and $\mathbf{a} : G \to A$ by

$$g \in \mathbf{k}(g)\mathbf{a}(g)N \qquad (g \in G).$$

The set of restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$ we denote by Σ and the positive system determined by the Iwasawa decomposition by Σ^+. We write W for the Weyl group of Σ.

Let κ be the Killing form on $\mathfrak{g}$ and let $\widetilde{\kappa}$ be a non-degenerate $\mathrm{Ad}(G)$-invariant symmetric bilinear form on $\mathfrak{g}$ such that its restriction to $[\mathfrak{g}, \mathfrak{g}]$ coincides with the restriction of κ and $-\widetilde{\kappa}(\,\cdot\,, \theta\,\cdot\,)$ is positive definite. We write $\|\cdot\|$ for the corresponding norm on $\mathfrak{g}$.

3. The complex crown of a Riemannian symmetric space

The Riemannian symmetric space $Z = G/K$ can be realized as a totally real subvariety of the Stein symmetric space $Z_\mathbb{C} = G_\mathbb{C}/K_\mathbb{C}$:

$$Z = G/K \hookrightarrow Z_\mathbb{C}, \quad gK \mapsto gK_\mathbb{C}.$$

In the following we view $Z \subset Z_\mathbb{C}$ and write $z_0 = K \in Z$ for the standard base point.

We define the subgroups $A_{\mathbb{C}} = \exp(\mathfrak{a}_{\mathbb{C}})$ and $N_{\mathbb{C}} = \exp(\mathfrak{n}_{\mathbb{C}})$ of $G_{\mathbb{C}}$. We note that $N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}}$ is a Zariski-open subset of $G_{\mathbb{C}}$. The maximal $G \times K_{\mathbb{C}}$-invariant domain in $G_{\mathbb{C}}$ containing e and contained in $N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}}$ is given by

$$\widetilde{\Xi} = G\exp(i\Omega)K_{\mathbb{C}}, \tag{3.1}$$

where $\Omega = \{Y \in \mathfrak{a} \mid (\forall\alpha \in \Sigma)\alpha(Y) < \pi/2\}$. Taking right cosets by $K_{\mathbb{C}}$, we obtain the G-domain

$$\Xi := \widetilde{\Xi}/K_{\mathbb{C}} \subset Z_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}, \tag{3.2}$$

commonly referred to as the *crown domain*. See [7] for the origin of the notion, [17, Cor. 3.3] for the inclusion $\widetilde{\Xi} \subset N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}}$ and [16, Th. 4.3] for the maximality.

We recall that Ξ is a contractible space. To be more precise, let $\widehat{\Omega} = \mathrm{Ad}(K)\Omega$ and note that $\widehat{\Omega}$ is an open convex subset of $\mathfrak{s}$. As a consequence of the Kostant convexity theorem it satisfies $\widehat{\Omega} \cap \mathfrak{a} = \Omega$ and $p_{\mathfrak{a}}\widehat{\Omega} = \Omega$, where $p_{\mathfrak{a}}$ is the orthogonal projection $\mathfrak{s} \to \mathfrak{a}$. The fiber map

$$G \times_K \widehat{\Omega} \to \Xi; \quad [g, X] \mapsto g\exp(iX) \cdot K_{\mathbb{C}},$$

is a diffeomorphism by [1, Prop. 4, 5 and 7]. Since $G/K \simeq \mathfrak{s}$ and $\widehat{\Omega}$ are both contractible, also Ξ is contractible. In particular, Ξ is simply connected.

We denote by $\mathbf{a} : G \to A$ the middle projection of the Iwasawa decomposition $G = KAN$ and note that $\mathbf{a}$ extends holomorphically to

$$\widetilde{\Xi}^{-1} := \{g^{-1} : g \in \widetilde{\Xi}\}.$$

Here the simply connectedness of Ξ plays a role to achieve $\mathbf{a} : \widetilde{\Xi}^{-1} \to A_{\mathbb{C}}$ uniquely: A priori $\mathbf{a}$ is only defined as a map to $A_{\mathbb{C}}/T_2$, where $T_2 := A_{\mathbb{C}} \cap K_{\mathbb{C}}$ is the 2-torsion subgroup of group $A_{\mathbb{C}}$. We denote the extension by the same symbol.

Likewise one defines $\mathbf{k} : G \to K$, which extends holomorphically to $\widetilde{\Xi}^{-1}$ as well.

For $R > 0$ we define a ball in A by

$$A_R := \{\exp(Y) \mid Y \in \mathfrak{a}, \ \|Y\| \le R\}.$$

Related to that we define the ball $B_R \subset G$ by $B_R = KA_RK$. We consider the following subset of $\mathfrak{k}$:

$$\mathfrak{k}(R) := \{Y \in \mathfrak{k} \mid \exp(iY)B_R \subset \widetilde{\Xi}\}_0. \tag{3.3}$$

Note that $\mathfrak{k}(R)$ is open, because $B_R \subset G$ is compact. Moreover, $\mathfrak{k}(R)$ is $\mathrm{Ad}(K)$-invariant. Hence it is uniquely determined by its intersection with a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$, i.e. $\mathfrak{k}(R)$ is determined by

$$\mathfrak{t}(R) := \mathfrak{t} \cap \mathfrak{k}(R).$$

Actually, it is sufficient to consider the intersection with a closed chamber of $\mathfrak{t}$, say $\mathfrak{t}^+$:

$$\mathfrak{t}(R)^+ := \mathfrak{t}^+ \cap \mathfrak{k}(R).$$

For $r > 0$ let $\mathfrak{k}_r := \{X \in \mathfrak{k} \mid \|X\| < r\}$ and define the domains in $K_{\mathbb{C}}$

$$K_{\mathbb{C}}(r) := K \exp(i\mathfrak{k}_r).$$

Note that $K_{\mathbb{C}}(r)$ is K-biinvariant, as $\mathfrak{k}_r$ is $\mathrm{Ad}(K)$-invariant. Note further that $K_{\mathbb{C}}(r) = \big(K_{\mathbb{C}}(r)\big)^{-1}$, since $\mathfrak{k}_r = -\mathfrak{k}_r$.

In general it is an interesting problem to determine $\mathfrak{k}(R)$ explicitly. We do this in Appendix A for two cases, namely $\mathfrak{g} = \mathfrak{so}(1,n)$ and $\mathfrak{g} = \mathfrak{su}(1,1)$, the latter being treated in a way so that the generalization to Hermitian symmetric spaces becomes apparent.

As a precise description of $\mathfrak{k}(R)$ may be difficult to obtain in general, one could instead determine the best possible $r = r(R) > 0$ with $\mathfrak{k}_r \subset \mathfrak{k}(R)$. The following proposition gives a first bound which, given the results in Appendix A, appears to be sharp up to constants.

Proposition 3.1. *There exist constants $C, c > 0$ such that for all $r, R > 0$ one has*

$$\mathfrak{k}_r \subset \mathfrak{k}(R) \qquad (r < Ce^{-cR}).$$

Proof. Let $G = \mathrm{GL}(n,\mathbb{R})$. We consider the standard Iwasawa decomposition of G, i.e. $K = \mathrm{O}(n,\mathbb{R})$, $A = \mathrm{diag}(n,\mathbb{R}_{>0})$ and N is the group of unipotent upper triangular matrices. It suffices to consider this case, as any real reductive group can be embedded into $G = \mathrm{GL}(n,\mathbb{R})$ with compatible Iwasawa decompositions. Here we remark that the possible incompatibility of the Cartan-Killing norms is taken care of by the presence of the constants C and c.

We recall that

$$Z = G/K \to \mathrm{Sym}(n,\mathbb{R})^+, \quad gK \mapsto gg^t,$$

identifies Z with the positive definite symmetric matrices. In this matrix picture $Z_{\mathbb{C}}$ is identified with $\mathrm{Sym}(n,\mathbb{C})_{\det\neq 0}$, the invertible symmetric matrices. In case $n \geq 3$, the crown domain $\Xi \subset \mathrm{Sym}(n,\mathbb{C})_{\det\neq 0}$ is not explicitly known. However, Ξ contains the so-called square root domain

$$\Xi^{\frac{1}{2}} = \mathrm{Sym}(n,\mathbb{R})^+ + i\mathrm{Sym}(n,\mathbb{R}) \subset \mathrm{Sym}(n,\mathbb{C})_{\det\neq 0}, \tag{3.4}$$

see [18, Sect. 8]. Let $m = \left[\frac{n}{2}\right]$ and define for $x \in \mathbb{R}^m$

$$D(x) = \begin{pmatrix} D_1(x) & & \\ & \ddots & \\ & & D_m(x) \end{pmatrix},$$

with

$$D_j(x) = \begin{pmatrix} \cos x_j & -\sin x_j \\ \sin x_j & \cos x_j \end{pmatrix}.$$

In case n is even we have $D(x) \in \mathrm{O}(n,\mathbb{R})$, and in case of n odd we view $D(x) \in \mathrm{O}(n,\mathbb{R})$ by means of the embedding

$$D(x) \mapsto \begin{pmatrix} D(x) & \\ & 1 \end{pmatrix}.$$

Our choice of maximal torus $T \subset K$ then is $T = \{D(x) \mid x \in \mathbb{R}^m\}$.

Let now $R > 0$ and $Y \in \mathrm{Sym}(n,\mathbb{R})^+$ with $\mathrm{spec}(Y) \subset [e^{-R}, e^R]$. We then seek an $r > 0$ such that for all $x \in \mathbb{R}^m$ with $\|x\| < r$ and Y as above we have $D(ix)YD(ix)^t \in \Xi^{\frac{1}{2}}$. If we decompose $D(ix) = U(x) + iV(x)$ into real and imaginary parts, this amounts to

$$U(x)YU(x) - V(x)YV(x)^t \in \mathrm{Sym}(n,\mathbb{R})^+,$$

by (3.4). With

$$S(x) = \begin{pmatrix} \begin{pmatrix} 0 & -\tanh x_1 \\ \tanh x_1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & -\tanh x_m \\ \tanh x_m & 0 \end{pmatrix} \end{pmatrix}$$

we can rewrite this as

$$Y - S(x)YS(x)^t \in \mathrm{Sym}(n,\mathbb{R})^+. \tag{3.5}$$

Now note that

$$\|S(x)YS(x)^t\|_{\mathrm{op}} \leq [\tanh r]^2 \|Y\|_{\mathrm{op}} \leq [\tanh r]^2 e^R.$$

On the other hand, the smallest eigenvalue of Y is at least e^{-R}. Hence (3.5) is satisfied, provided $[\tanh r]^2 e^{2R} < 1$. As $\tanh r \leq r$, (3.5) is implied by $r^2 < e^{-2R}$, and the assertion of the proposition follows. □

4. Generalities on the filtration of V^ω

4.1. Filtration by holomorphic extension

Let V be a Harish-Chandra module. The space V^ω is defined as the space of all analytic vectors in V^∞, i.e. $V^\omega := (V^\infty)^\omega$, equipped with its natural compact-open topology. In the following we provide various standard descriptions of V^ω.

The first one is in terms of holomorphic extensions. For $r > 0$ we define

$$V_r^\omega := \{v \in V^\infty \mid K \ni k \mapsto k \cdot v \in V^\infty \text{ extends holomorphically to } K_{\mathbb{C}}(r)\}$$

and endow this space with the Fréchet topology of uniform convergence on compacta in $K_{\mathbb{C}}(r)$.

Lemma 4.1. *For any Harish-Chandra module V every K-analytic vector is analytic. Moreover,*

$$V^\omega = \varinjlim_{r \to 0} V_r^\omega$$

as locally convex topological vector spaces.

Proof. From the definition it is easily checked that $\varinjlim_{r\to 0} V_r^\omega$ describes the space of analytic vectors for the representation on V^∞ restricted to K with the topology of uniform convergence on K.

We recall the notion of Δ-analytic vectors from [6, Sect. 5] and that the space of Δ-analytic vectors coincides with the space of analytic vectors for any F-representation of a Lie group, and in particular for V^∞. Let $\mathcal{C}$ be the Casimir element and let Δ_K and Δ_G be the standard Laplace elements in $\mathcal{U}(\mathfrak{k})$ and $\mathcal{U}(\mathfrak{g})$, respectively. Then $\Delta_G = \mathcal{C} + 2\Delta_K$. As Δ_G differs from $2\Delta_K$ by $\mathcal{C}$, which acts finitely on V, it follows that any Δ_K-analytic vector is Δ_G-analytic, and vice versa. This proves the first assertion.

The identity map from the space of G-analytic vectors to the space of K-analytic vectors is continuous. The second assertion now follows from the open mapping theorem (see [19, Theorem 24.30 and Remark 24.36]). □

4.2. Filtration by K-type decay

The next description of analytic vectors is by exponential decay of K-types. A norm p on V is called G-continuous provided that the completion V_p of the normed space (V,p) gives rise to a Banach-representation of G. We choose a G-continuous norm p on V. Let V^∞ be the up to isomorphism unique smooth completion of V with moderate growth, see [5], [23, Ch. 11] or [2]. We write a vector $v \in V^\infty$ as a convergent sum

$$v = \sum_{\tau \in \widehat{K}} v_\tau,$$

where v_τ is contained in the K-isotypical component $V[\tau]$ of V. For any $\tau \in \widehat{K}$ we denote by $|\tau|$ the norm of the highest weight of τ.

For $r > 0$ let us define

$$V^\omega(r) := \{v \in V^\infty \mid (\forall 0 < r' < r) \sum_{\tau \in \widehat{K}} e^{r'|\tau|} p(v_\tau) < \infty\}$$

and endow it with the Fréchet topology induced by the seminorms

$$v \mapsto \sum_{\tau \in \widehat{K}} e^{r'|\tau|} p(v_\tau) \qquad (0 < r' < r).$$

The space $V^\omega(r)$ is independent of the choice of the G-continuous norm p, as all these norms are polynomially comparable on the K-types, i.e. given two G continuous norms p and q on V there exists a constant $C > 0$, so that $p|_{V[\tau]} \leq C(1 + |\tau|)^C q|_{V[\tau]}$ for all $\tau \in \widehat{K}$, see [2, Th. 1.1].

Lemma 4.2. *For every Harish-Chandra module V we have*

$$V_r^\omega = V^\omega(r) \qquad (r > 0)$$

as topological vector spaces.

Proof. Let $r > 0$. We first prove the inclusion $V^\omega(r) \subset V_r^\omega$. For this let $v \in V^\omega(r)$ and $0 < r' < r$. Recall that $K_\mathbb{C}(r') = K \exp(i\mathfrak{t}_{r'})K$ with $\mathfrak{t}_{r'} = \{X \in \mathfrak{t} \mid \|X\| < r'\}$. Since the space $V^\omega(r)$ is independent of the choice of the G-continuous norm p, we may assume that p is Hermitian and K-unitary. Any element $t \in \exp(i\mathfrak{t}_{r'})$ acts semisimply on $V[\tau]$ with eigenvalues bounded by $e^{r'|\tau|}$. As p is K-unitary, it follows that

$$\sup_{k \in K_\mathbb{C}(r')} p(k \cdot v_\tau) \leq e^{r'|\tau|} p(v_\tau) \qquad (\tau \in \widehat{K}). \tag{4.1}$$

We recall that V_p is the Hilbert completion of V with respect to p and that V_p is a Hilbert representation of G. Because $v \in V^\omega(r)$, inequality (4.1) and the fact that $\dim V[\tau]$ is polynomially bounded in $|\tau|$ imply that the orbit map

$$f_v : K \to V_p, \quad k \mapsto k \cdot v,$$

extends holomorphically to $K_{\mathbb{C}}(r')$. Since this holds for all $r' < r$, the function f_v in fact extends holomorphically to $K_{\mathbb{C}}(r)$. The image of f_v is not only in V_p, but in the K-smooth vectors of V_p. Since the Fréchet spaces of K-smooth and G-smooth vectors in V_p are identical (see [2, Corollary 3.10]), we obtain that f_v is a holomorphic map with values in $V_p^\infty = V^\infty$. Thus we have shown that $V^\omega(r) \subset V_r^\omega$. The embedding is continuous in view of (4.1).

