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VANISHING VISCOSITY LIMIT TO RAREFACTION WAVE WITH VACUUM FOR 1-D COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH DENSITY-DEPENDENT VISCOSITY*

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Abstract. The vanishing viscosity limit of the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity $\epsilon(\rho)=\epsilon\rho^{\alpha}$ $(\alpha>0)$ is considered in the present paper. It is proven that given a rarefaction wave with one-side vacuum state to the compressible Euler equations, we can construct a sequence of solutions to the compressible Navier-Stokes equations which converge to the above rarefaction wave with vacuum as the viscosity tends to zero. Moreover, the convergence rate depending on α is obtained for all $\alpha>0$. The main difficulty in our proof lies in the degeneracies of the density and the density-dependent viscosity at the vacuum region in the vanishing viscosity limit.

Key words. Compressible Navier-Stokes equations, vanishing viscosity limit, density-dependent viscosity, rarefaction wave, vacuum.

AMS subject classifications. 35L60, 35L65, 76N15.

1. Introduction and main result

Consider the one-dimensional compressible Navier-Stokes equations in Euler coordinates:

$$\begin{cases} \rho_t+(\rho u)_x=0, & x\in\mathbf{R},\ t>0,\\ (\rho u)_t+\big(\rho u^2+p(\rho)\big)_x=(\epsilon(\rho)u_x)_x, \end{cases} \tag{1.1}$$

where the functions $\rho(t,x)\geq 0$, $u(t,x)$, and p represent the density, the velocity, and the pressure of the gas, respectively. Here we consider a $\gamma-$law with the pressure p and the viscosity given by:

$$p(\rho)=\frac{\rho^{\gamma}}{\gamma},\ \epsilon(\rho)=\epsilon\rho^{\alpha},$$

where $\gamma>1$ is the gas constant, and $\epsilon>0$, $\alpha>0$.

The asymptotic behavior of viscous flows, as the viscosity vanishes, is one of the important topics in the theory of compressible flows. It is expected that a general weak entropy solution to the Euler equations should be a (strong) limit of solutions to the corresponding Navier-Stokes equations with the same initial data as the viscosity tends to zero. However, the inviscid compressible flow contains singularities such as shock, contact discontinuity and the vacuum in general, which make it difficult to justify the vanishing viscosity limit to the Euler equations with basic wave patterns and the vacuum state.

The vanishing viscosity limit for the Cauchy problem has been studied by several researchers. For the system of the hyperbolic conservation laws with artificial viscosity

$$u_t+f(u)_x=\varepsilon u_{xx},$$

Goodman-Xin [5] first verified the viscous limit for piecewise smooth solutions separated by non-interacting shock waves using a matched asymptotic expansion method. Later

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Yu [29] proved it for the corresponding hyperbolic conservation laws with both shock and initial layers. In 2005, for general small BV initial data, Bianchini and Bressan [2] established the vanishing artificial viscosity limit via direct BV estimates with small oscillation and with general small BV initial data.

For the isentropic compressible Navier-Stokes equations with constant viscosity, Hoff-Liu [6] first proved the vanishing viscosity limit for piecewise constant shock even with initial layer. Later Xin [27] obtained the zero dissipation limit for rarefaction waves without vacuum for both rarefaction wave data and well-prepared smooth data. Then Wang [25] generalized the result of Goodman-Xin [5] to the isentropic Navier-Stokes equations.

For the full Navier-Stokes equations with constant viscosity, the results on the limits to the full Euler equations with basic wave patterns without vacuum, we refer to Wang [26] for the shock wave, Ma [20] for the contact discontinuity, Jiang-Ni-Sun [13] and Xin-Zeng [28] for the rarefaction wave, Huang-Wang-Yang [11] and Huang-Jiang-Wang [9] for the superposition of two rarefaction waves and a contact discontinuity, Huang-Wang-Yang [12] for the composite of one shock and one rarefaction wave, and Zhang-Pan-Wang-Tan [30] for the superposition of two shock waves with the initial layer. Recently, Huang-Wang-Wang-Yang [10] succeeded in justifying the vanishing viscosity limit of compressible Navier-Stokes equations in the setting of Riemann solutions for the superposition of shock wave, rarefaction wave, and contact discontinuity.

It is important and difficult to study the vacuum states in gas dynamics; the degeneracies and certain singularities caused by the vacuum states may yield some essential analytical difficulties. For example, the velocity can not even be defined in the vacuum region. In the setting of Riemann solutions, as pointed out by Liu-Smoller [18], among the two elementary hyperbolic waves, i.e., shocks and rarefaction waves to the one-dimensional isentropic compressible Euler equation (1.3), only the rarefaction wave can be connected to the vacuum states. We refer to Jiu-Wang-Xin [14] and Perepelitsa [22] for the time-asymptotic behavior toward rarefaction wave with vacuum for solutions to 1-d isentropic compressible Navier-Stokes equations. The results on zero dissipation limit of one-dimensional compressible Navier-Stokes equations with constant viscosity to the rarefaction wave with one-side vacuum state to the corresponding compressible Euler equations, more recently, Huang-Li-Wang [7] justified the isentropic case and Li-Wang [16] considered the non-isentropic case. Later, Liang [17] considered the density-dependent viscosity case, under a condition on the coefficient α and the gas constant γ:

$$1<\gamma\le 2,\quad \alpha\le\gamma,\quad \frac{3}{5}\le\alpha\le\frac{3}{2}. \tag{1.2}$$

We are now ready to formulate the problem. For the compressible Navier-Stokes equation (1.1), formally, as ϵ tends to zero, the limit system consists of the following inviscid Euler equations:

$$\begin{cases} \rho_t+(\rho u)_x=0,\\ (\rho u)_t+(\rho u^2+p(\rho))_x=0. \end{cases} \tag{1.3}$$

The Euler system (1.3) is a strictly hyperbolic one for $\rho>0$ whose characteristic fields are both genuinely nonlinear. Note that the eigenvalues of the Jacobi matrix of the Euler system (1.3) are

$$\lambda_1(\rho,u)=u-\sqrt{p'(\rho)},\qquad \lambda_2(\rho,u)=u+\sqrt{p'(\rho)},$$

with corresponding right eigenvectors

$$r_i(\rho,u)=(1,(-1)^i\frac{\sqrt{p'(\rho)}}{\rho})^t, \qquad i=1,2,$$

such that

$$r_i(\rho,u)\cdot\nabla_{\rho,u}\lambda_i(\rho,u)=(-1)^i\frac{\rho p''(\rho)+2p'(\rho)}{2\rho\sqrt{p'(\rho)}}\neq 0, \quad i=1,2.$$

The $i-$Riemann invariant $(i=1,2)$ can be defined by

$$\Sigma_i(\rho,u)=u+(-1)^{i+1}\int^{\rho}\frac{\sqrt{p'(s)}}{s}ds,$$

such that

$$\nabla_{(\rho,u)}\Sigma_i(\rho,u)\cdot r_i(\rho,u)\equiv 0, \qquad \forall\rho>0, \quad u.$$

The aim of the present paper is to remove the condition (1.2), that is, we want to justify the vanishing viscosity limit for any $\gamma>1$, $\alpha>0$. Given a rarefaction wave with one-side vacuum state to the isentropic Euler equations, we can construct a sequence of solutions to the isentropic Navier-Stokes equations which converge to the above rarefaction wave with vacuum as the viscosity tends to zero, and the uniform convergence rate is obtained for $\alpha\in(0,+\infty)$. We now give some comments on the analysis of this paper. Firstly, we construct a 2-rarefaction wave with one-side connected by vacuum state as in [7]: Given the Riemann initial data

$$\begin{cases}\rho(0,x)=0, & x<0,\\ (\rho,u)(0,x)=(\rho_+,u_+), & x>0,\end{cases} \tag{1.4}$$

where $\rho_+>0$, u_+ are given constants on the right state and the vacuum state on the left side, then the Riemann problem (1.3), (1.4) admits a $2-$rarefaction wave connected to the vacuum on the left side. From the fact that the $2-$Riemann invariant $\Sigma_2(\rho,u)$ is constant in (x,t) along the $2-$rarefaction wave curve, the velocity u_- can be computed by $u_-=\Sigma_2(\rho_+,u_+)$. Now we get a $2-$rarefaction wave $(\rho^{r_2},u^{r_2})(\xi)$, $(\xi=\frac{x}{t})$ of (1.3) connecting the vacuum state $\rho=0$ to (ρ_+,u_+), which can be expressed by

$$\lambda_2(\rho^{r_2}(\xi),u^{r_2}(\xi))=\begin{cases}\rho^{r_2}(\xi)\equiv 0, & \text{if } \xi<\lambda_2(0,u_-)=u_-,\\ \xi, & \text{if } u_-\leq\xi\leq\lambda_2(\rho_+,u_+),\\ \lambda_2(\rho_+,u_+), & \text{if } \xi>\lambda_2(\rho_+,u_+),\end{cases} \tag{1.5}$$

and

$$\Sigma_2(\rho^{r_2}(\xi),u^{r_2}(\xi))=\Sigma_2(0,u_-)=\Sigma_2(\rho_+,u_+). \tag{1.6}$$

Thus the momentum of 2-rarefaction wave can be defined by

$$m^{r_2}(\xi)=\begin{cases}\rho^{r_2}(\xi)u^{r_2}(\xi), & \text{if } \rho^{r_2}>0,\\ 0, & \text{if } \rho^{r_2}=0.\end{cases} \tag{1.7}$$

The vacuum state often yields degeneracies and certain singularities in the physical systems, which causes some essential analytical difficulties. Hence, to overcome the degeneracies caused by the vacuum in the rarefaction wave, we first cut off the 2-rarefaction wave with vacuum along the rarefaction wave curve suitably as in [7] (more details can be seen in Section 2). Compared with the previous work [7], for the constant viscosity case, the density-dependent viscosity term, such as (3.29), becomes subtle to deal with. In fact, we observe that the derivative estimates of the perturbation of the density may depend on the second order derivative estimates of velocity with some degenerate coefficients (see (3.42)), which is quite different from the constant viscosity case in [7] and [16]. Therefore, we succeed in removing the condition (1.2). By choosing the convergence rate a suitably as in (1.9) and then the parameters μ, δ as in (3.12), we close the a priori estimates and yield the desired result.

