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REGULARITY CRITERIA OF THE POROUS MEDIA EQUATION IN TERMS OF ONE PARTIAL DERIVATIVE OR PRESSURE FIELD*

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Abstract. We obtain new regularity criteria and smallness conditions for the global regularity of the N-dimensional supercritical porous media equation. In particular, it is shown that in order to obtain global regularity result, one only needs to bound a partial derivative in one direction or the pressure scalar field. Our smallness condition is also in terms of one direction, dropping conditions on $(N-1)$ other directions completely, or the pressure scalar field. The proof relies on key observations concerning the incompressibility of the velocity vector field and the special identity derived from Darcy's law.

Key words. Porous media equation, Darcy's law, regularity criteria.

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1. Introduction

We study the following active scalar convected by incompressible flow of Darcy's law, widely known as the incompressible porous media equation (IPM):

$$\begin{cases} \frac{\partial \theta}{\partial t} + (u\cdot\nabla)\theta + \nu\Lambda^{2\alpha}\theta = 0, \\ u = -\frac{\kappa}{\mu}(\nabla\pi + g\gamma\theta), \\ \theta(x,0) = \theta_0(x), \qquad \nabla\cdot u = 0, \qquad x\in\mathbb{R}^N, N\geq 2, N\in\mathbb{N}, \end{cases} \tag{1.1}$$

where θ represents the liquid density scalar field, u the velocity vector field and π the pressure scalar field. The parameters κ, μ, and g represent the matrix medium permeability, the dynamic viscosity, and the acceleration due to gravity, respectively, and we also denoted by γ the last canonical vector e_N. Together they model u governed by Darcy's law to describe the relationship between the liquid discharge and the pressure gradient (cf. [2]). Moreover, $\nu > 0$ is the dissipative coefficient, and $\Lambda = (-\Delta)^{\frac{1}{2}}$ and $\alpha \in (0,\frac{1}{2})$ are fixed parameters.

Without loss of generality throughout the rest of the paper, we set $\nu = 1$. Moreover, let us denote $\partial_t = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial x_i}$, $i=1,\ldots,N$ and L^p the Lebesgue spaces equipped with the norm $\|\cdot\|_{L^p}$ while those of Sobolev spaces H^s with $\|\cdot\|_{H^s}$. We also denote an operator $\nabla_{N-1} = (\partial_1,\ldots,\partial_{N-1},0)$.

The system (1.1) obeys the scaling invariance, namely that if $\theta(x,t)$ solves (1.1), then so does $\theta_\lambda(x,t) = \lambda^{2\alpha-1}\theta(\lambda x,\lambda^{2\alpha}t)$. Due to this scaling invariance and the fact that the L^p norms of the solution to (1.1) for $1\leq p\leq\infty$ are bounded by those of its initial data (cf. [7]), we call the case $\alpha > \frac{1}{2}$, $\alpha = \frac{1}{2}$, $\alpha < \frac{1}{2}$ the subcritical, critical, and supercritical cases, respectively.

In comparison to the velocity vector field in (1.1), which may be written as

$$\frac{\mu}{\kappa}u = -(\nabla\pi + g\gamma\theta) = -\nabla\pi - (0,\ldots,g\theta),$$

we recall in relevance that the authors in [29] studied the dynamics of the interface between two fluids with different viscosities and densities in a Hele-Shaw cell (cf. [20]).

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Moreover, the system (1.1) has caught much attention due to its similarity to the also intensely studied β-generalized surface quasi-geostrophic Equation (β-SQG) (e.g. [4, 12, 24, 25, 26, 36]):

$$\begin{cases}\partial_t\theta+(u\cdot\nabla)\theta+\nu\Lambda^{2\alpha}\theta=0,\\ \theta(x,0)=\theta_0(x), \quad u=\Lambda^{1-2\beta}(-\mathcal{R}_2\theta,\mathcal{R}_1\theta), \quad x\in\mathbb{R}^2 \text{ or } \mathbb{T}^2,\end{cases} \tag{1.2}$$

where θ represents temperature, $\alpha\in(0,\frac{1}{2})$, $\beta\in[\frac{1}{2},1)$, and $\mathcal{R}$ the Riesz transform.

The authors in [7] showed the existence of the unique global solution to (1.1) in the case $\alpha\in[\frac{1}{2},1)$, $N=3$. In case $N=3$, the author in [34] considered the case $\alpha\in(0,\frac{1}{2})$ and obtained local results using an iterative process, while, in [40], the authors considered the case $\alpha=\frac{1}{2}$ and obtained the global existence and uniqueness of the solution in critical Besov spaces. These results are similar to the works of [1] and [21] on the SQG. In [35] the author considered a modified critical porous media equation with $N=3$ and obtained global regularity results analogously to the modified critical SQG in [10].

On the other hand, in [8] the authors considered a singularly modified version of SQG and proved in particular its local well-posedness, while, in [19], the authors analogously considered a singularly modified version of IPM in the case $N=2$ and interestingly showed Lipschitz ill-posedness with smooth initial data. The key difference that led to these distinct results was that for the IPM (1.1), the Fourier symbol of the operator acting on the velocity term is an even function, while that of the β-SQG is odd. Moreover, the authors in [15] considered the case $N=2$, $\nu=0$, and showed the non-uniqueness for weak solutions to (1.1) in L^∞ in space and time. For more recent interesting studies on (1.1), we refer readers to [14, 16, 17, 28] and the references found therein.

There has been a significant amount of work on the regularity criteria of the SQG, mostly in the aim to bound $\nabla^\perp\theta$, and under a slight modification many of them can be transferred to the N-dimensional IPM. For example, in [18], the authors obtained a blow up criteria for the inviscid case of (1.1) with $N=2$:

$$\sup_{t\in[0,T]}\|\theta(t)\|_{H^m}<\infty \text{ for } m>2 \qquad \text{if and only if} \qquad \int_0^T\|\nabla\theta\|_{BMO}ds<\infty.$$

For $m>1+\frac{N}{2}$, it is clear that this result remains valid in the N-dimensional supercritical case. For an analogous blow up criterion for the solution to the SQG, we refer readers to [11]. Moreover, following the work of [33] on SQG, it is not difficult to obtain the following regularity criteria for the solution to (1.1) in terms of u:

$$\int_0^T\|\nabla u\|_{L^\infty}ds<\infty \quad \text{or} \quad \int_0^T\|\nabla u\|_{L^p}^r ds<\infty \quad \frac{N}{p}+\frac{2\alpha}{r}<2\alpha,\ \frac{N}{\alpha}\le p<\infty.$$

Recently, in [37] and subsequently in [38], it was shown that for any active scalar with a similar structure as β-SQG, specifically so that the velocity term is a Riesz type singular integral operator acting on θ, the regularity criteria can miss one direction. Indeed, for the two-dimensional supercritical β-SQG, it suffices to bound only a partial derivative, and for the system (1.1), the following result holds:

THEOREM 1.1. *Let $\alpha\in(0,\frac{1}{2})$. If $\theta\in C([0,T);H^s(\mathbb{R}^N))\cap L^2([0,T);H^{s+\alpha}(\mathbb{R}^N))$ solves (1.1) for a given $\theta_0\in H^s(\mathbb{R}^N)$, $s>1+\frac{N}{2}$, and*

$$\int_0^T\|\nabla_{N-1}\theta\|_{L^p}^r ds<\infty, \tag{1.3a}$$

$$\frac{N}{p}+\frac{2\alpha}{r}\leq 2\alpha, \frac{N(N-2\alpha)}{2\alpha^2}<p<\infty, \tag{1.3b}$$

or

$$\int_0^T \|\nabla_{N-1}\theta\|_{B^0_{\infty,1}} ds<\infty,$$

then θ remains in the same regularity class on $[0,T']$ for some $T'>T$.

The proof is similar to that of [38] and we sketch it in the Appendix for the reader's convenience.