For the converse inclusion $V_r^\omega \subset V^\omega(r)$, we note that for an irreducible Harish-Chandra module V the representation V^∞ can be embedded into the space of smooth vectors of a minimal principal series module $V_{\sigma,\lambda}$. The latter can be realized as the space of smooth functions $f : G \to V_\sigma$ satisfying

$$f(gman) = a^{-i\lambda-\rho}\sigma(m)^{-1}f(g) \qquad (g \in G, man \in MAN).$$

Note that $V_{\sigma,\lambda}^\infty$ is naturally a G-module, with G acting on $V_{\sigma,\lambda}^\infty$ by left displacements in the arguments, in symbols $\pi_{\sigma,\lambda}(g)(f) = f(g^{-1}\cdot)$. We write $\mathcal{H}$ for $C^\infty(K)$ equipped with the G-representation π_λ given by

$$\big(\pi_\lambda(g)f\big)(k) = \mathbf{a}(g^{-1}k)^{-i\lambda-\rho}f(\mathbf{k}\big(g^{-1}k)\big) \qquad \big(f \in \mathcal{H}, g \in G, k \in K\big).$$

We may embed V^∞ equivariantly into $\mathcal{H} \otimes V_\sigma$. It therefore suffices to prove that $\mathcal{H}_r^\omega \subset \mathcal{H}^\omega(r)$.

We let p be the L^2-norm on $\mathcal{H}$, which is G-continuous. Note that K acts also from the right on smooth functions on K, and therefore $\mathcal{H}$ carries a representation of $K \times K$. From now on we consider $\mathcal{H}$ as a $K \times K$ module. For $0 < r' < r$ we define a $K \times K$-invariant Hermitian norm on $\mathcal{H}_r^\omega$ by

$$q_{r'}(v) := \left[\int_{K_{\mathbb{C}}(r')} |v(k)|^2 \, d\mu(k)\right]^{\frac{1}{2}} \qquad (v \in \mathcal{H}_r^\omega).$$

Here $d\mu$ is the measure on $K_{\mathbb{C}}$ which in the polar decomposition $K_{\mathbb{C}} = K\exp(i\mathfrak{t}^+)K$ is given by

$$d\mu(k_1 \exp(it)k_2) = dk_1 \, dt \, dk_2,$$

with $dk_{1,2}$ the Haar measure on K and dt the Lebesgue measure on $\mathfrak{t}^+$. For $\tau \in \widehat{K}$ we define $\mathcal{H}[\tau]$ to be the $\tau \otimes \tau^\vee$-isotypical component of $\mathcal{H}$ and denote the restriction of p and $q_{r'}$ to $\mathcal{H}[\tau]$ by p_τ and $q_{r',\tau}$, respectively. Since $\mathcal{H}[\tau]$ is $K \times K$-irreducible, there exists a constant $c_{r',\tau} > 0$, so that $q_{r',\tau} = c_{r',\tau} \cdot p_\tau$.

We will estimate the constant from below by estimating $q_{r',\tau}(v)$ for a matrix coefficient

$$v = m_{w_1,w_2} : k \mapsto \langle w_1, \tau(k) w_2 \rangle,$$

where $w_1, w_2 \in V_\tau$. Using the Schur-Weyl orthogonality relations we obtain

$$q_{r',\tau}(v)^2 = \frac{\|w_1\|^2}{\dim \tau} \int_K \int_{\mathfrak{t}^+_{r'}} \|\tau\big(\exp(it)k\big) w_2\|^2 \, dk \, dt.$$

Next we pick an orthonormal basis of weight vectors $v_1, \ldots, v_n \in V_\tau$ and expand the integrand. We thus obtain that the right-hand side is equal to

$$\frac{\|w_1\|^2}{\dim \tau} \sum_{j=1}^n \int_K \int_{\mathfrak{t}^+_{r'}} |\langle \tau(k) w_2, \tau(\exp(it)) v_j \rangle|^2 \, dk \, dt.$$

Now we apply Schur-Weyl once more. This yields

$$\frac{\|w_1\|^2 \|w_2\|^2}{\dim(\tau)^2} \sum_{j=1}^n \int_{\mathfrak{t}^+_{r'}} \|\tau(\exp(it)) v_j\|^2 \, dt.$$

Again by Schur-Weyl we note

$$\frac{\|w_1\|^2 \|w_2\|^2}{\dim(\tau)} = p_\tau(v)^2.$$

Let μ_τ be the highest weight of τ, and assume that v_1 is a highest weight vector with weight μ_τ. Then for all $r'' < r'$ there exists a constant c, independent of τ, so that

$$q_{r',\tau}(v)^2 \geq \frac{p_\tau(v)^2}{\dim(\tau)} \int_{\mathfrak{t}^+_{r'}} e^{2\mu_\tau(it)} \, dt \geq c^2 e^{2\|\mu_\tau\| r''} p_\tau(v)^2.$$

As $\|\mu_\tau\| = |\tau|$, we conclude that for every $r'' < r'$ there exists a constant $c_{r''} > 0$, so that

$$c_{r',\tau} \geq c_{r''} e^{r''|\tau|} \qquad (\tau \in \widehat{K}).$$

If $v = \sum_{\tau\in\widehat{K}} v_\tau \in \mathcal{H}_r^\omega$, then for all $0 < r'' < r' < r$

$$\sum_{\tau\in\widehat{K}} e^{|\tau|r''} p_\tau(v_\tau) \le \frac{1}{c_{r''}} \sum_{\tau\in\widehat{K}} q_{r',\tau}(v_\tau) = \frac{1}{c_{r''}} q_{r'}(v) < \infty.$$

It follows that $v \in \mathcal{H}_r^\omega$ implies $v \in \mathcal{H}^\omega(r)$. Moreover, the embedding is continuous. □

4.3. Reduction to spherical principal series

It is our intention to show for a given Harish-Chandra module V and $r > 0$ that there is a continuous embedding

$$V^\omega(r) \subset V_R^{\min}$$

for some $R = R(r) > 0$. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we write V_λ for the spherical principal series representation $\operatorname{Ind}_P^G(\mathbb{C}_{i\lambda})$. We will first reduce the problem to the case in which $V = V_\lambda$ for some $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

Every irreducible Harish-Chandra module V is a quotient

$$V_\lambda \otimes F \twoheadrightarrow V \tag{4.2}$$

for some spherical principal series V_λ and finite dimensional representation F of G, see [22, Sect. 2]. We first recall how this arises. By the Casselman embedding theorem every irreducible Harish-Chandra module V is a quotient of some minimal principal series module $V_{\sigma,\lambda} = \operatorname{Ind}_P^G(V_\sigma \otimes \mathbb{C}_{i\lambda})$ with $(\sigma, V_\sigma) \in \widehat{M}$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$, i.e.

$$V_{\sigma,\lambda} \twoheadrightarrow V.$$

Now, by op. cit. the M-representation (σ, V_σ) can be realized as the quotient $F/\mathfrak{n}F$ of a finite dimensional module F of G, i.e. $V_\sigma = F/\mathfrak{n}F$. By the Mackey isomorphism we have

$$V_\lambda \otimes F = \operatorname{Ind}_P^G(\mathbb{C}_{i\lambda}) \otimes F \simeq \operatorname{Ind}_P^G(\mathbb{C}_{i\lambda} \otimes F|_P).$$

Hence the P-morphism $\mathbb{C}_{i\lambda} \otimes F|_P \to \mathbb{C}_{i\lambda} \otimes F/\mathfrak{n}F \simeq \mathbb{C}_{i\lambda} \otimes V_\sigma$ gives rise to the chain of quotients

$$V_\lambda \otimes F \twoheadrightarrow V_{\sigma,\lambda} \twoheadrightarrow V.$$

This proves (4.2).

Lemma 4.3. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and F a finite dimensional representation of G. The following assertions hold.*

(i) Let $r, R > 0$. If $V_\lambda^\omega(r)$ embeds continuously into $C_R^\infty(G) * V_\lambda$, then also $(V_\lambda \otimes F)^\omega(r)$ embeds continuously into $C_R^\infty(G) * (V_\lambda \otimes F)$.

(ii) Let V be a Harish-Chandra module so that there exists a quotient map $\varpi : V_\lambda \otimes F \to V$. Then for every $r > 0$

$$V^\omega(r) = \varpi\Big((V_\lambda \otimes F)^\omega(r)\Big),$$

where the symbol ϖ is also used for its the globalization to $(V_\lambda \otimes F)^\infty$.

Proof. First note that $(V_\lambda \otimes F)^\omega(r) = V_\lambda^\omega(r) \otimes F$, as F is in fact a $G_{\mathbb{C}}$-module. To prove (i), it thus suffices to show that $\big(C_R^\infty(G) * V_\lambda\big) \otimes F$ continuously embeds into $C_R^\infty(G) * (V_\lambda \otimes F)$. The proof for this is analogous to [2, Lemma 9.4].

We move on to (ii). Let p be a K-invariant G-continuous Hermitian norm on $V_\lambda \otimes F$. Let q be the corresponding quotient norm on V. Then q is G-continuous and K-invariant. Note that the definition of $V^\omega(r)$ does not depend on the choice of the G-continuous norm on V. Assertion (ii) now follows, since $V^\omega(r)$ as a K-module is a direct summand of $(V_\lambda \otimes F)^\omega(r)$. ☐

4.4. Kostant's condition

We would like to be more restrictive on the parameter λ of the quotient $V_\lambda \otimes F \twoheadrightarrow V$.

Lemma 4.4. *Every irreducible Harish-Chandra module V admits a quotient $V_\lambda \otimes F \twoheadrightarrow V$, where F is an irreducible finite dimensional representation of G and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfies the Kostant condition*

$$\operatorname{Re}(i\lambda)(\alpha^\vee) \geq 0 \qquad (\alpha \in \Sigma^+). \tag{4.3}$$

If (4.3) is satisfied, then $V_\lambda = \mathcal{U}(\mathfrak{g})v_{K,\lambda}$ is $\mathcal{U}(\mathfrak{g})$-cyclic for the K-fixed vector $v_{K,\lambda}$.

Proof. In view of (4.2), V admits a quotient $V_\lambda \otimes F \twoheadrightarrow V$, where F is a finite dimensional representation of G. Let F' be a K-spherical finite dimensional representation of lowest weight $-\mu \in \mathfrak{a}^*$, where μ is dominant. Then M acts trivially on the MA-module $F'/\mathfrak{n}F' \simeq \mathbb{C}_{-\mu}$ with A-weight $-\mu$. In particular, we obtain a quotient

$$V_{\lambda - i\mu} \otimes F' \twoheadrightarrow V_\lambda.$$

It follows that V admits a quotient $V_{\lambda-i\mu}\otimes F\otimes F'\twoheadrightarrow V$. The first assertion now follows by taking μ sufficiently large. The last assertion is [14, Th. 8]. $\square$

5. The geometric inclusion

The goal of this section is to show that $V_R^{\min}=C_R^\infty(G)*V$ embeds into $V^\omega(r)$ for any $r>0$ with $\mathfrak{k}_r\subset\mathfrak{k}(R)$. This reduces to the case where $V=V_\lambda$ is the spherical principal series representation with parameter $\lambda\in\mathfrak{a}_\mathbb{C}^*$. Elements in V_λ^∞ are uniquely determined by their restriction to K. This gives rise to the compact model, in which

- $V_\lambda^\infty=C^\infty(K/M)$,
- $V_\lambda^\omega=C^\omega(K/M)$,
- $V_\lambda=\mathbb{C}[K_\mathbb{C}/M_\mathbb{C}]$

as K-modules. The main result of this section is the following.

Proposition 5.1. *Let V be a Harish-Chandra module, and $R>0$. Let r be such that $\mathfrak{k}_r\subset\mathfrak{k}(R)$. Then we have the continuous embedding*

$$V_R^{\min}\subset V^\omega(r).$$

Proof. We first reduce to the case where $V=V_\lambda$ is a spherical principal series. We recall from Section 4.3 that V is a quotient of some $V_\lambda\otimes F$, with F a finite dimensional representation. Now all matrix coefficients of F extend holomorphically to $G_\mathbb{C}$, and this completes the reduction to $V=V_\lambda$.

We work in the compact model of V_λ. Let $v=\pi(f)w$ for some $w\in V$ and $f\in C_R^\infty(G)$. Then we note that for $k\in K$

$$v(k)=\pi(f)(w)(k)=\int_{B_R} f(g)w(g^{-1}k)\,dg. \tag{5.1}$$

Observe that $w(g^{-1}k)=w(\mathbf{k}(g^{-1}k))\mathbf{a}(g^{-1}k)^{-i\lambda-\rho}$. As $w\in\mathbb{C}[K_\mathbb{C}/M_\mathbb{C}]$, w is a holomorphic function on $K_\mathbb{C}/M_\mathbb{C}$. Thus with $B_RK_\mathbb{C}(r)\subset\widetilde{\Xi}^{-1}\subset K_\mathbb{C}A_\mathbb{C}N_\mathbb{C}$, we conclude that $\mathbf{a}$ and $\mathbf{k}$ are defined on $B_RK_\mathbb{C}(r)$ and holomorphic. Thus v extends to the holomorphic function on $K_\mathbb{C}(r)$ given by (5.1). This shows the continuous embedding for this case. $\square$

6. Preliminaries on the analytic inclusion

6.1. K-type expansion of functions on K/M

In the following we view functions on K/M as right M-invariant functions on K. For any $\tau\in\widehat{K}$ we fix a model (finite dimensional) Hilbert space V_τ.

For $\tau \in \widehat{K}$ we write $\tau^\vee$ for the dual representation. We then obtain for each $\tau \in \widehat{K}$ a $K \times K$-equivariant realization of $V_\tau \otimes V_{\tau^\vee}$ as polynomial functions on K:

$$V_\tau \otimes V_{\tau^\vee} \to \mathbb{C}[K_\mathbb{C}], \quad v \otimes v^\vee \mapsto m_{v,v^\vee};\ m_{v,v^\vee}(k) := v^\vee(k^{-1}v),$$

where $K \times K$ acts on $\mathbb{C}[K_\mathbb{C}]$ by the left-right regular representation. We arrive at the $K \times K$-isomorphism of $K \times K$-modules

$$\mathbb{C}[K_\mathbb{C}] = \bigoplus_{\tau \in \widehat{K}} V_\tau \otimes V_{\tau^\vee},$$

and taking right M-invariants at the K-isomorphism of K-modules

$$\mathbb{C}[K_\mathbb{C}/M_\mathbb{C}] = \bigoplus_{\tau \in \widehat{K}_M} V_\tau \otimes V_{\tau^\vee}^M,$$

where $\widehat{K}_M \subset \widehat{K}$ is the M-spherical part of $\widehat{K}$. Fix τ and identify $V_{\tau^\vee} \simeq V_\tau^*$. In particular, the unitary norm on V_τ induces the unitary dual norm on $V_{\tau^\vee}$ and we write $\|\cdot\|_\tau$ for the Hilbert-Schmidt norm on $V_\tau \otimes V_{\tau^\vee}$. We recall that $\|\cdot\|_\tau$ is independent of the particular unitary norm on V_τ (which is unique up to positive scalar by Schur's Lemma) and is thus intrinsically defined. Any function on $f \in \mathbb{C}[K_\mathbb{C}]$ we now expand into K-types $f = \sum_{\tau \in \widehat{K}} f_\tau$ with $f_\tau \in V_\tau \otimes V_{\tau^\vee}$. With that we record the well known Fourier characterizations of $C^\infty(K)$ and $C^\omega(K)$ as

$$C^\infty(K) = \{f = \sum_{\tau \in \widehat{K}} f_\tau \mid (\forall N \in \mathbb{N}) \sum_{\tau \in \widehat{K}} (1 + |\tau|)^N \|f_\tau\|_\tau < \infty\}$$

and

$$C^\omega(K) = \{f = \sum_{\tau \in \widehat{K}} f_\tau \mid (\exists r > 0) \sum_{\tau \in \widehat{K}} e^{r|\tau|} \|f_\tau\|_\tau < \infty\}.$$

Taking right M-invariants, we obtain corresponding Fourier characterizations of $C^\infty(K/M)$ and $C^\omega(K/M)$.

