

STOCHASTIC MODE-REDUCTION IN MODELS WITH CONSERVATIVE FAST SUB-SYSTEMS*

ANKITA JAIN[†], ILYA TIMOFEYEV[‡], AND ERIC VANDEN-EIJNDEN[§]

Abstract. A stochastic mode reduction strategy is applied to multiscale models with a deterministic energy-conserving fast sub-system. Specifically, we consider situations where the slow variables are driven stochastically and interact with the fast sub-system in an energy-conserving fashion. Because the stochastic terms only affect the slow variables, the fast sub-system evolves deterministically on a sphere of constant energy. However, in the full model the radius of the sphere slowly changes due to the coupling between the slow and fast dynamics. Therefore, the energy of the fast sub-system becomes an additional hidden slow variable that must be accounted for in order to apply the stochastic mode reduction technique to systems of this type.

Key words. Mode-reduction, conservative fast sub-system.

AMS subject classifications. 60J60, 62M99.

1. Introduction

The derivation of reduced models for various complex high-dimensional systems has been an active area of research for many decades, and it has recently received increased attention with particular emphasis on practical applications. For example, a stochastic mode reduction strategy was proposed in the context of atmospheric applications in [10, 12]. This strategy has been successfully applied to several prototype problems [11, 9, 1], and it also has been applied to more realistic geophysical examples [5, 4]. The technique builds on the idea of adiabatic elimination of fast variables in multiscale stochastic systems [7, 6, 8, 16] and assumes the existence of scale separation between the (low-dimensional) slow and (high-dimensional) fast variables. In the context of atmospheric applications, it is plausible to consider systems where the fast variables are driven stochastically and interact with the slow dynamics in an energy-conserving fashion. Since then, this technique has been extended to purely deterministic conservative systems with a chaotic deterministic heat bath constituting the fast variables [13, 14].

In this paper we present a mathematical formalism which extends the stochastic mode reduction to systems where the fast sub-system is deterministic and energy-conservative, but the slow variables can evolve in a stochastic or chaotic fashion. Therefore, unlike in [13], the total energy of the system is not conserved and the interactions of the slow variables with the fast bath affect the total energy of the fast sub-system. The most striking examples of such models are coupled parabolic-hyperbolic systems where the hyperbolic part plays the role of the fast sub-system. Such models arise, for example, in thermo-elasticity [18, 19, 3] and thermo-visco-elasticity [17, 2, 15], where they can be viewed as simplified versions of the true fluid-structure interaction models: the hyperbolic and the parabolic parts describe the fast-moving waves and the temperature in the domain, respectively.

*Received; December 18, 2014; accepted (in revised form): March 1, 2014. Communicated by Weinan E.

[†]University of Notre Dame, Department of Applied and Computational Mathematics and Statistics, 153 Hurley Hall, Notre Dame, IN 46556, USA (Ankita.Jain@nd.edu).

[‡]University of Houston, Department of Mathematics, 651 Philip G. Hoffman Hall, Houston, TX 77204-3008, USA (ilya@math.uh.edu).

[§]New York University, Courant Institute, 251 Mercer Street, New York, NY 10012, USA (eve2@cims.nyu.edu).

Applications of the stochastic mode reduction strategy typically require understanding the statistical behavior of the fast sub-system. Often, a mixed analytical-computational approach can be employed to evaluate the statistical behavior of the fast conservative sub-system on a shell of constant energy. However, because the energy of the fast sub-system is changing in time through the interactions with the slow modes, it is necessary to explicitly introduce the energy of the fast sub-system as an additional hidden slow variable in order to extend the stochastic mode reduction strategy to models described in the previous paragraph. We demonstrate that a closed-form stochastic differential equation for the evolution of the slow dynamics (including energy) can be derived and the coefficients in this equation can be evaluated from a single numerical simulation of the fast sub-system on an (arbitrary) shell of constant energy. Here we illustrate our approach on a simple prototype model – more realistic examples will be considered in subsequent papers. In particular, we consider an extended triad model which elucidates analytical and numerical issues of the stochastic-mode reduction for systems with deterministic conservative heat bath.

2. Prototype triad model

To illustrate our approach we consider a prototype extended triad model

$$\begin{aligned}\dot{x} &= \frac{1}{\varepsilon} \sum_{j,k=1}^n A_{1jk}^{xyy} y_j y_k - \gamma x + \sigma \dot{W}, \\ \dot{y}_j &= \frac{1}{\varepsilon} \sum_{k=1}^n A_{j1k}^{yxy} x y_k + \frac{1}{\varepsilon^2} \sum_{i,k=1}^n B_{jik}^{yyy} y_i y_k,\end{aligned}\tag{2.1}$$

where x is the slow variable, y_j , $j = 1, \dots, n$ are the fast variables, n is the total number of fast variables, and A_{1jk}^{xyy} , A_{j1k}^{yxy} , B_{jik}^{yyy} are interaction coefficients obeying the energy-conserving relationships

$$A_{1jk}^{xyy} + A_{j1k}^{yxy} + A_{k1j}^{yxy} = 0,\tag{2.2}$$

$$B_{jik}^{yyy} + B_{ikj}^{yyy} + B_{kji}^{yyy} = 0.\tag{2.3}$$

The fast variables are determined by the ε^2 timescale in the equation (2.1) and the virtual fast sub-system becomes

$$\dot{y}_j = \sum_{i,k=1}^n B_{jik}^{yyy} y_i y_k.\tag{2.4}$$

Condition (2.3) is essential for the formalism presented below. It ensures that the virtual fast sub-system in (2.4) is energy conservative which guarantees the existence of an invariant measure for this sub-system. The existence and uniqueness of this invariant measure is a necessary condition for the applicability of the approach presented in this paper. Condition (2.2), on the other hand, ensures that the energy is conserved by the interactions between the slow variable and the fast modes and, thus, (2.2) guarantees the existence of the invariant measure for the full model in (2.1). We stress that we made this assumption for simplicity: the existence of an invariant measure for the full system (2.1) is not necessary for our approach, and the formalism presented here is also potentially applicable to systems where the interaction between the slow variables and the fast sub-system is not energy-conservative.

One can attempt to apply the stochastic mode reduction strategy to the model in (2.1) directly. In the stochastic mode reduction procedure one needs to consider the behavior of the fast sub-system (2.4) and certain statistical properties of y -variables (e.g. correlations functions) typically enter as coefficients into the reduced model. The behavior of the fast sub-system depends drastically on the energy level, but the energy in the fast modes changes in time due to the interaction with the slow variables. Therefore, it is necessary to consider the energy of the fast sub-system

$$E = \sum_{k=1}^n y_k^2 \tag{2.5}$$

as an additional slow variable to explicitly keep track of the changes in the statistical behavior of y -variables. The extended multiscale triad model becomes

$$\begin{aligned} \dot{x} &= \frac{1}{\varepsilon} \sum_{j,k=1}^n A_{1jk}^{xyy} y_j y_k - \gamma x + \sigma \dot{W}, \\ \dot{E} &= -2 \frac{1}{\varepsilon} \sum_{j,k=1}^n A_{1jk}^{xyy} x y_j y_k, \\ \dot{y}_j &= \frac{1}{\varepsilon} \sum_{k=1}^n A_{j1k}^{yxy} x y_k + \frac{1}{\varepsilon^2} \sum_{i,k=1}^n B_{jik}^{yyy} y_i y_k, \end{aligned} \tag{2.6}$$

where we used the property (2.2) to simplify the equation for the energy.