However, it was not clear whether, like the two-dimensional $\beta-$SQG, the regularity criteria of the solution to a general active scalar in higher dimension such as (1.1) may be reduced to only one partial derivative. Moreover, while component reduction results of the regularity criteria for the three-dimensional Navier-Stokes equation (NSE) and the magnetohydrodynamics (MHD) system have caught much attention recently (e.g. [5, 27, 39]), the proofs require careful decomposition of the nonlinear terms. Hence, to the best of the author's knowledge, it is not clear if their regularity criteria remain valid in higher dimension or if we replace the Laplacian of the dissipative term by a fractional Laplacian with power arbitrarily close to zero, as in the case of (1.1).

In this paper, making use of the special structure of the velocity term governed by Darcy's law, we provide an affirmative answer and equivalently a regularity criteria in terms of the pressure scalar field. We present our main results.

Theorem 1.2. *Let $\alpha\in(0,\frac{1}{2})$. If $\theta\in C([0,T);H^s(\mathbb{R}^N))\cap L^2([0,T);H^{s+\alpha}(\mathbb{R}^N))$ solves (1.1) for a given $\theta_0\in H^s(\mathbb{R}^N)$, $s>1+\frac{N}{2}$, and*

$$\int_0^T \|\Delta\pi\|_{L^p}^r ds<\infty \qquad or \qquad \int_0^T \|\partial_N\theta\|_{L^p}^r ds<\infty, \tag{1.4a}$$

$$\frac{N}{p}+\frac{2\alpha}{r}\leq 2\alpha, \qquad \frac{N(N-2\alpha)}{2\alpha^2}<p\leq\infty, \tag{1.4b}$$

then θ remains in the same regularity class on $[0,T']$ for some $T'>T$. Moreover, if

$$\sup_{t\in[0,T]}\|\Delta\pi(t)\|_{L^{\frac{N}{2\alpha}}} \qquad or \qquad \sup_{t\in[0,T]}\|\partial_N\theta(t)\|_{L^{\frac{N}{2\alpha}}}$$

is sufficiently small, then θ remains in the same regularity class on $[0,T']$ for some $T'>T$.

Theorem 1.3. *Let $\alpha\in(0,\frac{1}{2})$. If $\theta\in C([0,T);H^s(\mathbb{R}^N))\cap L^2([0,T);H^{s+\alpha}(\mathbb{R}^N))$ solves (1.1) for a given $\theta_0\in H^s(\mathbb{R}^N)$, $s>1+\frac{N}{2}$, and*

$$\int_0^T \|\nabla_{N-1}u_N\|_{L^p}^r+\|\partial_N u\|_{L^p}^r ds<\infty, \tag{1.5a}$$

$$\frac{N}{p}+\frac{2\alpha}{r}\leq 2\alpha, \qquad \frac{N(N-2\alpha)}{2\alpha^2}<p\leq\infty, \tag{1.5b}$$

or

$$\sup_{t\in[0,T]}\|\nabla_{N-1}u_N(t)\|_{L^{\frac{N}{2\alpha}}}+\|\partial_N u(t)\|_{L^{\frac{N}{2\alpha}}}$$

is sufficiently small, then θ *remains in the same regularity class on* $[0,T']$ *for some* $T' > T$.

THEOREM 1.4. *Let* $\alpha \in (0,\frac{1}{2})$. *If* $\theta \in C([0,T);H^s(\mathbb{R}^N)) \cap L^2([0,T);H^{s+\alpha}(\mathbb{R}^N))$ *solves* (1.1) *for a given* $\theta_0 \in H^s(\mathbb{R}^N)$, $s > 1+\frac{N}{2}$, *and*

$$\int_0^T \left\| div\left(\frac{\partial_N u}{\partial_N \theta}\right)\right\|_{L^p}^r ds < \infty, \qquad or \qquad \int_0^T \left\| div\left(\frac{\partial_N u}{\Delta \pi}\right)\right\|_{L^p}^r ds < \infty, \tag{1.6a}$$

$$\frac{N}{p} + \frac{2\alpha}{r} \le 2\alpha, \qquad \frac{N(N-2\alpha)^2}{4\alpha^3} < p < \infty, \tag{1.6b}$$

then θ *remains in the same regularity class on* $[0,T']$ *for some* $T' > T$. *Moreover, if*

$$\sup_{t\in[0,T]} \left\| div\left(\frac{\partial_N u}{\partial_N \theta}\right)(t)\right\|_{L^{\frac{N}{2\alpha}}} < \infty \qquad or \qquad \sup_{t\in[0,T]} \left\| div\left(\frac{\partial_N u}{\Delta \pi}\right)(t)\right\|_{L^{\frac{N}{2\alpha}}} < \infty,$$

then there exists a constant $C = C(\alpha)$ *such that*

$$\|\theta_0\|_{L^\infty} < C \tag{1.7}$$

implies that θ *remains in the same regularity class on* $[0,T']$ *for some* $T' > T$.

REMARK 1.1.

1. Theorem 1.2 presents new results in comparison to the criteria of [7] and [18]; we recall that BMO is locally embedded in L^p, $p < \infty$.
2. We emphasize that for the endpoint case $p = \infty$, even though for the β-generalized SQG for which Theorem 1.1 applies we had to rely on the $B^0_{\infty,1}$ norm, in the case of IPM it is a a remarkable fact that we are able to directly extend to the L^∞-norm as shown in the theorems 1.2 and 1.3.
3. Theorems 1.1-1.4 show that ∂_N plays a distinct role from other partial derivatives in the case of IPM while in the work of [38] the role of ∂_2 of the β-SQG was no different from that of ∂_1. This is a remarkable fact due to the key identity derived from Darcy's law.
4. There have been numerous attempts to derive Darcy's law from the NSE through the homogenization process (cf. [32]). In this regard, it is of interest to recall that such regularity criteria that depend solely on the pressure scalar field exist for the NSE and the MHD system (e.g. [3, 6, 31]). Due to many differences, their proofs do not go through in the case of the N-dimensional IPM with fractional Laplacians.
5. In the theory of Serrin-type regularity criteria for the NSE and the MHD systems, the scaling invariance of the norms in (1.4a)-(1.4b), (1.5a)-(1.5b), and (1.6a)-(1.6b) is of much importance (cf. [30]). Due to the rescaling of the solution to (1.1),

$$\|\partial_N \theta_\lambda\|_{L^r(0,T;L^p(\mathbb{R}^N))} = \|\partial_N \theta\|_{L^r(0,\lambda^{2\alpha}T;L^p(\mathbb{R}^N))}$$

   if and only if $\frac{N}{p} + \frac{2\alpha}{r} = 2\alpha$.

In the subsequent sections, we prove our claims. By the standard argument of continuation of local theory, we only need to show H^s-bounds.

2. Proof of Theorem 1.2

We first prove a proposition:

PROPOSITION 2.1. *Let $\alpha \in (0,\frac{1}{2})$. If the solution $\theta(x,t)$ to (1.1) in $[0,T]$ satisfies the hypothesis of Theorem 1.2, then*

$$\begin{cases} \sup_{t\in[0,T]} \|\nabla_{N-1}\theta(t)\|_{L^{\frac{2p\alpha}{N}}}^{\frac{2p\alpha}{N}} + \|\partial_N\theta(t)\|_{L^{\frac{2p\alpha}{N}}}^{\frac{2p\alpha}{N}} \\ \qquad + \int_0^T \|\nabla_{N-1}\theta\|_{L^{\frac{2p\alpha}{N-2\alpha}}}^{\frac{2p\alpha}{N}} + \|\partial_N\theta\|_{L^{\frac{2p\alpha}{N-2\alpha}}}^{\frac{2p\alpha}{N}} ds < \infty \quad \text{if } p<\infty, \\ \\ \sup_{t\in[0,T]} \|\nabla_{N-1}\theta(t)\|_{L^{\frac{N-2\alpha}{\alpha}}}^{\frac{N-2\alpha}{\alpha}} + \|\partial_N\theta(t)\|_{L^{\frac{N-2\alpha}{\alpha}}}^{\frac{N-2\alpha}{\alpha}} \\ \qquad + \int_0^T \|\nabla_{N-1}\theta\|_{L^{\frac{N}{\alpha}}}^{\frac{N-2\alpha}{\alpha}} + \|\partial_N\theta\|_{L^{\frac{N}{\alpha}}}^{\frac{N-2\alpha}{\alpha}} ds < \infty \quad \text{if } p=\infty. \end{cases}$$

