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A REFINED EXISTENCE CRITERION FOR THE RELATIVISTIC VLASOV-MAXWELL SYSTEM*

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Abstract. We show that a smooth, compactly supported solution to the relativistic Vlasov-Maxwell system exists as long as the L^6 norm of the macroscopic density of particles remains bounded.

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1. Statement of the result

1.1. Glassey–Strauss' criterion. We are interested in the three-dimensional relativistic Vlasov-Maxwell system. This set of equations describes the behaviour of a collisionless plasma of identical particles interacting through a self-consistent electromagnetic field:

$$\begin{aligned}&\partial_t f + v(\xi)\cdot\nabla_x f + K(t,x,\xi)\cdot\nabla_\xi f = 0,\\&\partial_t E - \nabla_x \times B = -j, \quad \nabla_x \cdot E = \rho,\\&\partial_t B + \nabla_x \times E = 0, \quad \nabla_x \cdot B = 0.\end{aligned} \tag{1.1}$$

The unknown $f(t,x,\xi)\geq 0$ is the microscopic density of particles with position $x\in\mathbb{R}^3$ and momentum $\xi\in\mathbb{R}^3$. Here all physical constants, including the speed of light, were taken equal to one. In particular, the velocity of a particle of momentum ξ is simply given by

$$v(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}.$$

The charge and current density in Maxwell's equations are

$$\rho(t,x) = \int f(t,x,\xi)d\xi, \quad j(t,x) = \int f(t,x,\xi)v(\xi)d\xi, \tag{1.2}$$

while the Lorentz force generated by the electric field E and the magnetic field B is

$$K(t,x,\xi) = E(t,x) + v(\xi)\times B(t,x).$$

Although the Cauchy problem for this system has been the object of a number of works, see for instance [2] and references therein, the global well-posedness for general data in large time is still an open problem in the above full three-dimensional setting. In [4], Glassey and Strauss showed that classical $\mathcal{C}^1$ solutions generated by compactly supported initial data could break down only if some particles were accelerated at speeds arbitrarily close to the speed of light. Namely, suppose we are given initial data $f_{in}\in$

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$\mathcal{C}_c^1(\mathbb{R}^6)$ and $E_{in}, B_{in} \in \mathcal{C}^2(\mathbb{R}^3)$ which satisfy the compatibility conditions associated to Maxwell's equations:

$$\nabla_x \cdot E_{in} = \int f_{in}(x,\xi)d\xi, \qquad \nabla_x \cdot B_{in} = 0.$$

Let $f \in \mathcal{C}^1([0,T_{max}) \times \mathbb{R}^6)$ be the corresponding classical solution to the Cauchy problem, and denote the size of the support of f in the kinetic variable by

$$R(t) = \sup\{|\xi| : f(t,x,\xi) \neq 0 \text{ for some } x \in \mathbb{R}^3\}. \tag{1.3}$$

Then we have

$$\limsup_{t\to T_{max}^-} \Big(\|f(t,\cdot,\cdot)\|_{W^1_{x,\xi}(\mathbb{R}^3\times\mathbb{R}^3)} + \|(E,B)(t,\cdot)\|_{W^1_x(\mathbb{R}^3;\mathbb{R}^6)} \Big) = +\infty \Longrightarrow \limsup_{t\to T_{max}^-} R(t) = +\infty.$$

It was subsequently observed that seemingly weaker conditions preclude a blow-up of R in finite time. In [5], it was found that a uniform bound on the kinetic energy density is enough:

$$\limsup_{t\to T_{max}^-} R(t) = +\infty \quad \Longrightarrow \quad \limsup_{t\to T_{max}^-} \Big\| \int \sqrt{1+|\xi|^2} f(t,\cdot,\xi) d\xi \Big\|_{L^\infty(\mathbb{R}^3)} = +\infty.$$

