

LARGE TIME BEHAVIOR IN NONLINEAR SCHRÖDINGER EQUATIONS WITH TIME DEPENDENT POTENTIAL*

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Abstract. We consider the large time behavior of solutions to defocusing nonlinear Schrödinger equations in the presence of a time dependent external potential. The main assumption on the potential is that it grows at most quadratically in space, uniformly with respect to the time variable. We show a general exponential control of first order derivatives and momenta, which yields a double exponential bound for higher Sobolev norms and momenta. On the other hand, we show that if the potential is an isotropic harmonic potential with a time dependent frequency which decays sufficiently fast, then Sobolev norms are bounded, and momenta grow at most polynomially in time, because the potential becomes negligible for large time: there is scattering, even though the potential is unbounded in space for fixed time.

Key words. Nonlinear Schrödinger equation, time dependent potential, norm growth, lens transform, large time properties.

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1. Introduction

1.1. Motivation. For $x \in \mathbb{R}^d$, we consider the nonlinear Schrödinger equation with a defocusing nonlinearity and a time dependent external potential:

$$i\partial_t u + \frac{1}{2}\Delta u = V(t, x)u + |u|^{2\sigma}u; \quad u|_{t=0} = u_0. \quad (1.1)$$

Throughout this paper, we make the following assumption on the potential V :

ASSUMPTION 1.1. $V \in L_{\text{loc}}^\infty(\mathbb{R}_t \times \mathbb{R}_x^d)$ is real-valued and smooth with respect to the space variable: for (almost) all $t \in \mathbb{R}$, $x \mapsto V(t, x)$ is a C^∞ map. Moreover, it is at most quadratic in space, uniformly with respect to time:

$$\forall \alpha \in \mathbb{N}^d, |\alpha| \geq 2, \quad \partial_x^\alpha V \in L^\infty(\mathbb{R}_t \times \mathbb{R}_x^d).$$

In addition, $t \mapsto \sup_{|x| \leq 1} |V(t, x)|$ belongs to $L^\infty(\mathbb{R})$.

Observe that in this assumption — a global in time version of the one originally imposed in [15] — the final condition is required to ensure the boundedness in time of V and its first order derivatives at points within the unit ball to yield, after two integrations, the estimates $|\nabla V(t, x)| \lesssim \langle x \rangle$ and $|V(t, x)| \lesssim \langle x \rangle^2$, uniformly for (almost) all t . This condition could, of course, be equivalently substituted by demanding uniform boundedness in time for V and ∇V at fixed points in \mathbb{R}_x^d . It should also be pointed out that no spectral properties of V are imposed in this assumption.

A typical example that we have in mind is the time dependent harmonic potential:

$$V(t, x) = \frac{1}{2} \langle Q(t)x, x \rangle, \quad (1.2)$$

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where the matrix $Q(t) \in \mathbb{R}^{d \times d}$ is real-valued, bounded and symmetric. In this case, (1.1) appears for instance as an envelope equation in the propagation of coherent states; see [9]. The model (1.1) with (1.2) also appears in Bose–Einstein condensation, typically for $\sigma = 1$ (or $\sigma = 2$ sometimes in the one-dimensional case $d = 1$), with $Q(t)$ a diagonal matrix; see e.g. [10, 17, 24].

Throughout this paper, for $k \in \mathbb{N}$, we will use the notation

$$\Sigma^k = \left\{ f \in L^2(\mathbb{R}^d) ; \|f\|_{\Sigma^k} := \sum_{|\alpha|+|\beta| \leq k} \|x^\alpha \partial_x^\beta f\|_{L^2(\mathbb{R}^d)} < \infty \right\},$$

and $\Sigma^1 = \Sigma$. The main result in [9] relies on the property that the Σ^k norm of a solution u to (1.1)–(1.2) grows at most exponentially in time. This property has been established in some cases (see next subsection), and we present here several extensions.

Since, except possibly for the potential V , the equation is invariant under the transform $u(t, x) \mapsto \bar{u}(-t, x)$, from now on we consider (1.1) for $t \geq 0$ only.

1.2. Known results. It has been proved in [8] that under Assumption 1.1, with $\sigma > 0$, and if the nonlinearity is energy-subcritical ($\sigma < 2/(d-2)$ if $d \geq 3$), then, for all $u_0 \in \Sigma$, (1.1) has a unique, local solution, such that $u, xu, \nabla u \in C((-T, T); L^2) \cap L_{loc}^{(4\sigma+4)/(d\sigma)}(\mathbb{R}; L^{2\sigma+2})$. Moreover, its L^2 -norm is independent of time,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \forall t \in (-T, T).$$

It is also shown in [8] that the only obstruction to global existence is the unboundedness of $\|\nabla u(t)\|_{L^2}$ in finite time, a possibility which is ruled out either if $\sigma < 2/d$ or (when the nonlinearity is defocusing, which is the case in (1.1)) if V is C^1 in t and $\partial_t V$ satisfies Assumption 1.1; one can then let $T = \infty$ in the above statements. We will prove in this paper that, actually, V need not be C^1 in t : Assumption 1.1 seems to be the only relevant hypothesis.

Note that requiring symmetric properties in terms of regularity for xu and ∇u is natural, at least in the case of the linear harmonic potential, since the harmonic oscillator rotates the phase space. More generally, unless ∇V is bounded, (1.1) can be solved in Σ , but not merely in H^s , no matter how large s is; see [6].

In the energy critical case $\sigma = 2/(d-2)$ with $V(x) = \epsilon|x|^2$ a time independent isotropic quadratic potential ($\epsilon = -1$ or $+1$), it was proven in [23] that (1.1) has a unique global solution in Σ , like in the case $V = 0$ proven in [13, 26, 29] (see also [22, 30]).

Concerning the large time behavior and norm growth of the solutions, few results are available, and only for particular cases of harmonic potentials (1.2). If the nonlinearity is L^2 -subcritical and smooth (an assumption which boils down to the one-dimensional cubic case $d = \sigma = 1$), and $Q(t)$ (a real valued scalar function in $d = 1$) is locally Lipschitz and remains bounded, then the Sobolev norms and the momenta of u in L^2 grow at most exponentially in time under Assumption 1.1, ([8]): if $u_0 \in \Sigma^k$, there exists $C > 0$ such that

$$\|u(t)\|_{\Sigma^k} \leq C e^{Ct}, \quad \forall t \geq 0.$$

If the nonlinearity is L^2 -critical or supercritical ($\sigma \geq 2/d$) and the case of a time dependent isotropic repulsive quadratic potential is considered,

$$V(t, x) = \frac{1}{2} \Omega(t) |x|^2, \quad \text{with } \Omega(t) \leq 0,$$

($\Omega(t)$ also locally Lipschitz) then the same exponential control is available ([8]).

We note that if $V(t,x) = -|x|^2$, the nonlinearity in (1.1) is negligible for large time as there is scattering in Σ (see [5] for the energy-subcritical case and [23] for the energy-critical case),

$$\exists u_+ \in \Sigma, \quad \left\| u(t) - e^{i\frac{t}{2}(\Delta + |x|^2)} u_+ \right\|_{\Sigma} \xrightarrow{t \rightarrow +\infty} 0,$$

and the solutions to the linear equation (with potential) grow exponentially in time in the space Σ , since

$$e^{i\frac{t}{2}(\Delta + |x|^2)} u_+ \underset{t \rightarrow +\infty}{\sim} \frac{1}{\sinh t} \mathcal{F} \left(u_+ e^{|\cdot|^2/2} \right) \left(\frac{x}{\sinh t} \right) e^{i \frac{\cosh t}{\sinh t} \frac{|x|^2}{2}},$$

where we normalize the Fourier transform as

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \frac{1}{(2i\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx.$$

Scattering (in the L^2 topology) also holds for more general time dependent isotropic repulsive quadratic potentials in the L^2 -critical or supercritical ($\sigma \geq 2/d$) cases ([8]). Therefore, exponential growth of Sobolev norms for solutions of (1.1) does occur in the presence of these repulsive quadratic potentials, for the underlying reason that the corresponding linear solution has that property.

