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Existence of flips for generalized lc pairs

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We prove the existence of flips for $\mathbb{Q}$-factorial NQC generalized lc pairs, and the cone and contraction theorems for NQC generalized lc pairs. This answers a conjecture of Han-Li-Birkar. As an immediate application, we show that we can run the minimal model program for $\mathbb{Q}$-factorial NQC generalized lc pairs. In particular, we complete the minimal model program for $\mathbb{Q}$-factorial NQC generalized lc pairs in dimension ≤ 3 and pseudo-effective $\mathbb{Q}$-factorial NQC generalized lc pairs in dimension 4.

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1. Introduction

We work over the field of complex numbers $\mathbb{C}$, however many of the results also hold over any algebraically closed field k of characteristic zero.

The theory of *generalized pairs* (*g-pairs* for short) was introduced by C. Birkar and D.-Q. Zhang in [10] to tackle the effective Iitaka fibration conjecture. The structure of g-pairs naturally appears in the canonical bundle formula and sub-adjunction formulas [31, 18]. This theory has been used in an essential way in the proof of the Borisov-Alexeev-Borisov conjecture [6, 8]. We refer the reader to [7] for a more detailed introduction to the theory of g-pairs.

It has recently become apparent that the MMP for g-pairs is closely related to the MMP for usual pairs and varieties. In particular, the MMP for g-pairs has been used to prove the termination of pseudo-effective fourfold flips [22, 12, 20]. For this, and other reasons, it is important to study the minimal model program for generalized pairs. Although the MMP for gklt (generalized klt) g-pairs behaves very similar to the MMP for usual klt pairs ([10, Lemma 4.4], [22, Lemma 3.4]), there are several non-trivial issues when we study the MMP for glc (generalized lc) g-pairs: in order to complete the minimal model program, we need

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1. the cone and the contraction theorems,
2. the existence of flips, and
3. the termination of flips.

For the usual lc pairs, we know (1) (cf. [29, 34, 33, 3, 17]) and (2) (cf. [9, 5, 21]) completely. The difficult part for the MMP for usual pairs is (3) as we only know the termination of flips in dimension ≤ 3 [30, 39] and some special cases in dimension 4 [14, 15, 2, 4, 19, 20, 22, 12].

For glc g-pairs that are not gklt (more precisely, not even gdlt), the situation is completely different. First of all, we usually need to add the NQC (nef $\mathbb{Q}$-Cartier combination) condition for technical reasons (cf. [22, Example 3.19]), however this is a natural assumption and is contained in the original definition of g-pairs in [10]. Under the NQC assumption, the known results on the termination of flips are similar to the usual pair case (in particular, in full generality in dimension ≤ 3 [12] and in the pseudo-effective case in dimension 4 [12, 20]). However, the cone theorem, contraction theorem, and the existence of flips for glc g-pairs (cf. [7, 6.1] and [22, Conjectures 3.11, 3.12]), seem to be far more challenging.

In this paper, we prove the cone and the contraction theorems and the existence of flips for $\mathbb{Q}$-factorial NQC glc g-pairs in full generality, hence answering [7, 6.1] and [22, Conjectures 3.11, 3.12]:

Theorem 1.1 (Cone and contraction theorems for generalized lc pairs)**.** *Let $(X,B,\mathbf{M})/U$ be an NQC glc g-pair and $\pi : X \to U$ the associated morphism. Let $\{R_j\}_{j\in\Lambda}$ be the set of $(K_X+B+\mathbf{M}_X)$-negative extremal rays in $\overline{NE}(X/U)$ that are rational. Then:*

1. $$\overline{NE}(X/U)=\overline{NE}(X/U)_{K_X+B+\mathbf{M}_X\geq 0}+\sum_{j\in\Lambda}R_j.$$
   In particular, any $(K_X+B+\mathbf{M}_X)$-negative extremal ray in $\overline{NE}(X/U)$ is rational.
2. *Each R_j is spanned by a rational curve C_j such that $\pi(C_j)=\{pt\}$ and*
   $$0<-(K_X+B+\mathbf{M}_X)\cdot C_j\leq 2\dim X.$$
3. *For any ample/U $\mathbb{R}$-divisor A on X,*
   $$\Lambda_A:=\{j\in\Lambda\mid R_j\subset\overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A<0}\}$$

is a finite set. In particular, $\{R_j\}_{j\in\Lambda}$ *is countable, and is a discrete subset in* $\overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A<0}$*. Moreover, we may write*

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0} + \sum_{j\in\Lambda_A} R_j.$$

4. *Let* F *be a* $(K_X+B+\mathbf{M}_X)$*-negative extremal face in* $\overline{NE}(X/U)$*. Then* F *is a rational extremal face.*
5. *Assume that* $\mathbf{M}_X$ *is* $\mathbb{R}$*-Cartier. Let* F *be a* $(K_X+B+\mathbf{M}_X)$*-negative extremal face in* $\overline{NE}(X/U)$*. Then there exists a projective morphism* $\mathrm{cont}_F : X \to Y$ *over* U *satisfying the following:*
   (a) *Let* C *be an integral curve such that* $\pi(C)$ *is a point. Then* $\mathrm{cont}_R(C)$ *is a point if and only if* $[C]\in F$*.*
   (b) $\mathcal{O}_Y \cong (\mathrm{cont}_F)_*\mathcal{O}_X$*. In other words,* cont_F *is a contraction.*
   (c) *Let* L *be a line bundle on* X *such that* $L\cdot C=0$ *for any* C *such that* $[C]\in F$*. Then there exists a line bundle* L_Y *on* Y *such that* $L\cong f^*L_Y$*.*

Theorem 1.2 (Existence of flips for generalized lc pairs)**.** *Let* $(X,B,\mathbf{M})/U$ *be a* $\mathbb{Q}$*-factorial NQC glc g-pair and* $f : X\to Z$ *a* $(K_X+B+\mathbf{M}_X)$*-flipping contraction over* U*. Then the flip* $f^+ : X^+ \to Z$ *of* f *exists. Moreover,* X^+ *is* $\mathbb{Q}$*-factorial and* $\rho(X)=\rho(X^+)$*.*

We add the assumption "$\mathbb{Q}$-factorial" as it is a natural assumption which always appears in the minimal model program, and it is well-known that the non-$\mathbb{Q}$-factorial minimal model program may behave very differently from the $\mathbb{Q}$-factorial case (cf. [16, 4.4]).

Theorems 1.1 and 1.2 imply that we can run MMP for any $\mathbb{Q}$-factorial NQC glc g-pair:

Theorem 1.3. *We can run the MMP for* $\mathbb{Q}$*-factorial NQC glc g-pairs.*

Therefore, as long as we know the termination of flips, we can completely establish the minimal model program for $\mathbb{Q}$-factorial NQC glc g-pairs. In particular, we have:

Theorem 1.4. *The MMP for* $\mathbb{Q}$*-factorial NQC glc g-pairs in dimension* ≤ 3 *holds, and the MMP for pseudo-effective* $\mathbb{Q}$*-factorial NQC glc g-pairs in dimension* 4 *holds.*

Postscript remark. We refer the reader to [37, 41, 40] for further recent progress on the MMP for generalized lc pairs.

2. Preliminaries

We will freely use the notation and definitions from [35, 9]. For generalized pairs, we will follow the definitions in [22] and the notation as in [27, 13].

Definition 2.1. Let a be a real number, X a normal variety, and $D = \sum_i d_i D_i$ an $\mathbb{R}$-divisor on X, where D_i are the irreducible components of D. We define $D^{\leq a} := \sum_{i|d_i\leq a} d_i D_i$, $D^{=a} := \sum_{i|d_i=a} d_i D_i$, $D^{\geq a} := \sum_{i|d_i\geq a} d_i D_i$, $\lfloor D\rfloor := \sum_i \lfloor d_i\rfloor D_i$, and $\{D\} := \sum_i \{d_i\} D_i$.

Definition 2.2. Let $\phi : X \dashrightarrow Y$ be a birational map between normal varieties. We let $\mathrm{Exc}(\phi)$ be the union of the exceptional divisors of ϕ, and usually identify $\mathrm{Exc}(\phi)$ with the reduced exceptional divisor of ϕ. ϕ is called a *contraction* if ϕ is a projective morphism and $\phi_*\mathcal{O}_X = \mathcal{O}_Y$.

Lemma 2.3 (cf. [9, Lemma 3.2.1])**.** *Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. Let $\pi : X \to U$ be a projective morphism between normal quasi-projective varieties. Let D be a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor on X and let D' be its restriction to the generic fiber of π.*

If $D' \sim_{\mathbb{K}} B' \geq 0$ for some $\mathbb{K}$-divisor B' on the generic fiber of π, then $D \sim_{\mathbb{K},U} B \geq 0$ for some $\mathbb{K}$-divisor B, such that B' is the restriction of B to the generic fiber of π.

Definition 2.4 (**b**-divisors)**.** Let X be a normal quasi-projective variety. We call Y a *birational model* over X if there exists a projective birational morphism $Y \to X$.

Let $X \dashrightarrow X'$ be a birational map. For any valuation ν over X, we define $\nu_{X'}$ to be the center of ν on X'. A **b**-*divisor* $\mathbf{D}$ over X is a formal sum $\mathbf{D} = \sum_\nu r_\nu \nu$ where ν are valuations over X and $r_\nu \in \mathbb{R}$, such that ν_X is not a divisor except for finitely many ν. If in addition, $r_\nu \in \mathbb{Q}$ for every ν, then $\mathbf{D}$ is called a $\mathbb{Q}$-**b**-*divisor*. The *trace* of $\mathbf{D}$ on X' is the $\mathbb{R}$-divisor

$$\mathbf{D}_{X'} := \sum_{\nu_{i,X'} \text{ is a divisor}} r_i \nu_{i,X'}.$$

If $\mathbf{D}_{X'}$ is $\mathbb{R}$-Cartier and $\mathbf{D}_Y$ is the pullback of $\mathbf{D}_{X'}$ on Y for any birational model Y of X', we say that $\mathbf{D}$ *descends* to X', and also say that $\mathbf{D}$ is the *closure* of $\mathbf{D}_{X'}$, and write $\mathbf{D} = \overline{\mathbf{D}_{X'}}$.

Let $X \to U$ be a projective morphism and assume that $\mathbf{D}$ is a **b**-divisor over X such that $\mathbf{D}$ descends to some birational model Y over X. If $\mathbf{D}_Y$ is nef$/U$, then we say that $\mathbf{D}$ is *nef*$/U$. If $\mathbf{D}_Y$ is a Cartier divisor, then we say that $\mathbf{D}$ is **b**-*Cartier*. If $\mathbf{D}_Y$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then we say that $\mathbf{D}$

is $\mathbb{Q}$-**b**-*Cartier*. If $\mathbf{D}$ can be written as an $\mathbb{R}_{\geq 0}$-linear combination of nef$/U$ **b**-Cartier **b**-divisors, then we say that $\mathbf{D}$ is *NQC*$/U$.

We let $\mathbf{0}$ be the **b**-divisor $\bar{0}$.

Definition 2.5. Let $X \to U$ be a projective morphism such that X is a normal quasi-projective variety, U^0 a non-empty open subset of U, and $\mathbf{D}$ a **b**-divisor over X. We define a **b**-divisor $\mathbf{D}^0 := \mathbf{D} \times_U U^0$ in the following way. For any birational projective morphism $Y^0 \to X^0 = X \times_U U^0$, we may assume that $Y^0 = Y \times_U U^0$ where $Y \to X$ is a birational projective morphism. We let $\mathbf{D}^0_{Y^0} = \mathbf{D}_Y|_{Y_0}$. It is easy to see that this definition is independent of the choice of Y and defines a **b**-divisor.

It is easy to see that if $W \to X$ is a birational morphism such that $\mathbf{D}$ descends to W, then $\mathbf{D}^0$ is the closure of $\mathbf{D}_W \times_U U^0$. Since base change is compatible with pullbacks, $\mathbf{D}^0$ is well-defined and independent of the choice of W. We also note that if $\mathbf{D}$ is nef$/U$, then $\mathbf{D}^0$ is nef$/U^0$, and if $\mathbf{D}$ is NQC$/U$, then $\mathbf{D}^0$ is NQC$/U^0$.