In [6], one shows that for any $\theta > 4/p$ and $6 \leq p \leq +\infty$,

$$\limsup_{t\to T_{max}^-} R(t) = +\infty \quad \Longrightarrow \quad \limsup_{t\to T_{max}^-} \Big\| \int (1+|\xi|^2)^{\theta/2} f(t,\cdot,\xi) d\xi \Big\|_{L^p(\mathbb{R}^3)} = +\infty.$$

In [7], the limit case $p = +\infty$ and $\theta = 0$ in the above statement is obtained, that is,

$$\limsup_{t\to T_{max}^-} R(t) = +\infty \quad \Longrightarrow \quad \limsup_{t\to T_{max}^-} \|\rho(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} = +\infty.$$

1.2. A further improvement. The purpose of this note is to provide the following refinement, which shows that the exponent of the Lebesgue space measuring the singularity in the macroscopic density can be lowered.

THEOREM 1.1. *Suppose we are given a non-negative $f_{in} \in \mathcal{C}_c^1(\mathbb{R}^6)$ and $E_{in}, B_{in} \in \mathcal{C}^2(\mathbb{R}^3)$ satisfying the compatibility conditions associated to Maxwell's equations. Let $f \in \mathcal{C}^1([0,T_{max}) \times \mathbb{R}^6)$ be the corresponding classical solution to the Cauchy problem for (1.1), and let ρ and R be defined as in (1.2) and (1.3). Then we have*

$$\limsup_{t\to T_{max}^-} R(t) = +\infty \quad \Longrightarrow \quad \limsup_{t\to T_{max}^-} \|\rho(t,\cdot)\|_{L^6(\mathbb{R}^3)} = +\infty. \tag{1.4}$$

This improvement is a consequence of an additional cancellation, which appears in the fields expression, see the Lemma 2.1 below. Note however that the above criterion still remains far away from the available a priori estimates. Indeed, if we denote by $\mathcal{E}_a(t)$ the local energy in the ball $B(0,a) = \{x \in \mathbb{R}^3 : |x| \leq a\}$, that is

$$\mathcal{E}_a(t) = \int_{|x|\leq a} \int_{\mathbb{R}^3} \sqrt{1+|\xi|^2} f(t,x,\xi) d\xi dx + \int_{|x|\leq a} |E(t,x)|^2 + |B(t,x)|^2 dx,$$

then, it follows from the energy conservation law, cf. [3] for instance, that for any $t \geq 0$

$$\mathcal{E}_a(t) \leq \mathcal{E}_{a+t}(0). \tag{1.5}$$

In particular, the resulting upper bound on the kinetic energy term yields

$$\|\rho(t,\cdot)\|_{L^{4/3}(B(0,a))} \leq (\frac{4\pi}{3} \|f_{in}\|_{L^\infty(\mathbb{R}^6)})^{1/4} (\mathcal{E}_{a+t}(0))^{3/4},$$

to be compared with the exponent 6 in (1.4).

2. Proof of Theorem 1.1

In the sequel, we assume the Cauchy data (f_{in}, E_{in}, B_{in}) is fixed; we denote by (f, E, B) the corresponding solution and pick $T \in (0, T_{max})$. We use the notation $f \lesssim g$ as a shorthand for the inequality $f \leq Cg$, where $C > 0$ is some constant which depends only upon T and the initial data.

2.1. Time integral of the Lorentz force. Pick (x_0^*, ξ_0^*) in the support of f_{in}, and let us consider the characteristic curve (x^*, ξ^*) of the transport equation:

$$(x^*, \xi^*)'(t) = (v(\xi^*(t)), K(t, x^*(t), \xi^*(t))), \quad (x^*, \xi^*)(0) = (x_0^*, \xi_0^*).$$