On the other hand, if $V = 0$ and $\sigma \geq 2/d$ is an integer, there is scattering in Σ to the free linear case, thus leading to bounded Sobolev norms and momenta that grow polynomially ([31]): if $u_0 \in \Sigma^k$, there exists C such that

$$\|u(t)\|_{H^k} \leq C; \quad \| |x|^k u(t) \|_{L^2} \leq C \langle t \rangle^k, \quad \forall t \geq 0.$$

Finally, for confining harmonic potentials, one should start by noticing that in the time independent case

$$V(x) = \frac{1}{2} \sum_{j=1}^d \Omega_j x_j^2, \quad \text{with } \Omega_j > 0, \tag{1.3}$$

the conservation of energy (see (2.1)–(2.2) below) immediately implies boundedness of the Σ norm, $u \in L^\infty(\mathbb{R}; \Sigma)$. Moreover, the existence of periodic solutions of the form $u(t,x) = e^{-i\omega t} \psi(x)$ to the nonlinear problem, with isotropic confining harmonic potential $\Omega_j = \Omega > 0$ (see [7] for details), as well as the linear dynamics (which is time-periodic), naturally lead to the conjecture that we may also have $u \in L^\infty(\mathbb{R}; \Sigma^k)$, at least for localized and smooth enough data. In the case of the one-dimensional harmonic time independent oscillator $V(t,x) = x^2$, standard techniques yield an exponential control. Such bounds have been improved in [18], for small perturbations of x^2 , by adapting methods from finite dimensional dynamical systems to prove that, at least for small initial data, the Σ^k -norm may be bounded, because the solution is quasi-periodic in time ([18]). Such a conclusion is therefore expected to remain valid in a rather general setting for confining potentials. See also [3] for results in this direction.

A different perspective consists in considering the case where the potential decays rapidly in time. Such a case has been considered for potentials which are exactly quadratic in space:

PROPOSITION 1.2 (From Proposition 1.9 and Lemma 4.3 in [19]). *Let $1 \leq d \leq 3$ and $\sigma \in \mathbb{N}$, with $\sigma = 1$ if $d = 3$. Suppose that V is of the form (1.2), with*

$$|Q(t)| + \langle t \rangle \left| \frac{d}{dt} Q(t) \right| \leq \frac{C}{\langle t \rangle^\gamma}, \quad \text{for some } \gamma > 2. \tag{1.4}$$

If $u_0 \in \Sigma^k$ ($k \in \mathbb{N}$, $k \geq 1$), then there exist $\eta, C > 0$ such that

$$\|u(t)\|_{H^1} \leq C, \quad \|xu(t)\|_{L^2} \leq C \langle t \rangle^{1+\eta}, \quad \|u(t)\|_{\Sigma^k} \leq Ce^{Ct}, \quad \forall t \geq 0.$$

As a matter of fact, only the cubic nonlinearity case ($\sigma = 1$), in dimensions $d = 2$ or 3 , is considered in [19], but the proof remains valid under the above assumptions.

1.3. New results.

THEOREM 1.3. *Let $d \geq 1$ and $\sigma > 0$, with $\sigma < 2/(d - 2)$ if $d \geq 3$. If V satisfies Assumption 1.1 and $u_0 \in \Sigma$, then the solution u to (1.1) is global in time:*

$$u, \nabla u, xu \in C(\mathbb{R}; L^2(\mathbb{R}^d)).$$

Moreover, it grows at most exponentially in time: there exists $C > 0$ such that

$$\|u(t)\|_{\Sigma} \leq Ce^{Ct}, \quad \forall t \geq 0.$$

Note that, in general, this bound is (qualitatively) sharp, as shown by the repulsive harmonic potential case $V(t, x) = -|x|^2$ mentioned above. Unlike the previously known results, this norm growth conclusion is not restricted to harmonic potentials only. Using Strichartz estimates, we infer the following corollary, concerning the growth rate of the higher order Σ^k norms:

COROLLARY 1.4 (Double exponential bound). *Let $d \geq 1$, $k \geq 2$, and $\sigma > 0$, with $\sigma < 2/(d - 2)$ if $d \geq 3$. Suppose that the map $z \mapsto |z|^{2\sigma} z$ is C^k . If $u_0 \in \Sigma^k$, then there exists $C > 0$ such that*

$$\sup_{2 \leq |\alpha| + |\beta| \leq k} \|x^\alpha \partial_x^\beta u(t)\|_{L^2} \leq Ce^{e^{Ct}}, \quad \forall t \geq 0.$$

REMARK 1.5. For time independent confining harmonic potentials (1.3) we have boundedness of the Σ norm of the global solutions $u \in L^\infty(\mathbb{R}; \Sigma)$, rather than the general exponential growth of Theorem 1.3. Given this better starting point for the lower order derivatives and momenta, and using exactly the same method of proof by induction as in this corollary, we obtain then an exponential bound for the higher order norms rather than the double exponential

$$\sup_{2 \leq |\alpha| + |\beta| \leq k} \|x^\alpha \partial_x^\beta u(t)\|_{L^2} \leq Ce^{Ct}, \quad \forall t \geq 0.$$

See Remark 3.4 for details.

REMARK 1.6. In the case of the nonlinear Schrödinger equation without potential ($V = 0$), one can infer similarly that the \dot{H}^k ($k \geq 2$) norm of solutions which are globally bounded in $H^1(\mathbb{R}^d)$ grows at most exponentially in time. The use of Bourgain spaces (as initiated in [2, 27]) makes it possible to soften this exponential bound to a polynomial

bound. However, adapting these spaces to the present framework (which, in addition, is not Hamiltonian if $\partial_t V \neq 0$) seems to be a rather challenging issue.

REMARK 1.7. Another strategy might consist in resuming the pseudo-energy used in [25]. Note however that the pseudo-energy introduced in [25] turns out to be helpful in the context of the analysis of blowing-up solutions. Even in the absence of an external potential ($V = 0$), we have not been able to adjust this pseudo-energy to prove the boundedness of the H^2 -norm of u (nor even an exponential control), a property which is known by other arguments.

Using a (global) lens transform, we prove the following result, to be compared with Proposition 1.2.

THEOREM 1.8. *Let $d \leq 3$ and $\sigma \in \mathbb{N}$, with $\sigma \geq 2/d$ and $\sigma = 1$ if $d = 3$. Suppose that V is of the form*

$$V(t, x) = \frac{1}{2} \Omega(t) |x|^2, \quad \text{with} \quad |\Omega(t)| \leq \frac{C}{\langle t \rangle^\gamma} \quad \text{for some } \gamma > 2. \tag{1.5}$$

If $u_0 \in \Sigma^k$, then there exists $C > 0$ such that

$$\|u(t)\|_{H^k} \leq C, \quad \|\langle x \rangle^k u(t)\|_{L^2} \leq C \langle t \rangle^k, \quad \forall t \geq 0. \tag{1.6}$$

Finally, if $u_0 \in \Sigma$, then there exists $u_+ \in \Sigma$ such that

$$\left\| u(t) - e^{i \frac{t}{2} \Delta} u_+ \right\|_{L^2} \xrightarrow{t \rightarrow +\infty} 0.$$

REMARK 1.9. A consequence of the proof of this result is that if the potential satisfies (1.5), the Strichartz estimates associated to the linear evolution are *global in time* (while, as recalled above, this is not the case if $V(t, x) = |x|^2$).

Compared to Proposition 1.2, our assumptions seem to be more stringent on two aspects:

- The matrix Q is of the form $Q(t) = \Omega(t) I_d$, i.e. we consider isotropic potentials only.
- The nonlinearity is L^2 -critical or L^2 -supercritical ($\sigma \geq 2/d$).

However, it turns out that the second point rules out only one case compared to Proposition 1.2, and that is when $d = \sigma = 1$, for which exponential bounds in Σ^k for all k were already known under the mere assumption that Ω is bounded ([8]).