6.2. The Helgason Paley-Wiener theorem

We begin with a short review of the Fourier transform on $Z = G/K$ and recollect some notation. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we denote by V_λ the Harish-Chandra module of the K-spherical principal series with parameter λ as defined before. Recall that $V_\lambda^\infty = C^\infty(K/M)$ as K-module. We denote by $v_{K,\lambda} = \mathbf{1}_{K/M} \in V_\lambda$ the constant function.

For every $R > 0$ we let $\mathrm{PW}(\mathfrak{a}_\mathbb{C}^*, C^\infty(K/M))_R$ be the space of holomorphic functions $f : \mathfrak{a}_\mathbb{C}^* \to C^\infty(K/M)$, so that for every continuous semi-norm q on $C^\infty(K/M)$ and $N \in \mathbb{N}$ one has

$$\sup_{\lambda \in \mathfrak{a}_\mathbb{C}^*} q(f(\lambda))(1 + \|\lambda\|)^N e^{-R\|\operatorname{Im}\lambda\|} < \infty. \tag{6.1}$$

Further we denote

$$\mathrm{PW}(\mathfrak{a}_\mathbb{C}^*, C^\infty(K/M)) = \bigcup_{R>0} \mathrm{PW}(\mathfrak{a}_\mathbb{C}^*, C^\infty(K/M))_R$$

and refer to it as the Paley-Wiener space on $\mathfrak{a}_\mathbb{C}^*$ with values in $C^\infty(K/M)$.

The Fourier transform on Z is then defined by

$$\begin{aligned} &\mathcal{F} : C_c^\infty(Z) \to \mathrm{PW}(\mathfrak{a}_\mathbb{C}^*, C^\infty(K/M)), \\ &f \mapsto \mathcal{F}(f);\ \mathcal{F}(f)(\lambda) := \pi_\lambda(f) v_{K,\lambda}. \end{aligned}$$

Note that

$$\mathcal{F}(f)(\lambda)(kM) = \int_Z f(gK)\mathbf{a}(g^{-1}k)^{-i\lambda-\rho}\, d(gK) \qquad (k \in K).$$

It is convenient to write $\mathcal{F}(f)(\lambda, kM)$ for $\mathcal{F}(f)(\lambda)(kM)$.

In order to describe the image of $\mathcal{F}$, we recall the Weyl group W of the restricted root system $\Sigma = \Sigma(\mathfrak{a}, \mathfrak{g})$. Attached to $w \in W$ there is a meromorphic family of standard intertwining operators

$$I_{w,\lambda} : V_\lambda^\infty \to V_{w\lambda}^\infty.$$

Further we recall that $I_{w,\lambda}(v_{K,\lambda}) = \mathbf{c}_w(\lambda) v_{K,w\lambda}$ for a meromorphic and explicit function $\mathbf{c}_w$ (w-partial Harish-Chandra $\mathbf{c}$-function, calculated by Gindikin-Karpelevic). We define the normalized intertwining operator by $J_{w,\lambda} := \frac{1}{\mathbf{c}_w(\lambda)} I_{w,\lambda}$. We recall that $\lambda \mapsto J_{w,\lambda}$ is meromorphic on $\mathfrak{a}_\mathbb{C}^*$, and holomorphic on an open neighborhood of the cone

$$\{\lambda \in \mathfrak{a}_\mathbb{C}^* : \operatorname{Re}\big(i\lambda(\alpha^\vee)\big) \geq 0 \text{ for all } \alpha \in \Sigma^+ \cap w^{-1}\Sigma^-\}.$$

It is clear from the definitions that every Fourier transform $\phi = \mathcal{F}(f)$ satisfies the *intertwining relations*

$$J_{w,\lambda}(\phi(\lambda)) = \phi(w\lambda) \qquad (w \in W, \lambda \in \mathfrak{a}_\mathbb{C}^*). \tag{6.2}$$

Let $\mathrm{PW}_W(\mathfrak{a}_{\mathbb{C}}^*, C^\infty(K/M))$ be the subspace of $\mathrm{PW}(\mathfrak{a}_{\mathbb{C}}^*, C^\infty(K/M))$ of Paley-Wiener functions that satisfy all intertwining relations (6.2). Then a slight reformulation of Helgason's Paley-Wiener theorem [10, Theorem 8.3] states that

$$\mathcal{F}(C_R^\infty(Z)) = \mathrm{PW}_W(\mathfrak{a}_{\mathbb{C}}^*, C^\infty(K/M))_R \qquad (R > 0). \tag{6.3}$$

See [20, Lemma 2.2].

6.3. Intertwining relations on K-types

For any $\tau \in \widehat{K}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we have

$$V_\lambda[\tau] = C^\infty(K/M)[\tau] = V_\tau \otimes V_{\tau^\vee}^M \tag{6.4}$$

as K-modules, where $V_{\tau^\vee} = V_\tau^*$. We denote by $J_{w,\lambda}[\tau]$ the restriction of $J_{w,\lambda}$ to $V_\lambda[\tau]$ and observe that $J_{w,\lambda}[\tau] : V_\lambda[\tau] \to V_{w\lambda}[\tau]$. Within the identification (6.4) we then obtain

$$J_{w,\lambda}[\tau] \in \mathrm{End}_K(V_\tau \otimes V_{\tau^\vee}^M) \simeq \mathrm{End}(V_{\tau^\vee}^M).$$

Next we recall Kostant's factorization of $J_{w,\lambda}[\tau]$. In general, if $\mathfrak{e} \subset \mathfrak{g}$ is a subspace, we denote by $\mathcal{S}(\mathfrak{e})$ the symmetric algebra and by $\mathcal{S}^\star(\mathfrak{e})$ the image of $\mathcal{S}(\mathfrak{e})$ in $\mathcal{U}(\mathfrak{g})$ under the symmetrization map. From the Cartan decomposition $\mathfrak{g} = \mathfrak{s} + \mathfrak{k}$ and the PBW-theorem we thus obtain the direct sum decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{S}^\star(\mathfrak{s}) \oplus U(\mathfrak{g})\mathfrak{k}.$$

Next, according to [15, Th. 15] we have $\mathcal{S}(\mathfrak{s}) = \mathcal{H}(\mathfrak{s}) \otimes \mathcal{I}(\mathfrak{s})$, where $\mathcal{H}(\mathfrak{s})$ denotes the harmonic polynomials on $\mathfrak{s}_{\mathbb{C}}^*$ and $\mathcal{I}(\mathfrak{s})$ the K-invariant polynomials on $\mathfrak{s}_{\mathbb{C}}^*$. We derive the refined decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{H}^\star(\mathfrak{s})\mathcal{I}^\star(\mathfrak{s}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{k}. \tag{6.5}$$

Consequently we have for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ that

$$d\pi_\lambda(\mathcal{U}(\mathfrak{g}))v_{K,\lambda} = d\pi_\lambda(\mathcal{H}^\star(\mathfrak{s}))v_{K,\lambda}.$$

We recall from Lemma 4.4 that in case λ satisfies the Kostant condition (4.3), the vector $v_{K,\lambda}$ is cyclic in V_λ for $\mathcal{U}(\mathfrak{g})$. In general we have for each $\tau \in \widehat{K}$ the K-equivariant maps

$$Q_\tau(\lambda) : \mathcal{H}^\star(\mathfrak{s})[\tau] \to V_\lambda[\tau] = V_\tau \otimes V_{\tau^\vee}^M, \quad D \mapsto d\pi_\lambda(D)v_{K,\lambda},$$

which are isomorphisms if λ satisfies (4.3), see [14, Cor. to Prop. 4 and Cor. to Th. 7]. (In [14] the polynomials Q_τ are denoted by P^τ. Compared to the polynomials defined in [11, p. 238] there is a sign difference in the argument.)

For fixed $\tau \in \widehat{K}_M$ we recall that the assignment

$$\mathfrak{a}_\mathbb{C}^* \ni \lambda \mapsto Q_\tau(\lambda) \in \mathrm{Hom}_K(\mathcal{H}^\star(\mathfrak{s})[\tau], V_\tau \otimes V_{\tau^\vee}^M)$$

is polynomial. Since $J_{w,\lambda} v_{K,\lambda} = v_{K,w\lambda}$, we obtain the relation

$$J_{w,\lambda}[\tau] \circ Q_\tau(\lambda) = Q_\tau(w\lambda),$$

and as a consequence Kostant's factorization

$$J_{w,\lambda}[\tau] = Q_\tau(w\lambda) \circ Q_\tau(\lambda)^{-1}, \tag{6.6}$$

which exhibits $J_{w,\lambda}[\tau]$ for fixed $\tau \in \widehat{K}_M$ as a rational vector-valued function

$$\mathfrak{a}_\mathbb{C}^* \ni \lambda \mapsto J_{w,\lambda}[\tau] \in \mathrm{End}(V_{\tau^\vee}^M).$$

Remark 6.1. To understand the polynomial dependence of $\lambda \mapsto Q_\tau(\lambda)$ better, it proves useful to introduce a normalization. Set

$$\widetilde{Q}_\tau(\lambda) := Q_\tau(\lambda) \circ Q_\tau(0)^{-1} \in \mathrm{End}_K(V_\tau \otimes V_{\tau^\vee}^M) \simeq \mathrm{End}(V_{\tau^\vee}^M).$$

Hence $\widetilde{Q}_\tau(0) = \mathrm{id}$ and we can, upon fixing a basis of the vector space $V_{\tau^\vee}^M$, view $\widetilde{Q}_\tau$ as a polynomial function on $\mathfrak{a}_\mathbb{C}^*$ with values in the space of $l(\tau) \times l(\tau)$-matrices, where $l(\tau) := \dim V_{\tau^\vee}^M$.

Remark 6.2. In case G has real rank one, it was shown by Kostant in [14, Theorem 6] that $V_{\tau^\vee}^M$ is one-dimensional for all $\tau \in \widehat{K}_M$. In this case, for fixed $\tau \in \widehat{K}_M$ the map

$$\lambda \mapsto \widetilde{Q}_\tau(\lambda)$$

is an explicitly computable polynomial in λ (see [11, Ch. III, Cor. 11.3]), and consequently $\lambda \mapsto J_{w,\lambda}[\tau]$ is a scalar-valued rational function.

Specifically, let now $G = \mathrm{SL}(2,\mathbb{R})$ with $K = \mathrm{SO}(2,\mathbb{R})$ and A as before. We identify $\widehat{K}_M$ with $\mathbb{Z}$ and $\mathfrak{a}_\mathbb{C}^*$ with $\mathbb{C}$ via $\mathbb{C} \ni \lambda \mapsto \lambda\rho$. Then for $n = \tau \in \mathbb{Z}$

$$\widetilde{Q}_n(\lambda) = \frac{\Gamma\left(\frac{1}{2}(i\lambda+\rho)(\alpha^\vee) + |n|\right)\Gamma\left(\frac{1}{2}\rho(\alpha^\vee)\right)}{\Gamma\left(\frac{1}{2}(i\lambda+\rho)(\alpha^\vee)\right)\Gamma\left(\frac{1}{2}\rho(\alpha^\vee)+|n|\right)} = \frac{\Gamma\left(\frac{1}{2}(i\lambda+1)+|n|\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}(i\lambda+1)\right)\Gamma\left(\frac{1}{2}+|n|\right)}$$

$$= \frac{(1+i\lambda)(3+i\lambda)\cdot\ldots\cdot(2|n|-1+i\lambda)}{1\cdot 3\cdot\ldots\cdot(2|n|-1)}.$$

Then, for all $n \in \mathbb{Z} = \widehat{K}_M$ and $\lambda \in \mathbb{C} = \mathfrak{a}_{\mathbb{C}}^*$ and $w \in W$ the non trivial element, the map $J_{w,\lambda}[n]$ is given by the scalar

$$J_{w,\lambda}[n] = \frac{(1-i\lambda)(3-i\lambda)\cdot\ldots\cdot(2|n|-1-i\lambda)}{(1+i\lambda)(3+i\lambda)\cdot\ldots\cdot(2|n|-1+i\lambda)}.$$

7. Strategy of proof

In this section we describe the general strategy of proof for the analytic inclusion. The approach is simpler when G/K has rank one, and therefore we give a separate proof for that. The strategy for rank one is described through the following Ansatz 1. The general case is treated in Ansatz 2.

7.1. Ansatz 1

We consider a spherical principal series module V_{λ_0}, where λ_0 satisfies (4.3). Let $r > 0$ and $v \in V_{\lambda_0}^{\omega}(r)$, i.e. $v = \sum_{\tau\in\widehat{K}_M} v_\tau$ with $v_\tau \in V_{\lambda_0}[\tau] = V_\tau \otimes V_{\tau^\vee}^M$, so that

$$\sum_{\tau\in\widehat{K}_M} e^{r'|\tau|}\|v_\tau\|_\tau < \infty \qquad (0 < r' < r).$$

We make the following ansatz. First, let

$$F(\lambda) = F_v(\lambda) = \sum_{\tau\in\widehat{K}_M} u_\tau(\lambda),$$

where for each $\tau \in \widehat{K}_M$

$$\mathfrak{a}_{\mathbb{C}}^* \ni \lambda \to u_\tau(\lambda) \in V_\tau \otimes V_{\tau^\vee}^M$$

is a certain holomorphic function such that $u_\tau(\lambda_0) = v_\tau$. Specifically, we set

$$u_\tau(\lambda) = \phi_\tau(\lambda) Q_\tau(\lambda) \circ Q_\tau(\lambda_0)^{-1} v_\tau,$$

where $\phi_\tau \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*)^W$ is a W-invariant holomorphic function with $\phi_\tau(\lambda_0) = 1$. Suppose that the series defining $F(\lambda)$ converges locally uniformly with respect to λ, so that $F_v \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*, C^\infty(K/M))$. Then we observe with (6.6) and the W-invariance of $\lambda \mapsto \phi_\tau(\lambda)$ that

$$\begin{aligned} J_{w,\lambda} F(\lambda) &= \sum_{\tau \in \widehat{K}_M} \phi_\tau(\lambda) J_{w,\lambda}[\tau] \circ Q_\tau(\lambda) \circ Q_\tau(\lambda_0)^{-1} v_\tau \\ &= \sum_{\tau \in \widehat{K}_M} \phi_\tau(\lambda) Q_\tau(w\lambda) \circ Q_\tau(\lambda_0)^{-1} v_\tau = F(w\lambda). \end{aligned}$$

In other words $\lambda \mapsto F(\lambda)$ satisfies the intertwining relations. If we can now construct the ϕ_τ in such a way that $F \in \mathrm{PW}(\mathfrak{a}_{\mathbb{C}}^*, C^\infty(K/M))_R$ for some $R = R(r)$, then the Paley-Wiener theorem (6.3) implies the existence of an $f \in C_R^\infty(Z)$ such that $\mathcal{F}(f) = F$. In particular, we obtain $v = \pi_{\lambda_0}(f) v_{K,\lambda_0}$, that is

$$V_{\lambda_0}^\omega(r) \subset C_R^\infty(G) * V_{\lambda_0}.$$

We follow this ansatz for the rank one spaces in Section 9.

