COMMUN. MATH. SCI.
Vol. 13, No. 2, pp. 539–555

RANDOM ATTRACTOR AND STATIONARY MEASURE FOR STOCHASTIC LONG-SHORT WAVE EQUATIONS*

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Abstract. Asymptotic behaviors of stochastic long-short equations driven by a random force, which is smooth enough in space and white noise in time, are mainly considered. The existence and uniqueness of solutions for stochastic long-short equations are obtained via Galerkin approximation by the stopping time and the Borel-Cantelli Lemma on the basis of *a priori* estimates in the sense of expectation. A global random attractor and the existence of a stationary measure are investigated by the Birkhoff ergodic theorem and the Chebyshev inequality.

Key words. Stochastic long-short equations, existence and uniqueness, global random attractor, stationary measure.

AMS subject classifications. 76B03, 35Q35, 60H15.

1. Introduction and main results

Long-short wave equations describe the resonance interaction between the long wave and the short wave, and were first derived by Djordjević and Redekop in [5]. The long-short wave equations can be written as

$$iu_t + u_{xx} - nu = 0, \tag{1.1}$$

$$n_t + (|u|^2)_x = 0, \tag{1.2}$$

where u is the envelope of the short wave and complex, n is amplitude of the long wave and real. The physical significance of this system is represented by the dispersion of the short wave balanced by nonlinear interaction of the long wave with the short wave. This system also appears in the analysis of internal waves in [12] and in plasma physics, where it describes the resonance of high–frequency electron plasma oscillation and the associated low–frequency ion density perturbation as in [17]. Due to its important physical and mathematical properties, the long-short equations have drawn much attention from many mathematicians and physicists and have been quite extensively studied in theory. The existence of global solutions for long-short wave equations and generalized long-short wave equations are obtained by Guo in [8] and [9], respectively. The orbital stability of solitary waves and the existence of a global attractor have been studied in [6, 10, 11, 15, 22].

It is well known that stochastic partial differential equations (SPDEs) play an important role in understanding the dynamics of many interesting phenomena. Recently, the importance of taking random effects into account in modeling, analyzing, simulating, and predicting complex phenomena has been widely recognized in geophysical and climate dynamics, materials science, chemistry, biology, and other areas, see [7, 13].

*Received: June 13, 2013; accepted (in revised form): May 25, 2014. Communicated by Shi Jin.

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SPDEs are more realistic mathematical models for complex systems under random environmental effects or errors of measurement. The existence and uniqueness of the solution and of attractors for SPDEs have been studied by many authors, see [1, 2, 20, 21].

As is well known, there are mainly two sources of random effects coming from the random environmental effects and the errors of measurement. Hence, random effects now have a broad rang of applications. In particular, in 1960 Kalman and in 1961 Kalman and Bucy proved what is now known as the Kalman-Bucy filter which is an example of a recent mathematical discovery that gives some random procedure. It now has wide applications [18]. Therefore, when the random environmental effects or the errors of measurement are included into the model of long-short wave equations, the stochastic long-short wave equations can be obtained. So far, there are few papers considering it. Hence, it is significant to study the stochastic long-short wave equations.

In this paper, we consider the following stochastic long-short equations in $\mathbb{R}^1$ on a regular domain $D=[0,L]$ in the Itô sense:

$$n_t+(|u|^2)_x+\delta n=f+\dot{W}_1, \tag{1.3}$$

$$iu_t+u_{xx}-nu+i\alpha u=g+\dot{W}_2, \tag{1.4}$$

where W_1 and W_2 are independent $L^2(D)$ value Wiener processes which come from the random environmental effects or the errors of measurement and will be detailed in the next section.

There are two points that are worthy of note in this paper. Firstly, it is well known that there are some technical and tricky issues that make it difficult to treat higher dimensional noises. We consider only a one dimensional noise term, which is a natural choice for some particular applications. For the higher dimensional noises, the investigation will be further considered in the future. Secondly, this study differs from the deterministic literature in that we will use the Itô formula and the some stochastic inequalities and obtain the estimates in the meaning of the expectation. Therefore, we need more skills to deal with the difficulties coming from the coupled nonlinear terms in the stochastic literature than in the deterministic literature.

In Section 3, some techniques and the Itô formula are used for the coupled nonlinear terms in the meaning of the expectation. The maximal estimates for stochastic integrals are important, see Lemma 2.2. The random dynamical system (1.3)-(1.4) is investigated in Section 4. The existence and uniqueness of the solutions for the stochastic long-short equations via the standard Galerkin approximation method by the stopping time and the Borel-Cantelli Lemma on the basis of *a priori* estimates in different investigated spaces V_1 and V_2 are given. Usually, the Galerkin method for SPDEs leads to martingale solutions instead of weak solutions. However, in this approach we transform (1.3)-(1.4) into partial differential equations with random coefficients. Then by *a priori* estimates obtained in Section 3, we construct a full probability measure set $\tilde{\Omega}$. So the classic Galerkin approximation can be applied to obtain a unique solution for (1.3)-(1.4). This method has been applied to the stochastic porous medium equation in [14]. Now we give the main results of this paper. First, the existence and uniqueness of solutions for (1.3)-(1.4) is given as follows.

THEOREM 1.1. *If* $(n_0,u_0)\in V_1$, $q_1\in H^1(D)$, $q_2\in H^2(D)$, $f\in H^1(D)$, *and* $g\in H^1(D)$, *then there exists a unique solution* $(n,u)\in L^\infty(\mathbb{R}^+;V_1)$ *almost surely for equations (1.3)-(1.4). Moreover* (n,u) *is continuous from* $\mathbb{R}^+$ *to* V_1.

Here, q_1 and q_2 are coefficients of standard complex and real valued Wiener process, which are the parts of the Wiener process terms in (1.3)-(1.4) and are given in detail in Section 2. Similar to Theorem 1.1, using the same method, we can obtain the corresponding results in V_2 as follows.

THEOREM 1.2. *If* $(n_0,u_0)\in V_2$, $q_1\in H^2(D)$, $q_2\in H^3(D)$, $f,g\in H^2(D)$, *then there exists a unique solution* $(n,u)\in L^\infty(\mathbb{R}^+;V_2)$ *almost surely for equations (1.3)-(1.4). Moreover* (n,u) *is continuous from* $\mathbb{R}^+$ *to* V_2.

In fact, by theorems 1.1 and 1.2, we can define a continuous random dynamical system in V_1 and V_2, respectively. Then a random attractor endowed with the weak topology can be constructed for the continuous random dynamical system in V_1 and V_2, respectively.

THEOREM 1.3. *If* $(n_0,u_0)\in V_1$, $q_1\in H^1(D)$, $q_2\in H^2(D)$, *and* $f,g\in H^1(D)$, *then (1.3)-(1.4) have a global random weak attractor* $\mathcal{A}(\omega)$ *which is a random tempered compact set in* V_1 *endowed with the weak topology.*

THEOREM 1.4. *If* $(n_0,u_0)\in V_2$, $q_1\in H^2(D)$, $q_2\in H^3(D)$, *and* $f,g\in H^2(D)$. *Then (1.3)-(1.4) have a global random weak attractor* $\mathcal{A}(\omega)$ *which is a random tempered compact set in* V_2 *endowed with the weak topology.*

Although the random attractor is constructed in the weak topology, we can prove the existence of a stationary measure in some phase spaces endowed with the strong topology. But we cannot prove the existence of a random weak attractor in V_2 endowed with usual topology for the existence of noise term. By random dynamics theory, the existence of a random attractor with the usual topology yields the existence of one stationary measure. However, the random attractor is obtained in the weak topology of V_2 which does not yield the existence of a stationary measure in V_2. But in the following we can construct a stationary measure in V_2.