On the other hand, our assumptions demand only a certain minimum decay in time for Ω and impose no restriction on its time derivative (which, in our case, might not even exist). So a rapidly oscillatory potential for large time, as, for instance, is the case with

$$\Omega(t) = \frac{\cos(e^t)}{\langle t \rangle^3},$$

is eligible for Theorem 1.8, while it is not for Proposition 1.2. In fact, it does seem more natural to require a decay exclusively on the function Ω , rather than also adding conditions for its time derivative, as we will see from the proof of Theorem 1.8: in the linear case

$$i \partial_t u + \frac{1}{2} \Delta u = \frac{1}{2} \Omega(t) |x|^2 u,$$

if $|\Omega(t)| \lesssim \langle t \rangle^{-\gamma}$ for $\gamma > 2$, then u satisfies (1.6). As suggested by the last statement of the proposition, the potential is negligible for large time, even though, for fixed t , the harmonic potential cannot be treated as a perturbation. Heuristically, this can be seen through the standard asymptotics (in L^2)

$$e^{i\frac{1}{2}\Delta} f \underset{t \rightarrow +\infty}{\sim} \frac{1}{t^{d/2}} \widehat{f}\left(\frac{x}{t}\right) e^{i|x|^2/(2t)}. \tag{1.7}$$

Asymptotically, the right variable is x/t , and since by assumption

$$|\Omega(t)||x|^2 \lesssim \left|\frac{x}{t}\right|^2 \frac{1}{t^{\gamma-2}},$$

it is sensible to expect the external potential to be negligible for large time. The proof of Theorem 1.8 will make this intuition more precise. Also note that, compared to the conclusion of Proposition 1.2, the control of the momenta (even in the case of $\|xu\|_{L^2}$) and higher Sobolev norms is improved. The sharpness of the decay assumption on Ω is discussed in Remark 4.3.

1.4. Outline of the paper. Theorem 1.3 is proved in Section 2, and we infer Corollary 1.4 in Section 3. The case where V is an isotropic harmonic potential (1.5) is treated in Section 4, where Theorem 1.8 is established.

2. Proof of Theorem 1.3

First, we recall that from [8], under the assumptions of Theorem 1.3, (1.1) has a unique, local solution. The obstruction to global existence is the unboundedness of $\|\nabla u(t)\|_{L^2}$ in finite time. Thus, we simply have to prove a suitable *a priori* estimate.

A natural candidate for an energy in the case of (1.1) is

$$E(t) = \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{\sigma+1} \|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} + \int_{\mathbb{R}^d} V(t,x)|u(t,x)|^2 dx. \tag{2.1}$$

It was established in [8] that if V is C^1 with respect to t , and $\partial_t V$ satisfies Assumption 1.1, then $E \in C^1((-T, T); \mathbb{R})$, and its evolution is given by

$$\frac{dE}{dt} = \int_{\mathbb{R}^d} \partial_t V(t,x)|u(t,x)|^2 dx. \tag{2.2}$$

In the same spirit as in [1], introduce the pseudo-energy

$$\mathcal{E}(t) = \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{\sigma+1} \|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} + \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 dx.$$

From the relation

$$\mathcal{E}(t) = E(t) + \frac{1}{2} \int_{\mathbb{R}^d} (|x|^2 - 2V(t,x)) |u(t,x)|^2 dx,$$

we infer, at least formally,

$$\begin{aligned} \frac{d\mathcal{E}}{dt} &= \frac{1}{2} \int_{\mathbb{R}^d} (|x|^2 - 2V(t,x)) \partial_t |u(t,x)|^2 dx \\ &= \operatorname{Re} \int_{\mathbb{R}^d} (|x|^2 - 2V(t,x)) \bar{u}(t,x) \partial_t u(t,x) dx \end{aligned}$$

$$\begin{aligned} &= \operatorname{Im} \int_{\mathbb{R}^d} (|x|^2 - 2V(t, x)) \bar{u}(t, x) i \partial_t u(t, x) dx \\ &= -\frac{1}{2} \operatorname{Im} \int_{\mathbb{R}^d} (|x|^2 - 2V(t, x)) \bar{u}(t, x) \Delta u(t, x) dx \\ &= \operatorname{Im} \int_{\mathbb{R}^d} \bar{u}(t, x) (x - \nabla V(t, x)) \cdot \nabla u(t, x) dx. \end{aligned}$$

Now, from Assumption 1.1 and the observations that follow it, there exists C independent of t such that

$$|\nabla V(t, x)| \leq C \langle x \rangle.$$

Therefore, using the conservation of mass and the Cauchy–Schwarz inequality, we infer

$$\frac{d\mathcal{E}}{dt} \leq C_0 (1 + \|xu(t)\|_{L^2} \|\nabla u(t)\|_{L^2}).$$

From Young’s inequality, \mathcal{E} satisfies an inequality of the form $\dot{\mathcal{E}} \leq C_0(1 + \mathcal{E})$, with C_0 independent of t (but depending on the conserved mass $\|u_0\|_{L^2}$): Gronwall’s lemma yields an exponential bound.

Under the assumptions of Theorem 1.3, though, V need not be differentiable with respect to time, so the above computations cannot be followed step by step. To overcome this issue, simply note that the evolution of E was merely used as a shortcut in the above presentation, and that by using standard arguments (see e.g. [11]), one directly proves

$$\frac{d\mathcal{E}}{dt} = \operatorname{Im} \int_{\mathbb{R}^d} \bar{u}(t, x) (x - \nabla V(t, x)) \cdot \nabla u(t, x) dx, \quad \forall t \in (-T, T),$$

and Theorem 1.3 follows.

3. Double exponential control: proof of Corollary 1.4

Corollary 1.4 is proved by induction on k , applying the following lemma to the inequalities that result from the use of Strichartz estimates for the evolution equations of $x^\alpha \partial_x^\beta u$. In this lemma, thus, $\|w\|$ must be thought of as a placeholder for all the combinations of norms of $x^\alpha \partial_x^\beta u$ of order $k = |\alpha| + |\beta|$.

LEMMA 3.1. *For $0 \leq s \leq t$, denote $\tau = t - s$ and $I = [s, s + \tau] = [s, t]$. Let w satisfy the following property: there exist Lebesgue exponents $p, q \geq 1$, parameters $\alpha, \tau_0 > 0$, a non-decreasing function f , and a constant C such that, given any $s \geq 0$ and $\tau \in [0, \tau_0]$,*

$$\|w\|_{L^p(I; L^q) \cap L^\infty(I; L^2)} \leq C \|w(s)\|_{L^2} + C \tau^\alpha e^{Ct} \|w\|_{L^p(I; L^q) \cap L^\infty(I; L^2)} + f(t). \tag{3.1}$$

Then there exists C_1 , depending only on C , α , and τ_0 , but independent of $t \geq 0$, such that

$$\|w\|_{L^p([0, t]; L^q) \cap L^\infty([0, t]; L^2)} \leq C_1 e^{C_1 t} (\|w(0)\|_{L^2} + f(t)), \quad \forall t \geq 0.$$

Proof. Let us consider the interval $[0, t]$ and fix this t as an upper bound for the time variable.

Then, for this value of t , we take τ satisfying

$$C \tau^\alpha e^{Ct} = \frac{1}{10} \Leftrightarrow \tau = \left(\frac{1}{C10} \right)^{\frac{1}{\alpha}} e^{-\frac{Ct}{\alpha}}, \tag{3.2}$$

(without loss of generality, the total time t can be chosen initially to be large enough to have $\tau \leq \tau_0$ as well as $\tau \leq t$) such that the corresponding term in (3.1) gets absorbed by the left hand side of the inequality, as

$$\|w\|_{L^p(I;L^q) \cap L^\infty(I;L^2)} \leq \frac{10}{9} \left(C \|w(t')\|_{L^2} + f(t' + \tau) \right), \tag{3.3}$$

for any interval $I = [t', t' + \tau] \subset [0, t]$.

