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GLOBAL WEAK SOLUTION FOR A COUPLED COMPRESSIBLE NAVIER-STOKES AND Q-TENSOR SYSTEM*

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Abstract. In this paper, we study a coupled compressible Navier-Stokes/Q-tensor system modeling the nematic liquid crystal flow in a three-dimensional bounded spatial domain. The existence and long time dynamics of globally defined weak solutions for the coupled system are established, using weak convergence methods, compactness, and interpolation arguments. The symmetry and traceless properties of the Q-tensor play key roles in this process.

Key words. Navier-Stokes, Q-tensor, liquid crystals, global weak solution, symmetric, traceless.

AMS subject classifications. 35A05, 76A10, 76D03.

1. Introduction

In this paper we consider the following hydrodynamic system modeling the compressible nematic liquid crystal flow in a bounded domain, which is composed of coupled Navier-Stokes and Q-tensor equations (see [4, 38]):

$$\rho_t + \nabla\cdot(\rho u) = 0, \tag{1.1}$$

$$\begin{aligned}(\rho u)_t + \nabla\cdot(\rho u\otimes u) + \nabla(P(\rho)) = \mathcal{L}u - \nabla\cdot\big(L\nabla Q\odot\nabla Q - \mathcal{F}(Q)I_3\big)\\ + L\nabla\cdot(Q\mathcal{H}(Q) - \mathcal{H}(Q)Q),\end{aligned} \tag{1.2}$$

$$Q_t + u\cdot\nabla Q - \Omega Q + Q\Omega = \Gamma\mathcal{H}(Q). \tag{1.3}$$

The system (1.1)-(1.3) is subject to the following initial conditions:

$$(\rho,\rho u,Q)|_{t=0} = (\rho_0(x), q_0(x), Q_0(x)),\quad x\in U, \tag{1.4}$$

with

$$Q_0\in H^1(U),\quad Q_0\in S_0^{(3)}\quad \text{a.e. in } U, \tag{1.5}$$

and the following boundary conditions:

$$u(x,t) = 0,\ \ Q(x,t) = Q_0(x),\ \text{for } (x,t)\in\partial U\times(0,\infty). \tag{1.6}$$

The following compatibility condition is also imposed

$$\rho_0\in L^\gamma(U),\ \rho_0\ge 0; q_0\in L^1(U),\ q_0 = 0 \text{ if } \rho_0 = 0;\ \frac{|q_0|^2}{\rho_0}\in L^1(U). \tag{1.7}$$

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Here $U\subset\mathbb{R}^3$ is a smooth bounded domain, $\rho:U\times[0,+\infty)\to\mathbb{R}^1$ is the density function of the fluid, $u:U\times[0,+\infty)\to\mathbb{R}^3$ represents the velocity field of the fluid, $P=\rho^\gamma$ stands for the pressure function with the adiabatic constant $\gamma>1$, and $Q:U\times(0,+\infty)\to S_0^{(3)}$ is the order parameter, with $S_0^{(3)}\subset\mathbb{M}^{3\times 3}$ representing the space of Q-tensors in dimension 3, i.e.

$$S_0^{(3)}=\{Q\in\mathbb{M}^{3\times 3};Q_{ij}=Q_{ji},\mathrm{tr}(Q)=0,i,j=1,\cdots,3\}.$$

Throughout our paper, div stands for the divergence operator in $\mathbb{R}^3$ and $\mathcal{L}$ stands for the Lamé operator:

$$\mathcal{L}u=\nu\Delta u+(\nu+\lambda)\nabla\mathrm{div}u,$$

where ν and λ are shear viscosity and bulk viscosity coefficients of the fluid, respectively, which satisfy the following physical assumptions:

$$\nu>0,\quad 2\nu+3\lambda\geq 0. \tag{1.8}$$

The (i,j)-th entry of the tensor $\nabla Q\odot\nabla Q$ is $\sum_{k,l=1}^{3}\nabla_i Q_{kl}\nabla_j Q_{kl}$, and $I_3\subset\mathbb{M}^{3\times 3}$ stands for the 3×3 identity matrix. Furthermore, $\mathcal{F}(Q)$ represents the free energy density of the director field

$$\mathcal{F}(Q)=\frac{L}{2}|\nabla Q|^2+\frac{a}{2}\mathrm{tr}(Q^2)-\frac{b}{3}\mathrm{tr}(Q^3)+\frac{c}{4}\mathrm{tr}^2(Q^2), \tag{1.9}$$

and we denote

$$\mathcal{H}(Q)=L\Delta Q-aQ+b\Big[Q^2-\frac{I_3}{3}\mathrm{tr}(Q^2)\Big]-cQ\mathrm{tr}(Q^2). \tag{1.10}$$

Here $\Omega=\frac{\nabla u-\nabla^T u}{2}$ is the skew-symmetric part of the rate of strain tensor. $L>0$, $\Gamma>0$, $a\in\mathbb{R}$, $b>0$, and $c>0$ are material-dependent elastic constants (cf. [34]).

The celebrated hydrodynamic theory for nematic liquid crystals, namely the Ericksen-Leslie theory, was developed between 1958 and 1968. Afterwards Lin [22] and Lin-Liu [23, 24] added a penalization term to the Oseen-Frank energy functional to relax the nonlinear constraint of unit vector length, and made a series of important analytic works, such as existence of global weak solutions, partial regularity, etc. The corresponding compressible liquid crystal flow was studied in Wang-Yu [39]; also see [29]. On the other hand, quite recently, for a simplified Ericksen-Leslie system with the nonlinear constraint of unit vector length, Lin-Lin-Wang [25] proved the existence of global weak solutions that are smooth away from at most finitely many singular times in any bounded smooth domain of $\mathbb{R}^2$, and results on uniqueness of weak solutions were given in [26, 40]. Moreover, for the corresponding compressible flow in the one-dimensional case, the existence of global regular and weak solutions to the compressible flow of liquid crystals was obtained in [6, 7]. Strong solutions in the three-dimensional case were also discussed in [18–20].

Besides the Ericksen-Leslie theory, there are alternative theories that attempt to describe the nematic liquid crystal, among which the most comprehensive description is the Q-tensor theory proposed by P. G. De Gennes in [21]. Roughly speaking, a Q-tensor is a symmetric and traceless matrix which can be interpreted from the physical point of view as a suitably normalized second-order moment of the probability distribution function describing the orientation of rod-like liquid crystal molecules (see [1,2] for details). The static theory of the Q-tensor has been extensively studied in [1,2,30,34]. On the other hand, the mathematical analysis of the corresponding hydrodynamic system was studied in Paicu-Zarnescu [36,37]. More precisely, they establish the existence of global weak solutions to the coupled system of incompressible Navier-Stokes equations and Q-tensors in both two and three dimensionals, as well as the existence of global regular solutions in two-dimensions.

In this paper, we are interested in the compressible version of the model studied in [37]. In the current case, the fluid flow is governed by the compressible Navier-Stokes equations, and the motion of the order-parameter Q is described by a parabolic type equation. It combines a usual equation describing the flow of compressible fluid with extra nonlinear coupling terms. These extra terms are induced elastic stresses from the elastic energy through the transport, which is represented by the equation of motion for the tensor order parameter Q:

$$(\partial_t + u\cdot\nabla)Q - S(\nabla u, Q) = \Gamma\mathcal{H},$$

where $\Gamma > 0$ is a collective rotational diffusion constant. The first term on the left hand side of the above equation is the material derivative of Q, which is generalized by a second term

$$S(\nabla u, Q) = (\xi A + \Omega)\Big(Q + \frac{I_3}{3}\Big) + \Big(Q + \frac{I_3}{3}\Big)(\xi A - \Omega) - 2\xi\Big(Q + \frac{I_3}{3}\Big)\mathrm{tr}(Q\nabla u).$$

Here $A = \frac{\nabla u + \nabla^T u}{2}$ is the rate of strain tensor and $\Omega = \frac{\nabla u - \nabla^T u}{2}$ is the skew-symmetric part of ∇u. The term $S(\nabla u, Q)$ appears in the equation because the order parameter distribution can be both rotated and stretched by the flow gradients. ξ is a constant which depends on the molecular details of a given liquid crystal, which also measures the ratio between the tumbling and aligning effect that a shear flow would exert over the liquid crystal directors. Furthermore, it is noted that in the uniaxial nematic phase, when the magnitude of the order parameter Q remains constant, the coupled hydrodynamic system is reduced to the Ericksen-Leslie system with the validity of Parodi's relation (see [4]). For the sake of simplicity in mathematical analysis, we take $\xi = 0$ in our system. We want to point out that the case for $\xi \neq 0$ is mathematically much more challenging (cf. Remark 4.5). There are no existing results for the coupled system of compressible Navier-Stokes and Q-tensors, and the goal of this paper is to establish the existence of global weak solutions for the compressible coupled system. We note that due to higher nonlinearities in the coupled system (1.1)-(1.3), compared to earlier works in [29,39], it is more difficult to study the current system mathematically.

Note that when Q is absent in (1.1)-(1.3), the system is reduced to the compressible Navier-Stokes equations. For the multidimensional compressible Navier-Stokes equations, early work by Matsumura and Nishida [31–33] established the global existence with small initial data, and later by Hoff [14–16] for discontinuous initial data. To remove the difficulties of large oscillations, Lions in [27] introduced the concept of renormalized solutions and proved the global existence of finite energy weak solutions for $\gamma > 9/5$, where vacuum is allowed initially, and then Feireisl et al. in [10–12] extended the existence results to $\gamma > 3/2$. Because the compressible Navier-Stokes equations is a sub-system to (1.1)-(1.3), one cannot expect better results than those in [10–12]. To this end, in this paper we shall study the initial-boundary value problem for large initial data in certain functional spaces with $\gamma > 3/2$. To achieve our goal, we will use a three-level approximation scheme similar to that in [10,12], which consists of Faedo-Galerkin approximation, artificial viscosity, and artificial pressure (see also [8,9,29,39]). Then, following the idea in [10], we show that the uniform estimate of the density $\rho^{\gamma+\alpha}$ in L^1 for some $\alpha>0$ ensures the vanishing of artificial pressure and the strong compactness of the density. We will establish the weak continuity of the effective viscous flux for our systems similar to that for compressible Navier-Stokes equations as in Lions and Feireisl in [10,12,27] to remove the difficulty of possible large oscillation of the density. To obtain the related lemma on effective viscous flux, we have to make a delicate analysis to deal with the coupling and interaction between the $Q-$tensor and the fluid velocity, especially with certain higher order terms arising from equation (1.2). It is noted that we have to exploit the structure of the system (1.1)-(1.3), and make use of certain special properties of the Q-tensor, namely symmetry and trace-free, to obtain the necessary a priori bounds for Q and the weak continuity for the effective viscous flux (see for instance Proposition 2.1, Lemma 4.2, and Remark 4.4).

The remainder of this paper is organized as follows. In Section 2, after the introduction of some preliminaries, we state the main existence result of this paper, namely Theorem 2.5. In sections 3-5, we study the three-level approximations, namely Faedo-Galerkin, vanishing viscosity, and artificial pressure, respectively. Finally, in Section 6, we discuss briefly the long time dynamics of the global weak solution.

2. Preliminaries

Throughout this paper, we denote by $\langle\cdot,\cdot\rangle$ the scalar product between two vectors, and

$$A:B=\mathrm{tr}(A^TB)=\mathrm{tr}(AB^T)$$

represents the inner product between two 3×3 matrices A and B, $\|\cdot\|_{L^2(U)}$ will be written as $\|\cdot\|$ if necessary. Here and after, the Einstein summation convention will be used. We use the Frobenius norm of a matrix $|Q|=\sqrt{\mathrm{tr}(Q^2)}=\sqrt{Q_{ij}Q_{ij}}$ and Sobolev spaces for Q-tensors are defined in terms of this norm. For instance,

$$L^2(U,S_0^3)=\{Q:U\to S_0^3,\int_U|Q(x)|^2dx<\infty\}.$$

Meanwhile, we denote $\mathcal{D}$ as C_0^∞, and $\mathcal{D}'$ as the space of distributions. We denote by C and $C_i, i=0,1,\cdots$ generic constants which may depend only on U, the coefficients of the system (1.1)-(1.3), the initial data (ρ_0,u_0,Q_0), and T. Special dependence will be pointed out explicitly in the text if necessary. We also denote the total energy by

$$\mathcal{E}(t)=\int_U\Big(\frac{1}{2}\rho|u|^2(t)+\frac{\rho^\gamma(t)}{\gamma-1}\Big)dx+\mathcal{G}(Q(t)), \tag{2.1}$$

where

$$\mathcal{G}(Q(t))=\int_U\bigg(\frac{L}{2}|\nabla Q|^2+\frac{a}{2}\mathrm{tr}(Q^2)-\frac{b}{3}\mathrm{tr}(Q^3)+\frac{c}{4}\mathrm{tr}^2(Q^2)\bigg)dx. \tag{2.2}$$

An important property of the coupling system (1.1)–(1.6) is that it has a basic energy law, which indicates the dissipative nature of the system. It states that the total sum of the kinetic and internal energy are dissipated due to viscosity and internal elastic relaxation.

PROPOSITION 2.1. *If (ρ,u,Q) is a smooth solution of the problem (1.1)-(1.6), then for any $t>0$, the following energy dissipative law holds:*

$$\frac{d}{dt}\mathcal{E}(t)+\int_U\left(\nu|\nabla u|^2+(\nu+\lambda)|\mathrm{div}\,u|^2\right)dx+\Gamma\int_U \mathrm{tr}^2(\mathcal{H})dx=0. \tag{2.3}$$

Proof. Multiplying equation (1.2) with u and then integrating over U, using the density equation (1.1) and boundary condition (1.6) for u, we get after integration by parts that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_U\rho|u|^2dx=&-(\nu+\lambda)\int_U|\mathrm{div}u|^2dx-\nu\int_U|\nabla u|^2dx+\int_U\rho^\gamma\mathrm{div}u\,dx\\&-L\int_U(u\cdot\nabla Q):\Delta Q dx-L\int_U\nabla u:(Q\Delta Q)\,dx+L\int_U\nabla u:(\Delta QQ)\,dx\\&+\int_U\Big\langle u,\nabla\Big[\frac{a}{2}\mathrm{tr}(Q^2)-\frac{b}{3}\mathrm{tr}(Q^3)+\frac{c}{4}\mathrm{tr}^2(Q^2)\Big]\Big\rangle dx.\end{aligned} \tag{2.4}$$

Next, we multiply equation (1.3) with -$\mathcal{H}$, then take the trace and integrate over U. Because $\Omega+\Omega^T=0$, $Q^T=Q$, and $\mathrm{tr}(Q)=0$, after integration by parts we have

$$\begin{aligned}&\frac{d}{dt}\mathcal{G}(Q(t))\\&=-\Gamma\int_U\mathrm{tr}^2(\mathcal{H})dx+L\int_U(u\cdot\nabla Q):\Delta Q\,dx\\&\quad-\int_U\Big\langle u,\nabla\Big[\frac{a}{2}\mathrm{tr}(Q^2)-\frac{b}{3}\mathrm{tr}(Q^3)+\frac{c}{4}\mathrm{tr}^2(Q^2)\Big]\Big\rangle dx\\&\quad-\frac{L}{2}\int_U(\nabla uQ+Q\nabla^Tu):\Delta Qdx+\frac{L}{2}\int_U(\nabla^TuQ+Q\nabla u):\Delta Qdx \qquad (2.5)\\&=-\Gamma\int_U\mathrm{tr}^2(\mathcal{H})dx+L\int_U(u\cdot\nabla Q):\Delta Q\,dx\\&\quad-\int_U\Big\langle u,\nabla\Big[\frac{a}{2}\mathrm{tr}(Q^2)-\frac{b}{3}\mathrm{tr}(Q^3)+\frac{c}{4}\mathrm{tr}^2(Q^2)\Big]\Big\rangle dx\\&\quad-L\int_U\nabla u:(\Delta QQ)dx+L\int_U\nabla u:(Q\Delta Q)dx.\end{aligned} \tag{2.6}$$

Adding (2.4) and (2.6) together yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_U\rho|u|^2dx+\frac{d}{dt}\mathcal{G}(Q(t))\\&=-(\nu+\lambda)\int_U|\mathrm{div}\,u|^2dx-\nu\int_U|\nabla u|^2dx-\Gamma\int_U\mathrm{tr}^2(\mathcal{H})dx+\int_U\rho^\gamma\mathrm{div}u\,dx.\end{aligned} \tag{2.7}$$

Using the density equation again, it follows after integration by parts several times that

$$\begin{aligned}\int_U\rho^\gamma\mathrm{div}u\,dx&=-\int_U\langle\gamma\rho^{\gamma-2}\nabla\rho,\rho u\rangle\,dx=-\frac{\gamma}{\gamma-1}\int_U\langle\nabla\rho^{\gamma-1},\rho u\rangle\,dx\\&=\frac{\gamma}{\gamma-1}\int_U\rho^{\gamma-1}\mathrm{div}(\rho u)\,dx=-\frac{1}{\gamma-1}\frac{d}{dt}\int_U\rho^\gamma dx.\end{aligned} \tag{2.8}$$

Consequently, we finish the proof by combining (2.7) and (2.8). ∎

It is worth pointing that the assumption $c>0$ is necessary from a modeling point of view (see [30, 34]) so that the total energy $\mathcal{E}$ is bounded from below.