2.1. Stationary distribution of the triad model. The stationary distribution of the generalized triad model (2.6) can be computed explicitly for any ε and it is easy to show that the stationary distribution for x, y_1, \dots, y_n is a product of Gaussian distributions with mean zero and identical variances $\sigma^2/(2\gamma)$. In particular, the stationary distribution does not depend on the parameter ε .

The Fokker-Planck equation for the invariant density $\rho = \rho(x, y_1, \dots, y_n)$ for x, y_1, \dots, y_n in (2.6) can be written as

$$0 = \left(A_0 + \frac{1}{\varepsilon} A_1 + \frac{1}{\varepsilon^2} A_2 \right) \rho,$$

where

$$\begin{aligned} A_0 &= \gamma \partial_x x + \frac{\sigma^2}{2} \partial_{xx}, \\ A_1 &= - \sum_{j,k} A_{1jk}^{xyy} y_j y_k \partial_x - \sum_{j,k} A_{j1k}^{yxy} x \partial_{y_j} y_k, \\ A_2 &= - \sum_{i,j,k} B_{jik}^{yyy} \partial_{y_j} y_i y_k. \end{aligned}$$

The differential operator A_0 annihilates the Gaussian density

$$\rho_x(x) = \frac{\sqrt{2\gamma}}{\sqrt{2\pi\sigma}} \exp\left(-\frac{\gamma}{\sigma^2} x^2\right). \tag{2.7}$$

Operators A_1 and A_2 annihilate separately any function of the full energy $x^2 + E$ with E in (2.5) due to the conservation of energy properties in (2.2) and (2.3). Therefore,

they also annihilate the function

$$\rho = \left(\frac{\sqrt{2\gamma}}{\sqrt{2\pi\sigma}} \right)^{n+1} \exp \left(-\frac{\gamma}{\sigma^2} \left[x^2 + \sum_j y_j^2 \right] \right).$$

Thus, the invariant measure for x, y_1, \dots, y_n in (2.6) is a product measure with

$$x, y_1, \dots, y_n \sim N \left(0, \frac{\sigma^2}{2\gamma} \right).$$

The stationary distribution for the energy E in (2.5) can be easily derived because the joint stationary distribution of fast variables is a product of Gaussian densities and, therefore, the stationary distribution for E is a χ -squared distribution

$$\rho_E(s) = C s^{(n-2)/2} e^{-\frac{s\gamma}{\sigma^2}}. \quad (2.8)$$

The joint stationary distribution for the slow variables x and E is a product density of the corresponding marginal densities in (2.7) and (2.8).

2.2. Limit of full model as $\varepsilon \rightarrow 0$. Derivation of the reduced model for x in E in (2.6) proceeds in a typical fashion. The Kolmogorov backward equation associated with (2.6) for a scalar function $u = u(t, x, E, y_1, \dots, y_n)$ is given by

$$\partial_t u = L_0 u + \frac{1}{\varepsilon} L_1 u + \frac{1}{\varepsilon^2} L_2 u, \quad (2.9)$$

where the operators above are

$$\begin{aligned} L_0 &= -\gamma x \partial_x + \frac{\sigma^2}{2} \partial_{xx}, \\ L_1 &= \sum_{j,k} A_{1jk}^{xyy} y_j y_k \partial_x + \sum_{j,k} A_{j1k}^{yxy} x y_k \partial_{y_j} - 2 \sum_{j,k} A_{1jk}^{xyy} x y_j y_k \partial_E, \\ L_2 &= \sum_{i,j,k} B_{jik}^{yyy} y_i y_k \partial_{y_j}. \end{aligned}$$

Next, we introduce the projection operator

$$\mathbb{P} \cdot = \int \cdot d\mu(\vec{y}|E),$$

where $d\mu(\vec{y}|E)$ is the invariant measure of the fast sub-system (2.4). The energy of the fast sub-system does not change in time and, thus, the measure $d\mu(\vec{y}|E)$ is concentrated on the sphere of constant energy. In the absence of additional information, we assume that the measure $d\mu(\vec{y}|E)$ is the uniform measure on the sphere, i.e.

$$d\mu(\vec{y}|E) = S_n^{-1} E^{1-n/2} \delta(E - \sum_j y_j^2) d\vec{y}, \quad (2.10)$$

where n is the total number of fast variables, E is the fixed energy in the fast sub-system, $S_n^{-1} E^{1-n/2}$ is the normalizing factor such that S_n does not depend on E , and $\delta(\cdot)$ is the Dirac delta function.

Considering the expansion

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$$

and collecting powers of ε we obtain

$$\begin{aligned} O(\varepsilon^{-2}) : L_2 u_0 &= 0, \\ O(\varepsilon^{-1}) : L_2 u_1 &= -L_1 u_0, \\ O(1) : L_2 u_2 &= \partial_t u_0 - L_0 u_0 - L_1 u_1. \end{aligned} \tag{2.11}$$

The first equation implies that $u_0 = u_0(x, E, t)$ and is independent of the fast variables y_1, \dots, y_n because L_2 involves differentiation with respect to only y variables. Applying \mathbb{P} to the second equation in (2.11) we obtain the compatibility condition $\mathbb{P}L_1 = 0$ (because \mathbb{P} is the projection with respect to the invariant measure of the fast sub-system and thus $\mathbb{P}L_2 = L_2\mathbb{P} = 0$). The second equation in (2.11) implies that

$$u_1 = -L_2^{-1}L_1 u_0. \tag{2.12}$$

Finally, substituting (2.12) into the third equation and applying \mathbb{P} to both sides we obtain the reduced (homogenized) equation for u_0 :

$$\partial_t u_0 = L_0 u_0 + L u_0, \tag{2.13}$$

where L is the reduced operator given by

$$L = -\mathbb{P}L_1L_2^{-1}L_1\mathbb{P}. \tag{2.14}$$

Note that the stochastic terms in the equation for x which are not affected by the fast sub-system are simply “carried through” by the mode reduction procedure, i.e. the reduced backward equation involves the term $L_0 u_0$, where operator L_0 corresponds to the terms without ε in the equation for x . Derivation of the reduced operator proceeds in a formal fashion by manipulating the summations in the expression (2.14) and formally expressing the action of L_2^{-1} . Details of the derivation are presented in Appendix A.