Now, breaking up the full interval $[0, t]$ into $N \sim t/\tau$ small intervals of length τ , $I_j = [t_j, t_{j+1}] = [t_j, t_j + \tau]$, $j = 0, \dots, N-1$, such that $[0, t] = \cup_j I_j$, we have, for $1 \leq p < \infty$,

$$\begin{aligned} \|w\|_{L^p([0,t];L^q)}^p &= \int_0^t \|w\|_{L^q}^p dt' = \sum_{j=0}^{N-1} \int_{t_j}^{t_j+\tau} \|w\|_{L^q}^p dt' = \sum_{j=0}^{N-1} \|w\|_{L^p(I_j;L^q)}^p \\ &\leq \sum_{j=0}^{N-1} \|w\|_{L^p(I_j;L^q) \cap L^\infty(I_j;L^2)}^p \leq \left(\frac{20}{9}\right)^p \sum_{j=0}^{N-1} (C^p \|w(t_j)\|_{L^2}^p + f(t_{j+1})^p), \end{aligned}$$

We now use the fact that the $\|w\|_{L^\infty(I_{j-1};L^2)}$ norm, from the previous time step $j-1$, bounds the $\|w(t_j)\|_{L^2}$ norm at the initial time of the following one, to obtain from (3.3),

$$\|w(t_j)\|_{L^2}^p \leq \|w\|_{L^p(I_{j-1};L^q) \cap L^\infty(I_{j-1};L^2)}^p \leq \left(\frac{20}{9}\right)^p (C^p \|w(t_{j-1})\|_{L^2}^p + f(t_j)^p).$$

Applying this estimate repeatedly and bounding all the terms $f(t_j)$ uniformly by their maximum $f(t)$ over the whole interval $[0, t]$, we infer

$$\begin{aligned} \|w(t_j)\|_{L^2}^p &\leq \left(\frac{20}{9}C\right)^{jp} \|w(0)\|_{L^2}^p + \left(1 + \left(\frac{20}{9}C\right)^p + \dots + \left(\frac{20}{9}C\right)^{(j-1)p}\right) \left(\frac{20}{9}\right)^p f(t)^p \\ &\leq \tilde{C} \left(\frac{20}{9}C\right)^{jp} (\|w(0)\|_{L^2}^p + f(t)^p), \end{aligned}$$

so that

$$\|w\|_{L^p([0,t];L^q)}^p \leq \left(\frac{20}{9}\right)^p \sum_{j=0}^{N-1} C^p \tilde{C} \left(\frac{20}{9}C\right)^{jp} (\|w(0)\|_{L^2}^p + f(t)^p) + \left(\frac{20}{9}\right)^p N f(t)^p,$$

thus yielding

$$\|w\|_{L^p([0,t];L^q)} \leq \tilde{C} \left(\frac{20}{9}C\right)^N (\|w(0)\|_{L^2} + f(t)),$$

for a new \tilde{C} constant. Finally, using the fact that $N \sim t/\tau$ and (3.2), one obtains

$$\|w\|_{L^p([0,t];L^q)} \leq C_1 e^{e^{C_1 t}} (\|w(0)\|_{L^2} + f(t)),$$

with C_1 , a final constant, as in the statement of the lemma.

Of course the case $p = \infty$ is even simpler, as

$$\|w\|_{L^\infty([0,t];L^q)} = \max_{j=0, \dots, N-1} \|w\|_{L^\infty(I_j;L^q)} \leq \frac{10}{9} \left(C \max_{j=0, \dots, N-1} \|w(t_j)\|_{L^2} + f(t) \right),$$

and, as above, each $\|w(t_j)\|_{L^2}$ norm can be controlled by the previous time step $\|w\|_{L^\infty(I_{j-1};L^2)}$ norm. Thus, repeated application again of (3.3) yields

$$\|w\|_{L^\infty([0,t];L^q)} \leq \tilde{C} \left(\frac{10}{9}C\right)^N \left(\|w(0)\|_{L^2} + f(t)\right),$$

from which the double exponential bound for $p = \infty$ now follows as before.

To conclude, we only need to note that the $\|w\|_{L^\infty([0,t];L^2)}$ norm falls into this latter case. □

REMARK 3.2. Observe that, if some rate of growth in time is already known for the $\|w(t)\|_{L^2}$ norm — as in the case of the conservation of the mass or the exponential growth of the Σ norm in Theorem 1.3, for example — then the previous proof can be greatly simplified, bounding all of the $\|w(t_i)\|_{L^2}$ uniformly by $\|w(t)\|_{L^2}$, to yield the following exponential control of the $\|w\|_{L^p([0,t];L^q)}$ norm from its corresponding $\|w(t)\|_{L^2}$ norm growth, as well as $f(t)$,

$$\|w\|_{L^p([0,t];L^q)} \leq C_1 e^{C_1 t} (\|w(t)\|_{L^2} + f(t)).$$

We now return to the main proof of Corollary 1.4 by observing that, although Lemma 3.1 imposes no restrictions on the pair (p, q) of Lebesgue exponents, the Strichartz estimates require only admissible pairs suited to each particular equation which, for the Schrödinger equation (with or without potential), are the following.

DEFINITION 3.3. *A pair (p, q) is admissible if $2 \leq q < \frac{2d}{d-2}$ ($2 \leq q \leq \infty$ if $d = 1$, $2 \leq q < \infty$ if $d = 2$) and*

$$\frac{2}{p} = \delta(q) := d \left(\frac{1}{2} - \frac{1}{q} \right).$$

Strichartz estimates for the Schrödinger equation with potential satisfying Assumption 1.1 are a consequence of the results in [15, 16]. Indeed, the existence of a strongly continuous propagator, unitary on L^2 , for the linear Schrödinger equation with potential satisfying Assumption 1.1 is proved in [15]. In [16] it is proved that, for bounded time intervals, this propagator exhibits an $L^1 - L^\infty$ decay in time. As is now well known, these two properties are the crucial ingredients that lead to Strichartz estimates for the linear propagator (see e.g. [21]). A precise statement of these estimates, in our context, can be found in [8, Section 2].

Two points need careful attention at this point, though. The first one is that these Strichartz estimates are just local in time. Unlike the case of the Schrödinger equation without potential, whose estimates are global, only Strichartz estimates for finite time intervals can be expected when general potentials are considered. The typical counter-example is the linear Schrödinger equation with a confining harmonic potential (1.3), which exhibits time periodic solutions and thus cannot possibly satisfy global dispersive estimates in time. The second point requiring a careful observation has to do with the fact that the potentials being considered here also depend on time. The equation is therefore not autonomous and the propagator depends now on the initial time of the flow. In particular, the maximum size of the finite time interval $[s, s + \tau]$ for the Strichartz estimates to hold (i.e. the parameter τ_0 of the previous lemma) should thus depend generally on the overall time interval $[0, t]$ being considered. It can be shown, however, that for potentials whose spatial derivatives are uniformly bounded in time — which is the case we are imposing in Assumption 1.1, unlike the more frequent condition

of just local boundedness in time found in the literature, as in [15], [16], or [8] — one can indeed pick a uniform value of τ_0 that holds globally on $[0, \infty)$. See again [8, Section 2] for a careful discussion of these two issues.

We now have all the properties needed to go ahead with the induction procedure. For the sake of clarity, we will start with the cases $k = 0$ and $k = 1$, for which an exponential growth is already known from Theorem 1.3. However, the point is that we can easily write all the complete formulas and estimates for these simpler cases, which illustrate the essential features that remain for general higher values of k , whose computations and formulas then become much more cumbersome.