Now we state our main result as follows:

THEOREM 1.1. *Let* $(\rho^{r_2}, m^{r_2})(x/t)$ *be the 2-rarefaction wave with one-side vacuum state defined by* (1.5)-(1.7). *Then there exists a small positive constant* ϵ_0 *such that if* $\epsilon \in (0,\epsilon_0)$, *the compressible Navier-Stokes equation (1.1) admits a family of smooth solutions* $(\rho^\epsilon, m^\epsilon = \rho^\epsilon u^\epsilon)(x,t)$ *satisfying*

$$\begin{aligned}(\rho^\epsilon - \rho^{r_2}, m^\epsilon - m^{r_2}), (\rho^\epsilon, m^\epsilon)_x &\in C^0((0,+\infty); L^2(\mathbf{R})),\\ u^\epsilon_{xx} &\in L^2(0,+\infty; L^2(\mathbf{R})).\end{aligned}$$

Furthermore, when $\epsilon \to 0$, *for any given positive constant* $h > 0$, *there exists a constant* $C_h > 0$, *independent of* ϵ, *such that*

$$\begin{aligned}\sup_{t\geq h}\left\|\rho^\epsilon(\cdot,t) - \rho^{r_2}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty} &\leq C_h \epsilon^a |\ln\epsilon|,\\ \sup_{t\geq h}\left\|m^\epsilon(\cdot,t) - m^{r_2}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty} &\leq C_h \epsilon^a |\ln\epsilon|\end{aligned} \tag{1.8}$$

with the positive constant a *given by*

$$a = \frac{1}{18\alpha + 10\gamma + 12}. \tag{1.9}$$

The rest of the paper is organized as follows. In Section 2, a smooth 2-rarefaction wave is constructed based on the inviscid Burgers equation to approximate the cut-off rarefaction wave for the Euler equations. In Section 3, we will finish the proof of Theorem 1.1.

Throughout this paper, we let C denote generic positive constants which are independent of time t or τ and viscosity ϵ unless they need to be distinguished. For the function spaces, $H^l(\mathbf{R})$, $l = 0,1,2,...$, denotes the l-th order Sobolev space on the whole space $\mathbf{R}$ with its norm

$$\|f\|_l = \left(\sum_{j=0}^{l} \|\partial_y^j f\|^2\right)^{\frac{1}{2}}, \quad \text{and } \|\cdot\| := \|\cdot\|_{L^2(dz)},$$

while $L^2(dz)$ means the L^2 integral over $\mathbf{R}$ with respect to the Lebesgue measure dz, and $z = x$ or y.

2. Rarefaction waves

Since there is no exact rarefaction wave profile for the Navier-Stokes equation (1.1), the following approximate rarefaction wave profile satisfying the Euler equations was motivated by Matsumura-Nishihara [21] and Xin [27].

Consider the Riemann problem for the inviscid Burgers equation:

$$\begin{cases} w_t + w w_x = 0, \\ w(x,0) = \begin{cases} w_-, & x<0, \\ w_+, & x>0. \end{cases} \end{cases} \tag{2.1}$$

If $w_- < w_+$, then the Riemann problem (2.1) admits a rarefaction wave solution $w^r(x,t) = w^r(\frac{x}{t})$ given by

$$w^r\left(\frac{x}{t}\right) = \begin{cases} w_-, & \frac{x}{t} \le w_-, \\ \frac{x}{t}, & w_- \le \frac{x}{t} \le w_+, \\ w_+, & \frac{x}{t} \ge w_+. \end{cases} \tag{2.2}$$

As in [21] and [27], we can construct the approximate rarefaction wave to the compressible Navier-Stokes equation (1.1) by the solution of the Burgers equation

$$\begin{cases} w_t + w w_x = 0, \\ w(0,x) = w_\delta(x) = w\left(\frac{x}{\delta}\right) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} \tanh\frac{x}{\delta}, \end{cases} \tag{2.3}$$

where $\delta > 0$ is a small parameter to be determined and the hyperbolic tangent function is $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. In fact, we choose $\delta = \epsilon^a$ in (3.12) with a given by (1.9) in the following. Note that the solution $w_\delta^r(t,x)$ of the problem (2.3) is given by

$$w_\delta^r(t,x) = w_\delta(x_0(t,x)), \qquad x = x_0(t,x) + w_\delta(x_0(t,x))t. \tag{2.4}$$

Also note that $w_\delta^r(t,x)$ has the following properties:

Lemma 2.1. *([27, 7]) The problem* (2.3) *has a unique smooth global solution $w_\delta^r(x,t)$ for each $\delta > 0$ such that*

(1) $w_- < w_\delta^r(x,t) < w_+$, $\partial_x w_\delta^r(x,t) > 0$, for $x \in \mathbf{R}$, $t \ge 0$, $\delta > 0$.

(2) The following estimates hold for all $t > 0$, $\delta > 0$, and $p \in [1,\infty]$:

$$\|\partial_x w_\delta^r(\cdot,t)\|_{L^p} \le C(w_+ - w_-)^{1/p}(\delta + t)^{-1+1/p}, \tag{2.5}$$

$$\|\partial_x^2 w_\delta^r(\cdot,t)\|_{L^p} \le C(\delta + t)^{-1}\delta^{-1+1/p}, \tag{2.6}$$

$$\left|\frac{\partial^2 w_\delta^r(x,t)}{\partial x^2}\right| \le \frac{4}{\delta}\frac{\partial w_\delta^r(x,t)}{\partial x}. \tag{2.7}$$

(3) There exists a constant $\delta_0 \in (0,1)$ such that for $\delta \in (0,\delta_0]$, $t > 0$,

$$\left\| w_\delta^r(\cdot,t) - w^r\left(\frac{\cdot}{t}\right)\right\|_{L^\infty} \le C\delta t^{-1}\left[\ln(1+t) + |\ln\delta|\right].$$

Note that Lemma 2.1 appears in [7], but is a little different from one in [27]. For the detailed proof of Lemma 2.1, one can refer to [27] and [7] and we omit it here for brevity.

Following [7], we will cut off the 2-rarefaction wave with vacuum along the wave curve in order to overcome the difficulty caused by the vacuum. More precisely, for any

$\mu>0$ to be determined, a state $(\rho,u)=(\mu,u_\mu)$ belonging to the 2-rarefaction wave curve can be obtained. By the fact that along the 2-rarefaction wave curve, the 2-Riemann invariant $\Sigma_2(\rho,u)$ is constant in (x,t), we can compute directly the velocity

$$u_\mu=\Sigma_2(\rho_+,u_+)+\frac{2}{\gamma-1}\mu^{\frac{\gamma-1}{2}}.$$

Now this new 2-rarefaction wave $(\rho_\mu^{r_2},u_\mu^{r_2})(\xi)$, $(\xi=x/t)$ connecting the state (μ,u_μ) to the state (ρ_+,u_+) can be defined by

$$\lambda_2(\rho_\mu^{r_2},u_\mu^{r_2})(\xi)=\begin{cases}\lambda_2(\mu,u_\mu), & \xi<\lambda_2(\mu,u_\mu),\\ \xi, & \lambda_2(\mu,u_\mu)\le\xi\le\lambda_2(\rho_+,u_+),\\ \lambda_2(\rho_+,u_+), & \xi>\lambda_2(\rho_+,u_+)\end{cases} \tag{2.8}$$

and

$$\Sigma_2(\rho_\mu^{r_2},u_\mu^{r_2})=\Sigma_2(\mu,u_\mu)=\Sigma_2(\rho_+,u_+). \tag{2.9}$$

Correspondingly, we can define the momentum function $m_\mu^{r_2}=\rho_\mu^{r_2}u_\mu^{r_2}$. It is easy to show that the cut-off 2-rarefaction wave $(\rho_\mu^{r_2},m_\mu^{r_2})(x/t)$ converges to the original 2-rarefaction wave with vacuum $(\rho^{r_2},m^{r_2})(x/t)$ in sup-norm with the convergence rate μ as μ tends to zero. More precisely, we have

Lemma 2.2. *There exists a constant $\mu_0\in(0,1)$ such that for $\mu\in(0,\mu_0]$, $t>0$,*

$$\|(\rho_\mu^{r_2},m_\mu^{r_2})(\cdot/t)-(\rho^{r_2},m^{r_2})(\cdot/t)\|_{L^\infty}\le C\mu,$$

where the positive constant C is independent of μ.

The proof of Lemma 2.2 can be obtained directly from the explicit solution formula of a rarefaction wave, so we omit it for brevity.