The compatibility condition $\mathbb{P}L_1 = 0$ implies the following conditions on the stationary first and second moments of the y -variables in the statistical behavior of the fast sub-system (2.4):

$$\mathbb{E}_\mu y_j = 0, \quad \mathbb{E}_\mu y_j y_k = 0 \text{ for } j \neq k. \tag{2.15}$$

The first condition is quite plausible, it simply implies the symmetry of the distribution. The second condition is not obvious. While the stationary second mixed moment $\mathbb{E}y_j y_k$ is zero in the full model, the requirement in (2.15) is for the dynamics of the deterministic fast sub-system. Therefore, the conditions in (2.15) need to be verified numerically in the simulations of the fast sub-system on a shell of constant energy.

The backward reduced operator can be computed explicitly, and after substituting the action of L_2^{-1} and some formal manipulations it becomes

$$\begin{aligned} L = & -\frac{E^{1/2}}{n^{3/2}} M \left((n+1)x\partial_x + 2E\partial_E - 2x^2(n+1)\partial_E \right) \\ & + \frac{E^{3/2}}{n^{3/2}} M \left(\partial_{xx} - 2x\partial_{xE} - 2x\partial_{Ex} + 4x^2\partial_{EE} \right), \end{aligned} \tag{2.16}$$

where n is the number of fast variables and M is a numerical constant describing some bulk statistical properties of the fast sub-system. In particular, M is the area under

the fourth-order two-point moment of the fast sub-system (2.4) computed on the energy level $E = n$:

$$M = \sum_{jkj'k'} A_{1jk}^{xyy} A_{1j'k'}^{xyy} \int_0^\infty \int y_j y_k y_{j'}(\tau) y_{k'}(\tau) d\mu(\vec{y} | E = n) d\tau. \quad (2.17)$$

The numerical constant M cannot be deduced analytically and has to be computed numerically. Nevertheless, this computation needs to be done only once using time-averaging in the simulations of the fast sub-system. Moreover, the fast sub-system alone is not a multiscale system and, thus, can be simulated rather easily.

The effective SDE corresponding to the reduced operator (2.16) is

$$dx = -\gamma x dt - (n+1) M x \frac{E^{1/2}}{n^{3/2}} dt + \sigma dW_1 + \sqrt{2M} \left(\frac{E}{n}\right)^{3/4} dW_2, \quad (2.18)$$

$$dE = -2M \left(\frac{E}{n}\right)^{3/2} dt + 2(n+1) M x^2 \frac{E^{1/2}}{n^{3/2}} dt - 2\sqrt{2M} x \left(\frac{E}{n}\right)^{3/4} dW_2.$$

3. Numerical simulations

To illustrate the mode reduction approach we choose parameters of the full model (2.1) as

$$\gamma = 1, \quad \sigma = 2.236, \quad n = \dim(\vec{y}) = 10. \quad (3.1)$$

The interaction coefficients are given in tables B.1 and B.2 in Appendix B. The corresponding number of triads in the full model is

$$\text{Number of } xyy \text{ triads} = 10, \quad \text{Number of } yyy \text{ triads} = 19.$$

We use an analog of the split-step method to integrate the full model (2.1). In particular, due to the multiscale nature of the full model, the time-stepping for the deterministic part of the full model requires a relatively high order discretization formula in order to correctly represent the energy transfer between x and y -variables during a single time-step. We observed empirically that the full model is rather sensitive to the deterministic integrator; thus, we use a RK5 formula to integrate numerically the deterministic part of the system (2.1). Next, we use Euler discretization to add a Gaussian random variable which approximates ΔW . The overall (stochastic) scheme is of order Δt , but the energy transfer in this numerical scheme is represented much more accurately. This turns out to be essential. Indeed we observed empirically that numerical simulations of the full system with the low-order deterministic discretization (Euler and RK2) produce severe discrepancies with analytical predictions (e.g. the stationary distribution of the energy variable given by (2.8)). Both RK4 and RK5 discretizations of the deterministic terms produce similar results in this case, and we use a higher-order method to ensure the robustness of numerical results.

We compute the statistical properties of the slow variable, x , as well as the statistics of E , because E enters explicitly into the reduced model and stationary statistics of both variables should be used as criteria for the comparison between the full model and the reduced dynamics.

We use time-averaging combined with Monte-Carlo approach to accelerate computations of the full model in (2.6). In particular, we perform 10 independent runs

with different initial conditions of the full model with $T=40,000$ for time-averaging in each run and then average over these 10 realizations. We use the time-step $\Delta t = 10^{-4}, 2.5 \times 10^{-5}, 2 \times 10^{-5}, 10^{-6}$ for $\varepsilon = 1, 0.5, 0.25, 0.1$, respectively.

The reduced model is integrated using the same method with the RK5 deterministic integrator and Euler discretization for the noise. Compared to the full model, the reduced system is not as sensitive to the choice of the discrete scheme for the deterministic terms. In particular, the numerical results with the RK2 and RK5 discretizations are very similar. However, a much smaller time-step is required for the numerical scheme with the Euler discretization (i.e. the stochastic Euler-Maruyama scheme) to produce accurate results. We also perform a hybrid approach combining time-averaging and Monte-Carlo simulations to compute the stationary statistical properties of the reduced model. We run $K=10$ trajectories each with $T=100,000$, $\Delta t = 10^{-5}$, and different initial conditions, and then take the average of statistical quantities. The coefficient M in the reduced model (2.18) computed from the fast sub-system numerically is

$$M = 1.2759.$$

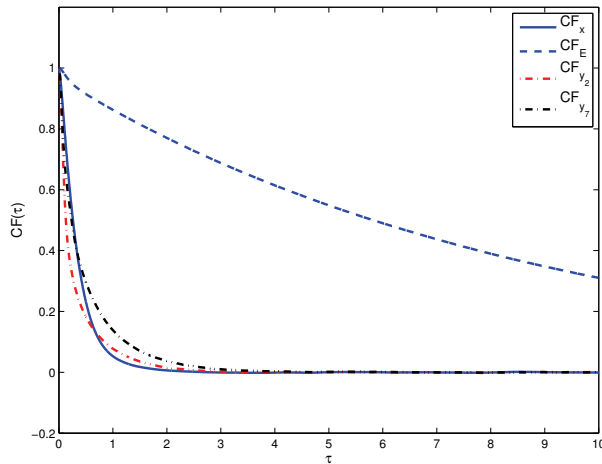


FIG. 3.1. Normalized correlation functions CF_x , CF_E and CF_{y_2} and CF_{y_7} ; modes y_2 and y_7 are among the slowest y -variables and their correlation functions illustrate the absence of scale separation for $\varepsilon=1$.