LEMMA 2.2. *For any smooth solution (ρ,u,Q) to the problem* (1.1)-(1.6),

$$\begin{aligned}\mathcal{E}(t)\geq &\int_U\left(\frac{\rho|u|^2}{2}+\frac{\rho^\gamma}{\gamma-1}\right)dx+\frac{L}{2}\|\nabla Q(t)\|^2+\frac{c}{8}\int_U\left[\operatorname{tr}(Q^2)+\frac{2a}{c}-\frac{2b^2}{c^2}\right]^2dx\\ &-\frac{1}{2c^3}(b^2-ca)^2|U|,\end{aligned}\tag{2.9}$$

where $|U|$ represents the Lebesgue measure of the domain U.

Proof. Because $Q\in S_0^3$, Q has three real eigenvalues at each point: λ_1, λ_2, and $-(\lambda_1+\lambda_2)$. Hence $\operatorname{tr}(Q^2)=2(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)$, $\operatorname{tr}(Q^3)=-3\lambda_1\lambda_2(\lambda_1+\lambda_2)$. Notice that

$$\begin{aligned}\operatorname{tr}(Q^3)=-3\lambda_1\lambda_2(\lambda_1+\lambda_2)&\leq 3(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)\left[\frac{\varepsilon(\lambda_1+\lambda_2)^2}{4}+\frac{1}{\varepsilon}\right]\\ &\leq 3(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)\left[\frac{\varepsilon(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)}{2}+\frac{1}{\varepsilon}\right]\\ &\leq\frac{3\varepsilon}{8}\operatorname{tr}^2(Q^2)+\frac{3}{2\varepsilon}\operatorname{tr}(Q^2).\end{aligned}\tag{2.10}$$

Taking $\varepsilon=\frac{c}{b}$ in (2.10), then we infer that

$$\begin{aligned}\mathcal{G}(Q)&\geq\frac{L}{2}\|\nabla Q\|^2+\int_U\frac{c}{8}\operatorname{tr}^2(Q^2)-\left(\frac{b^2}{2c}-\frac{a}{2}\right)\operatorname{tr}(Q^2)\ dx\\ &=\frac{L}{2}\|\nabla Q\|^2+\frac{c}{8}\int_U\left[\operatorname{tr}(Q^2)+\frac{2a}{c}-\frac{2b^2}{c^2}\right]^2dx-\frac{1}{2c^3}(b^2-ca)^2|U|.\end{aligned}\tag{2.11}$$

□

Consequently, using Proposition 2.1 and Lemma 2.2, it is straightforward to deduce the following a priori bounds for Q.

COROLLARY 2.3. *For any smooth solution (ρ,u,Q) to the problem* (1.1)-(1.6), *it holds*

$$Q\in L^{10}(0,T;U)\cap L^\infty([0,T];H^1(U))\cap L^2([0,T];H^2(U)),\quad \nabla Q\in L^{\frac{10}{3}}(0,T;U).\tag{2.12}$$

Proof. First, using Proposition 2.1 and Lemma 2.2, we have

$$\begin{aligned}&\frac{L}{2}\|\nabla Q(t)\|^2+\frac{c}{8}\int_U\left[\operatorname{tr}(Q^2)+\frac{2a}{c}-\frac{2b^2}{c^2}\right]^2dx\\ &\leq\frac{1}{2c^3}(b^2-ca)^2|U|+\mathcal{E}(t)\leq\frac{1}{2c^3}(b^2-ca)^2|U|+\mathcal{E}(0),\end{aligned}$$

hence $\nabla Q\in L^\infty(0,T;L^2(U))$. Meanwhile, using Hölder's inequality, it is easy to get from the above inequality that

$$\begin{aligned}\|Q(t)\|^4_{L^2(U)}&\leq|U|\int_U\operatorname{tr}^2(Q^2)\,dx\leq 2|U|\int_U\left[\operatorname{tr}(Q^2)+\frac{2a}{c}-\frac{2b^2}{c^2}\right]^2+\left(\frac{2a}{c}-\frac{2b^2}{c^2}\right)^2dx\\ &\leq\frac{16|U|}{c}\left[\frac{1}{2c^3}(b^2-ca)^2|U|+\mathcal{E}(0)\right]+\frac{8|U|^2}{c^4}(ac-b^2)^2,\end{aligned}$$

which indicates $Q\in L^\infty(0,T;L^2(U))$. Next, we observe that

$$
\begin{aligned}
&\frac{\Gamma}{2}\int_0^T\int_U L^2|\Delta Q(x,t)|^2\,dxdt\\
\le&\Gamma\int_0^T\int_U \mathrm{tr}^2(\mathcal{H})\,dxdt+\Gamma\int_0^T\int_U\Big|aQ-b\Big[Q^2-\frac{I_3}{3}\mathrm{tr}(Q^2)\Big]+cQ\mathrm{tr}(Q^2)\Big|^2\,dxdt\\
\le&\mathcal{E}(0)-\mathcal{E}(t)+C\Gamma\int_0^T\|Q\|^2_{H^1(U)}\,dt\\
\le&\frac{1}{2c^3}(b^2-ca)^2|U|+\mathcal{E}(0)+CT.
\end{aligned}
$$

Here $C>0$ depends on a, b, c, Γ, U, and $\mathcal{E}(0)$. Consequently, we know $\Delta Q\in L^2(0,T;L^2(U))$. Finally, we infer from the Gagliardo-Nirenberg inequality that

$$
\begin{aligned}
\|Q\|_{L^{10}(U)}&\le C\|Q\|^{\frac{4}{5}}_{L^6(U)}\|\Delta Q\|^{\frac{1}{5}}_{L^2(U)}+C\|Q\|_{L^6(U)}\\
&\le C\|Q\|^{\frac{4}{5}}_{H^1(U)}\|\Delta Q\|^{\frac{1}{5}}_{L^2(U)}+C\|Q\|_{H^1(U)},\\
\|\nabla Q\|_{L^{\frac{10}{3}}(U)}&\le C\|\nabla Q\|^{\frac{2}{5}}_{L^2(U)}\|\Delta Q\|^{\frac{3}{5}}_{L^2(U)}+C\|\nabla Q\|_{L^2(U)},
\end{aligned}
$$

thus the proof is complete by noting that $Q\in L^\infty(0,T;H^1(U))$ and $\Delta Q\in L^2(0,T;L^2(U))$. ∎

Next, we introduce the definition of finite energy weak solutions.

DEFINITION 2.4. *For any $T>0$, (ρ,u,Q) is called a finite energy weak solution to the problem* (1.1)-(1.6) *if the following conditions are satisfied.*

- $\rho\ge 0$, $\rho\in L^\infty([0,T];L^\gamma(U))$, $\quad u\in L^2([0,T];H_0^1(U))$, $Q\in L^\infty([0,T];H^1(U))\cap L^2([0,T];H^2(U))$, *and* $Q\in S_0^3$ *a.e. in* $U\times[0,T]$.
- *Equations* (1.1)-(1.3) *are valid in* $\mathcal{D}'((0,T),U)$. *Moreover,* (1.1) *is valid in* $\mathcal{D}'((0,T),\mathbb{R}^3)$ *if* ρ,u *are extended to be zero on* $\mathbb{R}^3\setminus U$.
- *The energy* $\mathcal{E}$ *is locally integrable on* $(0,T)$ *and the energy inequality*

$$\frac{d}{dt}\mathcal{E}(t)+\int_U\big(\nu|\nabla u|^2+(\nu+\lambda)|\mathrm{div}\,u|^2+\Gamma\mathrm{tr}^2(\mathcal{H})\big)\,dx\le 0 \qquad \text{holds in } D'(0,T).$$

- *For any function $g\in C^1(\mathbb{R}^+)$ with the property*

$$\text{there exists a positive constant } M=M(g) \text{ such that } g'(z)=0, \text{ for all } z\ge M, \tag{2.13}$$

the following renormalized form of the density equation holds in $\mathcal{D}'((0,T),U)$:

$$g(\rho)_t+\mathrm{div}(g(\rho)u)+(g'(\rho)\rho-g(\rho))\mathrm{div}\,u=0. \tag{2.14}$$

Now we can state the main result of this paper on the existence of global weak solutions.

THEOREM 2.5. *Suppose $\gamma>\frac{3}{2}$ and that the compatibility condition* (1.7) *is satisfied. Then for any $T>0$, the problem* (1.1)-(1.6) *admits a finite energy weak solution (ρ,u,Q) on $(0,T)\times U$.*

We shall prove Theorem 2.5 via a three-level approximation scheme which consists of Faedo-Galerkin approximation, artificial viscosity, and artificial pressure, as well as the weak convergence method.

3. The Faedo-Galerkin approximation

3.1. Approximate solutions. In this section, our goal is to solve the following problem:

$$\rho_t + \operatorname{div}(\rho u) = \varepsilon \Delta \rho, \tag{3.1}$$

$$\begin{aligned}&(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) + \delta \nabla \rho^\beta + \varepsilon \nabla \rho \cdot \nabla u \\ &= \mathcal{L}u - \nabla \cdot \big(L \nabla Q \odot \nabla Q - \mathcal{F}(Q) I_3\big) + L \nabla \cdot (Q\mathcal{H}(Q) - \mathcal{H}(Q)Q),\end{aligned} \tag{3.2}$$

$$Q_t + u \cdot \nabla Q - \Omega Q + Q\Omega = \Gamma \mathcal{H}(Q), \tag{3.3}$$

with modified initial conditions:

$$\rho|_{t=0} = \rho_0 \in C^3(\bar{U}),\ 0 < \underline{\rho} \le \rho_0(x) \le \bar{\rho},\ \frac{\partial \rho_0}{\partial n}\Big|_{\partial U} = 0, \tag{3.4}$$

$$\rho u|_{t=0} = q(x) \in C^2(\bar{U}, \mathbb{R}^3),\ Q|_{t=0} = Q_0(x),\quad Q_0 \in H^1(U),\quad Q_0 \in S_0^3 \text{ a.e. in } U. \tag{3.5}$$

Here $\underline{\rho}$ and $\bar{\rho}$ are two positive constants. The problem is subject to the following boundary conditions:

$$\frac{\partial \rho}{\partial \vec{n}}\Big|_{\partial U} = 0, \tag{3.6}$$

$$u|_{\partial U} = 0,\ Q|_{\partial U} = Q_0(x). \tag{3.7}$$

REMARK 3.1. It is noted that (cf. [12]) the extra term $\varepsilon\Delta\rho$ appearing on the right-hand side of equation (3.1) represents a "vanishing viscosity" without any physical meaning. On the other hand, such a mathematical operation converts the original hyperbolic equation (1.1) to a parabolic one such that one can expect better regularity results for ρ at this point. Meanwhile, the extra quantity $\varepsilon \nabla\rho \cdot \nabla u$ in equation (3.2) is added to cancel extra terms in order to establish necessary energy laws (see (3.22) below). The term $\delta\rho^\beta$ is added to achieve higher integrability for ρ, which is shown in the next section.

To begin with, using a standard argument shown in [10], we have the following existence result.

LEMMA 3.1. *For the initial-boundary value problem* (3.1), (3.4), *and* (3.6), *there exists a mapping* $\mathcal{S} = S(u) \colon C([0,T]; C^2(\bar{U}, \mathbb{R}^3)) \to C([0,T]; C^3(U))$ *with the following properties:*

(i) $\rho = \mathcal{S}(u)$ *is the unique classical solution of* (3.1), (3.4), *and* (3.6);

(ii) $\underline{\rho} \exp\big(-\int_0^t \|\operatorname{div} u(s)\|_{L^\infty(U)} ds\big) \le \rho(t,x) \le \bar{\rho} \exp\big(\int_0^t \|\operatorname{div} u(s)\|_{L^\infty(U)} ds\big)$;

(iii) For any u_1, u_2 *in the set*

$$\mathcal{M}_k = \{u \in C([0,T]; H_0^1(U)),\ s.t.\ \|u(t)\|_{L^\infty(U)} + \|\nabla u(t)\|_{L^\infty(U)} \le k, \forall t\},$$

it holds that

$$\|\mathcal{S}(u_1) - \mathcal{S}(u_2)\|_{C([0,T]; H^1(U))} \le Tc(k,T) \|u_1 - u_2\|_{C([0,T]; H_0^1(U))}. \tag{3.8}$$

Next, we shall provide the following lemma which is useful for subsequent arguments in the Faedo-Galerkin approximation scheme.

Lemma 3.2. *For each* $u\in C([0,T];C_0^2(\bar U,\mathbb{R}^3))$, *there exists a unique solution* $Q\in L^\infty([0,T];\ H^1(U))\cap L^2([0,T];H^2(U))$ *to the initial boundary value problem*

$$Q_t+u\cdot\nabla Q-\Omega Q+Q\Omega=\Gamma\mathcal{H}(Q), \tag{3.9}$$

$$Q|_{t=0}=Q_0(x),\quad Q|_{\partial U}=Q_0, \tag{3.10}$$

with Q_0 *satisfying* (1.5). *Moreover, the above mapping* $u\mapsto Q[u]$ *is continuous from each bounded set of* $C([0,T];C_0^2(\bar U,\mathbb{R}^3))$ *to* $L^\infty([0,T];\ H^1(U))\cap L^2([0,T];H^2(U))$. *Furthermore,* $Q[u]\in S_0^3$ *a.e. in* $U\times[0,T]$.

Proof. For each $u\in C([0,T];C_0^2(\bar U,\mathbb{R}^3))$, the existence of such Q is guaranteed by standard parabolic theory (cf. [28]). To prove that Q lies in $L^\infty([0,T];\ H^1(U))\cap L^2([0,T];H^2(U))$, suppose $\|u\|_{C(0,T;C_0^2(\bar U))}\le M$ for some positive constant M. We multiply equation (3.9) with $-\Delta Q$, then take the trace and integrate over U, using Young's inequality, to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla Q\|^2+\Gamma L\|\Delta Q\|^2&=\int_U(u\cdot\nabla Q):\Delta Q\,dx+\int_U(Q\Omega):\Delta Q\,dx-\int_U(\Omega Q):\Delta Q\,dx\\&\quad+\int_U\Big(aQ-bQ^2+\frac{b}{3}\mathrm{tr}(Q^2)I_3+cQ\mathrm{tr}(Q^2)\Big):\Delta Q\,dx\\&\le\frac{\Gamma L}{4}\|\Delta Q\|^2+C_1\|Q\|_{H^1}^2.\end{aligned}$$

Next, multiplying equation (3.9) with Q, in a similar way we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Q\|^2&=\Gamma L\int_U\Delta Q:Q\,dx-\int_U(u\cdot\nabla Q):Q\,dx+\int_U\frac{a}{2}\mathrm{tr}(Q^2)-\frac{b}{3}\mathrm{tr}(Q^3)+\frac{c}{4}\mathrm{tr}^2(Q^2)\,dx\\&\le\frac{\Gamma L}{4}\|\Delta Q\|^2+C_2\|Q\|_{H^1}^2.\end{aligned}$$

Here $C_1>0$ and $C_2>0$ are two constants which may depend on M, a, b, c, Γ and L. Summing up the above two equations, we obtain

$$\frac{d}{dt}\|Q\|_{H^1}^2+\Gamma L\|\Delta Q\|^2\le C\|Q\|_{H^1}^2.$$

Using Gronwall's inequality again, we infer that

$$\|Q\|_{L^\infty(0,T;H^1(U))}+\|Q\|_{L^2(0,T;H^2(U))}\le C^*, \tag{3.11}$$

where $C^*>0$ is a constant which may depend on M, $\|Q_0\|_{H^1(U)}$, a, b, c, Γ, L, and T.