7.2. Ansatz 2

For the second ansatz we need some terminology. The space of $C^\infty(K/M)$-valued meromorphic functions on $\mathfrak{a}_{\mathbb{C}}^*$ will be denoted by $\mathfrak{M}(\mathfrak{a}_{\mathbb{C}}^*, C^\infty(K/M))$. We recall that a vector-valued function f on $\mathfrak{a}_{\mathbb{C}}^*$ is called meromorphic provided that for all $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$ there exists an open neighborhood U of λ_0 and a polynomial $p(\lambda)$ so that $\lambda \mapsto p(\lambda) f(\lambda)$ extends to a holomorphic function on U. In this regard we recall that $V_\lambda^\infty = C^\infty(K/M)$ as K-modules for every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. We then view an element $f \in \mathfrak{M}(\mathfrak{a}_{\mathbb{C}}^*, C^\infty(K/M))$ as a section of the bundle $\coprod_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*} V_\lambda^\infty \to \mathfrak{a}_{\mathbb{C}}^*$, i.e. we consider $f(\lambda) \in V_\lambda^\infty$. The key observation is that the prescription

$$W \times \mathfrak{M}(\mathfrak{a}_{\mathbb{C}}^*, C^\infty(K/M)) \to \mathfrak{M}(\mathfrak{a}_{\mathbb{C}}^*, C^\infty(K/M)), \quad (w, f) \mapsto w \circ f;$$

$$(w \circ f)(\lambda) := J_{w, w^{-1}\lambda} f(w^{-1}\lambda) \qquad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*),$$

defines an action of W and, moreover, a meromorphic function f satisfies the intertwining relations if and only if it is W-invariant for this action.

Now we come to the ansatz proper. Fix $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$ which satisfies the Kostant condition (4.3), and let $W_{\lambda_0} \subset W$ be the stabilizer of λ_0. As λ_0 satisfies (4.3), it follows that $J_{w, w^{-1}\lambda_0} = J_{w,\lambda_0}$ is defined for all $w \in W_{\lambda_0}$ and constitutes an intertwining operator $J_{w,\lambda_0} : V_{\lambda_0}^\infty \to V_{\lambda_0}^\infty$ for which $J_{w,\lambda_0}(v_{K,\lambda_0}) = v_{K,\lambda_0}$. The fact that v_{K,λ_0} is fixed by J_{w,λ_0} and that v_{K,λ_0} is cyclic for V_{λ_0} (see Lemma 4.4) implies that J_{w,λ_0} is equal to the identity on V_{λ_0} and hence also on $V_{\lambda_0}^\infty$.

Let now $v \in V_{\lambda_0}^\omega(r)$. Let further $f_v : \mathfrak{a}_\mathbb{C}^* \to C^\infty(K/M)$ be a holomorphic function satisfying the properties

- $f_v(\lambda_0) = \frac{1}{|W_{\lambda_0}|} v$,
- $f_v(w\lambda_0) = 0$ if $w \in W \setminus W_{\lambda_0}$.

Given a choice for f_v, we define a meromorphic function by

$$\mathcal{A}(f_v) := \sum_{w \in W} w \circ f_v,$$

and note that $\mathcal{A}(f_v)$ automatically satisfies the intertwining relations. Moreover,

$$\mathcal{A}(f_v)(\lambda_0) = \sum_{w \in W} J_{w,w^{-1}\lambda_0} f_v(w^{-1}\lambda_0) = \sum_{w \in W_{\lambda_0}} J_{w,\lambda_0} f_v(\lambda_0) = v, \tag{7.1}$$

i.e. $\mathcal{A}(f_v)$ interpolates v at $\lambda = \lambda_0$.

The difficulty is that the operators $J_{w,w^{-1}\lambda}$ have poles and the function f_v has to be chosen carefully, so that $\mathcal{A}(f_v)$ is indeed holomorphic and satisfies the Paley-Wiener condition for some $R = R(r) > 0$. The overall strategy is to start with a simple minded function $f_v(\lambda) = p_{\lambda_0}(\lambda)v$ for some polynomial p_{λ_0} and then modify f_v along its K-isotypical components, i.e. for each $\tau \in \widehat{K}$ we replace $p_{\lambda_0}(\lambda)v_\tau$ by $\phi_\tau(\lambda)p_{\lambda_0}(\lambda)v_\tau$ for some appropriate holomorphic function ϕ_τ.

This ansatz is used for the general case in Section 10.

Remark 7.1. Compared to the first ansatz this approach is computationally more complex, as we have to average over the Weyl group W, and in addition the functions ϕ_τ have to be such that the poles of the rational functions $J_{w,\lambda}$ are canceled. However, the advantage of this ansatz is that intertwining operators, in contrast to the Q-polynomials, factor into rank one intertwiners, which can be explicitly computed and estimated.

7.3. An application of Helgason's Paley-Wiener theorem

The following proposition will be used in the implementation of both ansatzes.

We define $\mathcal{H}$ to be the K representation $L^2(K/M)$. Accordingly, we write $\mathcal{H}^\infty$ and $\mathcal{H}^\omega$ for $C^\infty(K/M)$ and $C^\omega(K/M)$, respectively. For each $r > 0$ we define a Fréchet space by

$$\mathcal{H}^\omega(r) := \{v \in \mathcal{H}^\omega \mid (\forall 0 < r' < r) \sum_{\tau \in \widehat{K}_M} e^{r'|\tau|} \|v_\tau\| < \infty\}$$

with the indicated seminorms. We write $\mathcal{H}_\tau$ for the τ-component of $\mathcal{H}$.

Proposition 7.2. *Let $r, R > 0$ and $\lambda_0 \in \mathfrak{a}_\mathbb{C}^*$. Consider a family $(F_\tau)_{\tau \in \widehat{K}_M}$ of holomorphic functions $F_\tau : \mathfrak{a}_\mathbb{C}^* \to \mathrm{End}(\mathcal{H}_\tau)$ satisfying the following conditions.*

(i) For every $\tau \in \widehat{K}_M$ we have $F_\tau(\lambda_0) = \mathrm{id}$.
(ii) For every $\tau \in \widehat{K}_M$, $w \in W$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$ the intertwining relation holds

$$F_\tau(w \cdot \lambda) = J_{w,\lambda} \circ F_\tau(\lambda).$$

(iii) There exist a real number $0 < r' < r$, integers $j, l \in \mathbb{N}_0$, and a constant $C' > 0$ so that for all $\tau \in \widehat{K}_M$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$

$$\|F_\tau(\lambda)\|_{\mathrm{op}} \leq C'(1 + |\tau|)^j \, e^{r'|\tau|} \, (1 + \|\lambda\|)^l \, e^{R \, \|\operatorname{Im} \lambda\|}.$$

Then for every $\epsilon > 0$ a continuous linear map $\varphi : \mathcal{H}^\omega(r) \to C^\infty_{R+\epsilon}(G/K)$ exists so that

$$v = \varphi(v) * v_{K,\lambda_0} \qquad \big(v \in \mathcal{H}^\omega(r)\big).$$

Proof. For $k \in \mathbb{N}_0$, let p_k be the continuous seminorm on $\mathcal{H}^\infty$ given by

$$p_k(u) = \sum_{\tau \in \widehat{K}_M} (1 + |\tau|)^k \|u_\tau\| \qquad \big(u \in \mathcal{H}^\infty\big).$$

Note that this family of seminorms determines the topology of $\mathcal{H}^\infty$.

Let $r' < r'' < r$. It follows from (iii) that for $\lambda \in \mathfrak{a}_\mathbb{C}^*$, $k \in \mathbb{N}_0$ and $v \in \mathcal{H}^\omega(r)$

$$(7.2) \qquad \sum_{\tau \in \widehat{K}_M} (1 + |\tau|)^k \|F_\tau(\lambda)(v)\| \leq C''(1 + \|\lambda\|)^l \, e^{R \, \|\operatorname{Im} \lambda\|} \sum_{\tau \in \widehat{K}_M} e^{r''|\tau|} \|v_\tau\|,$$

where

$$C'' := C' \sup_{\tau \in \widehat{K}_M} (1 + |\tau|)^{j+k} e^{(r' - r'')|\tau|} < \infty.$$

By definition $\sum_{\tau\in\widehat{K}_M} e^{r''|\tau|}\|v_\tau\| < \infty$ for each $v \in \mathcal{H}^\omega(r)$. It follows from (7.2) that for every $v \in \mathcal{H}^\omega(r)$ the series

$$F_v(\lambda) := \sum_{\tau\in\widehat{K}_M} F_\tau(\lambda)(v_\tau) \qquad \big(\lambda \in \mathfrak{a}_{\mathbb{C}}^*\big)$$

converges in $\mathcal{H}^\infty$. The convergence is uniform for λ in compacta, and hence F_v is a holomorphic $\mathcal{H}^\infty$-valued function depending linearly on v.

Let $\epsilon > 0$. We claim that there exists a $\theta \in C_\epsilon^\infty(K\backslash G/K)$ so that $\theta * v_{K,\lambda_0} = v_{K,\lambda_0}$. To see this, first note that

$$C_c^\infty(K\backslash G/K) * v_{K,\lambda_0} \subseteq \mathbb{C}v_{K,\lambda_0}.$$

Consider now a Dirac sequence $(\theta_n)_{n\in\mathbb{N}}$ of functions $\theta_n \in C_{1/n}^\infty(K\backslash G/K)$. Since $\theta_n * v_{K,\lambda_0}$ converges to v_{K,λ_0} for $n \to \infty$, there exists an $m \in \mathbb{N}$ so that $\theta_n * v_{K,\lambda_0} \neq 0$ for all $n > m$. Let now $n > m$ be so large that $\frac{1}{n} < \epsilon$. After a rescaling of θ_n we obtain a function with the claimed property.

Since θ is K-invariant, its Fourier transform $\widehat{\theta} = \mathcal{F}(\theta)$ is a W-invariant scalar-valued holomorphic function on $\mathfrak{a}_{\mathbb{C}}^*$, and by the Paley-Wiener Theorem (6.3) it satisfies for every $N \in \mathbb{N}_0$ the estimate

$$\sup_{\lambda\in\mathfrak{a}_{\mathbb{C}}^*} |\widehat{\theta}(\lambda)|(1+\|\lambda\|)^N e^{-\epsilon\|\operatorname{Im}\lambda\|} < \infty. \tag{7.3}$$

Moreover, since $\theta * v_{K,\lambda_0} = v_{K,\lambda_0}$ we have

$$\widehat{\theta}(\lambda_0) = 1. \tag{7.4}$$

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $v \in \mathcal{H}^\omega(r)$ we define $f_v(\lambda) := \widehat{\theta}(\lambda)F_v(\lambda) \in \mathcal{H}^\infty$. The function $f_v : \mathfrak{a}_{\mathbb{C}}^* \to \mathcal{H}^\infty$ thus obtained is holomorphic. It follows from (7.4) and assumption (i) that $f_v(\lambda_0) = F_v(\lambda_0) = v$. In view of assumption (ii) the function f_v satisfies the intertwining relations (6.2). Finally, it follows from the estimates (7.2) and (7.3) that there exist for every $N \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ a constant $C_{N,k} > 0$, so that for every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $v \in \mathcal{H}^\omega(r)$

$$p_k\big(f_v(\lambda)\big) \leq C_{N,k}(1+\|\lambda\|)^{-N} e^{(R+\epsilon)\|\operatorname{Im}\lambda\|} \sum_{\tau\in\widehat{K}_M} e^{r''|\tau|}\|v_\tau\|. \tag{7.5}$$

Now it follows from the Paley-Wiener theorem (6.3) that $f_v = \mathcal{F}(\varphi_v')$ for some $\varphi_v' \in C_{R+\epsilon}^\infty(G/K)$. Set $\varphi_v = \varphi_v' * \theta \in C_{R+2\epsilon}^\infty(G/K)$. Note that φ_v depends linearly on v and satisfies

$$\varphi_v * v_{K,\lambda_0} = \varphi_v' * v_{K,\lambda_0} = \mathcal{F}(\varphi_v')(\lambda_0) = f_v(\lambda_0) = v.$$

It remains to show continuity from $\mathcal{H}^\omega(r)$ to $C^\infty_{R+2\epsilon}(G/K)$ of the map $v \mapsto \varphi_v$. The space $\mathrm{PW}_W(\mathfrak{a}^*_\mathbb{C}, \mathcal{H}^\infty)_{R+\epsilon}$ is a subspace of $L^2\big(\mathfrak{a}^*, \mathcal{H}, \frac{d\lambda}{|c(\lambda)|^2}\big)$. By the Plancherel theorem for G/K and (7.5) we have

$$\begin{aligned}\|\varphi'_v\|^2_{L^2} &= \int_{\mathfrak{a}^*} \|f_v(\lambda)\|^2 \frac{d\lambda}{|c(\lambda)|^2} \\ &\le \int_{\mathfrak{a}^*} p_0\big(f_v(\lambda)\big)^2 \frac{d\lambda}{|c(\lambda)|^2} \le c_0 \left(\sum_{\tau \in \widehat{K}_M} e^{r''|\tau|}\|v_\tau\|\right)^2,\end{aligned}$$

with

$$c_0 = C^2_{N,0} \int_{\mathfrak{a}^*} (1 + \|\lambda\|)^{-2N} \frac{d\lambda}{|c(\lambda)|^2} < \infty$$

for a sufficiently large $N \in \mathbb{N}$. Finally for every continuous seminorm q on $C^\infty_{R+2\epsilon}(G/K)$ there exists a constant $c' > 0$, only depending on θ, so that

$$q(\varphi_v) \le c'\|\varphi'_v\|_{L^2}.$$

The continuity follows. □

8. An explicit construction in one variable

For every $n \in \mathbb{N}_0$ and $R > 0$ we define an entire function $f_{n,R}$ on $\mathbb{C}$ by

$$f_{n,R}(z) := \frac{\sin(zR\pi)}{zR\pi \cdot \prod_{j=1}^n \Big(1 - \big(\frac{Rz}{j}\big)^2\Big)} = \prod_{j=n+1}^\infty \left(1 - \left(\frac{Rz}{j}\right)^2\right) \tag{8.1}$$

for $z \in \mathbb{C}$, invoking the product expansion of the sine function. Next we define for $n \in \mathbb{N}_0$ a polynomial function q_n on $\mathbb{C}$ by

$$q_n(z) := \prod_{j=1}^n \left(1 + \frac{z}{j}\right) \qquad (z \in \mathbb{C}). \tag{8.2}$$

Proposition 8.1. *There exist $c, R_0 > 0$ and for every $r > 0$ a constant $C_r > 0$ so that the following assertion holds for every $n \in \mathbb{N}_0$.*

Let $r > 0$ and $R > R_0$ with

$$\frac{(\log R)^2}{R^2} < cr. \tag{8.3}$$

Let V be a finite dimensional inner product space and $P : \mathbb{C} \to \mathrm{End}(V)$ a polynomial map such that

$$\|P(z)\|_{\mathrm{op}} \leq q_n(|z|)^k \qquad (z \in \mathbb{C})$$

for some $k \in \mathbb{N}$. Then

$$\|f_{n,R}(z)^k P(z)\|_{\mathrm{op}} \leq \Big[C_r\, e^{rn} e^{R\pi\,|\operatorname{Im} z|}\Big]^k$$

for all $z \in \mathbb{C}$.

