COMMUN. MATH. SCI.
Vol. 13, No. 7, pp. 1649–1664

ON MULTISCALE MODELS OF PEDESTRIAN CROWDS FROM MESOSCOPIC TO MACROSCOPIC*

NICOLA BELLOMO† AND ABDELGHANI BELLOUQUID‡

Abstract. This paper deals with the derivation of macroscopic equations from the underlying mesoscopic description that is suitable for capturing the main features of pedestrian crowd dynamics. The interactions are modeled by means of the theoretical tools of game theory while the macroscopic equations are derived from asymptotic limits.

Key words. Pedestrian crowds, complexity, scaling, nonlinear interactions.

AMS subject classifications. 90B20, 92D50, 92-02, 35L65, 82-02.

1. Introduction

The modeling, qualitative analysis, and simulation of crowd dynamics appears to be a challenging research field which is rapidly growing. It provides mathematical tools that can be used to study and optimize some of the needs of our society such as evacuation dynamics in conditions of danger and panic. The modeling approach can be developed at three classical scales; namely, the microscopic scale where the dynamics of each pedestrian is modeled individually in relation to other pedestrians, the macroscopic scale which refers to the dynamics of locally averaged quantities such as density, momentum, and energy, and the mesoscopic (kinetic) scale which corresponds to the dynamics of a probability distribution over the microscopic state of the pedestrians.

The state-of-the-art has been documented in some review papers that report the pertinent literature as well as a critical analysis on the validity of several, rather different, models. Among others, the survey by Helbing [15] focuses on the physics of the system, viewed as a multi-particle system, while the more recent survey [4] is specifically devoted to crowd dynamics and presents a critical analysis of different mathematical approaches to crowd modeling. This paper points out that not only do different mathematical structures correspond to the aforementioned scales, but also that none of them is completely satisfactory since various technical and conceptual advantages and drawbacks are linked to modeling at each scale. At present, the literature in the field is not as extensive as that of individual-based and macro-scale models. However, some interesting recent research contributions can be brought to the reader's attention, for example [12], which deals with a hierarchy of models, and [1], which presents a kinetic theory approach to modeling evacuation dynamics. A detailed analysis of micro-scale interactions has been presented in [7] and [9].

A multiscale approach is therefore necessary to obtain a detailed description of the complex dynamics of pedestrians in unbounded and bounded domains. Useful information on the empirical data and the validation of models has been given in the ETH-report

*Received: January 28, 2014; accepted (in revised form): July 27, 2014. Communicated by Pierre Degond.
The first author is partially supported by the MIUR, PRIN 2012 Project “Mathematical problems in kinetic theory and applications” prot. 2012AZS52J. The second author is supported by Hassan II Academy of Sciences and Technology (Morocco).
†Department of Mathematics, Faculty Sciences, King Abdulaziz University, Jeddah, Saudi Arabia; and SiTI, Sistemi Territoriali per l'Innovazione, via Pier Carlo Boggio, 61, Città, Metropolitana di Torino, Italy (nicola.bellomo@polito.it).
‡Cadi Ayyad University, Ecole Nationale des Sciences Appliquées, BP Guéliz, Boulevards Abdelkrim Al Khattabi, Marrakech, Morroco (a.bellouquid@ uca.ma).

[8], and in some recent papers, concerning, among others, [19] self-organization ability and [20] heterogeneity and fundamental diagrams consisting of the mean velocity and flux versus the local density.

This present paper pursues two specific objectives: the derivation of a general mathematical structure based on appropriate developments of the kinetic theory suitable for capturing the main features of crowd dynamics and the derivation of macroscopic equations from the underlying mesoscopic description. The derivation of hydrodynamic-type models should in fact be obtained, according to the authors, from a suitable asymptotic limit of the kinetic description. Hence, the mathematical structure of macroscopic models should not be heuristically postulated "a priori". The contents of the paper are as follows: Section 2 deals with modeling at low scales where interactions are described in line with [2, 3] through stochastic games, and subsequently, it is shown how modeling at a low scale leads to kinetic-type models. Section 3 shows how macroscopic equations can be obtained from the underlying description that is offered by kinetic theory methods.