In view of the expression of the Lorentz force, we find the equalities

$$\begin{aligned}\sqrt{1+|\xi^*(T)|^2} &= \sqrt{1+|\xi_0^*|^2} + \int_0^T K(t, x^*(t), \xi^*(t)) \cdot v(\xi^*(t))dt \\ &= \sqrt{1+|\xi_0^*|^2} + \int_0^T E(t, x^*(t)) \cdot v(\xi^*(t))dt. \end{aligned} \tag{2.1}$$

The electric field may be written and decomposed, see [1] for instance, as

$$\begin{aligned} E_i &= E_i^0 + \int (\alpha_i^{-1} Y) * (f \mathbf{1}_{t \geq 0}) d\xi - \sum_{j=1}^{3} \int (\partial_{\xi_j} \alpha_i^0 Y) * (K_j f \mathbf{1}_{t \geq 0}) d\xi \\ &= E_i^0 + F_i - G_i, \quad i = 1,2,3. \end{aligned}$$

Here E^0 is the homogeneous part which depends only upon the initial data, $*$ denotes the convolution in the (t,x) variables and $Y(t,x) = (4\pi t)^{-1} \delta_{|x|=t}$ is the forward fundamental solution to the three-dimensional wave equation. The coefficients are given by

$$\alpha^{-1}(s, y, \xi) = \frac{(1 - |v(\xi)|^2)(y - sv(\xi))}{(s - v(\xi) \cdot y)^2}, \qquad \alpha^0(s, y, \xi) = \frac{y - sv(\xi)}{s - v(\xi) \cdot y}.$$

Denoting by (e_1, e_2, e_3) the canonical basis, we obtain for any $i = 1,2,3$,

$$\nabla_\xi \alpha_i^0 = \frac{1}{\sqrt{1+|\xi|^2}(s - v \cdot y)^2} [s(s - v \cdot y)(v_i v - e_i) + (y_i - sv_i)(y - (v \cdot y)v)].$$

In the sequel, we aim at estimating the contribution from F and G:

$$\begin{aligned} I_F &= \int_0^T F(t, x^*(t)) \cdot v(\xi^*(t))dt, \\ I_G &= \int_0^T G(t, x^*(t)) \cdot v(\xi^*(t))dt. \end{aligned}$$

2.2. An additional cancellation. Let us assume for now that $K \in \mathbb{R}^3$ is some given, fixed vector. Then, for any $\xi \in \mathbb{R}^3$ and any $(s, y) \in \operatorname{supp} Y = \{(s, y) \in \mathbb{R}_+ \times \mathbb{R}^3 : |y| = s\}$, we have

$$K \cdot \nabla_\xi \alpha_i^0(s, y, \xi) = \frac{s((K \cdot v)v_i - K_i)}{\sqrt{1+|\xi|^2}(s - v \cdot y)} + \frac{(y_i - sv_i)(K \cdot y - (K \cdot v)(v \cdot y))}{\sqrt{1+|\xi|^2}(s - v \cdot y)^2}, \tag{2.2}$$

for any $i = 1,2,3$. Besides, we check that $(s, y) \in \operatorname{supp} Y$ and $|v| = |v(\xi)| \leq 1$ imply

$$|y - sv|^2 = 2s(s - v \cdot y) + s^2(|v|^2 - 1) \leq 2s(s - v \cdot y),$$

$$|y-(v\cdot y)v|^2=|y|^2-2(v\cdot y)^2+(v\cdot y)^2|v|^2\leq s^2-(v\cdot y)^2.$$

The two terms in the right hand side of (2.2) are thus bounded above by

$$\left|\frac{s((K\cdot v)v_i-K_i)}{\sqrt{1+|\xi|^2}(s-v\cdot y)}\right|\leq\frac{2s|K|}{\sqrt{1+|\xi|^2}(s-v\cdot y)},$$
$$\left|\frac{(y_i-sv_i)K\cdot(y-(v\cdot y)v)}{\sqrt{1+|\xi|^2}(s-v\cdot y)^2}\right|\leq\frac{\sqrt{2s(s-v\cdot y)}|K|\sqrt{s^2-(v\cdot y)^2}}{\sqrt{1+|\xi|^2}(s-v\cdot y)^2}=\frac{\sqrt{2s(s+v\cdot y)}|K|}{\sqrt{1+|\xi|^2}(s-v\cdot y)},$$

and hence

$$|K\cdot\nabla_\xi\alpha^0(s,y,\xi)|\lesssim\frac{s|K|}{\sqrt{1+|\xi|^2}(s-v\cdot y)}. \tag{2.3}$$