Proof. The following decomposition of u in (1.1), observed in [7] in the case $N=3$, will be important in our proof:

$$u = c(\theta + \mathcal{P}\theta) \tag{2.1}$$

for some constant $c>0$ where $\mathcal{P}$ is a Riesz type singular integral operator bounded in L^p, $p\in(1,\infty)$ so that we immediately have

$$\|u\|_{L^p} \leq c\|\theta\|_{L^p}, \qquad p\in(1,\infty). \tag{2.2}$$

In higher dimension, we may see this fact from first noticing that

$$0 = \nabla\cdot u = -\frac{\kappa}{\mu}(\partial_{11}^2\pi + \ldots + \partial_{NN}^2\pi) - \frac{\kappa}{\mu}g\partial_N\theta,$$

so that we have the identity

$$-\Delta\pi = g\partial_N\theta \tag{2.3}$$

with which we may write

$$u = -\frac{\kappa}{\mu}g(\partial_1\partial_N(-\Delta)^{-1}\theta, \ldots, \partial_{N-1}\partial_N(-\Delta)^{-1}\theta, \partial_{NN}^2(-\Delta)^{-1}\theta + \theta).$$

Thus, the Fourier symbol of an operator acting on θ as u is a constant multiple of

$$\left(\frac{\xi_1\xi_N}{|\xi|^2}, \ldots, \frac{\xi_{N-1}\xi_N}{|\xi|^2}, \frac{\xi_{NN}^2}{|\xi|^2} + 1\right)$$

(cf. [34]). Finally, we recall that due to the continuity of the Riesz transform in L^p, $p\in(1,\infty)$,

$$\|\partial_i\partial_j f\|_{L^p} \leq c\|\Delta f\|_{L^p}, \qquad i,j=1,\ldots,N. \tag{2.4}$$

Now we first consider the case $p<\infty$. We fix (p,r) that satisfies (1.4b) and fix $q=\frac{2p\alpha}{N}$. It is easy to check that $q\in(2,\infty)$, considering (1.4b) and the range of α. We apply ∂_N to (1.1), multiply by $q|\partial_N\theta|^{q-2}\partial_N\theta$, and integrate in space to obtain

$$\partial_t\|\partial_N\theta\|_{L^q}^q + q\int(\Lambda^{2\alpha}\partial_N\theta)|\partial_N\theta|^{q-2}\partial_N\theta = -q\int\partial_N((u\cdot\nabla)\theta)|\partial_N\theta|^{q-2}\partial_N\theta. \tag{2.5}$$

We need the following lemma which is a generalization of the work in [13, 22]:

LEMMA 2.2. *For any* $\alpha\in[0,1]$, $x\in\mathbb{R}^N$ *or* $x\in\mathbb{T}^N$, $N\in\mathbb{N}$, *and* $f,\Lambda^{2\alpha}f\in L^s$, $s\geq 2$,

$$2\int|\Lambda^\alpha(f^{\frac{s}{2}})|^2\leq s\int|f|^{s-2}f\Lambda^{2\alpha}f.$$

We estimate the dissipative term with this lemma as follows:

$$2\|(\partial_N\theta)^{\frac{q}{2}}\|_{\dot{H}^\alpha}^2=2\int|\Lambda^\alpha(\partial_N\theta)^{\frac{q}{2}}|^2\leq q\int(\Lambda^{2\alpha}\partial_N\theta)|\partial_N\theta|^{q-2}\partial_N\theta.$$

By the homogeneous Sobolev embedding $\dot{H}^\alpha\hookrightarrow L^{\frac{2N}{N-2\alpha}}$, we have

$$c(q,\alpha)\|\partial_N\theta\|^q_{L^{\frac{qN}{N-2\alpha}}}=c(q,\alpha)\||\partial_N\theta|^{\frac{q}{2}}\|^2_{L^{\frac{2N}{N-2\alpha}}}\leq 2\||\partial_N\theta|^{\frac{q}{2}}\|^2_{\dot{H}^\alpha}.$$

On the right hand side of (2.5) we have, due to the incompressibility of u and Hölder's inequalities,

$$\begin{aligned}
&-q\int\partial_N((u\cdot\nabla)\theta)|\partial_N\theta|^{q-2}\partial_N\theta\\
&=-q\int\partial_Nu\cdot\nabla\theta|\partial_N\theta|^{q-2}\partial_N\theta+u\cdot(\nabla\partial_N\theta)|\partial_N\theta|^{q-2}\partial_N\theta\\
&\leq q\sum_{i=1}^{N-1}\|\partial_Nu_i\|_{L^{\frac{qN}{2\alpha}}}\|\partial_i\theta\|_{L^q}\|\partial_N\theta\|^{q-2}_{L^q}\|\partial_N\theta\|_{L^{\frac{qN}{N-2\alpha}}}\\
&\quad+\|\partial_Nu_N\|_{L^{\frac{qN}{2\alpha}}}\|\partial_N\theta\|^{q-1}_{L^q}\|\partial_N\theta\|_{L^{\frac{qN}{N-2\alpha}}}.
\end{aligned}$$

Now, by (2.2) and (2.3),

$$\|\partial_Nu_i\|_{L^{\frac{qN}{2\alpha}}}+\|\partial_Nu_N\|_{L^{\frac{qN}{2\alpha}}}\leq c\|\partial_N\theta\|_{L^{\frac{qN}{2\alpha}}}=c\|\Delta\pi\|_{L^{\frac{qN}{2\alpha}}}.$$

Using this and Young's inequalities we obtain

$$\begin{aligned}
&-q\int\partial_N((u\cdot\nabla)\theta)|\partial_N\theta|^{q-2}\partial_N\theta)\\
&\leq\epsilon\|\partial_N\theta\|^q_{L^{\frac{qN}{N-2\alpha}}}+c\|\Delta\pi\|^{\frac{q}{q-1}}_{L^{\frac{qN}{2\alpha}}}(\|\nabla_{N-1}\theta\|^q_{L^q}+\|\partial_N\theta\|^q_{L^q}.
\end{aligned}\tag{2.6}$$

Next, we apply ∇_{N-1} on (1.1), multiply by $q|\nabla_{N-1}\theta|^{q-2}\nabla_{N-1}\theta$, integrate in space, and by Lemma 2.2 and the homogeneous Sobolev embedding again obtain

$$\partial_t\|\nabla_{N-1}\theta\|^q_{L^q}+c(q,\alpha)\|\nabla_{N-1}\theta\|^q_{L^{\frac{qN}{N-2\alpha}}}\leq -q\int\nabla_{N-1}((u\cdot\nabla)\theta)|\nabla_{N-1}\theta|^{q-2}\cdot\nabla_{N-1}\theta.\tag{2.7}$$

On the other hand, due to the incompressibility of u,

$$\begin{aligned}
&-q\int\nabla_{N-1}u\cdot\nabla\theta|\nabla_{N-1}\theta|^{q-2}\cdot\nabla_{N-1}\theta+u\cdot\nabla\nabla_{N-1}\theta|\nabla_{N-1}\theta|^{q-2}\cdot\nabla_{N-1}\theta\\
&=-q\sum_{i,j=1}^{N-1}\int\partial_ju_i\partial_i\theta|\nabla_{N-1}\theta|^{q-2}\partial_j\theta+\partial_ju_N\partial_N\theta|\nabla_{N-1}\theta|^{q-2}\partial_j\theta
\end{aligned}$$

$$\leq q\sum_{i,j=1}^{N-1}\|\partial_j u_i\|_{L^{\frac{qN}{2\alpha}}}\|\partial_i\theta\|_{L^{\frac{qN}{N-2\alpha}}}\|\nabla_{N-1}\theta\|_{L^q}^{q-2}\|\partial_j\theta\|_{L^q}$$
$$+\|\partial_j u_N\|_{L^{\frac{qN}{N-2\alpha}}}\|\partial_N\theta\|_{L^{\frac{qN}{2\alpha}}}\|\nabla_{N-1}\theta\|_{L^q}^{q-1}$$

by Hölder's inequality. Now, by (1.1) and (2.4), we have

$$\sum_{i,j=1}^{N-1}\|\partial_j u_i\|_{L^{\frac{qN}{2\alpha}}}\leq c\sum_{j=1}^{N-1}\|\partial_j\nabla\pi\|_{L^{\frac{qN}{2\alpha}}}\leq c\|\Delta\pi\|_{L^{\frac{qN}{2\alpha}}}.$$

