Scattering rigidity for analytic metrics

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For analytic negatively curved Riemannian manifolds with analytic strictly convex boundary, we show that the scattering map for the geodesic flow determines the manifold up to isometry. In particular, one recovers both the topology and the metric. More generally our result holds in the analytic category under the no conjugate point and hyperbolic trapped set assumptions.

1. Introduction

On a compact connected Riemannian manifold (M, g) with boundary ∂M, denote by

$$\mathcal{G}(M, g) := \{\gamma : [0,1] \to M \,|\, \gamma \text{ is a geodesic for } g, (\gamma(0), \gamma(1)) \in (\partial M)^2\}$$

the set of geodesics with endpoints in the boundary. The *scattering data* of (M, g) is the set

$$\mathrm{Sc}(M, g) := \left\{ \left(\gamma(0), \frac{\dot{\gamma}(0)}{|\dot{\gamma}(0)|_g}, \gamma(1), \frac{\dot{\gamma}(1)}{|\dot{\gamma}(1)|_g}\right) \in (T_{\partial M} M)^2 \,\middle|\, \gamma \in \mathcal{G}(M, g) \right\},$$

of endpoints, together with the normalized tangent vector at the endpoints, of all geodesics in $\mathcal{G}(M, g)$. This is the graph of a map S_g, called scattering map, defined on a subset of the set of incoming tangent vectors at ∂M and mapping to a subset of the set of outgoing tangent vectors at ∂M; see Figure 1. A natural geometric inverse problem consists in determining the manifold M and the metric g from the scattering data $\mathrm{Sc}(M, g)$. This problem is called the *scattering rigidity* problem. A manifold (M, g) is called scattering rigid if it is the only Riemannian manifold, up to isometry being the identity on the boundary, with boundary equal (or isometric) to $(\partial M, g|_{\partial M})$ and with scattering data given by $\mathrm{Sc}(M, g)$. Here, we notice that the tangent space $T_{\partial M} M$ of M at ∂M can be identified with $T\partial M \oplus \mathbb{R}\nu$ where ν is the inward pointing unit normal vector field to ∂M for g, so that the scattering
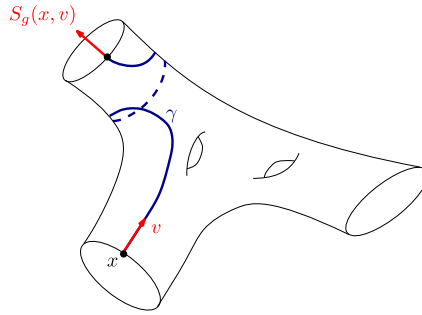
Figure 1: The point $(x, v; S_g(x, v)) \in \mathcal{S}](M, g)$ corresponds to the geodesic $\gamma \in \mathcal{G}(M, g)$. The map $(x, v) \mapsto S_g(x, v)$ is called the scattering map.

data $\mathrm{Sc}(M, g)$ and $\mathrm{Sc}(M', g')$ can be compared if the boundary are isometric as Riemannian manifold $(\partial M, g|_{\partial M}) \simeq (\partial M', g'|_{\partial M'})$ (by identifying the inward pointing normal ν and ν' for g and g'). To simplify notations, we always write $(\partial M, g|_{T\partial M}) = (\partial M', g'|_{T\partial M'})$ to mean that the boundary are isometric.

As far as we know, there are only very few cases of manifolds that are known to be scattering rigid:

- the product $B_{\mathbb{R}^n}(0, 1) \times \mathbb{S}^1$ equipped with the flat product metric — Croke [6],
- Simple Riemannian surfaces — Wen [49].

We recall that a Riemannian manifold (M, g) is called *simple* if M is topologically a ball, the boundary ∂M is strictly convex (i.e. its second fundamental form is positive) and g has no conjugate points (see e.g. [37]).

There are however more results of scattering rigidity within certain classes of Riemannian manifolds. To be precise, we say that a manifold (M, g) is scattering rigid within the class of Riemannian manifold with a given property $\mathcal{P}$ if (M, g) has the property $\mathcal{P}$ and (M, g) is the only (up to isometry equal to the identity on the boundary) Riemannian manifold with the property $\mathcal{P}$ that has the scattering data $\mathrm{Sc}(M, g)$. The scattering rigidity within the class of simple Riemannian manifolds is equivalent to the so-called *boundary rigidity problem* posed by Michel [32]:

> *For simple Riemannian manifolds, does the Riemannian distance between all pairs of boundary points determine the metric up to isometry equal to the identity on the boundary?*

In particular, scattering rigidity within the class of simple Riemannian surfaces with negative curvature (resp. non-positive curvature) is a consequence of the proof of boundary rigidity by Otal [35] (resp. Croke [5]), while the resolution of the problem within the class of general simple Riemannian surfaces is a result of Pestov–Uhlmann [38]. In dimension $n \geq 3$, the scattering rigidity among simple non-positively curved Riemannian metrics follows from the recent works of Stefanov–Uhlmann–Vasy [45]. We refer in general to the recent book by Paternain–Salo–Uhlmann [37] for an overview of the subject on this problem.

For what concerns Riemannian manifolds with boundary that are not simple, in particular those which are not simply connected and have geodesics of infinite length, the only results in the direction of scattering rigidity are the special example of [6] mentionned above, and the result of the second author [18] stating that two negatively curved surfaces with strictly convex boundary and the same scattering data are conformally equivalent.

In this work, we prove that scattering rigidity holds within the class of real analytic negatively curved manifolds:

Theorem 1. *Let (M_1, g_1) and (M_2, g_2) be two real analytic negatively curved compact connected Riemannian manifolds (of any dimension) with non-empty analytic strictly convex boundary. Assume that $(\partial M_1, h_1) = (\partial M_2, h_2)$ where $h_i := g_i|_{T\partial M_i}$ is the metric on the boundary, and that their scattering data $\mathrm{Sc}(M_1, g_1) = \mathrm{Sc}(M_2, g_2)$ agree. Then there exists an analytic diffeomorphism $\psi : M_1 \to M_2$ such that $\psi|_{\partial M_1} = \mathrm{Id}$ and $\psi^* g_2 = g_1$.*

In particular for analytic negatively curved surfaces, the scattering data determines both the topology and the geometry, and not only the conformal class as in [18].

We note that the class of manifolds we consider have in general trapped geodesics, that is infinite length geodesics that do not intersect the boundary. In fact, we prove a result for a more general class of manifolds, as we now explain. Let (M, g) be a compact Riemannian manifold with strictly convex boundary. Let $SM := \{(x, v) \in TM \,|\, |v|_{g_x} = 1\}$ be the unit tangent bundle and $\varphi_t^g : SM \to SM$ be the geodesic flow at time t, with X_g its generating vector field. The trapped set is the closed set defined by

$$\mathcal{K}^g := \{(x, v) \in SM \,|\, \forall t \in \mathbb{R}, \, \varphi_t^g(x, v) \in \mathring{M}\}.$$

This is an invariant set under φ_t^g. We will assume that the trapped set $\mathcal{K}^g$ is *hyperbolic* for the geodesic flow, i.e there exists a continuous flow-invariant

splitting of $T(SM)$ over $\mathcal{K}^g$:

$$\forall z \in \mathcal{K}^g, \quad T_z(SM) = \mathbb{R}X_g(z) \oplus E_s(z) \oplus E_u(z)$$

and constants $C > 0, \nu > 0$ such that $\|\mathrm{d}\varphi_t^g|_{E_s}\| \leq Ce^{-\nu t}$ and $\|\mathrm{d}\varphi_{-t}^g|_{E_u}\| \leq Ce^{-\nu t}$ for all $t > 0$; the distributions E_s and E_u are called the stable and unstable bundles. Another assumption we make is the absence of conjugate points: recall that x, x' are said to be conjugate points if there is $v \in S_x M$, and $t > 0$ such that $\pi_0(\varphi_t^g(x, v)) = x'$ and

$$\mathrm{d}\varphi_t^g(x, v)\mathcal{V} \cap \mathcal{V} \neq 0,$$

where $\mathcal{V} := \ker \mathrm{d}\pi_0 \subset T(SM)$ is the vertical bundle, $\pi_0 : SM \to M$ being the natural projection on the base.

Definition 1.1. Let (M, g) be a smooth compact, connected, Riemannian manifold with non-empty boundary. We say that (M, g) is of Anosov type if

1. the boundary ∂M is strictly convex,
2. the trapped set $\mathcal{K}^g$ is a hyperbolic set for the geodesic flow φ_t^g of g,
3. (M, g) does not have pairs of conjugate points.

Properties (2) and (3) hold when g has negative curvature. However this needs not be always the case. Indeed, one can either perturb a negatively curved manifold to create a small patch of positive curvature away from the trapped set, or consider a strictly convex set inside a closed manifold with Anosov geodesic flow. See [12] for a discussion of such manifolds.

We prove the following result, that contains Theorem 1:

Theorem 2. *Let (M_1, g_1) and (M_2, g_2) be two real analytic Riemannian manifolds of Anosov type with analytic boundary. Assume that $(\partial M_1, h_1) = (\partial M_2, h_2)$ where $h_i := g_i|_{T\partial M_i}$ is the metric on the boundary, and that their scattering data $\mathrm{Sc}(M_1, g_1) = \mathrm{Sc}(M_2, g_2)$ agree. Then there exists an analytic diffeomorphism $\psi : M_1 \to M_2$ such that $\psi|_{\partial M_1} = \mathrm{Id}$ and $\psi^* g_2 = g_1$.*

We emphasize that we do not require to know the travel time, i.e. the lengths of geodesics between boundary points, to obtain the rigidity. The corresponding problem of determining the metric from both the scattering data and the travel time is called the *lens rigidity problem*. For simple manifolds, the lens ridigity, scattering rigidity and boundary rigidity problems are all equivalent, but this is not the case for non-simple manifolds; we refer to the article of Croke–Wen [8] for a study of the difference between scattering rigidity and lens rigidity.

The lens rigidity problem has been considered in the non simple case. There are more results on lens rigidity than on scattering rigidity, although not many in the case with non-empty trapped set. Here are a few:

- Stefanov–Uhlmann [44] obtained a local[1] lens rigidity result for a certain class of Riemannian manifolds in dimension $n \geq 3$ that can have some mild trapping.
- Vargo [47] proved lens rigidity in the class of non-trapping analytic metrics, without assuming convexity of the boundary and in a class of metrics that can have a certain amount of conjugate points.
- Croke–Herreros [7] proved that a 2-dimensional cylinder in negative curvature is lens rigid.
- Guillarmou–Mazzucchelli–Tzou [22] prove lens rigidity in the class of non-trapping surfaces with no conjugate points (but the boundary is not assumed convex).
- Stefanov–Uhlmann–Vasy [45] prove lens rigidity in the class of Riemannian manifolds admitting strictly convex foliations in dimension $n \geq 3$. This class covers some cases with conjugate points and some mild trapping.
- Cekic–Guillarmou–Lefeuvre [2] prove a local lens rigidity result for negatively curved compact manifolds with strictly convex boundary.

A real difficulty for the lens rigidity problem in the case of trapped geodesics is that having same lens data for two metrics does not a priori imply that their geodesic flows are conjugate. To avoid that problem, one can also consider the *marked lens rigidity problem*, where we consider rather the lens data on the universal cover. For two negatively curved metrics on surface M with strictly convex boundary, having the same marked lens data is equivalent to having conjugate geodesic flows with a conjugacy homotopic to identity and equal to the identity on the boundary ∂SM of the unit tangent bundle. In this direction, we mention the following results:

- Guillarmou–Mazzucchelli [21] prove that negatively curved Riemannian surfaces with strictly convex boundary having the same marked length spectrum are isometric. This has been extended recently by Erchenko-Lefeuvre [13] for surfaces of Anosov type.
- Lefeuvre [31] prove that negatively curved Riemannian manifolds with strictly convex boundary having the same marked length spectrum and that are close enough in C^k norms are isometric.

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[1]Local in the sense that two metrics in that particular class that are close enough in some C^k norm and that have same lens data must be isometric.

This rigidity problem is somehow closer to the *marked length spectrum rigidity* on closed Riemannian manifolds with Anosov geodesic flow, where one asks whether two such metrics with conjugate geodesic flows (or equivalently same marked length spectrum) are actually isometric, see Otal, Croke or Guillarmou–Lefeuvre–Paternain [36, 5, 20] for a solution on surfaces and Guillarmou–Lefeuvre [19] for results in higher dimension. A close link between these problems on manifolds with boundary and closed manifold is shown in the recent work of Chen–Erchenko–Gogolev [4]. It turns out that for closed manifolds, the length spectrum rigidity and the marked length spectrum rigidity problem are not equivalent, as there exist, by Vignéras [48], Riemannian metrics with constant negative curvature that have the same length spectrum but are not isometric, while surfaces with negative curvatures and same marked length spectrum are isometric [36]. The result in Theorem 1 thus suggests that these different phenomena in the closed case do not seem to happen for manifolds with boundary in negative curvature, since we do not require to know the marking to obtain rigidity, while in the closed case this is necessary even in the analytic category.

In terms of difficulty of these rigidity problems, one has the chain of implications:

$$\text{scattering rigidity} \implies \text{lens rigidity} \implies \text{marked lens rigidity}$$

and, as mentionned above, they are not always equivalent.

In the case of obstacles in $\mathbb{R}^n$, there are related lens rigidity results by Noakes–Stoyanov [34] who show that the scattering sojourn times allow to recover finitely many disjoint strictly convex obstacles. In the case of analytic strictly convex obstacles with the non-eclipse condition, De Simoi–Kaloshin–Leguil [9] prove that the length of the marked periodic orbits generically determines the obstacles under a $\mathbb{Z}^2 \times \mathbb{Z}^2$ symmetry assumption (the marking is by the ordered sequence of obstacles hit by the orbit). This last result suggests that analytic negatively curved manifold with strictly convex boundary could also possibly be determined by their marked length spectrum (length of periodic orbits marked by their free homotopy classes).

To conclude, let us finally state a conjecture motivated by Theorems 1 and 2:

Conjecture. Two compact Riemannian manifolds of Anosov type with the same scattering data must be isometric by an isometry fixing the boundary.

Method of proof. Our proof shares some similarities with the solution of the Calderón problem for analytic Riemannian metrics due to Lassas–Uhlmann [29, 28]. In the Calderón problem case, we recall that the argument

is first to show that the Cauchy data for the equation $\Delta_g u = 0$ on M determines the Green's function of the Laplacian Δ_g near the boundary ∂M when the metrics are assumed to be analytic. Then one proves that two analytic metrics with the same Green's function near ∂M are isometric in dimension $n > 2$ (conformal in dimension $n = 2$).

In our case, we first show that the scattering data determines the metric near the boundary (see Lemma 2.3), and then that it determines the resolvent R_g of the geodesic vector field X_g near the boundary of SM (see the proof of Lemma 3.1). For this to hold, we use the analyticity, the strict convexity of ∂M and the fact that the trapped set has Lebesgue measure 0 (in order to have a proper notion of resolvent of X_g). The scattering data are essentially the Cauchy data at $\partial(SM)$ for the equation $X_g u = 0$ on SM.

Compared to the Calderón problem where the Green's function is a pseudo-differential operator (as inverse of an elliptic differential operator), the resolvent R_g is a much more singular operator. It was analyzed for smooth metrics with hyperbolic trapped set in [10] using microlocal methods. Its integral kernel has quite wild singularities at the trapped set, but the C^∞ wavefront set can still be described (even though it typically looks like a fractal conical set in $T(SM)$). Composing by the pull-back operator π_0^* and the push-forward operator π_{0*} (where $\pi_0 : SM \to M$), using the transversality between the vertical and stable/unstable bundles, and the absence of conjugate points, we obtain an operator $\Pi_0^g := \pi_{0*} R_g \pi_0^*$ that is much better behaved. Indeed, most of the singularities disappear and Π_0^g is an elliptic pseudo-differential operator of order -1 with principal symbol $c_n |\xi|_g^{-1}$ for some $c_n > 0$ depending only on $n = \dim(M)$, as was proved in [18]. It thus looks quite similar to the Green's function of the Laplacian (except for its order that is half of that of the Green's function).

We can then use the techniques from analytic microlocal analysis developed in [17] (building from fundamental works of Helffer-Sjöstrand and Sjöstrand [23, 41], see the work of Galkowski and Zworski [14, 15] for another recent example of application of these methods) to analyze the Schwartz kernel $\Pi_0^g(x, x')$ of Π_0^g: we are able to show in Section 4 that, if g is in addition analytic, the Schwartz kernel $\Pi_0^g(x, x')$ of Π_0^g is analytic outside the diagonal. This requires in particular to prove radial estimates for the analytic wave front set (Proposition 4.13); a related result was proved by Galkowski-Zworski [15] but rather for smooth radial manifolds. The fact that the operator Π_0^g is an analytic pseudo-differential operator was first proved for simple analytic metrics by Stefanov and Uhlmann [42]. In our case, the analysis involved is significantly more consequent due to the trapped set and the result can not be deduced from [42]; we rely on the recent monograph

[17]. We can then use a method developped (for the Green's function) in [28] to embed analytically (M, g) into $L^2(N_\epsilon)$ using Π_0, where N_ϵ is an ϵ-collar neighborhood of $N := \partial M$, and show that the image of the embedding in $L^2(N_\epsilon)$ is entirely determined by the scattering data. These embeddings for two metrics with the same scattering data allow to construct an isometry $\psi : M_1 \to M_2$.

Another corollary of this work is the following (see Proposition 2.7 for a precise statement)

Proposition 1.2. *On a real analytic Riemannian manifold of Anosov type, the X-ray transform on symmetric 2-tensors is solenoidal injective.*

Notations. $\mathcal{A}(M)$ denotes the set of analytic functions on an analytic manifold M with or without boundary, $\mathcal{D}'(M)$ denotes the set of distributions on a manifold M without boundary, defined as the topological dual to the space $C_c^\infty(\mathring{M})$ of smooth compactly supported functions in the interior $\mathring{M}$, $H^s(M)$ denotes the L^2-based Sobolev space of order s on a compact manifold with or without boundary. We shall denote by $d_M(\cdot, \cdot)$ a fixed distance on M, if M is a compact manifold.

2. Scattering map and the normal operator

2.1. Dynamical preliminaries

In this section, we introduce several important dynamical preliminaries, we refer to [18] for a more detailed exposition.

Let (M, g) be a real-analytic Riemannian manifold with analytic strictly convex boundary ∂M. We denote by SM its unit tangent bundle, which is

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also an analytic manifold with analytic boundary, and $\pi_0 : SM \to M$ the projection on the base. We shall use $z = (x, v)$ for the variable on SM, where $x = \pi_0(z)$ and $v \in S_x M$ is the tangent vector, and we denote by α_L the Liouville 1-form on SM defined by $\alpha_L(x, v)(V) = g_x(v, d\pi_0(V))$ for $V \in T_{(x,v)}(SM)$. Finally, we denote by $-z := (x, -v)$ the reflected vector.

The geodesic vector field X_g has analytic coefficients and its flow is denoted by $\varphi_t^g : SM \to SM$. The boundary ∂SM splits into

$$\partial SM = \partial_- SM \sqcup \partial_+ SM \sqcup \partial_0 SM$$

where $\partial_\pm SM := \{(x, v) \in \partial SM \mid \mp g_x(v, \nu) > 0\}$ and $\partial_0 SM = \{(x, v) \in \partial SM \mid g_x(v, \nu) = 0\}$, where ν denotes the inward unit normal vector to ∂M. Define the measure on $\partial_\pm SM$

$$d\mu_\nu(x, v) = |\iota_{\partial_\pm SM}^* i_{X_g} \omega_L|$$

where $\omega_L = \alpha_L \wedge d\alpha_L^{n-1}$ is the Liouville volume form, and $d\mu_L := |\omega_L|$ the Liouville measure.

We define the escape time by

$$\tau_g : SM \to \mathbb{R}_+ \cup \{\infty\}, \quad \tau_g(z) = \sup\{t \geq 0 \mid \forall s \in]0, t], \varphi_s^g(z) \in SM\}.$$

One has $\tau_g = 0$ on $\partial_+ SM \cup \partial_0 SM$ by strict convexity of the boundary. The forward (resp. backward) trapped set Γ_-^g (resp. Γ_+^g) are defined by

$$(1) \qquad \Gamma_\pm^g := \{z \in SM \mid \tau_g(\mp z) = \infty\}.$$

By the analytic implicit function theorem and the fact that ∂M is strictly convex, the map τ_g belongs to $\mathcal{A}(SM \setminus (\Gamma_-^g \cup \partial_0 SM))$.