2. Modeling interactions at the micro-scale

The derivation of kinetic type equations requires a detailed analysis of the interactions at a micro-scale, namely at the scale of individual pedestrians related to the statistical representation of the overall system; this requires a suitable probability distribution over the micro-state.

Let us consider the `representation` of a large system of pedestrians moving in a whole space, $\mathbf{R}^2$, starting at time $t=0$ in a domain $\Sigma \subset \mathbf{R}^2$. The micro-state can be defined by position $\mathbf{x}=\{x,y\}\in\mathbf{R}^2$ and by a vector velocity $\mathbf{v}=\{v_x,v_y\}$ where x and y are coordinates of a system of orthogonal axes and v_x,v_y are components of the velocity over the aforementioned axes. Dimensionless variables can be used by referring the space coordinates to the largest dimension ℓ of Σ and the velocity modulus v to the largest admissible velocity v_M that can be reached by a speedy pedestrian in optimal conditions of the environment in which the crowd moves.

Generally, the number of pedestrians is not large enough to justify the assumption of a continuous distribution of the probability distribution over the micro-state. Therefore, discrete velocities and activity variables are considered analogously to the case of vehicular traffic [5, 13]. More precisely, polar coordinates with discrete values are used for the velocity variable $\mathbf{v}=\{v,\theta\}$ where v is the velocity modulus and θ identifies the velocity directions with respect to the x-axis:

$$I_\theta=\{\theta_1=0,\dots,\theta_i,\dots,\theta_n=\frac{n-1}{n}2\pi\},\quad I_v=\{v_1=0,\dots,v_j,\dots,v_m=1\}, \tag{2.1}$$

while the vector velocities are denoted by $\mathbf{v}_{ij}=\{\theta_i,v_j\}$.

The corresponding statistical representation is delivered from the one particle distribution function,

$$f_{ij}(t,\mathbf{x},\mathbf{v})=f(t,\mathbf{x})\otimes\delta(\theta-\theta_i)\otimes\delta(v-v_j), \tag{2.2}$$

where f is divided by the maximal packing density ρ_M that corresponds to the maximum filling of pedestrians, based on their average size, per unit area.

Macro-scale quantities are obtained by weighted sums. For instance, the local density and mean velocity $\mathbf{q}$, referring to ρ_M and v_M, respectively, are given by

$$\rho(t,\mathbf{x})=\sum_{i=1}^{n}\sum_{j=1}^{m}f_{ij}(t,\mathbf{x}),\qquad \mathbf{q}(t,\mathbf{x})=\frac{1}{\rho(t,\mathbf{x})}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{v}_{ij}f_{ij}(t,\mathbf{x}). \tag{2.3}$$

Let us now consider `the dynamics of interactions` as, in the kinetic theory [10], the derivation of models viewed as evolution equations for the distribution function requiring a detailed modeling of the interactions at the micro-scale. This topic has already been treated in [2]; therefore, the hallmarks that lead to modeling interactions are simply summarized here in order to avoid repetition. In short:

1. Each pedestrian, viewed as a micro-scale individual, is an active particle, called the ij-particle, with a state which is given in each $\mathbf{x}$ at time t by $\{\theta_i, v_j\}$. The particles have an interaction domain Ω which could in general depend on $\mathbf{x}$ and a preferred velocity direction which corresponds to a well defined objective.
2. Active particles have the ability to perceive within Ω a density ρ^a that depends on both the local density and the density gradients in direction of the motion. This density, ρ^a, which differs from the true local density [11], is called the `perceived density`.
3. Interaction dynamics involve three types of particles: the hk-candidate particle, the rs-field particles, and the ij-test particle. The hk-candidate particle has state $\{\Theta_h, v_k\}$ which falls, in probability, into the ij-state after interactions with the rs-field particles. The rs-field particles have state $\{\Theta_r, v_s\}$ and lie in Ω. The ij-test particle, which is a representative of the whole subsystem, modifies its velocity when it interacts with all the field particles in its visibility zone.
4. The frequency of interaction between the candidate and the field particles in $\mathbf{x}^* \in \Omega$ is modeled by the `interaction rate` $\eta(\rho^a(t,\mathbf{x}^*))$ which depends on ρ^a.
5. The interactions that modify the velocity variable are modeled by the probability $\mathcal{A}_{hk}^{rs}(ij)$ that a candidate hk-particle adjusts its velocity and state to θ_i, v_j after interaction with a rs-field particle. The components of $\mathcal{A}_{hk}^{rs}(ij)$, called the `table of games`, depend on ρ^a and on a parameter $\alpha \in [0,1]$, which corresponds to physical conditions, such as light of road conditions, in the close proximity.