Thus, using (2.2), (2.3), and Young's inequality, we further bound by

$$-q\int\nabla_{N-1}((u\cdot\nabla)\theta)|\nabla_{N-1}\theta|^{q-2}\cdot\nabla_{N-1}\theta$$
$$\leq c(\|\Delta\pi\|_{L^{\frac{qN}{2\alpha}}}\|\nabla_{N-1}\theta\|_{L^{\frac{qN}{N-2\alpha}}}\|\nabla_{N-1}\theta\|_{L^q}^{q-1}+\|\nabla_{N-1}u_N\|_{L^{\frac{qN}{N-2\alpha}}}\|\partial_N\theta\|_{L^{\frac{qN}{2\alpha}}}\|\nabla_{N-1}\theta\|_{L^q}^{q-1})$$
$$\leq\epsilon\|\nabla_{N-1}\theta\|_{L^{\frac{qN}{N-2\alpha}}}^q+c\|\Delta\pi\|_{L^{\frac{qN}{2\alpha}}}^{\frac{q}{q-1}}\|\nabla_{N-1}\theta\|_{L^q}^q. \tag{2.8}$$

Summing (2.5)-(2.8), for $\epsilon>0$ sufficiently small, we have due to Gronwall's inequality

$$\sup_{t\in[0,T]}\|\nabla_{N-1}\theta(t)\|_{L^{\frac{2p\alpha}{N}}}^{\frac{2p\alpha}{N}}+\|\partial_N\theta(t)\|_{L^{\frac{2p\alpha}{N}}}^{\frac{2p\alpha}{N}}+\int_0^T\|\nabla_{N-1}\theta\|_{L^{\frac{2p\alpha}{N-2\alpha}}}^{\frac{2p\alpha}{N}}+\|\partial_N\theta\|_{L^{\frac{2p\alpha}{N-2\alpha}}}^{\frac{2p\alpha}{N}}ds$$
$$\leq c(\|\nabla_{N-1}\theta_0\|_{L^{\frac{2p\alpha}{N}}}^{\frac{2p\alpha}{N}}+\|\partial_N\theta_0\|_{L^{\frac{2p\alpha}{N}}}^{\frac{2p\alpha}{N}})e^{\int_0^T\|\Delta\pi\|_{L^p}^{\frac{2p\alpha}{2p\alpha-N}}ds}<\infty.$$

This completes the proof of Proposition 2.1 in the case $p<\infty$.

Next, let us consider the case $p=\infty$. Let $q=\frac{N-2\alpha}{\alpha}$. Clearly $q\in(2,\infty)$ considering that $\alpha\in\left(0,\frac{1}{2}\right)$. Thus, as before, we apply ∂_N to (1.1), multiply by $q|\partial_N\theta|^{q-2}\partial_N\theta$, integrate in space, and use Lemma 2.2 to obtain

$$\partial_t\|\partial_N\theta\|_{L^q}^q+c(q,\alpha)\|\partial_N\theta\|_{L^{\frac{qN}{N-2\alpha}}}^q$$
$$\leq -q\sum_{i=1}^{N}\int\partial_N u_i\partial_i\theta|\partial_N\theta|^{q-2}\partial_N\theta$$
$$=-q\sum_{i=1}^{N-1}\int\partial_N u_i\partial_i\theta|\partial_N\theta|^{q-2}\partial_N\theta+\partial_N u_N\partial_N\theta|\partial_N\theta|^{q-2}\partial_N\theta$$
$$=-q\sum_{i,j=1}^{N-1}\int\partial_N u_i\partial_i\theta|\partial_N\theta|^{q-2}\partial_N\theta-\partial_j u_j\partial_N\theta|\partial_N\theta|^{q-2}\partial_N\theta$$
$$\leq q\sum_{i,j=1}^{N-1}(\|\partial_N u_i\|_{L^q}\|\partial_i\theta\|_{L^q}\|\partial_N\theta\|_{L^q}^{q-2}\|\partial_N\theta\|_{L^\infty}+\|\partial_j u_j\|_{L^q}\|\partial_N\theta\|_{L^\infty}\|\partial_N\theta\|_{L^q}^{q-1})$$
$$\leq c(\|\partial_N\theta\|_{L^q}^{q-1}\|\nabla_{N-1}\theta\|_{L^q}\|\Delta\pi\|_{L^\infty}+\|\nabla_{N-1}\theta\|_{L^q}\|\Delta\pi\|_{L^\infty}\|\partial_N\theta\|_{L^q}^{q-1})$$
$$\leq c\|\Delta\pi\|_{L^\infty}(\|\nabla_{N-1}\theta\|_{L^q}^q+\|\partial_N\theta\|_{L^q}^q) \tag{2.9}$$

by incompressibility of u, Hölder's inequalities, (2.3), and Young's inequalities.

Next, we apply ∇_{N-1} on (1.1), multiply by $q|\nabla_{N-1}\theta|^{q-2}\nabla_{N-1}\theta$, integrate in space, and use Lemma 2.2 and the homogeneous Sobolev embedding to estimate

$$\partial_t\|\nabla_{N-1}\theta\|_{L^q}^q+c(q,\alpha)\|\nabla_{N-1}\theta\|_{L^{\frac{qN}{N-2\alpha}}}^q$$

$$
\begin{aligned}
&\le -q\int \nabla_{N-1}u\cdot\nabla\theta|\nabla_{N-1}\theta|^{q-2}\cdot\nabla_{N-1}\theta\\
&= -q\sum_{i,j=1}^{N-1}\int \partial_j u_i\partial_i\theta|\nabla_{N-1}\theta|^{q-2}\partial_j\theta+\partial_j u_N\partial_N\theta|\nabla_{N-1}\theta|^{q-2}\partial_j\theta\\
&\le q\sum_{i,j=1}^{N-1}(\|\partial_j u_i\|_{L^\infty}\|\partial_i\theta\|_{L^q}\|\nabla_{N-1}\theta\|_{L^q}^{q-2}\|\partial_j\theta\|_{L^q}+\|\partial_j u_N\|_{L^q}\|\partial_N\theta\|_{L^\infty}\|\nabla_{N-1}\theta\|_{L^q}^{q-2}\|\partial_j\theta\|_{L^q})\\
&\le c\|\Delta\pi\|_{L^\infty}\|\nabla_{N-1}\theta\|_{L^q}^q
\end{aligned}
\tag{2.10}
$$

by Hölder's inequalities, (2.3), and (2.4). We sum (2.9) and (2.10) and obtain by Gronwall's inequality

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|\nabla_{N-1}\theta(t)\|_{L^{\frac{N-2\alpha}{\alpha}}}^{\frac{N-2\alpha}{\alpha}}+\|\partial_N\theta(t)\|_{L^{\frac{N-2\alpha}{\alpha}}}^{\frac{N-2\alpha}{\alpha}}+\int_0^T\|\nabla_{N-1}\theta\|_{L^{\frac{N}{\alpha}}}^{\frac{N-2\alpha}{\alpha}}+\|\partial_N\theta\|_{L^{\frac{N}{\alpha}}}^{\frac{N-2\alpha}{\alpha}}ds\\
&\le c(\|\nabla_{N-1}\theta_0\|_{L^{\frac{N}{\alpha}}}^{\frac{N-2\alpha}{\alpha}}+\|\partial_N\theta_0\|_{L^{\frac{N}{\alpha}}}^{\frac{N-2\alpha}{\alpha}})e^{\int_0^T\|\Delta\pi\|_{L^\infty}ds}<\infty.
\end{aligned}
$$

This completes the proof of Proposition 2.1. □

Proof of Theorem 1.2.