The trapped set is defined by

$$(2) \qquad \mathcal{K}^g := \Gamma_+^g \cap \Gamma_-^g.$$

It is a closed subset of $S\mathring{M}$ invariant by the flow φ_t. We assume that $\mathcal{K}^g$ is hyperbolic. In that case, $\mathrm{Vol}(\Gamma_\pm^g) = 0$, see [18, Section 2]. We define the dual bundles $E_0^*, E_s^*, E_u^* \subset T^*(SM)$ over $\mathcal{K}^g$ by

$$E_0^*(E_u \oplus E_s) = 0, \quad E_u^*(E_u \oplus \mathbb{R}X_g) = 0, \quad E_s^*(E_s \oplus \mathbb{R}X_g) = 0.$$

By [10, Lemma 1.11], the bundle E_s^* over $\mathcal{K}^g$ admits a continuous extension E_-^* to the set Γ_-^g, and E_u^* admits a continuous extension E_+^* to the set Γ_+^g, and they satisfy

$$E_\pm^*(\mathbb{R}X_g) = 0.$$

As explained in [18, Section 2], since (M, g) has hyperbolic trapped set and no conjugate points, there is a small extension (M_e, g_e) of (M, g) which has the same properties, with the same trapped set $\mathcal{K}^{g_e} = \mathcal{K}^g$. The sets $\Gamma_{\pm}^{g_e}$ associated with g_e are extensions of $\Gamma_{\pm}^g$, and we still denote by $E_{\pm}^*$ the bundles associated with g_e, which are extensions of those associated with g. Finally, since (M, g) is analytic with analytic boundary, one can choose (M_e, g_e) to be also analytic with analytic boundary.

Next we define the scattering map.

Definition 2.1. The scattering map $S_g : \partial_- SM \setminus \Gamma_-^g \to \partial_+ SM \setminus \Gamma_+^g$ is defined by

$$S_g(z) := \varphi_{\tau_g(z)}^g(z).$$

The scattering operator $\mathcal{S}_g : C_c^\infty(\partial_+ SM \setminus \Gamma_+^g) \to C_c^\infty(\partial_- SM \setminus \Gamma_-^g)$ is defined by

$$\mathcal{S}_g f(z) = f(S_g(z)).$$

By [18, Lemma 3.4], it extends as a unitary map from $L^2(\partial_+ SM, \mathrm{d}\mu_\nu)$ to $L^2(\partial_- SM, \mathrm{d}\mu_\nu)$.

We also define the backward resolvent for the flow

$$(3) \quad R_g : C_c^\infty(S\mathring{M} \setminus \Gamma_-^g) \to C^\infty(SM), \quad R_g f(z) = \int_{-\tau_g(-z)}^0 f(\varphi_t^g(z)) \mathrm{d}t.$$

This operator yields a solution to the boundary value problem

$$X_g R_g f = f, \quad R_g f|_{\partial_- SM} = 0, \quad \forall f \in C_c^\infty(S\mathring{M} \setminus \Gamma_-^g).$$

According to [18, Proposition 4.2], R_g extends as a bounded map

$$(4) \qquad\qquad R_g : L^\infty(SM) \to L^p(SM), \quad \forall p \in [1, \infty),$$

and $X_g R_g = \mathrm{Id}$ in $\mathcal{D}'(S\mathring{M})$ with $(R_g f)|_{\partial_- SM} = 0$ if $f \in C_c^\infty(S\mathring{M})$. One can also define R_{g_e} the resolvent for the extension (M_e, g_e), and it is direct to see, by convexity of ∂M, that $(R_{g_e} f)|_{SM} = R_g f$ if $\mathrm{supp}(f) \subset SM$. A closely related operator is the *X-ray transform*, which is defined for $f \in C_c^\infty(SM \setminus (\Gamma_+^g \cup \Gamma_-^g))$ by the formula

$$(5) \quad I_g f(y) := R_g f(y) = \int_{-\tau_g(-y)}^0 f(\varphi_t^g(y)) \mathrm{d}t \in C_c^\infty(\partial_+ SM \setminus \Gamma_+^g).$$

It satisfies $I_g X_g f = 0$. Its adjoint using the $d\mu_\nu$ and $d\mu$ measures is denoted I_g^*, satisfies $X_g I_g^* = 0$ on $\mathcal{D}'(\partial_+ SM \setminus \Gamma_+^g)$. For $u \in C_c^\infty(\partial_+ SM \setminus \Gamma_+^g)$, it is given for $z \in SM \setminus \Gamma_-^g$ by

$$(6) \qquad I_g^* u(z) = u(\varphi_{\tau_g(z)}^g(z)).$$

By [18, Lemma 5.1] the operator I_g extends as a continuous map $I_g : L^p(SM) \to L^2(\partial_- SM)$ for all $p > 2$.

By abuse of notation we also denote by $(z, z') \mapsto R_g(z, z')$ (resp. $(z, z') \mapsto R_{g_e}(z, z')$) the Schwartz kernel of R_g (resp. R_{g_e}) which is viewed as an element in $\mathcal{D}'(S\mathring{M} \times S\mathring{M})$ (resp. $\mathcal{D}'(S\mathring{M}_e \times S\mathring{M}_e)$). We have $R_g(\cdot, \cdot) = R_{g_e}|_{S\mathring{M} \times S\mathring{M}}$. The Schwartz kernel of I_g is $R_{g_e}|_{\partial_+ SM \times S\mathring{M}}$.

We finally remark that the scattering map has a Schwartz kernel given by the restriction of the resolvent kernel of X_g to $\partial_\pm SM$:

$$\mathcal{S}_g = R_{g_e}|_{\partial_+ SM \times \partial_- SM}.$$

(See [3, Proposition 3.2] in dimension 2 and [2] in general.) In comparison with the Calderón problem, this is similar to the relation between the Dirichlet-to-Neumann map and the Green kernel of the Laplacian with Dirichlet condition.

We close this section by a description of the C^∞ wave front set of the Schwartz kernel of R_{g_e}, taken from [10]. This will be important in section 4. We have

$$(7) \qquad \mathrm{WF}(R_{g_e}) \subset N^* \Delta(SM_e) \cup \Omega_+(SM_e) \cup (E_+^* \times E_-^*),$$

where $N^* \Delta(SM_e)$ is the conormal bundle to the diagonal $\Delta(SM_e)$ of $SM_e \times SM_e$,

$$\Omega_\pm(SM_e) := \{(\varphi_t^g(z), (d\varphi_t^g(z)^{-1})^T \xi;\ z, -\xi) \in T^*(SM_e \times SM_e) \mid$$
$$\pm t \geq 0,\ \xi(X_g(z)) = 0\}.$$

The dual resolvents R_g^* and $R_{g_e}^*$ are also well-defined (for example as maps $L^2 \to L^1$ by (4)) and have Schwartz kernel

$$(8) \qquad R_g^*(z, z') = R_g(z', z), \quad R_{g_e}^*(z, z') = R_{g_e}(z', z).$$

They are given by the expressions

$$R_g^* f(z) = \int_0^{\tau_g(z)} f(\varphi_t^g(z)) \mathrm{d}t, \quad R_{g_e}^* f(z) = \int_0^{\tau_{g_e}(z)} f(\varphi_t^{g_e}(z)) \mathrm{d}t,$$

they satisfy $-X_g R_g^* f = f$ in $\mathcal{D}'(S\mathring{M})$ with $R_g^* f|_{\partial_+ SM} = 0$ for $f \in C^\infty(SM)$ (resp. $-X_{g_e} R_{g_e}^* f = f$ in $S\mathring{M}_e$ if $f \in C^\infty(SM_e)$) and the wave-front set of the Schwartz kernel of $R_{g_e}^*$ satisfies

$$(9) \qquad \mathrm{WF}(R_{g_e}^*) \subset N^* \Delta(SM_e) \cup \Omega_-(SM_e) \cup (E_-^* \times E_+^*).$$

2.2. Determination of the metric near the boundary

We start with a normal form for the metric near the boundary:

Lemma 2.2. *Let (M, g) be an analytic Riemannian manifold with analytic strictly convex boundary. There exists an analytic diffeomorphism $\psi : [0, \epsilon)_r \times \partial M \to U \subset M$, called normal form for g, such that*

$$\psi^* g = \mathrm{d}r^2 + h(r)$$

where $r \mapsto h(r) \in \mathcal{A}(\partial M; S^2 T^* M)$ is a one parameter analytic family of analytic metrics on ∂M.

Proof. It suffices to use the map $\psi(r, y) = \exp_y(r\nu(y))$ for ν the inward unit normal vector to ∂M. This map is analytic map since g is analytic. $\square$

Using a standard argument, we show that S_g determines the Taylor expansion of the map $r \mapsto h(r)$ at $r = 0$ and so it determines $h(r)$ for r near 0 by unique continuation:

Lemma 2.3. *Assume that (M_1, g_1) and (M_2, g_2) are two analytic Riemannian manifolds with strictly convex analytic boundary and no conjugate points, and let ψ_1 and ψ_2 their normal form diffeomorphisms and denote by $\psi_i^* g_i = \mathrm{d}r^2 + h_i(r)$. Assume that $(\partial M_1, h_1(0)) = (\partial M_2, h_2(0))$ and that $S_{g_1} = S_{g_2}$ (i.e. M_1 and M_2 have the same scattering data). Then $h_1(r) = h_2(r)$ near $r = 0$.*

Proof. Let us denote by $(N, h) := (\partial M_1, h_1(0)) = (\partial M_2, h_2(0))$. Let $b_i \in C^0(N \times N)$ be the boundary distance function defined by $b_i(y, y') := d_{g_i}(y, y')$ where $d_{g_i} \in C^0(M_i \times M_i)$ is the Riemannian distance of (M_i, g_i). First, we will show that the identity $S_{g_1} = S_{g_2}$ implies that the boundary distances b_1 and b_2 for g_1 and g_2 agree near the diagonal of $N \times N$. Then, we can apply Theorem 2.1 of Lassas–Sharafutdinov–Uhlmann [27], which says that $h_1(r) - h_2(r) = \mathcal{O}(r^\infty)$ at $r = 0$ (since their argument is purely local near the boundary) and deduce that $h_1(r) = h_2(r)$ by analyticity.

To show that $b_1 = b_2$ near the diagonal if $S_{g_1} = S_{g_2}$, first choose $\epsilon > 0$ small and denote the injectivity radius of g_i by r_i, $i = 1, 2$. If $x, x' \in N$

are such that their distance in (N, h) satisfies $d_h(x, x') < \epsilon$ with ϵ small enough, then $d_{g_i}(x, x') < r_i$. Pick such $x, x' \in N$. Take the minimal geodesic curve $\gamma : [0, 1] \to N$ in (N, h) so that $\gamma(0) = x$ and $\gamma(1) = x'$, in particular $d_{g_i}(\gamma(t), x) < \epsilon$. We define $v_i(t)$ to be the unit vector in the direction of $\gamma(t)$, that is $(\exp_x^{g_i})^{-1}(\gamma(t))/|(\exp_x^{g_i})^{-1}(\gamma(t))|_{g_{i,x}} \in S_x M$, so that $b_i(x, \gamma(t)) = \tau_{g_i}(x, v_i(t))$ (we extend $v_i(t)$ by continuity at $t = 0$).

We claim that $v_i(t)$ is the unique vector that has the smallest orthogonal projection to $(TN)^\perp \subset TM$ among all vectors $(x, v) \in \partial_- SM_i$ satisfying $\pi_0(S_{g_i}(x, v)) = \gamma(t)$, i.e

$$|g_i(v_i(t), \nu_i)| = \min \left\{ |g_i(v, \nu_i)| \,\big|\, v \in \partial_- S_x M_i, \pi_0(S_{g_i}(x, v)) = \gamma(t) \right\},$$

with ν_i the inward unit normal at N in M_i. In particular this claim implies that $v_i(t)$ is entirely determined by $h_i(0)$ and S_{g_i}, and since

$$b_i(x, x') = \int_0^1 \partial_t(d_{g_i}(x, \gamma(t)))dt = \int_0^1 g_i(\nabla^{g_i} d_{g_i}(x, \cdot)|_{\gamma(t)}, \dot{\gamma}(t))dt,$$

wih $\nabla^{g_i} d_{g_i}(x, \cdot)|_{\gamma(t)} = S_{g_i}(x, v_i(t))$ (by Gauss Lemma), we conclude that $b_1 = b_2$ if $S_{g_1} = S_{g_2}$.

Let us prove the claim above. We start by observing that the projection $|g_i(v_i(t), \nu_i)|$ is bounded by $C\epsilon$ for some $C > 0$ that does not depend on ϵ. Let us now consider other geodesic curves with endpoints x and $\gamma(t)$. Due to the absence of conjugate points, there is exactly one such curve in each homotopy class of curves with endpoints $(x, \gamma(t))$, so if $\tilde{v}_i(t) \neq v_i(t)$ is such that $\pi_0(S_{g_i}(x, \tilde{v}_i(t))) = \gamma(t)$, the corresponding curve cannot be homotopic to γ and therefore its length $\tau_{g_i}(x, \tilde{v}_i(t))$ must be larger than r_i. On the other hand, by strict convexity of the boundary, there is $C > 0$ such that for all $\delta > 0$ small and $(x, v) \in \partial_- SM_i$ such that $|g_i(v, \nu_i)| < \delta$, one has $\tau_{g_i}(x, v) \leq C|g_i(v, \nu_i)|$ ([39, Lemma 4.1.2.]). We deduce that $|g_i(\tilde{v}_i(t), \nu_i)| > \delta$ since otherwise one would have

$$r_i \leq \tau_{g_i}(x, \tilde{v}_i(t)) \leq C|g_i(\tilde{v}_i(t), \nu_i)| \leq C\delta$$

which is a contradiction if $\delta > 0$ is small enough. Taking ϵ small enough so that $C\epsilon < \delta$, the claim is then proved and the proof is complete. □

2.3. The normal operator

Let us now define the normal operator Π_0^g that was introduced in [18]. Let $\pi_0^* : C^\infty(M) \to C^\infty(SM)$ be the pullback by $\pi_0 : SM \to M$ (the projection

on the base). It is continuous as a map $\pi_0^* : \mathcal{A}(M) \to \mathcal{A}(SM)$. Consider also the dual map, called pushforward, $\pi_{0*} : \mathcal{D}'(S\mathring{M}) \to \mathcal{D}'(\mathring{M})$ and we use the same notations for the corresponding operators on the extended manifolds SM_e and M_e. Then we define

$$\Pi_0^g : C_c^\infty(\mathring{M}) \to \mathcal{D}'(\mathring{M}), \quad \Pi_0^g := \pi_{0*} R_g \pi_0^*.$$

The same definition holds on the extension M_e for the corresponding operator $\Pi_0^{g_e}$, and since R_g is the restriction of R_{g_e} on functions supported in SM, we directly deduce that $(\Pi_0^{g_e} f)|_M = \Pi_0^g f$ if $\text{supp}(f) \subset \mathring{M}$. Finally, using that $\pi_0(x, -v) = \pi_0(x, v)$, one also has $\Pi_0^{g_e} = \pi_{0*} R_{g_e}^* \pi_0^*$ and thus (this fact also follows from the identity $\Pi_0^{g_e} = \pi_{0*} I^{g_e *} I^{g_e} \pi_0^*$, see [18, Section 5.1])

$$(10) \qquad\qquad (\Pi_0^{g_e})^* = \Pi_0^{g_e}.$$

The operator $I_0^{g_e} := I^{g_e} \pi_0^*$ is called the X-ray transform on M, it is injective by [18, Theorem 5].

We recall from [18, Proposition 5.7]:

Lemma 2.4. *If (M, g) is a smooth manifold of Anosov type, then $\Pi_0^g \in \Psi^{-1}(\mathring{M})$ is an elliptic pseudo-differential operator of order -1 on $\mathring{M}$ with principal symbol $\sigma(\Pi_0^g) = c_n |\xi|_g^{-1}$ for some $c_n > 0$ depending only on $n = \dim M$. It is moreover the restriction to M of the elliptic pseudo-differential operator $\Pi_0^{g_e} \in \Psi^{-1}(\mathring{M}_e)$. For each $x_0 \in \mathring{M}_e$, its Schwartz kernel has asymptotic behaviour as $x \to x_0$*

$$(11) \qquad\qquad \Pi_0^{g_e}(x, x_0) \sim c_n' d_{g_e}(x, x_0)^{-(n-1)}, \quad c_n' \neq 0$$

where d_{g_e} denotes the Riemannian distance on (M_e, g_e). Similarly for any $v \in T_{x_0} M$, we have as $x \to x_0$

$$(12) \qquad D_{x_0} \Pi_0^{g_e}(x_0, x).v \sim -(n-1)c_n' \Big(D_{x_0}(d_{g_e}(x, x_0)).v \Big) d_{g_e}(x, x_0)^{-n}.$$

If in addition the metric is analytic, we show the following stronger result, the proof of which is deferred to Section 4. It is the main technical result of the paper.

Proposition 2.5. *If (M_e, g_e) is an analytic manifold of Anosov type, then $(x, x') \mapsto \Pi_0^{g_e}(x, x')$ is analytic in $\mathring{M}_e \times \mathring{M}_e \setminus \{\text{diag}\}$.*

Even though it is not required for the proof of Theorem 2, we can use the local analysis from [42] to get a more precise statement.

Proposition 2.6. *The operator $\Pi_0^{g_e}$ is an elliptic analytic pseudo-differential operator of order -1 on $\mathring{M}_e$ (in the sense of [46, Chapter V]).*

We give another consequence of the analysis performed in Section 4, namely that the X-ray transform on real analytic manifolds of Anosov type is injective on the space of divergence-free symmetric 2-tensors. First, define the operator mapping a symmetric tensor of order $m \geq 1$ on M_e to a function on SM_e

$$\pi_m^* : C_c^\infty(\mathring{M}_e; S^m T^* \mathring{M}_e) \to C_c^\infty(S\mathring{M}_e), \quad (\pi_m^* f)(x, v) := f_x(\otimes^m v).$$

This operator maps real analytic tensors to real analytic functions on SM_e. We define $\pi_{m*} : \mathcal{D}'(S\mathring{M}_e) \to \mathcal{D}'(\mathring{M}_e; S^m T^* \mathring{M}_e)$ to be its dual defined by $\langle \pi_{m*} u, \chi \rangle := \langle u, \pi_m^* \chi \rangle$ for $\chi \in C_c^\infty(\mathring{M}_e; S^m T^* \mathring{M}_e)$. We define the X-ray transform on symmetric m-tensors on (M_e, g_e) by

$$I_m^{g_e} := I^{g_e} \pi_m^*.$$

We obtain the same result as [42, Theorem 1] and [43, Theorem 1], but now in the setting of manifolds of Anosov type.

Proposition 2.7. *Assume that (M, g) is a real analytic Riemannian manifold of Anosov type. Let $f \in L^2(M; S^2 T^* M)$ and assume that $I_2^g f = 0$. Then $f = D_g h$ for some one-form $h \in H_0^1(M; T^* M)$ vanishing at ∂M, where D_g is the symmetrized Levi-Civita covariant derivative on tensors.*

The proof is defered to Section 4. We notice that it can be extended with minor modifications to m-tensors for all $m \geq 1$. We focus on the case $m = 2$ since we use some results from [42, 43] done for $m \leq 2$ and since this is the case of interest for lens and boundary rigidity problems.

3. Proof of Theorem 2

We consider two analytic extensions (M_e^1, g_e^1) and (M_e^2, g_e^2) of M_1 and M_2 respectively. Let us denote by $N := \partial M_1 = \partial M_2$ the common boundary of M_1, M_2. Let $\psi_1 : [0, \epsilon) \times N \to U_1 \subset M_e^1$ and $\psi_2 : [0, \epsilon) \times N \to U_2 \subset M_e^2$ be the two normal forms near the boundary for g_e^1, g_e^2 (for $\epsilon > 0$ small enough), i.e.

$$\psi_i^* g_e^i = \mathrm{d}r^2 + h_i(r)$$

with h_i an analytic family of analytic metrics on $N := \partial M_1 = \partial M_2$. By choosing the extension small enough, we can assume that $\psi_i(\{\epsilon/2\} \times N) =$

∂M_i and that $\psi_i([0, \epsilon) \times N)$ does not intersect $\pi_0(\mathcal{K}^{g_i})$. Recall that ψ_i is analytic, and by Lemma 2.3, $h_1 = h_2$ in $r \in [\epsilon/2, \epsilon)$, thus everywhere in $[0, \epsilon) \times N$ by analytic continuation. For $0 \leq \delta \leq \epsilon$, we denote $N_\delta := (0, \delta) \times N$.

Let us now consider the map

(13) $$\Phi_i : M_i \to L^2(N_{\epsilon/4}), \quad \Phi_i(x) = \Pi_0^{g_i}(x, \psi_i(\cdot))|_{N_{\epsilon/4}}.$$

where $\Pi_0^{g_i}(x, x')$ is the Schwartz kernel of the operator $\Pi_0^{g_e^i}$ associated with g_e^i. Here, since $x \in M_i$ and $\psi_i(N_{\epsilon/4}) \cap M_i = \emptyset$, the map Φ_i is valued into analytic functions on the small collar $N_{\epsilon/4}$. The choice of the space $L^2(N_{\epsilon/4})$ is irrelevant; it only needs to be a reasonable functional Banach space. The map Φ_i is analytic on M_i as a composition of analytic maps, using that $\Pi_0^{g_e^i}(x, x')$ is analytic outside the diagonal.

We first show the following:

Lemma 3.1. *If $S_{g_1} = S_{g_2}$, then $\Phi_1(\psi_1(x)) = \Phi_2(\psi_2(x))$ for each $x \in (\epsilon/2, \epsilon) \times N$.*

Proof. For $i = 1, 2$, let $\Psi_i : SN_\epsilon \to SM_i$ be the lift of ψ_i to the unit tangent bundles, i.e. $\Psi_i(x, v) = (\psi_i(x), \mathrm{d}\psi_i(x)v)$. We will show that $(\Psi_1 \otimes \Psi_1)^* R_{g_e^1} = (\Psi_2 \otimes \Psi_2)^* R_{g_e^2}$ on SN_ϵ^2 as distributions. Since (on the level of kernels)

$$\Pi_0^{g_e^i} = (\pi_{0*} \otimes \pi_{0*}) R_{g_e^i},$$

the announced claim follows.