Now the approximate rarefaction wave $(\bar\rho_{\mu,\delta},\bar u_{\mu,\delta})(x,t)$ of the cut-off 2-rarefaction wave $(\rho_\mu^{r_2},u_\mu^{r_2})(\frac{x}{t})$ to the compressible Euler equation (1.3) can be defined by

$$\begin{cases} w_+=\lambda_2(\rho_+,u_+), \quad w_-=\lambda_2(\mu,u_\mu),\\ w_\delta^r(t,x)=\lambda_2(\bar\rho_{\mu,\delta},\bar u_{\mu,\delta})(t,x),\\ \Sigma_2(\bar\rho_{\mu,\delta},\bar u_{\mu,\delta})(x,t)=\Sigma_2(\rho_+,u_+)=\Sigma_2(\mu,u_\mu),\end{cases} \tag{2.10}$$

where w_δ^r is the solution of Burger's equation (2.3) defined in (2.4). From now on, the subscription of $(\bar\rho_{\delta,\mu},\bar u_{\delta,\mu})(x,t)$ will be omitted as $(\bar\rho,\bar u)(x,t)$ for simplicity. Then the approximate cut-off 2-rarefaction wave $(\bar\rho,\bar u)$ defined above satisfies

$$\begin{cases}\bar\rho_t+(\bar\rho\bar u)_x=0,\\ (\bar\rho\bar u)_t+\big(\bar\rho\bar u^2+p(\bar\rho)\big)_x=0,\end{cases} \tag{2.11}$$

and the properties of the approximate rarefaction wave $(\bar\rho,\bar u)$ are listed without proof in the following lemma.

Lemma 2.3. *The approximate cut-off 2-rarefaction wave $(\bar\rho,\bar u)$ defined in* (2.10) *satisfies the following properties:*

(i) $\bar u_x(x,t)=\frac{2}{\gamma+1}(w_\delta^r)_x>0$, *for* $x\in\mathbf{R}$, $t\ge0$;
$\bar\rho_x=\bar\rho^{\frac{3-\gamma}{2}}\bar u_x$, *and* $\bar\rho_{xx}=\bar\rho^{\frac{3-\gamma}{2}}\bar u_{xx}+\frac{3-\gamma}{2}\bar\rho^{2-\gamma}(\bar u_x)^2$.

(ii) The following estimates hold for all $t>0$, $\delta>0$, *and* $p\in[1,\infty]$:

$$\|\bar{u}_x(\cdot,t)\|_{L^p}\le C(w_+-w_-)^{1/p}(\delta+t)^{-1+1/p},$$

$$\|\bar{u}_{xx}(\cdot,t)\|_{L^p}\le C(\delta+t)^{-1}\delta^{-1+1/p}.$$

(iii) There exists a constant $\delta_0\in(0,1)$ *such that for* $\delta\in(0,\delta_0]$, $t>0$,

$$\|(\bar{\rho}-\rho_\mu^{r_2},\bar{u}-u_\mu^{r_2})(\cdot,t)\|_{L^\infty}\le C\delta t^{-1}\big[\ln(1+t)+|\ln\delta|\big].$$

The proof of Lemma 2.3 can be got similarly as in [7] and will be omitted for brevity.

3. Proof of Theorem 1.1

To prove Theorem 1.1, the solution $(\rho^\epsilon,u^\epsilon)$ is constructed as the perturbation around the approximate rarefaction wave $(\bar{\rho},\bar{u})$. Consider the Cauchy problem for (1.1) with smooth initial data

$$(\rho^\epsilon,u^\epsilon)(x,t=0)=(\bar{\rho},\bar{u})(x,0). \tag{3.1}$$

Then we introduce the perturbation

$$(\phi,\psi)(y,\tau)=(\rho^\epsilon,u^\epsilon)(x,t)-(\bar{\rho},\bar{u})(x,t), \tag{3.2}$$

where y,τ are the scaled variables

$$y=\frac{x}{\epsilon},\quad \tau=\frac{t}{\epsilon}, \tag{3.3}$$

and $(\rho^\epsilon,u^\epsilon)$ is assumed to be the solution to the problem (1.1). For the simplicity of the notation, the superscript of $(\rho^\epsilon,u^\epsilon)$ will be omitted as (ρ,u) from now on if there is no confusion of the notation. Substituting (3.2) and (3.3) into (1.1) and using the definition for $(\bar{\rho},\bar{u})$, we obtain

$$\phi_\tau+\rho\psi_y+u\phi_y+f=0, \tag{3.4}$$

$$\rho\psi_\tau+\rho u\psi_y+p'(\rho)\phi_y-\bar{\rho}^\alpha\psi_{yy}+g=(\bar{\rho}^\alpha)_y u_y+\big((\rho^\alpha-\bar{\rho}^\alpha)u_y\big)_y, \tag{3.5}$$

$$(\phi,\psi)(y,0)=0, \tag{3.6}$$

where

$$\begin{cases} f=\bar{u}_y\phi+\bar{\rho}_y\psi,\\ g=\rho\psi\bar{u}_y+\bar{\rho}_y\left[p'(\rho)-\dfrac{\rho}{\bar{\rho}}p'(\bar{\rho})\right]-\bar{\rho}^\alpha\bar{u}_{yy}. \end{cases} \tag{3.7}$$

We seek a global-in-time solution (ϕ,ψ) to the reformulated problem (3.4) – (3.6). To this end, the solution space for (3.4) – (3.6) is defined by

$$\chi(0,\tau_1(\epsilon))=\Big\{(\phi,\psi)\Big|(\phi,\psi)\in C([0,\tau_1(\epsilon)];H^1(\mathbf{R})),\quad \phi_y\in L^2(0,\tau_1(\epsilon);L^2(\mathbf{R})),\\ \psi_y\in L^2(0,\tau_1(\epsilon);H^1(\mathbf{R}))\Big\},$$

with $0<\tau_1(\epsilon)\leq+\infty$.

THEOREM 3.1. *There exist positive constants ϵ_1 and C independent of ϵ, such that if $0<\epsilon\leq\epsilon_1$, then the problem* (3.4)-(3.6) *admits a unique global-in-time solution $(\phi,\psi)\in\chi(0,+\infty)$ satisfying*

$$\sup_{\tau\in[0,+\infty)}\int_{\mathbf{R}}\left(\bar\rho\psi^2+\bar\rho^{\gamma-2}\phi^2\right)(\tau,y)dy+\int_0^{+\infty}\int_{\mathbf{R}}\left[\bar\rho^{\alpha}\psi_y^2+\bar u_y(\bar\rho^{\gamma-2}\phi^2+\bar\rho\psi^2)\right]dyd\tau\leq\epsilon^{\frac{1}{3}},\tag{3.8}$$

$$\sup_{\tau\in[0,+\infty)}\int_{\mathbf{R}}\bar\rho^{2\alpha-3}\phi_y^2(\tau,y)dy+\int_0^{+\infty}\int_{\mathbf{R}}\bar\rho^{\gamma-3+\alpha}\phi_y^2dyd\tau\leq C\epsilon^{\frac{1}{3}-(4\alpha+2\gamma)a}|\ln\epsilon|^{1-4\alpha-2\gamma},\tag{3.9}$$

and

$$\sup_{\tau\in[0,+\infty)}\int_{\mathbf{R}}\psi_y^2(\tau,y)dy+\int_0^{+\infty}\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau\leq C\epsilon^{\frac{1}{3}-(6\alpha+2\gamma)a}|\ln\epsilon|^{1-6\alpha-2\gamma},\tag{3.10}$$

with a is given by (1.9).

In what follows, the analysis is always carried out under the a priori assumptions

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}\|(\phi,\psi)(\cdot,\tau)\|_{L^\infty}\leq\epsilon^a,\qquad\sup_{\tau\in[0,\tau_1(\epsilon)]}\|(\sqrt{\bar\rho^{2\alpha-3}}\phi_y,\psi_y)\|\leq1,\tag{3.11}$$

with a is given by (1.9), where $[0,\tau_1(\epsilon)]$ is the time interval in which the solution exists (which may depend on ϵ). Take

$$\mu=\epsilon^a|\ln\epsilon|,\qquad\delta=\epsilon^a\tag{3.12}$$

in the sequel. Then it follows that $\mu\geq2\epsilon^a$ if $\epsilon\ll1$. Under the a priori assumptions (3.11), we can get

$$\frac{\bar\rho}{2}\leq\rho\leq\frac{3\bar\rho}{2}.\tag{3.13}$$

In fact, if $\epsilon\ll1$, then one has

$$\rho=\bar\rho+\phi\geq\bar\rho-\|\phi\|_{L^\infty}\geq\bar\rho-\epsilon^a\geq\bar\rho-\frac{1}{2}\mu\geq\frac{\bar\rho}{2},\tag{3.14}$$

$$\rho=\bar\rho+\phi\leq\bar\rho+\|\phi\|_{L^\infty}\leq\bar\rho+\epsilon^a\leq\bar\rho+\frac{1}{2}\mu\leq\frac{3\bar\rho}{2}.\tag{3.15}$$

Moreover, under the a priori assumptions (3.11), it holds that

$$C_1\bar\rho^{\gamma-2}\phi^2\leq p(\rho)-p(\bar\rho)-p'(\bar\rho)\phi\leq C_2\bar\rho^{\gamma-2}\phi^2,\tag{3.16}$$

where C_1,C_2 are positive constants independent of ϵ.