Proof. We are now ready to complete the proof of Theorem 1.2. Let us consider the case $p<\infty$ first. One can readily check using (1.4b) and the range of α that

$$
\frac{4p\alpha^2}{4p\alpha^2-N^2+2\alpha N}\le\frac{2p\alpha}{N}.
\tag{2.11}
$$

We use the following elementary inequality several times; hence, for convenience let us state it as a lemma of which the proof is simple and hence omitted:

Lemma 2.3. *If $a,b\ge 0$, then for any $s\ge 0$,*

$$
(a+b)^s\le 2^s(a^s+b^s).
$$

By Lemma 2.3, (2.11), and Proposition 2.1, we obtain

$$
\begin{aligned}
&\int_0^T\|\nabla\theta\|_{L^{\frac{2p\alpha}{N-2\alpha}}}^{\frac{4p\alpha^2}{4p\alpha^2-N^2+2\alpha N}}ds\\
&\le\int_0^T\left(2^{\frac{p\alpha}{N-2\alpha}}\int[|\nabla_{N-1}\theta|^{\frac{2p\alpha}{N-2\alpha}}+|\partial_N\theta|^{\frac{2p\alpha}{N-2\alpha}}]\right)^{\frac{(N-2\alpha)2\alpha}{4p\alpha^2-N^2+2\alpha N}}ds\\
&\le 2^{\frac{2p\alpha^2}{4p\alpha^2-N^2+2\alpha N}+\frac{(N-2\alpha)2\alpha}{4p\alpha^2-N^2+2\alpha N}}\int_0^T\|\nabla_{N-1}\theta\|_{L^{\frac{2p\alpha}{N-2\alpha}}}^{\frac{4p\alpha^2}{4p\alpha^2-N^2+2\alpha N}}+\|\partial_N\theta\|_{L^{\frac{2p\alpha}{N-2\alpha}}}^{\frac{4p\alpha^2}{4p\alpha^2-N^2+2\alpha N}}ds\\
&<\infty.
\end{aligned}
\tag{2.12}
$$

Now we use the following commutator estimate due to [23]:

Lemma 2.4. *Let $f,g\in C_0^\infty(\mathbb{R}^N),p,p_2,p_3\in(1,\infty)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}$, $s>0$. Then there exists a constant $c\ge 0$ such that*

$$
\|\Lambda^s(fg)-f\Lambda^s g\|_{L^p}\le c(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}}+\|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}).
$$

We estimate (1.1) after applying Λ^s and taking an L^2-inner product with $\Lambda^s\theta$,

$$\begin{aligned}\frac{1}{2}\partial_t\|\Lambda^s\theta\|_{L^2}^2+\|\Lambda^{s+\alpha}\theta\|_{L^2}^2 &= -\int \Lambda^s((u\cdot\nabla)\theta)\Lambda^s\theta - u\cdot\nabla\Lambda^s\theta\Lambda^s\theta\\
&\le c(\|\nabla u\|_{L^{\frac{N}{\epsilon\alpha}}}\|\Lambda^{s-1}\nabla\theta\|_{L^2}+\|\Lambda^s u\|_{L^2}\|\nabla\theta\|_{L^{\frac{N}{\epsilon\alpha}}})\|\Lambda^s\theta\|_{L^{\frac{2N}{N-2\epsilon\alpha}}}\\
&\le c\|\nabla\theta\|_{L^{\frac{N}{\epsilon\alpha}}}\|\Lambda^s\theta\|_{L^2}^{2-\epsilon}\|\Lambda^{s+\alpha}\theta\|_{L^2}^{\epsilon}\\
&\le \frac{1}{2}\|\Lambda^{s+\alpha}\theta\|_{L^2}^2+c\|\nabla\theta\|_{L^{\frac{N}{\epsilon\alpha}}}^{\frac{2}{2-\epsilon}}\|\Lambda^s\theta\|_{L^2}^2\end{aligned}$$

by (2.2), Gagliardo-Nirenberg inequality, and Young's inequality. We choose

$$\epsilon=\frac{N}{\alpha}\left(\frac{N-2\alpha}{2p\alpha}\right);$$

note that $\epsilon\in(0,1)$ due to (1.4b). By Gronwall's inequality and (2.12),

$$\sup_{t\in[0,T]}\|\Lambda^s\theta(t)\|_{L^2}^2+\int_0^T\|\Lambda^{s+\alpha}\theta\|_{L^2}^2ds\le c\|\Lambda^s\theta_0\|_{L^2}^2 e^{\int_0^T\|\nabla\theta\|_{L^{\frac{2p\alpha}{N-2\alpha}}}^{\frac{4p\alpha^2}{4p\alpha^2-N^2+2\alpha N}}ds}<\infty.$$

Next, we consider the case $p=\infty$. Firstly, using Lemma 2.3 again,

$$\begin{aligned}\int_0^T\|\nabla\theta\|_{L^{\frac{N}{\alpha}}}^2ds &\le \int_0^T(2^{\frac{N}{2\alpha}}\int|\nabla_{N-1}\theta|^{\frac{N}{\alpha}}+|\partial_N\theta|^{\frac{N}{\alpha}})^{\frac{2\alpha}{N}}ds\\
&\le 2\int_0^T 2^{\frac{2\alpha}{N}}(\|\nabla_{N-1}\theta\|_{L^{\frac{N}{\alpha}}}^2+\|\partial_N\theta\|_{L^{\frac{N}{\alpha}}}^2)ds<\infty\end{aligned}$$

by Hölder's inequality and Proposition 2.1. On the other hand,

$$\begin{aligned}\frac{1}{2}\partial_t\|\Lambda^s\theta\|_{L^2}^2+\|\Lambda^{s+\alpha}\theta\|_{L^2}^2 &\le c(\|\nabla u\|_{L^{\frac{N}{\alpha}}}\|\Lambda^{s-1}\nabla\theta\|_{L^2}+\|\Lambda^s u\|_{L^2}\|\nabla\theta\|_{L^{\frac{N}{\alpha}}})\|\Lambda^s\theta\|_{L^{\frac{2N}{N-2\alpha}}}\\
&\le c\|\nabla\theta\|_{L^{\frac{N}{\alpha}}}\|\Lambda^s\theta\|_{L^2}\|\Lambda^{s+\alpha}\theta\|_{L^2}\\
&\le \frac{1}{2}\|\Lambda^{s+\alpha}\theta\|_{L^2}^2+c\|\nabla\theta\|_{L^{\frac{N}{\alpha}}}^2\|\Lambda^s\theta\|_{L^2}^2\end{aligned}$$

by Lemma 2.4, (2.2), the homogeneous Sobolev embedding, and Young's inequality. Therefore, by Gronwall's inequality, this completes the proof of the first claim of Theorem 1.2.

Next, we fix p that satisfies (1.4b) and notice that $p\ge 2$. We apply ∇_{N-1} on (1.1), multiply by $p|\nabla_{N-1}\theta|^{p-2}\nabla_{N-1}\theta$, integrate in space, and use Lemma 2.2 and the homogeneous Sobolev embedding again to obtain

$$\begin{aligned}&\partial_t\|\nabla_{N-1}\theta\|_{L^p}^p+c(p,\alpha)\|\nabla_{N-1}\theta\|_{L^{\frac{pN}{N-2\alpha}}}^p\\
&\le -p\int\nabla_{N-1}u\cdot\nabla\theta|\nabla_{N-1}\theta|^{p-2}\cdot\nabla_{N-1}\theta+u\cdot\nabla\nabla_{N-1}\theta|\nabla_{N-1}\theta|^{p-2}\cdot\nabla_{N-1}\theta\\
&=-p\sum_{i,j=1}^{N-1}\int\partial_j u_i\partial_i\theta|\nabla_{N-1}\theta|^{p-2}\partial_j\theta+\partial_j u_N\partial_N\theta|\nabla_{N-1}\theta|^{p-2}\partial_j\theta\\
&\le p\sum_{i,j=1}^{N-1}\|\partial_j u_i\|_{L^{\frac{N}{2\alpha}}}\|\partial_i\theta\|_{L^{\frac{pN}{N-2\alpha}}}\|\nabla_{N-1}\theta\|_{L^{\frac{pN}{N-2\alpha}}}^{p-2}\|\partial_j\theta\|_{L^{\frac{pN}{N-2\alpha}}}\end{aligned}$$