Definition 2.6 (Generalized pairs)**.** A *generalized sub-pair* (*g-sub-pair* for short) $(X, B, \mathbf{M})/U$ consists of a normal quasi-projective variety X associated with a projective morphism $X \to U$, an $\mathbb{R}$-divisor B on X, and a nef$/U$ **b**-divisor $\mathbf{M}$ over X, such that $K_X + B + \mathbf{M}_X$ is $\mathbb{R}$-Cartier. If $\mathbf{M}$ is NQC$/U$, then we say that $(X, B, \mathbf{M})/U$ is an *NQC g-sub-pair*. If B is a $\mathbb{Q}$-divisor and $\mathbf{M}$ is a $\mathbb{Q}$-**b**-divisor, then we say that $(X, B, \mathbf{M})/U$ is a $\mathbb{Q}$-*g-sub-pair*.

If $\mathbf{M} = \mathbf{0}$, a g-sub-pair $(X, B, \mathbf{M})/U$ is called a *sub-pair* and is denoted by (X, B) or $(X, B)/U$.

If $U = \{pt\}$, we usually drop U and say that $(X, B, \mathbf{M})$ is *projective*. If U is not important, we may also drop U.

A g-sub-pair (resp. NQC g-sub-pair, $\mathbb{Q}$-g-sub-pair) $(X, B, \mathbf{M})/U$ is called a *g-pair* (resp. *NQC g-pair*, $\mathbb{Q}$-*g-pair*) if $B \geq 0$. A sub-pair (X, B) is called a *pair* if $B \geq 0$.

Definition 2.7 (Singularities of generalized pairs)**.** Let $(X, B, \mathbf{M})/U$ be a g-(sub-)pair. For any prime divisor E and $\mathbb{R}$-divisor D on X, we define $\operatorname{mult}_E D$ to be the *multiplicity* of E along D. Let $h : W \to X$ be any log resolution of $(X, \operatorname{Supp} B)$ such that $\mathbf{M}$ descends to W, and let

$$K_W + B_W + \mathbf{M}_W := h^*(K_X + B + \mathbf{M}_X).$$

The *log discrepancy* of a prime divisor D on W with respect to $(X, B, \mathbf{M})$ is $1 - \operatorname{mult}_D B_W$ and it is denoted by $a(D, X, B, \mathbf{M})$.

We say that $(X,B,\mathbf{M})$ is *(sub-)glc* (resp. *(sub-)gklt*) if $a(D,X,B,\mathbf{M})\geq 0$ (resp. >0) for every log resolution $h:W\to X$ as above and every prime divisor D on W.

We say that $(X,B,\mathbf{M})$ is *gdlt* if $(X,B,\mathbf{M})$ is glc, and there exists a closed subset $V\subset X$, such that

1. $X\backslash V$ is smooth and $B_{X\backslash V}$ is simple normal crossing, and
2. for any prime divisor E over X such that $a(E,X,B,\mathbf{M})=0$, $\operatorname{center}_X E\not\subset V$ and $\operatorname{center}_X E\backslash V$ is an lc center of $(X\backslash V, B|_{X\backslash V})$.

If $\mathbf{M}=\mathbf{0}$ and $(X,B,\mathbf{M})$ is (sub-)glc (resp, gklt, gdlt), we say that (X,B) is (sub-)lc (resp. (klt, dlt).

Suppose that $(X,B,\mathbf{M})$ is sub-glc. A *glc place* of $(X,B,\mathbf{M})$ is a prime divisor E over X such that $a(E,X,B,\mathbf{M})=0$. A *glc center* of $(X,B,\mathbf{M})$ is the center of a glc place of $(X,B,\mathbf{M})$ on X. The *non-gklt locus* $\operatorname{Ngklt}(X,B,\mathbf{M})$ of $(X,B,\mathbf{M})$ is the union of all glc centers of $(X,B,\mathbf{M})$. If $\mathbf{M}=\mathbf{0}$, a glc place (resp. a glc center, the non-gklt locus) of $(X,B,\mathbf{M})$ will be called an lc place (resp. an lc center, the non-klt locus) of (X,B), and we will denote $\operatorname{Ngklt}(X,B,\mathbf{M})$ by $\operatorname{Nklt}(X,B)$.

We note that the definitions above are independent of the choice of U.

Theorem 2.8. *Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC glc g-pair such that X is klt, and $A\geq 0$ an ample/U $\mathbb{R}$-divisor on X such that $(X,B+A,\mathbf{M})$ is glc and $K_X+B+A+\mathbf{M}_X$ is nef/U. Let*

$$(X,B,\mathbf{M}):=(X_1,B_1,\mathbf{M})\dashrightarrow(X_2,B_2,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(X_i,B_i,\mathbf{M})\dashrightarrow\dots$$

be a $(K_X+B+\mathbf{M}_X)$-MMP/U with scaling of A, and let λ_i be the i-th scaling number of this MMP for each i, i.e.

$$\lambda_i:=\inf\{t\mid t\geq 0, K_{X_i}+B_i+tA_i+\mathbf{M}_{X_i}\text{ is nef/}U\},$$

where A_i is the strict transform of A on X_i for each i. Then $\lambda_i\geq\lambda_{i+1}$ for each i, and one of the following holds:

1. *This MMP terminates after finitely many steps.*
2. *$\lim_{i\to+\infty}\lambda_i=0$, and $(X,B,\mathbf{M})$ does not have a log minimal model (see Definition 3.2 below).*

In particular, if $(X,B,\mathbf{M})/U$ is gdlt and has a log minimal model, then this MMP terminates with log minimal model of $(X,B,\mathbf{M})/U$.

Proof. By [22, Remark 3.25, Theorem 4.1], $\lambda_i \geq \lambda_{i+1}$ for each i, and we may assume that this MMP does not terminate and $\lambda_i = \lambda_{i+1} > 0$ for any $i \gg 0$. Let $\lambda := \lim_{i\to+\infty} \lambda_i$, then $\lambda_i = \lambda > 0$ for all $i \gg 0$. Since X is $\mathbb{Q}$-factorial klt, by [22, Lemma 3.4], we may pick

$$0 \leq \Delta \sim_{\mathbb{R},U} B + \mathbf{M}_X + \frac{\lambda}{2} A$$

such that (X, Δ) is klt and Δ is big$/U$. Now this MMP is also a $(K_X + \Delta)$-MMP with scaling of $0 \leq A' \sim_{\mathbb{R},U} (1 - \frac{\lambda}{2})A$ for some A' such that $(X, \Delta + A')$ is klt. This MMP terminates by [9, Corollary 1.4.2], a contradiction.

The in particular part follows from the fact that $(X_i, B_i, \mathbf{M})$ is $\mathbb{Q}$-factorial gdlt for each i if $(X, B, \mathbf{M})$ is gdlt, and

$$a(D, X, B, \mathbf{M}) < a(D, X_i, B_i, \mathbf{M})$$

for any i and any prime divisor D on X that is exceptional over X_i. □

3. Models

In this sections, we will study different types of models of generalized pairs. For the case of models of usual pairs, we refer the reader to [5, Section 2], [25, Section 2].

3.1. Definitions

Definition 3.1 (Log smooth model)**.** Let $(X, B, \mathbf{M})/U$ be a glc g-pair and $h : W \to X$ a log resolution of $(X, \operatorname{Supp} B)$ such that $\mathbf{M}$ descends to W. Let $B_W \geq 0$ and $E \geq 0$ be two $\mathbb{R}$-divisors on W such that

1. $K_W + B_W + \mathbf{M}_W = h^*(K_X + B + \mathbf{M}_X) + E$,
2. (W, B_W) is log smooth dlt,
3. E is h-exceptional, and
4. for any h-exceptional prime divisor D such that $a(D, X, B, \mathbf{M}) > 0$, D is a component of E.

Then $(W, B_W, \mathbf{M})$ is called a *log smooth model* of $(X, B, \mathbf{M})$.

Definition 3.2 (Models)**.** Let $(X, B, \mathbf{M})/U$ be a glc g-pair, $\phi : X \dashrightarrow X'$ a birational map over U, and $E := \operatorname{Exc}(\phi^{-1})$ the reduced ϕ^{-1}-exceptional divisor. Let $B' := \phi_* B + E$.

1. $(X', B', \mathbf{M})/U$ is called a *log birational model* of $(X, B, \mathbf{M})/U$.

2. $(X', B', \mathbf{M})/U$ is called a *weak glc model* of $(X, B, \mathbf{M})/U$ if
   (a) $(X', B', \mathbf{M})/U$ is a log birational model of $(X, B, \mathbf{M})/U$,
   (b) $K_{X'} + B' + \mathbf{M}_{X'}$ is nef$/U$, and
   (c) for any prime divisor D on X which is exceptional over X', $a(D, X, B, \mathbf{M}) \leq a(D, X', B', \mathbf{M})$.
3. $(X', B', \mathbf{M})/U$ is called a *log minimal model* of $(X, B, \mathbf{M})/U$ if
   (a) $(X', B', \mathbf{M})/U$ is a weak glc model of $(X, B, \mathbf{M})/U$,
   (b) $(X', B', \mathbf{M})$ is $\mathbb{Q}$-factorial gdlt, and
   (c) for any prime divisor D on X which is exceptional over X', $a(D, X, B, \mathbf{M}) < a(D, X', B', \mathbf{M})$.
4. $(X', B', \mathbf{M})/U$ is called a *good minimal model* of $(X, B, \mathbf{M})/U$ if
   (a) $(X', B', \mathbf{M})/U$ is a log minimal model of $(X, B, \mathbf{M})/U$, and
   (b) $K_{X'} + B' + \mathbf{M}_{X'}$ is semi-ample$/U$.

Definition-Lemma 3.3 ([22, Proposition 3.10])**.** Let $(X, B, \mathbf{M})/U$ be a glc g-pair. Then there exists a birational morphism $f : Y \to X$ and a glc g-pair $(Y, B_Y, \mathbf{M})/U$, such that

1. $(Y, B_Y, \mathbf{M})$ is $\mathbb{Q}$-factorial gdlt,
2. $K_Y + B_Y + \mathbf{M}_Y = f^*(K_X + B + \mathbf{M}_X)$, and
3. any f-exceptional divisor is a component of $\lfloor B_Y \rfloor$.

For any birational morphism f and $(Y, B_Y, \mathbf{M})$ which satisfies (1-3), f will be called a *gdlt modification* of $(X, B, \mathbf{M})$, and $(Y, B_Y, \mathbf{M})$ will be called a *gdlt model* of $(X, B, \mathbf{M})$.

3.2. Models under some birational maps

Lemma 3.4. *Let $(X, B, \mathbf{M})/U$ be a glc g-pair, $(X', B', \mathbf{M})/U$ a weak glc model of $(X, B, \mathbf{M})/U$ with birational map $\phi : X \dashrightarrow X'$, and $p : W \to X$ and $q : W \to X'$ a common resolution of $(X, B, \mathbf{M})$ and $(X', B', \mathbf{M})$ such that $q = \phi \circ p$. Assume that*

$$p^*(K_X + B + \mathbf{M}_X) = q^*(K_{X'} + B' + \mathbf{M}_{X'}) + E,$$

then $E \geq 0$ is exceptional over X'.

Proof. For any prime divisor D that is an irreducible component of E,

$$\operatorname{mult}_D E = a(D, X', B', \mathbf{M}) - a(D, X, B, \mathbf{M}).$$

Thus if D is not exceptional over X, then

- if D is not exceptional over X', then $\operatorname{mult}_D E = 0$, and
- if D is exceptional over X', then $\operatorname{mult}_D E \geq 0$ by Definition 3.2(2.c).

Therefore, $p_* E \geq 0$. Since $K_{X'} + B' + \mathbf{M}_{X'}$ is nef$/U$, $q^*(K_{X'} + B' + \mathbf{M}_{X'})$ is nef$/X$, hence E is anti-nef$/X$. By the negativity lemma, $E \geq 0$.