THEOREM 1.5. *If* $(n_0,u_0)\in V_2$, $q_1\in H^2(D)$, $q_2\in H^4(D)$, *then (1.3)-(1.4) have one stationary measure on* V_1 *and* V_2.

The spaces V_1 and V_2 can be seen below. The paper is organized as follows. In Section 2, we give some functional sets for the problem and some conditions. In Section 3, we establish a series of *a* time uniform *priori* estimates in different energy spaces which are the key step for solving the problem. Section 4 is devoted to the proofs of theorems 1.1 and 1.2. In Section 5, the proofs of theorems 1.3, 1.4, and 1.5 will be given. In the paper the letters c and C are generic positive constants independent of T which may change their values from term to term. In addition, c_T and C_T are generic positive constants dependent on T which also may change values from term to term.

2. Preliminaries

We denote the bounded set by $D=[0,L]$ in $\mathbb{R}^1$. Consider the stochastic long-short equations (1.3)-(1.4) on D with Dirichlet boundary condition $u(0,t)=u(L,t)=0$, $n(0,t)=n(L,t)=0$, and initial condition $u(x,0)=u_0(x)$, $n(x,0)=n_0(x)$, $x\in D$. Here $\delta>0$, $\alpha>0$. Now we introduce some functional spaces. The scalar product on $L^2(D)$ will be denoted by $(u,v)=\int_D u(x)\overline{v(x)}dx$ and the norm by $\|u\|^2=(u,u)^{\frac{1}{2}}=\left(\int_D|u|^2dx\right)^{\frac{1}{2}}$. The general p-norm of $L^p(D)$ $(p\geq 1)$ is denoted by $\|u\|_{L^p}=\left(\int_D|u|^pdx\right)^{\frac{1}{p}}$. The norm in $H^s(D)$ (s is a nonnegative integer in normal Sobolev spaces) is denoted by $\|u\|_{H^s}=\left(\sum_{0\leq|\alpha|\leq s}\|D^\alpha u\|^2\right)^{\frac{1}{2}}$. The space $H_0^s(D)$ is the set of $H_0^s(D)=\{u:\|u\|_{H^s(D)}<\infty,\ u(0,t)=u(L,t)=0\}$. It is well known that the norm of u in $H_0^s(D)$, that is $\|u\|_{H_0^s}$, is equivalent of the norm of u in $H^s(D)$ for a bounded domain D. We define $H^{-s}(D)$

to be the dual space of $H^s(D)$. We know that the Poincaré inequality $\|u\| \leq \lambda_1^{-\frac{1}{2}} \|\nabla u\|$ can be used for bounded domains D and $\lambda_1 = \frac{4\pi^2}{L^2}$.

Here we give a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The expectation operator with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. Stochastic terms $W_1(t)$ and $W_2(t)$ are defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ by

$$W_1(t) = q_1(x)\omega_1(t), \ W_2(t) = q_2(x)\omega_2(t),$$

where $\omega_1(t)$ is a standard real valued Wiener process, $\omega_2(t)$ is a standard complex valued Wiener process which is independent of $\omega_1(t)$, and $q_1(x), q_2(x)$ are sufficiently smooth functions in some sense.

We also define the different product spaces for the solution (n,u) of the (1.3)-(1.4)

$$\begin{aligned}
&V_0 = L^2(D) \times H_0^1(D),\\
&V_1 = H_0^1(D) \times (H^2(D) \cap H_0^1(D)),\\
&V_2 = (H^2(D) \cap H_0^1(D)) \times \{\varphi \in H^3(D) \cap H_0^1(D) : \varphi_{xx} \in H_0^1(D)\}.
\end{aligned}$$

Endow each V_i $(i = 0,1,2)$ with the usual norm, and we have $V_2 \subset V_1 \subset V_0$ with compact embeddings.

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}}) \subset (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}}) \subset (\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$ be three Banach reflective spaces and $\mathcal{X} \subset \mathcal{Y}$ with compact and dense embedding. Define the Banach space

$$\mathcal{G} = \{v : v \in L^2(0,T;\mathcal{X}), \frac{dv}{dt} \in L^2(0,T;\mathcal{Z})\}$$

with norm

$$\|v\|_{\mathcal{G}}^2 = \int_0^T \|v\|_{\mathcal{X}}^2 ds + \int_0^T \|\frac{dv}{dt}\|_{\mathcal{Z}}^2 ds, \ v \in \mathcal{G}.$$

We have the following lemma about compactness result from [16].

Lemma 2.1. *If K is bounded in $\mathcal{G}$, then K is precompact in $L^2(0,T;\mathcal{Y})$.*

Another lemma is needed for some maximal estimates on stochastic integrals. Assume U and H are separable Hilbert spaces and W is a Q-Wiener process on U_0 with $U_0 = Q^{\frac{1}{2}}U$. Let $L_2^0 = L_2^0(U_0,H)$ be the space of Hilbert-Schmidt operators from U_0 to H. For such operators we have the following results from [3].

Lemma 2.2. *For any $r \geq 1$ and any L_2^0-valued predictable process $\Phi(t)$, $t \in [0,T]$, we have*

$$\mathbb{E}\int_0^t \Phi(s)dW(s) = 0$$

and

$$\begin{aligned}
\mathbb{E}\left(\sup_{s\in[0,t]}\left|\int_0^s \Phi(\sigma)dW(\sigma)\right|^{2r}\right) &\leq c_r \sup_{s\in[0,t]} \mathbb{E}\left(\left|\int_0^s \Phi(\sigma)dW(\sigma)\right|^{2r}\right)\\
&\leq C_r \mathbb{E}\left(\int_0^t \|\Phi(s)\|_{L_2^0}^2 ds\right)^r, \qquad (2.1)
\end{aligned}$$

where c_r and C_r are some positive constants dependent on r.

3. *A* time uniform *priori* estimates

3.1. *A priori* estimates in V_0.

LEMMA 3.1. *Assume that $u_0, g, q_2 \in L^2(D)$. Then for any $T>0$ and $p\geq 1$, we have $u\in L^{2p}(\Omega;L^\infty(0,T;L^2(D)))\cap L^\infty(0,\infty;L^{2p}(\Omega;L^2(D)))$.*

Proof. Applying the Itô formula to $\|u\|^2$, one gets

$$\frac{d}{dt}\|u\|^2 = -2\alpha\|u\|^2 + 2\mathrm{Im}\int_D g\bar{u}dx + 2\mathrm{Im}\int_D \bar{u}\dot{W}_2 dx + \|q_2\|^2. \tag{3.1}$$

From (3.1), using Hölder's and Young's inequalities, we can obtain

$$\frac{d}{dt}\|u\|^2 + \alpha\|u\|^2 \leq \frac{1}{\alpha}\|g\|^2 + \|q_2\|^2 + 2\mathrm{Im}\int_D \bar{u}\dot{W}_2 dx. \tag{3.2}$$

Then, multiplying by $e^{\alpha t}$, integrating from 0 to t on the both sides of (3.2), and taking the expectation, we can get

$$\mathbb{E}\|u\|^2 \leq e^{-\alpha t}\mathbb{E}\|u_0\|^2 + \frac{1}{\alpha^2}\|g\|^2 + \frac{1}{\alpha}\|q_2\|^2 \leq C, \quad t>0, \tag{3.3}$$

where C is independent of T.