Such an upper bound was used in [6, 7] to estimate terms involving G. However, it conceals the following useful fact:

LEMMA 2.1. *For any $K,\xi\in\mathbb{R}^3$ and $(s,y)\in suppY=\{(s,y)\in\mathbb{R}_+\times\mathbb{R}^3:|y|=s\}$, we have*

$$(K\cdot\nabla_\xi\alpha^0(s,y,\xi))\cdot y=0.$$

Proof. Suppose $(s,y)\in\operatorname{supp}Y$. In view of (2.2), we have

$$(K\cdot\nabla_\xi\alpha^0)\cdot y=\frac{s((K\cdot v)(v\cdot y)-K\cdot y)}{\sqrt{1+|\xi|^2}(s-v\cdot y)}+\frac{(|y|^2-sv\cdot y)(K\cdot y-(K\cdot v)(v\cdot y))}{\sqrt{1+|\xi|^2}(s-v\cdot y)^2}.$$

Besides,

$$\begin{aligned}&s((K\cdot v)(v\cdot y)-K\cdot y)(s-v\cdot y)+(|y|^2-sv\cdot y)(K\cdot y-(K\cdot v)(v\cdot y))\\=&(K\cdot v)(v\cdot y)(s(s-v\cdot y)-|y|^2+sv\cdot y)-(K\cdot y)(s(s-v\cdot y)-|y|^2+sv\cdot y)\\=&((K\cdot v)(v\cdot y)-K\cdot y)(s^2-|y|^2),\end{aligned}$$

so that

$$(K\cdot\nabla_\xi\alpha^0(s,y,\xi))=\frac{(K\cdot v)(v\cdot y)-K\cdot y}{\sqrt{1+|\xi|^2}(s-v\cdot y)^2}\left(s^2-|y|^2\right)=0,$$

since (s,y) belongs to $\operatorname{supp}Y$. ∎

This simple observation is now used to sharpen our estimates.

2.3. An upper bound on I_G. Using Fubini's theorem, we obtain:

$$\begin{aligned}I_G=&\sum_{j=1}^3\int_0^T\int(\partial_{\xi_j}\alpha^0Y)*(K_jf)(t,x^*(t),\xi)d\xi\cdot v(\xi^*(t))dt\\=&\int_0^T\int_0^t\int_{|y|=t-s}\int[(fK)(s,x^*(t)-y,\xi)\cdot\nabla_\xi\alpha^0(t-s,y,\xi)]\cdot v(\xi^*(t))d\xi\frac{d\sigma(y)ds}{4\pi|t-s|}dt.\end{aligned}$$

For any $y\in\mathbb{R}^3\setminus\{0\}$, we denote by $p_{y^\perp}(u)=u-(u\cdot\omega)\omega$ the orthogonal projection on $y^\perp$, setting $y=|y|\omega$. As a consequence of Lemma 2.1, we find

$$[(fK)(s,x^*(t)-y,\xi)\cdot\nabla_\xi\alpha^0(t-s,y,\xi))]\cdot v(\xi^*(t))$$

$$=p_{y^\perp}((fK)(s,x^*(t)-y,\xi)\cdot\nabla_\xi\alpha^0(t-s,y,\xi))\cdot p_{y^\perp}(v(\xi^*(t))),$$

for any $0\le s<t\le T$, and $y,\xi\in\mathbb{R}^3$ such that $|y|=t-s$. Considering now this new scalar product, we observe that on the one hand, the inequality (2.3) implies