To prove uniqueness, suppose Q_1 and Q_2 are two different solutions. Then $\bar Q=Q_1-Q_2$ satisfies

$$\begin{aligned}\bar Q_t+u\cdot\nabla\bar Q-\Omega\bar Q+\bar Q\Omega=\Gamma\Big(&L\Delta\bar Q-a\bar Q+b\big[Q_1^2-Q_2^2-\frac{I_3}{3}\mathrm{tr}(Q_1^2-Q_2^2)\big]\\&-cQ_1\mathrm{tr}(Q_1^2)+cQ_2\mathrm{tr}(Q_2^2)\Big),\end{aligned} \tag{3.12}$$

$$\bar Q|_{t=0}=0,\quad \bar Q|_{\partial U}=0. \tag{3.13}$$

Multiplying both sides of equation (3.12) with $\bar Q$, then taking its trace and integrating over U, due to the assumption $u\in C([0,T];C_0^2(\bar U,\mathbb{R}^3))$ and the fact that $\|Q\|_{L^\infty(0,T;H^1(U))}\le C^*$, for $Q=Q_1,Q_2$, we get

$$\frac{1}{2}\frac{d}{dt}\|\bar Q\|^2+\Gamma L\|\nabla\bar Q\|^2$$

$$
\begin{aligned}
&= -\int_U (u\cdot\nabla\bar{Q}):\bar{Q}\,dx - \Gamma a\|\bar{Q}\|^2 + \Gamma b\int_U [\bar{Q}(Q_1+Q_2)]:\bar{Q}\,dx \\
&\quad -\frac{\Gamma b}{3}\int_U [\bar{Q}(Q_1+Q_2)]\mathrm{tr}(\bar{Q})\,dx - \Gamma c\int_U \mathrm{tr}(\bar{Q}^2)\mathrm{tr}(Q_1^2) + (Q_2:\bar{Q})\big(\bar{Q}:(Q_1+Q_2)\big)\,dx \\
&\le M\|\nabla\bar{Q}\|\|\bar{Q}\| + \Gamma|a|\|\bar{Q}\|^2 + \frac{4\Gamma b}{3}\|\bar{Q}\|_{L^6(U)}\|\bar{Q}\|\|Q_1+Q_2\|_{L^3(U)} \\
&\quad +2\Gamma c\|\bar{Q}\|_{L^6(U)}\|\bar{Q}\|\big(\|Q_1\|^2_{L^6(U)} + \|Q_2\|^2_{L^6(U)}\big) \\
&\le \frac{\Gamma L}{2}\|\nabla\bar{Q}\|^2 + C\|\bar{Q}\|^2, \qquad (3.14)
\end{aligned}
$$

where we used the Sobolev embedding inequality, Poincaré inequality, and Young's inequality to obtain the last inequality. Here C is a positive constant which depends on U, M, a, b, c, Γ, and L. Hence we arrive at the uniqueness result by applying Gronwall's inequality.

Then we let $\{u_n\}$ be a bounded sequence in $C_0^2(\bar{U},\mathbb{R}^3)$, with $\|u_n\|_{C(0,T;C_0^2(\bar{U}))} \le M$, $\forall n\in\mathbb{N}$, and

$$
\lim_{n\to\infty}\|u_n - u\|_{C(0,T;C_0^2(\bar{U}))} = 0, \qquad (3.15)
$$

for some $u\in C(0,T;C_0^2(\bar{U}))$. For the mappings $u_n\mapsto Q_n$, $u\mapsto Q$, we denote $\bar{Q}_n = Q_n - Q$, and we are going to show that

$$
\lim_{n\to\infty}\|\bar{Q}_n\|_{L^\infty(0,T;H^1(U))} + \|\bar{Q}_n\|_{L^2(0,T;H^2(U))} = 0. \qquad (3.16)
$$

Take the difference of the equations given by Q_n and Q, and then take the inner product with $-\Delta\bar{Q}_n$. We have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\bar{Q}_n\|^2 + \Gamma L\|\Delta\bar{Q}_n\|^2 \\
&= \int_U (u_n\cdot\nabla Q_n - u\cdot\nabla Q):\Delta\bar{Q}_n\,dx - \int_U (Q_n\Omega_n - Q\Omega):\Delta\bar{Q}_n\,dx \\
&\quad + \int_U (\Omega_n\bar{Q}_n - \Omega Q):\Delta\bar{Q}_n\,dx + \Gamma a\int_U \bar{Q}_n:\Delta\bar{Q}_n\,dx - \Gamma b\int_U [\bar{Q}_n(Q_n+Q)]:\Delta\bar{Q}_n\,dx \\
&\quad +\Gamma c\int_U \big(Q_n\mathrm{tr}(Q_n^2) - Q\mathrm{tr}(Q^2)\big):\Delta\bar{Q}_n\,dx \\
&\doteq I_1 + \cdots + I_6, \qquad (3.17)
\end{aligned}
$$

with $\Omega_n = \frac{\nabla u_n - \nabla^T u_n}{2}$, $n = 1,2,\ldots$. Noting that $\|Q_n\|_{L^\infty(0,T;H^1(U))} + \|Q_n\|_{L^2(0,T;H^2(U))} \le C^*$ uniformly for $n\in\mathbb{N}$, we can estimate I_1 to I_6 as follows:

$$
\begin{aligned}
I_1 &\le \int_U \big(|u_n\cdot\nabla\bar{Q}_n||\Delta\bar{Q}_n| + |u_n - u||\nabla Q||\Delta\bar{Q}_n|\big)\,dx \\
&\le \int_U \big(\|u_n\|_{L^\infty(U)}|\nabla\bar{Q}_n||\Delta\bar{Q}_n| + \|u_n - u\|_{L^\infty(U)}|\nabla\bar{Q}||\Delta\bar{Q}_n|\big)\,dx \\
&\le M\|\nabla\bar{Q}_n\|\|\Delta\bar{Q}_n\| + \|u_n - u\|_{L^\infty(U)}\int_U \left(\frac{12M}{\Gamma L}|\nabla\bar{Q}|^2 + \frac{\Gamma L}{48M}|\Delta\bar{Q}_n|^2\right)dx \\
&\le \frac{\Gamma L}{12}\|\Delta\bar{Q}_n\|^2 + \frac{12M(C^*)^2}{\Gamma L}\|u_n - u\|_{L^\infty(U)} + C\|\nabla\bar{Q}_n\|^2.
\end{aligned}
$$

$$
\begin{aligned}
I_2 &\le \int_U \left(|\bar{Q}_n||\Omega_n||\Delta\bar{Q}_n| + |Q||\Omega_n - \Omega||\Delta\bar{Q}_n|\right) dx \\
&\le M\|\bar{Q}_n\|\|\Delta\bar{Q}_n\| + \|\nabla u_n - \nabla u\|_{L^\infty(U)} \int_U \left(\frac{12M}{\Gamma L}|Q|^2 + \frac{\Gamma L}{48M}|\Delta\bar{Q}_n|^2\right) dx \\
&\le \frac{\Gamma L}{12}\|\Delta\bar{Q}_n\|^2 + C\|\nabla u_n - \nabla u\|_{L^\infty(U)} + C\|\nabla\bar{Q}_n\|^2,
\end{aligned}
$$

where we used Poincaré's inequality in the last step because $\bar{Q}_n|_{\partial U} = 0$. As for I_2, we get

$$
I_3 \le \frac{\Gamma L}{12}\|\Delta\bar{Q}_n\|^2 + C\|\nabla u_n - \nabla u\|_{L^\infty(U)} + C\|\nabla\bar{Q}_n\|^2.
$$

For I_4 and I_5, using Poincaré's inequality again yields

$$
I_4 \le \frac{\Gamma L}{12}\|\Delta\bar{Q}_n\|^2 + C\|\nabla\bar{Q}_n\|^2,
$$

$$
I_5 \le \Gamma b\|Q_n + Q\|_{L^6(U)}\|\bar{Q}_n\|_{L^3(U)}\|\Delta\bar{Q}_n\| \le \frac{\Gamma L}{12}\|\Delta\bar{Q}_n\|^2 + C\|\nabla\bar{Q}_n\|^2.
$$

Similarly,

$$
\begin{aligned}
I_6 &\le \Gamma c\|Q_n\|^2_{L^6(U)}\|\bar{Q}_n\|_{L^6(U)}\|\Delta\bar{Q}_n\| + \Gamma c\|Q\|_{L^6(U)}\|Q_n + Q\|_{L^6(U)}\|\bar{Q}_n\|_{L^6(U)}\|\Delta\bar{Q}_n\| \\
&\le \frac{\Gamma L}{12}\|\Delta\bar{Q}_n\|^2 + C\|\nabla\bar{Q}_n\|^2.
\end{aligned}
$$

Putting all these estimates together, we get

$$
\frac{d}{dt}\|\nabla\bar{Q}_n\|^2 + \Gamma L\|\Delta\bar{Q}_n\|^2 \le C\|u_n - u\|_{L^\infty(0,T;C_0^2(U))} + C\|\nabla\bar{Q}_n\|^2.
$$

Therefore, we conclude from Gronwall's inequality that

$$
\|\nabla\bar{Q}_n\|^2(t) + \int_0^T \|\Delta\bar{Q}_n\|^2 dt \le e^{CT}\|u_n - u\|_{L^\infty(0,T;C_0^2(U))}, \quad \forall t \in [0,T]. \tag{3.18}
$$

Hence we can prove (3.16) by letting $n \to \infty$.

To finish the proof of this lemma, we finally show that $Q \in S_0^3$, namely, $Q = Q^T$ and $\mathrm{tr}(Q) = 0$ a.e. in $U \times [0,T]$. It is easy to observe that if Q is a solution to (3.9), so is Q^T. Hence $Q = Q^T$ a.e. by the aforementioned uniqueness result. Then taking the trace of both sides of equation (3.9), and using the properties $\Omega = -\Omega^T$ and $Q = Q^T$, we have

$$
\begin{gathered}
\frac{\partial}{\partial t}\mathrm{tr}(Q) - u\cdot\nabla\mathrm{tr}(Q) = \Gamma L\Delta\mathrm{tr}(Q) - \Gamma a\,\mathrm{tr}(Q) - \Gamma c\,\mathrm{tr}(Q)\mathrm{tr}(Q^2), \\
\mathrm{tr}(Q)|_{t=0} = 0, \quad \mathrm{tr}(Q)|_{\partial U} = 0.
\end{gathered}
$$

Consequently, after multiplying both sides of the above equation with $\mathrm{tr}(Q)$ and integration over U, we can complete the proof by applying the initial and boundary conditions and Gronwall's inequality. ∎

We proceed to solve (3.1)-(3.7) by the Faedo-Gelerkin approximation scheme. Let $\{\psi_n\}_{n=1}^\infty \subset C^\infty(U,\mathbb{R}^3)$ be the eigenfunctions of the Laplacian operator that vanish on the boundary:

$$
-\Delta\psi_n = \lambda_n\psi_n \quad \text{in } U, \quad \psi_n|_{\partial U} = 0.
$$

Here $0<\lambda_1\leq\lambda_2\leq...$ are eigenvalues and $\{\psi_n\}_{n=1}^{\infty}$ forms an orthogonal basis of $H_0^1(U)$. Let $X_n\doteq\text{span}\{\psi_1,\cdots,\psi_n\}$, $n=1,2,\ldots$ be a sequence of finite dimensional spaces.

Then we consider the following variational approximate problem for $u_n\in C([0,T],X_n)$: $\forall t\in[0,T],\forall\psi\in X_n$,

$$
\begin{aligned}
&\int_U\langle\rho u_n(t),\psi\rangle\,dx-\int_U\langle q,\psi\rangle\,dx\\
=&\int_0^t\int_U\big\langle\mathcal{L}u_n-\text{div}(\rho u_n\otimes u_n)-(\rho^\gamma+\delta\rho^\beta)-\varepsilon\nabla\rho\cdot\nabla u_n,\psi\big\rangle\,dxds\\
&-\int_0^t\int_U\big\langle\nabla\cdot\big(L\nabla Q_n\odot\nabla Q_n-\mathcal{F}(Q_n)I_3\big),\psi\big\rangle\,dxds\\
&-L\int_0^t\int_U\langle\nabla\cdot(Q_n\mathcal{H}(Q_n)-\mathcal{H}(Q_n)Q_n),\psi\rangle\,dxds. \qquad (3.19)
\end{aligned}
$$

Next, following the idea in [10], we introduce a family of operators

$$
\mathcal{M}[\rho]:X_n\mapsto X_n^*,\ \mathcal{M}[\rho]v(w)=\int_U\langle\rho v,w\rangle dx,\ \forall v,w\in X_n.
$$

Here the existence and uniqueness of the solution Q_n to (3.3) is guaranteed by Lemma 3.2, while $\rho=\mathcal{S}(u_n)$ is the unique classical solution to (3.1) given by Lemma 3.1.

It follows from the arguments in [10] that the map

$$
\rho\mapsto\mathcal{M}^{-1}[\rho]
$$

from $N_\eta=\{\rho\in L^1(U)\,|\,\inf_{x\in U}\rho\geq\eta>0\}$ is well defined and satisfies

$$
\big\|\mathcal{M}^{-1}[\rho^1]-\mathcal{M}^{-1}[\rho^2]\big\|_{\mathcal{L}(X_n^*,X_n)}\leq C(n,\eta)\|\rho^1-\rho^2\|_{L^1(U)}. \qquad (3.20)
$$

Meanwhile, due to Lemma 3.1, we may rewrite the variational problem (3.19) as: $\forall t\in[0,T],\forall\psi\in X_n$,

$$
u_n(t)=\mathcal{M}^{-1}[\mathcal{S}(u_n)(t)]\Big(q^*+\int_0^t\mathcal{N}[\mathcal{S}(\rho_n(s),u_n(s),Q_n(s)]\,ds\Big), \qquad (3.21)
$$

with

$$
\begin{aligned}
\langle\mathcal{N}[\rho_n,u_n,Q_n],\psi\rangle=&\int_U\big\langle\mathcal{L}u_n-\text{div}(\rho_n u_n\otimes u_n)-(\rho_n^\gamma+\delta\rho_n^\beta)-\varepsilon\nabla\rho_n\cdot\nabla u_n,\psi\big\rangle\,dx\\
&-\int_U\big\langle\nabla\cdot\big(L\nabla Q_n\odot\nabla Q_n-\mathcal{F}(Q_n)I_3\big),\psi\big\rangle\,dx\\
&-L\int_U\langle\nabla\cdot(Q_n\mathcal{H}(Q_n)-\mathcal{H}(Q_n)Q_n),\psi\rangle\,dx,\\
\rho_n=\mathcal{S}(u_n),\quad Q_n=&Q_n[S_n],\quad q^*\in X_n^*,\quad\text{and}\quad q^*(\psi)=\int_U\langle q,\psi\rangle\,dx.
\end{aligned}
$$

Therefore, in view of (3.8) and (3.20), using a standard fixed point theorem on $C([0,T],X_n)$, we obtain a local solution (ρ_n,u_n,Q_n) on a short time interval $[0,T_n],T_n\leq T$ to the problem (3.1), (3.3), (3.19), with initial and boundary conditions (3.4)-(3.7).

Now we shall extend the local existence time T_n to T. First we can derive an energy law in a similar manner as Proposition 2.1, namely $\forall t \in (0,T_n)$, it holds that

$$\begin{aligned}&\frac{d}{dt}\int_U \Big[\frac{\rho_n|u_n|^2}{2}+\frac{\rho_n^\gamma}{\gamma-1}+\frac{\delta\rho_n^\beta}{\beta-1}+\mathcal{G}(Q_n)\Big]dx\\&\quad+\int_U \big(\nu|\nabla u|^2+(\nu+\lambda)|\mathrm{div}\, u|^2+\Gamma\mathrm{tr}^2(\mathcal{H}_n)\big)\,dx+\varepsilon\int_U\big(\gamma\rho_n^{\gamma-2}+\delta\beta\rho_n^{\beta-2}\big)|\nabla\rho_n|^2dx\\&\le 0. \end{aligned} \tag{3.22}$$

Consequently, combined with Lemma 2.2, we have

$$\int_0^{T_n}\|\nabla u_n\|^2dt\le\frac{2}{\nu}E_\delta[\rho_0,q_0,Q_0],$$

with

$$E_\delta[\rho_0,q_0,Q_0]\doteq\int_U\Big(\frac{|q_0|^2}{2\rho_0}+\frac{\rho_0^\gamma}{\gamma-1}+\frac{\delta\rho_0^\beta}{\beta-1}+\mathcal{G}(Q_0)\Big)\,dx+\frac{(b^2-ca)^2}{2c^3}|U|. \tag{3.23}$$

Meanwhile, because the L^2 norm and H^2 norm are equivalent on each finite dimensional space X_n, we can deduce from Lemma 3.1 that there exists $C_2 = C_2(n,\rho_0,q_0,Q_0,a,b,c,U)$, such that

$$0<C_2\le\rho_n(t,x)\le\frac{1}{C_2},\quad \forall t\in(0,T_n),\ x\in U.$$

Therefore, using the energy inequality (3.22) again, we know that

$$\|u_n(t)\|_{L^\infty(U)}+\|\nabla u_n(t)\|_{L^\infty(U)}\le C_3=C_3(n,\rho_0,q_0,Q_0,a,b,c,U),\quad\forall t\in[0,T_n],$$

which allows us to extend the existence interval $(0,T_n)$ of u_n to [0, T]. Further, we know from Lemma 3.1 and Lemma 3.2 that the local solution Q_n and ρ_n can also be extended up to T.