The cases $k = 0$ and $k = 1$ correspond to the same estimates used to prove local well posedness in Σ . The solution u , whose global existence has been established in Theorem 1.3, is thus a fixed point of the Duhamel formulation for any initial time $t_0 = s$. It satisfies, therefore,

$$u(t) = U(t, s)u(s) - i \int_s^t U(t, t') (|u|^{2\sigma} u)(t') dt', \tag{3.4}$$

where $U(t, s)$ represents the linear propagator of (1.1) from time s to time t . We then pick the usual Lebesgue exponents

$$q = 2\sigma + 2; \quad p = \frac{4\sigma + 4}{d\sigma}; \quad \theta = \frac{2\sigma(2\sigma + 2)}{2 - (d - 2)\sigma}.$$

This choice of the pair (p, q) is admissible and we also have

$$\frac{1}{q'} = \frac{2\sigma}{q} + \frac{1}{q}; \quad \frac{1}{p'} = \frac{2\sigma}{\theta} + \frac{1}{p},$$

so that applying Strichartz estimates on (3.4), as long as the finite time interval is smaller than the uniform bound $|t - s| \leq \tau_0$, we obtain

$$\|u\|_{L^p([s, t]; L^q) \cap L^\infty([s, t]; L^2)} \leq C \|u(s)\|_{L^2} + C \| |u|^{2\sigma} u \|_{L^{p'}([s, t]; L^{q'})},$$

and then, using Hölder to handle the nonlinear term,

$$\begin{aligned} \|u\|_{L^p([s, t]; L^q) \cap L^\infty([s, t]; L^2)} &\leq C \|u(s)\|_{L^2} + C \|u\|_{L^\theta([s, t]; L^q)}^{2\sigma} \|u\|_{L^p([s, t]; L^q)} \\ &\leq C \|u(s)\|_{L^2} + C |t - s|^{\frac{2\sigma}{\theta}} \|u\|_{L^\infty([s, t]; L^q)}^{2\sigma} \|u\|_{L^p([s, t]; L^q)}. \end{aligned}$$

Finally, the H^1 subcritical condition $\sigma < 2/(d - 2)$ permits the use of the Sobolev embedding, from which we get

$$\|u\|_{L^p([s, t]; L^q) \cap L^\infty([s, t]; L^2)} \leq C \|u(s)\|_{L^2} + C |t - s|^{\frac{2\sigma}{\theta}} \|u\|_{L^\infty([s, t]; H^1)}^{2\sigma} \|u\|_{L^p([s, t]; L^q)}.$$

Now, for $k = 1$, one needs to develop similar estimates for ∇u and xu . Therefore, we start by differentiating (1.1) to obtain the evolution equation for ∇u and its corresponding Duhamel formula

$$\nabla u(t) = U(t, s)\nabla u(s) - i \int_s^t U(t, t') \nabla (|u|^{2\sigma} u)(t') dt' - i \int_s^t U(t, t') (\nabla V(t') u(t')) dt'.$$

The only novelty now is the second integral, with the term $\nabla V(t')$, because all the remaining terms are estimated exactly as in the previous $k = 0$ case. Assumption 1.1

implies that $|\nabla V| \lesssim \langle x \rangle$ uniformly for all time, and noting that (1,2) are conjugate exponents to the admissible pair $(\infty, 2)$, we then get

$$\begin{aligned} & \|\nabla u\|_{L^p([s,t];L^q) \cap L^\infty([s,t];L^2)} \\ & \leq C \|\nabla u(s)\|_{L^2} + C \|u\|_{L^\theta([s,t];L^q)}^{2\sigma} \|\nabla u\|_{L^p([s,t];L^q)} + C \|u \nabla V\|_{L^1([s,t];L^2)} \\ & \leq C \|\nabla u(s)\|_{L^2} + C |t-s|^{\frac{2\sigma}{\theta}} \|u\|_{L^\infty([s,t];H^1)}^{2\sigma} \|\nabla u\|_{L^p([s,t];L^q)} \\ & \quad + C |t-s| \|u\|_{L^\infty([s,t];L^2)} + C |t-s| \|xu\|_{L^\infty([s,t];L^2)}. \end{aligned}$$

Analogously, for the momentum xu ,

$$xu(t) = U(t,s)(xu)(s) - i \int_s^t U(t,t') (|u|^{2\sigma} xu)(t') dt' - i \int_s^t U(t,t') (\nabla u(t')) dt',$$

where the first order derivative ∇u in the second integral now appeared from writing

$$x \frac{1}{2} \Delta u = \frac{1}{2} \Delta(xu) - \nabla u,$$

when multiplying the whole Equation (1.1) by x , to obtain the evolution equation for the momentum. Following the same procedure as above,

$$\begin{aligned} \|xu\|_{L^p([s,t];L^q) \cap L^\infty([s,t];L^2)} & \leq C \|xu(s)\|_{L^2} + C |t-s|^{\frac{2\sigma}{\theta}} \|u\|_{L^\infty([s,t];H^1)}^{2\sigma} \|xu\|_{L^p([s,t];L^q)} \\ & \quad + C |t-s| \|\nabla u\|_{L^\infty([s,t];L^2)}. \end{aligned}$$

So that, summing up the estimates for the first derivative and momentum,

$$\begin{aligned} & \|(xu, \nabla u)\|_{L^p([s,t];L^q) \cap L^\infty([s,t];L^2)} \\ & \leq C \|(xu, \nabla u)(s)\|_{L^2} + C |t-s|^{\frac{2\sigma}{\theta}} \|u\|_{L^\infty([s,t];H^1)}^{2\sigma} \|(xu, \nabla u)\|_{L^p([s,t];L^q)} \\ & \quad + C |t-s| \|(xu, \nabla u)\|_{L^\infty([s,t];L^2)} + C |t-s| \|u\|_{L^\infty([s,t];L^2)}. \end{aligned}$$

For $k=2$, the first order of momenta and spatial derivatives for which we are really getting new information about its norm growth, let us denote by $\|w_2\|$ the sum of all corresponding norms of terms of order 2

$$\|w_2\| = \sum_{|\alpha|+|\beta|=2} \|x^\alpha \partial^\beta u\|.$$

Then, after estimating the corresponding Duhamel formulations as we have done above, we get

$$\begin{aligned} & \|w_2\|_{L^p([s,t];L^q) \cap L^\infty([s,t];L^2)} \\ & \leq C \|w_2(s)\|_{L^2} + C |t-s|^{\frac{2\sigma}{\theta}} \|u\|_{L^\infty([s,t];H^1)}^{2\sigma} \|w_2\|_{L^p([s,t];L^q)} \\ & \quad + C |t-s| \|w_2\|_{L^\infty([s,t];L^2)} + C |t-s|^{\frac{2\sigma-1}{\theta}} \|u\|_{L^\infty([s,t];H^1)}^{2\sigma-1} \|\nabla u\|_{L^p([s,t];L^q)}^2 \\ & \quad + C |t-s| \|(xu, \nabla u)\|_{L^\infty([s,t];L^2)} + C |t-s| \|u\|_{L^\infty([s,t];L^2)}, \end{aligned}$$

where the new term $|t-s|^{\frac{2\sigma-1}{\theta}} \|u\|_{L^\infty([s,t];H^1)}^{2\sigma-1} \|\nabla u\|_{L^p([s,t];L^q)}^2$ occurs from differentiating the nonlinear powers $|u|^{2\sigma} u$ twice, in (1.1), for the evolution equations of the second order derivatives of u .

Generally, then, for $k \geq 2$, if we write

$$\|w_k\| = \sum_{|\alpha|+|\beta|=k} \|x^\alpha \partial^\beta u\|,$$

then, after writing evolution equations for each of these k order momenta and spatial derivatives, obtained by differentiating and multiplying (1.1) by enough powers of x , and estimating the corresponding Duhamel formulations, we finally obtain

$$\|w_k\|_{L^p([s,t];L^q) \cap L^\infty([s,t];L^2)} \leq C \|w_k(s)\|_{L^2} + C |t-s|^{\frac{2\sigma}{\theta}} \|u\|_{L^\infty([s,t];H^1)}^{2\sigma} \|w_k\|_{L^p([s,t];L^q)} \tag{3.5}$$

$$+ C |t-s| \|w_k\|_{L^\infty([s,t];L^2)} \tag{3.6}$$

$$+ C \sum_{\substack{0 < j < k \\ 1 \leq l_1, \dots, l_{j+1} \leq k-1 \\ l_1 + \dots + l_{j+1} = k}} |t-s|^{\frac{2\sigma-j}{\theta}} \|u\|_{L^\infty([s,t];H^1)}^{2\sigma-j} \|w_{l_1}\|_{L^p([s,t];L^q)} \cdots \|w_{l_{j+1}}\|_{L^p([s,t];L^q)} \tag{3.7}$$

$$+ C \sum_{0 \leq j \leq k-1} |t-s| \|w_j\|_{L^\infty([s,t];L^2)}. \tag{3.8}$$

This completely generalizes the $k=2$ case and no qualitatively new terms appear anymore.