$$|p_{y^\perp}((fK)(s,x^*(t)-y,\xi)\cdot\nabla_\xi\alpha^0(t-s,y,\xi))|\lesssim\frac{|fK|(s,x^*(t)-y,\xi)}{\sqrt{1+|\xi|^2}(1-v(\xi)\cdot\omega)},$$

for $p_{y^\perp}$ is an orthogonal projection, while on the other hand

$$|p_{y^\perp}(v(\xi^*(t)))|=\sqrt{|v(\xi^*(t))|^2-(v(\xi^*(t))\cdot\omega)^2}\lesssim\sqrt{1-v(\xi^*(t))\cdot\omega},$$

which becomes small when $v(\xi^*(t))$ and ω are close. We set

$$m(t,x)=\int\sqrt{1+|\xi|^2}f(t,x,\xi)d\xi,$$

so that it yields, using Fubini's theorem,

$$|I_G|\lesssim\int_0^T\int_s^T\int_{|y|=t-s}(m|K|)(s,x^*(t)-y)\sqrt{1-v(\xi^*(t))\cdot\omega}\frac{d\sigma(y)dt}{4\pi|t-s|}ds.$$

Next, we split the integral on the right hand side into two parts

$$|I_G|\lesssim\int_0^T\int_s^{s+\varepsilon(s)}\dots dtds+\int_0^T\int_{s+\varepsilon(s)}^T\dots dtds=:I'_G+I''_G.$$

Here, we separated the most singular part I'_G, to be considered later, from the rest I''_G by setting

$$\varepsilon(s)=\frac{T-s}{1+R^8(s)}.$$

Using Hölder's inequality, we may write

$$I''_G\lesssim\int_0^T\Big|\int_{s+\varepsilon(s)}^T\int_{|y|=t-s}(m|K|)^{3/2}(s,x^*(t)-y)(1-v(\xi^*(t))\cdot\omega)d\sigma(y)dt\Big|^{2/3}$$
$$\cdot\Big|\int_{s+\varepsilon(s)}^T\int_{|y|=t-s}\frac{1}{\sqrt{1-v(\xi^*(t))\cdot\omega}}\cdot\frac{d\sigma(y)dt}{(4\pi)^3|t-s|^3}\Big|^{1/3}ds,$$

$$\int_{s+\varepsilon(s)}^T\int_{|y|=t-s}\frac{1}{\sqrt{1-v(\xi^*(t))\cdot\omega}}\cdot\frac{d\sigma(y)dt}{(4\pi)^3|t-s|^3}$$
$$=\int_{s+\varepsilon(s)}^T\int_0^\pi\int_0^{2\pi}\frac{(t-s)^2\sin\vartheta d\varphi d\vartheta}{\sqrt{1-|v(\xi^*(t))|\cos\vartheta}}\cdot\frac{dt}{(4\pi)^3(t-s)^3}.$$

Introducing $u=\cos\vartheta$, this yields

$$\int_{s+\varepsilon(s)}^T\int_{-1}^1\frac{du}{\sqrt{1-|v(\xi^*(t))|u}}\cdot\frac{dt}{32\pi^2(t-s)}\lesssim\ln\left(\frac{T-s}{\varepsilon(s)}\right).$$

Indeed, whenever $|v(\xi^*(t))|\neq 0$, we have

$$\int_{-1}^{1}\frac{du}{\sqrt{1-|v(\xi^*(t))|u}}=\left[-\frac{2\sqrt{1-|v(\xi^*(t))|u}}{|v(\xi^*(t))|}\right]_{u=-1}^{u=+1}$$
$$=\frac{4}{\sqrt{1-|v(\xi^*(t))|}+\sqrt{1+|v(\xi^*(t))|}}\lesssim 1.$$

Note that this upper bound also holds true when $|v(\xi^*(t))|=0$.