$$
\begin{aligned}
&+\|\partial_j u_N\|_{L^{\frac{pN}{N-2\alpha}}}\|\partial_N\theta\|_{L^{\frac{N}{2\alpha}}}\|\nabla_{N-1}\theta\|^{p-2}_{L^{\frac{pN}{N-2\alpha}}}\|\partial_j\theta\|_{L^{\frac{pN}{N-2\alpha}}}\\
\leq & c(\|\Delta\pi\|_{L^{\frac{N}{2\alpha}}}\|\nabla_{N-1}\theta\|^p_{L^{\frac{pN}{N-2\alpha}}}+\|\nabla_{N-1}u_N\|_{L^{\frac{pN}{N-2\alpha}}}\|\partial_N\theta\|_{L^{\frac{N}{2\alpha}}}\|\nabla_{N-1}\theta\|^{p-1}_{L^{\frac{pN}{N-2\alpha}}})\\
\leq & c\|\Delta\pi\|_{L^{\frac{N}{2\alpha}}}\|\nabla_{N-1}\theta\|^p_{L^{\frac{pN}{N-2\alpha}}}
\end{aligned}
$$

by Hölder's inequalities, (2.2), (2.3), and (2.4). Thus,

$$
\partial_t\|\nabla_{N-1}\theta\|^p_{L^p}\leq (c\sup_{t\in[0,T]}\|\Delta\pi\|_{L^{\frac{N}{2\alpha}}}-c(p,\alpha))\|\nabla_{N-1}\theta\|^p_{L^{\frac{pN}{N-2\alpha}}}.
$$

Therefore, for $\sup_{t\in[0,T]}\|\Delta\pi\|_{L^{\frac{N}{2\alpha}}}$ sufficiently small, for any r that satisfies (1.3b), we obtain

$$
\int_0^T\|\nabla_{N-1}\theta\|^r_{L^p}ds\leq\|\nabla_{N-1}\theta_0\|^r_{L^p}T<\infty.
$$

By Theorem 1.1, this completes the proof of Theorem 1.2. ∎

3. Proof of Theorem 1.3

We first consider the case $p<\infty$. Since $u_N=-\frac{\kappa}{\mu}(\partial_N\pi+g\theta)$, we may write

$$
\frac{1}{g}\left(-\frac{\mu}{\kappa}u_N-\partial_N\pi\right)=\theta. \tag{3.1}
$$

Therefore, because for any $i=1,\dots,N-1$,

$$
-\frac{\mu}{\kappa}u_i=\partial_i\pi,
$$

we see that, using (3.1),

$$
\partial_i\theta=\frac{1}{g}\partial_i\left(-\frac{\mu}{\kappa}u_N-\partial_N\pi\right)=-\frac{1}{g}\frac{\mu}{\kappa}(\partial_iu_N-\partial_Nu_i). \tag{3.2}
$$

Using Lemma 2.3 and (3.2) we can verify that

$$
\int_0^T\|\nabla_{N-1}\theta\|^r_{L^p}ds\leq c\int_0^T\|\nabla_{N-1}u_N\|^r_{L^p}+\|\partial_Nu\|^r_{L^p}ds.
$$

Thus, by Theorem 1.1, the proof of this case $p<\infty$ is complete.

Next, we consider the case $p=\infty$. Let $q=\frac{N-2\alpha}{\alpha}$. Applying ∂_N on (1.1), multiplying by $q|\partial_N\theta|^{q-2}\partial_N\theta$, integrating in space, and using Lemma 2.2 we obtain

$$
\begin{aligned}
&\partial_t\|\partial_N\theta\|^q_{L^q}+c(q,\alpha)\|\partial_N\theta\|^q_{L^{\frac{qN}{N-2\alpha}}}\\
\leq & -q\sum_{i=1}^{N-1}\int\partial_Nu_i\partial_i\theta|\partial_N\theta|^{q-2}\partial_N\theta+\partial_Nu_N\partial_N\theta|\partial_N\theta|^{q-2}\partial_N\theta\\
= & \sum_{i=1}^{N-1}\int\frac{q}{g}\frac{\mu}{\kappa}\partial_Nu_i(\partial_iu_N-\partial_Nu_i)|\partial_N\theta|^{q-2}\partial_N\theta-q\partial_Nu_N\partial_N\theta|\partial_N\theta|^{q-2}\partial_N\theta\\
\leq & \sum_{i=1}^{N-1}\left(\left|\frac{q}{g}\frac{\mu}{\kappa}\right|\|\partial_Nu_i\|_{L^q}\|\partial_iu_N-\partial_Nu_i\|_{L^\infty}+q\|\partial_Nu_N\|_{L^\infty}\|\partial_N\theta\|_{L^q}\right)\|\partial_N\theta\|^{q-1}_{L^q}
\end{aligned}
$$

$$\leq c(\|\nabla_{N-1}u_N\|_{L^\infty}+\|\partial_N u\|_{L^\infty})\|\partial_N\theta\|_{L^q}^q \tag{3.3}$$

by (3.2) and Hölder's inequality. Similarly, we estimate

$$\begin{aligned}
&\partial_t\|\nabla_{N-1}\theta\|_{L^q}^q+c(q,\alpha)\|\nabla_{N-1}\theta\|^q_{L^{\frac{qN}{N-2\alpha}}}\\
&\leq -q\int \nabla_{N-1}((u\cdot\nabla)\theta)|\nabla_{N-1}\theta|^{q-2}\nabla_{N-1}\theta\\
&=-q\sum_{i,j=1}^{N-1}\int \partial_j u_i\partial_i\theta|\nabla_{N-1}\theta|^{q-2}\partial_j\theta+\partial_j u_N\partial_N\theta|\nabla_{N-1}\theta|^{q-2}\partial_j\theta\\
&=\sum_{i,j=1}^{N-1}\int \frac{q}{g}\frac{\mu}{\kappa}\partial_j u_i(\partial_i u_N-\partial_N u_i)|\nabla_{N-1}\theta|^{q-2}\partial_j\theta-q\partial_j u_N\partial_N\theta|\nabla_{N-1}\theta|^{q-2}\partial_j\theta\\
&\leq \sum_{i,j=1}^{N-1}\left|\frac{q}{g}\frac{\mu}{\kappa}\right|\|\partial_j u_i\|_{L^q}\|\partial_i u_N-\partial_N u_i\|_{L^\infty}\|\nabla_{N-1}\theta\|_{L^q}^{q-2}\|\partial_j\theta\|_{L^q}\\
&\quad +q\|\partial_j u_N\|_{L^\infty}\|\partial_N\theta\|_{L^q}\|\nabla_{N-1}\theta\|_{L^q}^{q-2}\|\partial_j\theta\|_{L^q}\\
&\leq c\|\nabla_{N-1}\theta\|_{L^q}^q(\|\nabla_{N-1}u_N\|_{L^\infty}+\|\partial_N u\|_{L^\infty})+c\|\nabla_{N-1}u_N\|_{L^\infty}(\|\partial_N\theta\|_{L^q}^q+\|\nabla_{N-1}\theta\|_{L^q}^q)
\end{aligned} \tag{3.4}$$

by (3.2) and Hölder's and Young's inequalities. Thus, we sum (3.3) and (3.4) so that, by Gronwall's inequality,

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\nabla_{N-1}\theta(t)\|^{\frac{N-2\alpha}{\alpha}}_{L^{\frac{N-2\alpha}{\alpha}}}+\|\partial_N\theta(t)\|^{\frac{N-2\alpha}{\alpha}}_{L^{\frac{N-2\alpha}{\alpha}}}+\int_0^T\|\nabla_{N-1}\theta\|^{\frac{N-2\alpha}{\alpha}}_{L^{\frac{N}{\alpha}}}+\|\partial_N\theta\|^{\frac{N-2\alpha}{\alpha}}_{L^{\frac{N}{\alpha}}}ds\\
&\leq c(\|\nabla_{N-1}\theta_0\|^{\frac{N-2\alpha}{\alpha}}_{L^{\frac{N-2\alpha}{\alpha}}}+\|\partial_N\theta_0\|^{\frac{N-2\alpha}{\alpha}}_{L^{\frac{N-2\alpha}{\alpha}}})e^{\int_0^T\|\nabla_{N-1}u_N\|_{L^\infty}+\|\partial_N u\|_{L^\infty}ds}<\infty.
\end{aligned}$$

Thus, we have proven the implication of Proposition 2.1. That this implies the lack of singularity follows. This completes the proof of the first claim of Theorem 1.3.