First, we examine the scale separation between the slow and the fast variables in the full model (2.1). Comparison of correlation functions of the slow variables and a few fast modes for $\varepsilon=1$ is presented in figure 3.1. The corresponding correlation times for $\varepsilon=1$ are presented in table 3. Correlation times are computed as the inverse of the area under the graph of the normalized correlation function. This simulation shows that there is a group of y -variables with time-scales comparable to the time-scale of x . Therefore, we can conclude that there is no time-scale separation between the x - and y -variables for $\varepsilon=1$. However, we also observe that E is much slower than x . Moreover, table 3 demonstrates that the scale separation between the slow variables (x, E) and fast y -variables increases as $\varepsilon \rightarrow 0$. Therefore, although we expect that the stochastic mode reduction will perform quite well in the limit $\varepsilon \rightarrow 0$, the agreement with the numerical

Variable	CT $\varepsilon=1$	CT $\varepsilon=0.5$	CT $\varepsilon=0.25$	CT $\varepsilon=0.1$
x	0.3395	0.3433	0.3324	0.3258
E	8.0605	7.7616	7.336	7.218
y_1	0.2395	0.0645	0.0177	0.0052
y_2	0.3239	0.0828	0.022	0.006
y_3	0.3148	0.088	0.025	0.0063
y_4	0.1460	0.0403	0.0125	0.0045
y_5	0.1706	0.0461	0.0131	0.0046
y_6	0.0713	0.0209	0.0068	0.0041
y_7	0.4910	0.1255	0.0039	0.0076
y_8	0.0570	0.0156	0.0056	0.0041
y_9	0.3383	0.0876	0.0244	0.006
y_{10}	0.1126	0.0305	0.0094	0.0041

TABLE 3.1. Correlation Time (CT) in the simulations of the full model with different values of ε .

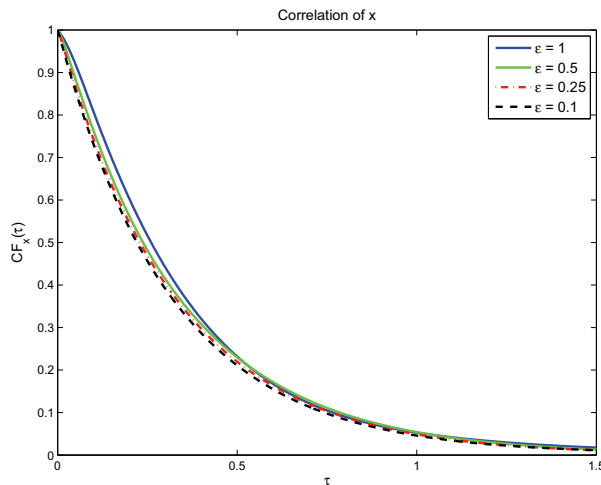


FIG. 3.2. Normalized correlation function of x in the full model (2.1) for $\varepsilon=1, 0.5, 0.25, 0.1$.

results for $\varepsilon=1$ is not guaranteed. In addition, figure 3.1 also demonstrates the necessity to include the energy E in the reduced description. The energy $E \equiv E(t)$ acts as a hidden stochastic variable which is essential for the derivation of the reduced model; figure 3.1 indicates that it is not possible, for instance, to average the reduced model (2.18) with respect to the stationary distribution for E , because E is much slower than x .

Next, we examine the behavior of the full model as $\varepsilon \rightarrow 0$ numerically. In particular, we consider four values of ε

$$\varepsilon = 1, 0.5, 0.25, 0.1$$

and perform simulations to illustrate changes in the statistical behavior of the full model as $\varepsilon \rightarrow 0$. The stationary density for x and E agrees very well with the analytical predictions in (2.7), (2.8) for any ε and we only depict the stationary correlation functions for x and E in figures 3.2 and 3.3, respectively. The stationary correlation function for

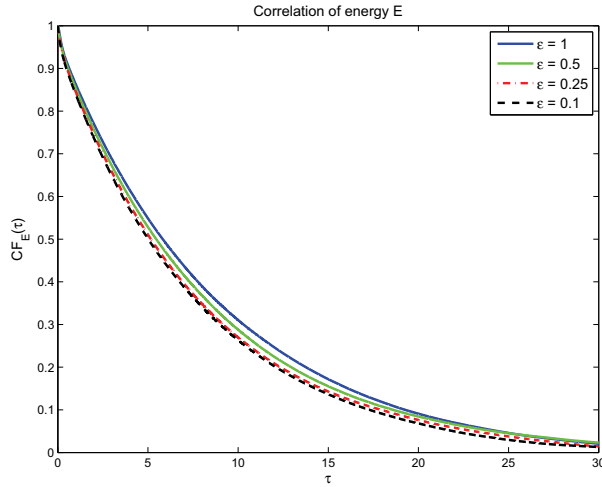


FIG. 3.3. Normalized correlation function of E in the full model (2.1) for $\varepsilon = 1, 0.5, 0.25, 0.1$.

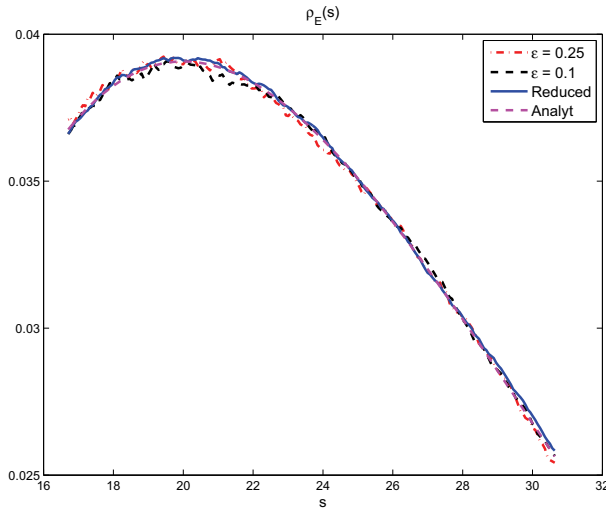


FIG. 3.4. Marginal Density of the slow variable E in the full model with $\varepsilon = 0.25, 0.1$, Reduced Model, and the analytical formula in (2.8).

a stochastic variable $z(t)$ is given by

$$CF_z(\tau) = \mathbb{E}z(t)z(t+\tau) - (\mathbb{E}z)^2,$$

and we normalize correlation functions by the variance, so that $CF(0) = 1$. Numerical simulations show that correlation functions of both x and E change only slightly as $\varepsilon \rightarrow 0$. Therefore, we expect that the stochastic mode reduction will be in good agreement with the full model even for $\varepsilon = 1$.

As a final step, we compare the equilibrium statistical behavior of the full and reduced models to validate the performance of the stochastic mode reduction. As ex-

pected, the marginal densities of x and E agree quite well in all simulations. Here we present only the numerical results for the marginal density of E , the marginal densities for x computed numerically completely overlap with each other and agree perfectly with the analytical prediction in (2.7). Comparison of the marginal density E is presented in figure 3.4.