On the other hand, integrating from 0 to t and taking the supremum and the expectation on the both sides of (3.2), we can obtain

$$\mathbb{E}\sup_{0\leq t\leq T}\|u\|^2 \leq \mathbb{E}\|u_0\|^2 + \left(\frac{1}{\alpha}\|g\|^2 + \|q_2\|^2\right)T + \mathbb{E}\sup_{0\leq t\leq T}|\int_0^t \mathrm{Im}\int_D \bar{u}\dot{W}_2 dxds|^2 + 1. \tag{3.4}$$

By Lemma 2.2, for some positive constant C, we have

$$\mathbb{E}\sup_{0\leq t\leq T}|\int_0^t \mathrm{Im}\int_D \bar{u}\dot{W}_2 dxds|^2 \leq C\|q_2\|^2\mathbb{E}\int_0^T \|u\|^2 ds. \tag{3.5}$$

So by (3.3), for any $T>0$, there exists a positive constant C_T such that

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}\|u\|^2 &\leq \mathbb{E}\|u_0\|^2 + \left(\frac{1}{\alpha}\|g\|^2 + \|q_2\|^2\right)T + C\|q_2\|^2\mathbb{E}\int_0^T \|u\|^2 ds + 1\\ &\leq C_T(\mathbb{E}\|u_0\|^4 + \|g\|^4 + \|q_2\|^4 + 1).\end{aligned} \tag{3.6}$$

By the above estimates, we can further give an estimate of $\|u(t)\|_0^{2p}$ for any $p\geq 1$. Now, applying the Itô formula and Hölder's inequality, we have

$$\frac{d}{dt}\|u\|^{2p} \leq -\frac{\alpha p}{2}\|u\|^{2p} + c(\|g\|^{2p} + \|q_2\|^{2p}) + 2p\|u\|^{2(p-1)}\mathrm{Im}\int_D \bar{u}\dot{W}_2 dx. \tag{3.7}$$

Multiplying by $e^{\frac{\alpha p}{2}t}$, integrating from 0 to t, and taking the expectation on the both sides of (3.7) yields

$$\mathbb{E}\|u\|^{2p} \leq e^{-\frac{\alpha p}{2}t}\mathbb{E}\|u_0\|^{2p} + c(\|g\|^{2p} + \|q_2\|^{2p}) \leq C, \qquad t>0, \tag{3.8}$$

where C is independent of T. On the other hand, from (3.7) we have

$$\frac{d}{dt}\|u\|^{2p} \leq c(\|g\|^{2p} + \|q_2\|^{2p}) + 2p\|u\|^{2(p-1)}\mathrm{Im}\int_D \bar{u}\dot{W}_2 dx. \tag{3.9}$$

Integrating from 0 to t for above inequality and taking the supremum and the expectation, we can obtain

$$\mathbb{E}\sup_{0\le t\le T}\|u(t)\|^{2p}\le \mathbb{E}\|u(0)\|^{2p}+c(\|g\|^{2p}+\|q_2\|^{2p})T+1$$
$$+\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t p\|u\|^{2(p-1)}\mathrm{Im}\int_D \bar{u}\dot{W}_2dxds\right|^2. \quad (3.10)$$

By Lemma 2.2, for some positive constant C, we obtain

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t p\|u\|^{2(p-1)}\mathrm{Im}\int_D \bar{u}\dot{W}_2dxds\right|^2\le Cp^2\|q_2\|^2\mathbb{E}\int_0^T\|u\|^{4p-2}ds, \quad (3.11)$$

where $2p-1\ge 1$. Then inserting (3.11) into (3.10), and by (3.8), for any $T>0$ we have

$$\mathbb{E}\sup_{0\le t\le T}\|u\|^{2p}\le C_T(u_0,g,q_2) \quad (3.12)$$

where $C_T(u_0,g,q_2)$ is a positive constant depending on u_0,g,q_2 and T. ◻

LEMMA 3.2. *Assume that* $(n_0,u_0)\in V_0$, $q_1\in L^2(D)$, $q_2\in H^1(D)$, $f,g\in L^2(D)$. *Then for any* $T>0$ *and* $p\ge 1$, *we have* $(n,u)\in L^{2p}(\Omega;L^\infty(0,T;V_0))\cap L^\infty(0,\infty;L^{2p}(\Omega;V_0))$.

Proof. Applying the Itô formula to $\|n\|^2$, since $n_t=-(|u|^2)_x-\delta n+f+\dot{W}_1$, we obtain

$$\frac{d}{dt}\|n\|^2-\|q_1\|^2+2\int_D n(|u|^2)_xdx+2\delta\|n\|^2-2\int_D fndx-2\int_D n\dot{W}_1dx=0. \quad (3.13)$$

Noticing that

$$\frac{d}{dt}\int_D i(u_x\bar{u}-u\bar{u}_x)dx=2\int_D i(u_x\bar{u}_t-\bar{u}_xu_t)dx+2\mathrm{Im}\int_D q_2\bar{q}_{2x}dx,$$

from (3.13), we can obtain

$$\frac{d}{dt}\left(\|n\|^2+\int_D i(u_x\bar{u}-u\bar{u}_x)dx\right)+2\delta\|n\|^2=-2\int_D i\alpha(u_x\bar{u}-\bar{u}_xu)dx+2\int_D fndx$$
$$+4\mathrm{Re}\int_D u\bar{g_x}dx+2\mathrm{Im}\int_D q_2\bar{q}_{2x}dx+\|q_1\|^2+4\mathrm{Re}\int_D u\bar{\dot{W}}_{2x}dx+2\int_D n\dot{W}_1dx. \quad (3.14)$$

Applying the Itô formula to $\|u_x\|^2$, and taking the inner product of (1.4) and $u_t+\alpha u$, we obtain

$$\frac{d}{dt}\|u_x\|^2+2\alpha\|u_x\|^2+2\mathrm{Re}\int_D nu\bar{u}_tdx+2\alpha\int_D n|u|^2dx$$
$$+2\mathrm{Re}\int_D g\bar{u}_tdx+2\alpha\mathrm{Re}\int_D g\bar{u}dx+2\mathrm{Re}\int_D(\bar{u}_t+\alpha\bar{u})\dot{W}_2dx=\|q_{2x}\|^2. \quad (3.15)$$

Notice that

$$\frac{d}{dt}\int_D n|u|^2dx=\int_D n_t|u|^2dx+2\mathrm{Re}\int_D nu_t\bar{u}dx+2\int_D q_1\mathrm{Im}(q_2\bar{u})dx.$$