Next, we fix p that satisfies (1.4b) and $p<\infty$. We apply ∂_N on (1.1), multiply by $p|\partial_N\theta|^{p-2}\partial_N\theta$, integrate in space, and apply Lemma 2.2 and the homogeneous Sobolev embedding as before to obtain

$$\begin{aligned}
&\partial_t\|\partial_N\theta\|_{L^p}^p+c(p,\alpha)\|\partial_N\theta\|^p_{L^{\frac{pN}{N-2\alpha}}}\\
&\leq -p\sum_{i=1}^{N-1}\int \partial_N u_i\partial_i\theta|\partial_N\theta|^{p-2}\partial_N\theta+\partial_N u_N\partial_N\theta|\partial_N\theta|^{p-2}\partial_N\theta\\
&=\sum_{i=1}^{N-1}\int \frac{p}{g}\frac{\mu}{\kappa}\partial_N u_i(\partial_i u_N-\partial_N u_i)|\partial_N\theta|^{p-2}\partial_N\theta-p\partial_N u_N\partial_N\theta|\partial_N\theta|^{p-2}\partial_N\theta\\
&\leq\left|\frac{p}{g}\frac{\mu}{\kappa}\right|\sum_{i=1}^{N-1}\|\partial_N u_i\|_{L^{\frac{pN}{N-2\alpha}}}\|\partial_i u_N-\partial_N u_i\|_{L^{\frac{N}{2\alpha}}}\|\partial_N\theta\|^{p-1}_{L^{\frac{pN}{N-2\alpha}}}+p\|\partial_N u_N\|_{L^{\frac{N}{2\alpha}}}\|\partial_N\theta\|^p_{L^{\frac{pN}{N-2\alpha}}}\\
&\leq c\|\partial_N\theta\|^p_{L^{\frac{pN}{N-2\alpha}}}(\|\nabla_{N-1}u_N\|_{L^{\frac{N}{2\alpha}}}+\|\partial_N u\|_{L^{\frac{N}{2\alpha}}})
\end{aligned}$$

by (3.2), Hölder's inequalities and (2.2). Therefore, we have

$$\partial_t\|\partial_N\theta\|_{L^p}^p\leq(c\sup_{t\in[0,T]}(\|\nabla_{N-1}u_N(t)\|_{L^{\frac{N}{2\alpha}}}+\|\partial_N u(t)\|_{L^{\frac{N}{2\alpha}}})-c(p,\alpha))\|\partial_N\theta\|^p_{L^{\frac{pN}{N-2\alpha}}}.$$

Hence, for

$$\sup_{t\in[0,T]}\|\nabla_{N-1}u_N(t)\|_{L^{\frac{N}{2\alpha}}}+\|\partial_N u(t)\|_{L^{\frac{N}{2\alpha}}}$$

sufficiently small, for any r that satisfies (1.4b), we have

$$\int_0^T\|\partial_N\theta\|_{L^p}^r ds\le\|\partial_N\theta_0\|_{L^p}^r T<\infty.$$

By Theorem 1.2, this completes the proof of Theorem 1.3.

4. Proof of Theorem 1.4

We fix (p,r) that satisfies (1.6b) and define $q=\frac{2p\alpha}{N}$ for which one may show $q>2$. Thus, applying ∂_N on (1.1), multiplying by $q|\partial_N\theta|^{q-2}\partial_N\theta$, integrating in space, and using Lemma 2.2 and the homogeneous Sobolev embedding again, we obtain

$$\begin{aligned}
&\partial_t\|\partial_N\theta\|_{L^q}^q+c(q,\alpha)\|\partial_N\theta\|^q_{L^{\frac{qN}{N-2\alpha}}}\\
&\le -q\sum_{i=1}^N\int\partial_N u_i\partial_i\theta|\partial_N\theta|^{q-2}\partial_N\theta\\
&=q(q-1)\sum_{i=1}^N\int\partial_N u_i\theta|\partial_N\theta|^{q-2}\partial^2_{Ni}\theta\\
&=-q(q-1)\sum_{i=1}^N\int\theta|\partial_N\theta|^q\left(\frac{\partial^2_{Ni}u_i\partial_N\theta-\partial_N u_i\partial^2_{Ni}\theta}{|\partial_N\theta|^2}\right)\\
&=-q(q-1)\int\theta|\partial_N\theta|^q\text{div}\left(\frac{\partial_N u}{\partial_N\theta}\right)\\
&\le q(q-1)\|\theta\|_{L^\infty}\|\partial_N\theta\|_{L^{\frac{Nq}{N-2\alpha}}}\|\partial_N\theta\|_{L^q}^{q-1}\left\|\text{div}\left(\frac{\partial_N u}{\partial_N\theta}\right)\right\|_{L^{\frac{Nq}{2\alpha}}}\\
&\le\epsilon\|\partial_N\theta\|^q_{L^{\frac{Nq}{N-2\alpha}}}+c\|\theta_0\|_{L^\infty}^{\frac{q}{q-1}}\|\partial_N\theta\|_{L^q}^q\left\|\text{div}\left(\frac{\partial_N u}{\partial_N\theta}\right)\right\|_{L^{\frac{Nq}{2\alpha}}}^{\frac{q}{q-1}}
\end{aligned}$$

by the incompressibility of u and Hölder's and Young's inequalities. For $\epsilon>0$ sufficiently small, Gronwall's inequality implies

$$\sup_{t\in[0,T]}\|\partial_N\theta(t)\|_{L^{\frac{2p\alpha}{N}}}^{\frac{2p\alpha}{N}}+\int_0^T\|\partial_N\theta\|_{L^{\frac{2p\alpha}{N-2\alpha}}}^{\frac{2p\alpha}{N}}ds\le c\|\partial_N\theta_0\|_{L^{\frac{2p\alpha}{N}}}^{\frac{2p\alpha}{N}}e^{\int_0^T\|\text{div}\left(\frac{\partial_N u}{\partial_N\theta}\right)\|_{L^p}^{\frac{2p\alpha}{2p\alpha-N}}ds}<\infty.$$

Thus, in particular we have shown

$$\int_0^T\|\partial_N\theta\|_{L^{\frac{2p\alpha}{N-2\alpha}}}^{\frac{2p\alpha}{N}}ds<\infty,\qquad\frac{N(N-2\alpha)^2}{4\alpha^3}<p<\infty.$$

By Theorem 1.2, this completes the proof of the first claim of Theorem 1.4.

Next, we take p that satisfies (1.4b). We apply ∂_N on (1.1), multiply by $p|\partial_N\theta|^{p-2}\partial_N\theta$, integrate in space, and apply Lemma 2.2 and the homogeneous Sobolev embedding to estimate similarly as before

$$\partial_t\|\partial_N\theta\|_{L^p}^p+c(p,\alpha)\|\partial_N\theta\|^p_{L^{\frac{pN}{N-2\alpha}}}$$

$$\leq -p\sum_{i=1}^{N}\int \partial_N u_i \partial_i \theta |\partial_N\theta|^{p-2}\partial_N\theta$$
$$= -p(p-1)\int \theta|\partial_N\theta|^p \operatorname{div}\left(\frac{\partial_N u}{\partial_N\theta}\right)$$
$$\leq p(p-1)\|\theta\|_{L^\infty}\|\partial_N\theta\|^p_{L^{\frac{Np}{N-2\alpha}}}\left\|\operatorname{div}\left(\frac{\partial_N u}{\partial_N\theta}\right)\right\|_{L^{\frac{N}{2\alpha}}}$$

by the incompressibility of u and Hölder's inequality. Thus,

$$\partial_t\|\partial_N\theta\|^p_{L^p} \leq \left(p(p-1)\|\theta_0\|_{L^\infty}\sup_{t\in[0,T]}\left\|\operatorname{div}\left(\frac{\partial_N u}{\partial_N\theta}\right)(t)\right\|_{L^{\frac{N}{2\alpha}}} - c(p,\alpha)\right)\|\partial_N\theta\|^p_{L^{\frac{pN}{N-2\alpha}}}.$$

Hence, if

$$\sup_{t\in[0,T]}\left\|\operatorname{div}\left(\frac{\partial_N u}{\partial_N\theta}\right)(t)\right\|_{L^{\frac{N}{2\alpha}}} < \infty,$$

then there exists $c>0$ such that $\|\theta_0\|_{L^\infty} < c$ implies that for any r that satisfies (1.4b) we have

$$\int_0^T \|\partial_N\theta\|^r_{L^p}ds \leq \|\partial_N\theta_0\|^r_{L^p}T < \infty,$$

and hence, by Theorem 1.2, this completes the proof of Theorem 1.4.