Comparison of the two-point statistical quantities is a much more severe test for the mode-reduction because the mode reduction does not, in general, preserve the time-scales of the slow variables, and the two-point statistics for the full and reduced models can fail to agree due to the insufficient scale separation or invalidity of the underlying assumptions (e.g. ergodicity, uniform distribution on the sphere, etc.).

Because the two-point statistical quantities cannot be computed analytically, we compare the results of numerical simulations of the full model for $\varepsilon = 0.25, 0.1$ and the reduced model. Comparison of the correlation functions for x and E is presented in figure 3.5. Correlation functions for both x and E in the simulations of the full model agree very well with the numerical results from the reduced model. This demonstrates the validity of the mode reduction procedure in the setup considered in this paper, i.e. when it is necessary to consider the energy of the fast sub-system as an additional slow variable.

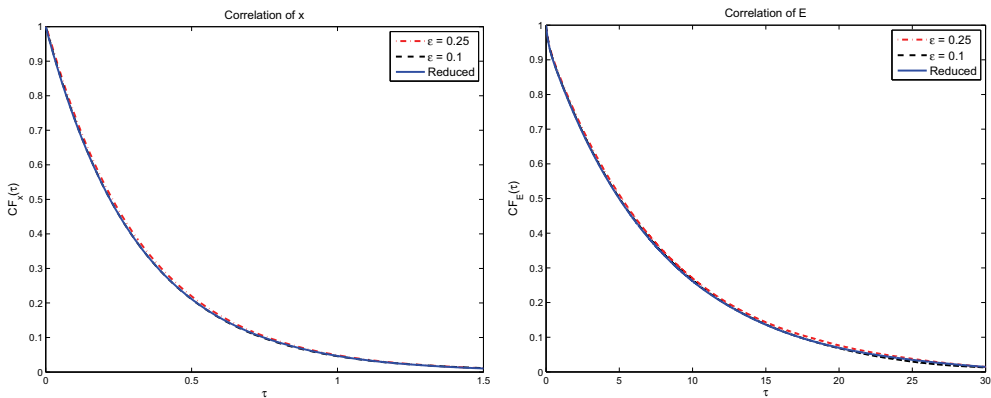


FIG. 3.5. Correlation function of CF_x and CF_E in the full model (2.1) with $\varepsilon = 0.25, 0.1$ and in the Reduced model (2.18). The solid blue line (reduced model) overlaps completely with the black dashed line (full mode with $\varepsilon = 0.1$),

It is also important to compare higher-order statistical quantities to analyze the non-Gaussian behavior of the system. Because the equilibrium distribution of x is Gaussian and the correlation function exhibits exponential decay, one can be easily deceived to believe that a linear Ornstein-Uhlenbeck model would be sufficient to accurately reproduce the statistical behavior of x . To illustrate the importance of the functional form of the reduced model in (2.18), we compare the behavior of the fourth-order two-point moment

$$K_z(\tau) = \frac{\mathbb{E}z^2(t+\tau)z^2(t)}{(\mathbb{E}z^2)^2 + 2(\mathbb{E}z(t)z(t+\tau))^2} \quad (3.2)$$

in figure 3.6. For Gaussian processes $K_z(\tau) \equiv 1$ for all lags τ . Therefore, $K_x(\tau)$ and $K_E(\tau)$ measure the non-Gaussian features of the corresponding stochastic processes.

$K_x(\tau)$ in the left part of figure 3.6 demonstrates the non-Gaussian feature of the stochastic variable $x(t)$. Moreover, this figure also suggests that the scale-separation affects the convergence of the higher-order statistics; while correlation functions of x in the full model for $\varepsilon=0.25, 0.1$ and the reduced model nearly overlap (figure 3.5), there is about 1% discrepancy between the full model with $\varepsilon=0.25$ and $\varepsilon=0.1$ and, also, between the reduced model and the full model with $\varepsilon=0.1$. The discrepancy with the full model with $\varepsilon=1$ and $\varepsilon=0.5$ (not shown here) is even more severe. On the other hand, the kurtosis for the slow variable E in the reduced stochastic model (2.18) is in a much better agreement with the full model (2.6) for all values of ε .

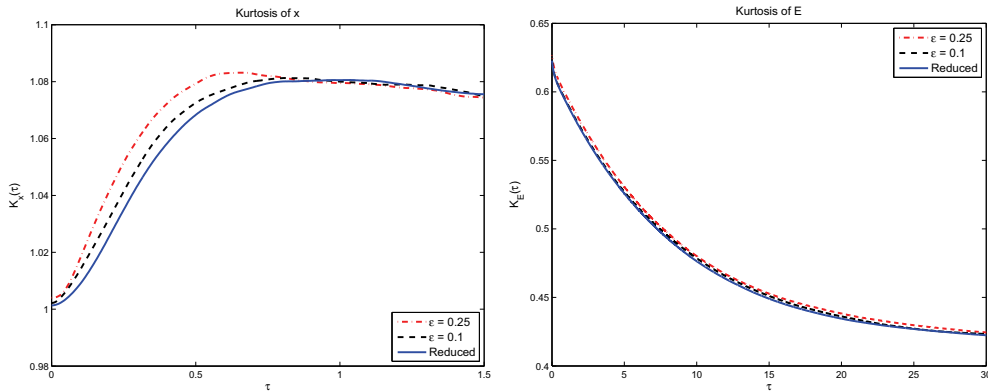


FIG. 3.6. Kurtosis (3.2) of slow variables x and E in full model with $\varepsilon=0.25, 0.1$ and in the Reduced model. The vertical scale for the Kurtosis of x is rather small; errors between the full and reduced model for the Kurtosis of x are approximately 1%.

Although the energy variable E is much slower than x in the reduced model (2.18) (e.g. compare the correlation functions CF_E and CF_x depicted in figure 3.5) for the particular choice of interaction coefficients discussed in this example, the structure of the reduced equation (2.18) does not allow to average the behavior of E with respect to the stationary measure of x . In particular, the structure of the diffusion term precludes one from further accelerating the x variable without affecting the joint stationary distribution for x, E . On the other hand, by the same argument, it would be impossible to average the behavior of x with respect to the stationary measure for E , if E was the faster variable. Therefore, the system in (2.18) is the correct reduced model regardless of the relative behavior of x and E .

4. Conclusions

We have presented an extension of the stochastic mode-reduction strategy for systems with stochastically perturbed slow variables and energy-conserving fast sub-system. The coupling between the slow and fast variables induces a slow energy exchange between the slow variables and fast sub-system, and it is necessary to include the slowly-evolving energy of the fast sub-system as an additional variable in the reduced description of the full model. Our numerical simulations demonstrate that the proposed formalism captures the equilibrium statistical behavior of the slow variables very well. Moreover, our example also demonstrates that the “hidden” energy variable can evolve on a much slower time-scale than the slow variables in the full model and, thus, the statistical behavior of these slow variables cannot be averaged over the distribution of the energy.