In addition, since $n_t=-((|u|^2)_x+\delta n-f-\dot{W}_1)$ and

$$-2\mathrm{Re}\int_D(\bar{u}_t+\alpha\bar{u})\dot{W}_2dx=2\mathrm{Im}\int_D\bar{u}_x\dot{W}_{2x}dx+2\mathrm{Im}\int_D(n\bar{u}+\bar{g})\dot{W}_2dx,$$

from (3.15), we can obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|u_x\|^2+\int_D n|u|^2dx+2\mathrm{Re}\int_D g\bar{u}dx\right)+2\alpha\|u_x\|^2\\&=-\delta\int_D n|u|^2dx+\int_D f|u|^2dx-2\alpha\int_D n|u|^2dx-2\alpha\mathrm{Re}\int_D g\bar{u}dx+2\int_D q_1\mathrm{Im}(q_2\bar{u})dx\\&\quad+\|q_{2x}\|^2+2\mathrm{Im}\int_D\bar{u}_x\dot{W}_{2x}dx+2\mathrm{Im}\int_D(n\bar{u}+\bar{g})\dot{W}_2dx+\int_D|u|^2\dot{W}_1dx.\end{aligned}\tag{3.16}$$

Then we can estimate each term using Hölder's inequality, the Gagliardo-Nirenberg inequality, and Young's inequality. Now, let $H_0(t)=\|u_x\|^2+\|n\|^2+\int_D n|u|^2dx+2\mathrm{Re}\int_D g\bar{u}dx+\int_D i(u_x\bar{u}-u\bar{u}_x)dx$ and take $\eta=\min\{\alpha,\delta\}$. From the sum of (3.14) and (3.16), it can be inferred that

$$\begin{aligned}\frac{d}{dt}H_0(t)+\eta H_0(t)\le\,&c(f,g,q_1,q_2)+c\|u\|^6+\int_D|u|^2\dot{W}_1dx+2\int_D n\dot{W}_1dx\\&+4\mathrm{Re}\int_D u\dot{\bar{W}}_{2x}dx+2\mathrm{Im}\int_D\bar{u}_x\dot{W}_{2x}dx+2\mathrm{Im}\int_D(n\bar{u}+\bar{g})\dot{W}_2dx.\end{aligned}\tag{3.17}$$

Multiplying by $e^{\eta t}$, integrating from 0 to t, and taking expectation on the both sides of (3.17) then yields

$$\mathbb{E}H_0(t)\le e^{-\eta t}\mathbb{E}H_0(0)+c(f,g,q_1,q_2)+c\mathbb{E}\int_0^t e^{-\eta(t-s)}\|u\|^6ds.\tag{3.18}$$

By (3.8), we can estimate (3.18) and obtain

$$\mathbb{E}H_0(t)\le e^{-\beta t}\mathbb{E}H_0(0)+c(f,g,q_1,q_2,u_0)\le c(f,g,q_1,q_2,u_0)\le C,\quad t>0,\tag{3.19}$$

where C is independent of T.

Since

$$H_0(t)\ge\frac{1}{2}(\|u_x\|^2+\|n\|^2)-c(\|u\|^2+\|g\|^2+\|u\|^6),\tag{3.20}$$

for any $t>0$, we obtain

$$\mathbb{E}(\|u_x\|^2+\|n\|^2)\le c\mathbb{E}(\|u\|^2+\|g\|^2+\|u\|^6)+c\mathbb{E}H_0(t)\le C,\tag{3.21}$$

where C is independent of T.

Further, we estimate $H_0^p(t)$ for $p\ge1$. Now, applying the Itô formula to $H_0^p(t)$, we have

$$\begin{aligned}\frac{d}{dt}H_0^p(t)\le&-\frac{\eta p}{2}H_0^p(t)+c(\|u\|^{6p}+c)+pH_0^{p-1}(t)\left(\int_D|u|^2\dot{W}_1dx+2\int_D n\dot{W}_1dx\right)\\&+pH_0^{p-1}(t)\left(4\mathrm{Re}\int_D u\dot{\bar{W}}_{2x}dx+2\mathrm{Im}\int_D\bar{u}_x\dot{W}_{2x}dx+2\mathrm{Im}\int_D(n\bar{u}+\bar{g})\dot{W}_2dx\right).\end{aligned}\tag{3.22}$$

Multiplying by $e^{\frac{\eta p}{2}t}$, integrating from 0 to t, and taking expectation on the both sides of (3.22) then yields

$$\mathbb{E}H_0^p(t)) \le e^{-\frac{\eta p}{2}t}\mathbb{E}H_0^p(0)+c+c\mathbb{E}\int_0^t e^{-\frac{\eta p}{2}(t-s)}\|u\|^{6p}ds. \tag{3.23}$$

By (3.8) and (3.23) we obtain

$$\mathbb{E}H_0^p(t)) \le e^{-\frac{\eta p}{2}t}\mathbb{E}H_0^p(0)+c \le C, \quad t>0, \tag{3.24}$$

where C is independent of T. Therefore, by (3.20), we have

$$\mathbb{E}(\|u_x\|^{2p}+\|n\|^{2p}) \le C, \qquad t>0. \tag{3.25}$$

On the one hand, by integrating from 0 to t on the both sides of (3.17), one deduces

$$\begin{aligned} H_0(t) \le\, & H_0(0)+c(f,g,q_1,q_2)t+c\int_0^t \|u\|^6 ds+\int_0^t\left(\int_D |u|^2\dot{W}_1dx+2\int_D n\dot{W}_1dx\right)ds \\ & +\int_0^t\left(4\text{Re}\int_D u\bar{\dot{W}}_{2x}dx+2\text{Im}\int_D \bar{u}_x\dot{W}_{2x}dx+2\text{Im}\int_D (n\bar{u}+\bar{g})\dot{W}_2dx\right)ds. \end{aligned} \tag{3.26}$$

Now, taking the supremum and expectation on the both sides of (3.26) yields

$$\begin{aligned} \mathbb{E}\sup_{0\le t\le T} H_0(t) \le\, & \mathbb{E}H_0(0)+c(f,g,q_1,q_2)T+c+c\mathbb{E}\sup_{0\le t\le T}\int_0^t \|u\|^6 ds \\ & +\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_D |u|^2\dot{W}_1dxds\right|^2+\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_D n\dot{W}_1dxds\right|^2 \\ & +\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t \text{Re}\int_D u\bar{\dot{W}}_{2x}dxds\right|^2+\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t \text{Im}\int_D \bar{u}_x\dot{W}_{2x}dxds\right|^2 \\ & +\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t \text{Im}\int_D (n\bar{u}+\bar{g})\dot{W}_2dxds\right|^2. \end{aligned} \tag{3.27}$$

After estimating each term of the right hand side of (3.27), we obtain

$$\mathbb{E}\sup_{0\le t\le T} H_0(t) \le C_T(E_0,f,g,q_1,q_2). \tag{3.28}$$

On the other hand, for $H_0^p(t)$ $(p\ge 1)$, integrating from 0 to t and taking the supremum and the expectation on the both sides of (3.22) yields

$$\begin{aligned} & \mathbb{E}\sup_{0\le t\le T} H_0^p(t) \le \mathbb{E}H_0^p(0)+c(q_1,q_2)T+c\mathbb{E}\sup_{0\le t\le T}\int_0^t \|u\|^{6p}ds+c \\ & +p^2\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t H_0^{p-1}(s)\int_D |u|^2\dot{W}_1dxds\right|^2+p^2\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t 2H_0^{p-1}(s)\int_D n\dot{W}_1dxds\right|^2 \\ & +p^2\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t H_0^{p-1}(s)4\text{Re}\int_D u\bar{\dot{W}}_{2x}dxds\right|^2+p^2\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t 2H_0^{p-1}(s)\text{Im}\int_D \bar{u}_x\dot{W}_{2x}dxds\right|^2 \\ & +p^2\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t 2H_0^{p-1}(s)\text{Im}\int_D (n\bar{u}+\bar{g})\dot{W}_2dxds\right|^2. \end{aligned} \tag{3.29}$$