The proof is divided into several lemmas. Let

$$F_{n,R}(z) := f_{n,R}(z)^k P(z).$$

The first two lemmas contain estimates of the function defined by

$$\widetilde{F}_{n,R}(z) := f_{n,R}(z) q_n(|z|),$$

for which we have

$$\|F_{n,R}(z)\|_{\mathrm{op}} \leq |\widetilde{F}_{n,R}(z)|^k \qquad (z \in \mathbb{C}). \tag{8.4}$$

Lemma 8.2. *There exists a constant $C > 0$, so that for all $R > 3$, $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$ with $|z| \geq \frac{n}{R}$ we have*

$$|\widetilde{F}_{n,R}(z)| \leq Ce^{R\pi|\operatorname{Im} z|}.$$

Proof. By symmetry, we may assume that $\operatorname{Re} z \geq 0$ without loss of generality. Note that

$$\widetilde{F}_{n,R}(z) = \frac{(1+|z|)\cdots(n+|z|)}{(1+Rz)\cdots(n+Rz)} \cdot \frac{1\cdot 2\cdots n}{(1-Rz)\cdots(n-Rz)} \cdot \frac{\sin(\pi Rz)}{\pi Rz}. \tag{8.5}$$

We claim that

$$\left|\frac{(1+|z|)\cdots(n+|z|)}{(1+Rz)\cdots(n+Rz)}\right| \leq 1.$$

To prove the claim it suffices to show that for all $1 \leq j \leq n$ we have

$$j + |z| \leq |j + Rz|.$$

Since $\operatorname{Re} z \geq 0$, we have

$$|j + Rz|^2 = R^2|z|^2 + 2Rj \operatorname{Re} z + j^2 \geq R^2|z|^2 + j^2.$$

For $R \geq 3$ and $|z| \geq \frac{n}{R}$ the condition $R^2|z|^2 \geq |z|^2 + 2n|z|$ is satisfied, and hence

$$R^2|z|^2 + j^2 \geq |z|^2 + 2n|z| + j^2 \geq (j + |z|)^2.$$

This proves the claim.

We further claim that

$$\left| \frac{1 \cdot 2 \cdots (n-1)}{(1 - Rz) \cdots (n - 1 - Rz)} \right| \leq 1$$

if $R|z| \geq n$. This is a direct consequence of the inequality $|j - Rz| \geq R|z| - j \geq n - j$.

Altogether, we obtain the estimate

$$\begin{aligned} |\widetilde{F}_{n,R}(z)| &\leq \left| \frac{n}{n - Rz} \right| \cdot \left| \frac{\sin(\pi Rz)}{\pi Rz} \right| \\ &= \frac{n}{R|z|} \left| \frac{\sin\left(\pi Rz\right)}{\pi(n - Rz)} \right| \leq \left| \frac{\sin\left(\pi(n - Rz)\right)}{\pi(n - Rz)} \right| \leq Ce^{R\pi|\operatorname{Im} z|}. \end{aligned}$$

□

Lemma 8.3. *There exists a constant $c > 0$ such that the following holds: For all $r > 0$ there exists $C > 0$, so that for all $n \in \mathbb{N}_0$ and $R > e$ with*

$$\frac{(\log R)^2}{R^2} < cr,$$

we have

$$|\widetilde{F}_{n,R}(z)| \leq Ce^{rn} \qquad (0 \leq z \leq \tfrac{n}{R}).$$

Proof. Let $n \in \mathbb{N}_0$, $R > 1$ and $z \geq 0$. We shall estimate $\widetilde{F}_{n,R}(z)$ using Stirling's approximation. Euler's reflection identity $\Gamma(1 - x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}$ and the functional equation of the Gamma function yield

$$\left(\prod_{j=1}^{n} (j + Rz) \right) \left(\prod_{j=1}^{n} (j - Rz) \right) \frac{\pi Rz}{\sin \pi Rz} = \Gamma(n + 1 - Rz)\Gamma(n + 1 + Rz).$$

This allows to rewrite (8.5) and express $\widetilde{F}_{n,R}$ in terms of Gamma functions:

$$\widetilde{F}_{n,R}(z) = \frac{\Gamma(n+1+z)\Gamma(n+1)}{\Gamma(1+z)\Gamma(n+1+Rz)\Gamma(n+1-Rz)}.$$

We recall Stirling's approximation

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}} e^{-x} x^x \left(1 + O(1/x)\right) \tag{8.6}$$

for $x \to \infty$. Applying Stirling to $\widetilde{F}_{n,R}$, we obtain that there exists a constant $c' > 0$, independent of n and R, so that for all $z \in [0, \frac{n}{R}]$

$$\widetilde{F}_{n,R}(z) \leq c' \sqrt{1 + \frac{n}{R}}\, e^{h_{n,R}(z)}, \tag{8.7}$$

where

$$\begin{aligned} h_{n,R}(z) := {} & (n+1+z)\log(n+1+z) + (n+1)\log(n+1) \\ & - (1+z)\log(1+z) - (n+1+Rz)\log(n+1+Rz) \\ & - (n+1-Rz)\log(n+1-Rz). \end{aligned}$$

Here we used the straightforward estimate for $z \in [0, \frac{n}{R}]$

$$\frac{(1+z)(n+1+Rz)(n+1-Rz)}{(n+1+z)(n+1)} = \frac{(1+z)\big((n+1)^2 - R^2 z^2\big)}{(n+1+z)(n+1)} \leq 1 + \frac{n}{R}$$

to estimate the square roots in (8.6).

The term $(1+z)\log(1+z)$ in $h_{n,R}(z)$ can be estimated below by $z\log(z)$. With the substitution $z = (n+1)x$ we then obtain

$$h_{n,R}(z) \leq (n+1) H_R(x) \tag{8.8}$$

where

$$\begin{aligned} H_R(x) := {} & (1+x)\log(1+x) - x\log(x) \\ & - (1+Rx)\log(1+Rx) - (1-Rx)\log(1-Rx). \end{aligned}$$

We will show

$$H_R(x) \leq 4\frac{(\log R)^2}{R^2} \qquad x \in (0, \tfrac{1}{R}), \tag{8.9}$$

for all $R \geq e$. We estimate the first two terms of $H_R(x)$ in a separate lemma.

Lemma 8.4. *For every* $b \geq e^2$ *and* $0 < x \leq 1$

$$(1+x)\log(1+x) - x\log x \leq \frac{(\log b)^2}{b} + bx^2.$$

Proof. Since $(1+x)\log(1+x) \leq 2x$ for $0 < x \leq 1$ it suffices to show

$$2x - x\log x =: \phi(x) \leq \psi(x) := \frac{(\log b)^2}{b} + bx^2.$$

The functions ϕ and ψ are concave and convex, respectively. We will prove the inequality by exhibiting a separating line of slope $\log b$.

We have $\phi'(x) = 1 - \log x$ and hence $\phi'(x) = \log b$ for $x = \frac{e}{b}$. Then

$$\phi(x) \leq \phi(\tfrac{e}{b}) + \log(b)(x - \tfrac{e}{b}) = \tfrac{e}{b} + \log(b)x.$$

On the other hand $\psi'(x) = 2bx = \log b$ for $x = \frac{\log b}{2b}$ and therefore

$$\psi(x) \geq \psi\Big(\frac{\log b}{2b}\Big) + \log(b)\Big(x - \frac{\log b}{2b}\Big) = \frac{3(\log b)^2}{4b} + \log(b)x.$$

Hence $\psi \geq \phi$ if $\frac{3}{4}(\log b)^2 \geq e$ and in particular if $b \geq e^2$. □

We proceed with the proof of (8.9). Let

$$\varphi(t) = (1+t)\log(1+t) + (1-t)\log(1-t)$$

for $0 \leq t < 1$. Then $\varphi(0) = \varphi'(0) = 0$ and $\varphi''(t) = \frac{1}{1+t} + \frac{1}{1-t} \geq 2$. Hence

$$\varphi(t) \geq t^2.$$

Then for $x \in (0, \frac{1}{R})$

$$H_R(x) \leq (1+x)\log(1+x) - x\log x - R^2x^2.$$

We obtain (8.9) from Lemma 8.4 with $b = R^2$.

We can now finish the proof of Lemma 8.3. Let $0 < c < \frac{1}{4}$. If $r > 0$, $R \geq e$ and $\frac{\log(R)^2}{R^2} \leq cr$, then

$$|\widetilde{F}_{n,R}(z)| \leq c'e^r\sqrt{1+n}\, e^{4crn} \leq Ce^{rn}, \quad n \in \mathbb{N}_0, z \in [0, \tfrac{n}{R}],$$

by (8.7) and (8.8), with a constant $C > 0$ depending only on c and r. □

Proof of Proposition 8.1. By Lemma 8.2 and (8.4) we have for all $R > 3$, $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$ with $|z| \geq \frac{n}{R}$

$$\|F_{n,R}(z)\|_{\mathrm{op}} \leq \left[Ce^{R\pi|\operatorname{Im} z|}\right]^k. \tag{8.10}$$

It therefore suffices to estimate $F_{n,R}(z)$ for z in $D = \left\{z \in \mathbb{C} : |z| \leq \frac{n}{R}\right\}$.

Note that

$$\|F_{n,R}(z)\|_{\mathrm{op}} = \sup_{\substack{v,w \in V \\ \|v\|=\|w\|=1}} |\langle F_{n,R}(z)v, w\rangle|,$$

and that the matrix coefficients $\langle F_{n,R}(z)v, w\rangle$ depend holomorphically on $z \in \mathbb{C}$.

Let $D_\pm = D \cap \mathbb{C}_\pm$ where $\mathbb{C}_\pm$ denotes the closed upper/lower half plane. By the maximum modulus principle a holomorphic function in $D_\pm$ assumes its maximum modulus on $\partial D_\pm$, i.e. on the union of the semicircle $\partial D \cap \mathbb{C}_\pm$ and the segment $D \cap \mathbb{R} = [-\frac{n}{R}, \frac{n}{R}]$. We apply the principle to the holomorphic function

$$\langle F_{n,R}(z)v, w\rangle\, e^{\pm iRk\pi z}$$

on $D_\pm$, which by (8.10) is bounded in absolute value by C^k on $\partial D \cap \mathbb{C}_\pm$.

On the other hand, with c as in Lemma 8.3 it follows that for all r satisfying (8.3) there exists a constant C_r such that $|\langle F_{n,R}(z)v, w\rangle\, e^{\pm iRk\pi z}|$ is bounded by $[C_r e^{rn}]^k$ for $z \in [-\frac{n}{R}, \frac{n}{R}]$. Assuming as we may that $C_r \geq C$, we obtain

$$|\langle F_{n,R}(z)v, w\rangle\, e^{\pm iRk\pi z}| \leq [C_r e^{rn}]^k$$

for all $z \in D_\pm$. This implies the proposition. □

The following lemma will be used in the next two sections.

Lemma 8.5. *Let $R > 0$ and $z_0 \in \mathbb{C}$. Assume $Rz_0 \notin \mathbb{Z}\backslash\{0\}$. Then*

$$\inf_{n \in \mathbb{N}_0} |f_{n,R}(z_0)| > 0.$$

Proof. With (8.1) we observe that $f_{n,R}(z) = 0$ if and only if $Rz \in \mathbb{Z}$ and $|z| > \frac{n}{R}$. In particular the assumption on z_0 implies $f_{n,R}(z_0) \neq 0$ for all $n \in \mathbb{N}_0$.

If $n \geq N := \lceil R|z_0|\rceil$ then

$$|f_{n,R}(z_0)| \geq \prod_{j=n+1}^{\infty} \left(1 - \left(\frac{R|z_0|}{j}\right)^2\right) \geq \prod_{j=N+1}^{\infty} \left(1 - \left(\frac{R|z_0|}{j}\right)^2\right) = f_{N,R}(|z_0|).$$

Hence

$$\inf_{n\in\mathbb{N}_0} |f_{n,R}(z_0)| \geq \min\left\{|f_{0,R}(z_0)|, \ldots, |f_{N-1,R}(z_0)|, f_{N,R}(|z_0|)\right\} > 0$$

by the first observation in the proof. □

We end this section with a remark that will be useful when Proposition 8.1 is applied in Sections 9 and 10.

Remark 8.6. Let $P(R,r)$ be any proposition depending on two variables $R, r > 0$. Then the proposition

$$\exists c, R_0 > 0\ \forall r > 0, R > R_0 : \quad \Big(\frac{(\log R)^2}{R^2} < cr \ \Rightarrow\ P(R,r)\Big)$$

possesses a scaling invariance. Let $a, A > 0$ and $B \in \mathbb{R}$. Then

$$\exists d, S_0 > 0\ \forall s > 0, S > S_0 : \quad \Big(\frac{(\log S)^2}{S^2} < ds \Rightarrow\ P(AS + B, as)\Big)$$

is an equivalent proposition. This follows from the observation that there exist constants $C_0, C_1, C_2 > 0$ so that for all $R > C_0$

$$C_1\frac{\log R}{R} \leq \frac{\log(AR+B)}{AR+B} \leq C_2\frac{\log R}{R}.$$

9. The rank one cases

Using the construction from the previous section we can now complete the argument in case G is of real rank one. Let $\alpha \in \Sigma^+$ be the indivisible root. Then

$$\mathfrak{g} = \mathfrak{g}^{2\alpha} + \mathfrak{g}^{\alpha} + \mathfrak{a} + \mathfrak{m} + \mathfrak{g}^{-\alpha} + \mathfrak{g}^{-2\alpha}$$

with $\mathfrak{a} = \mathbb{R}\alpha^\vee$. We set $m_\alpha := \dim \mathfrak{g}^\alpha$ and $m_{2\alpha} = \dim \mathfrak{g}^{2\alpha}$. Then

$$\rho = \frac{1}{2}(m_\alpha + 2m_{2\alpha})\alpha.$$

The goal of this section is to prove the following

Theorem 9.1. *Let G be a group of real rank one and V_{λ_0} a representation of the K-spherical principal series with λ_0 satisfying (4.3). Then there exist positive constants $c, R_0 > 0$ independent of λ_0, such that for all $R, r > 0$ with $R > R_0$ and $\frac{(\log R)^2}{R^2} < cr$, we have a continuous embedding*

$$V_{\lambda_0}^\omega(r) \subset C_R^\infty(G) * v_{K,\lambda_0} = \left(V_{\lambda_0}\right)_R^{\min}.$$

Remark 9.2. With the theorem above we can prove Theorem 1.2 for groups of real rank one. Let c and R_0 be as above. Then Theorem 9.1 and Lemma 4.2 imply that the conclusion of Theorem 1.2 is valid for $V = V_{\lambda_0}$. For a general irreducible Harish-Chandra module V the conclusion then follows from Lemmas 4.3 and 4.4.

In order to prepare for the proof of Theorem 9.1, we introduce some new notation. As mentioned in Remark 6.2, the normalized Q-matrices from Remark 6.1 are scalar-valued and are given by explicit formulas from [11, Ch. III, Cor. 11.3]. This corollary invokes non-negative integers $0 \leq r \leq s$ which are defined in [11, Ch. III, Th. 11.2] for each $\tau \in \widehat{K}_M$. From the proof of that theorem it follows that r and s have the same parity if $m_{2\alpha} \neq 0$. It further follows from the equation for s and r on page 346 in op. cit. that there exists an $m > 0$, independent of τ, such that

$$s \leq m|\tau|. \tag{9.1}$$

We may and will take $m \in \mathbb{N}$. In order to express the $\widetilde{Q}_\tau(\lambda)$ in an efficient way, we introduce some new notation.

For elements $0 < a \leq b$ with $b - a \in \mathbb{N}_0$, we define polynomials in the complex plane by

$$\Gamma_{a,b}(z) := \frac{\Gamma(z+b)\Gamma(a)}{\Gamma(z+a)\Gamma(b)} = \frac{(z+a)(z+a+1)\cdots(z+b-1)}{a(a+1)\cdots(b-1)}. \tag{9.2}$$

For later reference we note the following estimates by the polynomials introduced in (8.2):

$$|\Gamma_{a,b}(z)| \leq \Gamma_{a,b}(|z|) \leq \begin{cases} q_{b-a}(|z|) & \text{if } a \geq 1 \\ \frac{b}{a}\, q_{b-a}(|z|) & \text{if } a < 1. \end{cases} \tag{9.3}$$

The inequality for $a \geq 1$ follows from

$$\frac{|z|+a+j}{a+j} = 1 + \frac{|z|}{a+j} \leq 1 + \frac{|z|}{1+j}$$

for each $j \geq 0$, and the other one is then a consequence of

$$\Gamma_{a,b}(|z|) = \frac{(|z|+a)\,b}{(|z|+b)\,a}\,\Gamma_{a+1,b+1}(|z|).$$

We also note that

$$|\Gamma_{a,b}(z)| \geq 1 \qquad (\mathrm{Re}(z) \geq 0). \tag{9.4}$$

Next we define positive half integers. In case $m_{2\alpha} = 0$ we set

$$a_\tau := \rho(\frac{\alpha^\vee}{2}) = \frac{m_\alpha}{2} \quad \text{and} \quad b_\tau := \rho(\frac{\alpha^\vee}{2}) + s,$$

and note that $b_\tau - a_\tau = s \in \mathbb{N}_0$.