To finish this subsection, we summarize all the results in the following lemma, part of which is based on (3.22), arguments in Lemma 2.2, and Corollary 2.3, while (3.28) and (3.29) are due to interpolation inequalities (see [10] for details).

Lemma 3.3. *If $\beta\ge 4$, then there exists solution (ρ_n,u_n,Q_n) to* (3.1)*,* (3.19)*,* (3.9) *in $(0,T)\times U$, and*

$$\sup_{t\in[0,T]}\|\rho_n(t)\|^\gamma_{L^\gamma(U)}\le C(E_\delta[\rho_0,q_0,Q_0],\gamma), \tag{3.24}$$

$$\delta\sup_{t\in[0,T]}\|\rho_n(t)\|^\beta_{L^\beta(U)}\le C(E_\delta[\rho_0,q_0,Q_0],\beta), \tag{3.25}$$

$$\sup_{t\in[0,T]}\big\|\sqrt{\rho_n}(t)u_n(t)\big\|^2_{L^2(U)}\le 2E_\delta[\rho_0,q_0,Q_0], \tag{3.26}$$

$$\|u_n\|_{L^2(0,T;H^1_0(U))}\le C(E_\delta[\rho_0,q_0,Q_0],\lambda,\nu), \tag{3.27}$$

$$\|\rho_n\|_{L^{\beta+1}((0,T)\times U)}\le C(E_\delta[\rho_0,q_0,Q_0],\varepsilon,\delta,U), \tag{3.28}$$

$$\varepsilon\|\nabla\rho_n\|^2_{L^2(0,T;L^2(U))}\le C(E_\delta[\rho_0,q_0,Q_0],\beta,\delta,U,T), \tag{3.29}$$

$$\|Q_n\|_{L^{10}((0,T)\times U)}\le C(E_\delta[\rho_0,q_0,Q_0],a,b,c,L,\Gamma,U,T), \tag{3.30}$$

$$\|Q_n\|_{L^\infty(0,T;H^1(U))}\le\frac{2}{L}E_\delta[\rho_0,q_0,Q_0], \tag{3.31}$$

$$\|\nabla Q_n\|_{L^{\frac{10}{3}}((0,T)\times U)} \le C(E_\delta[\rho_0,q_0,Q_0],a,b,c,L,\Gamma,U,T), \tag{3.32}$$

$$\|Q_n\|_{L^2(0,T;H^2(U))} \le C(E_\delta[\rho_0,q_0,Q_0],a,b,c,L,\Gamma,U,T). \tag{3.33}$$

3.2. Passing to the limit. Now we shall employ the estimate in Lemma 3.3 to pass to the limit as $n\to\infty$ of the solution sequence (ρ_n,u_n,Q_n) to obtain a solution to the problem (3.1)-(3.7). To this end, we have to ensure that all of these a priori estimates are independent of n. Here and after, for the sake of convenience, we do not distinguish sequence convergence and subsequence convergence.

To begin with, it follows from Lemma 2.3 in [10] that if $\beta > 4$ and $\gamma > \frac{3}{2}$, then

$$u_n \to u \ \text{ weakly in } L^2(0,T;H_0^1(U,\mathbb{R}^3)), \tag{3.34}$$

$$\rho_n \to \rho \ \text{ in } L^4((0,T)\times U), \tag{3.35}$$

$$\rho_n^\gamma \to \rho^\gamma, \ \ \rho_n^\beta \to \rho^\beta \text{ in } L^1((0,T)\times U). \tag{3.36}$$

Meanwhile, using Sobolev's inequality, we deduce from (3.30)-(3.33) that

$$\begin{aligned}\Big\|\frac{\partial Q_n}{\partial t}\Big\|_{L^{\frac{3}{2}}(U)} &\le \|u_n\|_{L^6(U)}\|\nabla Q_n\|_{L^2(U)} + 2\|\nabla u_n\|_{L^2(U)}\|Q_n\|_{L^6(U)} + \Gamma\|\mathcal{H}_n\|_{L^{\frac{3}{2}}(U)} \\ &\le C\|\nabla u_n\| + C\|\mathcal{H}_n\|, \end{aligned} \tag{3.37}$$

which implies that

$$\Big\|\frac{\partial Q_n}{\partial t}\Big\|_{L^2(0,T;L^{\frac{3}{2}}(U))} \le C(E_\delta[\rho_0,q_0,Q_0],a,b,c,L,\Gamma,\lambda,\nu,U,T).$$

Combined with (3.33), we know from the well-known Aubin-Lions compactness theorem that

$$\{Q_n\} \text{ is precompact in } L^2(0,T;H^1(U)).$$

Therefore, we conclude that

$$Q_n \to Q \text{ weakly in } L^2(0,T;H^2(U)), \text{ strongly in } L^2(0,T;H^1(U)).$$

Hence it is easy to show that Q is a weak solution to (3.3). Furthermore, we get from (3.24), (3.26), and (3.27) that $\{\rho_n u_n\}$ is uniformly bounded in $L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(U))$. Consequently, using (3.34) and (3.35), we have

$$\rho_n u_n \to \rho u \ \text{ weakly star in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(U)), \tag{3.38}$$

and then we can pass to limit in the continuity equation (3.1).

Finally, in order the prove that the limit u satisfies equation (3.2), we need the following lemma in [10].

Lemma 3.4. *There exist $r>1$, $s>2$ such that $\partial_t\rho_n$, $\Delta\rho_n$ are uniformly bounded in $L^r((0,T)\times U)$, and $\nabla\rho_n$ is uniformly bounded in $L^s(0,T)\times U)$. The limit function ρ satisfies equation* (3.1) *almost everywhere on $(0,T)\times U$ and the boundary condition* (3.6) *in the trace sense.*

We now show that for any fixed test function ψ in (3.19), $\int_U \langle \rho_n u_n(t),\psi\rangle dx$ is equi-continuous in t. By Lemma 3.3 and Lemma 3.4, we get, for any $0<\zeta<1$,

$$\Big|\int_t^{t+\zeta}\int_U \langle L\nabla Q_n\odot\nabla Q_n - \mathcal{F}(Q_n)I_3 - LQ_n\mathcal{H}(Q_n) + L\mathcal{H}(Q_n)Q_n, \nabla\psi\rangle\, dxds\Big|$$

$$
\begin{aligned}
&\le C\zeta\|\nabla\psi\|_{L^\infty(U)}\sup_{0\le t\le T}(\|\nabla Q_n(t)\|^2+1)+C\int_t^{t+\zeta}\|Q_n\|\|\Delta Q_n\|\|\nabla\psi\|_{L^\infty(U)}\,ds\\
&\le C\zeta+C\|\nabla\psi\|_{L^\infty(U)}\Big(\int_t^{t+\zeta}\|\Delta Q_n\|^2\,ds\Big)^{\frac12}\Big(\int_t^{t+\zeta}\|Q_n\|^2ds\Big)^{\frac12}\\
&\le C\zeta^{\frac12},
\end{aligned}
$$

$$
\Big|\int_t^{t+\zeta}\int_U\langle\rho_n^\gamma+\delta\rho_n^\beta,\psi\rangle\,dxdt\Big|\le C\zeta\|\nabla\psi\|_{L^\infty(U)}\sup_{0\le t\le T}\int_U\big(\rho_n^\gamma+\delta\rho_n^\beta\big)\,dx\le C\zeta,
$$

$$
\Big|\int_t^{t+\zeta}\int_U\langle\mathcal{L}u_n,\psi\rangle\,dxds\Big|\le C\zeta\|\nabla\psi\|_{L^2(U)}\Big(\int_0^T\|\nabla u_n(t)\|^2\,dt\Big)^{\frac12}\le C\zeta,
$$

$$
\Big|\int_t^{t+\zeta}\int_U\langle\nabla\cdot(\rho_n u_n\otimes u_n),\psi\rangle\,dxds\Big|\le C\zeta\|\nabla\psi\|_{L^\infty}\sup_{0\le t\le T}\int_U\rho_n|u_n|^2dx\le C\zeta,
$$

$$
\begin{aligned}
&\Big|\int_t^{t+\zeta}\int_U\langle\varepsilon\nabla\rho_n\cdot\nabla u_n,\psi\rangle\,dxds\Big|\\
&\le\varepsilon\|\psi\|_{L^\infty(U)}\Big(\int_t^{t+\zeta}\int_U|\nabla u_n|^2dt\Big)^{\frac12}\Big(\int_t^{t+\zeta}\big(\int_U|\nabla\rho_n|^2\big)^{\frac{s}{2}}\Big)^{\frac1s}\zeta^{\frac12-\frac1s}\\
&\le C\zeta^{\frac12-\frac1s},
\end{aligned}
$$

where we used Lemma 3.4 for the last estimate. Hence we know (cf. Corollary 2.1 in [12])

$$
\rho_n u_n\to\rho u\ \text{ in }\ C\big([0,T];L^{\frac{2\gamma}{\gamma+1}}_{weak}\big). \tag{3.39}
$$

Due to the compact embedding $L^{\frac{2\gamma}{\gamma+1}}(U)\hookrightarrow H^{-1}(U)$ if $\gamma>\frac32$ (cf. [10]), we infer from (3.39) that

$$
\rho_n u_n\to\rho u\ \text{ in } C([0,T];H^{-1}(U)),
$$

which together with (3.34) indicates

$$
\rho_n u_n\otimes u_n\to\rho u\otimes u\ \text{ in } \mathcal{D}'((0,T)\times U).
$$

Finally, the convergence of the remaining term $\nabla\rho_n\cdot\nabla u_n\to\nabla\rho\cdot\nabla u$ in $\mathcal{D}'((0,T)\times U)$ follows [10].

In all, we summarize the above results as follows.

Proposition 3.5. *The problem* (3.1)-(3.7) *admits a weak solution* (ρ,u,Q) *which satisfies all estimates in Lemma 3.3. Moreover, the energy inequality* (3.22) *holds in* $\mathcal{D}'(0,T)$ *and there exists* $r>1$*, such that* $\rho_t,\Delta\rho\in L^r((0,T)\times U)$ *and the equation* (3.1) *is satisfied pointwise in* $(0,T)\times U$*. In addition,* $Q\in S_0^3$ *a.e. in* $[0,T]\times U$.

4. Vanishing artificial viscosity

Our next aim is to let $\varepsilon\to0$ in the modified continuity equation (3.1) and velocity equation (3.2). We denote by $(\rho_\varepsilon,u_\varepsilon,Q_\varepsilon)$ the corresponding solution of the problem (3.1)-(3.7). At this point, we are lacking in the bound of $\nabla\rho_\varepsilon$ (see (3.29)) and consequently, it is essential to study the strong compactness of $\{\rho_\varepsilon\}_{\varepsilon>0}$ in $L^1((0,T)\times U)$.

4.1. Density estimates independent of viscosity. To begin with, we deduce from (3.27) and (3.29) that

$$\varepsilon\nabla\rho_\varepsilon\cdot\nabla u_\varepsilon\to 0 \text{ in } L^1((0,T)\times U), \tag{4.1}$$

$$\varepsilon\Delta\rho_\varepsilon\to 0 \text{ in } L^2(0,T;H^{-1}(U)). \tag{4.2}$$

In the same way as last section, we get

$$Q_\varepsilon\to Q \text{ weakly in } L^2(0,T;H^2(U)) \text{ and strongly in } L^2(0,T;H^1(U)). \tag{4.3}$$

REMARK 4.1. Because $Q_\varepsilon\in S_0^3$ a.e. in $[0,T]\times U$, it is also true that its limit $Q\in S_0^3$ a.e. in $[0,T]\times U$ because of the above convergence result (4.3).

More importantly, we can prove the following estimate of density independent of ε.

LEMMA 4.1. *Suppose $(\rho_\varepsilon,u_\varepsilon,Q_\varepsilon)$ is a sequence of solutions to the problem* (3.1)-(3.7) *constructed in Proposition 3.5. Then*

$$\|\rho_\varepsilon\|_{L^{\gamma+1}((0,T)\times U)}+\|\rho_\varepsilon\|_{L^{\beta+1}((0,T)\times U)}\le C(E_\delta(\rho_0,q_0,Q_0),a,b,c,\delta,\beta,\lambda,\nu,L,U,T). \tag{4.4}$$

Proof. The proof is similar to [10] (cf. Lemma 3.1). We introduce an operator ([3,13])

$$\mathcal{B}:\{f\in L^p(U):\int_U f\,dx=0\}\mapsto[H_0^{1,p}(U)]^3,$$

such that $v=\mathcal{B}(f)$ solves the problem

$$\operatorname{div}v=f \text{ in } U,\ v|_{\partial U}=0.$$

Then we take the test function for (3.2) as

$$\psi(t)\mathcal{B}(\rho_\varepsilon-m_0),\ \psi\in\mathcal{D}(0,T),\ 0\le\psi\le 1,\ m_0=\frac{1}{|U|}\int_U\rho(t)dx.$$

We note that the total mass m_0 is a constant such that the test function is well defined. Then direct calculations lead to

$$\begin{aligned}
&\int_0^T\int_U\psi(\rho_\varepsilon^{\gamma+1}+\delta\rho_\varepsilon^{\beta+1})dxdt\\
&=m_0\int_0^T\psi\Big(\int_U\rho_\varepsilon^\gamma+\delta\rho_\varepsilon^\beta\,dx\Big)dt+(\lambda+\nu)\int_0^T\psi\int_U\rho_\varepsilon\operatorname{div}u_\varepsilon\,dxdt\\
&\quad-\int_0^T\psi_t\int_U\big\langle\rho_\varepsilon u_\varepsilon,\mathcal{B}(\rho_\varepsilon-m_0)\big\rangle dxdt+\nu\int_0^T\psi\int_U\nabla u_\varepsilon:\nabla\mathcal{B}(\rho_\varepsilon-m_0)\,dxdt\\
&\quad-\int_0^T\psi\int_U\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon:\nabla\mathcal{B}(\rho_\varepsilon-m_0)\,dxdt-\varepsilon\int_0^T\psi\int_U\big\langle\rho_\varepsilon u_\varepsilon,\mathcal{B}(\Delta\rho_\varepsilon)\big\rangle dxdt\\
&\quad-\int_0^T\psi\int_U\big\langle\rho_\varepsilon u_\varepsilon,\mathcal{B}(\operatorname{div}(\rho_\varepsilon u_\varepsilon))\big\rangle dxdt+\varepsilon\int_0^T\psi\int_U\nabla u_\varepsilon:\mathcal{B}(\rho_\varepsilon-m_0)\nabla\rho_\varepsilon\,dxdt\\
&\quad+\int_0^T\psi\int_U\big(\nabla Q_\varepsilon\otimes\nabla Q_\varepsilon-\mathcal{F}(Q_\varepsilon)I_3\big):\nabla\mathcal{B}(\rho_\varepsilon-m_0)\,dxdt
\end{aligned}$$

$$-L\int_0^T \psi \int_U \big(Q_\varepsilon \mathcal{H}(Q_\varepsilon) - \mathcal{H}(Q_\varepsilon)Q_\varepsilon\big) : \nabla \mathcal{B}(\rho_\varepsilon - m_0)\, dxdt$$
$$\doteq I_1 + \cdots + I_{10}.$$

Now we estimate $I_1, \cdots, I_{10}$. By (3.24), (3.25), (3.27), and (3.28), we get

$$|I_1| \le |m_0| T\Big(\sup_{t\in[0,T]} \|\rho_\varepsilon(t)\|^\gamma_{L^\gamma(U)} + \delta \sup_{t\in[0,T]} \|\rho_n(t)\|^\beta_{L^\beta(U)}\Big) \le C(E_\delta(\rho_0, q_0, Q_0), \gamma, \beta, T).$$

$$|I_2| \le (\lambda+\nu)\|\rho_\varepsilon\|_{L^2(0,T;L^2(U))}\|\nabla u_\varepsilon\|_{L^2(0,T;L^2(U))} \le C(E_\delta(\rho_0, q_0, Q_0), \delta, \gamma, \beta, \lambda, \nu, U).$$