Now we estimate each term of (3.29). For the third term on the right hand of (3.29),

$$c\mathbb{E}\sup_{0\le t\le T}\int_0^t \|u\|^{6p}ds \le c\mathbb{E}\int_0^T \|u\|^{6p}ds \le C_T.$$

For the fifth and sixth terms on the right hand of (3.29), we have

$$p^2\mathbb{E}\sup_{0\le t\le T}|\int_0^t H_0^{p-1}(s)\int_D |u|^2\dot{W}_1dxds|^2+p^2\mathbb{E}\sup_{0\le t\le T}|\int_0^t 2H_0^{p-1}(s)\int_D n\dot{W}_1dxds|^2$$
$$\le c\|q_1\|^2\mathbb{E}\int_0^T(H_0^{2p}(s)+\|u\|^{2p}+\|u\|^{6p}+\|g\|^{2p})ds\le C_T.$$

For the seventh and eighth terms on the right hand of (3.29), using a similar method, we estimate

$$p^2|\int_0^t H_0^{p-1}(s)4\mathrm{Re}\int_D u\bar{\dot{W}}_{2x}dxds|^2+p^2|\int_0^t 2H_0^{p-1}(s)\mathrm{Im}\int_D \bar{u}_x\dot{W}_{2x}dxds|^2$$
$$\le c\|q_{2x}\|^2\mathbb{E}\int_0^T(H_0^{2p}(s)+\|u\|^{2p}+\|u\|^{6p}+\|g\|^{2p})ds\le C_T.$$

For the last term on the right hand of (3.29), we estimate

$$p^2\mathbb{E}\sup_{0\le t\le T}|\int_0^t 2H_0^{p-1}(s)\mathrm{Im}\int_D(n\bar{u}+\bar{g})\dot{W}_2dxds|^2$$
$$\le c\|q_{2x}\|^2\mathbb{E}\int_0^T(H_0^{2p}(s)+\|u\|^{2p}+\|u\|^{6p}+\|g\|^{2p})ds+c\|q_2\|^2\mathbb{E}\int_0^T(H_0^{2p}(s)+\|g\|^{2p})ds$$
$$\le C_T.$$

Then, according to the above estimates, from (3.29) we obtain

$$\mathbb{E}\sup_{0\le t\le T}H_0^p(t)\le\mathbb{E}H_0^p(0)+C_T(q_1,q_2,E_0,n_1,n_0)\le C_T(q_1,q_2,E_0,n_1,n_0). \tag{3.30}$$

Moreover, from (3.20) it is inferred that for $p\ge1$

$$\begin{aligned}&\mathbb{E}\sup_{0\le t\le T}(\|u_x\|^{2p}+\|n\|^{2p})\\ \le& c\mathbb{E}\sup_{0\le t\le T}H_0^p(t)+c\mathbb{E}\sup_{0\le t\le T}(\|u\|^{2p}+\|u\|^{6p}+\|g\|^{2p})\le C_T.\end{aligned} \tag{3.31}$$

Then we obtain

$$\mathbb{E}\sup_{0\le t\le T}\|(n,u)\|_{V_0}^{2p}\le C_T(f,g,q_1,q_2,u_0,n_0). \tag{3.32}$$

Then the proof is complete. ∎

3.2. *A priori* estimates in V_1.

LEMMA 3.3. *Assume that $(n_0,u_0)\in V_1$, $q_1\in H^1(D)$, $q_2\in H^2(D)$, $f,g\in H^1(D)$. Then for any $T>0$ and $p\ge1$, we have $(n,u)\in L^{2p}(\Omega;L^\infty(0,T;V_1))\cap L^\infty(0,\infty;L^{2p}(\Omega;V_1))$.*

Proof. Applying the Itô formula to $\|u_{xx}\|^2$, we have

$$\frac{d}{dt}\|u_{xx}\|^2=2\mathrm{Re}\int_D\bar{u}_{txx}u_{xx}dx+\|q_{2xx}\|^2. \tag{3.33}$$

Since $u_t=i(u_{xx}-nu+i\alpha u-g-\dot{W}_2)$ and

$$\frac{d}{dt}\mathrm{Re}\int_D nu\bar{u}_{xx}dx$$

$$= \mathrm{Re}\int_D (n_t u\bar{u}_{xx} + nu_t\bar{u}_{xx} + nu\bar{u}_{xxt} + q_1\bar{q}_{2xx} + i(q_1\bar{q}_{2xx}u - q_1q_2\bar{u}_{xx}))dx, \tag{3.34}$$

we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|u_x\|^2 - 2\mathrm{Re}\int_D nu\bar{u}_{xx}dx - 2\mathrm{Re}\int_D g\bar{u}_{xx}dx\right) + 2\alpha\|u_{xx}\|^2 = \|q_{2xx}\|^2 + 2\|q_{2x}\|^2\\&+ 2\mathrm{Re}\int_D i(q_1q_2\bar{u}_{xx} - q_1\bar{q}_{2xx}u)dx + 2\alpha\mathrm{Re}\int_D nu\bar{u}_{xx}dx + 2\alpha\mathrm{Re}\int_D g\bar{u}_{xx}dx\\&- 2\mathrm{Re}\int_D n_t u\bar{u}_{xx}dx - 2\mathrm{Re}\int_D nu_t\bar{u}_{xx}dx + 2\mathrm{Re}\int_D \dot{W}_2(\bar{u}_{xxt} + \alpha\bar{u}_{xx})dx.\end{aligned} \tag{3.35}$$

As before, we can estimate each term on the right hand side of (3.35) using Höder's inequality, the Gagliardo-Nirenberg inequality, and Young's inequality. Then, from (3.35) we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|u_x\|^2 - 2\mathrm{Re}\int_D nu\bar{u}_{xx}dx - 2\mathrm{Re}\int_D g\bar{u}_{xx}dx\right) + 2\alpha\|u_{xx}\|^2\\&\le \frac{\alpha}{2}\|u_{xx}\|^2 + \frac{\delta}{2}\|n_x\|^2 + c(\|q_1\|^4_{H^1} + \|q_2\|^4_{H^2} + \|f\|^4 + \|g\|^8 + 1 + \|u\|^8_{H^1} + \|n\|^{12})\\&\quad + 2\mathrm{Re}\int_D \dot{W}_2(\bar{u}_{xxt} + \alpha\bar{u}_{xx})dx - 2\mathrm{Re}\int_D u\bar{u}_{xx}\dot{W}_1dx + 2\mathrm{Re}\int_D in\bar{u}_{xx}\dot{W}_2dx.\end{aligned} \tag{3.36}$$