Therefore I''_G is bounded above by

$$\int_0^T\Big|\int_{s+\varepsilon(s)}^T\int_{|y|=t-s}(m|K|)^{3/2}(s,x^*(t)-y)(1-v(\xi^*(t))\cdot\omega)d\sigma(y)dt\Big|^{2/3}\ln^{1/3}\left(\frac{T-s}{\varepsilon(s)}\right)ds.$$

Next, we set $\omega=\omega(\theta,\phi)=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$:

$$\int_{s+\varepsilon(s)}^T\int_{|y|=t-s}(m|K|)^{3/2}(s,x^*(t)-y)(1-v(\xi^*(t))\cdot\omega)d\sigma(y)dt$$
$$=\int_{s+\varepsilon(s)}^T\int_0^\pi\int_0^{2\pi}(m|K|)^{3/2}(s,x^*(t)-(t-s)\omega(\theta,\phi))$$
$$(1-v(\xi^*(t))\cdot\omega(\theta,\phi))(t-s)^2\sin\theta d\phi d\theta dt.$$

We follow now [6], and in particular use Lemma 2.1 in this reference:

LEMMA 2.2. *Given $0\le s<s_1<s_2$, define $\Omega_{s_1,s_2}=(s_1,s_2)\times(0,\pi)\times(0,2\pi)$. The map*

$$\begin{aligned}\pi:\ \Omega_{s_1,s_2}&\longrightarrow\pi(\Omega_{s_1,s_2})\\(t,\theta,\phi)&\longmapsto x^*(t)-(t-s)\omega(\theta,\phi)\end{aligned}$$

is a $\mathcal{C}^1-$diffeomorphism with Jacobian $J\pi(t,\theta,\phi)=(v(\xi^*(t))\cdot\omega-1)(t-s)^2\sin\theta$.

Using this change of variable, we obtain

$$I''_G\lesssim\int_0^T\Big|\int_{\pi(\Omega_{s+\varepsilon(s),T})}(m|K|)^{3/2}(s,z)dz\Big|^{2/3}\ln^{1/3}(1+R(s))ds.$$

Using Hölder's inequality again, we find, as a consequence of the conservation of energy (1.5),

$$\Big|\int_{\pi(\Omega_{s+\varepsilon(s),T})}(m|K|)^{3/2}(s,z)dz\Big|^{2/3}\le\|m\|_{L^6(\pi(\Omega_{s+\varepsilon(s),T}))}\|K\|_{L^2(\pi(\Omega_{s+\varepsilon(s),T}))}$$
$$\lesssim R(s)\|\rho(s,\cdot)\|_{L^6(\pi(\Omega_{s+\varepsilon(s),T}))}$$

and hence

$$I''_G\lesssim\int_0^T R(s)\ln^{1/3}(1+R(s))\|\rho(s,\cdot)\|_{L^6(\pi(\Omega_{s+\varepsilon(s),T}))}ds.$$

Besides, we get

$$I'_G=\int_0^T\int_s^{s+\varepsilon(s)}\int_{|y|=t-s}(m|K|)(s,x^*(t)-y)\sqrt{1-v^*(t)\cdot\omega}\frac{d\sigma(y)dt}{4\pi|t-s|}ds$$

$$\lesssim \int_0^T \Big|\int_s^{s+\varepsilon(s)} \int_{|y|=t-s} |K|^2(s,x^*(t)-y)(1-v^*(t)\cdot\omega)d\sigma(y)dt\Big|^{1/2} \|m(s,\cdot)\|_{L^\infty(\mathbb{R}^3)}\varepsilon^{1/2}(s)ds.$$

Using the conservation of energy (1.5) and the integration lemma, we have

$$\int_s^{s+\varepsilon(s)} \int_{|y|=t-s} |K|^2(s,x^*(t)-y)(1-v^*(t)\cdot\omega)d\sigma(y)dt = \int_{\pi(\Omega_{s,s+\varepsilon(s)})} |K|^2(s,z)dz \lesssim 1.$$