The derivation of the reduced equation follows a standard formalism which relies on the asymptotic expansion of the backward equation, and the reduced equation involves constants which describe some bulk equilibrium statistical properties of the fast sub-system. Although these constants cannot be deduced analytically, they can be computed from a single microcanonical simulation of the fast sub-system on a shell of constant energy. Such simulation is not particularly demanding because it is performed on the natural time scale of the fast sub-system, which is not multiscale.

The formalism presented in this paper extends the applicability of the stochastic mode-reduction strategy. Moreover, this formalism can be applied to purely deterministic systems with chaotic behavior of the slow variables. Such systems can arise, for instance, as spectral truncations of coupled hyperbolic-parabolic PDE models, which will be considered in future work.

5. Acknowledgment. The research of I.T. was partially supported by the NSF Grant DMS-1109582. I.T. also acknowledges support from SAMSI and IMA as a long-term visitor in the Fall of 2011 and Fall 2012, respectively. The research of E. V.-E. was partially supported by the DOE ASCR Grant DE-FG02-88ER25053, the NSF Grant DMS07-08140, and the ONR Grant N00014-11-1-0345.

Appendix A. Derivation of the reduced operator. Here, we outline the derivation of the reduced backward operator in (2.16). Substituting L_1 into (2.14) and neglecting derivatives ∂_{y_j} to the right of L_2^{-1} (because the effective operator is applied to the function $u_0(x, E, t)$ which does not involve the y -variables), we obtain the following expression for L :

$$\begin{aligned} -\mathbb{P}L_1L_2^{-1}L_1 &= -\sum_{jkj'k'} \int (A_{1jk}^{xyy} y_j y_k \partial_x + A_{j1k}^{yxy} x y_k \partial_{y_j} - 2A_{1jk}^{xyy} x y_j y_k \partial_E) L_2^{-1} \\ &\quad (A_{1j'k'}^{xyy} y_{j'} y_{k'} \partial_x - 2A_{1j'k'}^{xyy} x y_{j'} y_{k'} \partial_E) d\mu(\vec{y}|E) \\ &= -\sum_{jkj'k'} \int_0^\infty \int (A_{1jk}^{xyy} y_j y_k \partial_x + A_{j1k}^{yxy} x y_k \partial_{y_j} - 2A_{1jk}^{xyy} x y_j y_k \partial_E) \\ &\quad (A_{1j'k'}^{xyy} y_{j'}(\tau) y_{k'}(\tau) \partial_x - 2A_{1j'k'}^{xyy} x y_{j'}(\tau) y_{k'}(\tau) \partial_E) d\mu(\vec{y}|E) d\tau, \quad (\text{A.1}) \end{aligned}$$

where we substituted the action of L_2^{-1} which corresponds to propagating the initial conditions y_j in time using fast sub-system and integrating over all times. Here, $y_j(\tau)$ is the solution on the fast sub-system on the energy level E .

Deriving diffusion terms. The diffusion part of the reduced operator can be computed by collecting second derivatives in (A.1):

$$\begin{aligned} L_{diff} &= \sum_{jkj'k'} A_{1jk}^{xyy} A_{1j'k'}^{xyy} \int_0^\infty \int y_j y_k y_{j'}(\tau) y_{k'}(\tau) \rho(\vec{y}|E) d\vec{y} d\tau \\ &\quad [\partial_x^2 - 2x \partial_x \partial_E - 2x \partial_E \partial_x + 4x^2 \partial_E^2], \end{aligned}$$

where $\rho(\vec{y}|E)$ is given by (2.10). Let the double-integral in the expression above be denoted by a constant Q . This constant describes some bulk statistical properties of the fast sub-system. In particular, it is the area under the fourth two-point moment in the fast sub-system.

$$Q = \sum_{jkj'k'} A_{1jk}^{xyy} A_{1j'k'}^{xyy} \int_0^\infty \int y_j y_k y_{j'}(\tau) y_{k'}(\tau) \rho(\vec{y}|E) d\vec{y} d\tau.$$

Rescaling of the fast sub-system. Note that fast sub-system (2.4) is a deterministic model which conserves E . On the other hand, the fast sub-system is invariant under the following transformation:

$$y = \left(\frac{E}{n}\right)^{1/2} y^*, \quad t = \left(\frac{E}{n}\right)^{-1/2} t^*,$$

where y^* evolves on the energy shell $E = n$. Hence, the area under the fourth moment Q can be rewritten as

$$\begin{aligned} Q &= \left(\frac{E}{n}\right)^{3/2} \sum_{jkj'k'} A_{1jk}^{xyy} A_{1j'k'}^{xyy} \int_0^\infty \int y_j^* y_k^* y_{j'}^* (\tau) y_{k'}^* (\tau) \rho(\vec{y}|E = n) d\vec{y} d\tau \\ &= \left(\frac{E}{n}\right)^{3/2} M, \end{aligned} \tag{A.2}$$

where M is the area under the fourth moment in the fast sub-system (2.4) with fixed energy $E = n$:

$$M = \sum_{jkj'k'} A_{1jk}^{xyy} A_{1j'k'}^{xyy} \int_0^\infty \int y_j y_k y_{j'} (\tau) y_{k'} (\tau) \rho(\vec{y}|E = n) d\vec{y} d\tau. \tag{A.3}$$

The constant M cannot be derived analytically. Therefore, we have to estimate the constant M numerically. However, the constant M has to be computed only once from the numerical simulations of the fast sub-system alone, which is not a multiscale model.

The diffusion matrix. Let Σ be the diffusion matrix in the backward equation. The square of the diffusion matrix for reduced model can be written as

$$\Sigma^2 = 2Q \begin{pmatrix} 1 & -2x \\ -2x & 4x^2 \end{pmatrix} = 2M \left(\frac{E}{n}\right)^{3/2} \begin{pmatrix} 1 & -2x \\ -2x & 4x^2 \end{pmatrix}.$$

Deriving the drift and diffusion terms using uniform measure on the sphere.

The derivation of the drift terms in the reduced operator can proceed in two possible ways. First, because the diffusion in the reduced operator is known and the stochastic mode reduction preserves the invariant distribution of the slow variables (2.7), (2.8), one can derive the drift terms in the Fokker-Planck operator which are consistent with the stationary distribution of x and E in the full model. This derivation relies heavily on the known form of the stationary distribution for the slow variables and, thus, the particular form of the self-interactions of the slow variables given by the operator L_0 and the form of the coupling terms between the slow and fast variables.

As an alternative derivation, we proceed by formally manipulating the derivatives in the expression for the reduced operator (A.1) to obtain the drift terms. To this end, one also has to differentiate the invariant measure $\mu(\vec{y}|E)$ in the expression (A.1).