We will write $\partial_\sigma SN_\epsilon = \Psi_1^{-1} \partial_\sigma SM_e^1 = \Psi_2^{-1} \partial_\sigma SM_e^2$ for $\sigma \in \{+, -, 0\}$. For $\chi \in C_c^\infty(S\mathring{N}_\epsilon)$, call $\chi_i := \Psi_{i*}\chi$ and consider the distribution

$$u_i(z) = \langle R_{g_e^i}(\Psi_i(z), \Psi_i(\cdot)), \chi \rangle = (R_{g_e^i}\chi_i)(\Psi_i(z)).$$

It suffices now to prove that $u_1 = u_2$ for any such χ. A priori, by Proposition 4.2 of [18], $u_i \in L^p(SN_\epsilon)$ for all $p < \infty$. Additionnally, according to the support of χ_i, we have $u_i \in C^\infty(SN_\epsilon \setminus \Psi_i(\Gamma_+^{g_e^i}))$, and it satisfies

$$X_i u_i = \chi, \quad u_i|_{\partial_- SN_\epsilon} = 0$$

where X_i is the geodesic vector field of $\mathrm{d}r^2 + h_i(r)$. We have $X_1 = X_2$ in SN_ϵ and let φ_t be the flow of $X_1 = X_2$ there. From the strict convexity of the boundary, if $z \in SN_\epsilon$, we deduce that $\pi_0(\varphi_t(z))$ remains in N_ϵ in at least one of the intervals $[-\tau_{g_e^1}(\Psi_1(-z)), 0], [0, \tau_{g_e^1}(\Psi_1(z))]$. In the first case, from the expression (3) we deduce directly that $u_1(z) = u_2(z)$.

Since $\Gamma_+^{g_e^i}$ has measure 0, and u_i is in L^p, we concentrate thus on the z's whose trajectory remains in N_ϵ for $t \in [0, \tau_{g_e^1}(\Psi_1(z))]$, but for which $\tau_{g_e^1}(\Psi_1(-z)) < \infty$. According to the expression

$$u_i(z) = I_{g_e^i}\chi_i(\varphi_{\tau_{g_e^i}(\Psi_i(z))}^{g_e^i}(\Psi_i(z))) - \int_0^{\tau_{g_e^i}(\Psi_i(z))} \chi(\varphi_t(z))\mathrm{d}t,$$

it suffices thus to prove that the quantity $I_{g_e^i}\chi_i(\varphi_{\tau_{g_e^i}(\Psi_i(z))}^{g_e^i}(\Psi_i(z)))$, or equivalently $u_i|_{\partial_+ SN_\epsilon}$, does not depend on i (here we use the X-ray transform $I_{g_e^i}$ defined in (5)). However, the quantity $I_{g_e^i}\chi_i(\varphi_{\tau_{g_e^i}(\Psi_i(z))}^{g_e^i}(\Psi_i(z)))$ is the sum of two terms, the first corresponds to the trajectory of $\Psi_i(z)$ close to $\Psi_i(z)$, and the second to the trajectory near $\partial_- SM_e^i$. This is because χ is supported in N_ϵ. We already know that the first contribution does not depend on i. For the second contribution, we can use the main assumption that $S_{g_1} = S_{g_2}$ to deduce that they are also equal. Finally, $u_1(z) = u_2(z)$ on SN_ϵ as expected. $\square$

We next show the

Lemma 3.2. *The map Φ_i is an analytic embedding of M_i into $L^2(N_{\epsilon/4})$ for $i = 1, 2$.*

Proof. First, we show that Φ_i is an injective map. If $x_0, x_1 \in M_i$ are such that $\Phi_i(x_0) = \Phi_i(x_1)$, then $\Pi_0^{g_e^i}(x_0, \psi_i(x')) = \Pi_0^{g_e^i}(x_1, \psi_i(x'))$ for all $x' \in N_{\epsilon/4}$. In particular, this implies that $\Pi_0^{g_e^i}(x_0, x) = \Pi_0^{g_e^i}(x_1, x)$ for all $x \in \psi_i(N_{\epsilon/4})$ and by analytic continuation we get $\Pi_0^{g_i}(x_0, x) = \Pi_0^{g_i}(x_1, x)$ for all $x \in \mathring{M}_e^i \setminus \{x_0, x_1\}$. Using the asymptotic expansion (11) and letting respectively x tend to x_0 and to x_1, we deduce that $x_0 = x_1$.

Next, we show its derivative $D\Phi_i(x)$ is injective for $x \in M_i$. The map Φ_i is C^1 on M_i since $\Pi_0^{g_e^i}$ has a smooth kernel outside the diagonal (it is in fact analytic). Let $v \in T_x M_i$, and assume that $D\Phi_i(x).v = 0$. We then get $D_x\Pi_0^{g_e^i}(x, x').v = 0$ for all $x' \in M_e^i \setminus M_i$. Using the analytic extension of $\Pi_0^{g_e^i}(x, x')$ outside the diagonal we obtain $D_x\Pi_0^{g_e^i}(x, x').v = 0$ for all $x' \in M_e^i \setminus \{x\}$. By letting $x' \to x$ and using the asymptotic expansion (12), we deduce that $v = 0$, showing that Φ_i is an immersion, and thus an embedding. $\square$

We can now conclude the proof of Theorem 2.

Proposition 3.3. *The image of Φ_1 and Φ_2 agree, i.e. $\Phi_1(M_1) = \Phi_2(M_2)$, and the map $\Phi = \Phi_2^{-1} \circ \Phi_1 : M_1 \to M_2$ is an analytic map such that $\Phi^* g_2 = g_1$.*

Proof. Let us define

$$U := \{x \in M_1 \,|\, \exists x' \in M_2, \Phi_2(x') = \Phi_1(x)\}.$$

Since, according to Lemma 3.2, Φ_2 is injective, we can define the map $\Phi = \Phi_2^{-1} \circ \Phi_1$ on U. Let $U' \subset U$ be the set of points $x \in U$ such that Φ is well-defined and C^1 on a neighbourhood of x. Notice, by Lemma 3.1, that $\Psi_1((\epsilon/2, \epsilon) \times N) \cap M_1 \subset U'$, so that U' is non empty and open. We will prove that it is closed in M_1.

Let x be an accumulation point of U'. Since we already know that the boundary of M_1 belongs to U', we may assume that x belongs to the interior of M_1. Then, choose $x_n \in U'$ converging to x and let $x'_n := \Phi(x_n)$. Up to extracting a subsequence, $x'_n \to x'$ for some $x' \in M_2$. By continuity of Φ_1 and Φ_2, we get $\Phi_1(x) = \Phi_2(x')$, in particular $x \in U$. Since $\Phi_2 \circ \Phi = \Phi_1$ in U' with Φ smooth there, we have $D\Phi_2(x'_n)D\Phi(x_n) = D\Phi_1(x_n)$. In particular $D\Phi_2(x'_n)(T_{x'_n} M_2) = D\Phi_1(x_n)(T_{x_n} M_1)$, and since $D\Phi_i$ is continuous with constant rank, we obtain that (since $x' = \Phi(x)$)

$$T_{\Phi_2(x')}\Phi_2(M_2) = D\Phi_2(x')(T_{x'} M_2) = D\Phi_1(x)(T_x M_1) = T_{\Phi_1(x)}\Phi_1(M_1).$$

Next, we show that x is actually in U'. For this, let $u = \Phi_1(x)$ and $\mathcal{V} := T_u \Phi_1(M_1) \subset L^2(N_{\epsilon/4})$ and $\mathcal{V}^\perp$ the orthogonal to $\mathcal{V}$ for the $L^2(N_{\epsilon/4})$ scalar product. Denote also by $P : L^2(N_{\epsilon/4}) \to \mathcal{V}$ the orthogonal projection, and let

$$P\Phi_i : M_i \to \mathcal{V}.$$

This is an analytic map, and the differential $DP\Phi_1(x)$ and $DP\Phi_2(x')$ are surjective by construction, as $DP\Phi_i = PD\Phi_i$, and are also injective by the fact that Φ_i is an embedding. By the local inverse theorem in the analytic category, there is a connected open neighbourhood W of u in $\mathcal{V}$ and an analytic map $H_i : W \to M_i$, diffeomorphic on its image, such that

$$P\Phi_i \circ H_i(w) = w, \quad \forall w \in W.$$

Thus $\Phi_i \circ H_i(w) = w + (1-P)\Phi_i \circ H_i(w)$ and if we let $Q_i(w) := (1-P)\Phi_i \circ H_i(w) \in \mathcal{V}^\perp$, we see that $\Phi_i(M_i)$ is locally the graph of $Q_i|_W$. Moreover Q_i is analytic and the graph of Q_1 coincides with the graph of Q_2 in a non empty open subset $W' \subset W$ since x is an accumulation point of U', thus $Q_1 = Q_2$ on W'. This implies by analytic continuation that $Q_1 = Q_2$ in W. In particular there is an open neighbourhood $V := H_1(W)$ of x in U, and Φ is thus well-defined in V. Since $\Phi_1 \circ H_1 = \Phi_2 \circ H_2$ on W, we also get

$\Phi = \Phi_2^{-1} \circ \Phi_1 = H_2 \circ H_1^{-1}$ on V, in particular Φ is analytic on V. Hence, $x \in U'$, and thus U' is closed.

Hence, U' is open, closed and non-empty, so that $U' = M_1$. By definition, Φ is C^1 on U', the proof above show that it is actually analytic. The relation $\Phi_2 \circ \Phi = \Phi_1$ proves that Φ is injective (since Φ_1 is). Notice that the image of Φ is closed (since M_1 is compact), open (since Φ is a local diffeomorphism) and non-empty (since M_1 is non-empty). Hence, $\Phi(M_1) = M_2$, and Φ is a real-analytic diffeomorphism from M_1 to M_2.

It remains to prove that Φ is an isometry. For this, it suffices to observe that Φ is an isometry when restricted to $U_1 \cap M_1$, according to Lemma 2.3. By analytic continuation, Φ is thus an isometry everywhere. □

4. Analyticity of the kernel of the Normal Operator away from the diagonal

This section is dedicated to the proof of Proposition 2.5. The proof relies on an analytic wave front set computation, based on classical results (ellipticity, propagation of singularities, ...) and a new radial estimates in the analytic category, Proposition 4.13. The C^∞ analogue of this estimate is now a standard tool in scattering theory (see for instance [11, Theorem E.42]).

In §4.1, we recall the results from [17] that we will use. In §4.2, we discuss the notion of analytic wave front sets, and how one can investigate it using the FBI transform. In §4.3, we prove the microlocal radial estimates needed for the proof of Proposition 2.5, proof that is given in §4.4.

4.1. The FBI transform

4.1.1. Basic properties. Let $\mathcal{M}$ be a closed real-analytic manifold (in the proof of Proposition 2.5, $\mathcal{M}$ will be an extension of the unit tangent bundle SM_e of M_e, see §4.3). For convenience, we will endow $\mathcal{M}$ with a real-analytic Riemannian metric. We then have a notion of Grauert tube of radius $\epsilon > 0$ of $\mathcal{M}$, which we denote $(\mathcal{M})_\epsilon$. We endow $T^*\mathcal{M}$ with the corresponding real-analytic Kohn–Nirenberg metric $g_{\mathrm{KN}} = \mathrm{d}x^2 + \frac{\mathrm{d}\xi^2}{\langle \xi \rangle^2}$, and consider the associated Grauert tube $(T^*\mathcal{M})_\epsilon$. This is a conic, complex, pseudo-convex neighbourhood of $T^*\mathcal{M}$. We define on it a Japanese bracket $(T^*\mathcal{M})_\epsilon \ni \alpha \mapsto \langle |\alpha| \rangle \in [1, +\infty[$ and the distance d_{KN} associated with the Kohn–Nirenberg metric (see pages 23-24 in [17] for further discussion of these definitions). For a subset A of $\mathcal{M}$, we will write $T_A^*\mathcal{M}$ for the restriction of $T^*\mathcal{M}$ to A.

In order to study the microlocal analytic regularity of distributions on the closed real-analytic manifold $\mathcal{M}$, we introduce an *analytic FBI transform*. We follow the approach exposed in [17, Chapter 2], whose main features we recall here. An analytic FBI transform is an operator $T : \mathcal{D}'(\mathcal{M}) \to C^\infty(T^*\mathcal{M})$ with real-analytic kernel K_T:

$$Tu(\alpha) = \int_{\mathcal{M}} K_T(\alpha, y)u(y)\mathrm{d}y, \quad u \in \mathcal{D}'(\mathcal{M}), \quad \alpha = (\alpha_x, \alpha_\xi) \in T^*\mathcal{M}.$$

The kernel K_T depends on an (implicit) small semi-classical parameter $h > 0$ and has the following properties:

- real-analyticity: K_T has a holomorphic extension to $(T^*\mathcal{M})_\epsilon \times (\mathcal{M})_\epsilon$ for some small ϵ that does not depend on h.
- K_T is negligible when y and α_x are away from each other: for every $\delta > 0$, there are $\epsilon' \in]0, \epsilon[$ and $C > 0$ such that if $h > 0$ is small enough, $(\alpha, y) \in (T^*\mathcal{M})_{\epsilon'} \times (\mathcal{M})_{\epsilon'}$ and the distance between α_x and y is larger than δ then

$$|K_T(\alpha, y)| \leq C \exp\left(-\frac{\langle|\alpha|\rangle}{Ch}\right).$$

- local behaviour: for $(\alpha, y) \in (T^*\mathcal{M})_\epsilon \times (\mathcal{M})_\epsilon$, if α_x and y are close to each other, $K_T(\alpha, y)$ is given by

$$K_T(\alpha, y) \simeq e^{i\frac{\Phi_T(\alpha, y)}{h}} a(\alpha, y),$$

up to an error exponentially decaying in $\langle|\alpha|\rangle/h$. Here, Φ_T and $h^{3n/4}a$ are analytic symbols of orders respectively 1 and $n/4$ on a neighbourhood of uniform size for the Kohn–Nirenberg metric of $\{\alpha_x = y\}$ in $(T^*\mathcal{M})_\epsilon \times (\mathcal{M})_\epsilon$. See [17, Definition 1.1 and Lemma 1.6] for the definition of an analytic symbols (the class of analytic symbols of order m is called $S^{1,m}$ there).

Moreover, the phase Φ_T satisfies the following properties:

- if $(\alpha, y) \in T^*\mathcal{M} \times \mathcal{M}$ then the imaginary part of $\Phi_T(\alpha, y)$ is non-negative;
- $\Phi_T(\alpha, \alpha_x) = 0$ for $\alpha = (\alpha_x, \alpha_\xi) \in T^*\mathcal{M}$;
- for $\alpha \in T^*\mathcal{M}$, we have $\mathrm{d}_y\Phi_T(\alpha, \alpha_x) = -\alpha_\xi$;
- there is $C > 0$, such that for $(\alpha, y) \in T^*\mathcal{M} \times \mathcal{M}$ near $\{\alpha_x = y\}$ we have $\mathrm{Im}\,\Phi_T(\alpha, y) \geq C^{-1}\langle|\alpha|\rangle\, d(\alpha_x, y)^2$.

In the terminology of [17, Definition 1.7], Φ_T is an admissible phase.

Due to the real-analyticity condition on the kernel K_T of T, it is not clear *a priori* that such a transform T exists. However, from [17, Theorem 6], there is such a T, assuming h is small enough. Moreover, if we endow $T^*\mathcal{M}$ with the volume associated with the canonical symplectic form on $T^*\mathcal{M}$, then we may assume that T is an isometry from $L^2(\mathcal{M})$ into $L^2(T^*\mathcal{M})$. Hence, denoting by S the adjoint of T, the composition ST is the identity operator. Notice that S is an operator with real-analytic kernel given for α and y real by

$$K_S(y, \alpha) = \overline{K_T(\alpha, y)}.$$

Due to the real-analyticity of K_T, we see that if u is a distribution (or even a hyperfunction), then Tu is in fact real-analytic, with a holomorphic extension to $(T^*\mathcal{M})_\epsilon$. Hence, if Λ is a perturbation of $T^*\mathcal{M}$ in $(T^*\mathcal{M})_\epsilon$, we can define a FBI transform associated with Λ by restriction: $T_\Lambda u := (Tu)_{|_\Lambda}$. Of course, in order to have interesting properties for T_Λ, we need to make assumptions on Λ. We will only consider Λ's of the form

$$(14) \qquad \Lambda = \Lambda_G = e^{H_G^{\omega_I}} T^*\mathcal{M}.$$

Here, G is a real valued symbol of order at most 1 in the Kohn–Nirenberg class on $(T^*\mathcal{M})_\epsilon$, and $H_G^{\omega_I}$ is the Hamiltonian vector field of G for the real symplectic form ω_I, which is the imaginary part of the canonical complex symplectic form on the cotangent bundle of the complexification of $\mathcal{M}$. For practical purposes, we will always assume that G is a small symbol of order 1, so that Λ is C^∞ close to $T^*\mathcal{M}$. In particular, the real part ω_R of the canonical complex symplectic form defines a symplectic form on Λ.

From the geometric properties of Λ, there is a symbol H on Λ, of the same order as G such that $\mathrm{d}H = -\operatorname{Im}\theta_{|\Lambda}$, where θ denotes the canonical complex Liouville 1-form on the cotangent bundle of the complexification of $\mathcal{M}$. Letting $\mathrm{d}\alpha$ be the volume form associated with the symplectic form ω_R on Λ, we will work with the weighted L^2 spaces

$$L_k^2(\Lambda) := L^2\left(\Lambda, \langle|\alpha|\rangle^{2k} e^{-\frac{2H}{h}} \mathrm{d}\alpha\right), \quad k \in \mathbb{R},$$

and their equivalents on $\mathcal{M}$

$$\mathcal{H}_\Lambda^k := \{u \in \mathcal{O}((\mathcal{M})_\delta)' : T_\Lambda u \in L_k^2(\Lambda)\}.$$

Here, $\delta > 0$ is fixed, depending on the geometry of $\mathcal{M}$ and $\mathcal{O}((\mathcal{M})_\delta)$ is the closure of bounded holomorphic functions on $(M)_{2\delta}$, in the space of

holomorphic functions on $(\mathcal{M})_\delta$ endowed with the sup norm, and $\mathcal{O}((\mathcal{M})_\delta)'$ its topological dual. According to [17, Corollary 2.2], these are complex Hilbert spaces, and according to [17, Corollary 2.3], real analytic functions are dense in them. We will also use $\mathcal{H}_\Lambda^\infty = \bigcap_{k \in \mathbb{R}} \mathcal{H}_\Lambda^k$.

Later on, we will work with G's of order $\log\langle\xi\rangle$ instead of $\langle\xi\rangle$. In that case, the spaces $\mathcal{H}_\Lambda^k$ only contain distributions, and we can replace $\mathcal{O}((\mathcal{M})_\delta)'$ by $\mathcal{D}'(\mathcal{M})$ in the definition. We will thus avoid completely the discussion of hyperfunctions. However, the reader should keep in mind that as $h \to 0$, the distributions in the space $\mathcal{H}_\Lambda^k$ will potentially be of order C/h. The norm on $\mathcal{H}_\Lambda^k$ is given by

$$\text{(15)} \qquad \|u\|_{\mathcal{H}_\Lambda^k}^2 = \int_\Lambda |T_\Lambda u|^2 \, \langle|\alpha|\rangle^{2k} \, e^{-\frac{2H}{h}} \, \mathrm{d}\alpha$$

for $u \in \mathcal{D}'(\mathcal{M})$.

By restricting the kernel of S to $\mathcal{M} \times \Lambda$, we get an operator S_Λ that turns functions on Λ growing at a subexponential rate into elements of $\mathcal{O}((\mathcal{M})_\delta)'$. Moreover, the composition $S_\Lambda T_\Lambda$ is well-defined, and is the identity operator [17, Lemma 2.7]. The structure of the operator $\Pi_\Lambda \coloneqq T_\Lambda S_\Lambda$ is given in [17, Lemmata 2.9 and 2.10]. In particular [17, Proposition 2.4], Π_Λ is bounded on $L_k^2(\Lambda)$ for every $k \in \mathbb{R}$.

The motivation for working with such a Λ is the following central result (this is a particular case of [17, Theorem 7]).

Proposition 4.1 (Multiplication formula). *Let P be an analytic semiclassical differential operator of order $m \in \mathbb{N}$ on $\mathcal{M}$. Assume that G in the definition of Λ is small enough. Then there is a symbol q of order m on Λ such that for every $u \in \mathcal{H}_\Lambda^{m/2}$ we have*

$$\int_\Lambda T_\Lambda(Pu)\overline{T_\Lambda u}e^{-\frac{2H}{h}}\mathrm{d}\alpha = \int_\Lambda q\,|T_\Lambda u|^2\,e^{-\frac{2H}{h}}\,\mathrm{d}\alpha + \mathcal{O}(h^\infty).$$

More precisely the remainder is smaller than $C_N h^N \|u\|_{\mathcal{H}_\Lambda^{(m-N)/2}}^2$ for $N \geq 0$.