Observing that $\|m(s,\cdot)\|_{L^\infty(\mathbb{R}^3)} \lesssim R^4(s)$ and recalling the definition of $\varepsilon(s)$, we conclude that

$$I'_G \lesssim \int_0^T \frac{R^4(s)}{\sqrt{1+R^8(s)}}(T-s)^{1/2}ds \lesssim 1.$$

Eventually we obtain

$$|I_G| \lesssim 1 + \int_0^T R(s)\ln^{1/3}(1+R(s))\|\rho(s,\cdot)\|_{L^6(\mathbb{R}^3)}ds. \tag{2.4}$$

2.4. Final estimates. We also need an upper bound on I_F. We find

$$\begin{aligned} |I_F| &= \Big|\int_0^T \int ((\alpha^{-1}Y)*f)(t,x^*(t),\xi)d\xi\cdot v(\xi^*(t))dt\Big| \\ &\lesssim \int_0^T \int_0^t \int_{|y|=t-s} m(s,x^*(t)-y)\frac{d\sigma(y)ds}{4\pi|t-s|^2}dt. \end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned} |I_F| \lesssim \int_0^T &\Big|\int_s^T \int_{|y|=t-s} m^6(s,x^*(t)-y)(1-v(\xi^*(t))\cdot\omega)d\sigma(y)dt\Big|^{1/6} \\ &\cdot\Big|\int_s^T \int_{|y|=t-s} \frac{1}{(1-v(\xi^*(t))\cdot\omega)^{1/5}}\cdot\frac{d\sigma(y)dt}{|t-s|^{12/5}}\Big|^{5/6}ds. \end{aligned}$$

This last integral is bounded in the same fashion as above:

$$\begin{aligned} &\int_s^T \int_{|y|=t-s} \frac{1}{(1-v(\xi^*(t))\cdot\omega)^{1/5}}\cdot\frac{d\sigma(y)dt}{(t-s)^{12/5}} \\ &= \int_s^T \int_0^\pi \int_0^{2\pi} \frac{(t-s)^2\sin\vartheta d\varphi d\vartheta}{(1-|v(\xi^*(t))|\cos\vartheta)^{1/5}}\cdot\frac{dt}{(t-s)^{12/5}} \\ &= \int_s^T \int_{-1}^1 \frac{du}{(1-|v(\xi^*(t))|u)^{1/5}}\cdot\frac{dt}{(t-s)^{2/5}} \lesssim 1. \end{aligned}$$

Therefore

$$|I_F| \lesssim \int_0^T \Big|\int_s^T \int_{|y|=t-s} m^6(s,x^*(t)-y)(1-v^*(t)\cdot\omega)d\sigma(y)dt\Big|^{1/6}ds.$$

The same change of variable as above yields

$$|I_F| \lesssim \int_0^T \Big|\int_{\pi(\Omega_{s,T})} m^6(s,z)dz\Big|^{1/6}ds$$

$$\lesssim \int_0^T R(s)\|\rho(s,\cdot)\|_{L^6(\pi(\Omega_{s,T}))} ds. \tag{2.5}$$

Bringing (2.4) and (2.5) into (2.1), we infer we can find a constant $C > 0$, which depends only on T_{max} and the initial data, such that

$$\sqrt{1+|\xi^*(T)|^2} \leq \sqrt{1+|\xi_0^*|^2} + C\left(1+\int_0^T R(s)\ln^{1/3}(1+R(s))\|\rho(s,\cdot)\|_{L^6(\mathbb{R}^3)} ds\right).$$

Therefore, for any $T \in [0, T_{max})$, we have

$$R(T) \leq R(0) + C\left(1+\int_0^T R(s)\ln^{1/3}(1+R(s))\|\rho(s,\cdot)\|_{L^6(\mathbb{R}^3)} ds\right).$$

We then conclude using Gronwall's lemma.

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