If E is not exceptional over X', then there exists a component D of E that is not exceptional over X'. If D is not exceptional over X, then $\operatorname{mult}_D E = 0$, a contradiction. Thus D is exceptional over X. By the definition of weak glc models, $a(D, X', B', \mathbf{M}) = 0$. Since $E \geq 0$, $a(D, X, B, \mathbf{M}) \leq a(D, X, B', \mathbf{M}) = 0$. Since $(X, B, \mathbf{M})/U$ is a glc g-pair, $a(D, X, B, \mathbf{M}) \geq 0$. Thus $a(D, X, B, \mathbf{M}) = 0$, which implies that $\operatorname{mult}_D E = 0$, a contradiction. □

Lemma 3.5. *Let $(X, B, \mathbf{M})/U$ be a glc g-pair. Let $(X_1, B_1, \mathbf{M})/U$ and $(X_2, B_2, \mathbf{M})/U$ be two weak glc models of $(X, B, \mathbf{M})/U$ with induced birational map $\phi : X_1 \dashrightarrow X_2$, and $g_1 : W \to X_1$ and $g_2 : W \to X_2$ a common resolution such that $\phi \circ g_1 = g_2$. Then:*

1. $$g_1^*(K_{X_1} + B_1 + \mathbf{M}_{X_1}) = g_2^*(K_{X_2} + B_2 + \mathbf{M}_{X_2}).$$
   In particular, if $K_{X_2} + B_2 + \mathbf{M}_{X_2}$ is ample$/U$, then ϕ is a morphism.
2. *If $K_{X_1} + B_1 + \mathbf{M}_{X_1}$ is semi-ample$/U$, then for any weak glc model $(X', B', \mathbf{M})/U$ of $(X, B, \mathbf{M})/U$, $K_{X'} + B' + \mathbf{M}_{X'}$ is semi-ample$/U$.*

Proof. Let $\phi_1 : X \dashrightarrow X_1$ and $\phi_2 : X \dashrightarrow X_2$ be the induced birational maps. Possibly replacing W, we may assume that the induced birational map $h : W \to X$ is a morphism. Let

$$E_i := h^*(K_X + B + \mathbf{M}_X) - g_i^*(K_{X_i} + B_i + \mathbf{M}_{X_i})$$

for $i \in \{1, 2\}$. By Lemma 3.4, $E_i \geq 0$ and is exceptional over X_i for $i \in \{1, 2\}$. Thus $g_{1,*}(E_2 - E_1) \geq 0$ and $E_1 - E_2$ is nef$/X_1$, and $g_{2,*}(E_1 - E_2) \geq 0$ and $E_2 - E_1$ is nef$/X_2$. By the negativity lemma, $E_2 - E_1 \geq 0$ and $E_1 - E_2 \geq 0$. Thus $E_1 = E_2$, which implies (1). (2) immediately follows from (1). □

Lemma 3.6. *Let $(X,B,\mathbf{M})/U$ be a glc g-pair, $h: W \to X$ a log resolution of $(X, \operatorname{Supp} B)$ such that $\mathbf{M}$ descends to W, and $(W,B_W,\mathbf{M})$ a log smooth model of $(X,B,\mathbf{M})$. Then any weak glc model (resp. log minimal model, good minimal model) of $(W,B_W,\mathbf{M})/U$ is a weak glc model (resp. log minimal model, good minimal model) of $(X,B,\mathbf{M})/U$.*

Proof. Since $(W,B_W,\mathbf{M})$ is a log smooth model of $(X,B,\mathbf{M})$, we may write

$$K_W+B_W+\mathbf{M}_W=h^*(K_X+B+\mathbf{M}_X)+E$$

for some $E\geq 0$ that is h-exceptional.

Claim 3.7. *Let $(X',B',\mathbf{M})/U$ be a weak glc model of $(W,B_W,\mathbf{M})/U$. Then $a(D,X,B,\mathbf{M})\leq a(D,X',B',\mathbf{M})$ for any prime divisor D over X.*

Proof. Let $\phi_W: W \dashrightarrow X'$ be the induced birational map, and let $p: V\to W$ and $q: V\to X'$ be a common resolution such that $q=\phi_W\circ p$. By Lemma 3.4,

$$p^*(K_W+B_W+\mathbf{M}_W)=q^*(K_{X'}+B'+\mathbf{M}_{X'})+F$$

for some $F\geq 0$ that is exceptional over X'. Then we have

$$p^*h^*(K_X+B+\mathbf{M}_X)=q^*(K_{X'}+B'+\mathbf{M}_{X'})+F-p^*E,$$

thus

$$p^*E-F\sim_{\mathbb{R},X} q^*(K_{X'}+B'+\mathbf{M}_{X'})$$

is nef$/X$. Since $h_*p_*(F-p^*E)=h_*p_*F\geq 0$, by the negativity lemma, $F\geq p^*E$. Thus $a(D,X,B,\mathbf{M})\leq a(D,X',B',\mathbf{M})$ for any prime divisor D over X or X'. $\square$

Proof of Lemma 3.6 continued. First we prove the weak glc model case. Let $(X',B',\mathbf{M})/U$ be a weak glc model of $(W,B_W,\mathbf{M})/U$ with induced birational map $\phi_W: W\dashrightarrow X'$. We check Definition 3.2(2) for $(X,B,\mathbf{M})/U$ and $(X',B',\mathbf{M})/U$. Definition 3.2(2.b) holds by construction. For any prime divisor D on X which is exceptional over X', h^{-1}_*D is a prime divisor on W which is exceptional over X'. Thus

$$a(D,X,B,\mathbf{M})=a(D,W,B_W,\mathbf{M})\leq a(D,X',B',\mathbf{M}),$$

and we have Definition 3.2(2.c). We are only left to show that $(X',B',\mathbf{M})/U$ is a log birational model of $(X,B,\mathbf{M})/U$. Let $\phi: X\dashrightarrow X'$ be the induced morphism and $B'' := \phi_*B + \mathrm{Exc}(\phi^{-1})$, then we only need to show that $B' = B''$. By construction, $B' = (\phi_W)_*B_W + \mathrm{Exc}(\phi_W^{-1})$. Let D be a prime divisor on X'. There are three cases:

Case 1. D is not exceptional over X. In this case,

$$\begin{aligned}1 - \mathrm{mult}_D B'' &= a(D, X', B'', \mathbf{M}) = a(D, X, B, \mathbf{M})\\ &= a(D, W, B_W, \mathbf{M}) = a(D, X', B', \mathbf{M}) = 1 - \mathrm{mult}_D B',\end{aligned}$$

so $\mathrm{mult}_D B' = \mathrm{mult}_D B''$.

Case 2. D is exceptional over W. In this case, D is a component of $\mathrm{Exc}(\phi_W^{-1})$ and a component of $\mathrm{Exc}(\phi^{-1})$, hence

$$\mathrm{mult}_D B' = 1 = \mathrm{mult}_D B''.$$

Case 3. D is exceptional over X but not exceptional over W. In this case,

$$1 - \mathrm{mult}_D B' = a(D, X', B', \mathbf{M}) = a(D, W, B_W, \mathbf{M}).$$

Since $E \geq 0$, $a(D, W, B_W, \mathbf{M}) \leq a(D, X, B, \mathbf{M})$. By Claim 3.7,

$$a(D, X, B, \mathbf{M}) \leq a(D, X', B', \mathbf{M}).$$

Thus

$$a(D, X, B, \mathbf{M}) = a(D, X', B', \mathbf{M}) = a(D, W, B_W, \mathbf{M}).$$

By Definition 3.1(4),

$$a(D, X, B, \mathbf{M}) = a(D, X', B', \mathbf{M}) = a(D, W, B_W, \mathbf{M}) = 0,$$

which implies that

$$\mathrm{mult}_D B' = 1 = \mathrm{mult}_D \mathrm{Exc}(\phi^{-1}) = \mathrm{mult}_D B''.$$

Thus $B' = B''$, so $(X', B', \mathbf{M})/U$ is a log birational model of $(X, B, \mathbf{M})/U$, and we have proved the weak glc model case.

Next we prove the log minimal model case. Let $(X', B', \mathbf{M})/U$ be a log minimal model of $(W, B_W, \mathbf{M})/U$. We will check Definition 3.2(3) for $(X, B, \mathbf{M})/U$ and $(X', B', \mathbf{M})/U$. Definition 3.2(3.a) follows from (1). Definition 3.2(3.b) is immediate from the construction. For any prime divisor D

on X which is exceptional over X', $f_*^{-1}D$ is a prime divisor on W which is exceptional over X'. Thus

$$a(D,X,B,\mathbf{M}) = a(D,W,B_W,\mathbf{M}) < a(D,X',B',\mathbf{M}).$$

so we get Definition 3.2(3.c), and we have the log minimal model case.

The good minimal model case follows immediately from the log minimal model case. $\square$

3.3. Models under pullbacks

Lemma 3.8. *Let $(X,B,\mathbf{M})/U$ be a glc g-pair. If $(X,B,\mathbf{M})/U$ has a weak glc model, then $(X,B,\mathbf{M})/U$ has a log minimal model.*

Proof. Let $(X',B',\mathbf{M})/U$ be a weak glc model of $(X,B,\mathbf{M})/U$. Let $h: W \to X$ be a log resolution of $(X, \operatorname{Supp} B)$ such that the induced map $\phi_W : W \to X'$ is a morphism, and $\mathbf{M}$ descends to W. We may write

$$K_W + B_W + \mathbf{M}_W = h^*(K_X + B + \mathbf{M}_X) + E$$

for some log smooth pair (W,B_W), such that $B_W := h_*^{-1}B + \operatorname{Exc}(h)$ and $E \geq 0$ is exceptional over X. Then $(W,B_W,\mathbf{M})$ is a log smooth model of $(X,B,\mathbf{M})$. By Lemma 3.4, we have

$$h^*(K_X + B + \mathbf{M}_X) = \phi_W^*(K_{X'} + B' + \mathbf{M}_{X'}) + G$$

where $G \geq 0$ is exceptional over X'. Thus

$$K_W + B_W + \mathbf{M}_W \sim_{\mathbb{R},X'} G + E.$$

Claim 3.9. *E is exceptional over X'.*

Proof. Let D be a component of E. Then $a(D,X,B,\mathbf{M}) > 0$ and D is exceptional over X.

Assume that D is not exceptional over X'. Since $(X',B',\mathbf{M})/U$ is a log birational model of $(X,B,\mathbf{M})/U$, $a(D,X',B',\mathbf{M}) = 0$. Since $G \geq 0$, $a(D,X,B,\mathbf{M}) \leq a(D,X',B',\mathbf{M})$. Thus $a(D,X,B,\mathbf{M}) = 0$, hence D is not a component of E, a contradiction. $\square$

Proof of Lemma 3.8 continued. By Claim 3.9, $G+E$ is exceptional over X'. By [22, Proposition 3.9], we may run a $(K_W + B_W + \mathbf{M}_W)$-MMP$/X'$ with scaling of a general ample$/X'$ divisor, which terminates with a model Y such that $K_Y + B_Y + \mathbf{M}_Y \sim_{\mathbb{R},X'} 0$, where B_Y is the strict transform of B on Y. By the negativity lemma, $K_Y+B_Y+\mathbf{M}_Y$ is the pullback of $K_{X'}+B'+\mathbf{M}_{X'}$. Thus $K_Y + B_Y + \mathbf{M}_Y$ is nef$/U$. Since $(W, B_W, \mathbf{M})$ is $\mathbb{Q}$-factorial gdlt and $W \dashrightarrow Y$ is a $(K_W + B_W + \mathbf{M}_W)$-MMP$/X'$, $(Y, B_Y, \mathbf{M})$ is $\mathbb{Q}$-factorial gdlt. Thus $(Y, B_Y, \mathbf{M})/U$ is a log minimal model of $(W, B_W, \mathbf{M})/U$. The lemma follows from Lemma 3.6. □

Lemma 3.10. *Let $(X, B, \mathbf{M})/U$ and $(Y, B_Y, \mathbf{M})/U$ be two glc g-pairs, and $f : Y \to X$ a projective birational morphism such that*

$$K_Y + B_Y + \mathbf{M}_Y = f^*(K_X + B + \mathbf{M}_X) + E$$

for some $E \geq 0$ that is exceptional over X. Then

1. *any weak glc model of $(X, B, \mathbf{M})/U$ is a weak glc model of $(Y, B_Y, \mathbf{M})/U$, and*
2. *if $(X, B, \mathbf{M})/U$ has a weak glc model (resp. log minimal model, good minimal model), then $(Y, B_Y, \mathbf{M})/U$ has a weak glc model (resp. log minimal model, good minimal model).*

Proof. (1) Let $(X', B', \mathbf{M})/U$ be a weak glc model of $(X, B, \mathbf{M})/U$, $\phi : X \dashrightarrow X'$ the induced birational map, and $\phi_Y := \phi \circ f$. Let $p : W \to Y$ and $q : W \to X'$ be a common resolution and let $h := f \circ p$. By Lemma 3.4,

$$h^*(K_X + B + \mathbf{M}_X) = q^*(K_{X'} + B' + \mathbf{M}_{X'}) + F$$

for some $F \geq 0$ that is exceptional over X'. Thus

$$p^*(K_Y + B_Y + \mathbf{M}_Y) = q^*(K_{X'} + B' + \mathbf{M}_{X'}) + p^*E + F.$$

Thus $a(D, Y, B_Y, \mathbf{M}) \leq a(D, X', B', \mathbf{M})$ for any prime divisor D over X'. In particular, if $a(D, X', B', \mathbf{M}) = 0$, then $a(D, Y, B_Y, \mathbf{M}) = 0$.