In addition, applying the Itô formula to $\|n_x\|^2$, we have

$$\frac{d}{dt}\|n_x\|^2 = 2\int_D n_x n_{xt}dx + \|q_{1x}\|^2. \tag{3.37}$$

Since $n_t = -(|u|^2)_x - \delta n + f + \dot{W}_1$ and

$$\frac{d}{dt}i\int_D (u_x\bar{u}_{xx} - \bar{u}_x u_{xx})dx = 2i\int_D (u_{xt}\bar{u}_{xx} - \bar{u}_{xt}u_{xx})dx + 2i\mathrm{Re}\int_D q_{2x}\bar{q}_{2xx}dx,$$

we obtain

$$\begin{aligned}&\frac{d}{dt}(\|n_x\|^2 + i\int_D (u_x\bar{u}_{xx} - \bar{u}_x u_{xx})dx) + 2\delta\|n_x\|^2\\&\le \frac{\alpha}{2}\|u_{xx}\|^2 + \frac{\delta}{2}\|n_x\|^2 + c(\|g_x\|^2 + \|f_x\|^2 + \|q_2\|^2_{H^2} + \|q_{1x}\|^2)\\&\quad + 2\int_D n_x\dot{W}_{1x}dx + 4\mathrm{Re}\int_D \bar{u}_{xx}\dot{W}_{2x}dx.\end{aligned} \tag{3.38}$$

So from (3.36) and (3.38), taking $\eta = \min\{\alpha, \delta\}$, and letting

$$H_1(t) = \|u_x\|^2 + \|n_x\|^2 - 2\mathrm{Re}\int_D nu\bar{u}_{xx}dx - 2\mathrm{Re}\int_D g\bar{u}_{xx}dx + i\int_D (u_x\bar{u}_{xx} - \bar{u}_x u_{xx})dx,$$

we obtain

$$\begin{aligned}&\frac{d}{dt}H_1(t) + \eta H_1(t)\\&\le c(\|u\|^8_{H^1} + \|n\|^{12} + \|q_1\|^4_{H^1} + \|q_2\|^4_{H^2} + \|f\|^4_{H^1} + \|g\|^8_{H^1} + 1)\end{aligned}$$

$$+2\mathrm{Re}\int_D \dot{W}_2(\bar{u}_{xxt}+\alpha\bar{u}_{xx})dx-2\mathrm{Re}\int_D u\bar{u}_{xx}\dot{W}_1dx+2\mathrm{Re}\int_D in\bar{u}_{xx}\dot{W}_2dx$$
$$+2\int_D n_x\dot{W}_{1x}dx+4\mathrm{Re}\int_D \bar{u}_{xx}\dot{W}_{2x}dx. \tag{3.39}$$

Multiplying by $e^{\eta t}$, integrating from 0 to t, and taking expectation on the both sides of (3.39), by (3.21) we obtain

$$\mathbb{E}H_1(t)\le e^{-\beta t}\mathbb{E}H_1(0)+c(f,g,q_1,q_2,u_0)\le c(f,g,q_1,q_2,u_0)\le C,\quad t>0, \tag{3.40}$$

where C is independent of T.

Since

$$H_1(t)\ge\frac{1}{2}(\|u_{xx}\|^2+\|n_x\|^2)-c(\|u_x\|^8+\|n\|^4+1+\|g\|^2), \tag{3.41}$$

for any $t>0$, we obtain

$$\mathbb{E}(\|u_{xx}\|^2+\|n_x\|^2)\le c\mathbb{E}(\|u_x\|^8+\|n\|^4+1+\|g\|^2)+c\mathbb{E}H_1(t)\le C, \tag{3.42}$$

where C is independent of T.

Further, we estimate $H_1^p(t)$ for $p\ge1$. As before, applying the Itô formula to $H_1^p(t)$ and taking expectation, one gets

$$\mathbb{E}H_1^p(t))\le e^{-\frac{\eta p}{2}t}\mathbb{E}H_1^p(0)+c+c\mathbb{E}\int_0^t e^{-\frac{\eta p}{2}(t-s)}(\|u_x\|^{8p}+\|n\|^{12p}+1)ds. \tag{3.43}$$

By (3.21) we obtain

$$\mathbb{E}H_0^p(t))\le e^{-\frac{\eta p}{2}t}\mathbb{E}H_0^p(0)+c\le C,\quad t>0, \tag{3.44}$$

where C is independent of T. Therefore, by (3.41), for any $t>0$, we have

$$\mathbb{E}(\|u_{xx}\|^{2p}+\|n_x\|^{2p})\le C. \tag{3.45}$$

After integrating from 0 to t and taking the supremum and expectation on both sides of (3.39), as with the estimates in Lemma 3.2 for each term, we deduce

$$\mathbb{E}\sup_{0\le t\le T}H_1(t)\le C_T(u_0,f,g,q_1,q_2). \tag{3.46}$$

For $H_0^p(t)$ $(p\ge1)$, integrating from 0 to t, taking the supremum and the expectation, and estimating each term, we obtain

$$\mathbb{E}\sup_{0\le t\le T}H_1^p(t)\le\mathbb{E}H_1^p(0)+C_T(f,g,q_1,q_2,u_0,n_0)\le C_T(f,g,q_1,q_2,u_0,n_0). \tag{3.47}$$

Moreover, from (3.41) it is inferred that for $p\ge1$

$$\mathbb{E}\sup_{0\le t\le T}(\|u_{xx}\|^{2p}+\|n_x\|^{2p})\le C_T(f,g,q_1,q_2,u_0,n_0). \tag{3.48}$$

Then we obtain

$$\mathbb{E}\sup_{0\le t\le T}\|(n,u)\|_{V_1}^{2p}\le C_T(f,g,q_1,q_2,u_0,n_0), \tag{3.49}$$

and the proof is complete. □

3.3. *A priori* estimates in V_2. Similarly to subsections 3.1 and 3.2, using the same method and idea, we can obtain *a priori* estimates in V_2. For convenience we only give the idea of the proof. Applying the Itô formula to $\|n_{xx}\|^2$ and $\|u_{xxx}\|^2$ respectively, and using (1.3)-(1.4), we can obtain some inequalities by Hölder's inequality, Young's inequality, and the Gagliardo-Nirenberg inequality together. Then, taking the supremum and the expectation for corresponding inequalities and using Gronwall-type estimates on $\|n_{xx}\|^2$ and $\|u_{xxx}\|^2$, it is easy to obtain the following lemma.

LEMMA 3.4. *Assume that $(n_0,u_0)\in V_2$, $q_1\in H^2(D)$, $q_2\in H^3(D)$, and $f,g\in H^2(D)$. Then for any $T>0$ and $p\geq 1$, we have $(n,u)\in L^{2p}(\Omega;L^\infty(0,T;V_2))\cap L^\infty(0,\infty;L^{2p}(\Omega;V_2))$.*

4. Proofs of Theorem 1.1 and Theorem 1.2

By the above *a priori* estimates, we prove the existence and uniqueness of a solution for stochastic equations (1.3)-(1.4) in spaces V_1 as in Theorem 1.1.

The proof of the Theorem 1.1.