Appendix A. proof of Theorem 1.1.

In this Appendix we sketch the proof of Theorem 1.1, which is an N-dimensional generalization of the result in [38] added by the endpoint case. We first prove the following proposition:

PROPOSITION A.1. *Let $\alpha\in(0,\frac{1}{2})$. If the solution $\theta(x,t)$ to* (1.1) *satisfied the hypothesis of Theorem 1.1, then*

$$\begin{cases} \sup_{t\in[0,T]}\|\nabla_{N-1}\theta(t)\|^{\frac{2p\alpha}{N}}_{L^{\frac{2p\alpha}{N}}} + \|\partial_N\theta(t)\|^{\frac{2p\alpha}{N}}_{L^{\frac{2p\alpha}{N}}} \\ \qquad + \int_0^T \|\nabla_{N-1}\theta\|^{\frac{2p\alpha}{N}}_{L^{\frac{2p\alpha}{N-2\alpha}}} + \|\partial_N\theta\|^{\frac{2p\alpha}{N}}_{L^{\frac{2p\alpha}{N-2\alpha}}} ds < \infty \qquad \text{if } p<\infty, \\ \\ \sup_{t\in[0,T]}\|\nabla_{N-1}\theta(t)\|^{\frac{N-2\alpha}{\alpha}}_{L^{\frac{N-2\alpha}{\alpha}}} + \|\partial_N\theta(t)\|^{\frac{N-2\alpha}{\alpha}}_{L^{\frac{N-2\alpha}{\alpha}}} \\ \qquad + \int_0^T \|\nabla_{N-1}\theta\|^{\frac{N-2\alpha}{\alpha}}_{L^{\frac{N}{\alpha}}} + \|\partial_N\theta\|^{\frac{N-2\alpha}{\alpha}}_{L^{\frac{N}{\alpha}}} ds < \infty \qquad \text{if } p=\infty. \end{cases}$$

Proof. Let us first consider the case $p<\infty$. We fix (p,r) that satisfies (1.3b) and define $q=\frac{2p\alpha}{N}$. Similarly as in the proof of Proposition 2.1, one can show that

$$\partial_t\|\partial_N\theta\|^q_{L^q} + c(q,\alpha)\|\partial_N\theta\|^q_{L^{\frac{qN}{N-2\alpha}}} \leq \epsilon\|\partial_N\theta\|^q_{L^{\frac{qN}{N-2\alpha}}} + c\|\nabla_{N-1}\theta\|^{\frac{q}{q-1}}_{L^{\frac{qN}{2\alpha}}}\|\partial_N\theta\|^q_{L^q} \qquad \text{(A.1)}$$

and

$$\partial_t\|\nabla_{N-1}\theta\|^q_{L^q} + c(q,\alpha)\|\nabla_{N-1}\theta\|^q_{L^{\frac{qN}{N-2\alpha}}}$$

$$\leq \epsilon \|\nabla_{N-1}\theta\|^q_{L^{\frac{qN}{N-2\alpha}}} + c\|\nabla_{N-1}\theta\|^{\frac{q}{q-1}}_{L^{\frac{qN}{2\alpha}}}(\|\nabla_{N-1}\theta\|^q_{L^q} + \|\partial_N\theta\|^q_{L^q}). \tag{A.2}$$

Summing (A.1) and (A.2), taking $\epsilon > 0$ sufficiently small, and applying Gronwall's inequality completes the proof in case $p < \infty$.

Next, let us consider the case $p = \infty$. We recall the inhomogeneous Besov space

$$B^m_{p,r} := \{f \in \mathcal{S}' : \|f\|_{B^m_{p,r}} < \infty\}$$

where

$$\|f\|_{B^m_{p,r}} := \begin{cases} \left(\sum_{j=-1}^{\infty}(2^{jm}\|\Delta_j f\|_{L^p})^r\right)^{\frac{1}{r}} & \text{if } r < \infty, \\ \sup_{-1 \leq j < \infty} 2^{jm}\|\Delta_j f\|_{L^p} & \text{if } r = \infty, \end{cases}$$

$$\Delta_j f = \begin{cases} 0 & \text{if } j \leq -2, \\ \Psi * f & \text{if } j = -1, \\ \Phi_j * f & \text{if } j = 0,1,2,\ldots, \end{cases}$$

so that

$$\text{supp}\hat{\Phi}_j \subset A_j = \{\xi \in \mathbb{R}^N : 2^{j-1} < |\xi| < 2^{j+1}\}, \quad \Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f.$$

(For further discussion on Besov space, we refer the reader to [9].) Now let $q = \frac{N-2\alpha}{\alpha} \geq 2$, as verified in the proof of Proposition 2.1. We apply ∂_N to (1.1), multiply by $q|\partial_N\theta|^{q-2}\partial_N\theta$, integrate in space, and use Lemma 2.2 to obtain

$$\begin{aligned} &\partial_t\|\partial_N\theta\|^q_{L^q} + c(q,\alpha)\|\partial_N\theta\|^q_{L^{\frac{qN}{N-2\alpha}}} \\ &\leq q\sum_{i,j=1}^{N-1}(\|\partial_N u_i\|_{L^q}\|\partial_i\theta\|_{L^\infty}\|\partial_N\theta\|^{q-1}_{L^q} + \|\partial_j u_j\|_{L^\infty}\|\partial_N\theta\|^q_{L^q}) \\ &\leq c(\|\partial_N\theta\|^q_{L^q}\|\nabla_{N-1}\theta\|_{B^0_{\infty,1}} + \|\nabla_{N-1}u\|_{B^0_{\infty,1}}\|\partial_N\theta\|^q_{L^q}) \\ &\leq c(1 + \|\nabla_{N-1}\theta\|_{B^0_{\infty,1}})\|\partial_N\theta\|^q_{L^q} \end{aligned} \tag{A.3}$$

by Hölder's inequalities and $B^0_{\infty,1} \subset L^\infty$ as well as

$$\|\nabla_{N-1}u\|_{B^0_{\infty,1}} \leq \|\nabla_{n-1}\Psi\|_{L^2}\|u\|_{L^2} + \sum_{j=0}^{\infty}\|\Delta_j\nabla_{N-1}u\|_{L^\infty} \leq c(\|\theta_0\|_{L^2} + \|\nabla_{N-1}\theta\|_{B^0_{\infty,1}}).$$

Similarly, one can show

$$\begin{aligned} &\partial_t\|\nabla_{N-1}\theta\|^q_{L^q} + c(q,\alpha)\|\nabla_{N-1}\theta\|^q_{L^{\frac{qN}{N-2\alpha}}} \\ &\leq c(1 + \|\nabla_{N-1}\theta\|_{B^0_{\infty,1}})(\|\nabla_{N-1}\theta\|^q_{L^q} + \|\partial_N\theta\|^q_{L^q}). \end{aligned} \tag{A.4}$$

Summing (A.3) and (A.4) and applying Gronwall's inequality complete the proof of Proposition A.1. ∎

Identically to how the implication of Proposition 2.1 led to the H^s bound, we know that we have the H^s bound as a consequence of Proposition A.1. This completes the proof of Theorem 1.1.

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