Since the proof for the local existence of the solution to (3.4)-(3.6) is standard, we omit it for brevity. Note that in order to get the convergence rate of the local solution with respect to ϵ as in (3.11), if we denote the local existence time interval by $[0,\tau_0]$, then τ_0 may depend on ϵ, that is, $\tau_0=\tau_0(\epsilon)$. The next step for the proof of Theorem 3.1 is to extend the local solution to the global solution in $[0,\infty)$ for small but fixed viscosity coefficient ϵ. To do so, it is sufficient to show the following a priori estimates for fixed ϵ with $0<\epsilon\ll 1$.

LEMMA 3.2 (A priori estimates). *Let* $(\phi,\psi)\in\chi(0,\tau_1(\epsilon))$ *be a solution to the problem* $(3.4)-(3.6)$, *where* $\tau_1(\epsilon)$ *is the maximum existence time of the solution satisfying a priori assumptions* (3.11). *Then there exists a positive constant* ϵ_2 *such that if* $0<\epsilon\le\epsilon_2$, *then*

$$\begin{aligned}&\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}}\left(\bar\rho\psi^2+\bar\rho^{\gamma-2}\phi^2\right)(\tau,y)dy\\&\quad+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\left[\bar\rho^\alpha\psi_y^2+\bar u_y(\bar\rho^{\gamma-2}\phi^2+\bar\rho\psi^2)\right]dyd\tau\le\epsilon^{\frac13},\end{aligned}\tag{3.17}$$

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}}\bar\rho^{2\alpha-3}\phi_y^2(\tau,y)dy+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\bar\rho^{\gamma-3+\alpha}\phi_y^2dyd\tau\le C\epsilon^{\frac13-(4\alpha+2\gamma)a}|\ln\epsilon|^{1-4\alpha-2\gamma},\tag{3.18}$$

and

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}}\psi_y^2(\tau,y)dy+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau\le C\epsilon^{\frac13-(6\alpha+2\gamma)a}|\ln\epsilon|^{1-6\alpha-2\gamma},\tag{3.19}$$

with a *is given by* (1.9), *and the constant* C *independent of* ϵ *and* $\tau_1(\epsilon)$.

Proof. The proof of Lemma 3.2 consists of the following steps.

Step 1. First, define

$$E:=\Phi(\rho,\bar\rho)+\frac{\psi^2}{2},$$

where

$$\Phi(\rho,\bar\rho):=\int_{\bar\rho}^{\rho}\frac{p(\xi)-p(\bar\rho)}{\xi^2}d\xi=\frac{1}{(\gamma-1)\rho}\big(p(\rho)-p(\bar\rho)-p'(\bar\rho)\phi\big).\tag{3.20}$$

Direct computations yield

$$\begin{aligned}&\big(\rho E\big)_\tau+\rho^\alpha\psi_y^2+\bar u_y\big(p(\rho)-p(\bar\rho)-p'(\bar\rho)\phi+\rho\psi^2\big)\\&=\bar\rho^\alpha\bar u_{yy}\psi+(\bar\rho^\alpha)_y\psi\bar u_y-(\rho^\alpha-\bar\rho^\alpha)\psi_y\bar u_y+(\cdots)_y.\end{aligned}$$

Hereafter $(\cdots)_y$ represents the term which vanishes after integration. Then integrating the above equation over $\mathbf{R}^1\times[0,\tau]$ and using (3.13), (3.16), and (3.20) imply

$$\begin{aligned}&\int_{\mathbf{R}}\left(\bar\rho\psi^2+\bar\rho^{\gamma-2}\phi^2\right)dy+\int_0^\tau\int_{\mathbf{R}}\left(\bar\rho^\alpha\psi_y^2+\bar\rho^{\gamma-2}\bar u_y\phi^2+\bar\rho\bar u_y\psi^2\right)dyd\tau\\&\le C\textstyle\int_0^\tau\int_{\mathbf{R}}|\bar\rho^\alpha\psi\bar u_{yy}|+|(\bar\rho^\alpha)_y\psi\bar u_y|+|(\rho^\alpha-\bar\rho^\alpha)\psi_y\bar u_y|dyd\tau:=\sum_{i=1}^3I_i.\end{aligned}\tag{3.21}$$

Now we estimate the terms on the right hand side of (3.21) one by one. By Sobolev's inequality and Lemma 2.3, one has

$$\begin{aligned}
I_1 &= C\int_0^\tau \int_{\mathbf{R}} |\bar{u}_{yy}\bar{\rho}^\alpha \psi| dy d\tau \le C\int_0^\tau \|\bar{u}_{yy}\|_{L^1}\|\psi\|^{1/2}\|\psi_y\|^{1/2} d\tau \\
&\le C\mu^{-\frac{\alpha}{4}}\int_0^\tau \frac{1}{\tau+\delta/\epsilon}\|\psi\|^{1/2}\|\sqrt{\bar{\rho}^\alpha}\psi_y\|^{1/2} d\tau \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^\alpha \psi_y^2 dy d\tau + C\mu^{-\frac{\alpha}{3}}\int_0^\tau \left(\frac{1}{\tau+\delta/\epsilon}\right)^{4/3}\|\psi\|^{2/3} d\tau \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^\alpha \psi_y^2 dy d\tau + C\mu^{-\frac{\alpha+1}{3}}\sup_{\tau\in[0,\tau_1(\epsilon)]}\|\sqrt{\bar{\rho}}\psi\|^{\frac{2}{3}}\int_0^\tau \left(\frac{1}{\tau+\delta/\epsilon}\right)^{4/3} d\tau \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^\alpha \psi_y^2 dy d\tau + \frac{1}{8}\sup_{\tau\in[0,\tau_1(\epsilon)]}\|\sqrt{\bar{\rho}}\psi\|^2 + C\mu^{-\frac{\alpha+1}{2}}\left(\int_0^\tau \left(\frac{1}{\tau+\delta/\epsilon}\right)^{4/3} d\tau\right)^{3/2} \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^\alpha \psi_y^2 dy d\tau + \frac{1}{8}\sup_{\tau\in[0,\tau_1(\epsilon)]}\|\sqrt{\bar{\rho}}\psi\|^2 + C\left(\frac{\epsilon}{\mu^{\alpha+1}\delta}\right)^{\frac{1}{2}} \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^\alpha \psi_y^2 dy d\tau + \frac{1}{8}\sup_{\tau\in[0,\tau_1(\epsilon)]}\|\sqrt{\bar{\rho}}\psi\|^2 + \epsilon^{\frac{1}{3}},
\end{aligned} \tag{3.22}$$

where we have used the fact that

$$C\left(\frac{\epsilon}{\mu^{\alpha+1}\delta}\right)^{\frac{1}{2}} = C\epsilon^{\frac{1-a(\alpha+2)}{2}}|\ln\epsilon|^{-\frac{\alpha+1}{2}} = C\epsilon^{\frac{17\alpha+10\gamma+10}{2(18\alpha+10\gamma+12)}}|\ln\epsilon|^{-\frac{\alpha+1}{2}} \le \epsilon^{\frac{1}{3}}, \quad \text{if} \quad \epsilon \ll 1.$$

From Lemma 2.3 (i), one has

$$\bar{\rho}_y = \bar{\rho}^{\frac{3-\gamma}{2}}\bar{u}_y, \tag{3.23}$$

and one can get

$$\begin{aligned}
I_2 &= C\int_0^\tau \int_{\mathbf{R}} |(\bar{\rho}^\alpha)_y \psi \bar{u}_y| dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{u}_y \bar{\rho}\psi^2 dy d\tau + C\mu^{-\gamma}\int_0^\tau \int_{\mathbf{R}} \bar{u}_y^3 dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{u}_y \bar{\rho}\psi^2 dy d\tau + \mu^{-\gamma}\int_0^\tau \left(\frac{1}{\tau+\delta/\epsilon}\right)^2 d\tau \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{u}_y \bar{\rho}\psi^2 dy d\tau + C\frac{\epsilon}{\mu^\gamma\delta}.
\end{aligned} \tag{3.24}$$

Similarly, it holds that,

$$\begin{aligned}
I_3 &= C\int_0^\tau \int_{\mathbf{R}} |(\rho^\alpha - \bar{\rho}^\alpha)\psi_y \bar{u}_y| dy d\tau \le C\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^{\frac{\alpha-\gamma}{2}}\bar{u}_y|\sqrt{\bar{\rho}^\alpha}\psi_y||\sqrt{\bar{\rho}^{\gamma-2}}\phi| dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^\alpha \psi_y^2 dy d\tau + C\frac{\epsilon}{\mu^\gamma\delta}\int_0^\tau \int_{\mathbf{R}} \bar{u}_y \bar{\rho}^{\gamma-2}\phi^2 dy d\tau \\
&\le \frac{1}{8}\int_0^\tau \int_{\mathbf{R}} \bar{\rho}^\alpha \psi_y^2 dy d\tau + \frac{1}{16}\int_0^\tau \int_{\mathbf{R}} \bar{u}_y \bar{\rho}^{\gamma-2}\phi^2 dy d\tau,
\end{aligned} \tag{3.25}$$

where in the last inequality we have used the fact that

$$C\frac{\epsilon}{\mu^\gamma\delta} = C\epsilon^{1-a(\gamma+1)}|\ln\epsilon|^{-\gamma} \le \frac{1}{16}, \quad \text{if} \quad \epsilon \ll 1.$$