By the property of the operator $\mathcal{B}$ ([3]), we know

$$\|\mathcal{B}(\rho_\varepsilon - m_0)\|_{H_0^{1,\beta}(U)} \le C(\beta, U)\|\rho_\varepsilon - m_0\|_{L^\beta(U)}.$$

Using the Sobolev embedding theorem for $\beta > 4$, (3.25), and (3.26), we get

$$|I_3| \le C\int_0^T \|\sqrt{\rho_\varepsilon}\|_{L^2(U)}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^2(U)}\|\mathcal{B}(\rho_\varepsilon - m_0)\|_{L^\infty(U)} \le C(E_\delta(\rho_0, q_0, Q_0), \delta, \gamma, \beta, U).$$

Similar to the estimate for I_3,

$$|I_4| \le C\|u_\varepsilon\|_{L^2(0,T;H^1(U))}\|\rho_\varepsilon\|_{L^2(0,T;L^2(U))} \le C(E_\delta(\rho_0, q_0, Q_0), \delta, \beta, \lambda, \nu, U).$$

$$|I_5| \le \int_0^T \|\rho_\varepsilon\|_{L^3(U)}\|u_\varepsilon\|^2_{L^6(U)}\|\rho_\varepsilon\|_{L^3(U)}\, dt \le C(E_\delta(\rho_0, q_0, Q_0), \delta, \beta, \lambda, \nu, U).$$

By (3.27), (3.28), (3.29), the property of operator $\mathcal{B}$ and the Sobolev embedding theorem, we deduce that for $\varepsilon \le 1$, it holds that

$$|I_6| \le C\varepsilon\int_0^T \|\rho_\varepsilon\|_{L^3(U)}\|u_\varepsilon\|_{L^6(U)}\|\nabla\rho_\varepsilon\|_{L^2(U)}\, dt \le C(E_\delta(\rho_0, q_0, Q_0), \delta, \beta, \lambda, \nu, U, T).$$

Next, because the operator $\mathcal{B}$ ([3]) enjoys the property

$$\|\mathcal{B}(f)\|_{L^2(U)} \le C(U)\|g\|_{L^2(U)} \text{ for } \mathcal{B}(f) = \operatorname{div} g$$

with

$$g \cdot \vec{n}|_{\partial U} = 0,$$

we infer from (3.27) and (3.28) that

$$|I_7| \le \int_0^T \|\rho_\varepsilon\|_{L^3(U)}\|u_\varepsilon\|_{L^6(U)}\|\rho_\varepsilon u_\varepsilon\|_{L^2(U)}\, dt \le \int_0^T \|\rho_\varepsilon\|^2_{L^3(U)}\|\nabla u_\varepsilon\|^2_{L^2(U)}\, dt$$
$$\le C(E_\delta(\rho_0, q_0, Q_0), \delta, \beta, \lambda, \nu, U).$$

Furthermore, by (3.27) and (3.29), we obtain

$$|I_8| \le \sqrt{\varepsilon}\|\sqrt{\varepsilon}\nabla\rho_\varepsilon\|_{L^2(0,T;L^2(U))}\|\nabla u_\varepsilon\|_{L^2(0,T;L^2(U))}\|\mathcal{B}(\rho_\varepsilon - m_0)\|_{L^\infty(0,T;L^\infty(U))}$$
$$\le C(\beta, U)\sqrt{\varepsilon}\|\sqrt{\varepsilon}\nabla\rho_\varepsilon\|_{L^2(0,T;L^2(U))}\|\nabla u_\varepsilon\|_{L^2(0,T;L^2(U))}\|\rho_\varepsilon\|_{L^\infty(0,T;L^\beta(U))}$$

$$\leq C(E_\delta(\rho_0,q_0,Q_0),\delta,\beta,\lambda,\nu,U,T), \quad \text{for } \varepsilon \leq 1. \tag{4.5}$$

Then by (3.25), (3.30), and (3.32), we know

$$\begin{aligned}
|I_9| \leq & \int_0^T \|\nabla Q_\varepsilon\|^2_{L^{\frac{10}{3}}(U)} \|\nabla \mathcal{B}(\rho_\varepsilon - m_0)\|_{L^{\frac{5}{2}}(U)} dt \\
& + \int_0^T \|\mathcal{F}(Q_\varepsilon)\|_{L^{\frac{5}{2}}(U)} \|\nabla \mathcal{B}(\rho_\varepsilon - m_0)\|_{L^{\frac{5}{3}}(U)} dt \\
\leq & C(L,U) \int_0^T \big(\|\nabla Q_\varepsilon\|^2_{L^{\frac{10}{3}}(U)} + 1\big) \|\nabla \mathcal{B}(\rho_\varepsilon - m_0)\|_{L^{\frac{5}{2}}(U)} dt \\
& + C(a,b,c) \int_0^T \big(\|\mathrm{tr}^2(Q^2)\|_{L^{\frac{5}{2}}(U)} + 1)\|\nabla \mathcal{B}(\rho_\varepsilon - m_0)\|_{L^{\frac{5}{3}}(U)} dt \\
\leq & C(E_\delta(\rho_0,q_0,Q_0),a,b,c,\delta,\beta,\lambda,\nu,L,U,T).
\end{aligned}$$

Finally, we deduce from (3.25), (3.30), and (3.33) that

$$\begin{aligned}
|I_{10}| \leq & 2L \int_0^T \|Q_\varepsilon\|_{L^{10}(U)} \|\Delta Q_\varepsilon\|_{L^2(U)} \|\nabla \mathcal{B}(\rho_\varepsilon - m_0)\|_{L^{\frac{5}{2}}(U)} dt \\
\leq & C(E_\delta(\rho_0,q_0,Q_0),a,b,c,\delta,\beta,\lambda,\nu,L,U,T).
\end{aligned}$$

Hence we finish the proof by summing up all previous results for $I_1,\cdots,I_{10}$. □

Lemma 4.1 together with (3.25) imply that

$$\rho_\varepsilon \to \rho \text{ in } C(0,T;L^\beta_{weak}(U)) \text{ and weakly in } L^{\beta+1}((0,T)\times U). \tag{4.6}$$

Here the definition of $L^\beta_{weak}(U)$ is taken from [12] (see Subsection 2.2 therein for details). Moreover,

$$u_\varepsilon \to u \text{ weakly in } L^2(0,T;H^1_0(U)), \tag{4.7}$$

which together with (3.24) and (4.6) yield

$$\rho_\varepsilon u_\varepsilon \to \rho u \text{ in } C\big([0,T];L^{\frac{2\gamma}{\gamma+1}}_{weak}(U)\big). \tag{4.8}$$

Applying the same arguments as in the last section, and noting that $\frac{2\gamma}{\gamma+1} > \frac{6}{5}$, it then follows from (4.7) and (4.8) that

$$\rho_\varepsilon u_\epsilon \otimes u_\varepsilon \to \rho u \otimes u \text{ in } \mathcal{D}'((0,T)\times U). \tag{4.9}$$

Meanwhile, (4.3) implies that

$$\begin{aligned}
& -\nabla\cdot(\nabla Q_\varepsilon \odot \nabla Q_\varepsilon - \mathcal{F}(Q_\varepsilon)I_3) + L\nabla\cdot(Q_\varepsilon \mathcal{H}(Q_\varepsilon) - \mathcal{H}\big(Q_\varepsilon)Q_\varepsilon\big) \\
\to & -\nabla\cdot(\nabla Q \odot \nabla Q - \mathcal{F}(Q)I_3) + L\nabla\cdot\big(Q\mathcal{H}(Q) - \mathcal{H}(Q)Q\big) \quad \text{in } \mathcal{D}'((0,T)\times U).
\end{aligned} \tag{4.10}$$

In conclusion, we prove that the limit (ρ,u,Q) satisfies the following equations in $\mathcal{D}'((0,T)\times U)$:

$$\begin{aligned}
\rho_t + \nabla\cdot(\rho u) = 0, \qquad\qquad & \qquad (4.11) \\
(\rho u)_t + \nabla\cdot(\rho u \otimes u) + \nabla p = \mathcal{L}u - \nabla\cdot\big(L\nabla Q \odot \nabla Q - \mathcal{F}(Q)I_3\big)
\end{aligned}$$

$$+L\nabla\cdot\big(Q\mathcal{H}(Q)-\mathcal{H}(Q)Q\big), \tag{4.12}$$

$$Q_t+u\cdot\nabla Q-\Omega Q+Q\Omega=\Gamma\mathcal{H}(Q), \tag{4.13}$$

with the initial data

$$\rho(0)=\rho_0,\quad (\rho u)(0)=q_0,\quad Q(0)=Q_0.$$

REMARK 4.2. Using Lemma 4.1 and the assumption $\beta>\gamma$ we know the pressure p in the above system (4.11)-(4.13) has the property

$$\rho_\varepsilon^\gamma+\delta\rho_\varepsilon^\beta\to p \text{ weakly in } L^{\frac{\beta+1}{\beta}}((0,T)\times U). \tag{4.14}$$

The remaining part of this section is to improve the convergence in (4.14) to be strong in $L^1((0,T)\times U)$, such that

$$p=\rho^\gamma+\delta\rho^\beta.$$

4.2. The effective viscous flux. The quantity $\rho^\gamma+\delta\rho^\beta-(\lambda+2\nu)\mathrm{div}\,u$ is usually referred to as the effective viscous flux. We shall find that it plays an essential role on our coupled system (see also [15, 27]).

LEMMA 4.2. *Let $(\rho_\varepsilon,u_\varepsilon,Q_\varepsilon)$ be a sequence of solutions constructed in Proposition 3.5, and (ρ,u,Q) be its limit satisfying* (4.11)-(4.13)*, respectively. Then for any $\psi\in\mathcal{D}(0,T)$, $\phi\in\mathcal{D}(U)$, it holds that*

$$\begin{aligned}&\lim_{\varepsilon\to0^+}\int_0^T\psi\int_U\phi\big(\rho_\varepsilon^\gamma+\delta\rho_\varepsilon^\beta-(\lambda+2\nu)\mathrm{div}\,u_\varepsilon\big)\rho_\varepsilon\,dxdt\\&=\int_0^T\psi\int_U\phi\big(p-(\lambda+2\nu)\mathrm{div}\,u\big)\rho\,dxdt.\end{aligned} \tag{4.15}$$

REMARK 4.3. It is worth pointing out that from the fluid mechanics point of view, the quantity $P-(\lambda+2\nu)\mathrm{div}\,u$ appearing in (4.15) is the amplitude of the normal viscous stress augmented by the hydrostatic pressure.

Proof. We consider the singular integral operator

$$\mathcal{A}_i=\partial_{x_i}\Delta^{-1},$$

or equivalently in terms of its Fourier symbol

$$\mathcal{A}_j(\xi)=\frac{-\sqrt{-1}\xi_j}{|\xi|^2}.$$

By Proposition 3.5, $\rho_\varepsilon,u_\varepsilon$ satisfy (3.1) a.e. on $(0,T)\times U$ with the boundary condition (3.6). In particular, we extend ρ_ε, u_ε to be zero outside U. Then we have

$$\partial_t\rho_\varepsilon+\mathrm{div}\,(\rho_\varepsilon u_\varepsilon)=\varepsilon\mathrm{div}(1_U\nabla\rho_\varepsilon)\quad\text{in } \mathcal{D}'((0,T)\times\mathbb{R}^3), \tag{4.16}$$

with 1_U the characteristic function on U. Next, we consider the vector-valued test function

$$\varphi(t,x)=\psi(t)\phi(x)\mathcal{A}(\rho_\varepsilon)\doteq\psi(t)\phi(x)\big(\mathcal{A}_1(\rho_\varepsilon),\mathcal{A}_2(\rho_\varepsilon),\mathcal{A}_3(\rho_\varepsilon)\big),$$

where $\psi\in\mathcal{D}(0,T)$, $\phi\in\mathcal{D}(U)$. Analogously, after direct calculations we derive

$$\begin{aligned}
&\int_0^T\psi\int_U\phi\big(\rho_\varepsilon^\gamma+\delta\rho_\varepsilon^\beta-(\lambda+2\nu)\operatorname{div}u_\varepsilon\big)\rho_\varepsilon\,dxdt\\
&=(\lambda+\nu)\int_0^T\psi\int_U\operatorname{div}u_\varepsilon\langle\nabla\phi,\mathcal{A}(\rho_\varepsilon)\rangle\,dxdt-\int_0^T\psi\int_U(\rho_\varepsilon^\gamma+\delta\rho_\varepsilon^\beta)\langle\nabla\phi,\mathcal{A}(\rho_\varepsilon)\rangle\,dxdt\\
&\quad-\int_0^T\psi\int_U\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon:\nabla\phi\otimes\mathcal{A}(\rho_\varepsilon)\,dxdt-\int_0^T\psi_t\int_U\phi\langle\rho_\varepsilon u_\varepsilon,\mathcal{A}(\rho_\varepsilon)\rangle\,dxdt\\
&\quad+\nu\int_0^T\psi\int_U\mathcal{A}(\rho_\varepsilon)\otimes\nabla\phi:\nabla u_\varepsilon\,dxdt-\nu\int_0^T\psi\int_U\nabla\mathcal{A}(\rho_\varepsilon):u_\varepsilon\otimes\nabla\phi\,dxdt\\
&\quad+\nu\int_0^T\psi\int_U\rho_\varepsilon(u_\varepsilon\cdot\nabla\phi)\,dxdt+\int_0^T\psi\int_U\phi\big[\rho_\varepsilon\nabla_j\mathcal{A}_i(\rho_\varepsilon u_\varepsilon^j)-\rho_\varepsilon u_\varepsilon^j\nabla_j\mathcal{A}_i(\rho_\varepsilon)\big]\,dxdt\\
&\quad-\int_0^T\psi\int_U\big(L\nabla Q_\varepsilon\odot\nabla Q_\varepsilon-\mathcal{F}(Q_\varepsilon)I_3\big):\nabla\big(\phi\mathcal{A}(\rho_\varepsilon)\big)\,dxdt\\
&\quad+L\int_0^T\psi\int_U\big(Q_\varepsilon\mathcal{H}(Q_\varepsilon)-\mathcal{H}(Q_\varepsilon)Q_\varepsilon\big):\nabla\big(\phi\mathcal{A}(\rho_\varepsilon)\big)\,dxdt\\
&\quad-\varepsilon\int_0^T\psi\int_U\phi\big\langle\rho_\varepsilon u_\varepsilon,\mathcal{A}(\operatorname{div}(1_U\nabla\rho_\varepsilon))\big\rangle\,dxdt+\varepsilon\int_0^T\psi\int_U\phi\nabla u_\varepsilon:\mathcal{A}(\rho_\varepsilon)\otimes\nabla\rho_\varepsilon\,dxdt\\
&\doteq I_1+\cdots+I_{12}.
\end{aligned}\tag{4.17}$$

In the meantime, we can repeat the above procedures to the limit equations (4.11) and (4.12), because we have the following result from [10].