Since $(X', B', \mathbf{M})/U$ is a log birational model of $(X, B, \mathbf{M})/U$, $B' = \phi_*B + \operatorname{Exc}(\phi^{-1})$. Let $B'' := (\phi_Y)_*B_Y + \operatorname{Exc}(\phi_Y^{-1})$. For any prime divisor D on X', there are two cases:

Case 1. D is not exceptional over X. In this case,

$$\begin{aligned} 1 - \operatorname{mult}_D B' &= a(D, X', B', \mathbf{M}) = a(D, X, B, \mathbf{M}) \\ &= a(D, Y, B_Y, \mathbf{M}) = a(D, X', B'', \mathbf{M}) = 1 - \operatorname{mult}_D B'', \end{aligned}$$

so $\operatorname{mult}_D B' = \operatorname{mult}_D B''$.

Case 2. D is exceptional over X. In this case,

$$a(D, X', B', \mathbf{M}) = 1 - \mathrm{mult}_D B' = 0.$$

Since $a(D, Y, B_Y, \mathbf{M}) \leq a(D, X', B', \mathbf{M})$, $a(D, Y, B_Y, \mathbf{M}) = 0$. Thus if D is not exceptional over Y, then

$$\mathrm{mult}_D B'' = \mathrm{mult}_D B_Y = 1 - a(D, Y, B_Y, \mathbf{M}) = 1 = \mathrm{mult}_D B',$$

and if D is exceptional over Y, then

$$\mathrm{mult}_D B'' = \mathrm{mult}_D \mathrm{Exc}(\phi_Y^{-1}) = 1 = \mathrm{mult}_D B'.$$

Thus $B' = B''$, hence $(X', B', \mathbf{M})/U$ is a log birational model of $(Y, B_Y, \mathbf{M})/U$. Since $K_{X'} + B' + \mathbf{M}_{X'}$ is nef$/U$, and $a(D, Y, B_Y, \mathbf{M}) \leq a(D, X', B', \mathbf{M})$ for any prime divisor D over X', $(X', B', \mathbf{M})/U$ is a weak glc model of $(Y, B_Y, \mathbf{M})/U$, and we get (1).

(2) follows from (1) and Lemmas 3.8 and 3.5. □

Lemma 3.11. *Let $(X, B, \mathbf{M})/U$ and $(Y, B_Y, \mathbf{M})/U$ be two NQC glc g-pairs and let $f : Y \to X$ be a projective birational morphism such that*

1. *$K_Y + B_Y + \mathbf{M}_Y = f^*(K_X + B + \mathbf{M}_X)$, and*
2. *for any prime f-exceptional divisor E, $a(E, X, B, \mathbf{M}) = 0$.*

Then $(X, B, \mathbf{M})/U$ has a weak glc model (resp. log minimal model, good minimal model) if and only if $(Y, B_Y, \mathbf{M})/U$ has a weak glc model (resp. log minimal model, good minimal model).

Proof. By Lemma 3.10 we only need to prove the if part. Notice that as $a(D, X, B, \mathbf{M}) = a(D, Y, B_Y, \mathbf{M})$ for any prime divisor D over X and f^{-1} does not contract any divisor, we only need to show that any log birational model $(X', B', \mathbf{M})/U$ of $(Y, B_Y, \mathbf{M})/U$ is also a log birational model of $(X, B, \mathbf{M})/U$. In this case, if $(X', B', \mathbf{M})/U$ is a weak glc model (resp. log minimal model, good minimal model) of $(Y, B_Y, \mathbf{M})/U$ then $(X', B', \mathbf{M})/U$ will also be a weak glc model (resp. log minimal model, good minimal model) of $(X, B, \mathbf{M})/U$.

Let $(X', B', \mathbf{M})/U$ be a log birational model of $(Y, B_Y, \mathbf{M})/U$ with induced birational maps $\phi_Y : Y \dashrightarrow X'$ and $\phi : X \dashrightarrow X'$. Let $B'' := \phi_* B + \mathrm{Exc}(\phi^{-1})$, then for any prime divisor D on X', there are three cases:

Case 1. D is not exceptional over X. In this case,

$$1 - \mathrm{mult}_D B'' = a(D, X', B'', \mathbf{M}) = a(D, X, B, \mathbf{M})$$

$$= a(D, Y, B_Y, \mathbf{M}) = a(D, X', B', \mathbf{M}) = 1 - \operatorname{mult}_D B',$$

so $\operatorname{mult}_D B' = \operatorname{mult}_D B''$.

Case 2. D is exceptional over Y. In this case, D is a component of $\operatorname{Exc}(\phi_Y^{-1})$ and a component of $\operatorname{Exc}(\phi^{-1})$, hence

$$\operatorname{mult}_D B' = 1 = \operatorname{mult}_D B''.$$

Case 3. D is exceptional over X but not exceptional over Y. In this case, $a(D, X, B, \mathbf{M}) = a(D, Y, B_Y, \mathbf{M}) = 0$. Thus

$$\begin{aligned} \operatorname{mult}_D B' &= 1 - a(D, X', B', \mathbf{M}) = 1 - a(D, Y, B_Y, \mathbf{M}) \\ &= 1 = \operatorname{mult}_D \operatorname{Exc}(\phi^{-1}) = \operatorname{mult}_D B''. \end{aligned}$$

Thus $B' = B''$, so $(X', B', \mathbf{M})/U$ is a log birational model of $(X, B, \mathbf{M})/U$, and the lemma follows. □

4. A special good minimal model

In this section we prove the following theorem. When $\mathbf{M} = \mathbf{0}$, it is [25, Theorem 1.2] ([21, Theorem 1.1] for the $\mathbb{Q}$-pair case).

Theorem 4.1. *Let $(X, B, \mathbf{M})/U$ be an NQC glc g-pair and $U^0 \subset U$ a non-empty open subset. Let $X^0 := X \times_U U^0$, $B^0 := B \times_U U^0$, and $\mathbf{M}^0 := \mathbf{M} \times_U U^0$. Assume that*

1. *the morphism $X \to U$ is a projective morphism between normal quasi-projective varieties,*
2. *$(X^0, B^0, \mathbf{M}^0)/U^0$ has a good minimal model,*
3. *all glc centers of $(X, B, \mathbf{M})$ intersect X^0, and*
4. *$\mathbf{M}^0$ descends to X^0 and $\mathbf{M}^0_{X^0} \sim_{\mathbb{R}, U^0} 0$.*

Then $(X, B, \mathbf{M})/U$ has a good minimal model.

We need the following two lemmas:

Lemma 4.2. *Let Let $(X, B, \mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC gdlt g-pair. Assume that there exists a non-empty open subset $U^0 \subset U$, such that*

1. *the image of any strata of $S := \lfloor B \rfloor$ in U intersects U^0, and*
2. *$\mathbf{M}^0 := \mathbf{M} \times_U U^0$ descends to $X^0 := X \times_U U^0$ and $\mathbf{M}^0_{X^0} \sim_{\mathbb{R}, U^0} 0$.*

Then there exists an $\mathbb{R}$*-divisor* $0 \leq G \sim_{\mathbb{R},U} \mathbf{M}_X$ *such that* $(X, B+G)$ *is lc and* $\mathrm{Nklt}(X, B+G) = \mathrm{Ngklt}(X, B, \mathbf{M})$.

Proof. By the theory of Shokurov-type rational polytopes (cf. [22, Proposition 3.20]) and the theory of uniform rational polytopes (cf. [23, Lemma 5.3], [11, Theorem 1.4]), we may assume that $(X, B, \mathbf{M})$ is a $\mathbb{Q}$-g-pair. Possibly shrinking U^0, we may assume that U^0 is affine.

By [33, Proposition 6-1-3, Remark 6-1-4] (see also [38, Lemma 6]) and standard semi-stable reduction arguments (cf. [1], [28, Theorem B.6], [32, Theorem 2], [25, Step 2 of Proof of Lemma 3.2]), we may let $f : X' \to X$ be a resolution with morphisms $\pi' : X' \to U'$ and $\varphi : U' \to U$, such that

- $\mathbf{M}$ descends to X'.
- We may write
$$K_{X'} + B_{X'} + \mathbf{M}_{X'} = f^*(K_X + B + \mathbf{M}_X) + E_{X'},$$
where $B_{X'}, E_{X'} \geq 0$, $B_{X'} \wedge E_{X'} = 0$, $(X', \mathrm{Supp}(B_{X'} + E_{X'}))$ is quasi-smooth.
- $p := \pi \circ f = \varphi \circ \pi' : X' \to U$ where U' is smooth, π' and φ are projective, f is birational, and π' has connected equidimensional fibers.

We show that there is a φ-nef $\mathbb{Q}$-divisor $M_{U'}$ on U' such that $\mathbf{M}_{X'} \sim_{\mathbb{Q},U} \pi'^* M_{U'}$. By our construction, $\mathbf{M}_{X'}|_{X'_\eta} \sim_{\mathbb{Q}} 0$ where X'_η is the generic fiber of p. Thus $\mathbf{M}_{X'} \sim_{\mathbb{Q}} 0$ over the generic point $\eta_{U'}$ of U'. By Lemma 2.3, $\mathbf{M}_{X'} \sim_{\mathbb{Q},U'} D$ where $D \geq 0$ is vertical over U'. Since π' is equidimensional, $\pi'(D)$ is a $\mathbb{Q}$-divisor on U'. Since U' is smooth, for any prime divisor P on U', we may define

$$\nu_P := \sup\{\nu \mid \nu \geq 0, D - \nu \pi'^* P \geq 0\},$$

then $\nu_P > 0$ for only finitely many prime divisors P on U'. Let $D' := D - \pi'^*(\sum_P \nu_P P)$, then $\mathbf{M}_{X'} \sim_{\mathbb{Q},U'} D' \geq 0$ and D' is very exceptional over U. By the general negativity lemma [5, Lemma 3.3], $\mathbf{M}_{X'} \sim_{\mathbb{Q},U'} 0$. In particular, since $\mathbf{M}_{X'}$ is nef$/U$, $\mathbf{M}_{X'} \sim_{\mathbb{Q},U} \pi'^* M_{U'}$ for some $\mathbb{Q}$-divisor $M_{U'}$ that is nef$/U$.

Let $X'^0 := X' \times_U U^0$ and $U'^0 := U' \times_U U^0$. Since $\mathbf{M}_{X'}|_{X'^0} \sim_{\mathbb{Q},U^0} 0$, we have that $M_{U'^0} := M_{U'}|_{U'^0} \sim_{\mathbb{Q},U^0} 0$.