To conclude the proof, we only need to argue that for (3.5) and (3.6) we can do

$$C |t-s|^{\frac{2\sigma}{\theta}} \|u\|_{L^\infty([s,t];H^1)}^{2\sigma} \|w_k\|_{L^p([s,t];L^q)} + C |t-s| \|w_k\|_{L^\infty([s,t];L^2)} \leq C \tau^\alpha e^{Ct} \|w_k\|_{L^p([s,t];L^q) \cap L^\infty([s,t];L^2)},$$

by making $|t-s| = \tau \leq \tau_0$, $\alpha = \frac{2\sigma}{\theta}$, and using the exponential growth of the Σ norm, from Theorem 1.3, to bound the $\|u\|_{L^\infty([s,t];H^1)}$ norm (the H^1 subcritical condition $\sigma < \frac{2}{d-2}$ guarantees that $0 < \alpha < 1$). Whereas, for (3.7) and (3.8), these involve exclusively the norms of the previous induction steps $\leq k-1$, whose growth is known by the induction hypothesis, and which can thus be bounded by a non-decreasing double exponential function $f(t) = Ce^{e^{Ct}}$.

Therefore, we have finally established that this general Strichartz type inequality for the norm of the derivatives and momenta of order k , $\|w_k\|_{L^p([s,t];L^q) \cap L^\infty([s,t];L^2)}$, suits exactly the hypotheses of Lemma 3.1 from which we infer its double exponential growth, ending the proof.

REMARK 3.4. This result is a corollary of the exponential growth rate of the Σ norm, coming from Theorem 1.3, because that rate controls the norm $\|u\|_{L^\infty([s,t];H^1)}$ in (3.5) and, consequently, the estimate (3.1) in the hypotheses of Lemma 3.1. If the norm $\|u\|_{L^\infty([s,t];H^1)}$, however, is known to grow at a different rate, then a different final result is obtained for the growth rate of the norms of the higher order derivatives and momenta. In particular, for confining time independent harmonic potentials of the type (1.3), the conservation (2.2) of the energy (2.1) implies that $u \in L^\infty(\mathbb{R}, \Sigma)$ and, thus, that this crucial term is uniformly bounded in time

$$\|u\|_{L^\infty([s,t];H^1)} \leq C, \quad \forall 0 \leq s \leq t.$$

Then, if one were to follow the exact same steps of the previous corollary's proof, the only difference would occur in that, instead of an exponential term in (3.1) in the hypotheses of Lemma 3.1, we would now have

$$\|w\|_{L^p(I;L^q) \cap L^\infty(I;L^2)} \leq C \|w(s)\|_{L^2} + C \tau^\alpha \|w\|_{L^p(I;L^q) \cap L^\infty(I;L^2)} + f(t),$$

which yields a single exponential growth

$$\|w\|_{L^p([0,t];L^q) \cap L^\infty([0,t];L^2)} \leq C_1 e^{C_1 t} (\|w(0)\|_{L^2} + f(t)), \quad \forall t \geq 0.$$

Indeed, in the proof of Lemma 3.1, the size of the small time intervals $\tau \leq \tau_0$ would now be just a uniform constant and not depend on t . Thus, the number of these intervals $N \sim t/\tau$ would merely be proportional to t , and not exponential. As everything else follows in exactly the same way as the general case, proved above, this then implies the final result, of the single exponential growth of norms of the higher derivatives and momenta, in the time independent confining potential case.

4. Asymptotically vanishing potential: proof of Theorem 1.8

The proof is based on a lens transform as in [8]. The main idea is that the decay of Ω as $t \rightarrow \infty$ allows us to avoid the compactification of time, which is one of the features of the lens transform in the case $\Omega = 1$ (see e.g. [4, 28]).

4.1. Lens transform. Since some adaptations will be needed, we resume the approach presented in [8, Section 4]. Suppose that v solves a non-autonomous equation

$$i\partial_t v + \frac{1}{2}\Delta v = H(t)|v|^{2\sigma}v. \tag{4.1}$$

We want v and u (solution of (1.1)) to be related by the formula

$$u(t,x) = \frac{1}{b(t)^{d/2}} v\left(\zeta(t), \frac{x}{b(t)}\right) e^{\frac{i}{2}a(t)|x|^2}, \tag{4.2}$$

with a, b, ζ real-valued, and, for some time $t_0 \geq 0$,

$$b(t_0) > 0; \quad \zeta(t_0) > 0. \tag{4.3}$$

Apply the Schrödinger differential operator to the formula (4.2), and identify the terms so that u solves (1.1). We find:

$$\dot{b} = ab; \quad \dot{a} + a^2 + \Omega = 0; \quad \dot{\zeta} = \frac{1}{b^2}; \quad b(t)^{d\sigma-2} H(\zeta(t)) = 1. \tag{4.4}$$

Introduce a solution to

$$\dot{\mu} + \Omega(t)\mu = 0; \quad \dot{\nu} + \Omega(t)\nu = 0, \tag{4.5}$$

such that the (constant) Wronskian is $W := \nu\dot{\mu} - \dot{\nu}\mu \equiv 1$. The solutions to (4.4) can be expressed as

$$a = \frac{\dot{\nu}}{\nu}; \quad b = \nu; \quad \zeta = \frac{\mu}{\nu}.$$

Note that ζ is locally invertible, since $\zeta(t_0) > 0$ and

$$\dot{\zeta} = \frac{1}{b^2} = \frac{1}{\nu^2}, \text{ hence } \dot{\zeta}(t_0) > 0.$$

Therefore, the lens transform is locally invertible. Moreover, we can write, as long as $\nu > 0$,

$$H(t) = b(\zeta^{-1}(t))^{2-d\sigma} = \nu \left(\left(\frac{\mu}{\nu} \right)^{-1}(t) \right)^{2-d\sigma}.$$

4.2. A particular fundamental solution. The idea is then to construct a suitable solution (μ, ν) , so that ζ is invertible in a neighborhood of $t = +\infty$. This scattering problem is solved by adapting [14, Lemma A.1.2]:

LEMMA 4.1. *Suppose that there exists $\gamma > 2$ such that $|\Omega(t)| \lesssim \langle t \rangle^{-\gamma}$. Then there exist μ_∞, ν_∞ solving*

$$\ddot{\mu}_\infty + \Omega(t)\mu_\infty = 0; \quad \ddot{\nu}_\infty + \Omega(t)\nu_\infty = 0,$$

with $\nu_\infty(T) = 1, \mu_\infty(T) = T$ for some $T > 0$, and such that, as $t \rightarrow \infty$,

$$|\nu_\infty(t) - 1| = \mathcal{O}\left(\frac{1}{t^{\gamma-2}}\right), \quad |\dot{\nu}_\infty(t)| = \mathcal{O}\left(\frac{1}{t^{\gamma-1}}\right), \tag{4.6}$$

$$|\mu_\infty(t) - t| = \mathcal{O}(t^{3-\gamma}), \quad |\dot{\mu}_\infty(t) - 1| = \mathcal{O}\left(\frac{1}{t^{\gamma-2}}\right). \tag{4.7}$$

The Wronskian of ν_∞ and μ_∞ is $W := \nu_\infty \dot{\mu}_\infty - \dot{\nu}_\infty \mu_\infty \equiv 1$.