Combining (3.21) and (3.22)-(3.25) yields

$$\begin{aligned}&\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}}\left(\bar\rho\psi^2+\bar\rho^{\gamma-2}\phi^2\right)dy\\&+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\left(\bar\rho^\alpha\psi_y^2+\bar\rho^{\gamma-2}\bar u_y\phi^2+\bar\rho\bar u_y\psi^2\right)dyd\tau\le\epsilon^{\frac13},\qquad\text{if}\quad\epsilon\ll1.\end{aligned}\tag{3.26}$$

Step 2. Next we derive the estimation of ϕ_y. Substituting (3.4) into (3.5) and recalling $(3.7)_1$ yields

$$\begin{aligned}&\rho\psi_\tau+\rho u\psi_y+p'(\rho)\phi_y+\frac{\bar\rho^\alpha}{\rho}(\phi_{y\tau}+u\phi_{yy}+\bar u_{yy}\phi+\bar\rho_{yy}\psi+\bar\rho_y\psi_y\\&\quad+u_y\phi_y+\bar u_y\phi_y+\rho_y\psi_y)=-g+(\bar\rho^\alpha)_yu_y+\left((\rho^\alpha-\bar\rho^\alpha)u_y\right)_y.\end{aligned}\tag{3.27}$$

Multiplying the above equation by $\bar\rho^\alpha\phi_y/\rho^2$ gives

$$\begin{aligned}&\left(\frac{\bar\rho^{2\alpha}}{2\rho^3}\phi_y^2+\bar\rho^\alpha\frac{\psi\phi_y}{\rho}\right)_\tau+\left(\bar\rho^{2\alpha}\frac{u\phi_y^2}{2\rho^3}-\bar\rho^\alpha\frac{\psi\phi_\tau}{\rho}\right)_y-\bar\rho^\alpha\psi_y^2+\bar\rho^\alpha p'(\rho)\frac{\phi_y^2}{\rho^2}\\&=\bar\rho^\alpha\left(-\bar\rho_y\frac{\psi^2}{\rho^2}(\phi_y+\bar\rho_y)+\bar u_y\frac{\psi_y\phi}{\rho}-\bar\rho_y\bar u_y\frac{\psi\phi}{\rho^2}+\bar\rho\bar u_y\frac{\psi\phi_y}{\rho^2}\right)\\&\quad-\bar\rho^{2\alpha}\frac{\phi_y}{\rho^3}(\bar u_{yy}\phi+\bar\rho_{yy}\psi+2\bar\rho_y\psi_y)+\frac{\phi_y^2}{2\rho^3}\left((\bar\rho^{2\alpha})_\tau+(\bar\rho^{2\alpha})_yu\right)\\&\quad-\bar\rho^\alpha g\frac{\phi_y}{\rho^2}+\bar\rho^\alpha(\bar\rho^\alpha)_y\frac{u_y\phi_y}{\rho^2}+(\bar\rho^\alpha)_\tau\frac{\psi\phi_y}{\rho}-(\bar\rho^\alpha)_y\frac{\psi\phi_\tau}{\rho}+\left((\rho^\alpha-\bar\rho^\alpha)u_y\right)_y\frac{\bar\rho^\alpha}{\rho^2}\phi_y.\end{aligned}\tag{3.28}$$

Integrating the resulting equation over $\mathbf{R}^1\times[0,\tau]$ and combing with (3.26) and the fact that

$$\begin{aligned}&\left((\rho^\alpha-\bar\rho^\alpha))u_y\right)_y=(\rho^\alpha-\bar\rho^\alpha)u_{yy}+(\rho^\alpha-\bar\rho^\alpha)_yu_y\\&=(\rho^\alpha-\bar\rho^\alpha)(\psi_{yy}+\bar u_{yy})+\alpha(\rho^{\alpha-1}\rho_y-\bar\rho^{\alpha-1}\bar\rho_y)(\psi_y+\bar u_y)\\&=(\rho^\alpha-\bar\rho^\alpha)(\psi_{yy}+\bar u_{yy})+\alpha\big[\rho^{\alpha-1}\phi_y\psi_y+\rho^{\alpha-1}\phi_y\bar u_y\\&\quad+(\rho^{\alpha-1}-\bar\rho^{\alpha-1})\bar\rho_y\psi_y+(\rho^{\alpha-1}-\bar\rho^{\alpha-1})\bar\rho_y\bar u_y\big],\end{aligned}\tag{3.29}$$

gives

$$\begin{aligned}&\int_{\mathbf{R}}\left(\bar\rho^{2\alpha-3}\phi_y^2+\bar\rho\psi^2+\bar\rho^{\gamma-2}\phi^2\right)dy\\&\quad+\int_0^\tau\int_{\mathbf{R}}\left(\bar\rho^{\gamma-3+\alpha}\phi_y^2+\bar\rho^\alpha\psi_y^2+\bar u_y(\bar\rho^{\gamma-2}\phi^2+\bar\rho\psi^2)\right)dyd\tau\\&\le C\left|\int_0^\tau\int_{\mathbf{R}}\left(\bar\rho^\alpha\bar u_y\frac{\psi_y\phi}{\rho}+\bar\rho^{\alpha+1}\bar u_y\frac{\psi\phi_y}{\rho^2}-2\bar\rho^{2\alpha}\frac{\phi_y}{\rho^3}\bar\rho_y\psi_y\right)dyd\tau\right|\\&\quad+C\left|\int_0^\tau\int_{\mathbf{R}}\left(-\frac{\bar\rho^{2\alpha}}{\rho^3}\phi_y(\bar u_{yy}\phi+\bar\rho_{yy}\psi)+\frac{\bar\rho^\alpha}{\rho^2}\phi_y(\rho^\alpha-\bar\rho^\alpha)\bar u_{yy}\right)dyd\tau\right|\\&\quad+C\left|\int_0^\tau\int_{\mathbf{R}}\bar\rho^\alpha g\frac{\phi_y}{\rho^2}dyd\tau\right|+C\left|\int_0^\tau\int_{\mathbf{R}}\frac{\phi_y^2}{2\rho^3}\left((\bar\rho^{2\alpha})_\tau+(\bar\rho^{2\alpha})_yu\right)dyd\tau\right|\\&\quad+C\Big|\int_0^\tau\int_{\mathbf{R}}\Big(-\bar\rho^\alpha\bar\rho_y\bar u_y\frac{\psi\phi}{\rho^2}+\bar\rho^\alpha(\bar\rho^\alpha)_y\frac{\psi_y\phi_y}{\rho^2}-(\bar\rho^\alpha)_y\frac{\psi\phi_\tau}{\rho}-\bar\rho^\alpha\bar\rho_y^2\frac{\psi^2}{\rho^2}\end{aligned}$$

$$
\begin{aligned}
&+\alpha\big(\rho^{\alpha-1}\phi_y\bar{u}_y+(\rho^{\alpha-1}-\bar{\rho}^{\alpha-1})\bar{\rho}_y\psi_y\big)\frac{\bar{\rho}^\alpha}{\rho^2}\phi_y\Big)dyd\tau\Big|\\
&+C\left|\int_0^\tau\int_{\mathbf{R}}\left((\bar{\rho}^\alpha)_\tau\frac{\psi\phi_y}{\rho}-\bar{\rho}^\alpha\bar{\rho}_y\frac{\psi^2\phi_y}{\rho^2}\right)dyd\tau\right|\\
&+C\left|\int_0^\tau\int_{\mathbf{R}}\left(\bar{\rho}^\alpha(\bar{\rho}^\alpha)_y\frac{\bar{u}_y\phi_y}{\rho^2}+\alpha(\rho^{\alpha-1}-\bar{\rho}^{\alpha-1})\bar{\rho}_y\bar{u}_y\bar{\rho}^\alpha\frac{\phi_y}{\rho^2}\right)dyd\tau\right|\\
&+C\left|\int_0^\tau\int_{\mathbf{R}}(\rho^\alpha-\bar{\rho}^\alpha)\psi_{yy}\frac{\bar{\rho}^\alpha}{\rho^2}\phi_ydyd\tau\right|+C\left|\alpha\int_0^\tau\int_{\mathbf{R}}\rho^{\alpha-1}\phi_y\psi_y\frac{\bar{\rho}^\alpha}{\rho^2}\phi_ydyd\tau\right|+C\epsilon^{\frac{1}{3}}\\
:=&\sum_{i=1}^9 J_i+C\epsilon^{\frac{1}{3}},
\end{aligned}
\tag{3.30}
$$

In the following each term on the right-hand side of (3.30) will be estimated one by one. By Cauchy's inequality and (3.13), we can get

$$
\begin{aligned}
J_1\le& C\int_0^\tau\Big(\bar{\rho}^{\frac{\alpha-\gamma}{2}}\bar{u}_y\Big(|\sqrt{\bar{\rho}^\alpha}\psi_y||\sqrt{\bar{\rho}^{\gamma-2}}\phi|+|\sqrt{\bar{\rho}^{\gamma+\alpha-3}}\phi_y||\sqrt{\bar{\rho}}\psi|\Big)\\
&+\bar{\rho}^{\alpha-\gamma}\bar{u}_y|\sqrt{\bar{\rho}^{\gamma+\alpha-3}}\phi_y||\sqrt{\bar{\rho}^\alpha}\psi_y|\Big)dyd\tau\\
\le&\left(\frac{1}{16}+C\frac{\epsilon}{\mu^\gamma\delta}\right)\int_0^\tau\int_{\mathbf{R}}\big(\bar{\rho}^\alpha\psi_y+\bar{\rho}^{\gamma+\alpha-3}\phi_y^2+\bar{u}_y(\bar{\rho}\psi^2+\bar{\rho}^{\gamma-2}\phi^2)\big)dyd\tau\\
\le&\frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau+\epsilon^{\frac{1}{3}}.
\end{aligned}
\tag{3.31}
$$