Additionally, up to a $\mathcal{O}(h)$ in the space of Kohn–Nirenberg symbols of order $m - 1$ on Λ, the symbol q is the restriction to Λ of the holomorphic continuation of the principal symbol of P.

In order to prove Proposition 2.5, we would like to apply this formula to the generator of the geodesic flow on SM_e. However, once SM_e is embedded into a closed real-analytic manifold, the generator of the geodesic flow is only analytic near SM_e, so that we need a local version of the multiplication formula. We explain now how this difficulty can be overcome.

4.1.2. Pseudo-locality properties. The pseudo-locality property of the FBI transform is the first thing to notice.

Lemma 4.2. *Let K_0 and K_1 be compact subsets of $\mathcal{M}$ that do not intersect. There exist $C_0 > 0$ and a conic neighbourhood V of $T^*_{K_1}\mathcal{M}$ in $(T^*\mathcal{M})_\epsilon$ such that, for every $N \in \mathbb{R}$, there is a constant C such that for every $u \in H^{-N}(\mathcal{M})$ supported in K_0 and $\alpha \in V$, we have*

$$|Tu(\alpha)| \leq C \|u\|_{H^{-N}} \exp\left(-\frac{\langle|\alpha|\rangle}{C_0 h}\right).$$

Proof. This is an immediate consequence of the assumption that the kernel K_T of T is small away from the diagonal (the second assumption on K_T). $\quad\square$

This consequence of Lemma 4.2 will be used several times.

Lemma 4.3. *Let K_0 be a compact subset of $\mathcal{M}$. Let K_1 be a compact subset of $\mathcal{M}$ that does not intersect K_0. Assume that G is a small enough symbol of order 1 on $(T^*\mathcal{M})_\epsilon$ and let $\Lambda = \Lambda_G$. Let $N, k \in \mathbb{R}$. There is a constant $C > 0$ such that for every $u \in H^{-N}(\mathcal{M})$ supported in K_0 we have*

$$\int_{\Lambda \cap (T^*_{K_1}\mathcal{M})_\epsilon} \langle|\alpha|\rangle^{2k} |Tu(\alpha)|^2 \, e^{-\frac{2H(\alpha)}{h}} \, \mathrm{d}\alpha \leq C \|u\|^2_{H^{-N}} \exp\left(-\frac{1}{Ch}\right).$$

We end this section with the adaptation of Proposition 4.1 to differential operators that are analytic only on a part of $\mathcal{M}$. We use here the norm introduced in (15).

Proposition 4.4. *Let P be a semi-classical differential operator of order m on $\mathcal{M}$. Let K_0 be a compact subset of $\mathcal{M}$. Assume that P has real-analytic coefficients on a neighbourhood of K_0. Assume that G is a small enough symbol of order 1 on $(T^*\mathcal{M})_\epsilon$ and let $\Lambda = \Lambda_G$. There exists a Kohn–Nirenberg symbol q of order m on Λ such that for every $N, m_1, m_2 \in \mathbb{R}$ such that $m = m_1 + m_2 - 1$, there exists a constant $C > 0$ such that for every $u \in H^{-N}(\mathcal{M}) \cap \mathcal{H}^\infty_\Lambda$ supported in K_0 we have*

$$\left|\int_\Lambda T_\Lambda(Pu)\overline{T_\Lambda u}e^{-\frac{2H}{h}} \, \mathrm{d}\alpha - \int_\Lambda q \, |Tu|^2 \, e^{-\frac{2H}{h}} \, \mathrm{d}\alpha\right|$$

$$\leq C \exp\left(-\frac{1}{Ch}\right) \|u\|^2_{H^{-N}} + Ch \|u\|_{\mathcal{H}^{m_1}_\Lambda} \|u\|_{\mathcal{H}^{m_2}_\Lambda}.$$

Moreover, near $T^*_{K_0}\mathcal{M}$, the symbol q is the restriction of the principal symbol of P to Λ.

In order to prove Proposition 4.4, let $\chi \in C^\infty((\mathcal{M})_\epsilon)$ be such that $\chi \equiv 1$ on a neighbourhood of K_0, and P has a holomorphic extension on a neighbourhood of the support of χ. We identify χ with the function $\alpha \mapsto \chi(\alpha_x)$ on $(T^*\mathcal{M})_\epsilon$, and with the corresponding multiplication operator. Proposition 4.4 will be deduced from the two following lemmas.

Lemma 4.5. *Under the assumption of Proposition 4.4, we have*

$$T_\Lambda Pu(\alpha) = \chi T_\Lambda PS_\Lambda \chi T_\Lambda u(\alpha) + \mathcal{O}\left(\exp\left(-\frac{\langle|\alpha|\rangle}{Ch}\right)\|u\|_{H^{-N}}\right),$$

for $\alpha \in \Lambda$.

Lemma 4.6. *Under the assumption of Proposition 4.4, let B_Λ denote the orthogonal projector from $L_0^2(\Lambda)$ to its closed subspace $T_\Lambda \mathcal{H}_\Lambda^0$. Then there exists a Kohn–Nirenberg symbol $\tilde{q}$ of order m on Λ such that*

$$B_\Lambda \chi T_\Lambda PS_\Lambda \chi B_\Lambda = B_\Lambda \tilde{q} B_\Lambda + R,$$

where R is $\mathcal{O}(h^\infty)$ as an operator from $L_k^2(\Lambda)$ to $L_{-k}^2(\Lambda)$ for every $k \in \mathbb{R}$. Moreover, $\tilde{q} = \chi^2 q \mod h S_{KN}^{m-1}(\Lambda)$ where q is the restriction of the principal symbol of P to Λ near $T_{K_0}^*\mathcal{M}$.

Before proving Lemmas 4.5 and 4.6, let us explain how they can be used to prove Proposition 4.4.

Proof of Proposition 4.4. Applying Lemma 4.5 and then Lemma 4.6, we find that:

$$\int_\Lambda T_\Lambda Pu.\overline{T_\Lambda u}e^{-\frac{2H}{h}}\,\mathrm{d}\alpha$$

$$= \int_\Lambda \chi T_\Lambda PS_\Lambda \chi T_\Lambda u.\overline{T_\Lambda u}e^{-\frac{2H}{h}}\,\mathrm{d}\alpha + \mathcal{O}\left(\exp\left(-\frac{1}{Ch}\right)\|u\|_{H^{-N}}^2\right)$$

$$= \int_\Lambda \chi T_\Lambda PS_\Lambda \chi B_\Lambda T_\Lambda u.\overline{B_\Lambda T_\Lambda u}e^{-\frac{2H}{h}}\,\mathrm{d}\alpha + \mathcal{O}\left(\exp\left(-\frac{1}{Ch}\right)\|u\|_{H^{-N}}^2\right)$$

$$= \int_\Lambda B_\Lambda \chi T_\Lambda PS_\Lambda \chi B_\Lambda T_\Lambda u.\overline{T_\Lambda u}e^{-\frac{2H}{h}}\,\mathrm{d}\alpha + \mathcal{O}\left(\exp\left(-\frac{1}{Ch}\right)\|u\|_{H^{-N}}^2\right)$$

$$= \int_\Lambda B_\Lambda \tilde{q} B_\Lambda T_\Lambda u.\overline{T_\Lambda u}e^{-\frac{2H}{h}}\,\mathrm{d}\alpha$$

$$+ \mathcal{O}\left(\exp\left(-\frac{1}{Ch}\right)\|u\|_{H^{-N}}^2 + h^\infty \|u\|_{\mathcal{H}_\Lambda^{m_1}}\|u\|_{\mathcal{H}_\Lambda^{m_2}}\right)$$

$$= \int_\Lambda \tilde{q} \, |T_\Lambda u|^2 \, e^{-\frac{2H}{h}} \, d\alpha$$

$$+ \mathcal{O}\left(\exp\left(-\frac{1}{Ch}\right) \|u\|_{H^{-N}}^2 + h^\infty \|u\|_{\mathcal{H}_\Lambda^{m_1}} \|u\|_{\mathcal{H}_\Lambda^{m_2}} \right)$$

$$= \int_\Lambda q \, |T_\Lambda u|^2 \, e^{-\frac{2H}{h}} \, d\alpha$$

$$+ \mathcal{O}\left(\exp\left(-\frac{1}{Ch}\right) \|u\|_{H^{-N}}^2 + h \|u\|_{\mathcal{H}_\Lambda^{m_1}} \|u\|_{\mathcal{H}_\Lambda^{m_2}} \right).$$

On the first line, we have not used $u \in \mathcal{H}_\Lambda^\infty$, but rather that $T_\Lambda u(\alpha)$ is at most of the size $\mathcal{O}(e^{\delta \langle |\alpha| \rangle / h} \|u\|_{H^{-N}})$ on Λ, with δ that can be made arbitrarily small by taking G small enough (as follows from a direct inspection of K_T). □

Proof of Lemma 4.5. We start by applying Lemma 4.2 to find that, for $u \in H^{-N}(\mathcal{M}) \cap \mathcal{H}_\Lambda^\infty$ supported in K_0,

$$T_\Lambda P u(\alpha) = \chi T_\Lambda P u(\alpha) + \mathcal{O}\left(\exp\left(-\frac{\langle |\alpha| \rangle}{Ch}\right) \|u\|_{H^{-N}} \right)$$

$$= \chi T_\Lambda P S_\Lambda T_\Lambda u(\alpha) + \mathcal{O}\left(\exp\left(-\frac{\langle |\alpha| \rangle}{Ch}\right) \|u\|_{H^{-N}} \right)$$

$$= \chi T_\Lambda P S_\Lambda \chi T_\Lambda u(\alpha) + \chi T_\Lambda P S_\Lambda (1 - \chi) T_\Lambda u(\alpha)$$

$$+ \mathcal{O}\left(\exp\left(-\frac{\langle |\alpha| \rangle}{Ch}\right) \|u\|_{H^{-N}} \right).$$

Now, we claim that $\chi T_\Lambda P S_\Lambda (1 - \chi) T_\Lambda u(\alpha)$ can in fact be put in the error term. The Schwartz kernel of the operator $\chi T_\Lambda P S_\Lambda (1 - \chi)$ is given by

$$\chi(\alpha_x)(1 - \chi(\beta_x)) \int_\mathcal{M} K_T(\alpha, y) P_y(K_S(y, \beta)) dy.$$

We can split the integral here into the y's that are close to α_x and those that are not. If y is away from α_x then by assumption the kernel $K_T(\alpha, y)$ is $\mathcal{O}\left(\exp(-\langle |\alpha| \rangle / Ch)\right)$. On the other hand $K_S(y, \beta)$ is always $\mathcal{O}(\exp(\delta \langle \beta \rangle / h))$ where $\delta > 0$ can be chosen arbitrarily small by imposing that β remains close to real (which will be verified if G is small enough). Hence, the y's away from α_x contribute to the kernel of $\chi T_\Lambda P S_\Lambda (1 - \chi)$ by an

(16) $$\mathcal{O}\left(\exp\left(-\frac{\langle |\alpha| \rangle}{Ch} + \frac{\delta \langle |\beta| \rangle}{h}\right) \right).$$

When y is close to α_x, then $P_y(K_S(y, \beta))$ is analytic in y, with a holomorphic extension bounded by $\mathcal{O}(\exp(\frac{\delta\langle\beta\rangle}{h}))$ on a neighbourhood of α_x of fixed size. Hence, we can apply the non-stationary phase method [17, Proposition 1.1], as in the proof of [17, Lemma 2.4] for instance (see the estimate on $T^D u$ defined by (2.25) there), to find that the y's close to α contribute to the kernel of $\chi T_\Lambda P S_\Lambda (1 - \chi)$ by a term of the same size as (16). The kernel itself is then of size at most (16). Using Lemma 4.2 to estimate the size of $Tu(\beta)$ when $\chi(\beta) \neq 1$, and choosing $\delta > 0$ small enough, we find indeed that $\chi T_\Lambda P S_\Lambda (1 - \chi) T_\Lambda u(\alpha)$ is an $\mathcal{O}(\exp(-\frac{\langle|\alpha|\rangle}{Ch}) \|u\|_{H^{-N}})$, so that we have

$$(17) \quad T_\Lambda P u(\alpha) = \chi T_\Lambda P S_\Lambda \chi T_\Lambda u(\alpha) + \mathcal{O}\left(\exp\left(-\frac{\langle|\alpha|\rangle}{Ch}\right) \|u\|_{H^{-N}}\right). \qquad \square$$

Proof of Lemma 4.6. Lemma 4.6 is the equivalent of [17, Proposition 2.10]. The proof of the latter can be rewritten here without changing a word of it, once we prove that the operator $\chi T_\Lambda P S_\Lambda \chi$ satisfies [17, Lemma 2.9 and 2.10] (see also [17, Remark 2.15]). Thus, let us explain how to prove that the operator $\chi T_\Lambda P S_\Lambda \chi$ satisfies [17, Lemma 2.9 and 2.10].

The kernel of the operator $\chi T_\Lambda P S_\Lambda \chi$ is given for $\alpha, \beta \in (T^*\mathcal{M})_\epsilon$ by

$$\chi T_\Lambda P S_\Lambda \chi(\alpha, \beta) = \chi(\alpha_x)\chi(\beta_x) \int_{\mathcal{M}} K_T(\alpha, y) P_y(K_T(y, \beta)) \mathrm{d}y.$$

Since the kernel $K_T(\alpha, y)$ and $K_S(y, \beta)$ are negligible when y is away from α_x and β_x, the y's in this integral that are away from the support of χ only contributes by a negligible term: if U is a neighbourhood of $\mathrm{supp}\,\chi \cap \mathcal{M}$, then there is $\epsilon' > 0$ such that

$$(18)$$
$$\chi T_\Lambda P S_\Lambda \chi(\alpha, \beta) = \chi(\alpha_x)\chi(\beta_x) \int_U K_T(\alpha, y) P_y(K_T(y, \beta)) \mathrm{d}y$$
$$+ \mathcal{O}\left(\exp\left(-\frac{\langle|\alpha|\rangle + \langle|\beta|\rangle}{Ch}\right)\right),$$

for $\alpha, \beta \in (T^*\mathcal{M})_{\epsilon'}$. The remaining integral over U only involves real-analytic functions that are exponentially small on the boundary of U, and can consequently be dealt with following exactly the same steps as in the proof of Lemmas 2.9 and 2.10 in [17] in the analytic case pp.110–111 (here we do not need the much more complicated proof required for the Gevrey case). For the convenience of the reader, we recall here the main lines of the argument.

The equivalent of [17, Lemma 2.9] is that the kernel of $\chi T_\Lambda P S_\Lambda \chi$ is negligible away from the diagonal: for every $\eta > 0$, there are $C, \epsilon' > 0$ such that if $\alpha, \beta \in (T^*\mathcal{M})_{\epsilon'}$ are such that $d_{KN}(\alpha, \beta) \geq \eta/2$ then

$$\chi T_\Lambda P S_\Lambda \chi(\alpha, \beta) = \mathcal{O}\left(\exp\left(-\frac{\langle|\alpha|\rangle + \langle|\beta|\rangle}{Ch}\right)\right).$$

The proof of this fact is very similar to the proof of Proposition 4.7 given in Appendix A. Notice that we only need to consider the case of α_x and β_x in the support of χ. We will take $\epsilon' \ll \eta \ll 1$ and let $\tilde{\alpha}_x$ and $\tilde{\beta}_x$ denote elements of $\mathcal{M}$ that are at distance at most ϵ' of α_x and β_x respectively. If the distance between α_x and β_x is larger than $\eta/4$, we write

$$\int_U K_T(\alpha, y) P_y(K_T(y, \beta)) \mathrm{d}y$$

$$= \int_{D(\tilde{\alpha}_x, \eta/100)} K_T(\alpha, y) P_y(K_T(y, \beta)) \mathrm{d}y$$

$$+ \int_{D(\tilde{\beta}_x, \eta/100)} K_T(\alpha, y) P_y(K_T(y, \beta)) \mathrm{d}y$$

$$+ \int_{U \setminus (D(\tilde{\alpha}_x, \eta/100) \cup D(\tilde{\beta}_x, \eta/100))} K_T(\alpha, y) P_y(K_T(y, \beta)) \mathrm{d}y.$$

The third integral is negligible because $K_T(\alpha, y)$ and $K_S(y, \beta)$ are negligible when y is away from α_x and β_x (this is the second assumption we made on K_T). The other two integral are dealt with similarly. For the first one for instance, we use the local behaviour of K_T to rewrite it, up to negligible terms, as

$$(19) \qquad \int_{D(\tilde{\alpha}_x, \eta/100)} e^{i\frac{\Phi_T(\alpha, y)}{h}} a(\alpha, y) P_y(K_T(y, \beta)) \mathrm{d}y.$$

Here, since α_x is away from β_x, the function $y \mapsto P_y(K_T(y, \beta))$ is analytic with a holomorphic extension bounded by $\mathcal{O}(\exp(-\langle|\beta|\rangle/Ch))$ on a neighbourhood of fixed size of the domain of integration. Moreover, for η and ϵ' small enough the phase Φ_T is non-stationary (as $\mathrm{d}_y\Phi_T(\alpha, \alpha_x) = -\alpha_\xi$) and has positive imaginary part on the boundary of the domain of integration. We can consequently use the non-stationary phase method [17, Proposition 1.5] to see that the integral (19) is indeed negligible.

If α_x and β_x are close, but α_ξ and β_ξ are away from each other, we can use local coordinates and rewrite up to negligible (exponentially decaying)

terms
(20)
$$\int_U K_T(\alpha, y) P_y(K_T(y, \beta)) \mathrm{d}y \simeq \int_{D(\tilde{\alpha}_x, \eta/100)} e^{i\frac{\Phi_T(\alpha, y) + \Phi_S(y, \beta)}{h}} a(\alpha, y)\tilde{b}(y, \beta)\mathrm{d}y,$$

where $\tilde{b}$ is an analytic symbol. Since $\mathrm{d}_y\Phi_T(\alpha, \alpha_x) = -\alpha_\xi$ and $\mathrm{d}_y\Phi_S(\beta_x, \beta) = \beta_\xi$, we see that when α_ξ and β_ξ are away from each other, the phase in (20) is non-stationary again, and the non-stationary phase method gives that (20) is negligible (see the proof of Proposition 4.7 in Appendix A for a similar argument).

The equivalent of [17, Lemma 2.10] in our context is a description of the structure of the kernel of $\chi T_\Lambda P S_\Lambda \chi$ near the diagonal: there are small $\eta > 0$ and $\epsilon' > 0$, such that for $\alpha, \beta \in (T^*\mathcal{M})_{\epsilon'}$ with $d_{KN}(\alpha, \beta) < \eta$ we have, up to exponentially decaying terms

$$(21) \qquad \chi T_\Lambda P S_\Lambda \chi(\alpha, \beta) \simeq e^{i\frac{\Phi_{TS}(\alpha, \beta)}{h}} e(\alpha, \beta),$$

where $\Phi_{TS}(\alpha, \beta)$ is the critical value of $y \mapsto \Phi_T(\alpha, y) + \Phi_S(y, \beta)$ and e is a Kohn–Nirenberg symbol supported near the diagonal. Moreover, e is given at first order on the diagonal by

$$e(\alpha, \alpha) = \frac{1}{(2\pi h)^n}\chi(\alpha)^2 p(\alpha) + \mathcal{O}\left(h^{-n+1}\langle|\alpha|\rangle^{m-1}\right),$$

where p is the (holomorphic extension of the) principal symbol of P. In order to prove (21), we just notice that the only relevant term in (18) is the integral over U, and since α and β are close to each other, this integral is given up to negligible terms by (20). Then, we observe that when $\alpha = \beta$ the phase $y \mapsto \Phi_T(\alpha, y) + \Phi_S(y, \beta)$ has a non-degenerate critical point at $y = \alpha_x = \beta_x$. Moreover, provided $\epsilon' \ll \eta \ll 1$, the phase is positive on the boundary of the domain of integration. We can consequently apply the holomorphic stationary phase method [17, Poposition 1.6] (see also [40, Théorème 2.8 and Remarque 2.10]) to get (21). $\qquad\square$

4.2. Some properties of the analytic wavefront set

4.2.1. General facts about analytic wave front set. We will use the notion of analytic wave front set $\mathrm{WF}_a(u)$ of a distribution u on a real-analytic manifold. See for instance [25, §8.4 & 8.5], [40, §6] or [24] for the definition and basic properties of this notion, the definition from [40] is recalled in Appendix A. These references define differently the analytic wave front set, but

all the classical definitions of the analytic wave front set coincide according to a result of Bony [1]. Notice also that these references deal mainly with the case of distributions on open subsets of $\mathbb{R}^n$, but the case of distributions on a real-analytic manifold follows immediately since the notion is invariant by real-analytic change of coordinates.

Let us start by recalling that if u is a distribution on a real-analytic manifold $\mathcal{M}$ then $\mathrm{WF}_a(u)$ is a closed conic subset of $T^*\mathcal{M} \setminus \{0\}$ whose projection to $\mathcal{M}$ is the analytic singular support of u, that is the set of points $x \in \mathcal{M}$ such that u is not analytic on a neighbourhood of x (see for instance [25, Theorem 8.4.5] or [40, Theorem 6.3]).