To prove the claim it suffices to show that for a general element $G' \in |\mathbf{M}_{X'}/U|_{\mathbb{Q}}$, the pair $(X', B_{X'} + G')$ is lc and its lc centers coincide with the lc centers of $(X', B_{X'})$, i.e. the strata of $\lfloor B_{X'} \rfloor$. If this is the case, then $(X', B_{X'} - E_{X'} + G')$ is sub-lc and $K_{X'} + B_{X'} - E_{X'} + G' \sim_{\mathbb{Q}} f^*(K_X + B + G)$

where $G = f_*G' \in |\mathbf{M}_X/U|_{\mathbb{Q}}$ and $(X, B+G)$ is log canonical and its log canonical places coincide with the glc places of $(X, B, \mathbf{M})$.

Let $E \geq 0$ be an effective divisor on U' such that $-E$ is ample over U (note that E is not necessarily exceptional, but its support can be chosen to avoid any point not in the exceptional locus). It follows that $|M_{U'}/U|_{\mathbb{Q}} \supset |M_{U'} - \epsilon E/U|_{\mathbb{Q}} + \epsilon E$. Since $M_{U'} - \epsilon E$ is ample over U, for a general element $G' \in |\mathbf{M}_{X'}/U|_{\mathbb{Q}}$ we have that the set of nklt places of $(X', B_{X'} + G')$ are contained in the set of nklt places of $(X', B_{X'} + \epsilon\pi'^*E)$. Thus, the only non-klt centers of $(X', B_{X'} + G')$ are strata of $\lfloor B_{X'} \rfloor$.

To prove the claim, it suffices to show that the support of a general element $G' \in |\mathbf{M}_{X'}/U|_{\mathbb{Q}}$ does not contain any stratum S' of $\lfloor B_{X'} \rfloor$ or equivalently that there exist one element $G' \in |\mathbf{M}_{X'}/U|_{\mathbb{Q}}$ whose support does not contain any given stratum S' of $\lfloor B_{X'} \rfloor$. Note that $f(S')$ is a glc center of $(X, B, \mathbf{M})$. As $(X, B, \mathbf{M})$ is gdlt, its glc centers are the strata of $\lfloor B \rfloor$ which intersect X^0 by assumption. Pick a point $x \in f(S') \cap X^0$ and let $u = \pi(x) \in U^0$ and $u' \in U'^0$ such that $\varphi(u') = u$. Since $M_{U'^0} \sim_{\mathbb{Q},U^0} 0$, we have that $mM_{U'^0} \sim_{U^0} 0$ for some integer $m > 0$. It follows that $\mathcal{O}_{U'}(mM_{U'})$ is generated over U^0 i.e. $\varphi^*\varphi_*\mathcal{O}_{U'}(mM_{U'})|_{U'^0} \to \mathcal{O}_{U'}(mM_{U'})|_{U'^0}$ is surjective. Since $\varphi_*\mathcal{O}_{U'}(mM_{U'}) \otimes \mathcal{O}_U(H)$ is globally generated for any sufficiently ample line bundle H on U, then $\mathcal{O}_{U'}(mM_{U'} + \varphi^*H)$ is globally generated at any point of U'^0. In particular we can pick a divisor $\Gamma \in |mM_{U'} + \varphi^*H|$ whose support does not contain u'. If $G' = \pi'^*\Gamma/m \sim_{\mathbb{Q},U} \mathbf{M}_{X'}$, then the support of G' does not contain S', and this concludes the proof. $\square$

Lemma 4.3. *Let $(X, B, \mathbf{M})/U$ and $(X, B', \mathbf{M}')/U$ be two NQC glc g-pairs, $f : Y \to X$ a birational morphism, $K_Y + B_Y + \mathbf{M}_Y := f^*(K_X + B + \mathbf{M}_X)$ and $K_Y + B'_Y + \mathbf{M}'_Y := f^*(K_X + B' + \mathbf{M}'_X)$, such that Y is $\mathbb{Q}$-factorial klt, $(Y, B_Y, \mathbf{M})/U$ and $(Y, B'_Y, \mathbf{M}')/U$ are glc g-pairs, and $a(E, X, B, \mathbf{M}) = a(E, X, B', \mathbf{M}') = 0$ for any prime f-exceptional divisor E.*

Assume that there exists a positive real number r such that $K_X + B + \mathbf{M}_X \sim_{\mathbb{R},U} r(K_X + B' + \mathbf{M}'_X)$. Then $(X, B, \mathbf{M})/U$ has a good minimal model if and only if $(X, B', \mathbf{M}')$ has a good minimal model.

Proof. Let A_Y be a general ample$/U$ divisor on Y such that $(Y, B_Y + A_Y, \mathbf{M})/U$ and $(Y, B'_Y + A_Y, \mathbf{M}')/U$ are glc, and $K_Y + B_Y + A_Y + \mathbf{M}_Y$ and $K_Y + B'_Y + rA_Y + \mathbf{M}'_Y$ are nef$/U$.

Without loss of generality, we may assume that $(X, B, \mathbf{M})/U$ has a good minimal model and only need to show that $(X, B', \mathbf{M}')/U$ has a good minimal model. By Lemma 3.11, $(Y, B_Y, \mathbf{M})/U$ has a good minimal model. By Theorem 2.8 and Lemma 3.5(2), we may let $\phi : Y \dashrightarrow Z$ be a $(K_Y + B_Y + \mathbf{M}_Y)$-MMP$/U$ with scaling of A_Y, such that $(Z, B_Z, \mathbf{M})/U$ is a weak glc model of $(Y, B_Y, \mathbf{M})/U$ and $K_Z + B_Z + \mathbf{M}_Z$ is semi-ample$/U$, where B_Z is the strict transform of B_Y on Z. Then ϕ is also a $(K_Y + B'_Y + \mathbf{M}'_Y)$-MMP$/U$ with scaling of rA_Y. We let B'_Z be the strict transform of B'_Y on Z, then $K_Z + B_Z + \mathbf{M}_Z \sim_{\mathbb{R},U} r(K_Z + B'_Z + \mathbf{M}'_Z)$. Thus $(Z, B'_Z, \mathbf{M}')/U$ is a weak glc model of $(Y, B'_Y, \mathbf{M}')/U$ and $K_Z + B'_Z + \mathbf{M}'_Z$ is semi-ample$/U$. By Lemmas 3.8 and 3.5(2), $(Y, B'_Y, \mathbf{M}')$ has a good minimal model. By Lemma 3.11, $(X, B', \mathbf{M}')/U$ has a good minimal model. □

Proof of Theorem 4.1. By Definition-Lemma 3.3 and Theorem 3.11, possibly replacing $(X, B, \mathbf{M})$ with a gdlt modification, we may assume that $(X, B, \mathbf{M})$ is $\mathbb{Q}$-factorial gdlt. By Lemma 4.2, we may find an $\mathbb{R}$-divisor $0 \leq G \sim_{\mathbb{R}} \mathbf{M}_X$ such that $(X, B+G)$ is lc and $\mathrm{Nklt}(X, B+G) = \mathrm{Ngklt}(X, B, \mathbf{M})$. By [25, Theorem 1.2] (see also [21, Theorem 1.1]), $(X, B+G)/U$ has a good minimal model. By Lemma 4.3, $(X, B, \mathbf{M})/U$ has a good minimal model. □

5. Base-point-free, contraction, and cone theorems for generalized pairs

In this section, we prove Theorem 1.1. For the reader's convenience, we will prove Theorem 1.1(1-4) (the cone theorem) and Theorem 1.1(5) (the contraction theorem) separately, and we will also prove a base-point-free theorem, stated as follows:

Theorem 5.1 (Base-point-free theorem for glc g-pairs)**.** *Let $(X, B, \mathbf{M})/U$ be an NQC glc g-pair and $\pi : X \to U$ the associated projective morphism. Assume that $\mathbf{M}_X$ is $\mathbb{R}$-Cartier. Let L be a π-nef Cartier divisor on X such that $L-(K_X+B+\mathbf{M}_X)$ is π-ample. Then mL is π-generated for any integer $m \gg 0$.*

5.1. Preliminary results on non-lc pairs

Before we give the proof, let us first recall some results on non-lc pairs.

Definition 5.2. Let (X, Δ) be a sub-pair. A *non-lc place* of (X, Δ) is a prime divisor D over X such that $a(D, X, \Delta) < 0$. A *non-lc center* of (X, Δ) is the center of a non-lc place of (X, Δ) on X. The *non-lc locus* $\mathrm{Nlc}(X, \Delta)$ of (X, Δ) is the union of all non-lc centers of (X, Δ).

Definition 5.3 (cf. [3, Definition 5.2], [17, Theorem 4.5.2(1), Definition 6.7.1])**.** Let (X, Δ) be a (not necessarily lc) pair. We define

$$\overline{NE}(X/U)_{\mathrm{Nlc}(X,\Delta)} := \mathrm{Im}(\overline{NE}(\mathrm{Nlc}(X, \Delta)/U) \to \overline{NE}(X/U)).$$

Definition 5.4 (cf. [3, Definition 5.3], [17, Definition 6.7.2])**.** Let (X, Δ) be a (not necessarily lc) pair, $\pi : X \to U$ a projective morphism, and F an extremal face of $\overline{NE}(X/U)$.

1. A *supporting function* of F is a π-nef $\mathbb{R}$-divisor H such that $F = \overline{NE}(X/U) \cap H^{\perp}$. If H is a $\mathbb{Q}$-divisor, we say that H is a *rational supporting function.* Since F is an extremal face of $\overline{NE}(X/U)$, F always has a supporting function.
2. For any $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X, we say that F is D*-negative* if
$$F \cap \overline{NE}(X/U)_{D \geq 0} = \{0\}.$$
3. We say that F is *rational* if F has a rational supporting function.
4. We say that F is *relatively ample at infinity with respect to* (X, Δ) if
$$F \cap \overline{NE}(X/U)_{\mathrm{Nlc}(X,\Delta)} = \{0\}.$$
Equivalently, $H|_{\mathrm{Nlc}(X,\Delta)}$ is $\pi|_{\mathrm{Nlc}(X,\Delta)}$-ample for any supporting function H of F.

5.2. Proof of the cone theorem

In this subsection, we prove the cone theorem (Theorem 1.1(1-4)). We first prove a useful lemma which allows us to associate a (not necessarily lc) pair to a glc g-pair.

Lemma 5.5. *Let $(X, B, \mathbf{M})/U$ be a glc g-pair and A a nef and big/U $\mathbb{R}$-divisor on X. Then there exists a pair (X, Δ), such that*

1. $\Delta \sim_{\mathbb{R},U} B + \mathbf{M}_X + A$, *and*
2. $\mathrm{Nlc}(X, \Delta) = \mathrm{Ngklt}(X, B, \mathbf{M})$.

Proof. Let $h : W \to X$ be a log resolution of $(X, \operatorname{Supp} B)$ such that $\mathbf{M}$ descends to W, and suppose that
$$K_W + B_W + \mathbf{M}_W = h^*(K_X + B + \mathbf{M}_X)$$
for some sub-glc g-sub-pair $(W, B_W, \mathbf{M})/U$. Since $\mathbf{M}_W$ is nef/U, $\mathbf{M}_W + h^*A$ is nef and big/U. Thus there exists an $\mathbb{R}$-divisor $E \geq 0$ such that
$$\mathbf{M}_W + h^*A = H_n + \frac{1}{n}E$$

for any positive integer n and some ample$/U$ $\mathbb{R}$-divisors H_n on W. Since $h: W \to X$ is a log resolution of $(X, \operatorname{Supp} B)$, we may pick $n \gg 0$ such that $\operatorname{Nlc}(W, B_W + \frac{1}{n}E) \subset \operatorname{Supp} B_W^{=1}$. In particular, for any positive real number ϵ, $\operatorname{Nlc}(W, B_W + \epsilon B_W^{=1} + \frac{1}{n}E) = \operatorname{Supp} B_W^{=1}$.