Recalling (2.7) in Lemma 2.1 and Lemma 2.3, one can get that

$$
\begin{aligned}
J_2\le& C\int_0^\tau\frac{\epsilon}{\delta}\bar{\rho}^{\frac{3\alpha-3\gamma}{2}}\bar{u}_y|\sqrt{\bar{\rho}^{\gamma+\alpha-3}}\phi_y|\big(|\sqrt{\bar{\rho}^{\gamma-2}}\phi|+|\sqrt{\bar{\rho}}\psi|\big)dyd\tau\\
\le&\frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau+C\left(\frac{\epsilon}{\mu^\gamma\delta}\right)^3\int_0^\tau\int_{\mathbf{R}}\bar{u}_y(\bar{\rho}^{\gamma-2}\phi^2+\bar{\rho}\psi^2)dyd\tau\\
\le&\frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau+\epsilon^{\frac{1}{3}}.
\end{aligned}
\tag{3.32}
$$

By Cauchy's inequality, it holds that

$$
J_3\le\frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau+C\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\alpha-\gamma-1}g^2dyd\tau.
\tag{3.33}
$$

Recalling (3.7), (3.13), and Lemma 2.3 (i), one can get

$$
\begin{aligned}
|g|\le&\bar{\rho}^\alpha|\bar{u}_{yy}|+|\bar{\rho}\bar{u}_y\psi|+C|\bar{\rho}^{\gamma-2}\bar{\rho}_y\phi|\\
\le&\bar{\rho}^\alpha|\bar{u}_{yy}|+|\bar{\rho}\bar{u}_y\psi|+C|\bar{\rho}^{\frac{\gamma-1}{2}}\bar{u}_y\phi|.
\end{aligned}
\tag{3.34}
$$

Thus the last term in (3.33) can be estimated by

$$
\begin{aligned}
&\left|\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\alpha-\gamma-1}g^2dyd\tau\right|\\
\le& C\mu^{-1-\gamma}\int_0^\tau\|\bar{u}_{yy}\|^2d\tau+C\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\alpha-\gamma}\bar{u}_y^2(\bar{\rho}^{\gamma-2}\phi^2+\bar{\rho}\psi^2)dyd\tau
\end{aligned}
$$

$$\leq C\mu^{-1-\gamma}\left(\frac{\epsilon}{\delta}\right)^2 + C\frac{\epsilon}{\mu^\gamma\delta}\int_0^\tau\int_{\mathbf{R}}\bar{u}_y(\bar{\rho}^{\gamma-2}\phi^2+\bar{\rho}\psi^2)dyd\tau \leq \epsilon^{\frac{1}{3}}. \tag{3.35}$$

By Cauchy's inequality, it holds that

$$\begin{aligned} J_4 &\leq C\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{2\alpha-4}\bar{\rho}_y\phi_y^2dyd\tau \leq C\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2\cdot\bar{\rho}^{\frac{2\alpha+1-3\gamma}{2}}\bar{u}_ydyd\tau \\ &\leq C\mu^{\frac{1-3\gamma}{2}}\frac{\epsilon}{\delta}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau \leq \frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau. \end{aligned} \tag{3.36}$$

Recalling (3.4), (3.13), and Lemma 2.3 (i), we can compute that

$$\begin{aligned} J_5 &\leq C\int_0^\tau\int_{\mathbf{R}}\Big(\bar{\rho}^{\alpha-\gamma}\bar{u}_y\big(|\sqrt{\bar{u}_y\bar{\rho}}\psi||\sqrt{\bar{u}_y\bar{\rho}^{\gamma-2}}\phi| + |\sqrt{\bar{\rho}^{\gamma+\alpha-3}}\phi_y||\sqrt{\bar{\rho}^{\alpha}}\psi_y| + \bar{u}_y\bar{\rho}\psi^2 \\ &\quad + \bar{\rho}^{\gamma+\alpha-3}\phi_y^2\big) + \bar{\rho}^{\frac{\alpha-\gamma}{2}}\bar{u}_y|\sqrt{\bar{\rho}^{\alpha}}\psi_y||\sqrt{\bar{\rho}}\psi| + \bar{\rho}^{\frac{\alpha+1-2\gamma}{2}}\bar{u}_y|\sqrt{\bar{\rho}^{\gamma-3+\alpha}}\phi_y||\sqrt{\bar{\rho}}\psi|\Big)dyd\tau \\ &\leq \left(\frac{1}{16}+C\frac{\epsilon}{\mu^\gamma\delta}\right)\int_0^\tau\int_{\mathbf{R}}\big(\bar{\rho}^{\gamma-3+\alpha}\phi_y^2+\bar{\rho}^{\alpha}\psi_y^2\big)dyd\tau \\ &\quad + C\frac{\epsilon}{\mu^{2\gamma}\delta}\int_0^\tau\int_{\mathbf{R}}\bar{u}_y(\bar{\rho}\psi^2+\bar{\rho}^{\gamma-2}\phi^2)dyd\tau \leq \frac{1}{8}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau + \epsilon^{\frac{1}{3}}. \end{aligned} \tag{3.37}$$

Similarly, it holds that

$$\begin{aligned} J_6 &\leq C\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\frac{\alpha+1-2\gamma}{2}}\bar{u}_y\left|\sqrt{\bar{\rho}^{\gamma-3+\alpha}}\phi_y\right|\left|\sqrt{\bar{\rho}}\psi\right|dyd\tau \\ &\leq \frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma-3+\alpha}\phi_y^2dyd\tau + C\frac{\epsilon}{\mu^{2\gamma}\delta}\int_0^\tau\int_{\mathbf{R}}\bar{u}_y\bar{\rho}\psi^2dyd\tau. \end{aligned} \tag{3.38}$$

By Cauchy's inequality and Lemma 2.3, we can get

$$\begin{aligned} J_7 &\leq C\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\frac{4\alpha-3-\gamma}{2}}\bar{u}_y^2|\phi_y|dyd\tau \\ &\leq \frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau + C\mu^{-2\gamma}\int_0^\tau\|\bar{u}_y\|_{L^4}^4d\tau \\ &\leq \frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau + C(\frac{\epsilon}{\mu^\gamma\delta})^2. \end{aligned} \tag{3.39}$$

By Lemma 2.3, the a priori assumptions (3.11), (3.26), and Cauchy's inequality, it holds that

$$\begin{aligned} J_8 &\leq C\int_0^\tau\int_{\mathbf{R}}\left|(\rho^\alpha-\bar{\rho}^\alpha)\psi_{yy}\frac{\bar{\rho}^\alpha}{\rho^2}\phi_y\right|dyd\tau \\ &\leq \frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau + C\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\alpha-1}\psi_{yy}^2\phi^2\bar{\rho}^{2\alpha-\gamma-2}dyd\tau \\ &\leq \frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau + C\mu^{-\gamma-2}\sup_{[0,\tau_1(\epsilon)]}\|\phi\|\sup_{[0,\tau_1(\epsilon)]}\|\phi_y\|\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\alpha-1}\psi_{yy}^2dyd\tau \\ &\leq \frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau \\ &\quad + C\mu^{-\frac{3\gamma+2\alpha+4}{2}}\sup_{[0,\tau_1(\epsilon)]}\left\|\sqrt{\bar{\rho}^{\gamma-2}}\phi\right\|\sup_{[0,\tau_1(\epsilon)]}\left\|\sqrt{\bar{\rho}^{2\alpha-3}}\phi_y\right\|\int_0^\tau\int_{\mathbf{R}}\bar{\rho}^{\alpha-1}\psi_{yy}^2dyd\tau \end{aligned}$$

$$\leq \frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\gamma+\alpha-3}\phi_y^2dyd\tau+C\left(\mu^{-3\gamma-2\alpha-4}\epsilon^{\frac{1}{3}}\right)^{\frac{1}{2}}\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau. \tag{3.40}$$

Recalling (3.11) and Sobolev's inequality, it holds that

$$\begin{aligned}
J_9 \leq& C\int_0^\tau\int_{\mathbf{R}}\left|\rho^{\alpha-1}\phi_y\psi_y\frac{\bar\rho^\alpha}{\rho^2}\phi_y\right|dyd\tau\\
\leq& C\int_0^\tau\|\psi_y\|^{1/2}\|\psi_{yy}\|^{1/2}\left\|\sqrt{\bar\rho^{2\alpha-3}}\phi_y\right\|^2d\tau\\
\leq& C\mu^{-\frac{\alpha}{4}}\int_0^\tau\|\psi_y\|^{1/2}\left\|\sqrt{\bar\rho^{\alpha-1}}\psi_{yy}\right\|^{1/2}\left\|\sqrt{\bar\rho^{2\alpha-3}}\phi_y\right\|^2d\tau\\
\leq& \mu^{2\alpha}|\ln\epsilon|^{-1}\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-\alpha}|\ln\epsilon|^{\frac{1}{3}}\int_0^\tau\|\psi_y\|^{\frac{2}{3}}\left\|\sqrt{\bar\rho^{2\alpha-3}}\phi_y\right\|^{\frac{8}{3}}d\tau\\
\leq& \mu^{2\alpha}|\ln\epsilon|^{-1}\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau\\
&+C\mu^{-\frac{4\alpha+2\gamma}{3}}|\ln\epsilon|^{\frac{1}{3}}\sup_{[0,\tau_1(\epsilon)]}\left\|\sqrt{\bar\rho^{2\alpha-3}}\phi_y\right\|^{\frac{4}{3}}\int_0^\tau\left\|\sqrt{\bar\rho^{\alpha}}\psi_y\right\|^{\frac{2}{3}}\left\|\sqrt{\bar\rho^{\gamma-3+\alpha}}\phi_y\right\|^{\frac{4}{3}}d\tau\\
\leq& \mu^{2\alpha}|\ln\epsilon|^{-1}\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+\frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\gamma+\alpha-3}\phi_y^2dyd\tau\\
&+C\mu^{-4\alpha-2\gamma}|\ln\epsilon|\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha}\psi_y^2dyd\tau\\
\leq& \mu^{2\alpha}|\ln\epsilon|^{-1}\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+\frac{1}{16}\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\gamma+\alpha-3}\phi_y^2dyd\tau+C\mu^{-4\alpha-2\gamma}|\ln\epsilon|\epsilon^{\frac{1}{3}},
\end{aligned} \tag{3.41}$$

where we have used the a priori assumption

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}\left\|\sqrt{\bar\rho^{2\alpha-3}}\phi_y\right\|\leq 1.$$