In this paper, we will use the following characterization of the analytic wave front set in term of the FBI transform. There is no surprise in the proof of this theorem, that can be found in Appendix A. We use here the same notation as in §4.1, in particular $\mathcal{M}$ is a closed real-analytic manifold, and T an analytic FBI transform on $\mathcal{M}$.

Proposition 4.7. *Let $u \in \mathcal{D}'(\mathcal{M})$ be independent of h. Let $\alpha_0 \in T^*\mathcal{M} \setminus \{0\}$. Then the following assertions are equivalent:*

   (i) *α_0 does not belong to $\mathrm{WF}_a(u)$;*
   (ii) *there is neighbourhood $\Omega \subseteq T^*\mathcal{M}$ of α_0 and a constant $C > 0$ such that for h small enough and $\alpha \in \Omega$ we have*

$$(22) \qquad |Tu(\alpha)| \leq C \exp\left(-\frac{1}{Ch}\right).$$

 (iii) *there is conic neighbourhood $\Omega \subseteq T^*\mathcal{M}$ of α_0 and a constant $C > 0$ such that for h small enough and $\alpha \in \Omega$ large enough we have*

$$(23) \qquad |Tu(\alpha)| \leq C \exp\left(-\frac{\langle \alpha \rangle}{Ch}\right).$$

Remark 1. One could add a fourth equivalent property in Proposition 4.7 by letting α tends to $+\infty$ while h is fixed. Using this characterization of the analytic wave front set, one could prove without much trouble the basic properties of the analytic wave front set (behaviour under pull-back and push-forward for instance). Some other standard properties of the analytic wave front set, such as the elliptic estimates and propagation of singularities may also be proved using the FBI transform, in the spirit of the proof of Propositions 4.13 and 4.14 below.

We will also need the following result, which follows easily by using Sjöstrand's definition [40, Definition 6.1]. See Appendix A for a proof.

Lemma 4.8. *Let $u \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$. Let $(x, y) \in \mathcal{M} \times \mathcal{M}$. Assume that u is C^∞ near (x, y) and that u is analytic in the first variable, uniformly in the second variable, on a neighbourhood of (x, y). Then $\mathrm{WF}_a(u) \cap T_x^* \mathcal{M} \times T_y^* \mathcal{M} \subseteq \{0\} \times T_y^* \mathcal{M}$.*

4.2.2. Lagrangian deformation and analytic regularity. We explain now how the spaces $\mathcal{H}_\Lambda^k$ from §4.1 associated with a complex Lagrangian Λ are related to regularity issues. In order to do so, we will use a Lagrangian Λ defined by (14) using a function G which is a symbol of logarithmic order, meaning that in local coordinates $\tilde{x} = x + iy$ and $\tilde{\xi} = \xi + i\eta$, for every $\gamma, \delta, \rho, \sigma \in \mathbb{N}^n$ there is a constant $C_{\gamma,\delta,\rho,\sigma}$ such that

$$\left| \partial_x^\gamma \partial_y^\delta \partial_\xi^\rho \partial_\eta^\sigma G(\tilde{x}, \tilde{\xi}) \right| \leq C_{\gamma,\delta,\rho,\sigma} (1 + |\xi|)^{-|\rho|-|\sigma|} \log\left(2 + |\xi|\right)$$

for $\tilde{x}$ in a coordinate patch for $(\mathcal{M})_\epsilon$ and $|\eta| \leq \epsilon(1 + |\xi|)$.

In particular, G will be of order 1, so that the theory from [17] will apply, but since Λ is now much closer to $T^*\mathcal{M}$, the spaces $\mathcal{H}_\Lambda^k$ will contain $C^\infty(\mathcal{M})$ (see Lemma 4.10). However, G will not depend on h, so that a distribution that belongs to $\mathcal{H}_\Lambda^0$ with some uniform estimates in h must have its analytic wave front set localized outside of the set of negative ellipticity of G (see Lemma 4.12). To put it shortly, we are doing C^∞ microlocal analysis from the classical point of view, but real-analytic microlocal analysis in the semiclassical perspective.

Notice that an advantage of working with a symbol of logarithmic order G is that the assumption that G is small enough as a symbol of order 1 (needed to apply the results from [17]) can easily be satisfied by multiplying G by a function that vanishes on a large neighbourhood of 0 and takes value 1 near infinity. As we do not care about the value of G near 0 in the applications below, this is particularly practical for us.

We start with an estimate which will be useful to discard the part of the phase space that is not of interest for our purpose.

Lemma 4.9. *Let G be a symbol of logarithmic order on $(T^*\mathcal{M})_\epsilon$. Assume that G is small enough as a symbol of order 1. Let $\Lambda = \Lambda_G$ be the associated Lagrangian defined by (14). Let $\Omega \subseteq T^*\mathcal{M}$ be such that there is $\delta > 0$ such that for $\alpha \in \Omega$ such that $\langle \alpha \rangle \geq \delta^{-1}$, we have:*

$$(24) \qquad\qquad G(\alpha) \geq \delta \log \langle \alpha \rangle .$$

Write $\Omega' = e^{H_G^{\omega_I}} \Omega \subseteq \Lambda$. There is a constant $C_0 > 0$ such that, for every $N, k \in \mathbb{R}$, there is a constant $C > 0$ such that for all h small enough and

every $u \in H^{-N}(\mathcal{M})$ we have

$$(25) \qquad \int_{\Omega'} |T_\Lambda u|^2 \, e^{-\frac{2H}{h}} \, \langle |\alpha| \rangle^{2k} \, \mathrm{d}\alpha \leq C e^{\frac{C_0}{h}} \, \|u\|_{H^{-N}}^2 \, .$$

Here the constant C_0 can be made arbitrarily small taking G arbitrarily small, but we will not need this. Actually, this growth only comes from small frequencies.

Proof. We can assume that u is real-analytic, as the general result then follows by an approximation argument. Indeed, using for instance the eigenfunctions of the Lapalce operator associated to a real-analytic metric, one can produce a sequence $(u_n)_{n \geq 0}$ of real-analytic functions converging to u in $H^{-N}(\mathcal{M})$. Then if each u_n satisfies (25) (with constants C and C_0 that does not depend on u_n), it follows from Fatou's lemma that u also satisfies (25) since $(T_\Lambda u_n)_{n \geq 0}$ converges pointwise to $T_\Lambda u$. Without loss of generality, we may also assume that N is a positive integer.

Denote by K_{TS} the kernel of TS and its holomorphic extension. Recall that $T_\Lambda u = (Tu)_{|\Lambda}$ where Tu is considered as a function on $(T^*\mathcal{M})_\epsilon$ by holomorphic extension. Then, since u is real-analytic, Tu decays exponentially fast, and we can use the formula

$$(26) \qquad T_\Lambda u(\alpha) = \int_{T^*\mathcal{M}} K_{TS}(\alpha, \beta) Tu(\beta) \mathrm{d}\beta \quad \text{for } \alpha \in \Lambda$$

to bound $T_\Lambda u$. The formula (26) follows from the fact that ST is the identity operator.

According to Lemma 2.9 in [17], given $\eta > 0$, if G is small enough, and $d_{KN}(\alpha, \beta) > \eta$, with $\alpha \in \Lambda$ and $\beta \in T^*\mathcal{M}$, the kernel $K_{TS}(\alpha, \beta)$ is exponentially small in $(\langle |\alpha| \rangle + \langle |\beta| \rangle)/h$. In particular,

$$T_\Lambda u(\alpha) = \int_{\beta \in T^*\mathcal{M}, \ d_{KN}(\alpha, \beta) \leq \eta} K_{TS}(\alpha, \beta) Tu(\beta) \mathrm{d}\beta + \mathcal{O}(e^{-\langle |\alpha| \rangle / Ch} \|u\|_{H^{-N}}).$$

Here we used the fact that $\mathcal{H}_{T^*M}^k$ is the semi-classical Sobolev space of order k (see Corollary 2.4 in [17]).

Let us turn now to the points close to α. Here we can use Lemma 2.10 of [17]. For $\eta > 0$ small enough and $d_{KN}(\alpha, \beta) < \eta$,

$$(27) \qquad K_{TS}(\alpha, \beta) = e^{i \frac{\Phi_{TS}(\alpha, \beta)}{h}} e(\alpha, \beta) + \mathcal{O}\left(\exp\left(-\frac{\langle |\alpha| \rangle}{h} \right) \right),$$

where $h^n e$ is a symbol of order 0 and the phase $\Phi_{TS}(\alpha, \beta)$ is the critical value of $y \mapsto \Phi_T(\alpha, y) + \Phi_S(\beta, y)$. This formula is only stated for $\alpha, \beta \in \Lambda$ in [17, Lemma 2.10], but since we can rely here on the proof given in the analytic case, the formula actually holds as soon as α and β are close enough to the diagonal of $T^*\mathcal{M}$ (for the same reason, e is actually analytic near the diagonal). This is explained at the start of the proof (p.111) and also in the remark before figure 1 therein.

Now using the same argument as before to deal with the remainder, we can concentrate on

$$I(\alpha) := \int_{d_{KN}(\alpha, \beta) < \eta} e^{i\frac{\Phi_{TS}(\alpha, \beta)}{h}} e(\alpha, \beta) Tu(\beta) \mathrm{d}\beta$$

in order to bound $T_\Lambda u(\alpha)$. Choose $R > \delta^{-1}$ large. For $|\alpha| < R$, the integral $I(\alpha)$ is controlled by

$$(28) \qquad\qquad\qquad C e^{C_0/h} \|u\|_{H^{-N}},$$

where the constant C_0 only depends on Λ and R.

Let us assume now that $|\alpha| > R$. Then we estimate the imaginary part of the phase $\Phi_{TS}(\alpha, \beta)$, assuming that $\alpha \in \Omega'$ and $\beta \in T^*\mathcal{M}$ are at distance at most η. Write $\alpha = e^{H_G^{\omega_I}}\gamma$ for some $\gamma \in \Omega$. Using that the imaginary part of $\Phi_{TS}(\gamma, \beta)$ is non-negative (see [17, Lemma 2.13]), we find

$$\mathrm{Im}\,\Phi_{TS}(\alpha, \beta) \geq \mathrm{Im}\left(\Phi_{TS}(e^{H_G^{\omega_I}}\gamma, \beta) - \Phi_{TS}(\gamma, \beta)\right)$$

$$\geq \mathrm{d}_\alpha \,\mathrm{Im}\,\Phi_{TS}(\gamma, \beta) \cdot H_G^{\omega_I}(\gamma) + \mathcal{O}\left(\frac{(\log \langle |\alpha| \rangle)^2}{\langle |\alpha| \rangle}\right)$$

$$\geq \mathrm{d}_\alpha \,\mathrm{Im}\,\Phi_{TS}(\alpha, \alpha) \cdot H_G^{\omega_I}(\alpha) + \mathcal{O}\left(\frac{(\log \langle |\alpha| \rangle)^2}{\langle |\alpha| \rangle} + \eta \log \langle |\alpha| \rangle\right)$$

$$\geq \mathrm{Im}(\theta(H_G^{\omega_I})(\alpha)) + \mathcal{O}\left(\frac{(\log \langle |\alpha| \rangle)^2}{\langle |\alpha| \rangle} + \eta \log \langle |\alpha| \rangle\right).$$

Here, we used the fact that G has logarithmic order (which implies that both $H_G^{\omega_I}$ and the distance between α and γ are $\mathcal{O}(\log\langle |\alpha| \rangle / \langle |\alpha| \rangle)$) and that $\mathrm{d}_\alpha \Phi_{TS}(\alpha, \alpha) = \theta$, where θ is the canonical 1-form on the cotangent bundle of the complexification $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ (see [17, (2.73)]). We recall the explicit formula [17, (2.9)] for the weight H:

$$H = \int_0^1 \left(e^{(\tau-1)H_G^{\omega_I}}\right)^* (G - \mathrm{Im}\,\theta(H_G^{\omega_I})) \mathrm{d}\tau,$$

in order to see that

$$H(\alpha) = G(\alpha) - \text{Im}(\theta(H_G^{\omega_I})(\alpha)) + \mathcal{O}\left(\frac{(\log\langle|\alpha|\rangle)^2}{\langle|\alpha|\rangle}\right)$$
$$= G(\gamma) - \text{Im}(\theta(H_G^{\omega_I})(\alpha)) + \mathcal{O}\left(\frac{(\log\langle|\alpha|\rangle)^2}{\langle|\alpha|\rangle}\right).$$

It follows, using assumption (24), that

$$\text{Im}\,\Phi_{TS}(\alpha,\beta) \geq G(\gamma) - H(\alpha) + \mathcal{O}\left(\frac{(\log\langle|\alpha|\rangle)^2}{\langle|\alpha|\rangle} + \eta\log\langle|\alpha|\rangle\right)$$
$$\geq \delta\log\langle|\alpha|\rangle - H(\alpha) + \mathcal{O}\left(\frac{(\log\langle|\alpha|\rangle)^2}{\langle|\alpha|\rangle} + \eta\log\langle|\alpha|\rangle\right)$$
$$\geq \frac{\delta}{2}\log\langle|\alpha|\rangle - H(\alpha),$$

provided η is small enough and R is large enough. We insert this estimate in our formula

$$|I(\alpha)| \leq e^{\frac{H(\alpha)}{h}}\int_{d_{KN}(\alpha,\beta)<\eta} e^{-\frac{\delta\log\langle|\alpha|\rangle}{2h}}|e(\alpha,\beta)||Tu(\beta)|\mathrm{d}\beta,$$
$$\leq e^{\frac{H(\alpha)}{h}}\langle|\alpha|\rangle^{-\frac{\delta}{2h}}h^{-n}\int_{d_{KN}(\alpha,\beta)<\eta}|Tu(\beta)|\mathrm{d}\beta,$$
$$\leq e^{\frac{H(\alpha)}{h}}\langle|\alpha|\rangle^{\frac{n}{2}+N-\frac{\delta}{2h}}h^{-n-N}\|u\|_{H^{-N}}.$$

Since, $T_\Lambda u(\alpha)$ coincides with $I(\alpha)$ up to exponentially decaying terms, we deduce that

$$\int_{\alpha\in\Omega',\,|\alpha|>R}|T_\Lambda u(\alpha)|^2 e^{-2\frac{H(\alpha)}{h}}\langle|\alpha|\rangle^{2k}\mathrm{d}\alpha \leq C_N e^{-1/Ch}\|u\|_{H^{-N}}^2.$$

The result then follows since for low frequencies $e^{-H(\alpha)/h}|T_\Lambda u(\alpha)|$ is bounded by (28). $\qquad\square$

The point of working with a logarithmic weight G is that C^∞ regularity is enough to prove that a distribution belongs to $\mathcal{H}_\Lambda^k$, as expressed in the following lemma. However, notice that the value of the integral in (29) could *a priori* blows up when h tends to 0. The main point in the proof of Proposition 4.13 below is to prove that it does not happen.

Lemma 4.10. *Let G be a symbol of logarithmic order on $(T^*\mathcal{M})_\epsilon$. Assume that G is small enough as a symbol of order 1. Let Ω be a closed conic subset*

of $T^*\mathcal{M}$. Let $\Omega' = e^{H_G^{\omega_I}}\Omega$. Let $u \in \mathcal{D}'(\mathcal{M})$ be such that $\mathrm{WF}(u) \cap \Omega = \emptyset$, where $\mathrm{WF}(u)$ denotes the C^∞ wave front set of u. Then, for every $k \in \mathbb{R}$, and $h > 0$ small enough

$$(29) \qquad \int_{\Omega'} |T_\Lambda u|^2 \, e^{-\frac{2H}{h}} \, \langle |\alpha| \rangle^{2k} \, d\alpha < +\infty.$$

Proof. Since the wave front set of u does not intersect Ω, it follows for instance from [50, Theorem 4.8] that for every $N > 0$, there is a constant $C_N > 0$ such that for $\alpha \in \Omega$, we have

$$(30) \qquad |Tu(\alpha)| \le C_N h^N \, \langle |\alpha| \rangle^{-N}.$$

Then, we can estimate the imaginary part of $\Phi_{TS}(\alpha, \beta)$ as in the proof of Lemma 4.9. Since we do not make an ellipticity assumption on G anymore, we only find that $\mathrm{Im}\,\Phi_{TS}(\alpha, \beta) + H(\alpha)$ is a $\mathcal{O}(\log\langle |\alpha| \rangle)$ for $\alpha \in \Lambda$ and $\beta \in T^*\mathcal{M}$ close to each other. Hence, using (26) to estimate $e^{-\frac{H}{h}} |T_\Lambda u|$, we find that there is a constant C_0, that does not depend on N (while C_N does), such that for $\alpha \in \Omega'$ large, we have

$$(31) \qquad e^{-\frac{H(\alpha)}{h}} |T_\Lambda u(\alpha)| \le C_N \, \langle |\alpha| \rangle^{\frac{C_0}{h} - N}.$$

The bound (29) follows by taking N large enough (depending on h). $\qquad\square$

If we consider the analytic wave front set instead of the C^∞ wave front set, then the dependence on h of the bound in Lemma 4.10 becomes explicit.

Lemma 4.11. *Let G be a symbol of logarithmic order on $(T^*\mathcal{M})_\epsilon$. Assume that G is small enough as a symbol of order 1. Let Ω be a closed conic subset of $T^*\mathcal{M}$. Let $\Omega' = e^{H_G^{\omega_I}}\Omega$. Let $u \in \mathcal{D}'(\mathcal{M})$ be such that $\mathrm{WF}_a(u) \cap \Omega = \emptyset$. Then for every $k \in \mathbb{R}$ there is a constant $C > 0$ such that, for h small enough, we have*

$$\int_{\Omega'} |T_\Lambda u|^2 \, e^{-\frac{2H}{h}} \, \langle |\alpha| \rangle^{2k} \, d\alpha \le C e^{\frac{C}{h}}.$$

Proof. The proof is the same as for Lemma 4.10 with (30) replaced by (23). Then (31) becomes

$$e^{-\frac{H(\alpha)}{h}} |Tu(\alpha)| \le C \exp\left(\frac{C_0}{h} \log\langle |\alpha| \rangle - \frac{C_1}{h} \langle |\alpha| \rangle \right),$$

for some $C, C_0, C_1 > 0$ and $\alpha \in \Omega'$ large enough. The result follows (small frequencies give rise at most to an $\mathcal{O}(e^{C/h})$). $\qquad\square$

Finally, we give a result that allows to get regularity estimates from estimates in the space $\mathcal{H}_\Lambda^0$.

Lemma 4.12. *Let G be a symbol of logarithmic order. Assume that G is small enough as a symbol of order 1. Let Ω be a conic subset of $T^*\mathcal{M} \setminus \{0\}$. Assume that there is $\delta > 0$ such that for every $\alpha \in \Omega$ large enough we have*

$$G(\alpha) \leq -\delta \log \langle \alpha \rangle .$$

Let $u \in \mathcal{D}'(\mathcal{M})$ not depend on h. Assume that $u \in \mathcal{H}_\Lambda^0$ and that there is a constant $C > 0$ such that, for h small enough, we have

$$\|u\|_{\mathcal{H}_\Lambda^0} \leq C \exp\left(\frac{C}{h}\right),$$

then the analytic wave front set of u does not intersect Ω.

Proof. We will work as in the proofs of Lemmas 4.9 and 4.10, except that we go from Λ to $T^*\mathcal{M}$ instead of going from $T^*\mathcal{M}$ to Λ. The formula (26) becomes for $\alpha \in \Omega$

$$(32) \qquad Tu(\alpha) = \int_\Lambda K_{TS}(\alpha, \beta) T_\Lambda u(\beta) \mathrm{d}\beta$$

This formula is deduced from (26) by a contour shift, and the convergence is ensured by the decay of K_{TS} away from the diagonal, and the estimate we assumed on $T_\Lambda u$. Then, we work as in the proof of Lemma 4.9 to estimate the imaginary part of the phase, taking into account the sign shifts (notice in particular that $\mathrm{d}_\beta \Phi_{TS}(\beta, \beta) = -\theta$, where θ is the canonical 1-form on $(T^*\mathcal{M})_\epsilon$), we find that for $\alpha \in \Omega$ and $\beta \in \Lambda$ close to each other and large enough we have

$$\mathrm{Im}\, \Phi_{TS}(\alpha, \beta) - H(\beta) \geq \frac{\delta}{2} \log \langle |\alpha| \rangle .$$

Since Ω is closed, this estimate actually holds on a conic neighbourhod of Ω (up to taking δ slightly smaller). Neglecting off-diagonal terms as always, it follows from (32) and the fact that $e^{-\frac{H}{h}} Tu$ is square integrable on Λ that

$$(33) \qquad |Tu(\alpha)| \leq C e^{\frac{C}{h} - \frac{\delta}{4} \frac{\log \langle |\alpha| \rangle}{h}}$$

for α large enough in a conic neighbourhood of Ω. Let then Ω_0 be a compact part of $T^*\mathcal{M}$ such that $\Omega \subseteq \bigcup_{t>0} t\Omega_0$ and the elements of Ω_0 are large

enough so that (33) gives that $|Tu(\alpha)|$ is a $\mathcal{O}(e^{-1/Ch})$ for $\alpha \in \Omega_0$. According to Proposition 4.7, the set Ω_0 does not intersect $\mathrm{WF}_a(u)$. The same is true for Ω since $\mathrm{WF}_a(u)$ is conic. □

4.3. Analytic radial estimates

We leave now the generality from §4.1–4.2 to come back to the geometric context of Proposition 2.5. Our analysis will be based on the framework from [10], and we will try to use notations that are coherent with this reference.