Lemma 4.3. *Suppose $\rho\in L^2((0,T)\times U)$, $u\in L^2(0,T;H_0^1(U))$ is a solution of (4.11) in $\mathcal{D}'((0,T)\times U)$. Then the equation (4.11) still holds in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, provided (ρ,u) are extended to be 0 in $\mathbb{R}^3\backslash U$.*

Consequently, the counterpart to (4.17) is

$$\begin{aligned}
&\int_0^T\psi\int_U\phi(p-(\lambda+2\nu)\operatorname{div}u)\rho\,dxdt\\
&=(\lambda+\nu)\int_0^T\psi\int_U\operatorname{div}u\langle\nabla\phi,\mathcal{A}(\rho)\rangle\,dxdt-\int_0^T\psi\int_U P\langle\nabla\phi,\mathcal{A}(\rho)\rangle\,dxdt\\
&\quad-\int_0^T\psi\int_U\rho u\otimes u:\nabla\phi\otimes\mathcal{A}(\rho)\,dxdt-\int_0^T\psi_t\int_U\phi\langle\rho u,\mathcal{A}(\rho)\rangle\,dxdt\\
&\quad+\nu\int_0^T\psi\int_U\mathcal{A}(\rho)\otimes\nabla\phi:\nabla u\,dxdt-\nu\int_0^T\psi\int_U\nabla\mathcal{A}(\rho):u\otimes\nabla\phi\,dxdt\\
&\quad+\nu\int_0^T\psi\int_U\rho(u\cdot\nabla\phi)\,dxdt+\int_0^T\psi\int_U\phi\big[\rho\nabla_j\mathcal{A}_i(\rho u^j)-\rho u^j\nabla_j\mathcal{A}_i(\rho)\big]\,dxdt\\
&\quad-\int_0^T\psi\int_U\big(L\nabla Q\odot\nabla Q-\mathcal{F}(Q)I_3\big):\nabla\big(\phi\mathcal{A}(\rho)\big)\,dxdt\\
&\quad+L\int_0^T\psi\int_U\big(Q\mathcal{H}(Q)-\mathcal{H}(Q)Q\big):\nabla\big(\phi\mathcal{A}(\rho)\big)\,dxdt\\
&\doteq J_1+\cdots+J_{10}.
\end{aligned}\tag{4.18}$$

Due to the classical L^p-theory for elliptic problems, we have

$$\|\mathcal{A}(v)\|_{H^{1,s}(U)} \leq C(s,U)\|v\|_{L^s(\mathbb{R}^3)}, \quad 1<s<\infty, \tag{4.19}$$

which combined with (4.6) leads to

$$\mathcal{A}(\rho_\varepsilon) \to \mathcal{A}(\rho) \ \text{ in } C\big(\overline{(0,T)\times U}\big), \tag{4.20}$$

and hence

$$\nabla\mathcal{A}(\rho_\varepsilon) \to \nabla\mathcal{A}(\rho) \ \text{ in } C\big([0,T];L^\beta_{weak}(U)\big). \tag{4.21}$$

Therefore, direct derivations from (4.7) and (4.20) show that

$$I_1 \to J_1, \quad I_5 \to J_5, \quad \text{as} \quad \varepsilon \to 0.$$

Meanwhile, (4.8) and (4.20) indicate that

$$I_4 \to J_4, \quad \text{as} \quad \varepsilon \to 0.$$

By (4.7) and (4.8), we know $\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon \in L^2\big(0,T;L^{\frac{6\gamma}{3+4\gamma}}(U)\big)$. Then we infer from (4.9) that

$$\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon \to \rho u \otimes u \ \text{ weakly in } L^2\big(0,T;L^{\frac{6\gamma}{3+4\gamma}}(U)\big).$$

Consequently, we infer from (4.20) that

$$I_3 \to J_3, \quad \text{as} \quad \varepsilon \to 0,$$

provided $\beta \geq \frac{6\gamma}{2\gamma-3}$. Note that (4.21) indicates $\nabla\mathcal{A}(\rho_\varepsilon) \to \nabla\mathcal{A}(\rho)$ strongly in $C([0,T],H^{-1}(U))$, hence we get from (4.7) that

$$I_6 \to J_6, \quad \text{as} \quad \varepsilon \to 0.$$

Analogously, because $\beta > 4$, we can apply a similar argument to conclude that

$$I_7 \to J_7, \quad \text{as} \quad \varepsilon \to 0.$$

For I_8, it follows from (4.6), (4.8), and (4.19) that if $\beta > \frac{6\gamma}{2\gamma-3}$, then

$$\rho_\varepsilon \nabla_j \mathcal{A}_i(\rho_\varepsilon u_\varepsilon^j) - \rho_\varepsilon u_\varepsilon^j \nabla_j \mathcal{A}_i(\rho_\varepsilon) \in L^\infty(0,T;L^\alpha(U)), \ \text{ with } \ \frac{\gamma+1}{2\gamma} + \frac{1}{\beta} = \frac{1}{\alpha} < \frac{5}{6}.$$

Hence we infer from the celebrated Div-Curl Lemma and compact embedding $L^\alpha(U) \hookrightarrow H^{-1}(U)$ that

$$\rho_\varepsilon \nabla_j \mathcal{A}_i(\rho_\varepsilon u_\varepsilon^j) - \rho_\varepsilon u_\varepsilon^j \nabla_j \mathcal{A}_i(\rho_\varepsilon) \to \rho \nabla_j \mathcal{A}_i(\rho u) - \rho u \nabla_j \mathcal{A}_i(\rho) \ \text{ strongly in } H^{-1}(U).$$

Then applying the Lebesgue convergence theorem, we obtain

$$\rho_\varepsilon \nabla_j \mathcal{A}_i(\rho_\varepsilon u_\varepsilon^j) - \rho_\varepsilon u_\varepsilon^j \nabla_j \mathcal{A}_i(\rho_\varepsilon) \to \rho \nabla_j \mathcal{A}_i(\rho u) - \rho u \nabla_j \mathcal{A}_i(\rho)$$

strongly in $L^2\big(0,T;H^{-1}(U)\big)$, which combined with (4.7) yields

$$I_8 \to J_8, \quad \text{as} \quad \varepsilon \to 0.$$

Using (3.25), (3.27), (3.29), and (4.19), we get

$$I_{11}\to 0,\quad I_{12}\to 0,\quad \text{as}\quad \varepsilon\to 0.$$

It remains to prove the corresponding convergence results for I_9 and I_{10}, which are related to the order parameter Q. Notice that both I_9 and J_9 can be decomposed in the following manner:

$$\begin{aligned}
I_9 &= -\int_0^T \psi\int_U \big(L\nabla Q_\varepsilon\odot\nabla Q_\varepsilon - \mathcal{F}(Q_\varepsilon)I_3\big):(\mathcal{A}(\rho_\varepsilon)\otimes\nabla\phi)\,dxdt\\
&\quad -\int_0^T \psi\int_U \phi\big(L\nabla Q_\varepsilon\odot\nabla Q_\varepsilon - \mathcal{F}(Q_\varepsilon)I_3\big):\nabla\mathcal{A}(\rho_\varepsilon)\,dxdt\\
&\doteq I_{9a}+I_{9b}, \qquad (4.22)\\
J_9 &= -\int_0^T \psi\int_U \big(L\nabla Q\odot\nabla Q - \mathcal{F}(Q)I_3\big):\mathcal{A}(\rho)\otimes\nabla\phi\,dxdt\\
&\quad -\int_0^T \psi\int_U \phi\big(L\nabla Q\odot\nabla Q - \mathcal{F}(Q)I_3\big):\nabla\mathcal{A}(\rho)\,dxdt\\
&\doteq J_{9a}+J_{9b}. \qquad (4.23)
\end{aligned}$$

Due to (4.3) and (4.20), the convergence of I_{9a} to J_{9a} is straightforward. While for I_{9b} and J_{9b}, by the properties of the singular integral operator $\mathcal{A}$, it holds

$$\begin{aligned}
&I_{9b}-J_{9b}\\
&= -L\int_0^T \psi\int_U \phi(\nabla Q_\varepsilon-\nabla Q)\odot\nabla Q_\varepsilon:\nabla\mathcal{A}(\rho_\varepsilon)\,dxdt\\
&\quad -L\int_0^T \psi\int_U \phi\nabla Q\odot(\nabla Q_\varepsilon-\nabla Q):\nabla\mathcal{A}(\rho_\varepsilon)\,dxdt\\
&\quad -L\int_0^T \psi\int_U \phi\nabla Q\odot\nabla Q:\nabla\big(\mathcal{A}(\rho_\varepsilon)-\mathcal{A}(\rho)\big)\,dxdt\\
&\quad +\int_0^T \psi\int_U \phi\big(\mathcal{F}(Q_\varepsilon)-\mathcal{F}(Q)\big)\rho_\varepsilon\,dxdt+\int_0^T \psi\int_U \phi\mathcal{F}(Q)(\rho_\varepsilon-\rho)\,dxdt\\
&\doteq K_{9ba}+K_{9bb}+K_{9bc}+K_{9bd}+K_{9be}. \qquad (4.24)
\end{aligned}$$

Using (3.25), (4.3), and (4.19), we find $K_{9ba}\to 0$, $K_{9bb}\to 0$. By (3.30), (3.31), (4.3), (4.6), and Lemma 4.1, we know $K_{9bd}\to 0$, $K_{9be}\to 0$. As for K_{9bc}, we deduce from (4.21) that for a.e. fixed $t\in[0,T]$,

$$\psi(t)\int_U \phi(x)\nabla Q\odot\nabla Q:\big(\nabla\mathcal{A}(\rho_\varepsilon)-\nabla\mathcal{A}(\rho)\big)\,dx\to 0,\quad \text{as } \varepsilon\to 0.$$

Meanwhile, because $\beta>4$, using Hölder's inequality, we obtain from (4.19) and Lemma 4.1 that $\forall\varepsilon>0,\forall t\in[0,T]$,

$$\begin{aligned}
&\Big|\psi(t)\int_U \phi(x)\nabla Q\odot\nabla Q:\big(\nabla\mathcal{A}(\rho_\varepsilon)-\nabla\mathcal{A}(\rho)\big)\,dx\Big|\\
&\le C\|\nabla Q\|^2_{L^{\frac{10}{3}}(U)}\|\nabla\mathcal{A}(\rho_\varepsilon)-\nabla\mathcal{A}(\rho)\|_{L^{\frac{5}{2}}(U)}\\
&\le C(E_\delta(\rho_0,q_0,Q_0),a,b,c,\beta,L,U,T)\|\nabla Q\|^2_{L^{\frac{10}{3}}(U)},
\end{aligned}$$

with the right hand side term being integrable on $(0,T)$ due to (3.32). Hence we conclude that $K_{9bc}\to 0$ after applying Lebesgue's convergence theorem. In all, we prove

$$I_9\to J_9 \quad \text{as } \varepsilon\to 0.$$

For I_{10}, we have

$$\begin{aligned} I_{10} &= L\int_0^T \psi\int_U \big(Q_\varepsilon\mathcal{H}(Q_\varepsilon)-\mathcal{H}(Q_\varepsilon)Q_\varepsilon\big):\mathcal{A}(\rho_\varepsilon)\otimes\nabla\phi\, dxdt \\ &\quad + L\int_0^T \psi\int_U \big(Q_\varepsilon\mathcal{H}(Q_\varepsilon)-\mathcal{H}(Q_\varepsilon)Q_\varepsilon\big):\phi\nabla\mathcal{A}(\rho_\varepsilon)\, dxdt \\ &\doteq I_{10a}+I_{10b}. \end{aligned} \tag{4.25}$$

Notice that $Q_\varepsilon = Q_\varepsilon^T$, hence $Q_\varepsilon\mathcal{H}(Q_\varepsilon)-\mathcal{H}(Q_\varepsilon)Q_\varepsilon$ is skew-symmetric. We observe that $\nabla\mathcal{A}$ is symmetric. Therefore, we conclude

$$I_{10b}=0. \tag{4.26}$$

Remark 4.4. We want to point out that the special property of the Q-tensor is of great importance here, for otherwise we are not able to control the higher order terms in I_{10b}.

Remark 4.5. For the full system in the case $\xi\neq 0$, however, we cannot apply the above argument to eliminate the higher order terms. As a consequence, we are not able to keep control of the remaining terms with high nonlinearity.

We proceed to show the convergence of I_{10} to J_{10}.

$$\begin{aligned} I_{10}-J_{10} &= I_{10a}-J_{10} \\ &= L\int_0^T \psi\int_U \big(Q_\varepsilon\Delta Q_\varepsilon-\Delta Q_\varepsilon Q_\varepsilon\big):\big(\mathcal{A}(\rho_\varepsilon)-\mathcal{A}(\rho)\big)\otimes\nabla\phi\, dxdt \\ &\quad + L\int_0^T \psi\int_U \big((Q_\varepsilon-Q)\Delta Q_\varepsilon-\Delta Q_\varepsilon(Q_\varepsilon-Q)\big):\mathcal{A}(\rho)\otimes\nabla\phi\, dxdt \\ &\quad + L\int_0^T \psi\int_U \big(Q_\varepsilon(\Delta Q_\varepsilon-\Delta Q)-(\Delta Q_\varepsilon-\Delta Q)Q_\varepsilon\big):\mathcal{A}(\rho)\otimes\nabla\phi\, dxdt \\ &\doteq K_{10aa}+K_{10ab}+K_{10ac}. \end{aligned}$$

By (3.30), (3.31), (3.33), (4.3), and (4.20), it is easy to see that

$$K_{10aa}\to 0,\quad K_{10ab}\to 0,\quad K_{10ac}\to 0,\quad \text{as } \varepsilon\to 0,$$

hence

$$I_{10}\to J_{10},\quad \text{as } \varepsilon\to 0.$$

Summing up all the above convergence results, we finish the proof of Lemma 4.2. ∎

4.3. Strong convergence of density. In this subsection we shall show that

$$p=\rho^\gamma+\delta\rho^\beta,$$

and consequently the strong convergence of ρ_ε in $L^1((0,T)\times U)$. By Lemma 4.3, we can take the standard mollifier $\vartheta_m=\vartheta_m(x)$ to equation (4.11), such that

$$\partial_t S_m(\rho)+\operatorname{div}(S_m(\rho)u)=r_m, \quad \text{on } (0,T)\times\mathbb{R}^3, \tag{4.27}$$

with $S_m(\rho)=\vartheta*\rho$ and $r_m\to 0$ in $L^1((0,T)\times U)$ (cf. [27]). Then for any g satisfying (2.13), we can multiply (4.27) with $g'(S_m(\rho))$ and pass to the limit as $m\to\infty$. Then we may argue that ([5]) (ρ,u) solve (4.11) in the sense of renormalized solutions, namely, (2.14) holds in $\mathcal{D}'((0,T)\times U)$. Instead of the strong restrictions on g in (2.13), one can use the Lebesgue convergence theorem to relax the assumptions in Definition 2.4 to any function $g\in C^1(0,\infty)\cap C[0,\infty)$ with

$$|g'(z)z|\le C(z^\theta+z^{\frac{\gamma}{2}}), \quad \forall z>0 \text{ and some } 0<\theta<\frac{\gamma}{2}.$$

Hence we may choose $g(z)=z\ln(z)$ and integrate (2.14) to obtain

$$\int_0^T\int_U \rho\operatorname{div}u\,dxdt=\int_U\rho_0\ln(\rho_0)dx-\int_U\rho(T)\ln(\rho(T))dx. \tag{4.28}$$

Meanwhile, using Lemma 3.4 and the convexity of $g(z)=z\ln(z)$, we know

$$\partial_t g(\rho_\varepsilon)+\operatorname{div}(g(\rho_\varepsilon)u_\varepsilon)+\rho_\varepsilon\operatorname{div}u_\varepsilon-\varepsilon\Delta g(\rho_\varepsilon)\le 0,$$

which leads to

$$\int_0^T\int_U \rho_\varepsilon\operatorname{div}u_\varepsilon\,dxdt=\int_U\rho_0\ln(\rho_0)dx-\int_U\rho_\varepsilon(T)\ln(\rho_\varepsilon(T))dx. \tag{4.29}$$

Take two nondecreasing sequences $\psi_n\in\mathcal{D}(0,T)$, $\phi_n\in\mathcal{D}(U)$ of nonnegative functions with $\psi_n\to 1,\phi_n\to 1$ as $n\to\infty$. By Lemma 4.2, (4.28) and (4.29), one can apply standard arguments to show that

$$\limsup_{\varepsilon\to 0^+}\int_0^T\psi_n\int_U\phi_n\rho_\varepsilon^\gamma+\delta\rho_\varepsilon^\beta)\rho_\epsilon\,dxdt\le\int_0^T\int_U P\rho\,dxdt, \quad \text{for all } n=1,2,\ldots.$$

Noting that $P(z)=z^\gamma+\delta z^\beta$ is monotone, by Minty's trick [35], we have

$$\int_0^T\psi_m(t)\int_U\phi_m(x)\big(P(\rho_\varepsilon)-P(v)\big)(\rho_\epsilon-v)dxdt\ge 0.$$

Consequently, taking $n\to\infty$, we obtain after rearrangement that for any $v=\rho+\kappa\phi$, $\phi\in\mathcal{D}(U)$,

$$\int_0^T\int_U\big(p-P(v)\big)(\rho-v)dxdt\ge 0.$$

Letting $\kappa\to 0$, we come to the conclusion

$$p=\rho^\gamma+\delta\rho^\beta.$$

In all, we may summarize the above results in the following proposition.