We assume that the fast sub-system is ergodic on the hypersphere defined by the energy of the fast sub-system with respect to the uniform distribution on the sphere. Then the stationary distribution of the fast sub-system can be written as (2.10):

$$d\mu(\vec{y}|E) = S_n^{-1} E^{1-n/2} \delta(E - E_0) d\vec{y}, \tag{A.4}$$

where n is the total number of fast variables, E_0 is the given fixed energy in the fast sub-system, $S_n^{-1} E^{1-n/2}$ is the normalizing factor such that S_n does not depend on E ,

and $\delta(\cdot)$ is the Dirac delta function. The normalization factor $S_n^{-1}E^{1-n/2}$ is derived from the condition

$$\int d\bar{y} \rho(\bar{y}|E) = 1.$$

Because the stationary measure of the fast sub-system depends on the energy E , the operator L_2^{-1} also depends on the energy level E . Moreover, L_2^{-1} also depends on y_i and the behavior of the time-dependent variables $y_j(\tau)$ depends on the energy level as well. Therefore, to understand the derivatives $\partial_E L_2^{-1}$ and $\partial_{y_i} L_2^{-1}$, we split the effective operator L in (A.1) into three parts

$$L = I_1 + I_2 + I_3, \quad (\text{A.5})$$

where

$$\begin{aligned} I_1 &= - \int \rho(\bar{y}|E) \sum_{j,k} A_{1jk}^{xyy} y_j y_k \partial_x L_2^{-1} \left(\sum_{j,k} A_{1jk}^{xyy} y_j y_k \partial_x - 2 \sum_{j,k} A_{1jk}^{xyy} x y_j y_k \partial_E \right) d\bar{y} \\ &= \sum_{j,k} A_{1jk}^{xyy} \int_0^\infty \int \rho(\bar{y}|E) y_j y_k \partial_x \left(\sum_{j',k'} A_{1j'k'}^{xyy} y_{j'}(\tau) y_{k'}(\tau) [\partial_x - 2x\partial_E] \right) d\bar{y} d\tau, \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} I_2 &= - \int \rho(\bar{y}|E) \sum_{j,k} A_{j1k}^{yxy} x y_k \partial_{y_j} L_2^{-1} \left(\sum_{j,k} A_{1jk}^{xyy} y_j y_k \partial_x - 2 \sum_{j,k} A_{1jk}^{xyy} x y_j y_k \partial_E \right) d\bar{y} \\ &= x \sum_{j,k,j',k'} A_{j1k}^{yxy} A_{1j'k'}^{xyy} \int_0^\infty \int \rho(\bar{y}|E) y_k \partial_{y_j} (y_{j'}(\tau) y_{k'}(\tau)) [\partial_x - 2x\partial_E] d\bar{y} d\tau, \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned} I_3 &= \int \rho(\bar{y}|E) 2 \sum_{j,k} A_{1jk}^{xyy} x y_j y_k \partial_E L_2^{-1} \left(\sum_{j,k} A_{1jk}^{xyy} y_j y_k \partial_x - 2 \sum_{j,k} A_{1jk}^{xyy} x y_j y_k \partial_E \right) d\bar{y} \\ &= -2x \sum_{j,k,j',k'} A_{1jk}^{xyy} A_{1j'k'}^{xyy} \int_0^\infty \int \rho(\bar{y}|E) y_j y_k \partial_E (y_{j'}(\tau) y_{k'}(\tau)) [\partial_x - 2x\partial_E] d\bar{y} d\tau. \end{aligned} \quad (\text{A.8})$$

Computing I_1 . Because the fast sub-system does not depend on the value of the slow variable x , the calculation of I_1 is straightforward and we obtain

$$I_1 = \sum_{j,k} A_{1jk}^{xyy} \sum_{j',k'} A_{1j'k'}^{xyy} \int_0^\infty \int y_j y_k y_{j'}(\tau) y_{k'}(\tau) \rho(\bar{y}|E) d\bar{y} d\tau \partial_x [\partial_x - 2x\partial_E],$$

where $y_j(\tau)$ is the solution of the fast sub-system with $y_j(0) = y_j$. The expression for I_1 involves the area under the fourth two-points moment, similar to the discussion before:

$$I_1 = Q(\partial_{xx} - 2x\partial_{xE} - 2\partial_E),$$

where and $Q_{j,k,j',k'}$ is the integrated fourth-order two-point moment in the fast sub-system (2.4) on the energy level E . I_1 can be further simplified after rescaling of the fast sub-system to the energy shell $E = n$:

$$I_1 = \left(\frac{E}{n}\right)^{3/2} M [\partial_{xx} - 2x\partial_{xE} - 2\partial_E], \quad (\text{A.9})$$

where M is given by (2.17).

Computing I_2 . The operator I_2 involves differentiation with respect to y_j which cannot be evaluated in a straightforward manner. Instead, this differentiation can be switched onto the $\rho(\vec{y}|E)$ by integrating by parts. Therefore, we obtain that I_2 can be expressed as

$$I_2 = -2xS_n^{-1}E^{1-n/2}J[\partial_x - 2x\partial_E], \tag{A.10}$$

where J is defined as

$$J = \sum_{j,k,j',k'} A_{j1k}^{yx} A_{1j'k'}^{xy} \int_0^\infty \int y_j y_k y_{j'}(\tau) y_{k'}(\tau) \delta' \left(E - \sum_{r=1}^n y_r^2 \right) d\vec{y} d\tau. \tag{A.11}$$

Note that the expression above involves the derivative of the Dirac delta function.

Computing I_3 . We can use the product rule to express the I_3 as follows:

$$I_3 = -2x \sum_{j,k,j',k'} A_{1jk}^{xy} A_{1j'k'}^{xy} \int_0^\infty \int \rho(\vec{y}|E) y_j y_k \partial_E [y_{j'}(\tau) y_{k'}(\tau)] d\vec{y} d\tau [\partial_x - 2x\partial_E] - 2xQ[\partial_{E_x} - 2x\partial_{E_E}],$$

where Q is the integrated fourth-order two-point moment in (A.2). Because the behavior of $y(\tau)$ depends on the energy level, E , the first term is non-zero. Let the first term on the left in I_3 be denoted by \tilde{I}_3 ,

$$\tilde{I}_3 = \int_0^\infty \int \rho(\vec{y}|E) y_j y_k \partial_E [y_{j'}(\tau) y_{k'}(\tau)] d\vec{y} d\tau, \tag{A.12}$$

so that I_3 can be written as

$$I_3 = -2x \sum_{j,k,j',k'} A_{1jk}^{xy} A_{1j'k'}^{xy} \tilde{I}_3 [\partial_x - 2x\partial_E] - 2Qx[\partial_{E_x} - 2x\partial_{E_E}]. \tag{A.13}$$

In order to understand the structure if the term \tilde{I}_3 Let us compute the derivative of the area under the fourth-order moment.