We work on the unit tangent bundle SM_e of the extension (M_e, g_e) of (M, g). In order to be consistent with [10], we will write $\overline{\mathcal{U}}$ instead of SM_e. Notice that $\overline{\mathcal{U}}$ is a real-analytic manifold with boundary. We denote the interior of $\overline{\mathcal{U}}$ by $\mathcal{U}$ and its boundary by $\partial\mathcal{U}$. We embed $\overline{\mathcal{U}}$ into a closed real-analytic manifold $\mathcal{M}$. To see that it is possible, notice that the Sasaki metric on $\overline{\mathcal{U}}$ is analytic, hence there is an analytic inward pointing vector field defined on the neighbourhood of $\partial\mathcal{U}$ given by the gradient of the distance to $\partial\mathcal{U}$. Thus, $\partial\mathcal{U}$ admits a collar neighbourhood that identifies with $\partial\mathcal{U} \times [0, \epsilon[$ *via* real-analytic coordinates. We can use this collar neighbourhood to put a real-analytic structure on the double manifold $\mathcal{M}$ of $\overline{\mathcal{U}}$.

As in §4.1–4.2, we endow $\mathcal{M}$ with a real-analytic Riemannian metric (*a priori* unrelated to the metric g_e on M_e) and define an analytic FBI transform T on $\mathcal{M}$. For notational simplicity, we drop in this section the index g_e and denote by X the generator of the geodesic flow on SM_e, by φ_t its flow, by $\Gamma_\pm$ the incoming/outgoing trapped sets, etc. The vector field X is real-analytic on $\overline{\mathcal{U}}$ and satisfies the assumptions (A1)–(A5) from [10] (see [10, §5.2]). As in [10], we extend X to a vector field on $\mathcal{M}$ such that $\mathcal{U}$ and $\overline{\mathcal{U}}$ are convex for the flow of X. In order to have the intermediate results from [10] available, we assume that the extension of X is precisely the one constructed in [10]. There is *a priori* no reason for X to be analytic away from $\overline{\mathcal{U}}$, however, it is clear from the proof of [10, Lemma 1.1] that we can assume that X is analytic on a neighbourhood of $\overline{\mathcal{U}}$. Thanks to Theorem 4.4, this is enough in order to apply the methods from [17].

We will still denote by $(\varphi_t)_{t\in\mathbb{R}}$ the flow generated by X and by $(\Phi_t)_{t\in\mathbb{R}}$ the lift of φ_t to $T^*\mathcal{M}$:

$$\Phi_t(\alpha) = \left(\varphi_t(\alpha_x), (\mathrm{d}\varphi_t(\alpha_x)^{-1})^\top \alpha_\xi \right), \quad \alpha = (\alpha_x, \alpha_\xi) \in T^*\mathcal{M}.$$

We denote by $\widetilde{X}$ the generator of the flow $(\Phi_t)_{t\in\mathbb{R}}$. We also recall that the $\Gamma_\pm = \Gamma_\pm^g$ and $\mathcal{K} = \mathcal{K}^g$ are defined by (1) and (2). Finally, we denote by

p the principal symbol of X, which is also the symbol of the semi-classical differential operator hX:

$$p(\alpha) = i\alpha_\xi(X(\alpha_x)) \quad \text{for } \alpha \in T^*\mathcal{M}.$$

Notice that this formula also defines a holomorphic extension for p.

This section is dedicated to the proof of the following estimate which is crucial in the proof of Proposition 2.5.

Proposition 4.13. *Let u be a distribution on $\mathcal{U}$. Assume that Xu is compactly supported in $\mathcal{U}$, that the analytic wave front set $\mathrm{WF}_a(Xu)$ of Xu does not intersect E_+^*, that the C^∞ wave front set $\mathrm{WF}(u)$ of u does not intersect E_+^* and that $u_{|\partial_+\mathcal{U}} = 0$. Then the analytic wave front set $\mathrm{WF}_a(u)$ of u does not intersect E_+^*.*

Notice that replacing X by $-X$, we also get:

Proposition 4.14. *Let u be a distribution on $\mathcal{U}$. Assume that Xu is compactly supported in $\mathcal{U}$, that the analytic wave front set $\mathrm{WF}_a(Xu)$ of Xu does not intersect E_-^*, that the C^∞ wave front set $\mathrm{WF}(u)$ of u does not intersect E_-^* and that $u_{|\partial_-\mathcal{U}} = 0$. Then the analytic wave front set $\mathrm{WF}_a(u)$ of u does not intersect E_-^*.*

Remark 2. We refer to Propositions 4.13 and 4.14 as radial estimates since their proof relies on the source/sink structure of E_-^* and E_+^* for the Hamiltonian flow $(\Phi_t)_{t\in\mathbb{R}}$. The C^∞ analogues of Propositions 4.13 and 4.14 in the case of Anosov flows follow from the radial estimates [11, Theorems E.52 and E.54], and the case of open systems is dealt with in [10].

The analytic radial estimates that we prove here for open systems also apply to Anosov flows. The same proof using the tools from [17] would also give a Gevrey version of Propositions 4.13 and 4.14. It is also likely that one could deal with more general operators with source/sink structure.

See [15] for a similar statement near some smooth radial submanifolds.

4.3.1. Construction of an escape function. The main tool in the proof of Proposition 4.13 is the construction of an escape function.

Lemma 4.15. *Let Ω_0 be a conic neighbourhood of E_+^* in $T^*\mathcal{M}$ and K_0 be a compact subset of $\mathcal{U}$. There is a symbol G of logarithmic order on $(T^*\mathcal{M})_\epsilon$, a conic neighbourhood $\Omega \subseteq \Omega_0$ of E_+^* in $T^*\mathcal{M}$ and a constant $C > 0$ such that for $\alpha \in T^*\mathcal{M}$ with $\langle\alpha\rangle \geq C$ and $\alpha_x \in K_0$, we have:*

(i) *if $\alpha \notin \Omega$, then $G(\alpha) \geq C^{-1}\log\langle\alpha\rangle$;*

(ii) if $\alpha \in \Omega$, then $-H^{\omega_I}_{\text{Re}\,p} G(\alpha) \leq -C^{-1}$;
(iii) if $\alpha \in E^*_+$, then $G(\alpha) \leq -C^{-1} \log \langle \alpha \rangle$.

The proof of Lemma 4.15 will require the following fact, that follows from [10, Lemma 1.11].

Lemma 4.16. *Let $\kappa : T^*\mathcal{M} \setminus \{0\} \to S^*\mathcal{M}$ be the canonical projection on the cosphere bundle $S^*\mathcal{M}$ of $\mathcal{M}$. Let U_+ be a small enough neighbourhood of $\kappa(E^*_+)$ in $S^*\mathcal{M}$. Then, there is a function $\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ that tends to 0 in $+\infty$ and constants $C, \tilde{\gamma} > 0$ such that, for all $\alpha \in T^*\mathcal{M} \setminus \{0\}$ and $t \geq 0$ such that $\kappa(\alpha) \in U_+, \alpha_x \in \overline{\mathcal{U}}$ and $\varphi_t(\alpha_x) \in \overline{\mathcal{U}}$, we have*

$$d_{S^*\mathcal{M}}(\kappa(\Phi_t(\alpha)), \kappa(E^*_+)) \leq \epsilon(t) \text{ and } |\Phi_t(\alpha)| \geq C^{-1} e^{\tilde{\gamma} t} |\alpha|.$$

Proof. The main difference between Lemma 4.16 and [10, Lemma 1.11] is that we do not require here that α belongs to the kernel of the principal symbol p of X. This restriction can be removed by noticing that the Liouville 1-form α_L is invariant by the flow $(\phi_t)_{t\in\mathbb{R}}$ and $p(\alpha_L) = i$. Hence, we can write $\alpha = \beta + c\alpha_L$ with β in the kernel of p and then apply [10, Lemma 1.11] to β. $\square$

With Lemma 4.16 at our disposal, we can adapt the proof of [10, Lemma 1.12] to prove the following. Here, we let $\widetilde{X}$ act on $S^*\mathcal{M}$ by pushing it forward by the action of κ.

Lemma 4.17. *Let U_+ be a small enough neighbourhood of $\kappa(E^*_+)$ in $S^*\mathcal{M}$. Then, there is $m_+ \in C^\infty(S^*\mathcal{M})$ with the following properties:*

1. *$m_+ \equiv 1$ on a neighbourhood of $\kappa(E^*_+)$ and $0 \leq m_+ \leq 1$ everywhere;*
2. *$\text{supp}\, m_+ \cap \kappa(T^*_{\mathcal{U}}\mathcal{M}) \subseteq U_+$;*
3. *if $\alpha \in T^*_{\mathcal{U}}\mathcal{M}$, then $\widetilde{X}m_+(\kappa(\alpha)) \geq 0$;*
4. *there is $\delta > 0$ such that if $\alpha \in T^*_{\mathcal{U}}\mathcal{M} \setminus \{0\}$ and $\frac{1}{4} \leq m_+(\kappa(\alpha)) \leq \frac{3}{4}$ then $\widetilde{X}m_+(\kappa(\alpha)) \geq \delta$.*

Proof. First, [10, Lemma 1.4] says that if W is a neighbourhood of the trapped set $\mathcal{K}^g$, $\exists T > 0$ large enough so that

$$(34) \qquad\qquad x \in \mathcal{U}, \varphi_{2T}(x) \in \mathcal{U} \implies \varphi_T(x) \in W$$

and [10, Lemma 1.3] says that if $x \in \Gamma_+$, then $d_{\mathcal{M}}(\varphi_t(x), \mathcal{K}^g) \to 0$ as $t \to -\infty$. Let us choose a C^∞ function $m_0 : S^*\mathcal{M} \to [0,1]$, supported in $U_+ \cap \kappa(T^*_{\mathcal{U}}\mathcal{M})$ and such that $m_0 \equiv 1$ on a neighbourhood of $\kappa(E^*_u) = \kappa(E^*_+ \cap$

$T^*_{\mathcal{K}^g}\mathcal{M}$). Denote by W the projection of this neighbourhood to $\mathcal{M}$. Then, for $T > 0$ large, define the function

$$m_+ : \alpha \mapsto \frac{1}{T}\int_{-2T}^{-T} m_0(\kappa(\Phi_t(\alpha)))\mathrm{d}t.$$

Let us show that m_+ has the required properties when T is chosen large enough. The proof is the same as [10, Lemma 1.12], we summarize the argument for the convenience of the reader:

(1) It follows from a continuity argument and the fact that for W as above and all $\alpha \in \kappa(E^*_+)$, we have $\Phi_t(\alpha) \in \kappa(E^*_+ \cap T^*_W\mathcal{M})$ for all $t \in [-2T, -T]$ if $T > 0$ is large enough.

(2) If $\alpha \in \mathrm{supp}(m_+)$ and $\alpha_x \in \mathcal{U}$, then $\exists t \in [-T, -2T]$ such that $\Phi_t(\alpha) \in \mathrm{supp}(m_0)$, and $\alpha_x, \varphi_t(\alpha_x) \in \mathcal{U}$ while $\kappa(\Phi_t(\alpha)) \in U_+$, then Lemma 4.16 implies that $\alpha \in U_+$, assuming T is large enough.

(3) For $\alpha_x \in \mathcal{U}$, one has $T\widetilde{X}m_+(\alpha) = m_0(\kappa(\Phi_{-T}(\alpha)) - m_0(\kappa(\Phi_{-2T}(\alpha)))$. Assume that $m_0(\kappa(\Phi_{-T}(\alpha)) < 1$ and $m_0(\kappa(\Phi_{-2T}(\alpha))) > 0$, then $\varphi_{-2T}(\alpha_x) \in \mathcal{U}$ and $\varphi_{-T}(\alpha_x) \in W$ if T is large enough (by (34)), but $d_{S^*\mathcal{M}}(\kappa(\Phi_{-T}(\alpha)), \kappa(E^*_u)) > \epsilon$ for some $\epsilon > 0$ depending only on m_0. Since also $\kappa(\Phi_{-2T}(\alpha)) \in U_+$, Lemma 4.16 gives a contradiction with $d_{S^*\mathcal{M}}(\kappa(\Phi_{-T}(\alpha)), \kappa(E^*_u)) > \epsilon$ if $T > 0$ is chosen large enough. Thus $\widetilde{X}m_+(\alpha) \geq 0$.

(4) Let $\alpha \in T^*_{\mathcal{U}}\mathcal{M}\setminus\{0\}$ be such that $\frac{1}{4} \leq m_+(\kappa(\alpha)) \leq \frac{3}{4}$. Since $m_+(\kappa(\alpha)) \leq \frac{3}{4}$, then the proportion of t's in $[-2T, -T]$ such that $m_0(\kappa(\Phi_t(\alpha))) = 1$ is at most $\frac{3}{4}$. Hence, there is a $t \in [-\frac{7}{4}T, -T]$ such that $m_0(\kappa(\Phi_t(\alpha))) < 1$. Thus, if $T > 0$ is large enough, the same argument as for (3) based on Lemma 4.16 shows that $m_0(\kappa(\Phi_{-2T}(\alpha))) = 0$. Similarly, we get $m_0(\kappa(\Phi_{-T}(\alpha))) = 1$, so that $\widetilde{X}m_+(\kappa(\alpha)) = \frac{1}{T}$. $\qquad\square$

Remark 3. Before we give the proof of Lemma 4.15, let us link $H^{\omega_I}_{\mathrm{Re}\,p}$ to Φ_t. From equation (2.3) in [17], we see that for a smooth function f, we have in local coordinates $\tilde{x} = x + iy, \tilde{\xi} = \xi + i\eta$

$$H^{\omega_I}_f = \sum_{j=1}^{n} \partial_{\eta_j}f\partial_{x_j} - \partial_{x_j}f\partial_{\eta_j} + \partial_{\xi_j}f\partial_{y_j} - \partial_{y_j}f\partial_{\xi_j}.$$

If f vanishes on the reals, we see that $H^{\omega_I}_f$ is tangent to the reals. If additionally $f = \mathrm{Re}\,q$, q being holomorphic and pure imaginary on the reals, we

can use the Cauchy–Riemann equations to find

$$H_f^{\omega_I} = \sum_{j=1}^{n} \partial_{\eta_j} f \partial_{x_j} - \partial_{y_j} f \partial_{\xi_j} + \mathcal{O}(|y| + |\eta|)$$

$$= \sum_{j=1}^{n} \mathrm{Re}(\partial_{\eta_j} q)\partial_{x_j} - \mathrm{Re}(\partial_{y_j} q)\partial_{\xi_j} + \mathcal{O}(|y| + |\eta|)$$

$$= -\sum_{j=1}^{n} \partial_{\xi_j} \mathrm{Im}(q)\partial_{x_j} - \partial_{x_j} \mathrm{Im}(q)\partial_{\xi_j} + \mathcal{O}(|y| + |\eta|)$$

In the last line, we recognize the usual Hamilton vector field of $-\mathrm{Im}\, q$ on the reals. In particular,

$$\tag{35} -H_{\mathrm{Re}\,p}^{\omega_I} = \tilde{X} \quad \text{on } T^*\mathcal{M}.$$

Proof of Lemma 4.15. Let r be a non-negative symbol of logarithmic order on $T^*\mathcal{M}$ such that $r(\alpha) = \log \langle \alpha \rangle$ when α is large. Let

$$G_0(\alpha) = \int_{-T}^{0} r(\Phi_t(\alpha))\mathrm{d}t,$$

for T large enough. Thanks to Lemma 4.16, we see that there is a neighbourhood U_+ of $\kappa(E_+^*)$ in $S^*\mathcal{M}$ and a constant $C > 0$ such that

$$\tag{36} \forall \alpha \in \kappa^{-1}(U_+) \cap T_{\mathcal{U}}^*\mathcal{M}, \quad C \geq \tilde{X}G_0(\alpha) \geq C^{-1}.$$

Then, up to making U_+ smaller, we can assume that $U_+ \subseteq \kappa(\Omega_0)$ and apply Lemma 4.17. Let m_+ be the resulting function. We let $\tilde{m}$ be a symbol of order 0 on $T^*\mathcal{M}$ that coincides with $1 - 2m_+ \circ \kappa$ outside of a bounded set. Define then

$$G = \tilde{m}G_0.$$

For $\alpha \in T_{\mathcal{U}}^*\mathcal{M}$ large enough, using (35), we have

$$-H_{\mathrm{Re}\,p}^{\omega_I}G(\alpha) = -2\tilde{X}m_+(\kappa(\alpha))G_0(\alpha) + (1 - 2m_+ \circ \kappa(\alpha))\tilde{X}G_0(\alpha).$$

Let $\Omega = \kappa^{-1}(\{m_+ \geq \frac{1}{4}\}) \subset \kappa^{-1}(U_+)$, and notice that $G \geq \frac{1}{2}\log\langle\alpha\rangle$, if $|\alpha|$ is large enough and not in Ω; this gives point *(i)*.

If $\alpha \in E_+^*$ then $m_+(\kappa(\alpha)) = 1$, so that $G(\alpha) = -G_0(\alpha)$, and (iii) follows. Now, if $\alpha \in \Omega$ is large, either $m_+(\kappa(\alpha)) \geq \frac{3}{4}$, in which case we have by point

(3) of Lemma 4.17 and (36)

$$-H^{\omega_I}_{\operatorname{Re} p} G(\alpha) \leq -\frac{1}{2} C^{-1},$$

or $m_+(\kappa(\alpha)) \leq \frac{3}{4}$, in which case $\widetilde{X} m_+(\kappa(\alpha)) \geq \delta$ by point (4) of Lemma 4.17, and thus

$$-H^{\omega_I}_{\operatorname{Re} p} G(\alpha) \leq -2\delta G_0(\alpha) + C$$

and (ii) follows since $G_0 \geq \epsilon \log\langle\alpha\rangle$ for large $|\alpha|$. $\qquad\square$

4.3.2. Proof of Proposition 4.13. Set $f = Xu$. We start by reducing to the case of u compactly supported in $\mathcal{U}$. If $x \in \Gamma_+ \cap \partial\mathcal{U}$, then u vanishes on a neighbourhood of x, since Xu does, $u_{|\partial_+\mathcal{U}} = 0$ and X is transversal to the outgoing boundary. By compactness of $\Gamma_+ \cap \partial\mathcal{U}$, we find a compact subset K_1 of $\mathcal{U}$ such that $\operatorname{supp} f \subseteq K_1$ and $\overline{\mathcal{U} \setminus K_1}$ does not intersect $\Gamma_+ \cap \operatorname{supp} u$. Let $\chi \in C^\infty(\mathcal{M})$ be supported in $\mathcal{U}$ and such that $\chi \equiv 1$ on a neighbourhood of K_1. Then χu is a distribution on $\mathcal{M}$, supported in $\mathcal{U}$ and

$$X(\chi u) = \chi f + (X\chi)u = f + (X\chi)u.$$

Notice that $X\chi$ is supported in $\mathcal{U} \setminus K_1$, so that $(X\chi)u$ is supported away from Γ_+. In particular the analytic wave front set of $(X\chi)u$, and hence of $f + (X\chi)u$ does not intersect E_+^*. Since χ is C^∞, and since $\operatorname{WF}(u) \cap E_+^* = \emptyset$ by the assumptions on u, then $\operatorname{WF}(\chi u) \cap E_+^* = \emptyset$. Since $\chi \equiv 1$ on a neighbourhood of $\Gamma_+ \cap \operatorname{supp} u$, we have $\operatorname{WF}_a(u) \cap E_+^* = \operatorname{WF}_a(\chi u) \cap E_+^*$. Consequently, we may replace u by χu and f by $f + (X\chi)u$ and assume that u is compactly supported in $\mathcal{U}$. We let $K_0 \subseteq \mathcal{U}$ denote a compact neighbourhood of $\operatorname{supp} u$ and $\operatorname{supp} f$.