PROPOSITION 4.4. *Suppose $\beta > \max\{\frac{6\gamma}{2\gamma-3},\gamma,4\}$. Then for any given $T>0$ and $\delta>0$, there exists a finite energy weak solution (ρ,u,Q) to the problem*

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{4.30}$$

$$(\rho u)_t + \nabla\cdot(\rho u\otimes u) + \nabla(\rho^\gamma + \delta\rho^\beta) = \mathcal{L}u - \nabla\cdot\big(L\nabla Q\odot\nabla Q - \mathcal{F}(Q)I_3\big) + L\nabla\cdot\big(Q\mathcal{H}(Q) - \mathcal{H}(Q)Q\big), \tag{4.31}$$

$$Q_t + u\cdot\nabla Q - \Omega Q + Q\Omega = \Gamma\mathcal{H}(Q), \tag{4.32}$$

with initial and boundary conditions (3.4)-(3.7). *Furthermore, $\rho\in L^{\beta+1}((0,T)\times U)$ and the equation* (4.30) *is satisfied in the sense of renormalized solutions on $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ provided ρ,u are extended to be zero on $\mathbb{R}^3\setminus U$. In addition, the following estimates are valid:*

$$\sup_{t\in[0,T]}\|\rho(t)\|^\gamma_{L^\gamma(U)} \le C(E_\delta(\rho_0,q_0,Q_0),\gamma), \tag{4.33}$$

$$\delta\sup_{t\in[0,T]}\|\rho(t)\|^\beta_{L^\beta(U)} \le C(E_\delta(\rho_0,q_0,Q_0),\beta), \tag{4.34}$$

$$\sup_{t\in[0,T]}\big\|\sqrt{\rho}(t)u(t)\big\|^2_{L^2(U)} \le 2E_\delta(\rho_0,q_0,Q_0), \tag{4.35}$$

$$\|u\|_{L^2(0,T;H^1_0(U))} \le C(E_\delta(\rho_0,q_0,Q_0),\lambda,\nu), \tag{4.36}$$

$$\|Q\|_{L^{10}((0,T)\times U)} \le C(E_\delta(\rho_0,q_0,Q_0),a,b,c,L,\Gamma,U,T), \tag{4.37}$$

$$\|Q\|_{L^\infty(0,T;H^1(U))} \le \frac{2}{L}E_\delta(\rho_0,q_0,Q_0) \tag{4.38}$$

$$\|\nabla Q\|_{L^{\frac{10}{3}}((0,T)\times U)} \le C(E_\delta(\rho_0,q_0,Q_0),a,b,c,L,\Gamma,U,T) \tag{4.39}$$

$$\|Q\|_{L^2(0,T;H^2(U))} \le C(E_\delta(\rho_0,q_0,Q_0),a,b,c,L,\Gamma,U,T). \tag{4.40}$$

REMARK 4.6. The initial conditions (3.4)-(3.5) are satisfied in the weak sense, because we infer from (4.6) and (4.8) that

$$\rho_\varepsilon \to \rho \text{ in } C(0,T;L^\beta_{weak}(U)), \quad \rho_\varepsilon u_\varepsilon \to \rho u \text{ in } C\big([0,T];L^{\frac{2\gamma}{\gamma+1}}_{weak}(U)\big).$$

5. Vanishing artificial pressure

In this section, we denote by $(\rho_\delta,u_\delta,Q_\delta)$ the corresponding approximate solutions constructed in Proposition 4.4. We are going to finish the third level approximation, namely, we shall provide the convergence of solutions of $(\rho_\delta,u_\delta,Q_\delta)$ to the solution of the original problem (1.1)-(1.3) as δ goes to 0. The entire idea comes from [10] except the last part on propagation of oscillations. For the readers' convenience and the completeness of our whole proof, we shall retain it in our paper.

To begin with, we relax the conditions on the general initial data (ρ_0,u_0,Q_0). It is easy to find a sequence $\rho_\delta\in C^3_0(\bar{U})$ with the property

$$0\le\rho_\delta(x)\le\frac{1}{2}\delta^{-\frac{1}{\beta}}, \quad\text{and}\quad \|\rho_\delta-\rho_0\|_{L^2(U)}<\delta.$$

Taking $\rho_{0,\delta}=\rho_\delta+\delta$, due to (3.4), then we have

$$0<\delta\le\rho_{0,\delta}\le\delta^{-\frac{1}{\beta}}, \quad \frac{\partial\rho_{0,\delta}}{\partial\vec{n}}=0, \tag{5.1}$$

with

$$\rho_{0,\delta} \to \rho_0 \text{ in } L^\gamma(U) \text{ as } \delta \to 0. \tag{5.2}$$

Set

$$\tilde{q}_\delta(x) = \begin{cases} q(x)\sqrt{\frac{\rho_{0,\delta}}{\rho_0}}, & \text{if } \rho_0(x) > 0, \\ 0, & \text{if } \rho_0(x) = 0. \end{cases} \tag{5.3}$$

Then it follows from (1.7) that $\frac{|\tilde{q}_\delta|^2}{\rho_{0,\delta}}$ is uniformly bounded in $L^1(U)$. At the same time, it is easy to find $h_\delta \in C^2(\bar{U})$ such that

$$\left\| \frac{\tilde{q}_\delta}{\sqrt{\rho_{0,\delta}}} - h_\delta \right\|_{L^2(U)} < \delta.$$

Consequently, we choose $q_\delta = h_\delta \sqrt{\rho_{0,\delta}}$ and one can readily check that

$$\frac{|q_\delta|^2}{\rho_{0,\delta}} \text{ are uniformly bounded in } L^1(U), \tag{5.4}$$

and

$$q_\delta \to q \text{ in } L^1(U) \text{ as } \delta \to 0. \tag{5.5}$$

In what follows, we shall deal with the sequence of approximate solutions $(\rho_\delta, u_\delta, Q_\delta)$ to the problem (4.30)-(4.32) with the initial data $(\rho_\delta, q_\delta, Q_0)$.

Remark 5.1. We want to point out that due to the above modifications, the estimates (4.33)-(4.40) are independent of δ because the constant $E_\delta(\rho_{0,\delta}, q_{0,\delta}, Q_0)$ defined in (3.23) is independent of δ.

Now we shall develop some pressure estimates independent of $\delta > 0$. Notice that the continuity equation (4.30) is satisfied in the sense of renormalized solutions in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$, hence we may apply the standard mollifying operator to both sides of (2.14) and get

$$\partial_t S_m[g(\rho)] + \operatorname{div}\big(S_m[g(\rho)u]\big) + S_m\left[(g'(\rho)\rho - g(\rho))\operatorname{div} u\right] = r_m, \tag{5.6}$$

with

$$r_m \to 0 \text{ in } L^2(0,T; L^2(\mathbb{R}^3)) \text{ as } m \to \infty.$$

Using the operator $\mathcal{B}$ introduced in the proof of Lemma 4.1, we take the test function to (4.31) to be

$$\phi_i(t,x) = \psi(t)\mathcal{B}_i\Big\{S_m[g(\rho_\delta)] - \frac{1}{|U|}\int_U S_m[g(\rho_\delta)]dx\Big\}, \quad i = 1,2,3, \ \psi \in \mathcal{D}(0,T).$$

Next, we can approximate the function $g(z)$ by a sequence of functions $\{z^\theta \chi_n(z)\}$, where each $\chi_n(z)$ is a cutoff function such that $\chi_n(z) = 1$ on $[0,n]$ and $\chi_n(z) = 0$ on $z > 2n$. Then using all the estimates (4.33)-(4.40), we have the next result.

Lemma 5.1. *For $\gamma > \frac{3}{2}$, there exists a constant θ that only depends on γ, such that*

$$\int_0^T \int_U \left(\rho_\delta^{\gamma+\theta} + \delta \rho_\delta^{\beta+\theta}\right) dxdt \le C(\rho_0, q_0, Q_0, a, b, c, \lambda, \nu, \gamma, \beta, \Gamma, L, U, T),$$

provided $0<\theta<\min\{1,\frac{\gamma}{3},\frac{2\gamma}{3}-1\}$.

Proof. Because the technique is quite similar to Lemma 4.1, we shall skip the details of proof and leave it to interested readers. It is noted that the right hand side bound is independent of δ. ∎

5.1. The limit passage and the effective viscous flux. We conclude from the uniform estimates (4.33)-(4.40) in Proposition 4.4 and Lemma 5.1 that

$$\rho_\delta \to \rho \ \text{ in } C\big([0,T];L^\gamma_{weak}(U)\big), \tag{5.7}$$
$$\rho_\delta \to \overline{\rho^\gamma} \ \text{ weakly in } L^{\frac{\gamma+\theta}{\gamma}}((0,T)\times U), \tag{5.8}$$
$$u_\delta \to u \ \text{ weakly in } L^2(0,T;H^1_0(U)), \tag{5.9}$$
$$\rho_\delta u_\delta \to \rho u \ \text{ in } C\big([0,T];L^{\frac{2\gamma}{\gamma+1}}_{weak}(U)\big), \tag{5.10}$$
$$Q_\delta \to Q \ \text{ weakly in } L^2(0,T;H^2(U)), \tag{5.11}$$
$$Q_\delta \to Q \ \text{ strongly in } L^2(0,T;H^1(U)), \tag{5.12}$$

which implies

$$\rho_\delta u_\delta \otimes u_\delta \to \rho u\otimes u \ \text{ in } \mathcal{D}'((0,T)\times U), \tag{5.13}$$

and

$$\begin{aligned}&\nabla Q_\delta \odot \nabla Q_\delta - \mathcal{F}(Q_\delta)I_3 - L\big(Q_\delta\mathcal{H}(Q_\delta)-\mathcal{H}(Q_\delta)Q_\delta\big)\\ &\to \nabla Q\odot\nabla Q-\mathcal{F}(Q)I_3-L\big(Q\mathcal{H}(Q)-\mathcal{H}(Q)Q\big) \ \text{ in } \ L^1((0,T)\times U).\end{aligned} \tag{5.14}$$

Further, Lemma 5.1 implies that

$$\delta\rho_\delta^\beta \to 0 \ \text{ in } L^1((0,T)\times U). \tag{5.15}$$

Therefore, the limit (ρ,u,Q) satisfies

$$\rho_t+\mathrm{div}(\rho u)=0, \ \text{ in } \ \mathcal{D}'\big((0,T)\times\mathbb{R}^3\big), \tag{5.16}$$
$$\begin{aligned}(\rho u)_t+\nabla\cdot(\rho u\otimes u)+\nabla\overline{\rho^\gamma}&=\mathcal{L}u-\nabla\cdot\big(L\nabla Q\odot\nabla Q-\mathcal{F}(Q)I_3\big)\\ &\quad+L\nabla\cdot\big(Q\mathcal{H}(Q)-\mathcal{H}(Q)Q\big),\end{aligned} \tag{5.17}$$
$$Q_t+u\cdot\nabla Q-\Omega Q+Q\Omega=\Gamma\mathcal{H}(Q), \tag{5.18}$$

in $\mathcal{D}'\big((0,T)\times U\big)$. The initial data (1.4) is satisfied due to (5.2) and (5.5).

In what follows, our ultimate goal is to show $\overline{\rho^\gamma}=\rho^\gamma$, or equivalently, the strong convergence of ρ_δ in L^1. Consider a family of cut-off functions defined by $T_k(z)=kT(\frac{z}{k})$ for $z\in\mathbb{R}$, $k=1,2,3\ldots$, where $T\in C^\infty(R)$ is chosen to be

$$T(z)=z \text{ for } z\le 1, \ \ T(z)=2 \text{ for } z\ge 3, \ \ T \text{ is } \textit{concave}.$$

Because (ρ_δ,u_δ) is a normalized solution to (5.16),

$$T_k(\rho_\delta)_t+\mathrm{div}\big(T_k(\rho_\delta)u_\delta\big)+\big(T_k'(\rho_\delta)-T_k(\rho_\delta)\big)\mathrm{div}u_\delta=0, \ \text{ in } \ \mathcal{D}'\big((0,T)\times\mathbb{R}^3\big), \tag{5.19}$$

from which we get after passing to limit for $\delta\to0$ that

$$\overline{T_k(\rho)}_t+\mathrm{div}\big(\overline{T_k(\rho)}u\big)+\overline{\big(T_k'(\rho)-T_k(\rho)\big)\mathrm{div}u}=0, \ \text{ in } \ \mathcal{D}'\big((0,T)\times\mathbb{R}^3\big). \tag{5.20}$$

Here

$$\big(T_k'(\rho_\delta)-T_k(\rho_\delta)\big)\mathrm{div}u_\delta \to \overline{\big(T_k'(\rho)-T_k(\rho)\big)\mathrm{div}u} \quad \text{weakly in} \quad L^2((0,T)\times U), \tag{5.21}$$

and

$$T_k(\rho_\delta)\to \overline{T_k(\rho)} \quad \text{in} \quad C\big(0,T;L^p_{weak}(U)\big), \quad \forall 1\le p<\infty. \tag{5.22}$$

By similar arguments to those in the proof of Lemma 4.2, we have the following auxiliary result.

Lemma 5.2. *If (ρ_δ,u_δ) is a sequence of approximate solutions constructed in Proposition 4.4, then for any $\psi\in\mathcal{D}(0,T)$, $\phi\in\mathcal{D}(U)$, it holds that*

$$\begin{aligned}
&\lim_{\delta\to0^+}\int_0^T\psi(t)\int_U\phi(x)\big(\rho_\delta^\gamma-(\lambda+2\nu)\mathrm{div}\,u_\delta\big)T_k(\rho_\delta)\,dxdt\\
&=\int_0^T\psi(t)\int_U\phi(x)\big(\overline{\rho^\gamma}-(\lambda+2\nu)\mathrm{div}\,u\big)\overline{T_k(\rho)}\,dxdt.
\end{aligned}\tag{5.23}$$

5.2. The renormalized solutions and strong convergence of density. As in [10], we introduce a quantity called the oscillations defect measure. To consider the weak convergence of the sequence $\{\rho_\delta\}_{\delta>0}$ in $L^1((0,T)\times U)$, we define

$$\mathbf{osc}_{\gamma+1}[\rho_\delta-\rho]\equiv\sup_{k\ge1}\left(\limsup_{\delta\to0}\int_0^T\int_U\big|T_k(\rho_\delta)-T_k(\rho)\big|^{\gamma+1}dxdt\right), \tag{5.24}$$

where T_k are the cut-off functions defined above. First, by virtue of Lemma 5.2, we claim the following result concerning the oscillation defect measure.

Lemma 5.3. *There exists a constant C independent of k, such that*

$$\mathbf{osc}_{\gamma+1}[\rho_\delta-\rho]\le C.$$

Proof. Noting that z^γ is a convex function for $\gamma>\frac{3}{2}$, we have (see Theorem 2.11 in [12])

$$\rho^\gamma\le\overline{\rho^\gamma}, \quad z^\gamma-y^\gamma\ge(z-y)^\gamma, \quad \text{for} \quad z\ge y\ge0.$$

Meanwhile, because $T_k(z)$ is concave, we know

$$|T_k(z)-T_k(y)|\le|z-y|, \quad T_k(\rho)\ge\overline{T_k(\rho)}, \quad \forall k\ge1,$$

and hence

$$|T_k(z)-T_k(y)|^{\gamma+1}\le|z-y|^\gamma|T_k(z)-T_k(y)|\le(z^\gamma-y^\gamma)\big(T_k(z)-T_k(y)\big).$$

Consequently,

$$\begin{aligned}
&\limsup_{\delta\to0}\int_0^T\int_U|T_k(\rho_\delta)-T_k(\rho)|^{\gamma+1}dxdt\\
&\le\lim_{\delta\to0}\int_0^T\int_U(\rho_\delta^\gamma-\rho^\gamma)\big(T_k(\rho_\delta)-T_k(\rho)\big)\,dxdt+\int_0^T\int_U(\overline{\rho^\gamma}-\rho^\gamma)\big(T_k(\rho)-\overline{T_k(\rho)}\big)\,dxdt
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\delta\to 0}\int_0^T\int_U \rho_\delta^\gamma T_k(\rho_\delta) - \overline{\rho^\gamma}\,\overline{T_k(\rho)}\,dxdt\\
&= \nu\lim_{\delta\to 0}\int_0^T\int_U \operatorname{div}u_\delta T_k(\rho_\delta) - \operatorname{div}u T_k(\rho)\,dxdt\\
&\le \nu\lim_{\delta\to 0}\int_0^T\int_U \big(T_k(\rho_\delta)-T_k(\rho)+T_k(\rho)-\overline{T_k(\rho)}\big)\operatorname{div}u_\delta\,dxdt\\
&\le C\sup_{\delta>0}\|\operatorname{div}u_\delta\|_{L^2((0,T)\times U)}\limsup_{\delta\to 0}\|T_k(\rho_\delta)-T_k(\rho)\|_{L^{\gamma+1}((0,T)\times U)}, \qquad (5.25)
\end{aligned}
$$

where we applied Lemma 5.2 in the third step. ∎

Based on the uniform bound for the oscillation defect measure shown in Lemma 5.3, we can apply the same argument in [10] to show that the limit functions (ρ,u) satisfy (5.16) in the sense of renormalized solutions.