Proof. First, we know that $(n_0,u_0)\in V_1$. Consider $\{e_i(x)\}_{i=1}^\infty$ an orthonormal basis of eigenvectors of the Laplace operator on D, which is an orthonormal basis of $L^2(D)$. Let P^k be the projection from $L^2(D)$ onto the space spanned by $\{e_i : i=1,2,\cdots,k\}$. Then the approximation solution (n^k,u^k) solves the approximation problem

$$n_t^k + P^k(|u^k|^2)_x + \delta n^k = f^k + \dot{W_1}^k, \tag{4.1}$$

$$iu_t^k + u_{xx}^k - P^k(n^k u^k) + i\alpha u^k = \dot{W_2}^k, \tag{4.2}$$

where P^k is the projector onto the first k vectors e_i, $\dot{W_1}^k = P^k\dot{W}_1$, $\dot{W_2}^k = P^k\dot{W}_2$, and P^k commutes with the operator Δ. Now we will treat the above equations pathwise by introducing the following random processes solving

$$\eta_t^k + \delta\eta^k = \dot{W_1}^k, \tag{4.3}$$

$$i\xi_t^k + \xi_{xx}^k + i\alpha\xi^k = \dot{W_2}^k, \tag{4.4}$$

with Dirichlet boundary conditions $\eta(0,t)=\eta(L,t)=0$, $\xi(0,t)=\xi(L,t)=0$, and initial conditions $\eta_t(x,0)=0$, $\eta(x,0)=0$, $\xi(x,0)=0$, $x\in D$, where $\delta>0$, $\alpha>0$.

Following the same methods as in Section 3, for any $T>0$ and almost all $\omega\in\Omega$, we have

$$\eta\in C(0,T;H_0^1(D)),\quad \xi\in C(0,T;H^2(D)\cap H_0^1(D)). \tag{4.5}$$

Also, there is the following estimate

$$\mathbb{E}(\|\eta_x\|^2 + \|\xi_{xx}\|^2)\leq C, \tag{4.6}$$

for a positive constant C independent of T. In addition, for any $T>0$,

$$\mathbb{E}\sup_{0\leq t\leq T}(\|\eta_x\|^2 + \|\xi_{xx}\|^2)\leq C_T \tag{4.7}$$

holds for a positive constant C_T dependent on T. Now let $R^{k,M}=(N^{k,M},U^{k,M})$ be the solution of the following equations

$$N_t^{k,M} + \chi_M(\|R^{k,M}\|_{V_1})P^k(|u^{k,M}|^2)_x + \delta N^{k,M} = f^k, \tag{4.8}$$

$$iU_t^{k,M} + U_{xx}^{k,M} - \chi_M(\|R^{k,M}\|_{V_1}) P^k(n^{k,M}u^{k,M}) + i\alpha U^{k,M} = g^k, \tag{4.9}$$
$$N^{k,M}(x,0) = P^k n_0, U^{k,M}(x,0) = P^k u_0.$$

Here $n^{k,M} = N^{k,M} + P^k\eta$, $u^{k,M} = U^{k,M} + P^k\xi$, and $\chi_M \in C_0^\infty(\mathbb{R})$ such that $\chi_M(r) = 1$ for $|r| \le M$ and $\chi_M(r) = 0$ for $|r| \ge 2M$. Notice that (4.8)-(4.9) are random differential equations with Lipschitz nonlinearity in finite dimension. Then, for almost all $\omega \in \Omega$, we have a unique solution $(N^{k,M}, U^{k,M})$ for (4.8)-(4.9). Define the stopping time by

$$\tau_M = \inf\{t > 0 : \|R\|_{V_1}^{k,M} \ge M\}$$

if the set $\{\|R\|_{V_1}^{k,M} \ge M\}$ is nonempty, otherwise $\tau_M = \infty$. Since τ_M is increasing in M, let $\tau_\infty = \lim_{M\to\infty} \tau_M$ almost surely. For $t < \tau_M$, we have

$$(N^{k,M}, U^{k,M}) + (P^k\eta, P^k\xi)$$

satisfing (4.1)-(4.2). By the estimates given in Subsection 3.2 and (4.6)-(4.7), for any $t \ge 0$ we have

$$\mathbb{E}\|(N^{k,M}, U^{k,M})\|_{V_1}^2 \le C(f, g, q_1, q_2, n_0, u_0) \tag{4.10}$$

where the positive constant $C(f, g, q_1, q_2, n_0, u_0)$ is independent of T and M. And for $T > 0$ we have

$$\mathbb{E} \sup_{0 \le t \le T \wedge \tau_M} \|(N^{k,M}, U^{k,M})\|_{V_1}^2 \le C_T(f, g, q_1, q_2, n_0, u_0) \tag{4.11}$$

with the positive constant $C(f, g, q_1, q_2, n_0, u_0)$, which is dependent on T but independent of M. Here $T \wedge \tau_M = \min\{T, \tau_M\}$. On the other hand we have

$$\begin{aligned} &\mathbb{E}\|(N^{k,M}(T\wedge\tau_M), U^{k,M}(T\wedge\tau_M))\|_{V_1}^2 \\ \ge &\mathbb{E}[I(\tau_M \le T)|(N^{k,M}(T\wedge\tau_M), U^{k,M}(T\wedge\tau_M))|_{V_1}^2] \ge M^2\mathbb{P}(\tau_M \le T), \end{aligned}$$

where $I(\tau_M \le T) = 1$ for $\tau_M \le T$ and $I(\tau_M \le T) = 0$ for $\tau_M > T$. Then, according to (4.10), we have

$$\mathbb{P}(\tau_M \le T) \le \frac{1}{M^2} C(f, g, q_1, q_2, n_0, u_0).$$

According to the above estimate and the Borel-Cantelli lemma, for any $T > 0$ we have

$$\mathbb{P}(\tau_\infty > T) = 1.$$

So we know that

$$(N^k, U^k) = \lim_{M\to\infty} (N^{k,M}, U^{k,M})$$

satisfies the following random differential equations

$$N_t^k + P^k(|u^{k,M}|^2)_x + \delta N^k = f^k, \tag{4.12}$$
$$iU_t^k + U_{xx}^k - P^k(n^k u^k) + i\alpha U^k = g^k, \tag{4.13}$$

with initial conditions $N^k(0) = P^k n_0$, $U^k(0) = P^k u_0$. Then (N^k, U^k) satisfies the estimates (4.10) and (4.11), and for any $t \ge 0$ we get $(n^k, u^k) = (N^k, U^k) + (P^k\eta, P^k\xi)$ is the unique global solution of (4.1)-(4.2).

Now, we will consider (4.12)-(4.13) for fixed ω. Firstly, by (4.11), for any $T>0$ we have

$$\mathbb{P}\left(\bigcap_{L=1}^{\infty}\bigcup_{l=1}^{\infty}\cap_{k=l}^{\infty}\left\{\sup_{0\le t\le T}\|(N^k,U^k)\|_{V_1}^2\ge L\right\}\right)=0.$$

We let

$$\tilde{\Omega}=\bigcap_{L=1}^{\infty}\bigcup_{l=1}^{\infty}\cap_{k=l}^{\infty}\left\{\sup_{0\le t\le T}\|(N^k,U^k)\|_{V_1}^2\le L\right\}.$$

Then $\mathbb{P}(\Omega\backslash\tilde{\Omega})=0$. Therefore, for any fixed $\omega\in\tilde{\Omega}$, there is an $r(\omega)$ with $0<r(\omega)<\infty$ such that (ref [14])

$$\sup_{0\le t\le T}\|(N^k,U^k)\|_{V_1}^2\le r(\omega). \tag{4.14}$$

Then we can extract a subsequence still denoted by (N^k,U^k), such that, for any $T>0$, N^k converges to N weakly star in $L^\infty(0,T;H_0^1(D))$ and U^k converges to U weakly star in $L^\infty(0,T;H^2(D)\cap H_0^1(D))$. These convergences are sufficient to pass the limit $k\to\infty$ in linear terms, but we need a strong convergence of U^k for nonlinear terms. In fact, by (4.13) and the estimate (4.14), we know $U_t^k\in L^\infty(0,T;L^2(D))$. Then we can further extract a subsequence still denoted by U^k such that U^k converges to U strongly in $L^\infty(0,T;H_0^1(D))$. Then by a standard procedure we can pass the limit $k\to\infty$ for the nonlinear term. So we can show that $(N,U)\in L^\infty(0,T;V_1)$ is a weak solution of

$$N_t+(|u|^2)_x+\delta N=f, \tag{4.15}$$
$$iU_t+U_{xx}-nu+i\alpha U=g, \tag{4.16}$$

with initial conditions $N(0)=n_0(x)$, $U(0)=u_0(x)$, $x\in D$.