For $m_{2\alpha} > 0$ we first note that $\rho(\frac{\alpha^\vee}{2}) = \frac{m_\alpha}{2} + m_{2\alpha} =: d \in \mathbb{N}$ is a positive integer greater or equal to 2, as m_α is even when $m_{2\alpha} > 0$. Further we define positive half integers by

$$a_\tau^1 := \rho(\frac{\alpha^\vee}{4}) = \frac{d}{2} \quad \text{and} \quad b_\tau^1 := \frac{1}{2}(s + r + d)$$

and

$$a_\tau^2 := \frac{1}{2}(d + 1 - m_{2\alpha}) \quad \text{and} \quad b_\tau^2 := \frac{1}{2}(s - r + d + 1 - m_{2\alpha}).$$

Then both $b_\tau^1 - a_\tau^1 = \frac{1}{2}(s + r)$ and $b_\tau^2 - a_\tau^2 = \frac{1}{2}(s - r)$ are non-negative integers.

Having defined these constants, we rephrase [11, Ch. III, Cor. 11.3] as follows:

Lemma 9.3. *Let G be a group of real rank one and $\tau \in \widehat{K}_M$. Then the following assertions hold:*

1. *If $m_{2\alpha} = 0$, then $a_\tau \geq \frac{1}{2}$ and*

$$\widetilde{Q}_\tau(\lambda) = \Gamma_{a_\tau, b_\tau}(i\lambda(\frac{\alpha^\vee}{2})).$$

2. *If $m_{2\alpha} > 0$, then $a_\tau^1 \geq a_\tau^2 \geq 1$ and*

$$\widetilde{Q}_\tau(\lambda) = \Gamma_{a_\tau^1, b_\tau^1}(i\lambda(\frac{\alpha^\vee}{4}))\, \Gamma_{a_\tau^2, b_\tau^2}(i\lambda(\frac{\alpha^\vee}{4})).$$

9.1. Proof of Theorem 9.1 in case of $m_{2\alpha} = 0$

We identify $\mathfrak{a}_\mathbb{C}^*$ with $\mathbb{C}$ via

$$\mathbb{C} \mapsto \mathfrak{a}_\mathbb{C}^*, \quad z \mapsto z\alpha,$$

i.e. $\lambda = z\alpha \in \mathfrak{a}_{\mathbb{C}}^*$ identifies with $z \in \mathbb{C}$. In these coordinates we then have

$$\widetilde{Q}_\tau(z) = \Gamma_{\frac{m_\alpha}{2}, \frac{m_\alpha}{2}+s}(iz),$$

and it follows from (9.3) that $|\widetilde{Q}_\tau(z)| \leq (1+2s)q_s(|z|)$.

We recall from (9.1) that $s \leq m|\tau|$ for every $\tau \in \widehat{K}_M$, and that $m \in \mathbb{N}$. We write $\lceil \tau \rceil \in \mathbb{N}$ for the smallest integer greater or equal than $|\tau|$. We thus obtain the bound

$$|\widetilde{Q}_\tau(z)| \leq (1 + 2m\lceil\tau\rceil) q_{m\lceil\tau\rceil}(|z|). \tag{9.5}$$

We recall the functions $f_{n,R}$, depending on $R > 0$, as defined in (8.1). Let $z_0 \in \mathbb{C}$ be so that $\lambda_0 = z_0\alpha$. We assume that Rz_0 is not a non-zero integer. Then $f_{n,R}(z_0) \neq 0$ for all $n \in \mathbb{N}_0$. For $\tau \in \widehat{K}_M$ we define the W-symmetric entire function

$$\phi_\tau : \mathbb{C} \to \mathbb{C}; \quad z \mapsto \frac{f_{m\lceil\tau\rceil,R}(z)}{f_{m\lceil\tau\rceil,R}(z_0)}.$$

Now given $\lambda_0 = z_0\alpha \in \mathfrak{a}_{\mathbb{C}}^*$ satisfying (4.3), we follow Ansatz 1 in Section 7.1 and define

$$\widetilde{F}_\tau(z) = \phi_\tau(z)\widetilde{Q}_\tau(z)\widetilde{Q}_\tau(z_0)^{-1} \in \operatorname{End}_K(V_\tau \otimes V_{\tau^\vee}^M) \simeq \mathbb{C} \quad (z \in \mathbb{C}).$$

Let $F_\tau : \mathfrak{a}_{\mathbb{C}}^* \to \operatorname{End}_K(V_\tau \otimes V_{\tau^\vee}^M)$ be given by

$$F_\tau(z\alpha) = \widetilde{F}_\tau(z) \qquad (z \in \mathbb{C}).$$

It is immediate that F_τ satisfies the conditions (i) and (ii) in Proposition 7.2 with $\mathcal{H}_\tau = V_\tau \otimes V_{\tau^\vee}^M$.

We continue by investigating condition (iii). For that we need to control the normalizing factors $\widetilde{Q}_\tau(z_0)$ and $f_{m\lceil\tau\rceil,R}(z_0)$. By (4.3) the real part of iz_0 is non-negative, and hence it follows from (9.4) that

$$|\widetilde{Q}_\tau(z_0)| \geq 1 \qquad (\tau \in \widehat{K}_M).$$

Likewise, Lemma 8.5 gives a positive lower bound for $|f_{m\lceil\tau\rceil,R}(z_0)|$, uniformly in τ.

Let $c, R_0 > 0$ be as in Proposition 8.1, and assume that $R > R_0$ and $\frac{(\log R)^2}{R^2} < cr$. By perturbing R to a slightly smaller value we can ensure Rz_0 is not an integer, as assumed before. Let $r' < r$ be such that $\frac{(\log R)^2}{R^2} < cr'$.

From Proposition 8.1 and (9.5) it follows that there exists a constant $C > 0$ so that

$$|F_\tau(\lambda)| \leq C(1+|\tau|)\, e^{r'm|\tau|}\, e^{\pi R \|\frac{1}{2}\alpha^\vee\| \, \|\operatorname{Im}\lambda\|} \qquad \big(\lambda \in \mathfrak{a}_\mathbb{C}^*, \tau \in \widehat{K}_M\big).$$

By Proposition 7.2 this implies $V_{\lambda_0}^\omega(mr) \subset C_{AR+\epsilon}^\infty(G/K) * v_{K,\lambda_0}$, where $A = \pi\|\frac{1}{2}\alpha^\vee\|$. By Remark 8.6 the continuous embedding in the theorem follows. Finally, as v_{K,λ_0} is $\mathcal{U}(\mathfrak{g})$-cyclic by Lemma 4.4, we have $C_R^\infty(G) * v_{K,\lambda_0} = (V_{\lambda_0})_R^{\min}$.

9.2. Proof of Theorem 9.1 in case of $m_{2\alpha} > 0$

We now identify $\mathfrak{a}_\mathbb{C}^*$ with $\mathbb{C}$ via

$$\mathbb{C} \mapsto \mathfrak{a}_\mathbb{C}^*, \quad z \mapsto 2z\alpha.$$

From Lemma 9.3 we then have

$$\widetilde{Q}_\tau(z) = \Gamma_{a_\tau^1, b_\tau^1}(iz)\Gamma_{a_\tau^2, b_\tau^2}(iz).$$

As before we apply (9.3). The result is now

$$\widetilde{Q}_\tau(-i|z|) \leq [q_{m\lceil\tau\rceil}(|z|)]^2.$$

Next we define the W-symmetric entire function

$$\phi_\tau(z) := \frac{[f_{m\lceil\tau\rceil}(z)]^2}{[f_{m\lceil\tau\rceil}(z_0)]^2}$$

and argue along the same lines as before. This concludes the proof of Theorem 9.1.

10. The general higher rank case

The goal of this section is to prove the following

Theorem 10.1. *Let V_{λ_0} be a representation of the K-spherical principal series with λ_0 satisfying (4.3). Then there exist positive constants c, R_0 independent of λ_0, such that for all $R, r > 0$ with $R > R_0$ and $\frac{(\log R)^2}{R^2} < cr$ we have a continuous embedding*

$$V_{\lambda_0}^\omega(r) \subset C_R^\infty(G) * v_{K,\lambda_0} = \big(V_{\lambda_0}\big)_R^{\min}.$$

Remark 10.2. By the arguments in Remark 9.2 we obtain Theorem 1.2 of the introduction from Theorem 10.1 together with the reduction in Section 4.

To prepare for the proof of Theorem 10.1 we determine some estimates of the intertwining operators $J_{w,\lambda}$. We start by recalling the standard procedure by which the study of $J_{w,\lambda}$ is reduced to rank one.

10.1. Factorization of intertwining operators

Let $w \in W$ and write

$$w = s_1 s_2 \cdots s_n$$

as a reduced expression with simple reflections s_i associated to simple roots $\alpha_i \in \Pi$. Set

$$w_j := s_{j+1} \cdots s_n \in W \qquad (1 \leq j \leq n).$$

Then the reduced expression of w satisfies the condition

$$w_j^{-1}\alpha_j \in \Sigma^+ \qquad (1 \leq j \leq n) \tag{10.1}$$

and

$$w_j^{-1}\alpha_j \neq w_k^{-1}\alpha_k \qquad (1 \leq j < k \leq n). \tag{10.2}$$

See [3, VI.1.6 Corollaire 2]. Essential for our reasoning is the factorization

$$J_{w,\lambda} = J_{s_1,w_1\lambda} \circ J_{s_2,w_2\lambda} \circ \cdots J_{s_{n-1},w_{n-1}\lambda} \circ J_{s_n,\lambda}, \tag{10.3}$$

with each

$$J_{s_j,w_j\lambda} : V^\infty_{w_j\lambda} \to V^\infty_{w_{j-1}\lambda}$$

a rank one intertwiner.

10.2. Rank one intertwining operators

Let $s_\alpha \in W$ be the reflection in a simple root $\alpha \in \Sigma^+$. When restricted to a specific K-type $\tau \in \widehat{K}_M$, each $J_{s_\alpha,\lambda}[\tau]$ is an element of $\mathrm{End}(V_{\tau^\vee}^M)$ depending rationally on λ. We will describe the entries of a diagonal matrix for it.

Let $\mathfrak{g}_\alpha$ be the semisimple rank one subalgebra of $\mathfrak{g}$ generated by α. Then

$$\mathfrak{g}_\alpha = \mathfrak{g}^{2\alpha} \oplus \mathfrak{g}^{\alpha} \oplus \mathfrak{a}_\alpha \oplus \mathfrak{m}_\alpha \oplus \mathfrak{g}^{-\alpha} \oplus \mathfrak{g}^{-2\alpha}$$

with $\mathfrak{a}_\alpha = \mathbb{R}\alpha^\vee$ and $\mathfrak{m}_\alpha \triangleleft \mathfrak{m}$ an ideal. In particular, the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ descends to $\mathfrak{g}_\alpha$, and we obtain with $\mathfrak{k}_\alpha := \mathfrak{g}_\alpha \cap \mathfrak{k}$ a maximal compact subalgebra of $\mathfrak{g}_\alpha$. We denote by $G_\alpha := \langle \exp(\mathfrak{g}_\alpha) \rangle$ the analytic subgroup of G associated to $\mathfrak{g}_\alpha$, by $K_\alpha := \exp(\mathfrak{k}_\alpha)$ the maximal compact subgroup of G_α with Lie algebra $\mathfrak{k}_\alpha$, and by M_α the group $M \cap K_\alpha$. Note that M normalizes G_α. Hence if we branch $V_{\tau^\vee}$ with respect to K_α, then

$$V_{\tau^\vee}^M = \bigoplus_{\delta \in \widehat{K_\alpha}_{M_\alpha}} m(\delta) V_\delta^{M_\alpha},$$

where $m(\delta)$ denotes the multiplicity of δ in $\tau^\vee|_{K_\alpha}$. By [14, Theorem 6] each $V_\delta^{M_\alpha}$ is one-dimensional. We choose an orthonormal basis (depending on j) of $V_{\tau^\vee}^M$ of vectors from these one-dimensional subspaces.

For elements $\frac{1}{2} \leq a \leq b$ with $a, b \in \frac{1}{2}\mathbb{N}$ and $b - a \in \mathbb{N}_0$, we recall the polynomials $\Gamma_{a,b}(z)$ from (9.2). With respect to the chosen basis the operator $J_{s_\alpha,\lambda}[\tau]$ is of diagonal form, say

$$D_\tau(\lambda) = \mathrm{diag}(d_\tau^1(\lambda), \ldots, d_\tau^{l(\tau)}(\lambda)),$$

and each diagonal entry is of the form (see (6.6), Lemma 9.3)

$$d_\tau^k(\lambda) = \frac{\Gamma_{a,b_k}(-i\lambda(\frac{\alpha^\vee}{\gamma_\alpha}))\Gamma_{a',b'_k}(-i\lambda(\frac{\alpha^\vee}{\gamma_\alpha}))}{\Gamma_{a,b_k}(i\lambda(\frac{\alpha^\vee}{\gamma_\alpha}))\Gamma_{a',b'_k}(i\lambda(\frac{\alpha^\vee}{\gamma_\alpha}))} \qquad (1 \leq k \leq l(\tau)), \tag{10.4}$$

where $\gamma_\alpha = 2$ and $a' = b'_k$ if $m_{2\alpha} = 0$, and otherwise $\gamma_\alpha = 4$. The parameters a and a' depend only on α, and for all $\tau \in \widehat{K}$ the parameters b_k and b'_k satisfy

$$b'_k - a' \leq b_k - a \leq m|\tau| \qquad (1 \leq k \leq l(\tau))$$

for some $m \in \mathbb{N}$ independent of τ and α. Therefore we may and shall assume that $b_k, b'_k \leq m|\tau|$ for all non-trivial τ.

10.3. Cancellation of poles and estimate

Let $\alpha \in \Sigma^+$ be a simple root and let $\tau \in \widehat{K}_M$. In the following lemma we determine a polynomial on $\mathfrak{a}_\mathbb{C}^*$ which cancels the poles of $J_{s_\alpha,\lambda}[\tau]$. Moreover, we give an estimate of the product of $J_{s_\alpha,\lambda}[\tau]$ with this polynomial.

As in Section 9 we write $\lceil\tau\rceil = \lceil|\tau|\rceil$. We define the following polynomial on $\mathbb{C}$,

$$e_\tau(z) := \Gamma_{1,m\lceil\tau\rceil+1}(z)^2\, \Gamma_{\frac{1}{2},m\lceil\tau\rceil+\frac{1}{2}}(z)^2, \tag{10.5}$$

and recall the polynomials $q_n(z)$ from (8.2). In particular, we see from (9.3) that we can estimate the polynomial $e_\tau(z)$ by

$$|e_\tau(z)| \le (1 + 2\lceil\tau\rceil)^2\, q_{m\lceil\tau\rceil}(|z|)^4 \tag{10.6}$$

for all $z \in \mathbb{C}$ and all $\tau \in \widehat{K}$.

Lemma 10.3. *Let $\alpha \in \Sigma^+$ be simple.*

(i) *The map $\mathfrak{a}^*_{\mathbb{C}} \to \mathrm{End}(V^M_{\tau^\vee})$ given by*

$$\lambda \mapsto e_\tau\big(i\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})\big)\, J_{s_\alpha,\lambda}[\tau]$$

is polynomial for every τ in $\widehat{K}$.