Lemma 5.4. *The limit functions (ρ,u) satisfy equation (5.16) in the sense of renormalized solutions, namely,*

$$
g(\rho)_t + \operatorname{div}(g(\rho)u) + (g'(\rho)\rho - g(\rho))\operatorname{div}u = 0 \qquad (5.26)
$$

holds in $\mathcal{D}\big((0,T)\times\mathbb{R}^3\big)$ for any g satisfying (2.13).

Finally, we shall discuss the propagation of oscillations which comes from [12], whose amplitude in the sequence $\{\rho_\delta\}_{\delta>0}$ is measured by the following quantity:

$$
\mathbf{dft}[\rho_\delta\to\rho](t) \equiv \int_U \Big(\overline{\rho\ln(\rho)} - \rho\ln(\rho)\Big)(t,x)dx, \quad t\in[0,T].
$$

To this end, we introduce the auxiliary functions

$$
L_k(\rho) = \rho\int_1^\rho \frac{T_k(z)}{z^2}dz,
$$

where T_k are cutoff functions defined above. Now the equation

$$
\partial_t L_k(\rho_\delta) + \operatorname{div}\big(L_k(\rho_\delta)u_\delta\big) + T_k(\rho_\delta)\operatorname{div}u_\delta = 0
$$

holds in $\mathcal{D}'\big((0,T)\times\mathbb{R}^3\big)$. Letting $\delta\to 0$ we obtain

$$
\partial_t\overline{L_k(\rho)} + \operatorname{div}\big(\overline{L_k(\rho)}u\big) + \overline{T_k(\rho)\operatorname{div}u} = 0. \qquad (5.27)
$$

Here $L_k(\rho)\in C([0,T];L^1(U))$ and

$$
L_k(\rho_\delta)\to\overline{L_k(\rho)} \text{ in } C\big(0,T;L^\gamma_{weak}(U)\big), \quad T_k(\rho_\delta)\operatorname{div}u_\delta \to \overline{T_k(\rho)\operatorname{div}u}
$$

weakly in $L^2((0,T)\times U)$. By Lemma 5.4, the limits (ρ,u) satisfy

$$
\partial_t L_k(\rho) + \operatorname{div}(L_k(\rho)u) + T_k(\rho)\operatorname{div}u = 0 \quad \text{in } \mathcal{D}'\big((0,T)\times\mathbb{R}^3\big). \qquad (5.28)
$$

Taking the difference between (5.27) and (5.28), then taking the inner product of the resultant with a test function $\psi(t)\phi(x)$, with $\psi\in\mathcal{D}(0,T)$ and $\phi\in\mathcal{D}(\mathbb{R}^3)$ with $\phi\equiv 1$ on an open neighborhood of $\bar{U}$, we get after integrating from 0 to t that

$$
\int_U \big(\overline{L_k(\rho)} - L_k(\rho)\big)(t)dx
$$

$$=\int_0^t\int_U\left(\overline{T_k(\rho)}\mathrm{div}\,u-\overline{T_k(\rho)\mathrm{div}\,u}\right)dxdt+\int_0^t\int_U\left(T_k(\rho)-\overline{T_k(\rho)}\right)\mathrm{div}\,u\,dxdt. \quad (5.29)$$

Noting that $T_k(z)$ is a convex function of $z\geq 0$, by Lemma 5.2 we again deduce from (5.29) that for all $t\in[0,T]$,

$$\begin{aligned}
0&\leq\int_U\left(\overline{L_k(\rho)}-L_k(\rho)\right)(t)dx\\
&=\lim_{\delta\to0^+}\frac{1}{\nu}\int_0^t\int_U\left(\rho_\delta^\gamma T_k(\rho_\delta)-\overline{\rho^\gamma}\overline{T_k(\rho)}\right)dxd\tau+\int_0^t\int_U\left(T_k(\rho)-\overline{T_k(\rho)}\right)\mathrm{div}\,u\,dxd\tau\\
&\leq\int_0^t\int_U\left(T_k(\rho)-\overline{T_k(\rho)}\right)\mathrm{div}\,u\,dxd\tau\\
&\leq\|\mathrm{div}u\|_{L^2((0,T)\times U)}\|\overline{T_k(\rho)}-T_k(\rho)\|_{L^1((0,T)\times U)}^{\frac{\gamma-1}{2\gamma}}\|\overline{T_k(\rho)}-T_k(\rho)\|_{L^{\gamma+1}((0,T)\times U)}^{\frac{\gamma+1}{2\gamma}}\\
&\doteq I.
\end{aligned}\quad (5.30)$$

By (4.33), (4.36), Lemma 5.3, and letting $k\to\infty$ in (5.30), we get

$$\begin{aligned}
0&\leq\mathbf{dft}[\rho_\delta\to\rho](t)\leq I\\
&\leq C\lim_{k\to\infty}\|\overline{T_k(\rho)}-T_k(\rho)\|_{L^1((0,T)\times U)}^{\frac{\gamma-1}{2\gamma}}\\
&\leq C\lim_{k\to\infty}\|\overline{T_k(\rho)}-\rho\|_{L^1((0,T)\times U)}^{\frac{\gamma-1}{2\gamma}}+C\lim_{k\to\infty}\|T_k(\rho)-\rho\|_{L^1((0,T)\times U)}^{\frac{\gamma-1}{2\gamma}}\\
&\leq C\lim_{k\to\infty}\lim_{\delta\to0^+}\|T_k(\rho_\delta)-\rho_\delta\|_{L^1((0,T)\times U)}^{\frac{\gamma-1}{2\gamma}}\\
&\leq C\lim_{k\to\infty}2^{\frac{\gamma-1}{2\gamma}}k^{-\frac{(\gamma-1)^2}{2\gamma}}\lim_{\delta\to0^+}\|\rho_\delta\|_{L^\gamma((0,T)\times U)}^{\frac{\gamma-1}{2}}\\
&=0,
\end{aligned}\quad (5.31)$$

which indicates

$$\overline{\rho\ln\rho}(t)=\rho\ln\rho(t),\quad\text{for all}\quad t\in[0,T].$$

Hence we manage to prove the strong convergence of $\rho_\delta\to\rho$ in $L^1((0,T)\times U)$.

6. Long time dynamics

Finally, in this section we discuss briefly the long time behavior of any finite energy global weak solution (ρ,u,Q). The main result is as follows.

THEOREM 6.1. *Suppose $\gamma>\frac{3}{2}$. For any finite weak energy solution to the problem* (1.1)-(1.6)*, there exists a steady state solution $(\rho_s,0,Q_s)$ with*

$$\rho_s=\frac{m_0}{|U|},\quad \mathcal{H}(Q_s)=0 \ \textit{for}\ x\in U,\ Q_s|_{\partial_U}=Q_0, \quad (6.1)$$

where $m_0=\int_U\rho_0\,dx$, such that

$$\rho(t)\to\rho_s\quad \textit{weakly in}\ L^\gamma(U)\ \textit{as}\ t\to\infty, \quad (6.2)$$

and

$$\lim_{t\to\infty}\mathcal{E}(t)=\mathcal{E}_s, \quad (6.3)$$

where $\mathcal{E}_s$ is defined in (6.23)*. Furthermore, there exists an increasing sequence $\{t_n\}$ tending to infinity such that for $t\in[0,1]$, it holds as $n\to\infty$ that*

$$u(t+t_n)\to 0 \quad \text{weakly in } L^2(0,1;H^1(U)), \tag{6.4}$$

$$Q(t_n)\to Q_s \quad \text{strongly in } L^2(0,1;H^1(U)) \text{ and weakly in } L^2(0,1;H^2(U)). \tag{6.5}$$

REMARK 6.1. The existence of a classical solution Q_s in (6.1) is guaranteed from elliptic PDE theory. The infimum energy of $\mathcal{G}(Q)$ can be achieved, due to the weak lower semi-continuity and coercivity of $\mathcal{G}(Q)$.

REMARK 6.2. Because the structure of the set of steady states of the Q-equation is a continuum, we cannot deduce the uniqueness of the asymptotic limit.

Proof. To begin with, we obtain from Theorem 2.5 that

$$\operatorname*{ess\,sup}_{t>0}\mathcal{E}(t)+\int_0^\infty\int_U\big(\nu|\nabla u|^2+(\lambda+\nu)|\mathrm{div}u|^2+\Gamma\mathrm{tr}^2(\mathcal{H})\big)\,dxdt\le\mathcal{E}(0). \tag{6.6}$$

Consequently, we know from Corollary 2.3 that

$$\operatorname*{ess\,sup}_{t>0}\big(\|\rho\|_{L^\gamma(U)}+\|\sqrt{\rho}u\|_{L^2(U)}+\|Q\|_{H^1(U)}\big) + \int_0^\infty\int_U\|\nabla u\|^2_{L^2(U)}+\mathrm{tr}^2(\mathcal{H})\,dxdt\le C(\mathcal{E}_0,a,b,c,U). \tag{6.7}$$

For the sake of convenience, we introduce the sequences

$$\rho_n(x,t)\doteq\rho(x,t+n),\quad u_n(x,t)\doteq u(x,t+n),\quad Q_n(x,t)\doteq Q(x,t+n),$$
$$\mathcal{H}_n(x,t)=L\Delta Q_n-aQ_n-cQ_n\mathrm{tr}(Q_n^2),$$

for all integer n and $t\in(0,1)$, $x\in U$. Then it follows immediately from (6.7) that for any n, we have

$$\rho_n\in L^\infty(0,1;L^\gamma(U)),\quad \sqrt{\rho_n}u_n\in L^\infty(0,1;L^2(U)),\quad Q_n\in L^\infty(0,1;H^1(U)), \tag{6.8}$$

$$\lim_{n\to\infty}\int_0^1\Big(\|\nabla u_n\|^2_{L^2(U)}+\|\mathrm{tr}^2(\mathcal{H}_n)\|_{L^1(U)}\Big)\,dt=0. \tag{6.9}$$

Therefore, choosing a subsequence if necessary, we know as $n\to\infty$ that

$$\rho_n(x,t)\to\rho_s \quad \text{weakly in } L^\gamma\big((0,1)\times U\big), \tag{6.10}$$

$$u_n(x,t)\to 0 \quad \text{weakly in } L^2\big(0,1;H_0^1(U)\big), \tag{6.11}$$

$$Q_n(x,t)\to Q_s \quad \text{weakly in } L^2\big(0,1;H^2(U)\big), \tag{6.12}$$

$$H_n(x,t)\to 0 \quad \text{weakly in } L^2\big(0,1;L^2(U)\big). \tag{6.13}$$

On the other hand, it is easy to deduce from (6.7) and (6.9) that

$$\lim_{n\to\infty}\int_0^1\left(\|\rho_n|u_n|^2\|_{L^{\frac{3\gamma}{\gamma+3}}(U)}+\|\rho_n u_n\|^2_{L^{\frac{6\gamma}{\gamma+6}}(U)}\right)dt=0. \tag{6.14}$$

Because ρ,u are solutions to (1.1) in the sense of renormalized solutions, we take the test function sequence $\eta(x,t)=\psi(t)\phi(x)$ in (1.1), with $\phi(x)\in\mathcal{D}(U)$, $\psi(t)\in\mathcal{D}(0,1)$, to satisfy

$$\int_0^1\Big(\int_U\rho_n(x,t)\phi(x)dx\Big)\psi'(t)\,dt+\int_0^1\int_U\rho_n(x)u_n(x)\nabla\phi(x)\psi(t)\,dxdt=0.$$

Taking $n\to\infty$ and using (6.14), we get

$$\int_0^1 \Big(\int_U \rho_s \phi(x)dx\Big)\psi'(t)\,dt = 0,$$

which indicates that ρ_s is a function independent of t, and hence $m(\rho)\doteq\int_U \rho(x,t)dx$ is a constant. On the other hand, by (6.9), (6.12), and (6.13), we have

$$\mathcal{H}(Q_s)=0. \tag{6.15}$$

Hence if we apply the test function $\eta(x,t)$ again to equation (1.3), we know that Q_s is also a function independent of t. Moreover, we infer from equations (1.3) and (6.7) that

$$\partial_t Q_n \in L^2\big((0,1);L^{\frac{3}{2}}(U)\big),$$

which, combined with (6.12), allows us to deduce by the Aubin-Lions compactness theorem that

$$Q_n \to Q_s \ \text{ strongly in } L^2(0,1,H^1(U)), \tag{6.16}$$

with Q_s satisfying

$$\mathcal{H}(Q_s)=0, \quad Q_s\in S_0^3, \ \text{ a.e. in } U, \quad Q_s|_{\partial U}=Q_0. \tag{6.17}$$

Next, similar to arguments in previous sections, we can establish the following higher integrability result for ρ in $3D$.

Lemma 6.2. *For $\gamma>1$, there exists $\theta>0$ such that for all n, it holds that*

$$\int_0^1\int_U \rho_n^{\gamma+\theta}(x,t)\,dxdt \le C.$$

By Lemma 6.2, we may assume

$$\rho_n^\gamma \to \overline{\rho^\gamma} \ \text{ weakly in } L^{\frac{\gamma+\theta}{\gamma}}\big((0,1)\times U\big). \tag{6.18}$$

Thus, passing to the limit in equation (1.2), and using (6.8), (6.9), and (6.14), we obtain

$$\begin{aligned}
\nabla\overline{\rho^\gamma} &= -\nabla\cdot\big(L\nabla Q_s\odot\nabla Q_s - \mathcal{F}(Q_s)I_3\big)\\
&= -\nabla Q_s : \big[L\Delta Q_s - aQ_s + bQ_s^2 - cQ_s\mathrm{tr}(Q_s^2)\big]\\
&= -\nabla Q_s : \big[\mathcal{H}(Q_s) + \frac{b}{3}\mathrm{tr}(Q_s^2)I_3\big]\\
&= -\nabla Q_s : \mathcal{H}(Q_s) - \frac{b}{3}\mathrm{tr}(Q_s^2)\nabla\mathrm{tr}(Q_s)\\
&= 0 \quad \text{in } \mathcal{D}'\big((0,1)\times U\big).
\end{aligned} \tag{6.19}$$

Next, following the same argument as in [11], that is, using the L^p-version of the celebrated Div-Curl Lemma argument as in [11], we can actually show that the convergence in (6.18) is strong, and hence

$$\rho_n \to \rho_s \ \text{ strongly in } L^\gamma\big((0,1)\times U\big). \tag{6.20}$$

Note that we already claim that ρ_s is a function independent of t, thus (6.19)-(6.20) indicate that

$$\rho_s = \frac{m_0}{|U|}, \tag{6.21}$$

where we used the fact that $m(\rho) = \int_U \rho \, dx$ is a constant, and $m_0 = \int_U \rho_0 \, dx$.

On the other hand, by the basic energy law (2.3) and Lemma 2.2, we may assume

$$\mathcal{E}_\infty \doteq \lim_{t\to\infty} \mathcal{E}(t) = \lim_{t\to\infty} \left(\int_U \left[\frac{1}{2}\rho|u|^2(t) + \frac{\rho^\gamma(t)}{\gamma-1} \right] dx + \mathcal{G}(Q(t)) \right). \tag{6.22}$$

We define the energy for the limit functions $(\rho_s, 0, Q_s)$ by

$$\mathcal{E}_s \doteq \int_U \frac{\rho_s^\gamma}{\gamma-1} \, dx + \mathcal{G}(Q_s). \tag{6.23}$$

Using (6.14), (6.16), and (6.20), we get

$$\mathcal{E}_\infty = \lim_{n\to\infty} \int_0^1 \mathcal{E}(\tau+n) \, d\tau = \lim_{n\to\infty} \int_0^1 \left\{ \int_U \left[\frac{1}{2}\rho_n |u_n|^2 + \frac{\rho_n^\gamma}{\gamma-1} \right] dx + \mathcal{G}(Q_n) \right\} d\tau = \mathcal{E}_s. \tag{6.24}$$

Finally, it is easy to derive from equation (1.1) that

$$\rho(t) \to \rho_s \quad \text{weakly in } L^\gamma(U), \text{ as } t \to \infty.$$

$\square$

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