By assumption, there is a conic neighbourhood Ω_0 of E_+^* in $T^*\mathcal{M}$ such that $\operatorname{WF}(u) \cap \Omega_0 = \emptyset$ and $\operatorname{WF}_a(f) \cap \Omega_0 = \emptyset$. Let G and Ω be as in Lemma 4.15. Multiplying G by a smooth function equal to 1 outside $\{|\alpha| > R\}$ and supported in $\{|\alpha| > R/2\}$, we can assume that G is small enough as a symbol of order 1. Let $\Lambda := \Lambda_G$ and decompose Λ as

$$\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3,$$

where Λ_1, Λ_2 and Λ_3 are respectively the images of $T^*_{\mathcal{M}\setminus K_0}\mathcal{M}$, $T^*_{K_0}\mathcal{M} \setminus \Omega$ and $T^*_{K_0}\mathcal{M} \cap \Omega$ by $e^{H^{\omega_I}_G}$. According to Lemma 4.3 and since u is supported

in the interior of K_0, for every $N > 0$ large enough and all $k \in \mathbb{R}$, there is $C = C_k > 0$ such that

$$(37) \qquad \int_{\Lambda_1} |T_\Lambda u|^2 \langle|\alpha|\rangle^{2k} e^{-\frac{2H}{h}} \mathrm{d}\alpha \leq C \exp\left(-\frac{C}{h}\right) \|u\|_{H^{-N}}^2.$$

According to the properties of G (Lemma 4.15), G is larger than $\delta \log\langle|\alpha|\rangle$ for α large enough in $T_{K_0}^*\mathcal{M} \setminus \Omega$, so we can apply Lemma 4.9 and obtain

$$(38) \qquad \int_{\Lambda_2} |T_\Lambda u|^2 \langle|\alpha|\rangle^{2k} e^{-\frac{2H}{h}} \mathrm{d}\alpha \leq C \exp\left(\frac{C}{h}\right) \|u\|_{H^{-N}}^2.$$

Now, since the C^∞ wave front set of u does not intersect Ω_0, we see from Lemma 4.10, that for $h > 0$ small enough and every $k \in \mathbb{R}$, we have

$$\int_{\Lambda_3} |T_\Lambda u|^2 e^{-\frac{2H}{h}} \langle|\alpha|\rangle^{2k} \mathrm{d}\alpha < +\infty.$$

Hence, we have $u \in \mathcal{H}_\Lambda^\infty$, and we may apply the multiplication formula of Proposition 4.4 with $P = hX$ to find that

$$\mathrm{Re}\left(\int_\Lambda T_\Lambda u \overline{T_\Lambda Pu} e^{-\frac{2H}{h}} \mathrm{d}\alpha\right) \leq \int_\Lambda \mathrm{Re}\, q \, |T_\Lambda u|^2 e^{-\frac{2H}{h}} \mathrm{d}\alpha$$
$$+ Ch \int_\Lambda |T_\Lambda u|^2 e^{-\frac{2H}{h}} \mathrm{d}\alpha + C\|u\|_{H^{-N}}^2 e^{-\frac{C}{h}},$$

where q is the restriction to Λ of the principal symbol p of P. Since q is a symbol of order 1, we may apply (37) and (38) with $k = \frac{1}{2}$ to find that

$$\int_\Lambda \mathrm{Re}\, q \, |T_\Lambda u|^2 e^{-\frac{2H}{h}} \mathrm{d}\alpha \leq \int_{\Lambda_3} \mathrm{Re}\, q \, |T_\Lambda u|^2 e^{-\frac{2H}{h}} \mathrm{d}\alpha + Ch \int_{\Lambda_3} |T_\Lambda u|^2 e^{-\frac{2H}{h}} \mathrm{d}\alpha$$
$$+ Ce^{\frac{C}{h}} \|u\|_{H^{-N}}^2.$$

Now, if $\alpha \in \Lambda_3$, we have $\alpha = e^{H_G^{\omega_I}} \beta$ with $\beta \in T_{K_0}^*\mathcal{M} \cap \Omega$. Hence, recalling that p denotes the holomorphic extension of the symbol of P, we have

$$\mathrm{Re}\, q(\alpha) = \mathrm{Re}\, p(\alpha) = \mathrm{Re}\, p(\beta) + H_G^{\omega_I} \mathrm{Re}\, p(\beta) + \mathcal{O}\left(\frac{\log\langle|\alpha|\rangle^2}{\langle|\alpha|\rangle}\right)$$

$$= -H_{\mathrm{Re}\, p}^{\omega_I} G + \mathcal{O}\left(\frac{\log\langle|\alpha|\rangle^2}{\langle|\alpha|\rangle}\right)$$

$$\leq -C^{-1},$$

where the last inequality holds for α large enough and we used the fact that $\mathrm{Re}(p(\beta)) = 0$ for $\beta \in T^*\mathcal{M}$ a real covector. Since small frequencies always produce a term that grows at most exponentially fast when h tends to 0, we have

$$\int_{\Lambda_3} \mathrm{Re}\, q\, |T_\Lambda u|^2\, e^{-\frac{2H}{h}}\, \mathrm{d}\alpha \leq -C^{-1} \int_{\Lambda_3} |T_\Lambda u|^2\, e^{-\frac{2H}{h}}\, \mathrm{d}\alpha + Ce^{\frac{C}{h}} \|u\|^2_{H^{-N}}.$$

It follows that

$$\mathrm{Re}\left(\int_\Lambda T_\Lambda u \overline{T_\Lambda Pu} e^{-\frac{2H}{h}}\, \mathrm{d}\alpha \right) \leq (-C^{-1} + Ch) \int_{\Lambda_3} |T_\Lambda u|^2 e^{-\frac{2H}{h}}\, \mathrm{d}\alpha$$
$$+ Ce^{\frac{C}{h}} \|u\|^2_{H^{-N}},$$

so that for h small enough, we have

$$\int_{\Lambda_3} |T_\Lambda u|^2\, e^{-\frac{2H}{h}}\, \mathrm{d}\alpha \leq -C\, \mathrm{Re}\left(\int_\Lambda T_\Lambda u \overline{T_\Lambda Pu} e^{-\frac{2H}{h}}\, \mathrm{d}\alpha \right) + Ce^{\frac{C}{h}} \|u\|^2_{H^{-N}}$$
$$\leq C\delta \int_\Lambda |T_\Lambda u|^2\, e^{-\frac{2H}{h}}\, \mathrm{d}\alpha + C\delta^{-1} \int_\Lambda |T_\Lambda Pu|^2\, e^{-\frac{2H}{h}}\, \mathrm{d}\alpha$$
$$+ Ce^{\frac{C}{h}} \|u\|^2_{H^{-N}}$$
$$\leq C\delta \int_{\Lambda_3} |T_\Lambda u|^2\, e^{-\frac{2H}{h}}\, \mathrm{d}\alpha + C\delta^{-1} \int_{\Lambda_3} |T_\Lambda Pu|^2\, e^{-\frac{2H}{h}}\, \mathrm{d}\alpha$$
$$+ Ce^{\frac{C}{h}} \|u\|^2_{H^{-N}}.$$

Here δ is any small positive number, and we noticed that (37) and (38) still holds when u is replaced by Pu, for the same reasons. Since the analytic wave front set of $Pu = hf$ does not intersect Ω_0, we deduce from Lemma 4.11 that there is a constant $C(u) > 0$ depending on u such that

$$\int_{\Lambda_3} |T_\Lambda Pu|^2\, e^{-\frac{2H}{h}}\, \mathrm{d}\alpha \leq C(u) e^{\frac{C(u)}{h}}.$$

So that by taking δ small enough and using (37) and (38) again, we see that

$$\|u\|^2_{\mathcal{H}^0_\Lambda} \leq C(u) e^{\frac{C(u)}{h}}.$$

Hence, since G is elliptic on E^*_+, it follows from Lemma 4.12 that $\mathrm{WF}_a(u) \cap E^*_+ = \emptyset$. $\qquad \square$

4.4. Proof of Proposition 2.5

We now have at our disposal all the microlocal estimates required from the proof of Proposition 2.5, which will follow from the following result.

Lemma 4.18. *Let $v \in \mathcal{D}'(\mathring{M}_e)$ be supported in the interior of M_e. If x does not belong to the support of v then $\Pi_0^{g_e} v$ is analytic on a neighbourhood of x.*

In addition to microlocal estimates, the proof of Lemma 4.18 will require to understand the dynamics of the geodesic flow lifted to $T^*(SM_e) \subseteq T^*\mathcal{M}$. To do so, introduce the horizontal bundle, defined for $x \in SM_e$ by

$$H^*(x) = \{\xi \in T_x^*(SM_e) \,|\, \xi_{|\ker \mathrm{d}\pi_0(x)} = 0\} \quad \text{for } x \in SM_e.$$

Here, we recall that π_0 is the canonical projection $SM_e \to M_e$. The importance of the bundle H^* is due to the following two results.

Lemma 4.19. *Let $u \in \mathcal{D}'(\mathring{M}_e)$ be compactly supported, then the analytic wave front set of $\pi_0^* u$ is contained in H^*. The same holds true for $\pi_2^* u$ if $u \in \mathcal{D}'(\mathring{M}_e; S^2 T^* \mathring{M}_e)$.*

Lemma 4.20. *Let $u \in \mathcal{D}'(\mathcal{M})$ be supported in the interior of SM_e. Let x be a point in the interior of M and assume that for every $y \in \pi_0^{-1}(\{x\})$ the horizontal direction $H^*(y)$ does not intersect $\mathrm{WF}_a(u)$. Then $\pi_{0*}u$ is analytic on a neighbourhood of x. The same holds true for $\pi_{2*}u$ if $u \in \mathcal{D}'(\mathring{M}_e; S^2 T^* \mathring{M}_e)$.*

Lemma 4.19 is a consequence of [25, Theorem 8.5.1] and the fact that π_0 is analytic. Lemma 4.20 follows from [30, §3.b)] (see also [25, Theorem 8.5.4]), and the fact that the projection of the analytic wave front set to the physical space is the analytic singular support [40, Theorem 6.3]. The arguments apply as well for π_2^* and π_{2*}.

Most of the geometric information needed for the proof of Lemma 4.18 is contained in the following lemma, which is a consequence of the hyperbolicity of the trapped set and the absence of conjugate points.

Lemma 4.21. *For $x \in \Gamma_\pm^{g_e}$ one has $H^*(x) \cap E_\pm^*(x) = \{0\}$. Moreover, if $\alpha = (\alpha_x, \alpha_\xi) \in T^*(SM_e)$ is such that $\alpha_\xi(X(x)) = 0$, $\alpha_\xi \in H^*(x)$ and $(d\varphi_t^{g_e}(x)^{-1})^\top \alpha_\xi \in H^*(\varphi_t^{g_e}(x))$, then $\alpha_\xi = 0$.*

Proof. The proof is contained in the proof of [18, Proposition 5.1]: the fact that $H^*(x) \cap E_\pm^*(x) = \{0\}$ follows from the hyperbolicity of $\mathcal{K}^g$ and a result of Klingenberg [26, Proposition p.6], and the second statement is equivalent to the fact that $(\pi_0(\varphi_s^{g_e}(x)))_{s \in [0,t]}$ is free of conjugate points. $\qquad\square$

We are now ready to prove Lemma 4.18.

Proof of Lemma 4.18. Let $v \in \mathcal{D}'(\overset{\circ}{M}_e)$ be supported in the interior of M_e. Write $f = \pi_0^* v$ and $u = R_{g_e}^* f$. Then $\Pi_0^{g_e} v = \pi_{0*} u$, see (10). Notice that we have $X_{g_e} u = -f$ with $u = 0$ near $\partial_+ SM_e$, and by (9) and the composition rules for wave-front sets we have $\mathrm{WF}(u) \cap E_+^* = \emptyset$.

Let x be a point that does not belong to the support of v. According to Lemma 4.20, we only need to prove that the analytic wave front set of u does not intersect $H^*(y)$ for all $y \in \pi_0^{-1}(\{x\})$.

Let y be a point of $\pi_0^{-1}(\{x\})$ and $\alpha \in H^*(y)$. From Lemma 4.19, we know that the analytic wave front set of f is contained in $H^* \cap \pi_0^{-1}(\mathrm{supp}\, v)$. In particular, $\alpha \notin \mathrm{WF}_a(f)$ (since $\pi_0(y) = x \notin \mathrm{supp}\, v$). If $p(\alpha) \neq 0$, with p the principal symbol of X, we know that $\alpha \notin \mathrm{WF}_a(u)$ by ellipticity of X (see [25, Theorem 8.6.1] or [40, Theorem 6.4]).

Let us then consider the case $p(\alpha) = 0$. Thanks to Lemma 4.21, we know that the orbit of α for the lift of the geodesic flow to $T^*(SM_e)$ never intersects the analytic wave front set of f. Hence, by propagation of singularity (see for instance [24, Theorem 2.9.1]), we only need to prove that this orbit leaves the analytic wave front set of u. If $y \notin \Gamma_-^{g_e}$, then $\varphi_t^{g_e}(y)$ must intersect $\partial_+ \mathcal{U}$, and u vanishes near $\partial_+ \mathcal{U}$. If $y \in \Gamma_-^{g_e}$, then $\alpha \notin E_-^*$ due to Lemma 4.21, and thus $\Phi_t(\alpha)$ converges to E_+^* due to [10, Lemma 1.10]. Hence, we only need to prove that the analytic wave front set of u does not intersect E_+^* (since the analytic wave front set is closed). This fact follows from Proposition 4.13 since $Xu = -f$, the analytic wave front set of f does not intersect E_+^* by Lemmas 4.19 and 4.21, the C^∞ wave front set of u does not intersect E_+^* and $u = 0$ near $\partial_+ \mathcal{U}$. $\square$

From Lemma 4.18 and some functional analysis, we deduce the following:

Corollary 4.22. *Let K_0 and K_1 be disjoint compact subsets of $\overset{\circ}{M}_e$. Let $N \in \mathbb{R}$. Then there is a constant $C > 0$ and a complex neighbourhood V of K_1 such that if $u \in H^{-N}(\overset{\circ}{M}_e)$ is supported in K_0, then $\Pi_0^{g_e} u$ has a holomorphic extension to V bounded by $C \|u\|_{H^{-N}}$.*

Proof. Let $H^{-N}(K_0)$ denote the subspace of $H^{-N}(\overset{\circ}{M}_e)$ made of the elements that are supported in K_0. Let $(V_n)_{n \geq 0}$ be a decreasing family of complex neighbourhoods of K_1 such that $K_1 = \bigcap_{n \geq 0} V_n$. For every $m, n \geq 0$, introduce

$$F_{m,n} = \{u \in H^{-N}(K_0) : \Pi_0^{g_e} u \text{ has a holomorphic extension}$$
$$\text{to } V_n \text{ bounded by } m\}.$$

Then, Lemma 4.18 implies that

$$H^{-N}(K_0) = \bigcup_{m,n \geq 0} F_{m,n}.$$

Let us prove that the $F_{m,n}$'s are closed in $H^{-N}(K_0)$. Fix m and n and let $(u_p)_{p \geq 0}$ be a sequence of elements of $F_{m,n}$ that converges to $u \in H^{-N}(K_0)$. Set $v_p = \Pi_0^{g_e} u_p$ and let denote by $\tilde{v}_p$ the holomorphic extension of v_p to V_n. By Montel's Theorem, we may assume, up to extracting a subsequence, that $\tilde{v}_p$ converges uniformly on all compact subsets of V_n to a holomorphic function $\tilde{v}$ bounded by m. Since $\Pi_0^{g_e}$ is a (C^∞) pseudo-differential operator, we know that v_p converges to $\Pi_0^{g_e} u$ as a distribution, so that the restrictions of $\tilde{v}$ and $\Pi_0^{g_e} u$ to $V_n \cap M_e$ coincides, proving that $u \in F_{m,n}$.

Since the $F_{m,n}$'s are closed, it follows from Baire's Theorem that one of them has non-empty interior. Hence, for some $n \geq 0$, the linear subspace

$$\bigcup_{m \geq 0} F_{m,n}$$

has non-empty interior, and is thus equal to $H^{-N}(K_0)$. The result follows then from the uniform boundedness theorem applied to the family of linear forms $u \mapsto \Pi_0^{g_e} u(x)$ for $x \in V_n$. $\square$

Proof of Proposition 2.5. Apply $\Pi_0^{g_e}$ to a Dirac mass to deduce from Corollary 4.22 that, away from the diagonal, the kernel of $\Pi_0^{g_e}$ is analytic in x uniformly in y. The result follows then from Lemma 4.8, the fact that $\Pi_0^{g_e}$ is symmetric and the fact that the projection of the analytic wave front set to the physical space is the analytic singular support. $\square$

Proof of Proposition 2.6. Since we already know that the kernel of $\Pi_0^{g_e}$ is analytic away from the diagonal, we only need to understand the kernel of $\Pi_0^{g_e}$ near the diagonal. Let us choose $x_0 \in \mathring{M}_e$ and $\epsilon > 0$ very small so that the ball $B_\epsilon := \{x \in M_e \,|\, d_{g_e}(x, x_0) < \epsilon\}$ satisfies $d_{g_e}(B_\epsilon, \partial M_e) > 5\epsilon$, the radius of injectivity of (M_e, g_e) is larger than 10ϵ and the boundary of B_ϵ is strictly convex. In particular B_ϵ is a simple manifold, in the terminology of the introduction. Let us consider $A : C_c^\infty(B_\epsilon) \to C^\infty(B_\epsilon)$ defined by $Au := (\Pi_0^{g_e} u)|_{B_\epsilon}$. Thanks to Proposition 2.5, it is enough to prove that A is analytic pseudo-differential operator. By the proof of [18, Proposition 5.7], we can write

$$R_{g_e}^* = e^{3\epsilon X_{g_e}} R_{g_e}^* + \int_0^{3\epsilon} e^{t X_{g_e}} dt, \quad \Pi_0^{g_e} = \pi_{0*} e^{3\epsilon X_{g_e}} R_{g_e}^* \pi_0^* + \pi_{0*} \int_0^{3\epsilon} e^{t X_{g_e}} dt \pi_0^*$$

Let us write $A = A_1 + A_2$ where $A_1 u = (\pi_{0*} \int_0^{3\epsilon} e^{tX_{g_e}} dt \pi_0^* u)|_{B_\epsilon}$ for all $u \in C_c^\infty(B_\epsilon)$. By [18, Section 5.1], $A_1 = (I_0^{B_\epsilon})^* I_0^{B_\epsilon}$ is the normal operator for the X-ray transform $I_0^{B_\epsilon}$ on the simple manifold (B_ϵ, g_e). In [42, Proposition 3.2], Stefanov and Uhlmann proved that a similar operator is a real-analytic pseudo-differential operator. The same analysis applies here if we replace the formula [42, (2.1)] by the formula for the kernel of the normal operator given in [38, Lemma 3.1]. Hence, we find that A_1 is an elliptic analytic pseudo-differential on B_ϵ. Next we consider A_2. We can use the same argument as above, in the proof of Proposition 2.5 to see that A_2 has an analytic kernel. The only difference is that, adapting the proof of Lemma 4.18, we work with $u = e^{3\epsilon X_{g_e}} R_{g_e}^* f$ for $f = \pi_0^* v$ and v a distribution supported in B_ϵ, and $A_2 v = (\pi_{0*} u)|_{B_\epsilon}$. We only need to use that $e^{3\epsilon X_{g_e}} : f \mapsto f \circ \varphi_{3\epsilon}^{g_e}$, with $\varphi_{3\epsilon}^{g_e}$ an analytic diffeomorphism, and that 10ϵ is smaller than the radius of injectivity of (M_e, g_e) to deduce that $A_2 v$ is analytic in B_ϵ (the singular support of $\pi_{0*} u$ is at least at distance ϵ from B_ϵ). $\qquad \square$

Proof of Proposition 2.7. By using again [42, Proposition 3.2] (which contains the analysis for symmetric 2-tensors on simple manifolds) to deal with the operator

(39)

$$u \in C_c^\infty(B_\epsilon; S^2 T^* B_\epsilon) \mapsto \left(\pi_{2*} \int_0^{3\epsilon} e^{tX_{g_e}} dt \pi_2^* u \right)\Big|_{B_\epsilon} \in C^\infty(B_\epsilon; S^2 T^* B_\epsilon),$$

the same exact argument as in the proof of Proposition 2.6 shows that the normal operator $\Pi_2^{g_e} = (I_2^{g_e})^* I_2^{g_e}$ is an analytic pseudo-differential operator with principal symbol being the same as for simple manifolds. The only difference consists in changing π_0^*, π_{0*} by π_2^*, π_{2*}, which is harmless for the wave-front set analysis as these operators share the same exact (analytic) microlocal properties. Once we know this fact and that $\Pi_2^g = (I_2^g)^* I_2^g$, the proof of solenoidal injectivity is an adaptation of the proof of [43, Theorem 1]. We summarize the argument of that proof for the convenience of the reader and since our geometric assumption is not the same:

1) Any $L^2(M)$ symmetric 2-tensor can be decomposed as $f = f^s + D_g w$ with $D_g^* f^s = 0$ and $w \in H^1(M; T^* M)$ with $w|_{\partial M} = 0$, with $D_g : C^\infty(M; T^* M) \to C^\infty(M; S^2 T^* M)$ the symmetrized Levi-Civita covariant derivative, D_g^* is its formal adjoint (called divergence). The tensor f^s is called the solenoidal projection of f and one has $I_2^g f = I_2^g f^s$ since $I_2^g D_g w = I^g X_g \pi_1^* w = 0$. This decomposition can be done by setting $w = (\Delta_{D_g})^{-1} D_g^* f$ where $\Delta_{D_g} = D_g^* D_g$ is the symmetric Laplacian with Dirichlet condition. See [43, §2.3] or [39, Theorem 3.3.2] for details.

2) The operator Π_2^g is elliptic on solenoidal tensors in the sense that its principal symbol $p(x,\xi)$ is injective on $\{f \in S^2T_x^*M \mid f(\xi^\sharp, \cdot) = 0\}$ if $\xi^\sharp \in T_xM$ is the dual vector to ξ by g_x. This follows from the local description (39) of Π_2^g up to smoothing operators. Then by analytic ellipticity of Π_2^g (since Π_2^g is an analytic pseudodifferential operator as mentioned above), we find that if $I_2^g f^s = 0$, then f^s is analytic in the interior of M. Then, working as in [43, Lemma 6] (see also [42, Proposition 3.4]), we find $f^s \in \mathcal{A}(M)$, that is f^s is analytic up to the boundary. Note that to get the property near the boundary, we use $\Pi_2^{g_e}$ and the fact that g_e is analytic on M_e.