(ii) *There exists a constant $C > 0$ such that*

$$\|\, e_\tau\big(i\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})\big)\, J_{s_\alpha,\lambda}[\tau]\,\|_{\mathrm{op}} \le C(1 + |\tau|)^4\, q_{m\lceil\tau\rceil}\big(\big|\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})\big|\big)^8$$

for every $\tau \in \widehat{K}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

Proof. We may assume $|\tau| \ne 0$ since $J_{s_\alpha,\lambda}[\tau] = 1$ for the trivial K-type. We fix a simple root $\alpha \in \Sigma^+$ and define polynomials $d^{k,+}_\tau$ and $d^{k,-}_\tau$ to be the numerator and denominator in (10.4), respectively. Then

$$d^k_\tau(\lambda) = \frac{d^{k,+}_\tau(\lambda)}{d^{k,-}_\tau(\lambda)}.$$

Next we make the following observation: $\Gamma_{a,b}(z)$ divides $\Gamma_{a,b+n}(z)$ for all $n \in \mathbb{N}_0$, and $\Gamma_{a,b}(z)$ divides $\Gamma_{a-n,b}(z)$ for all $n \in \mathbb{N}_0$ such that $a - n \ge \frac{1}{2}$. It follows that

$$d^{k,-}_\tau(\lambda)\big|e_\tau\big(i\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})\big)$$

for all k, and this implies (i). Moreover, together with (9.4) it implies

$$\left|\frac{e_\tau\big(i\lambda(\frac{\alpha^\vee}{\gamma_\alpha})\big)}{d^{k,-}_\tau(\lambda)}\right| \le \frac{e_\tau\big(|\lambda(\frac{\alpha^\vee}{\gamma_\alpha})|\big)}{d^{k,-}_\tau(|\lambda|)} \le e_\tau\big(|\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})|\big)$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. For the numerator $d^{k,+}_\tau(\lambda)$ we find

$$\big|d^{k,+}_\tau(\lambda)\big| \le e_\tau\big(|\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})|\big)$$

also for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Hence

$$\left|e_\tau\big(i\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})\big)\, d_\tau^k(\lambda)\right| \leq e_\tau\big(|\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})|\big)^2$$

for all indices k. By (10.6) this implies (ii). □

10.4. Application of Proposition 8.1

The following lemma contains the main estimate for the proof of Theorem 10.1. Recall from (8.1) the functions $f_{n,R}$. We will determine an estimate for a product of these functions with $J_{w,\lambda}$. For this we use Lemma 10.3, Proposition 8.1 and the factorization (10.3).

Let

$$h := 8\pi \max_{\alpha\in\Sigma^+} \|\tfrac{\alpha^\vee}{\gamma_\alpha}\|.$$

Lemma 10.4. *There exist $c, R_0 > 0$ and for every $r > 0$ a constant $C_r > 0$ so that for every $R > R_0$ with*

$$\frac{(\log R)^2}{R^2} < cr \tag{10.7}$$

one has

$$\Big\|\Big(\prod_{\alpha\in\Sigma^+} f_{m\lceil\tau\rceil,R}\big(\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})\big)^8\, e_\tau\big(i\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})\big)\Big)\, J_{w,\lambda}[\tau]\,\Big\|_{\mathrm{op}} \leq C_r\left[(1+|\tau|)^4\, e^{8mr|\tau|}e^{hR\,\|\mathrm{Im}\,\lambda\|}\right]^{|\Sigma^+|}$$

for all $\tau \in \widehat{K}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $w \in W$.

Proof. Let c, R_0 be as in Proposition 8.1 and let $r > 0$. We first show that there exists a constant $C_r > 0$ such that if $R > R_0$ satisfies (10.7) then

$$\big\|f_{m\lceil\tau\rceil,R}\big(\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})\big)^8\, e_\tau\big(i\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})\big)\, J_{s_\alpha,\lambda}[\tau]\big\|_{\mathrm{op}} \leq C_r(1+|\tau|)^4\, e^{8mr|\tau|}e^{hR\,\|\mathrm{Im}\,\lambda\|} \tag{10.8}$$

for all $\tau \in \widehat{K}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and all simple roots $\alpha \in \Sigma^+$.

We apply Proposition 8.1 with $n = m\lceil\tau\rceil$ and

$$P(z) = (1+|\tau|)^{-4}\, e_\tau(iz)\, J_{s_\alpha,z\mu}[\tau],$$

where

$$\mu = \lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})^{-1}\lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

The estimate in Lemma 10.3 ensures the proposition is applicable. Hence

$$\big\| f_{m\lceil\tau\rceil,R}(z)^8\, e_\tau(iz)\, J_{s_\alpha,z\mu}[\tau]\big\|_{\mathrm{op}} \le (1+|\tau|)^4\,[C_r\, e^{rm\lceil\tau\rceil} e^{\pi R\,\|\mathrm{Im}\, z\|}]^8.$$

By inserting $z=\lambda(\frac{\alpha^\vee}{\gamma_\alpha})$ and $\lceil\tau\rceil \le |\tau|+1$ we obtain (10.8) for some $C_r>0$.

Let $w\in W$ and consider the factorization (10.3) of $J_{w,\lambda}$. By submultiplicativity of the operator norm we obtain from (10.8) that

$$\Big\|\Big(\prod_{j=1}^n f_{m\lceil\tau\rceil,R}\big(\lambda(\tfrac{w_j^{-1}\alpha_j^\vee}{\gamma_{\alpha_j}})\big)^8\, e_\tau\big(i\lambda(\tfrac{w_j^{-1}\alpha_j^\vee}{\gamma_{\alpha_j}})\big)\Big)\, J_{w,\lambda}[\tau]\,\Big\|_{\mathrm{op}} \\ \le \big[C_r(1+|\tau|)^4\, e^{8mr|\tau|} e^{hR\,\|\mathrm{Im}\,\lambda\|}\big]^n.$$

The $w_j^{-1}\alpha_j$ are all distinct and positive by (10.2) and (10.1), respectively. Hence each factor of the above product over j occurs exactly once in

$$\prod_{\alpha\in\Sigma^+}\big[f_{m\lceil\tau\rceil,R}\big(\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})\big)^8\, e_\tau\big(i\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha})\big)\big].$$

On the other hand, since by (10.6) the scalar valued polynomial e_τ satisfies the estimate

$$|e_\tau(z)| \le (1+2|\tau|)^2 q_{m\lceil\tau\rceil}(|z|)^4 \le (1+2|\tau|)^4 q_{m\lceil\tau\rceil}(|z|)^8,$$

we obtain in analogy with (10.8) that

$$\big|f_{m\lceil\tau\rceil,R}(\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha}))^8\, e_\tau(\lambda(\tfrac{\alpha^\vee}{\gamma_\alpha}))\big| \le C_r(1+|\tau|)^4 e^{8mr|\tau|} e^{hR\,\|\mathrm{Im}\,\lambda\|}$$

for every $\alpha\in\Sigma^+$. We apply this estimate to the roots $\alpha\in\Sigma^+$ which are not of the form $w_j^{-1}\alpha_j$ for any j and obtain the estimate as stated in the lemma. □

10.5. Conclusion of proof

We can now give the proof of Theorem 10.1, following Ansatz 2 from Section 7.2. Recall that λ_0 satisfies (4.3), that is,

$$\mathrm{Re}(i\lambda_0(\alpha^\vee)) \ge 0$$

for all $\alpha\in\Sigma^+$. We define the following functions on $\mathfrak{a}_{\mathbb{C}}^*$.

I. We choose a polynomial $p_{\lambda_0} : \mathfrak{a}^*_{\mathbb{C}} \to \mathbb{C}$ such that

$$\begin{cases} p_{\lambda_0}(\lambda_0) = \frac{1}{|W_{\lambda_0}|}, \\ p_{\lambda_0}(w\lambda_0) = 0 \qquad (w \in W \setminus W_{\lambda_0}), \end{cases}$$

where $W_{\lambda_0} \subset W$ is the stabilizer of λ_0.

II. For each $\tau \in \widehat{K}_M$ we define a polynomial

$$p_\tau(\lambda) := \frac{\prod_{\alpha\in\Sigma^+} e_\tau\big(i\lambda(\frac{\alpha^\vee}{\gamma_\alpha})\big)}{\prod_{\alpha\in\Sigma^+} e_\tau\big(i\lambda_0(\frac{\alpha^\vee}{\gamma_\alpha})\big)}.$$

It follows from (10.5) and (9.4) that

$$|e_\tau\big(i\lambda_0(\tfrac{\alpha^\vee}{\gamma_\alpha})\big)| \geq 1 \tag{10.9}$$

for all $\alpha \in \Sigma^+$.

III. For every $R > 0$ for which

$$\forall \alpha \in \Sigma^+ : \quad R\lambda_0(\tfrac{\alpha^\vee}{\gamma_\alpha}) \notin \mathbb{Z}\backslash\{0\}, \tag{10.10}$$

we define for each $n \in \mathbb{N}_0$ an entire function on $\mathfrak{a}^*_{\mathbb{C}}$ by

$$\psi_{n,R}(\lambda) := \frac{\prod_{\alpha\in\Sigma^+} f_{n,R}\big(\lambda(\frac{\alpha^\vee}{\gamma_\alpha})\big)}{\prod_{\alpha\in\Sigma^+} f_{n,R}\big(\lambda_0(\frac{\alpha^\vee}{\gamma_\alpha})\big)}.$$

By (10.10) and Lemma 8.5 there exists a constant $c_R > 0$ so that

$$|f_{n,R}\big(\lambda_0(\tfrac{\alpha^\vee}{\gamma_\alpha})\big)| \geq c_R \tag{10.11}$$

for all $n \in \mathbb{N}_0$ and $\alpha \in \Sigma^+$.

After these definitions we let

$$\phi_\tau(\lambda) := p_\tau(\lambda)[\psi_{m\lceil\tau\rceil,R}(\lambda)]^8$$

for $\tau \in \widehat{K}_M$, and we define $F_\tau : \mathfrak{a}^*_{\mathbb{C}} \to \mathrm{End}(V_\tau)$ by

$$F_\tau(\lambda) = \sum_{w\in W} \phi_\tau(w^{-1}\lambda)p_{\lambda_0}(w^{-1}\lambda)J_{w,w^{-1}\lambda}[\tau] \qquad (\lambda \in \mathfrak{a}^*_{\mathbb{C}}).$$

We are going to apply Proposition 7.2 to F_τ, and for that we need to verify its conditions (i)-(iii). As explained in Section 7.2, condition (i) follows from the fact that v_{K,λ_0} is cyclic for V_{λ_0} (see (7.1)), and (ii) is an automatic consequence of the cocycle condition

$$J_{w_2,w_1\lambda} \circ J_{w_1,\lambda} = J_{w_2w_1,\lambda}$$

for the intertwining operators.

Let c, R_0 be as in Lemma 10.4, and let $r > 0$ and $R > R_0$ satisfy (10.7). Let $r' < r$ be such that

$$\frac{(\log R)^2}{R^2} < cr'.$$

By perturbing R to a slightly smaller value we may assume that (10.10) is valid. It follows from Lemma 10.4 together with the denominator estimates (10.9)-(10.11) that there exists a constant $C > 0$ so that for every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\tau \in \widehat{K}_M$

$$\|F_\tau(\lambda)\|_{\mathrm{op}} \leq C(1+|\tau|)^{4|\Sigma^+|}\, e^{ar'|\tau|}\,(1+\|\lambda\|)^{\deg p_{\lambda_0}} e^{AR\,\|\operatorname{Im}\lambda\|},$$

where $a = 8m|\Sigma^+|$ and $A = h|\Sigma^+|$. This gives the remaining condition (iii) of Proposition 7.2, and with that can conclude that there is a continuous embedding

$$V_{\lambda_0}^{\omega}(ar) \subset C_{AR+\epsilon}^{\infty}(G) * v_{K,\lambda_0}.$$

By Remark 8.6 this implies the continuous embedding in Theorem 10.1. Finally, as v_{K,λ_0} is $\mathcal{U}(\mathfrak{g})$-cyclic by Lemma 4.4, we have $C_R^{\infty}(G) * v_{K,\lambda_0} = (V_{\lambda_0})_R^{\min}$.

Appendix A. The domains $\mathfrak{k}(R)$

We recall the open $\mathrm{Ad}(K)$-invariant domains $\mathfrak{k}(R)$, with $R > 0$, from (3.3). In this appendix we describe these in two interesting examples.

A.1. The unit disc: $G = \mathrm{SU}(1,1)$

While treating this example we use a notation so that the generalization to general Hermitian symmetric spaces becomes straightforward. First note that $G_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$ acts transitively on the projective space $\mathbb{P}^1(\mathbb{C})$. We identify $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \{\infty\}$ via the map

$$\mathbb{P}^1(\mathbb{C}) \to \mathbb{C} \cup \{\infty\}, \quad \mathbb{C}\begin{pmatrix} z \\ 1 \end{pmatrix} \mapsto z.$$

We define the subgroups of G

$$K = \left\{ k_\theta := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in \mathbb{R} \right\}, \quad A = \left\{ a_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

and note that $K_\mathbb{C} = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C}^* \right\}$. Further we define unipotent abelian subgroups of $G_\mathbb{C}$ by

$$P^+ := \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\} \quad \text{and} \quad P^- := \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

Note that P^+ and P^- are the stabilizers of ∞ and 0, respectively. Then both $K_\mathbb{C}P^\pm$ are Borel subgroups of $G_\mathbb{C}$ with $K_\mathbb{C}P^+ \cap K_\mathbb{C}P^- = K_\mathbb{C}$. Hence, $Z_\mathbb{C} = G_\mathbb{C}/K_\mathbb{C}$ is realized as an open affine subvariety of the projective variety $G_\mathbb{C}/K_\mathbb{C}P^+ \times G_\mathbb{C}/K_\mathbb{C}P^-$ via

$$gK_\mathbb{C} \mapsto (gK_\mathbb{C}P^+, gK_\mathbb{C}P^-).$$

More concretely, if we identify $G_\mathbb{C}/K_\mathbb{C}P^+ \times G_\mathbb{C}/K_\mathbb{C}P^-$ with $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ via

$$G_\mathbb{C}/K_\mathbb{C}P^+ \times G_\mathbb{C}/K_\mathbb{C}P^- \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$$
$$(g_1K_\mathbb{C}P^+, g_2K_\mathbb{C}P^-) \mapsto (g_1^{-t}(0), g_2(0)),$$

then $Z_\mathbb{C}$ is given by

$$Z_\mathbb{C} = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \setminus \{(z, w) : w \neq \phi(z)\},$$

where ϕ is the automorphism of $\mathbb{P}^1(\mathbb{C})$ which is induced from the linear map $\mathbb{C}^2 \ni (z_1, z_2) \mapsto (-z_1, z_2) \in \mathbb{C}^2$.

Let us denote by $\mathcal{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ the open unit disk (i.e. the bounded realization of G/K) and note that

$$Z = G/K = \{(z, \overline{z}) : z \in \mathcal{D}\} \subset Z_\mathbb{C}.$$

Now one has that that the crown domain is given by

$$\Xi = \mathcal{D} \times \mathcal{D} \subset Z_\mathbb{C}.$$

(A similar result holds for general Hermitian symmetric spaces, see [4, Sect. 3] or [18, Th. 7.7].) For $R > 0$ we note that

$$A_R = \left\{ a_t \in A \mid |t| \leq R/\sqrt{8} \right\}.$$

Now we calculate

$$k_{i\theta}a_t \cdot z_0 = (e^{2\theta}\tanh t, e^{-2\theta}\tanh t) \in Z_{\mathbb{C}}.$$

This is contained in $\Xi = \mathcal{D} \times \mathcal{D}$ precisely if $\theta \in (-r, r)$, when $r > 0$ is determined by $e^{2r}\tanh\frac{R}{\sqrt{8}} = 1$. Thus we have shown:

Proposition A.1. *Let $G = \mathrm{SU}(1,1)$, $R > 0$. Then*

$$\mathfrak{k}(R) = \{Y \in \mathfrak{k} \mid \|Y\| < \beta_R/\sqrt{8}\},$$

where $\beta_R = \frac{1}{2}\log\big(\coth(\frac{R}{\sqrt{8}})\big)$.