Proof. For $T > 0$, consider the problem

$$\ddot{z} + \Omega(t)z + g(t) = 0; \quad z(T) = 0; \quad \lim_{t \rightarrow +\infty} \dot{z}(t) = 0.$$

The integral formulation of this problem reads

$$z = r_T + \mathcal{P}_T z, \tag{4.8}$$

with

$$\begin{aligned} r_T(t) &= \int_T^t (s-T)g(s)ds + (t-T) \int_t^\infty g(s)ds, \\ \mathcal{P}_T z(t) &= \int_T^t (s-T)\Omega(s)z(s)ds + (t-T) \int_t^\infty \Omega(s)z(s)ds. \end{aligned}$$

Consider the two Banach spaces

$$\begin{aligned} Z_T^0 &= \{z \in C([T, \infty); \mathbb{R}) \mid \|z\|_0 := \sup_{t \geq T} |z(t)| < \infty\}, \\ Z_T^1 &= \left\{z \in C([T, \infty); \mathbb{R}) \mid \|z\|_1 := \sup_{t \geq T} \frac{|z(t)|}{t-T} < \infty\right\}. \end{aligned}$$

We readily check that \mathcal{P}_T is bounded on Z_T^0 and Z_T^1 , respectively, and that its norm on either of these spaces equals

$$\int_T^\infty (t-T)\Omega(t)dt.$$

By assumption, it is estimated by

$$\int_T^\infty (t-T)\Omega(t)dt \lesssim \int_T^\infty \frac{dt}{\langle t \rangle^{\gamma-1}} + T \int_T^\infty \frac{dt}{\langle t \rangle^\gamma} \lesssim T^{2-\gamma} \xrightarrow{T \rightarrow +\infty} 0.$$

Fixing T sufficiently large, this norm is smaller than one, and (4.8) has a unique solution, given by

$$z = (1 - \mathcal{P}_T)^{-1} r_T \in Z_T^j,$$

with $j = 0$ or 1 , according to the case considered, provided that $r_T \in Z_T^j$.

In view of the statement of the proposition, we start by constructing ν_∞ : $z = \nu_\infty - 1$ must solve

$$\ddot{z} + \Omega(t)z + \Omega(t) = 0.$$

Therefore, we use the above general result with $g = \Omega$: the function r_T belongs to Z_T^0 , hence a function $\nu_\infty \in Z_T^0$ (since $1 \in Z_T^0$). Since $r_T(t) = \mathcal{O}(t^{2-\gamma})$ as $t \rightarrow \infty$, we readily get

$$|z(t)| = |\nu_\infty(t) - 1| = \sum_{j=0}^\infty \mathcal{P}_T^j r_T(t) = \mathcal{O}\left(\frac{1}{t^{\gamma-2}}\right).$$

Differentiating (4.8), we infer

$$\dot{z}(t) = \int_t^\infty g(s)ds + \int_t^\infty \Omega(s)z(s)ds = \int_t^\infty \Omega(s)ds + \int_t^\infty \Omega(s)z(s)ds.$$

Since $z \in Z_T^0$, we obtain, for $t \geq T$,

$$|\dot{z}(t)| = |\dot{\nu}_\infty(t)| = \mathcal{O}\left(\frac{1}{t^{\gamma-1}}\right).$$

In the case of μ_∞ , we work in Z_T^1 instead: $z(t) = \mu_\infty(t) - t$ must satisfy

$$\ddot{z} + \Omega(t)z + t\Omega(t) = 0,$$

that is, $g(t) = t\Omega(t)$. Since we now have the pointwise estimate

$$r_T(t) = \mathcal{O}(T^{3-\gamma}) + \mathcal{O}(t^{3-\gamma}),$$

we have $r_T \in Z_T^1$, and for T sufficiently large,

$$z = (1 - \mathcal{P}_T)^{-1} r_T \in Z_T^1.$$

Proceeding as for ν_∞ , we infer

$$|\mu_\infty(t) - t| = \mathcal{O}(t^{3-\gamma}), \quad |\dot{\mu}_\infty(t) - 1| = \mathcal{O}\left(\frac{1}{t^{\gamma-2}}\right).$$

Finally, the Wronskian W does not depend on time, and goes to one as t goes to infinity, so $W \equiv 1$. □

4.3. End of the proof. In the pseudo-conformal invariant case $\sigma = 2/d$, we see that (4.1) is the standard autonomous equation, for which there is scattering. In addition, the Sobolev norms are bounded in time, and the momenta grow polynomially in time as in the statement of Theorem 1.8, as established in [31] (see also the appendix in [8]). In view of the asymptotic properties of μ_∞ and ν_∞ , u satisfies the same properties. We will be more precise concerning these statement by considering more generally the case $\sigma \geq 2/d$, where (4.1) must be thought of as a non-autonomous equation. We readily check the analogue of the conservation of energy and the pseudo-conformal conservation law in the present case. Let $J(t) = x + it\nabla$. The solution to (4.1) satisfies

$$\frac{d}{dt} \left(\frac{1}{2} \|\nabla v\|_{L^2}^2 + \frac{H(t)}{\sigma+1} \|v(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \right) = \frac{\dot{H}(t)}{\sigma+1} \|v(t)\|_{L^{2\sigma+2}}^{2\sigma+2}, \tag{4.9}$$

$$\frac{d}{dt} \left(\frac{1}{2} \|J(t)v\|_{L^2}^2 + \frac{t^2 H(t)}{\sigma+1} \|v(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \right) = \frac{tH(t)}{\sigma+1} (2-d\sigma) \|v(t)\|_{L^{2\sigma+2}}^{2\sigma+2} + \frac{t^2 \dot{H}(t)}{\sigma+1} \|v(t)\|_{L^{2\sigma+2}}^{2\sigma+2}, \tag{4.10}$$

where we recall that

$$H(t) = \nu_\infty \left(\left(\frac{\mu_\infty}{\nu_\infty} \right)^{-1} (t) \right)^{2-d\sigma}$$

is well-defined for $t \geq t_0$ sufficiently large:

$$|\nu_\infty(t) - 1| \leq \frac{1}{2} \quad \text{for } t \geq t_0.$$

In addition,

$$\dot{H}(t) = (2-d\sigma)\nu_\infty \left(\left(\frac{\mu_\infty}{\nu_\infty} \right)^{-1} (t) \right)^{3-d\sigma} \dot{\nu}_\infty \left(\left(\frac{\mu_\infty}{\nu_\infty} \right)^{-1} (t) \right),$$

hence $\dot{H}(t) = \mathcal{O}\left(\frac{1}{t^{\gamma-1}}\right)$ as $t \rightarrow \infty$. We infer from [12, Lemma 3.1] (see also [11, Theorem 4.11.1]) that (4.1) has a unique, global solution $v \in C(\mathbb{R}_+; \Sigma)$. Therefore, the lens transform (4.2) is well-defined and bijective in a neighborhood of $t = +\infty$.

Set, for t sufficiently large,

$$y(t) = \frac{t^2 H(t)}{\sigma+1} \|v(t)\|_{L^{2\sigma+2}}^{2\sigma+2}.$$

The relation (4.10) yields

$$y(t) \leq C(\|v(t_0)\|_\Sigma) + \int_{t_0}^t \frac{|\dot{H}(s)|}{H(s)} y(s) ds \lesssim 1 + \int_{t_0}^t y(s) \frac{ds}{s^{\gamma-1}},$$

and we infer from Gronwall’s lemma that $y \in L^\infty([t_0, \infty))$. Using (4.10) again, we deduce that $J(t)v \in L^\infty([t_0, \infty); L^2)$. Since

$$J(t) = ite^{i\frac{|x|^2}{2t}} \nabla \left(e^{-i\frac{|x|^2}{2t}} \cdot \right),$$

the Gagliardo–Nirenberg inequality yields, for $2 \leq r \leq 2/(d-2)$ ($r \leq \infty$ if $d=1$, $r < \infty$ if $d=2$),

$$\|v(t)\|_{L^r} \lesssim \frac{1}{|t|^\delta} \|v\|_{L^2}^{1-\delta} \|J(t)v\|_{L^2}^\delta, \quad \text{where } \delta = d \left(\frac{1}{2} - \frac{1}{r} \right), \tag{4.11}$$

giving a decay rate (in time) for Lebesgue norms (in space). Mimicking the proof of [8, Proposition A.4], we conclude:

PROPOSITION 4.2. *Let $\sigma \geq 2/d$ be an integer, with $\sigma \leq 2/(d-2)$ if $d \geq 3$. Suppose $v|_{t=0} \in \Sigma^k$ for some $k \in \mathbb{N}$, $k \geq 1$. Then $v \in C(\mathbb{R}; \Sigma^k)$. In addition, for all admissible pairs (p, q) , $v \in L^p(\mathbb{R}; W^{k,q}(\mathbb{R}^d))$, and*

$$\forall \alpha \in \mathbb{N}^d, |\alpha| \leq k, \quad \|x^\alpha v\|_{L^p([0,t]; L^q)} \lesssim \langle t \rangle^{|\alpha|}.$$