Then $(n,u)=(N,U)+(\eta,\xi)$ is a solution of (1.3)-(1.4) and satisfies the estimates given in Subsection 3.2. In the following we prove the continuity of the solution. For any $\omega\in\tilde{\Omega}$, $(|u|^2)_x\in L^\infty(0,T;H_0^1(D))$, we can obtain $N_t=-(|u|^2)_x-\delta N+f\in H_0^1(D)$. Then by Lemma 3.2 in [19], we can see that for almost all $\omega\in\Omega$ there is $N\in C(0,T;H_0^1(D))$. Using similar methods and noticing $U_t\in L^\infty(0,T;L^2(D))$ almost surely, according to [16] we can obtain $U\in C(0,T;H^2(D)\cap H_0^1(D))$. Then by definition of N and U we have $(n,u)\in C(0,T;V_1)$ almost surely. So the solution (n,u) is continuous from $[0,T]$ to V_1 almost surely.

As the noise is additive, we can follow the same approach as [11]. So the solution (n,u) is unique in $L^\infty(0,T;V_1)$ almost surely. □

The proof of Theorem 1.2 is similar to that of Theorem 1.1. It is omitted.

5. Proofs of Theorem 1.3–Theorem 1.5

Now we study the asymptotic behavior of the solution for (1.3)-(1.4). We will construct a random attractor for stochastic long-short wave equations in V_1 equipped with the weak topology. Some basic concepts related to random attractors for random dynamical systems are found in [1, 2, 20, 21].

The following existence result for a random attractor for a continuous RDS can be found in [1, 2]. It gives a sufficient condition for the existence of a random attractor.

THEOREM 5.1. *(see [1, 2]) Suppose Φ is a RDS on a Polish space (E,d) and there exists a random compact set $K(\omega)$ absorbing every bounded deterministic set $D \subset E$. Then the set*

$$\mathcal{A}(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))},$$

is a global random attractor for RDS Φ.

Next, using (5.1), we consider the random attractors for the stochastic long-short wave equations in V_1 and V_2 on the basis of *a priori* estimates in Section 3.

The proof of Theorem 1.3

Proof. According to the former analysis, we can consider the properties of solution a (N,U) of the system (4.15)-(4.16) instead of (n,u) of stochastic long-short wave equations. For any (n_0,u_0) and any $T>0$, the system (4.15)-(4.16) has a unique solution $(N,U) \in C(0,T;V_1)$ for almost all $\omega \in \Omega$. Noticing the system (4.15)-(4.16) has coefficients driven by θ_t, (N,U) defines a random dynamical system on V_1. Therefore, $(n,u) = (N+\eta, U+\xi)$ also defines a continuous random dynamical system on V_1, which is denoted by $\Phi(t,\omega)$, and $\Phi(t,\omega)$ is weakly continuous almost surely on V_1.

Denoted the ball center at 0 with radius r in V_1 by $B(0,r)$. Then, by the estimates given in Section 3, there is a random variable $R(\omega)$ such that, for any $r>0$, $(n,u) \in B(0,r)$. So there exists a random time $t_r(\omega)>0$, such that, for all $t>t_r(\omega)$ and almost all $\omega \in \Omega$,

$$\|\Phi(t,\theta_{-t}\omega)(n_0,u_0)\|_{V_1} \leq R(\omega).$$

Let

$$\mathcal{A}(\omega) = \cap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi(t,\theta_{-t}\omega) B(0,R(\omega))}^{V_1^w},$$

where the closure is taken with respect to the weak topology of V_1. Now we show that the random attractor $\mathcal{A}(\omega)$ is tempered. By the estimates obtained in Section 3, we have

$$\mathbb{E} \sup_{0 \leq t \leq 1} R^2(\theta_t \omega) < \infty.$$

Then, by Birkhoff's ergodic theorem [4],

$$\lim_{s \to \pm\infty} \frac{\sup_{t \in [0,1]} R^2(\theta_{t+s}\omega)}{s} = 0$$

on a $\theta-$invariant subset of Ω with full probability measure, i.e., $R(\omega)$ is tempered. Then we know $\mathcal{A}(\omega)$ is tempered. The proof is completed. □

Similarly to Theorem 1.3, using the same methods and ideas of proof, we can prove Theorem 1.4. That is to say, there is a random attractor for the stochastic long-short wave equations in V_2.

Next, the proof of Theorem 1.5 is given.

The proof of Theorem 1.5

Proof. If $(n_0,u_0)\in V_2$, by the results given in Section 3 and Section 4, (1.3)-(1.4) has a unique solution (n,u) with $(n(0),u(0))=(n_0,u_0)$, which, for any $t>0$, satisfies

$$\mathbb{E}(\|n_{xx}\|^2+\|u_{xxx}\|^2)\leq C, \tag{5.1}$$

for a positive constant $C>0$ which is independent of $t>0$.

Now let μ_t be the distribution of (n_t,n,E) for $t\geq 0$. Following the classical Bogolyubov-Krylov argument [3], we define

$$\bar{\mu}_t=\frac{1}{t}\int_0^t \mu_s ds$$

as

$$\bar{\mu}_t(\Gamma)=\frac{1}{t}\int_0^t \mu_s(\Gamma)ds$$

for any Borel set Γ of V_1, that is $\Gamma\in\mathscr{B}(V_1)$. By (5.1) we have

$$\int_{V_1}\|(n,u)\|_{V_2}^2\bar{\mu}_t(dv)=\frac{1}{t}\int_0^t \mathbb{E}\|(n(s),u(s))\|_{V_2}^2 ds\leq C.$$

By Chebyshev's inequality and the fact that V_2 has a compact embedding into V_1, $\{\bar{\mu}_t\}_{t\geq 0}$ is tight in V_1. Then there is a sequence $\{\bar{\mu}_{t_k}\}$ with $t_k\to\infty$ as $k\to\infty$ and a probability measure μ on V_1 such that $\{\bar{\mu}_{t_k}\}\to\mu$ weakly as $k\to\infty$. Therefore, μ is a stationary measure for stochastic long-short wave equations on V_1 by using the standard arguments as in [4]. Moreover, by (5.1), μ is in fact supported on V_2, namely, μ is a stationary measure for stochastic long-short wave equations on V_2. The proof is completed. □

Acknowledgement. The work is supported by NNSFC(No.11301097), CSC, GXNSF Grant(No.2013GXNSFAA019001, 2014GXNSFAA118016), Guangxi Education Institution Scientific Research Item(No.2013YB170), Guangxi University of Science and Technology Grant(No.03081587,03081588).

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