Now we may pick a real number $0 < \epsilon_0 \ll 1$ such that $H_n - \epsilon_0 B_W^{=1}$ is ample$/U$. Then we may pick $0 \leq A_W \sim_{\mathbb{R},U} H_n - \epsilon_0 B_W^{=1}$ such that $(W, \Delta_W := B_W + \epsilon_0 B_W^{=1} + \frac{1}{n}E + A_W)$ is a sub-pair and $\operatorname{Nlc}(W, \Delta_W) = \operatorname{Supp} B_W^{=1}$. The pair $(X, \Delta := h_*\Delta_W)$ satisfies our requirements. $\square$

Lemma 5.6. *Let $d \geq 2$ be an integer. Assume Theorem 1.1(1-4) in dimension $\leq d-1$.*

Let $(X, B, \mathbf{M})/U$ be an NQC glc g-pair of dimension d and $\pi: X \to U$ the associated projective morphism. Let A be an ample$/U$ $\mathbb{R}$-divisor on X and $\{R_j\}_{j \in \Lambda'_A}$ the set of $(K_X + B + \mathbf{M}_X + A)$-negative extremal rays (that are not necessarily rational) in $\overline{NE}(X/U)$. Then:

1. *Λ'_A is a finite set. In particular,*

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A \geq 0} + \sum_{j \in \Lambda'_A} R_j.$$

2. *For any $j \in \Lambda'_A$, R_j is spanned by a rational curve C_j such that $\pi(C_j) = \{pt\}$ and*

$$0 < -(K_X + B + \mathbf{M}_X + A) \cdot C_j \leq 2 \dim X.$$

Proof. By Lemma 5.5, we may pick $0 \leq \Delta \sim_{\mathbb{R},U} B + \mathbf{M}_X + A$ such that $\operatorname{Nlc}(X, \Delta) = \operatorname{Ngklt}(X, B, \mathbf{M})$.

For any glc center $\tilde{W}$ of $(X, B, \mathbf{M})$ with normalization $W \to \tilde{W}$, we let $(W, B_W, \mathbf{M}^W)/U$ be the NQC glc g-pair given by the sub-adjunction

$$K_W + B_W + \mathbf{M}^W_W \sim_{\mathbb{R}} (K_X + B + \mathbf{M}_X)|_W$$

as in [24, Theorem 5.1], and let $A_W := A|_W$. By Theorem 1.1(1-4) in dimension $\leq d-1$, we have

$$\overline{NE}(W/U) = \overline{NE}(W/U)_{K_W+B_W+\mathbf{M}^W_W+A_W \geq 0} + \sum_{j \in \Lambda_{A_W}} R_{j,W},$$

where $\{R_{j,W}\}_{j \in \Lambda_{A_W}}$ is the set of $(K_W + B_W + \mathbf{M}^W_W + A_W)$-negative extremal rays in $\overline{NE}(W/U)$ that are rational, where Λ_{A_W} is a finite set. For any

$j \in \Lambda_{A_W}$, we let R_j be the image of $R_{j,W}$ in X under the map

$$\cup_W \overline{NE}(W/U) \to \overline{NE}(\mathrm{Nlc}(X,\Delta)/U) \to \overline{NE}(X/U)$$

and let $\Lambda_A^0 := \cup_W \Lambda_{A_W}$. Then Λ_A^0 is a finite set. Finally, we let $\{R_j\}_{j\in\Lambda_A^1}$ be the set of $(K_X+B+\mathbf{M}_X+A)$-negative extremal rays in $\overline{NE}(X/U)$ that are relatively ample at infinity with respect to (X,Δ). By [3, Theorem 5.10(ii)], [17, Theorems 4.5.2(3), 6.7.4(2)], Λ_A^1 is a finite set.

Claim 5.7.

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0} + \sum_{j\in\Lambda_A^0} R_j + \sum_{j\in\Lambda_A^1} R_j.$$

Proof. For simplicity, we let

$$V := \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0} + \sum_{j\in\Lambda_A^0} R_j + \sum_{j\in\Lambda_A^1} R_j.$$

For any curve C on X, we will write $[C]$ for its class in $\overline{NE}(X/U)$, and for any glc center $\tilde{W}$ of $(X,B,\mathbf{M})$ with normalization W, if $C\subset W$, then we will write $[C]_W$ for its class in $\overline{NE}(W/U)$.

Suppose that $\overline{NE}(X/U)\neq V$. By [3, Theorem 5.10], [17, Theorems 4.5.2, 6.7.4], we have

$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0} + \overline{NE}(X/U)_{\mathrm{Nlc}(X,\Delta)} + \sum_{j\in\Lambda_A^1} R_j.$$

Thus there exists an integral curve $C\subset \mathrm{Nlc}(X,\Delta)=\mathrm{Ngklt}(X,B,\mathbf{M})$, such that $[C]$ is not contained in V. We may write

$$C = \sum_{W\mid W \text{ is a glc center of } (X,B,\mathbf{M})} C_W,$$

where each C_W is an integral curve in W. For any C_W, we have

$$[C_W]_W = c_W^0 R_W^0 + \sum_{j\in\Lambda_{A_W}} c_{j,W}R_{j,W}$$

where c_W^0 and each $c_{j,W}$ are non-negative real numbers, and

$$R_W^0 \in \overline{NE}(W/U)_{K_W+B_W+\mathbf{M}_W^W+A_W\geq 0}.$$

Since the image of R_W^0 in X is contained in $\overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0}$, $[C_W]$ is contained in $\overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0}+\sum_{j\in\Lambda_A^0}R_j$. Thus $[C_W]$ is contained in V, hence $[C]$ is contained in V, a contradiction. □

Proof of Lemma 5.6 continued. By Claim 5.7, any $(K_X+B+\mathbf{M}_X+A)$-negative extremal ray in $\overline{NE}(X/U)$ must be contained in $\{R_j\}_{j\in\Lambda_A^0\cup\Lambda_A^1}$, so $\Lambda_A'\subset\Lambda_A^0\cup\Lambda_A^1$. Since $\Lambda_A^0\cup\Lambda_A^1$ is a finite set, Λ_A' is a finite set, and we get (1).

By Theorem 1.1(1-4) in dimension $\leq d-1$, for any $j\in\Lambda_{A_W}$, $R_{j,W}$ is spanned by a rational curve C_j such that the image of C_j in U is a point, and

$$0<-(K_W+B_W+\mathbf{M}_W^W+A_W)\cdot C_j\leq 2\dim W<2\dim X.$$

Therefore, for any $j\in\Lambda_A^0=\cup_W\Lambda_{A_W}$, R_j is spanned by the curve C_j such that $\pi(C_j)=\{pt\}$ and

$$0<-(K_X+B+\mathbf{M}_X+A)\cdot C_j\leq 2\dim X.$$

By [17, Theorems 4.5.2(5)], for any $j\in\Lambda_A^1$, R_j is spanned by a rational curve C_j such that $\pi(C_j)=\{pt\}$ and

$$0<-(K_X+B+\mathbf{M}_X+A)\cdot C_j\leq 2\dim X.$$

Thus (2) holds and the proof is complete. □

Lemma 5.8. *Let $d\geq 2$ be an integer. Assume Theorem 1.1(1-4) in dimension $\leq d-1$.*

Let $(X,B,\mathbf{M})/U$ be an NQC glc g-pair of dimension d and $\pi:X\to U$ the associated projective morphism. Let A be an ample/U $\mathbb{R}$-divisor on X and $\{R_j\}_{j\in\Lambda_A}$ the set of $(K_X+B+\mathbf{M}_X+A)$-negative extremal rays in $\overline{NE}(X/U)$ that are rational. Then Λ_A is a finite set, and

$$\overline{NE}(X/U)=\overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0}+\sum_{j\in\Lambda_A}R_j.$$

Proof. Let Λ_A' be the set of $(K_X+B+\mathbf{M}_X+A)$-negative extremal rays in $\overline{NE}(X/U)$ (that are not necessarily rational). By Lemma 5.6, Λ_A' is a finite set. Possibly perturbing the coefficients, by the theory of Shokurov-type rational polytopes (cf. [22, Proposition 3.20]), and the theory of uniform rational polytopes (cf. [23, Lemma 5.3], [11, Therem 1.4]), we may assume that $(X, B, \mathbf{M})/U$ is a $\mathbb{Q}$-g-pair and A is a $\mathbb{Q}$-divisor. Moreover, $\Lambda_A \subset \Lambda'_A$ is a finite set.

For simplicity, we let $V := \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0} + \sum_{j\in\Lambda_A} R_j$. Suppose that $V \neq \overline{NE}(X/U)$. Since $\dim_{\mathbb{R}} N^1(X/U) \geq 2$, there exists a Cartier divisor N on X satisfying the following:

- N is not numerically equivalent to a multiple of $K_X + B + \mathbf{M}_X + A$ over U,
- N is positive on $V\backslash\{0\}$, and
- $N \cdot z_0 < 0$ for some $z_0 \in \overline{NE}(X/U)$.

Let Q be the dual cone of $\overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0}$, i.e.,

$$Q = \{D \in N^1(X/U) \mid D \cdot z \geq 0 \text{ for any } z \in \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0}\},$$

then Q is generated by π-nef divisors and $K_X + B + \mathbf{M}_X + A$. Since N is positive on $\overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0}\backslash\{0\}$, N is in the interior of Q. By Kleiman's Criterion, there exists an ample$/U$ $\mathbb{Q}$-divisor H on X and a positive real number p, such that

$$N = H + p(K_X + B + \mathbf{M}_X + A).$$

Since $N \cdot z_0 < 0$ and H is ample$/U$, we may let

$$t := \sup\{s \mid H + s(K_X + B + \mathbf{M}_X + A) \text{ is nef}/U\}.$$

Then $0 < t < p$. Since $(H + t(K_X + B + \mathbf{M}_X + A)) \cdot z \geq 0$ for any $z \in \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0}$, by Lemma 5.6,

$$t = \max\{s \mid (H + s(K_X + B + \mathbf{M}_X + A)) \cdot R_j \geq 0, \forall j \in \Lambda'_A\}$$

where $\{R_j\}_{j\in\Lambda'_A}$ the set of $(K_X + B + \mathbf{M}_X + A)$-negative extremal rays in $\overline{NE}(X/U)$ and is a finite set. Thus t is a rational number. Since N is not a multiple of $K_X + B + \mathbf{M}_X + A$, $H + t(K_X + B + \mathbf{M}_X + A)$ is a rational supporting function of a $(K_X + B + \mathbf{M}_X + A)$-negative extremal face F_N, which is spanned by $(K_X + B + \mathbf{M}_X + A)$-negative extremal rays. By Lemma 5.6, F_N is spanned by finitely many $(K_X + B + \mathbf{M}_X + A)$-negative extremal rays $R^1, \dots, R^n$ in $\overline{NE}(X/U)$ for some positive integer n. In particular, we may pick a Cartier divisor L on X such that $L \cdot R^1 > 0$ and $L \cdot R^i < 0$ for any $i \geq 2$. Since H is ample$/U$ and N is not numerically equivalent to a multiple of $K_X + B + \mathbf{M}_X + A$ over U, we may pick a rational number $\epsilon \in (0, 1)$ such that

- $N_\epsilon := (H-\epsilon L)+p(K_X+B+\mathbf{M}_X+A)$ is not numerically equivalent to a multiple of $K_X+B+\mathbf{M}_X+A$ over U for any $\epsilon\in(0,\epsilon_0)$,
- $H-\epsilon_0 L$ is ample$/U$, and
- $N_{\epsilon_0}\cdot z_0<0$.