$$\begin{aligned} & \partial_E \int_0^\infty \int y_j y_k y_{j'}(\tau) y_{k'}(\tau) \rho(\vec{y}|E) d\vec{y} d\tau \\ &= \int_0^\infty \int \left(y_j y_k y_{j'}(\tau) y_{k'}(\tau) \partial_E [\rho(\vec{y}|E)] + \rho(\vec{y}|E) y_j y_k \partial_E [y_{j'}(\tau) y_{k'}(\tau)] \right) d\vec{y} d\tau \\ &= \int_0^\infty \int y_j y_k y_{j'}(\tau) y_{k'}(\tau) \partial_E [\rho(\vec{y}|E)] d\vec{y} d\tau + \tilde{I}_3. \end{aligned}$$

Therefore, the required term can be rewritten as

$$\begin{aligned} & \sum_{j,k,j',k'} A_{1jk}^{xy} A_{1j'k'}^{xy} \tilde{I}_3 \\ &= \partial_E Q - \sum_{j,k,j',k'} A_{1jk}^{xy} A_{1j'k'}^{xy} \int_0^\infty \int y_j y_k y_{j'}(\tau) y_{k'}(\tau) \partial_E [\rho(\vec{y}|E)] d\vec{y} d\tau. \end{aligned} \tag{A.14}$$

Because the stationary measure of the fast variables $\mu(\vec{y}|E)$ given by (2.10) we can compute the derivative of $\rho(\vec{y}|E)$ as follows:

$$\partial_E [\rho(\vec{y}|E)] = (1 - n/2)E^{-1} \rho(\vec{y}|E) + S_n^{-1} E^{1-n/2} \delta' \left(E - \sum_{i=1}^n y_i^2 \right),$$

where $\delta'(\cdot)$ represent the distributional derivative of Dirac delta function. Substituting the above expression into \tilde{I}_3 we obtain

$$\begin{aligned} & -2x \sum_{j,k,j',k'} A_{1jk}^{xyy} A_{1j'k'}^{xyy} \tilde{I}_3 \\ & = -2x \partial_E Q + x(2-n)E^{-1}Q + 2xS_n^{-1}E^{1-n/2}J, \end{aligned} \quad (\text{A.15})$$

where Q and J are given by (A.2) and (A.11), respectively. Therefore, putting everything together, we obtain

$$\begin{aligned} I_3 & = -x \frac{n+1}{n^{3/2}} E^{1/2} M (\partial_x - 2x \partial_E) - 2x \frac{E^{3/2}}{n^{3/2}} M (\partial_{Ex} - 2x \partial_{EE}) \\ & \quad + 2x S_n^{-1} E^{1-n/2} J (\partial_x - 2x \partial_E). \end{aligned} \quad (\text{A.16})$$

Substituting I_1 , I_2 , and I_3 into the effective operator we obtain that the terms involving J (the derivative $\delta'()$) cancel, and the effective operator becomes

$$\begin{aligned} L & = - \frac{E^{1/2}}{n^{3/2}} M ((n+1)x \partial_x + 2E \partial_E - 2x^2(n+1) \partial_E) \\ & \quad + \frac{E^{3/2}}{n^{3/2}} M (\partial_{xx} - 2x \partial_{xE} - 2x \partial_{Ex} + 4x^2 \partial_{EE}), \end{aligned} \quad (\text{A.17})$$

where M is the summation of the area under the fourth-order two-point moment given by (2.17). The diffusion part of the operator agrees with the previously computed matrix, i.e. the diffusion part of L can be rewritten as

$$L_{diff} = \frac{1}{2} \sum_{i,k=1,2} (DD^T)_{ik} \partial_{z_i} \partial_{z_k}, \quad (\text{A.18})$$

where $(z_1, z_2) \equiv (x, E)$ and

$$D = \sqrt{2M} \left(\frac{E}{n} \right)^{3/4} \begin{pmatrix} 1 & 0 \\ -2x & 0 \end{pmatrix}. \quad (\text{A.19})$$

Using the effective L operator in (A.17) and diffusion in (A.19), we obtain a reduced SDE model for x and E given by

$$\begin{aligned} dx & = -\gamma x dt - (n+1)Mx \frac{E^{1/2}}{n^{3/2}} + \sigma dW_1 + \sqrt{2M} \left(\frac{E}{n} \right)^{3/4} dW_2, \\ dE & = -2M \left(\frac{E}{n} \right)^{3/2} + 2(n+1)Mx^2 \frac{E^{1/2}}{n^{3/2}} - 2\sqrt{2M}x \left(\frac{E}{n} \right)^{3/4} dW_2, \end{aligned}$$

where M is a numerical constant (2.17) which needs to be computed numerically; M represents the integrated fourth-order two-point moment computed in the fast subsystem (2.4) on the energy level $E = n$.

Appendix B. Interaction coefficients in the full model.

(j, k)	A_{1jk}	A_{j1k}	A_{k1j}
1, 2	1.2	-0.55	-0.65
8, 9	0.525	0.25	-0.775
4, 10	1.35	-0.725	-0.625
5, 6	1.125	-0.5	-0.625
3, 7	1.35	-0.725	-0.625
1, 10	0.525	0.25	-0.775
2, 4	1.2	-0.55	-0.65
5, 8	1.125	-0.5	-0.625
7, 9	0.875	-0.3	-0.575
3, 6	1.25	-0.625	-0.625

TABLE B.1. Coefficients $A_{1jk}^{xyy}, A_{j1k}^{yxy}, A_{jk1}^{yyx}$ used in coupling of x and y variables in the full model (2.1).

(i, j, k)	B_{ijk}	B_{jki}	B_{kij}
1, 2, 3	2	2.5	-4.5
1, 2, 4	4.2426	2.8284	-7.071
1, 2, 9	-1.2247	2.9393	-1.7146
1, 2, 10	2.1166	2.9103	-5.0269
1, 3, 4	1.7321	2.5981	-4.3302
1, 5, 6	3.8013	4.9193	-8.7206
1, 9, 10	3.9598	-2.2627	-1.6971
2, 3, 4	-2	4	-2
2, 5, 6	-4.5	2.1	2.4
2, 9, 10	1.7393	1.4230	-3.1623
3, 7, 8	1.1608	2.3217	-3.4825
4, 7, 8	-1.7321	-2.0785	3.8106
5, 6, 7	2.9566	2.0912	-5.0478
5, 6, 8	-2.6192	-1.4966	4.1158
5, 7, 8	4.6476	2.7111	-7.3587
5, 6, 9	-3	-1.8	4.8
5, 6, 10	1.8554	2.2677	-4.1231
6, 7, 8	4.6669	2.9698	-7.6367
8, 9, 10	3.923	2.3974	-6.3204

TABLE B.2. Coefficients B_{ijk}^{yyy} used in coupling of y variables in the full model (2.1).

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