Substituting (3.31)-(3.41) into (3.30) and recalling (3.12), it holds that

$$\begin{aligned}
&\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}}\left(\bar\rho^{2\alpha-3}\phi_y^2+\bar\rho\psi^2+\bar\rho^{\gamma-2}\phi^2\right)dy\\
&\quad+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\left(\bar\rho^{\gamma-3+\alpha}\phi_y^2+\bar\rho^\alpha\psi_y^2+\bar u_y\left(\bar\rho^{\gamma-2}\phi^2+\bar\rho\psi^2\right)\right)dyd\tau\\
\leq&\left(\mu^{2\alpha}|\ln\epsilon|^{-1}+C\left(\mu^{-3\gamma-2\alpha-4}\epsilon^{\frac{1}{3}}\right)^{\frac{1}{2}}\right)\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau\\
&\quad+C\mu^{-4\alpha-2\gamma}|\ln\epsilon|\epsilon^{\frac{1}{3}},\qquad \text{if}\quad \epsilon\ll 1.
\end{aligned} \tag{3.42}$$

Step 3. As the last step, we estimate $\sup\limits_{\tau\in[0,\tau_1(\epsilon)]}\|\psi_y\|$. For this, we rewrite (3.5) as

$$\rho\psi_\tau+\rho u\psi_y+p'(\rho)\phi_y-\rho^\alpha\psi_{yy}=-\bar g+(\rho^\alpha)_yu_y, \tag{3.43}$$

where

$$\bar g=\rho\psi\bar u_y+\bar\rho_y\left[p'(\rho)-\frac{\rho}{\bar\rho}p'(\bar\rho)\right]-\rho^\alpha\bar u_{yy}. \tag{3.44}$$

Multiplying the above equation by $-\dfrac{\psi_{yy}}{\rho}$ and integrating over $\mathbf{R}^1\times[0,\tau]$ yield

$$\begin{aligned}&\int_{\mathbf{R}}\frac{\psi_y^2}{2}dy+\int_0^\tau\int_{\mathbf{R}}\left(\frac{\bar u_y\psi_y^2}{2}+\frac{\rho^\alpha\psi_{yy}^2}{\rho}\right)dyd\tau\\&=\int_0^\tau\int_{\mathbf{R}}\left(p'(\rho)\frac{\psi_{yy}\phi_y}{\rho}+\bar g\frac{\psi_{yy}}{\rho}-\frac{\psi_y^3}{2}-(\rho^\alpha)_yu_y\frac{\psi_{yy}}{\rho}\right)dyd\tau.\end{aligned}\tag{3.45}$$

First, one has

$$\begin{aligned}\left|\int_0^\tau\int_{\mathbf{R}}p'(\rho)\frac{\psi_{yy}\phi_y}{\rho}dyd\tau\right|&\le\frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\gamma+\alpha-3}\phi_y^2\bar\rho^{\gamma-2\alpha}dyd\tau\\&\le\frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-2\alpha}\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\gamma+\alpha-3}\phi_y^2dyd\tau.\end{aligned}\tag{3.46}$$

Recalling (3.44), Lemma 2.3, and Cauchy's inequality, it holds that

$$\begin{aligned}&\left|\int_0^\tau\int_{\mathbf{R}}\bar g\frac{\psi_{yy}}{\rho}dyd\tau\right|\\&\le\frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\int_0^\tau\int_{\mathbf{R}}\bar\rho^{-\alpha-1}\bar g^2dyd\tau\\&\le\frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-1}\int_0^\tau\|\bar u_{yy}\|^2d\tau+C\int_0^\tau\int_{\mathbf{R}}\bar\rho^{-\alpha}\bar u_y^2(\bar\rho\psi^2+\bar\rho^{\gamma-2}\phi^2)dyd\tau\\&\le\frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-1}\left(\frac{\epsilon}{\delta}\right)^2+C\mu^{-\alpha}\frac{\epsilon}{\delta}\int_0^\tau\int_{\mathbf{R}}\bar u_y\left(\bar\rho\psi^2+\bar\rho^{\gamma-2}\phi^2\right)dyd\tau\\&\le\frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+\epsilon^{\frac13}.\end{aligned}\tag{3.47}$$

By Sobolev's and Cauchy's inequalities, we can get

$$\begin{aligned}\left|\int_0^\tau\int_{\mathbf{R}}\frac{\psi_y^3}{2}dyd\tau\right|&\le C\int_0^\tau\|\psi_{yy}\|^{\frac12}\|\psi_y\|^{\frac52}d\tau\\&\le\frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-\frac{\alpha}{3}}\int_0^\tau\|\psi_y\|^{\frac{10}{3}}d\tau\\&\le\frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-\frac{4\alpha}{3}}\sup_{\tau\in[0,\tau_1]}\|\psi_y\|^{\frac43}\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha}\psi_y^2dyd\tau\\&\le\frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-\frac{4\alpha}{3}}\epsilon^{\frac13},\end{aligned}\tag{3.48}$$

where in the last inequality we have used the a priori assumption

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}\|\psi_y\|\le1.$$

Finally, one has,

$$\begin{aligned}&\left|\int_0^\tau\int_{\mathbf{R}}(\rho^\alpha)_yu_y\frac{\psi_{yy}}{\rho}dyd\tau\right|\\&\le C\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-2}|\psi_{yy}|(|\phi_y\psi_y|+\bar u_y|\phi_y|+\bar\rho_y|\psi_y|+\bar\rho_y\bar u_y)dyd\tau\end{aligned}$$

$$\leq C\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-2}|\psi_{yy}|\Big(|\phi_y\psi_y|+\bar u_y\Big(|\phi_y|+\bar\rho^{\frac{3-\gamma}{2}}|\psi_y|\Big)+\bar\rho^{\frac{3-\gamma}{2}}\bar u_y^2\Big)dyd\tau:=\sum_{i=1}^3 K_i. \tag{3.49}$$

Now we estimate the terms on the right-hand side of (3.49) one by one. By Sobolev's inequality, it holds that

$$\begin{aligned}
K_1=&C\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-2}|\psi_{yy}\phi_y\psi_y|dyd\tau\\
\leq& C\mu^{-\frac{\alpha}{2}}\int_0^\tau\left\|\sqrt{\bar\rho^{\alpha-1}}\psi_{yy}\right\|\left\|\sqrt{\bar\rho^{2\alpha-3}}\phi_y\right\|\|\psi_y\|^{\frac12}\|\psi_{yy}\|^{\frac12}d\tau\\
\leq& C\mu^{-\alpha}\int_0^\tau\left\|\sqrt{\bar\rho^{\alpha-1}}\psi_{yy}\right\|^{\frac32}\left\|\sqrt{\bar\rho^{2\alpha-3}}\phi_y\right\|\left\|\sqrt{\bar\rho^{\alpha}}\psi_y\right\|^{\frac12}d\tau\\
\leq& \frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-4\alpha}\int_0^\tau\left\|\sqrt{\bar\rho^{2\alpha-3}}\phi_y\right\|^4\left\|\sqrt{\bar\rho^{\alpha}}\psi_y\right\|^2d\tau\\
\leq& \frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-4\alpha}\sup_{[0,\tau_1(\epsilon)]}\left\|\sqrt{\bar\rho^{2\alpha-3}}\phi_y\right\|^4\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha}\psi_y^2dyd\tau\\
\leq& \frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-4\alpha}\epsilon^{\frac13},
\end{aligned}\tag{3.50}$$

where in the last inequality we have used the a priori assumption

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}\left\|\sqrt{\bar\rho^{2\alpha-3}}\phi_y\right\|\leq 1.$$