Since

$$u(t, x) = \frac{1}{\nu_\infty(t)^{d/2}} v\left(\frac{\mu_\infty(t)}{\nu_\infty(t)}, \frac{x}{\nu_\infty(t)}\right) e^{\frac{i}{2}|x|^2 \dot{\nu}_\infty(t)/\nu_\infty(t)}, \tag{4.12}$$

we have

$$\begin{aligned} \|u(t)\|_{H^k} &\lesssim \frac{1}{|\nu_\infty(t)|} \left\| v\left(\frac{\mu_\infty(t)}{\nu_\infty(t)}\right) \right\|_{H^k} + |\dot{\nu}_\infty(t)|^k \left\| |x|^k v\left(\frac{\mu_\infty(t)}{\nu_\infty(t)}\right) \right\|_{L^2}, \\ \||x|^k u(t)\|_{L^2} &\lesssim |\nu_\infty(t)|^k \left\| |x|^k v\left(\frac{\mu_\infty(t)}{\nu_\infty(t)}\right) \right\|_{L^2}. \end{aligned}$$

Gathering (4.6), (4.7), and Proposition 4.2 together, we obtain Theorem 1.8, up to the scattering result.

In view of (4.11), scattering for (4.1) follows from the standard approach: from Duhamel’s formula,

$$e^{-i\frac{t}{2}\Delta} v(t) - e^{-i\frac{\tau}{2}\Delta} v(\tau) = -i \int_\tau^t e^{-i\frac{s}{2}\Delta} (H(s)|v(s)|^{2\sigma} v(s)) ds,$$

we infer that $\left(e^{-i\frac{t}{2}\Delta} v(t) \right)_{t \geq 0}$ is a Cauchy sequence in Σ as $t \rightarrow \infty$, hence converges to some $v_+ \in \Sigma$. Taking into account (4.6), (4.7), (4.12), and the asymptotics for $e^{i\frac{t}{2}\Delta}$ recalled in (1.7), the last point of Theorem 1.8 follows.

REMARK 4.3 (Sharpness of the decay assumption on Ω). The key property that we have used to define a lens transform in the neighborhood of $t = +\infty$ is that the function $\zeta = \mu/\nu$ is bijective from $[T, \infty)$ to $[\zeta(T), \infty)$ for T sufficiently large, where (μ, ν) solves (4.5) with a Wronskian equal to one. Note that if we had, instead of the above property, $\zeta(T) < 0$ and ζ bijective from $[T, \infty)$ to $(-\infty, \zeta(T)]$, it would be straightforward to adapt the above analysis (simply replace ν with $-\nu$). The point is that, unlike in the case $\Omega = 1$, the lens transform does not compactify time, because ν only has a finite number of zeroes and $\dot{\zeta} = 1/\nu^2$ ($\nu(t) = \cos t$ in the case $\Omega = 1$). This property would remain under the mere assumption (see for instance [20, Chapter XI, Theorem 7.1])

$$-\infty \leq \limsup_{t \rightarrow +\infty} t^2 \Omega(t) < \frac{1}{4}.$$

In the opposite case, ν has infinitely many oscillations, and even if it may sound sensible to expect most of Theorem 1.8 to remain valid, the last conclusion (scattering) should fail: the dynamics has a different nature.

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REFERENCES

- [1] P. Antonelli and C. Sparber, *Global well-posedness for cubic NLS with nonlinear damping*, Commun. Part. Diff. Eqs., 35, 2310–2328, 2010.
- [2] J. Bourgain, *On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE*, Internat. Math. Res. Notices, 277–304, 1996.
- [3] N. Burq, L. Thomann, and N. Tzvetkov, *Long time dynamics for the one dimensional non linear Schrödinger equation*, Ann. Inst. Fourier, 63, 2137–2198, 2013.
- [4] R. Carles, *Critical nonlinear Schrödinger equations with and without harmonic potential*, Math. Models Meth. Appl. Sci., 12, 1513–1523, 2002.
- [5] R. Carles, *Nonlinear Schrödinger equations with repulsive harmonic potential and applications*, SIAM J. Math. Anal., 35, 823–843, 2003.
- [6] R. Carles, *On the Cauchy problem in Sobolev spaces for nonlinear Schrödinger equations with potential*, Portugal. Math. (N. S.), 65, 191–209, 2008.
- [7] R. Carles, *Rotating points for the conformal NLS scattering operator*, Dyn. Part. Diff. Eqs., 6, 35–51, 2009.
- [8] R. Carles, *Nonlinear Schrödinger equation with time dependent potential*, Commun. Math. Sci., 9, 937–964, 2011.
- [9] R. Carles and C. Fermanian Kammerer, *Nonlinear coherent states and Ehrenfest time for Schrödinger equation*, Commun. Math. Phys., 301, 443–472, 2011.
- [10] Y. Castin and R. Dum, *Bose–Einstein condensates in time dependent traps*, Phys. Rev. Lett., 77, 5315, 1996.
- [11] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, New York University Courant Institute of Mathematical Sciences, New York, 10, 2003.
- [12] T. Cazenave and F. Weissler, *Rapidly decaying solutions of the nonlinear Schrödinger equation*, Commun. Math. Phys., 147, 75–100, 1992.
- [13] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in \mathbb{R}^3* , Ann. of Math., (2), 167, 767–865, 2008.
- [14] J. Dereziński and C. Gérard, *Scattering Theory of Classical and Quantum N-Particle Systems*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997.
- [15] D. Fujiwara, *A construction of the fundamental solution for the Schrödinger equation*, J. Analyse Math., 35, 41–96, 1979.
- [16] D. Fujiwara, *Remarks on the convergence of the Feynman path integrals*, Duke Math. J., 47, 559–600, 1980.
- [17] J.J. García-Ripoll, V.M. Pérez-García, and V. Vekslerchik, *Construction of exact solutions by spatial translations in inhomogeneous Schrödinger equations*, Phys. Rev. E, 64, 056602, 2001.
- [18] B. Grébert and L. Thomann, *KAM for the quantum harmonic oscillator*, Commun. Math. Phys., 307, 383–427, 2011.
- [19] L. Hari, *Coherent states for systems of L^2 -supercritical nonlinear Schrödinger equations*, Commun. Part. Diff. Eqs., 38, 529–573, 2013.
- [20] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons Inc., New York, 1964.
- [21] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math., 120, 955–980, 1998.
- [22] R. Killip and M. Viřan, *Global well-posedness and scattering for the defocusing quintic NLS in three dimensions*, Anal. PDE, 5, 855–885, 2012.
- [23] R. Killip, M. Viřan, and X. Zhang, *Energy-critical NLS with quadratic potentials*, Commun. Part. Diff. Eqs., 34, 1531–1565, 2009.
- [24] V. Ramesh Kumar, R. Radha, and P.K. Panigrahi, *Dynamics of Bose–Einstein condensates in a time-dependent trap*, Phys. Rev. A, 77, 023611, 2008.
- [25] P. Raphaël and J. Szeftel, *Standing ring blow up solutions to the N -dimensional quintic nonlinear Schrödinger equation*, Commun. Math. Phys., 290, 973–996, 2009.
- [26] E. Ryckman and M. Viřan, *Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in \mathbb{R}^{1+4}* , Amer. J. Math., 129, 1–60, 2007.
- [27] G. Staffilani, *On the growth of high Sobolev norms of solutions for KdV and Schrödinger equations*, Duke Math. J., 86, 109–142, 1997.
- [28] T. Tao, *A pseudoconformal compactification of the nonlinear Schrödinger equation and applications*, New York J. Math., 15, 265–282, 2009.
- [29] M. Viřan, *The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions*, Duke Math. J., 138, 281–374, 2007.
- [30] M. Viřan, *Global well-posedness and scattering for the defocusing cubic nonlinear Schrödinger equation in four dimensions*, Internat. Math. Res. Notices, 1037–1067, 2012.
- [31] B. Wang, *Nonlinear scattering theory for a class of wave equations in H^s* , J. Math. Anal. Appl., 296, 74–96, 2004.