A.2. The hyperboloids: $G = \mathrm{SO}_o(1,n)$

Let $G = \mathrm{SO}_o(1,n)$ with $K = \mathrm{SO}(n,\mathbb{R})$ being embedded into G as the lower right corner. (The group G does not satisfy the condition that it is the group of real points of a connected algebraic reductive group defined over $\mathbb{R}$. Instead one could consider the group $SO(1,n)$, which would satisfy this condition, but for convenience of notation we rather work with its connected component.) Consider the following quadratic form on $\mathbb{C}^{n+1}$

$$\square(u) = u_0^2 - u_1^2 - \ldots - u_n^2$$

and let $u \cdot v$ be the bilinear pairing obtained by polarization. Then

$$Z = G/K = \{x \in \mathbb{R}^{n+1} \mid \square(x) = 1, x_0 > 0\},$$

$$Z_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}} = \{u \in \mathbb{C}^{n+1} \mid \square(u) = 1\}$$

and

$$\Xi = \{u = x + iy \in Z_{\mathbb{C}} \mid \square(x) > 0, x_0 > 0\},$$

see [7, p. 96]. The canonical base point in $Z_{\mathbb{C}}$ is given by $z_0 = (1, 0 \ldots, 0)^T \in Z_{\mathbb{C}}$.

Set $l = \left[\frac{n}{2}\right]$ and note that l is the rank of K. Our choice and parametrization of $\mathfrak{t}$ are as follows:

$$\mathbb{R}^l \ni \beta = (\beta_1, \ldots, \beta_l) \mapsto T_\beta := \mathrm{diag}(0, \beta_1 U, \ldots, \beta_l U) \in \mathfrak{t} \tag{A.1}$$

where

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the first zero in the diagonal matrix means the zero 1×1-matrix in case n is even and the zero 2×2-matrix if n is odd.

With the standard choice of A and $R' := R/\sqrt{2(n-1)}$ we have

$$A_R = \left\{ \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & \mathbf{1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \mid |t| \leq R' \right\}$$

and an easy computation yields

$$KA_R \cdot z_0 = \left\{ \begin{pmatrix} \cosh t \\ u \end{pmatrix} \mid u \in \mathbb{R}^n, t \in [-R', R'],\ \|u\|_2 = |\sinh t| \right\}.$$

In the sequel we only treat the case of $n = 2l$ being even; the odd case requires just a small modification.

With $k_\beta = \exp(iT_\beta)$ we obtain from (A.1) that

$$k_\beta \begin{pmatrix} \cosh t \\ u \end{pmatrix} = \begin{pmatrix} \cosh t \\ u_1 \cosh \beta_1 - iu_2 \sinh \beta_1 \\ iu_1 \sinh \beta_1 + u_2 \cosh \beta_1 \\ \vdots \end{pmatrix}.$$

The right hand side is now in the crown domain if and only if

$$\Box\Big(\operatorname{Re} k_\beta \begin{pmatrix} \cosh t \\ u \end{pmatrix} \Big) > 0,$$

i.e.

$$\cosh^2 t - \cosh^2 \beta_1 (u_1^2 + u_2^2) - \ldots - \cosh^2 \beta_l (u_{n-1}^2 + u_n^2) > 0.$$

There is no loss of generality in restricting our attention to the closure $\mathfrak{t}^+$ of a chamber in $\mathfrak{t}$, i.e. we may assume that $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_{l-1} \geq |\beta_l| \geq 0$. Then the condition from above for all u with $\|u\|_2 = |\sinh t|$ means nothing else as

$$\cosh^2 t - (\cosh^2 \beta_1) \sinh^2 t > 0$$

for all $t \in [-R', R']$. A short calculation reformulates that in

$$|\sinh \beta_1| < \frac{1}{\sinh R'}.$$

We have thus shown:

Proposition A.2. *For $G = \mathrm{SO}_o(1,n)$, $R > 0$ and the notation introduced from above one has that*

$$\mathfrak{t}(R)^+ = \left\{ T_\beta \in \mathfrak{t}^+ \mid |\sinh \beta_1| < \frac{1}{\sinh R'} \right\},$$

where $R' = R/\sqrt{2(n-1)}$.

Appendix B. The Helgason conjecture

In this appendix we briefly describe how (1.1) implies the Helgason conjecture. We are essentially following Schmid's approach from [21].

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we define the Poisson transform

$$\mathcal{P}_\lambda : V_\lambda^\infty \to C^\infty(G/K), \quad v \mapsto \left(g \mapsto \int_K v(gk)\, dk \right).$$

This map admits a continuous extension to the space $V_\lambda^{-\omega} := (V_{-\lambda}^{\omega})'$. Let $\mathbb{D}(G/K)$ be the commutative algebra of G-invariant differential operators on G/K. As before, let $v_{K,\lambda}$ be the K-fixed vector in V_λ with $v_{K,\lambda}(e) = 1$. Note that

$$\mathcal{P}_\lambda(v)(g) = \langle g^{-1} \cdot v, v_{K,-\lambda} \rangle \qquad (v \in V_\lambda). \tag{B.1}$$

The algebra $\mathcal{U}(\mathfrak{g})^K / \mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k}$ acts from the right on smooth functions on G/K. In fact $\mathbb{D}(G/K)$ is isomorphic to $\mathcal{U}(\mathfrak{g})^K / \mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k}$. Note that $\mathcal{U}(\mathfrak{g})^K$ acts by scalars on $\mathbb{C}v_{K,-\lambda}$, and hence $\mathbb{D}(G/K)$ acts by a character χ_λ on the image of $\mathcal{P}_\lambda$. We write $C^\infty(G/K)_\lambda$ for the space of joint eigenfunctions of $\mathbb{D}(G/K)$ with eigencharacter χ_λ.

The following theorem is the Helgason conjecture, which was first proven in [12].

Theorem B.1. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be so that the K-spherical vector $v_{K,-\lambda}$ in $V_{-\lambda}$ is $\mathcal{U}(\mathfrak{g})$-cyclic. Then $\mathcal{P}_\lambda$ defines a G-equivariant isomorphism*

$$V_\lambda^{-\omega} \to C^\infty(G/K)_\lambda \tag{B.2}$$

of topological vector spaces.

Remark B.2. By Lemma 4.4 $v_{K,-\lambda}$ is $\mathcal{U}(\mathfrak{g})$-cyclic if $-\lambda$ satisfies (4.3).

We derive the theorem from (1.1). We recall Schmid's maximal globalization of a Harish-Chandra module V,

$$V_{\max} = \operatorname{Hom}_{(\mathfrak{g},K)}\big(V^\vee, C^\infty(G)\big),$$

where $V^\vee$ is the dual Harish-Chandra module of V, i.e. the space of K-finite vectors in the algebraic dual of V. Further, $C^\infty(G)$ is considered as a $(\mathfrak{g}, K)$-module, where $\mathfrak{g}$ and K act via the right-regular representation. We provide $V_{\max}$ with a topology as follows. The space

$$E := \operatorname{Hom}_{\mathbb{C}}\big(V^\vee, C^\infty(G)\big) \tag{B.3}$$

is a countable product of copies of the Fréchet space $C^\infty(G)$ and hence is a Fréchet space. Now $V_{\max} = \operatorname{Hom}_{(\mathfrak{g},K)}(V^\vee, C^\infty(G))$ is a closed subspace and as such inherits the structure of a Fréchet space. Moreover, the G-action on $V_{\max}$ is continuous.

Lemma B.3. *For any Harish-Chandra module V, the maximal globalization $V_{\max}$ is a reflexive Fréchet space.*

Proof. First we recall that $C^\infty(G)$ is reflexive. As the space E from (B.3) is a countable product of reflexive Fréchet spaces, it is reflexive by [19, Prop. 24.3]. Now $V_{\max}$ is a closed subspace of E and as such reflexive by [19, Prop. 23.26]. □

By taking matrix coefficients one sees that any globalization of V embeds continuously into $V_{\max}$. Here by globalization we understand a completion of V to a representation of G on a complete Hausdorff topological vector space $E = \overline{V}$. Note that the assignment $V \mapsto V_{\max}$ is a functor from the category of Harish-Chandra modules to the category of continuous representations. We define $V^{-\omega}$ as the continuous dual of $(V^\vee)^\omega$ equipped with the strong topology.

Proposition B.4. *For every Harish-Chandra module V we have*

$$V_{\max} = V^{-\omega}$$

as topological G-modules.

Proof. We now use Schmid's identity (1.1). As $V_{\min} = V^\omega$ for all Harish-Chandra modules V, it suffices to show that $V_{\max} = (V^\vee)'_{\min}$.

We recall from Lemma B.3 that $V_{\max}$ is reflexive. Since $V'_{\max}$ is a globalization of $V^\vee$ there exists an embedding $(V^\vee)_{\min} \to V'_{\max}$. Taking duals we obtain a map $V_{\max} \to (V^\vee)'_{\min}$. On the other hand $(V^\vee)'_{\min}$ is a globalization of V and hence embeds into $V_{\max}$. As these maps restrict to the identity on V, it follows that $V_{\max} = (V^\vee)'_{\min}$ as asserted. □

Proposition B.5. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be so that the K-spherical vector $v_{K,-\lambda}$ in $V_{-\lambda}$ is $\mathcal{U}(\mathfrak{g})$-cyclic. Then*

$$(V_\lambda)_{\max} = C^\infty(G/K)_\lambda$$

as topological G-modules.

For the proof of the proposition we need the following lemma.

Lemma B.6. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. If $v_{K,-\lambda}$ is $\mathcal{U}(\mathfrak{g})$-cyclic in $V_{-\lambda}$, then*

$$V_{-\lambda} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g})\mathfrak{k}+\mathcal{U}(\mathfrak{g})^K} \mathbb{C}v_{K,-\lambda}$$

as $(\mathfrak{g}, K)$-modules.

Proof. By assumption the natural map $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g})\mathfrak{k}+\mathcal{U}(\mathfrak{g})^K} \mathbb{C}v_{K,-\lambda} \to V_{-\lambda}$ of $(\mathfrak{g}, K)$-modules is surjective. It remains to prove injectivity. Recall from (6.5) that $\mathcal{U}(\mathfrak{g}) = \mathcal{H}^\star(\mathfrak{s})\mathcal{I}^\star(\mathfrak{s}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{k}$. Since $\mathcal{I}^\star(\mathfrak{s}) = \mathcal{H}^\star(\mathfrak{s})\mathcal{I}^\star(\mathfrak{s}) \cap \mathcal{U}(\mathfrak{g})^K$, we have as K-modules

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g})\mathfrak{k}+\mathcal{U}(\mathfrak{g})^K} \mathbb{C}v_{K,-\lambda} = \mathcal{H}^\star(\mathfrak{s})\mathcal{I}^\star(\mathfrak{s}) \otimes_{\mathcal{I}^\star(\mathfrak{s})} \mathbb{C}v_{K,-\lambda} \simeq \mathcal{H}^\star(\mathfrak{s}).$$

By Kostant-Rallis [15] the right-hand side is K-isomorphic to $\mathbb{C}[K/M]$. Since $V_{-\lambda}$ is K-isomorphic to $\mathbb{C}[K/M]$ as well, the assertion follows from the finite dimensionality of the K-isotypes. □

Proof of Proposition B.5. By Lemma B.6, we have the following equalities of G-modules,

$$\begin{aligned}(V_\lambda)_{\max} &= \mathrm{Hom}_{(\mathfrak{g},K)}\big(V_{-\lambda}, C^\infty(G)\big)\\ &= \mathrm{Hom}_{(\mathfrak{g},K)}\big(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g})\mathfrak{k}+\mathcal{U}(\mathfrak{g})^K} \mathbb{C}v_{K,-\lambda}, C^\infty(G)\big)\\ &= \mathrm{Hom}_{(\mathcal{U}(\mathfrak{g})\mathfrak{k}+\mathcal{U}(\mathfrak{g})^K,K)}\big(\mathbb{C}v_{K,-\lambda}, C^\infty(G)\big)\\ &= \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})^K}\big(\mathbb{C}v_{K,-\lambda}, C^\infty(G/K)\big).\end{aligned}$$

The assertion now follows from the definition of $C^\infty(G/K)_\lambda$. □

Proof of Theorem B.1. In view of Proposition B.4 and Proposition B.5, both sides of (B.2) are isomorphic to $(V_\lambda)_{\max}$. Furthermore, as $v_{K,-\lambda}$ is $\mathcal{U}(\mathfrak{g})$-cyclic, it follows from (B.1) that $\mathcal{P}_\lambda$ is injective, and hence bijective, on the space of K-finite vectors. The theorem now follows from the functoriality of the maximal globalizations. □

Appendix C. An application to eigenfunctions on $Z = G/K$

We recall the crown domain $\Xi \subset Z_{\mathbb{C}}$, the natural G-extension of Z inside of $Z_{\mathbb{C}}$. Also we recall from the preceding appendix that $V_{\lambda,\max} = C^\infty(Z)_\lambda$ for every spherical principal series V_λ, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. We mentioned in the introduction that for every K-spherical Harish-Chandra module V with K-spherical vector v_K that the orbit map

$$f_v : G/K \to V^\infty, \quad gK \mapsto g \cdot v_K$$

extends holomorphically to Ξ, see [18, Th. 1.1]. Therefore, every $\mathbb{D}(Z)$-eigenfunction extends holomorphically to Ξ, and thus we obtain $C^\infty(Z)_\lambda = \mathcal{O}(\Xi)_\lambda$, i.e.

$$V_{\lambda,\max} = \mathcal{O}(\Xi)_\lambda$$

by Prop. B.5. Now for every $r > 0$ we define K-invariant enlargements of Ξ inside of $Z_{\mathbb{C}}$ by

$$Z_{\mathbb{C}}(r) := K_{\mathbb{C}}(r) \cdot \Xi = \exp(i\mathfrak{k}_r) \cdot \Xi \subset Z_{\mathbb{C}}.$$

It is not clear whether $Z_{\mathbb{C}}(r)$ is simply connected. Out of precaution we pass to the simply connected cover $\widetilde{Z}_{\mathbb{C}}(r)$ of $Z_{\mathbb{C}}(r)$. Note that K acts naturally on the complex manifold $\widetilde{Z}_{\mathbb{C}}(r)$. From the definition of $V_{\lambda,r}^\omega$ and $V_\lambda^\omega \subset V_{\lambda,\max} = \mathcal{O}(\Xi)_\lambda$ we thus obtain

$$V_{\lambda,r}^\omega = \mathcal{O}(\widetilde{Z}_{\mathbb{C}}(r))_\lambda.$$

Hence the fact that $V_{\lambda,r}^\omega \subset V_{\lambda,\min}(R)$ for $\frac{(\log R)^2}{R^2} < cr$ (see Theorem 10.1) implies the following

Theorem C.1. *Let $-\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfying (4.3) and $r, R > 0$ such that $\frac{(\log R)^2}{R^2} < cr$. Then any $f \in \mathcal{O}(\widetilde{Z}_{\mathbb{C}}(r))_\lambda$ can be factorized as*

$$f = \psi * \phi_\lambda$$

where ϕ_λ is the Harish-Chandra spherical function in $C^\infty(G/K)_\lambda$ and $\psi \in C_R^\infty(G)$.

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Heiko Gimperlein
Leopold-Franzens-Universität Innsbruck
Engineering Mathematics
Technikerstrasse 13, 6020 Innsbruck
E-mail address: heiko.gimperlein@uibk.ac.at

Bernhard Krötz
Institute für Mathematik
Universität Paderborn
Warburger Strasse 100
D-33098 Paderborn
E-mail address: bkroetz@gmx.de

Job Kuit
Institut für Mathematik
Universität Paderborn
Warburger Strasse 100
D-33098 Paderborn
E-mail address: jobkuit@math.upb.de

Henrik Schlichtkrull
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen Ø
E-mail address: schlicht@math.ku.dk

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