By Cauchy's inequality and (3.26), we can get

$$\begin{aligned}
K_2=&C\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-2}\bar u_y|\psi_{yy}|\big(|\phi_y|+\bar\rho^{\frac{3-\gamma}{2}}|\psi_y|\big)dyd\tau\\
\leq& C\int_0^\tau\int_{\mathbf{R}}\bar\rho^{-\frac{\gamma}{2}}\bar u_y\left|\sqrt{\bar\rho^{\alpha-1}}\psi_{yy}\right|\Big(\left|\sqrt{\bar\rho^{\gamma+\alpha-3}}\phi_y\right|+\left|\sqrt{\bar\rho^{\alpha}}\psi_y\right|\Big)dyd\tau\\
\leq& \frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-\gamma}\left(\frac{\epsilon}{\delta}\right)^2\int_0^\tau\int_{\mathbf{R}}\big(\bar\rho^{\gamma+\alpha-3}\phi_y^2+\bar\rho^{\alpha}\psi_y^2\big)dyd\tau\\
\leq& \frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-\gamma}\left(\frac{\epsilon}{\delta}\right)^2\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\gamma+\alpha-3}\phi_y^2dyd\tau+\epsilon^{\frac13}.
\end{aligned}\tag{3.51}$$

Similarly,

$$\begin{aligned}
K_3=&C\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-2}|\psi_{yy}|\bar\rho^{\frac{3-\gamma}{2}}\bar u_y^2dyd\tau\\
\leq& \frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-\gamma}\bar u_y^4dyd\tau\\
\leq& \frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-\gamma}\int_0^\tau\|\bar u_y\|_{L^4}^4d\tau\\
\leq& \frac18\int_0^\tau\int_{\mathbf{R}}\bar\rho^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-\gamma}\left(\frac{\epsilon}{\delta}\right)^2.
\end{aligned}\tag{3.52}$$

Substituting (3.46)-(3.52) into (3.45), it holds that

$$\begin{aligned} &\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}}\psi_y^2dy+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\bar{\rho}^{\alpha-1}\psi_{yy}^2dyd\tau \\ &\le C\mu^{-2\alpha}\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau+C\mu^{-4\alpha}\epsilon^{\frac{1}{3}}, \qquad \text{if } \ \epsilon\ll 1. \end{aligned} \tag{3.53}$$

Substituting (3.53) into (3.42), it holds that

$$\begin{aligned} &\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}}\bar{\rho}^{2\alpha-3}\phi_y^2dy+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau \\ &\le \left(\mu^{2\alpha}|\ln\epsilon|^{-1}+C(\mu^{-3\gamma-2\alpha-4}\epsilon^{\frac{1}{3}})^{\frac{1}{2}}\right)\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\bar{\rho}^{\alpha-1}\psi_{yy}^2dyd\tau+C\mu^{-4\alpha-2\gamma}|\ln\epsilon|\epsilon^{\frac{1}{3}} \\ &\le C\left(|\ln\epsilon|^{-1}+(\mu^{-3\gamma-6\alpha-4}\epsilon^{\frac{1}{3}})^{\frac{1}{2}}\right)\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau+C\mu^{-4\alpha-2\gamma}|\ln\epsilon|\epsilon^{\frac{1}{3}} \\ &=C\left(|\ln\epsilon|^{-1}+\epsilon^{\frac{\gamma}{6}a}|\ln\epsilon|^{-\frac{3\gamma+6\alpha+4}{2}}\right)\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau \\ &\quad +C\epsilon^{\frac{1}{3}-(4\alpha+2\gamma)a}|\ln\epsilon|^{1-4\alpha-2\gamma}. \end{aligned} \tag{3.54}$$

So we can get

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}}\bar{\rho}^{2\alpha-3}\phi_y^2dy+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\bar{\rho}^{\gamma+\alpha-3}\phi_y^2dyd\tau\le C\epsilon^{\frac{1}{3}-(4\alpha+2\gamma)a}|\ln\epsilon|^{1-4\alpha-2\gamma} \tag{3.55}$$

and

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}\int_{\mathbf{R}}\psi_y^2dy+\int_0^{\tau_1(\epsilon)}\int_{\mathbf{R}}\bar{\rho}^{\alpha-1}\psi_{yy}^2dyd\tau\le C\epsilon^{\frac{1}{3}-(6\alpha+2\gamma)a}|\ln\epsilon|^{1-6\alpha-2\gamma}. \tag{3.56}$$

Therefore, (3.17), (3.18), and (3.19) can be derived directly from (3.26), (3.55), and (3.56). It follows from (3.17)-(3.19) that if ϵ is suitably small, then

$$\begin{aligned} \sup_{0\le\tau\le\tau_1(\epsilon)}\|\phi(\cdot,\tau)\|_{L^\infty}&\le\sqrt{2}\sup_{0\le\tau\le\tau_1(\epsilon)}\|\phi(\cdot,\tau)\|^{1/2}\|\phi_y(\cdot,\tau)\|^{1/2} \\ &\le C\sup_{0\le\tau\le\tau_1(\epsilon)}\left(\mu^{-\gamma-2\alpha}\int_{\mathbf{R}}\bar{\rho}^{\gamma-2}\phi^2dy\int_{\mathbf{R}}\bar{\rho}^{2\alpha-3}\phi_y^2dy\right)^{\frac{1}{4}} \\ &\le C\left(\mu^{-\gamma-2\alpha}\epsilon^{\frac{1}{3}}\cdot\epsilon^{\frac{1}{3}-(4\alpha+2\gamma)a}|\ln\epsilon|^{1-4\alpha-2\gamma}\right)^{\frac{1}{4}} \\ &=C\epsilon^{\frac{18\alpha+11\gamma+24}{12}a}|\ln\epsilon|^{\frac{1-6\alpha-3\gamma}{4}}\le\epsilon^a, \end{aligned} \tag{3.57}$$

and

$$\begin{aligned} &\sup_{0\le\tau\le\tau_1(\epsilon)}\|\psi(\cdot,\tau)\|_{L^\infty} \\ &\le\sqrt{2}\sup_{0\le\tau\le\tau_1(\epsilon)}\|\psi(\cdot,\tau)\|^{1/2}\|\psi_y(\cdot,\tau)\|^{1/2} \\ &\le C\sup_{0\le\tau\le\tau_1(\epsilon)}\left(\mu^{-1}\int_{\mathbf{R}}\bar{\rho}\psi^2dy\int_{\mathbf{R}}\psi_y^2dy\right)^{\frac{1}{4}} \end{aligned} \tag{3.58}$$

$$
\begin{aligned}
&\le C\left(\mu^{-1}\epsilon^{\frac{1}{3}}\cdot\epsilon^{\frac{1}{3}-(6\alpha+2\gamma)a}|\ln\epsilon|^{1-6\alpha-2\gamma}\right)^{\frac{1}{4}}\\
&= C\epsilon^{\frac{18\alpha+14\gamma+21}{12}a}|\ln\epsilon|^{-\frac{6\alpha+2\gamma}{4}}\le\epsilon^{a}. \qquad (3.59)
\end{aligned}
$$

Thus the a priori assumptions (3.11) are verified and the proof of Lemma 3.2 is completed. □

It is noted that the obtained a priori estimates are better than the a priori assumptions (3.11) in the maximum time interval $[0,\tau_1(\epsilon)]$. Based on these a priori estimates, we can claim $\tau_1(\epsilon)=\infty$. If $\tau_1(\epsilon)<\infty$, then by again using the local existence at time $\tau=\tau_1(\epsilon)$, we can find anther time $\tau_2(\epsilon)>\tau_1(\epsilon)$ so that the solution satisfies the assumptions (3.11) in the time interval $[0,\tau_2(\epsilon)]$ which contradicts the assumptions that $\tau_1(\epsilon)$ is the maximum time. Therefore we extend the local solution to the global solution in $[0,\infty)$ for small but fixed ϵ.

Proof of Theorem 1.1.

Proof. It remains to prove (1.8) with a given in (1.9). From Lemma 2.2, Lemma 2.3 (iii), (3.8)-(3.10), and recalling that $\mu=\epsilon^a|\ln\epsilon|$ and $\delta=\epsilon^a$, it holds that for any given positive constant h, there exists a constant $C_h>0$ which is independent of ϵ such that

$$
\begin{aligned}
&\sup_{t\ge h}\left\|\rho(\cdot,t)-\rho^{r_2}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\\
&\le \sup_{\tau\in[0,+\infty)}\|\phi(\cdot,\tau)\|_{L^\infty}+\sup_{t\ge h}\left\|\bar\rho(\cdot,t)-\rho_\mu^{r_2}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}+\sup_{t\ge h}\left\|\rho_\mu^{r_2}\left(\frac{\cdot}{t}\right)-\rho^{r_2}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\\
&\le C_h(\epsilon^a+\delta|\ln\delta|+\mu)\le C_h\epsilon^a|\ln\epsilon|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{t\ge h}\left\|m(\cdot,t)-m^{r_2}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\\
&\le \sup_{t\ge h}\left(\|m(\cdot,t)-\bar m(\cdot,t)\|_{L^\infty}+\left\|\bar m(\cdot,t)-m_\mu^{r_2}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}+\left\|m_\mu^{r_2}\left(\frac{\cdot}{t}\right)-m^{r_2}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\right)\\
&\le C\sup_{\tau\in[0,+\infty)}\left(\|\psi(\cdot,\tau)\|_{L^\infty}+\|\phi(\cdot,\tau)\|_{L^\infty}\right)\\
&\quad+\sup_{t\ge h}\left(\left\|\bar m(\cdot,t)-m_\mu^{r_2}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}+\left\|m_\mu^{r_2}\left(\frac{\cdot}{t}\right)-m^{r_2}\left(\frac{\cdot}{t}\right)\right\|_{L^\infty}\right)\\
&\le C_h(\epsilon^a+\delta|\ln\delta|+\mu)\le C_h\epsilon^a|\ln\epsilon|.
\end{aligned}
$$

Thus the proof of Theorem 1.1 is completed. □

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