3) According to [43, Lemma 4], if $I_2^g f^s = 0$, there exists an analytic one form $w_0 \in \mathcal{A}(U; T^*M)$ vanishing on ∂M, so that $h := f^s - D_g w_0$ satisfies $h(\psi_* \partial_r, \cdot) = 0$ and h vanishes to infinite order at ∂M, if $\psi : [0, \epsilon] \times \partial M \to U \subset M$ is the normal form of Lemma 2.2. Since h is analytic in U, this implies that $h = 0$ and thus $f^s = D_g w_0$ near ∂M.

4) The analytic 1-form w_0 defined in U admits an analytic extension to M and $f^s = D_g w_0$ on M, which implies that $w_0 = 0$ since f^s is the solenoidal projection of f. To prove the existence of the extension of w_0, we proceed as in the proof of [43, Theorem 1]. For $x \in M \setminus U$, take $x_0 \in \partial M$ and γ a geodesic between x_0 and x. Let $S \subset U$ be a local codimension 1 submanifold orthogonal to γ, we can then use geodesic normal coordinates (t, y) associated to S, covering a neighborhood V_γ of γ. We can then solve the equation $(f^s - D_g w_1)(\partial_t, \cdot) = 0$ in V_γ with initial condition $w_1|_S = w_0|_S$. As shown in the proof of [43, Lemma 4], this is an ODE which has a unique analytic solution, and this solution is equal to w_0 in U; this gives an extension of w_1 in V_γ, in particular near x. Moreover $f^s = D_g w_1$ in V_γ by unique continuation from $U \cap V_\gamma$. We need to check that this extension is independent of the choice of x_0 and γ. Let x_0' and γ' another pair as above and consider the analytic continuation of w_1 from $V_\gamma \cap V_{\gamma'}$ to $V_{\gamma'}$ by the process describes above, but starting from a submanifold S' passing through x orthogonal to γ'. We need to show that $w_1 = w_0$ in $U \cap V_{\gamma'}$. First, we observe that by the process above, w_1 extends analytically in a neighborhood of all broken geodesic γ'' with endpoints x_0, x_0', that is obtained by a homotopy of broken geodesics from $\gamma \cup \gamma'$, and the value of w_1 near x_0' is independent of such curve by unique continuation. Let us then take the unique (non broken) geodesic $\gamma_0 : [0, \ell_{\gamma_0}]$ in this homotopy class (it is unique by [21, Lemma 2.2] since g has no conjugate points). Let us consider the finite dimensional manifold $G_k(x_0, x_0', \gamma_0)$ of piecewise geodesics $c : [0, 1] \to M$ in the homotopy class of γ_0 with $(c(0), c(1)) = (x_0, x_0')$, with k pieces and with each geodesic segment of size less than the injectivity radius (k is chosen large enough). A standard argument using the gradient flow of the energy

functional $E(c) = \int_0^1 \|\dot{c}(t)\|_g^2 dt$ on $G_k(x_0, x_0', \gamma_0) \cap E^{-1}([0, E(\gamma \cup \gamma')])$ shows that there is a homotopy from $\gamma \cup \gamma'$ to γ_0 in $G_k(x_0, x_0', \gamma_0)$ (see for example [33, Theorem 16.3] and its adaptation to the case with strictly convex boundary as in the proof of [21, Lemma 2.2]). Let us define $v_0 := \dot{\gamma}_0(0) \in \partial_- SM$, $v_0' := \dot{\gamma}_0(\ell(\gamma_0))$ and $S_g(x_0, v_0) = (x_0', v_0')$, then we have

$$I_2^g f^s(x_0', v_0') = 0 = w_1(x_0')v_0' - w_1(x_0)v_0 = w_1(x_0')v_0' - w_0(x_0)v_0 = w_1(x_0')v_0'$$

since w_0 vanishes on ∂M. Since similarly one has $I_2^g f^s(x_0', v) = 0 = w_1(x_0')v$ for all $(x_0', v) \in \partial_+ SM$ near (x_0', v_0'), we obtain $w_1(x_0') = 0$. Moving now x_0' slightly, the same argument shows that $w_1 = 0$ on ∂M near x_0'. Since $f^s = D_g w_0 = D_g w_1$ near x_0' and $w_1 = w_0 = 0$ on ∂M near x_0', we deduce that $w_1 = w_0$ in a neighborhood of x_0' in M (the solution of this PDE being unique locally). We have thus proved the result. $\square$

Appendix A. Technical results on the analytic wave front set

For the convenience of the reader, we recall here the definition of the analytic wave front set of a distribution [40, Definition 6.1]. As the notion is invariant by analytic change of coordinates, we give the definition on an open subset of $\mathbb{R}^n$.

Definition A.1. Let U be an open susbet of $\mathbb{R}^n$ and $u \in \mathcal{D}'(U)$. Let $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus \{0\} \simeq \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Let:

- $\varphi(x, \alpha)$ be an analytic function on a neighbourhood of (x_0, x_0, ξ_0) such that $\varphi(x, \alpha) = 0$ and $d_x \varphi(x, \alpha) = \alpha_\xi$ when $x = \alpha_x$, and $\text{Im}\, \varphi(x, \alpha_x) \geq C^{-1}|x - \alpha_x|^2$ when x and α are real.
- $a_\lambda(x, \alpha)$ be an analytic function on a neighbourhood of (x_0, x_0, ξ_0) depending on a parameter $\lambda \geq \lambda_0$. We assume that $a_\lambda(x, \alpha)$ is bounded and bounded away from zero, uniformly in $\lambda \geq \lambda_0$ on a fixed complex neighbourhood of (x_0, x_0, ξ_0).
- $\chi \in \mathcal{C}_c^\infty(U)$ supported near x_0 and identically equal to 1 on a neighbourhood of x_0.

Then (x_0, ξ_0) does not belong to the analytic wave front set of u if and only if

$$\int_{\mathbb{R}^n} e^{i\lambda\varphi(x,\alpha)} a_\lambda(x, \alpha)\chi(x)u(x)\mathrm{d}x$$

decays exponentially fast when λ tends to $+\infty$, uniformly for α in a neighbourhood of (x_0, ξ_0).

It follows from [40, Proposition 6.2] that this definition does not depend on the choice of φ, a_λ or χ.

Proof of Proposition 4.7. The equivalence between (i) and (ii) is an immediate consequence of [40, Proposition 6.2]. It is clear that (iii) implies (ii), so let us prove that (ii) implies (iii).

We consider thus Ω_0 a bounded neighbourhood of $\alpha_0 \in T^*\mathcal{M} \setminus \{0\}$ where Tu is exponentially small in h. Then we pick Ω_1 another neighbourhood of α_0, contained and relatively compact in Ω_0, and let $\Omega = \bigcup_{t>0} t\Omega_1$. For $\alpha \in \Omega$, we write $\alpha = t\alpha'$ with $\alpha' \in \Omega_1$ and $t \in \mathbb{R}_+^*$.

Assuming $t > 1$ and letting $\tilde{h} = h/t$, we will need to distinguish the FBI transform with semi-classical paramater h and $\tilde{h}$, so that we will write T_h, $T_{\tilde{h}}$ instead of just T and S_h, $S_{\tilde{h}}$ instead of S to make it clear. Let us then use the inversion formula to write $T_h u = T_h S_{\tilde{h}} T_{\tilde{h}} u$ for h and $\tilde{h}$ small enough. The action of the operator $S_{\tilde{h}}$ on functions that grows at most polynomially is defined by duality. Hence, for $\alpha \in T^*\mathcal{M}$, we have

$$(40) \qquad T_h S_{\tilde{h}} T_{\tilde{h}} u(\alpha) = \int_{T^*\mathcal{M}} K_{h,\tilde{h}}(\alpha, \beta) T_{\tilde{h}} u(\beta) \mathrm{d}\beta,$$

where the kernel $K_{h,\tilde{h}}$ is defined by

$$K_{h,\tilde{h}}(\alpha, \beta) = \int_{\mathcal{M}} K_{T_h}(\alpha, y) K_{S_{\tilde{h}}}(y, \beta) \mathrm{d}y.$$

The integral in (40) is convergent as $T_{\tilde{h}} u$ grows at most polynomially and $K_{h,\tilde{h}}(\alpha, \beta)$ decays exponentially fast in β when α is fixed, since $y \mapsto K_{T_h}(\alpha, y)$ is real-analytic.

Let us start with a very crude estimate on the kernel $K_{h,\tilde{h}}$. It follows from the definition of K_{T_h} and $K_{S_{\tilde{h}}}$ that there is C such that if $\alpha, \beta \in T^*\mathcal{M}$ and $y \in \mathcal{M}$, we have

$$|K_{T_h}(\alpha, y)| \leq C h^{-3n/4} \langle \alpha \rangle^{n/4} \ \text{and} \ \left| K_{S_{\tilde{h}}}(y, \beta) \right| \leq C \tilde{h}^{-3n/4} \langle \beta \rangle^{n/4}.$$

Hence, up to making C larger, we have for $\alpha, \beta \in T^*\mathcal{M}$ that

$$(41) \qquad \left| K_{h,\tilde{h}}(\alpha, \beta) \right| \leq C(h\tilde{h})^{-3n/4} \langle \alpha \rangle^{n/4} \langle \beta \rangle^{n/4}.$$

With these notations, we can split

$$(42) \quad T_h u(\alpha) = \int_{\Omega_0} K_{h,\tilde{h}}(\alpha, \beta) T_{\tilde{h}} u(\beta) d\beta + \int_{T^*\mathcal{M} \setminus \Omega_0} K_{h,\tilde{h}}(\alpha, \beta) T_{\tilde{h}} u(\beta) d\beta.$$

We can use the crude estimate (41) and the assumption (ii) that we made on u to deduce that the first integral is a $\mathcal{O}(e^{-1/C\tilde{h}}) = \mathcal{O}(e^{-\langle\alpha\rangle/Ch})$. Since we assumed that u is a distribution, Tu grows at most polynomially, and it suffices in order to get (iii) to prove that for $\alpha \in \Omega$ and $\beta \notin \Omega_0$,

$$\tag{43} \left|K_{h,\tilde{h}}(\alpha,\beta)\right| \leq C\exp\left(-C^{-1}\left(\frac{\langle\alpha\rangle}{h} + \frac{\langle\beta\rangle}{\tilde{h}}\right)\right),$$

which implies that the second integral in (42) is also $\mathcal{O}(e^{-1/C\tilde{h}})$.

Now, $\alpha \in \Omega$ and $\beta \notin \Omega_0$ implies that α/h and $\beta/\tilde{h}$ are not close to each other, which is the right condition for (43) to hold, as we will now prove. The argument is very close to the proof of [17, Lemma 2.9].

Let us first assume that $\alpha, \beta \in T^*\mathcal{M}$ are such that the distance between α_x and β_x is larger than some small $\eta > 0$. Then, we may split the integral defining $K_{h,\tilde{h}}(\alpha,\beta)$ into

$$\tag{44}
\begin{aligned}
K_{h,\tilde{h}}(\alpha,\beta) = &\int_{D(\alpha_x,\eta/100)} K_{T_h}(\alpha,y)K_{S_{\tilde{h}}}(y,\beta)\mathrm{d}y \\
&+ \int_{D(\beta_x,\eta/100)} K_{T_h}(\alpha,y)K_{S_{\tilde{h}}}(y,\beta)\mathrm{d}y \\
&+ \int_{\mathcal{M}\setminus(D(\alpha_x,\eta/100)\cup D(\beta_x,\eta/100))} K_{T_h}(\alpha,y)K_{S_{\tilde{h}}}(y,\beta)\mathrm{d}y.
\end{aligned}$$

The third integral in (44) is easily dealt with: by assumptions K_{T_h} and $K_{S_{\tilde{h}}}$ are small there. Let us look at the first integral (the second one is dealt with similarly). It may be rewritten, up to a neglectible term, as

$$\int_{D(\alpha_x,\eta/100)} e^{i\frac{\Phi_T(\alpha,y)}{h}} a(\alpha,y)K_{S_{\tilde{h}}}(y,\beta)\mathrm{d}y.$$

Here, the phase $\Phi_T(\alpha,y)$ is non-stationary, as y is close to α_x and we have $\mathrm{d}_y\Phi_T(\alpha,\alpha_x) = -\alpha_\xi$, and the function $y \mapsto K_{S_{\tilde{h}}}(y,\beta)$ is real-analytic and has a holomorphic extension bounded by $\mathcal{O}(\exp(-\langle|\beta|\rangle/C\tilde{h}))$ on a complex neighbourhood of $D(\alpha_x,\eta/100)$ whose size is uniform in α, β, h and $\tilde{h}$. We can consequently apply the non-stationary phase method [17, Proposition 1.5] to see that this integral is indeed of the size (43). Let us mention that we apply here [17, Proposition 1.4] in the case $s = 1$, and that the assumption of positivity of $\mathrm{Im}\,\Phi_T$ on the boundary of $D(\alpha_x,\eta/100)$ follows from the fourth assumptions we made on Φ_T.

We want now to understand $K_{h,\tilde{h}}(\alpha,\beta)$ when the distance between α_x and β_x is less than 2η, but $\frac{\alpha}{h}$ and $\frac{\beta}{\tilde{h}}$ are away from each other in the Kohn–Nirenberg metric. As α_x and β_x are close to each other, we may work in coordinates, and the fact that $\frac{\alpha}{h}$ and $\frac{\beta}{\tilde{h}}$ are away from each other will write $|\frac{\alpha_\xi}{h} - \frac{\beta_\xi}{\tilde{h}}| \geq A\eta(\frac{\langle\alpha\rangle}{h} + \frac{\langle\beta\rangle}{\tilde{h}})$, for a large constant A to be determined later. We split the integral defining $K_{h,\tilde{h}}(\alpha,\beta)$ in two:

$$K_{h,\tilde{h}}(\alpha,\beta) = \left(\int_{D(\alpha_x,100\eta)} + \int_{\mathcal{M}\backslash D(\alpha_x,100\eta)} \right) K_{T_h}(\alpha,y) K_{S_{\tilde{h}}}(y,\beta)\mathrm{d}y.$$

The second integral is dealt with as the third integral in (44), it is at most of the size (43). The first integral may be rewritten as

$$\int_{D(\alpha_x,100\eta)} e^{i\left(\frac{\Phi_T(\alpha,y)}{h} + \frac{\Phi_S(y,\beta)}{\tilde{h}}\right)} a(\alpha,y)b(\beta,y)\mathrm{d}y,$$

where a and b are analytic symbols. The phase is non-stationary as

$$\mathrm{d}_y \left(\frac{\Phi_T(\alpha,y)}{h} + \frac{\Phi_S(y,\beta)}{\tilde{h}} \right)$$
$$= \frac{\mathrm{d}_y\Phi_T(\alpha,\alpha_x)}{h} + \frac{\mathrm{d}_y\Phi_S(\beta_x,\beta)}{\tilde{h}} + \mathcal{O}\left(\left(\frac{\langle\alpha\rangle}{h} + \frac{\langle\beta\rangle}{\tilde{h}} \right)\eta \right)$$
$$= \frac{\beta_\xi}{\tilde{h}} - \frac{\alpha_\xi}{h} + \mathcal{O}\left(\left(\frac{\langle\alpha\rangle}{h} + \frac{\langle\beta\rangle}{\tilde{h}} \right)\eta \right).$$

Taking A large enough, we get

$$\left| \mathrm{d}_y \left(\frac{\Phi_T(\alpha,y)}{h} + \frac{\Phi_S(y,\beta)}{\tilde{h}} \right) \right| \geq C^{-1} \left| \frac{\langle\alpha\rangle}{h} + \frac{\langle\beta\rangle}{\tilde{h}} \right|.$$

Hence, the non-stationary phase method gives again that (43) holds. Here, we cannot use [17, Proposition 1.4] directly since Φ_T is divided by h and Φ_S by $\tilde{h}$. However, we can apply [17, Proposition 1.1] and a rescaling argument. $\quad\square$

Proof of Lemma 4.8. We use here Definition A.1 of the analytic wave front set. Let $(\alpha,\beta) \in T_x^*\mathcal{M} \times T_y^*\mathcal{M}$ be such that $\alpha_\xi \neq 0$. Choose then φ,ψ be real-analytic functions defined respectively on a neighbourhood of (x,α) and

(y, β) in $\mathcal{M} \times T^*\mathcal{M}$, that satisfy

$$\begin{cases} \varphi(x', \alpha') = 0, \ \mathrm{d}_x \varphi(x', \alpha') = \alpha'_\xi & \text{when } \alpha'_x = x', \\ \psi(y', \beta') = 0, \ \mathrm{d}_y \psi(y', \beta') = \beta'_\xi & \text{when } \beta'_x = y'. \end{cases}$$

We also assume that there is $C > 0$ so that $\operatorname{Im} \varphi(x', \alpha') \geq C^{-1} d(x', \alpha'_x)^2$ and $\operatorname{Im} \psi(y', \beta') \geq C^{-1} d(y', \beta'_x)^2$ for x', α', y', β' real, respectively near x, α, y, β. Let then χ and ρ be C^∞ functions on $\mathcal{M}$ that are identically equal to 1, respectively near x and y. We want to prove that the integral

$$(45) \qquad \int_{\mathcal{M} \times \mathcal{M}} e^{i\lambda(\varphi(x', \alpha') + \psi(y', \beta'))} \chi(x') \rho(y') u(x', y') \mathrm{d}x' \mathrm{d}y'$$

decays exponentially fast when λ tends to $+\infty$ and α' and β' are near α and β. Notice that here we take $a_\lambda = 1$ in the notation from Definition A.1, which we can do since the definition does not depend on the choice of a_λ. Let us write the integral (45) as

$$\int_{\mathcal{M}} e^{i\lambda\psi(y', \beta')} \rho(y') \left(\int_{\mathcal{M}} e^{i\lambda\varphi(x', \alpha')} \chi(x') u(x', y') \mathrm{d}x' \right) \mathrm{d}y'.$$

In the inner integral, the x' that are away from x are negligible by positivity of $\operatorname{Im} \varphi$, and the x' near x are dealt with by the non-stationary phase method [17, Proposition 1.1], using that $\alpha_\xi \neq 0$. We find that the inner integral decays exponentially fast with λ uniformly in y'. Since the imaginary part of $\psi(y', \beta')$ is non-negative when y' and β' are real, we find that (45) decays exponentially fast with λ, which ends the proof of the lemma. $\square$

Appendix B. Application to lens rigidity for analytic asymptotically hyperbolic manifolds

We notice that Theorem 2 implies a *renormalized lens rigidity* result for analytic asymptotically hyperbolic manifolds. These are complete manifolds with curvature tending to -1 at infinity, and conformal to compact Riemannian manifolds with a conformal factor blowing up at infinity: (M, g) is the interior of a compact manifold with boundary $\overline{M}$ and there is a smooth boundary defining function $\rho \in C^\infty(\overline{M}, [0, 1])$ such that $(\overline{M}, \rho^2 g)$ is a smooth Riemannian manifold with boundary with $|d\rho|_{\rho^2 g} = 1$ on $\partial\overline{M}$. Once a conformal representative is chosen in the scattering infinity for g (see for instance the introduction of [16]), the scattering map for the geodesic flow on an

asymptotically hyperbolic manifold (M, g) with hyperbolic traped set can be defined as map $S_g : T^*\partial\overline{M} \to T^*\partial\overline{M}$, see [16, Proposition 2.6], and the renormalized length L_g can be defined by taking the constant term in the asymptotic expansion as $\epsilon \to 0$ of the length $\ell_g(\gamma \cap \{\rho \geq \epsilon\})$ for complete geodesics γ with both endpoints at infinity (see [16, Definition 4.2]). In the proof of [16, Theorem 1.4], it is shown in particular that two analytic asymptotically hyperbolic manifolds (M_1, g_1) and (M_2, g_2) with $(L_{g_1}, S_{g_1}) = (L_{g_2}, S_{g_2})$ must be isometric near $\partial\overline{M}$, with an isometry $\psi : U_1 \to U_2$ for U_i an open neighborhood of $\partial\overline{M}_i$. Up to reducing this neighborhood we can choose U_1 (and thus U_2) to be strictly convex in M_i since the level sets $\{\rho_i = \epsilon\}$ are easily seen to be strictly convex if $\epsilon > 0$ is small for the boundary defining functions ρ_i as above (i.e. $|d\rho_i|_{\rho_i^2 g} = 1$ on $\partial\overline{M}_i$). The fact that $S_{g_1} = S_{g_2}$ implies that the compact manifolds $(M_1 \setminus U_1, g_1)$ and $(M_2 \setminus U_2, g_2)$ have the same scattering maps. If we assume in addition that (M_i, g_i) have negative curvature, or more generally no conjugate points and a hyperbolic trapped set, then we can use Theorem 2 to deduce the following:

Corollary B.1. *Let (M_1, g_1) and (M_2, g_2) be two real analytic asymptotically hyperbolic manifolds with no conjugate points and with hyperbolic trapped set. If, after a choice of conformal representatives in the conformal infinity of g_1 and g_2, we have $(L_{g_1}, S_{g_1}) = (L_{g_2}, S_{g_2})$, then (M_1, g_1) is isometric to (M_2, g_2).*

This improves Theorem 1.4 in [16], which was showing the same result as Corollary B.1 under the extra assumption that the relative fundamental group $\pi_1(\overline{M}_1, \partial\overline{M}_1) = 0$.

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