Thus N_ϵ is positive on $\overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0}$. Since Λ_A is a finite set and $N\cdot R_j>0$ for any $j\in\Lambda_A$, we may pick a rational number $\epsilon_1\in(0,\epsilon_0)$ such that $N_{\epsilon_1}\cdot R_j>0$ for any $j\in\Lambda_A$. In particular, N_{ϵ_1} is positive on $V\backslash\{0\}$. Now we let

$$t_1:=\sup\{s\mid H-\epsilon_1 L+s(K_X+B+\mathbf{M}_X+A)\text{ is nef}/U\}.$$

By our construction,

$$t_1=\frac{(H-\epsilon_1 L)\cdot R^1}{-(K_X+B+\mathbf{M}_X+A)\cdot R^1}$$

is a rational number, $0<t_1<t<p$, and $H-\epsilon_1 L+t_1(K_X+B+\mathbf{M}_X+A)$ is a rational supporting function of R^1. Thus $R^1\in\Lambda_A$, and so $N_{\epsilon_1}\cdot R^1>0$. Therefore, $p<t_1$, a contradiction. □

Proof of Theorem 1.1(1-4). We apply induction on dimension of X. The $\dim X=1$ case is obviously true. So we may assume that $\dim X=d$ where $d\geq 2$ is an integer and Theorem 1.1(1-4) holds in dimension $\leq d-1$.

For any $(K_X+B+\mathbf{M}_X)$-negative extremal ray R in $\overline{NE}(X/U)$, R is also a $(K_X+B+\mathbf{M}_X+A)$-negative extremal ray for some ample$/U$ $\mathbb{R}$-divisor A on X. By Lemma 5.8, R is rational. By Lemma 5.6(2), R is generated by a rational curve C such that $\pi(C)=\{pt\}$ and

$$0<-(K_X+B+\mathbf{M}_X+A)\cdot C\leq 2\dim X.$$

Since R is also a $(K_X+B+\mathbf{M}_X+\epsilon A)$-negative extremal ray for any $\epsilon\in(0,1)$, by Lemma 5.6(2) again, we have

$$0<-(K_X+B+\mathbf{M}_X+\epsilon A)\cdot C\leq 2\dim X$$

for any $\epsilon\in(0,1)$. Thus

$$0<-(K_X+B+\mathbf{M}_X)\cdot C\leq 2\dim X,$$

and we get (2). (3) follows from Lemma 5.8 and the fact that

$$\{R_j\}_{j\in\Lambda}\subset\cup_{n=1}^{+\infty}\{R_j\}_{j\in\Lambda_{\frac{1}{n}A}}$$

for any ample$/U$ $\mathbb{R}$-divisor A on X. (1) follows from (3).

We now prove (4). For any $(K_X+B+\mathbf{M}_X)$-negative extremal face F in $\overline{NE}(X/U)$, F is also a $(K_X+B+\mathbf{M}_X+A)$-negative extremal face for some ample$/U$ $\mathbb{R}$-divisor A on X. Let $V := F^{\perp} \subset N^1(X/U)$. Then since F is spanned by a subset of $\{R_j\}_{j\in\Lambda_A}$, V is defined over $\mathbb{Q}$. We let

$$W_F := \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0} + \sum_{j|j\in\Lambda_A, R_j\not\subset F} R_j.$$

Then W_F is a closed cone, $\overline{NE}(X/U) = W_F + F$, and $W_F \cap F = \{0\}$. The supporting functions of F are the elements in V that are positive on $W_F\backslash\{0\}$, which is a non-empty open subset of V, and hence contains a rational element H. In particular, $F = H^{\perp}\cap\overline{NE}(X/U)$, hence F is rational, and we get (4). $\square$

5.3. Proof of the base-point-free theorem and the contraction theorem

Now we prove the base-point-free theorem (Theorem 5.1) for glc g-pairs. First we prove an auxiliary lemma.

Lemma 5.9. *Let $(X,B,\mathbf{M})/U$ be a glc g-pair such that $\mathbf{M}_X$ is $\mathbb{R}$-Cartier and* $\mathrm{Ngklt}(X,B,\mathbf{M}) = \mathrm{Nklt}(X,B)$. *Then there exists a birational morphism $h: W \to X$ such that $\mathbf{M}$ descends to W and* $\mathrm{Supp}(h^*\mathbf{M}_X - \mathbf{M}_W) = \mathrm{Exc}(h)$.

Proof. Let $f: Y \to X$ be a log resolution of (X,B) such that $\mathbf{M}$ descends to Y. Let $F = \mathrm{Exc}(f)$ be the reduced exceptional divisor. Write $K_Y + f^{-1}_*B + G = f^*(K_X+B)$ and $\mathbf{M}_Y + E = f^*\mathbf{M}_X$. Write $\mathrm{Supp}\,E = \cup_i E_i$. Note that $E = f^{-1}(f(E))$. If this is not the case, then since the fibers of f are connected, there is a curve C contained in a fiber $f^{-1}(x)$ such that C intersects the support of E but is not contained in the support of E. But then $-E\cdot C < 0$ contradicting the fact that $-E$ is nef over X. Let $Y^0 = Y \setminus \mathrm{Supp}\,E$ and let $X^0 = X \setminus f(\mathrm{Supp}\,E)$, then $Y^0 = f^{-1}(X^0)$.

Since $\mathrm{Ngklt}(X,B,\mathbf{M}) = \mathrm{Nklt}(X,B)$, the support of E does not contain any strata of $G^{=1}$. In particular $E \wedge G^{=1} = 0$, and no element in $\mathrm{Ngklt}(X,B,\mathbf{M})$ is contained in $X\setminus X^0$.

We now consider the generalized pair

$$(Y, f^{-1}_*B + eG^{=1} + (1-e)F + \sum_i s_iE_i, t\overline{\mathbf{M}}_Y)/X$$

where $0 < s_i \ll e \ll 1$, $t \gg 1$, and the real numbers s_i are sufficiently general (i.e. their representatives in $\mathbb{R}/\mathbb{Q}$ are sufficiently general). We have

$$K_Y + f_*^{-1}B + eG^{=1} + (1-e)F + \sum_i s_iE_i + t\mathbf{M}_Y$$
$$\sim_{\mathbb{R},X} eG^{=1} + (1-e)F - G - tE + \sum_i s_iE_i \sim_{\mathbb{R},X} F' - E'$$

where the coefficients of E' are sufficiently general real numbers, $\operatorname{Supp} E' = \operatorname{Supp} E$, and $\operatorname{Supp} F'$ consists of the set of exceptional divisors not contained in the support of $E \vee G^{=1}$.

We will now apply Theorem 4.1 to this generalized pair. To check the hypothesis, we consider the open subset Y^0 and X^0 defined above. (1) clearly holds, (3) has been checked above, and (4) holds since $\mathbf{M}_Y|_{Y^0} = (f|_{Y^0})^*\mathbf{M}_X|_{X^0}$ as $E|_{Y^0} = 0$. For (2), we must check that

$$(Y^0, (f_*^{-1}B + eG^{=1} + (1-e)F + \sum_i s_iE_i)|_{Y^0}, t\overline{\mathbf{M}}_Y|_{Y^0})$$
$$= (Y^0, (f_*^{-1}B + eG^{=1} + (1-e)F)|_{Y^0}, 0)$$

has a good minimal model over X^0. Since $K_{Y^0} + (f_*^{-1}B + eG^{=1} + (1-e)F)|_{Y^0} \sim_{\mathbb{R},X^0} F'|_{Y^0}$ where $F'|_{Y^0}$ is effective and exceptional over X^0, by [22, Proposition 3.9], $(Y^0, (f_*^{-1}B + eG^{=1} + (1-e)F)|_{Y^0})/X^0$ has a good minimal model and (2) holds. Therefore, by Theorems 4.1 and 2.8, we can run a $(K_Y + f_*^{-1}B + eG^{=1} + (1-e)F + \sum_i s_iE_i + t\mathbf{M}_Y)$-MMP$/X$, say $Y \dashrightarrow Z$ which contracts F' and obtain a good minimal model$/X$.

By [10, Lemma 4.4(3)], $\overline{\mathbf{M}_Y}$ descends to Z, hence $\mathbf{M}$ descends to Z. Let E_Z, E'_Z be the strict transforms of E, E' on the minimal model Z respectively. Then $-E'_Z$ is semi-ample$/X$ and we can then take the corresponding ample model $g : Z \to W$ of $-E'_Z/X$. Since $-E'_W$ is ample over X, the only $h : W \to X$ exceptional divisors are the components of $-E'_W$.

Since the coefficients of E'_Z are sufficiently general, no component of $\operatorname{Supp} E'_Z = \operatorname{Supp} E_Z$ is contracted by $h : Z \to W$. To see this, note that if $E'_Z \cdot C = 0$ for any curve C over X, then the same is true for every component of E'_Z. Since $E'_Z \equiv_W 0$, it follows that $P \equiv_W 0$ for any component P of the support of E'_Z. By the negativity lemma, P is not exceptional. Note that $g : Z \to W$ is also the ample model of any small perturbation of $-E'_Z$ and so g_*P is $\mathbb{Q}$-Cartier and $P = g^*g_*P$. But then $\mathbf{M}_Z \sim_{\mathbb{R},X} -E_Z = -g^*(E_W)$ where $E_W = g_*E_Z$. Thus $\mathbf{M}_Z = g^*g_*\mathbf{M}_Z = g^*\mathbf{M}_W$, so $\mathbf{M}$ descends to W.

Therefore, W satisfies our requirements. $\square$

Proof of Theorem 5.1. Since $L-(K_X+B+\mathbf{M}_X)$ is ample, $L-(K_X+B+(1-\epsilon)\mathbf{M}_X)$ is π-ample for any $0<\epsilon\ll 1$. Possibly replacing $\mathbf{M}$ with $(1-\epsilon)\mathbf{M}$ for some $0<\epsilon\ll 1$, we may assume that $\mathrm{Ngklt}(X,B,\mathbf{M})=\mathrm{Nklt}(X,B)$. Let $A:=\frac{1}{2}(L-(K_X+B+\mathbf{M}_X))$, then A is π-ample$/U$.

Let $f:Y\to X$ be a birational morphism such that $\mathbf{M}$ descends to Y. By the negativity lemma, we may assume that $\mathbf{M}_Y=f^*\mathbf{M}_X-E$ for some $E\geq 0$ that is exceptional over X. By Lemma 5.9, we may then assume that $\mathrm{Exc}(f)=\operatorname{Supp}E$.

Let $K_Y+B_Y:=f^*(K_X+B)$. By our construction, $\mathrm{Exc}(f)=\operatorname{Supp}E$ does not contain any lc place of (X,B). Thus we may pick $E'\geq 0$ on Y such that $-E'$ is ample$/X$ and E' does not contain any lc place of (X,B). Since $\mathrm{Ngklt}(X,B,\mathbf{M})=\mathrm{Nklt}(X,B)$, we may find $0<\epsilon\ll 1$ such that $f^*A-\epsilon E'$ is ample$/U$ and $(Y,B_Y+\epsilon E')$ is sub-lc. In particular, we may find an ample$/U$ $\mathbb{R}$-divisor $0\leq H_Y\sim_{\mathbb{R},U}\mathbf{M}_Y+f^*A-\epsilon E'$ on Y such that $(Y,B_Y+H_Y+\epsilon E')$ is sub-lc. Let $\Delta:=B+f_*H_Y$, then (X,Δ) is lc and $\Delta\sim_{\mathbb{R},U}B+\mathbf{M}_X+A$.

In particular, $L-(K_X+\Delta)\sim_{\mathbb{R},U}A$ is ample$/U$. The theorem follows from [3, Theorem 5.3], [17, Theorems 4.5.5, 6.5.1]. □

The contraction theorem (Theorem 1.1(5)) immediately follows from the base-point-free theorem:

Proof of Theorem 1.1(5). By Theorem 1.1(1-4), F has a supporting function H that is a π-nef Cartier divisor. In particular, we may assume that $H-(K_X+B+\mathbf{M}_X)$ is π-ample. By Theorem 5.1, H is semi-ample$/U$, hence defines a contraction $\mathrm{cont}_F:X\to Y$ over U. (a) and (b) immediately follow.

Since $-(K_X+B+\mathbf{M}_X)$ is ample$/Y$, for any line bundle L on X such that $L\cdot C=0$ for any C such that $[C]\in F$, $L-(K_X+B+\mathbf{M}_X)$ is ample$/Y$. By Theorem 5.1, mL is cont_F-generated and $mL\equiv_Y 0$ for any $m\gg 0$. Therefore, cont_F is defined by $|mL|$ and $|(m+1)L|$ over Y for any $m\gg 0$, which implies that $mL\cong f^*L_{Y,m}$ and $(m+1)L\cong f^*L_{Y,m+1}$ for some line bundles $L_{Y,m}$ and $L_{Y,m+1}$ on Y. We may let $L_Y:=L_{Y,m+1}-L_{Y,m}$, and we obtain (c). □

5.4. Corollaries

With the cone and contraction theorems proven, we can prove the following three corollaries, which guarantee that negative extremal contractions associated with NQC glc g-pairs behave similarly to negative extremal contractions associated with usual pairs. We omit the proofs as they are very similar to [35, Corollaries 3.17, 3.18]. These corollaries are necessary for us to run the minimal model program.

Corollary 5.10. *Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC glc g-pair and $f:X\to Z$ a contraction of a $(K_X+B+\mathbf{M}_X)$-negative extremal ray R over U. Then $\rho(X)=\rho(Z)+1$.*

Corollary 5.11. *Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC glc g-pair and $f:X\to Z$ a contraction of a $(K_X+B+\mathbf{M}_X)$-negative extremal ray R over U. Assume that f is a divisorial contraction, i.e.* $\dim X=\dim Z$ *and the exceptional locus of f is an irreducible divisor. Then Z is $\mathbb{Q}$-factorial.*

Corollary 5.12. *Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC glc g-pair and $f:X\to Z$ a contraction of a $(K_X+B+\mathbf{M}_X)$-negative extremal ray R over U. Assume that f is a Fano contraction, i.e.* $\dim X>\dim Z$*. Then Z is $\mathbb{Q}$-factorial.*

The following corollary will allow us to run the $\mathbb{Q}$-factorial generalized MMP with scaling (once the existence of flips is proven in the next section). The proof is very similar to [22, Lemma 3.23] so we omit it.

Corollary 5.13. *Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC glc g-pair, $D\geq 0$ an $\mathbb{R}$-divisor on X, and $\mathbf{N}$ an NQC/U $\mathbf{b}$-divisor over X, such that $(X,B+D,\mathbf{M}+\mathbf{N})$ is glc and $K_X+B+D+\mathbf{M}_X+\mathbf{N}_X$ is nef/U. Then either $K_X+B+\mathbf{M}_X$ is nef/U, or there exists an extremal ray R of $\overline{NE}(X/U)$, such that $(K_X+B+\mathbf{M}_X)\cdot R<0$ and $(K_X+B+tD+\mathbf{M}_X+t\mathbf{N}_X)\cdot R=0$, where*

$$t:=\inf\{s\geq 0\mid K_X+B+sD+\mathbf{M}_X+s\mathbf{N}_X \textit{ is nef}/U\}.$$

In particular, $K_X+B+tD+\mathbf{M}_X+t\mathbf{N}_X$ is nef/U.

We also refer the reader to [26, 36] for related results and further applications.

6. Proof of Theorems 1.2, 1.3, and 1.4

Now we are ready to prove the rest of our main theorems. We start with Theorem 1.2. In fact, we can prove a slightly stronger result only assuming that $\mathbf{M}_X$ is $\mathbb{R}$-Cartier.

Theorem 6.1. *Let $(X,B,\mathbf{M})/U$ be an NQC glc g-pair and $f:X\to Z$ a $(K_X+B+\mathbf{M}_X)$-flipping contraction over U. Assume that $\mathbf{M}_X$ is $\mathbb{R}$-Cartier. Then the flip $f^+:X^+\to Z$ of f exists. In particular, $\mathbf{M}_{X^+}$ is $\mathbb{R}$-Cartier, and if X is $\mathbb{Q}$-factorial, then X^+ is $\mathbb{Q}$-factorial and $\rho(X)=\rho(X^+)$.*

Proof. We prove the theorem in three steps. In Step 1, we construct the morphism $f^+ : X^+ \to Z$. In Step 2, we show that the morphism f^+ constructed in Step 1 is a $(K_X + B + \mathbf{M}_X)$-flip. In Step 3, we prove the in particular part of the theorem.

Step 1. In this step, we construct the morphism $f^+ : X^+ \to Z$.

Let $h : \tilde{X} \to X$ be a birational morphism such that $\mathbf{M}$ descends to $\tilde{X}$. Since $\mathbf{M}_X$ is $\mathbb{R}$-Cartier and $\mathbf{M}_{\tilde{X}}$ is nef$/X$, we have

$$\mathbf{M}_{\tilde{X}} + E = h^*\mathbf{M}_X$$

for some $E \geq 0$ that is exceptional over X. Let $T \subset X$ be the flipping locus and let C be any flipping curve contracted by f. There are two cases:

Case 1. $\mathbf{M}_X \cdot C \geq 0$. Then $(K_X + B) \cdot C < 0$, and f is also a $(K_X + B)$-flipping contraction. Thus there exists an ample$/Z$ $\mathbb{R}$-divisor $A \geq 0$ on X such that $K_X + B + A \sim_{\mathbb{R},Z} 0$ and $(X, B + A)$ is lc. By [25, Theorem 1.1], $(X, B)/U$ has a good minimal model. By Theorem 1.1(5.c), we have $K_X + B \sim_{\mathbb{R},Z} r(K_X + B + \mathbf{M}_X)$ for some positive real number r. We let $g : Y \to X$ be a dlt modification of (X, B) and let $K_Y + B_Y = g^*(K_X + B)$, then $K_Y + B_Y + \mathbf{M}_Y = g^*(K_X + B + \mathbf{M}_X)$, and $(Y, B_Y, \mathbf{M})/U$ and $(Y, B_Y, \mathbf{0})/U$ are glc g-pairs such that Y is $\mathbb{Q}$-factorial klt. By Lemma 4.3, $(X, B, \mathbf{M})/Z$ has a good minimal model $(X', B', \mathbf{M})/Z$, and we may let $X' \to X^+$ be the contraction induced by $K_{X'} + B' + \mathbf{M}_{X'}$ over Z and let $f^+ : X^+ \to Z$ be the induced morphism.

Case 2. $\mathbf{M}_X \cdot C < 0$. In this case, $C \subset h(E)$, hence $T \subset h(E)$. Let $Z^0 := Z \backslash \{f(h(E))\}$, $X^0 := X \times_Z Z^0$, $B^0 := B \times_Z Z^0$, and $\mathbf{M}^0 := \mathbf{M} \times_Z Z^0$. Since $\operatorname{center}_X E$ does not contain any glc center of $(X, B, (1-\epsilon)\mathbf{M})$, for any $\epsilon \in (0, 1)$,

- all glc centers of $(X, B, (1 - \epsilon)\mathbf{M})$ intersect X^0,
- $(X^0, B^0, (1 - \epsilon)\mathbf{M}^0)/Z^0$ is a good minimal model of itself (this is because $X^0 \cong Z^0$), and
- $\mathbf{M}^0$ descends to X^0 and $\mathbf{M}^0_{X^0} \sim_{\mathbb{R},Z^0} 0$.

Let $\epsilon_0 \in (0, 1)$ be a real number such that f is also a $(K_X + B + (1 - \epsilon_0)\mathbf{M}_X)$-flipping contraction. By Theorem 4.1, $(X, B, (1 - \epsilon_0)\mathbf{M})/Z$ has a good minimal model. Since $\rho(X/Z) = 1$, there exists a positive real number r such that $K_X + B + \mathbf{M}_X \equiv_Z r(K_X + B + (1 - \epsilon_0)\mathbf{M}_X)$. By Theorem 1.1(5.c), $K_X + B + \mathbf{M}_X \sim_{\mathbb{R},Z} r(K_X + B + (1 - \epsilon_0)\mathbf{M}_X)$. Let $g : Y \to X$

be a dlt modification of (X,B) and let $K_Y+B_Y:=g^*(K_X+B)$, then $K_Y+B_Y+(1-\epsilon_0)\mathbf{M}_Y=g^*(K_X+B+(1-\epsilon_0)\mathbf{M}_X)$ and $K_Y+B_Y+\mathbf{M}_Y=g^*(K_X+B+\mathbf{M}_X)$, and $(Y,B_Y,(1-\epsilon_0)\mathbf{M})/U$ and $(Y,B_Y,\mathbf{M})/U$ are glc g-pairs such that Y is $\mathbb{Q}$-factorial klt. By Lemma 4.3, $(X,B,\mathbf{M})/Z$ has a good minimal model $(X',B',\mathbf{M})/Z$, and we may let $X'\to X^+$ be the contraction induced by $K_{X'}+B'+\mathbf{M}_{X'}$ over Z and let $f^+:X^+\to Z$ be the induced morphism.

Step 2. In this step, we show that the f^+ we constructed in Step 1 is a $(K_X+B+\mathbf{M}_X)$-flip. Let B^+ be the strict transform of B on X^+. We only need to check the following two conditions by the definition of a flip:

(I) $K_{X^+}+B^++\mathbf{M}_{X^+}$ is $\mathbb{R}$-Cartier and ample$/Z$.
(II) f^+ is small.

(I) is immediate from our construction. Since f is small, to prove (II), we only need to show that the rational map $X\dashrightarrow X^+$ does not extract any divisor.

Let $p:W\to X$ and $q:W\to X'$ be a resolution of indeterminacy of $X\dashrightarrow X'$. By Lemma 3.4, $p^*(K_X+B+\mathbf{M}_X)=q^*(K_{X'}+B'+\mathbf{M}_{X'})+F$ where $F\geq 0$ is exceptional over X'. Let D be a prime divisor on X' that is exceptional over X and D_W its strict transform on W. Then D_W is covered by a family of p-vertical curves Σ_t such that $\Sigma_t\cdot p^*(K_X+B_X+\mathbf{M}_X)=0$. Since $F\cdot\Sigma_t\geq 0$, then $\Sigma_t\cdot q^*(K_{X'}+B'+\mathbf{M}_{X'})\leq 0$. Let $\Sigma_t'=q_*\Sigma_t$, then $\Sigma'_t\cdot(K_{X'}+B'+\mathbf{M}_{X'})\leq 0$ so that Σ'_t are contracted by $X'\to X^+$ and hence D is also contracted. Thus $X\dashrightarrow X^+$ does not extract any divisor, which implies (II). Thus f^+ is a $(K_X+B+\mathbf{M}_X)$-flip.

Step 3. Now we prove the in particular part of the theorem. Pick any $\mathbb{R}$-divisor D^+ on X^+, and let D be the strict transform of D^+ on X.

Assume that D is $\mathbb{R}$-Cartier. Since $\rho(X/Z)=1$, there exists a real number t such that $D+t(K_X+B+\mathbf{M}_X)\equiv_Z 0$. By Theorem 1.1(5.c), $D+t(K_X+B+\mathbf{M}_X)\sim_{\mathbb{R},Z}0$. Thus $D+t(K_X+B+\mathbf{M}_X)\sim_{\mathbb{R}}f^*D_Z$ for some $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D_Z on Z. Therefore, $D^++t(K_{X^+}+B^++\mathbf{M}_{X^+})\sim_{\mathbb{R}}(f^+)^*D_Z$. Since $K_{X^+}+B^++\mathbf{M}_{X^+}$ is $\mathbb{R}$-Cartier, D^+ is $\mathbb{R}$-Cartier. Therefore, if $\mathbf{M}_X$ is $\mathbb{R}$-Cartier, then $\mathbf{M}_{X^+}$ is $\mathbb{R}$-Cartier, and if X is $\mathbb{Q}$-factorial, then X^+ is $\mathbb{Q}$-factorial.

Since $X\dashrightarrow X^+$ is an isomorphism in codimension 1, there is a natural isomorphism between the groups of Weil divisors on X and X^+. When X and X^+ are both $\mathbb{Q}$-factorial, we have $\rho(X)=\rho(X^+)$, and the proof is concluded. $\square$

Proof of Theorem 1.2. It immediately follows from Theorem 6.1. $\square$

Proof of Theorem 1.3. It immediately follows from Theorems 6.1, 1.1, and Corollaries 5.10 and 5.11. $\square$

Proof of Theorem 1.4. It immediately follows from Theorem 1.3 and [12, Theorems 1.2, 1.3] ([20, Corollary 1] for the $\mathbb{Q}$-coefficient case). $\square$

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