REMARK 2.1. The derivation of the macro-scale models here does not refer to a specific model, but to all those that can be captured by a general structure, which includes model [2] with the addition of the use, in the table of games, of the perceived local density $\rho^a \neq \rho$. Said density is different from the real one; namely, $\rho^a > \rho$ for positive gradients along the direction of the velocity, and $\rho^a < \rho$ when the gradients are negative. The aforementioned structure can also include, as shown in the model proposed in [1], nonlocal interactions with walls, obstacles, and exits.

REMARK 2.2. The parameter $\alpha \in [0,1]$ models the physical quality of the ambient environment where pedestrians move between the two limits from $\alpha = 0$, which corresponds to the worst conditions (such as lack of light and/or low quality floor) that prevent motion, to $\alpha = 1$ which corresponds to the best conditions that favor high speeds.

REMARK 2.3. The term defined in the last item satisfies the following probability constraint:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,x);\alpha) = 1, \quad \forall h,r = 1,\ldots,n, \quad k,s = 1,\ldots,m, \tag{2.4}$$

for all admissible values of the density and parameter α.

Some hallmarks are now given for the modeling of the aforementioned terms suitable to depict micro-scale interactions.

• `Perceived density:` Let us consider the direction ν of the velocity of each particle and the density gradients $\partial_\nu \rho$ along the mentioned direction; positive gradients increase the value of ρ^a from ρ to the maximum admissible value $\rho = 1$, and negative gradients decrease this value from from ρ to the lowest admissible value $\rho = 0$. Accordingly, the following model can be adopted:

$$\rho^a = \rho^a(\rho) = \rho + \frac{\partial_\nu \rho}{\sqrt{1 + (\partial_\nu \rho)^2}} \big((1-\rho) H(\partial_\nu \rho) + \rho H(-\partial_\nu \rho)\big), \tag{2.5}$$

where $H(\cdot)$ is the Heaviside function $H(\cdot \geq 0) = 1$ and $H(\cdot < 0) = 0$. Therefore, positive gradients increase the perceived density to the limit $\rho = 1$, and negative gradients decrease it to the limit $\rho = 0$.

• `Visibility domain and Interaction rate:` The visibility domain is an arc of a circle symmetric with respect to the direction of motion. In unbounded domains, it can be assumed to be independent of $\mathbf{x}$. In contrast, the frequency by which pedestrians, viewed as active particles, interact can depend on the density perceived in their interaction domain. For instance, the interaction density increases linearly with the local perceived density, $\eta(\rho^a(t, \mathbf{x})) = \eta^0 (1 + \rho^a(\rho)(t, \mathbf{x}))$.

• `Table of games:` $\mathcal{A}_{hk}^{rs}(ij)$ refers, as already mentioned, to the interactions that involve candidate and field particles. The candidate particle develops a decision process by which both the direction and velocity modules are modified. A specific model, limited to the case of $\rho^a = \rho$, was proposed in [2] based on the assumption that pedestrians play a game at each interaction with the stream and a prescribed direction and choose, with probability, the direction that is conditioned by the two trends. The dynamics depend on the local density; namely, the trend towards the target increases with $1 - \rho$, and the opposite occurs for the trend to the stream which increases with ρ. Paper [1] proposes an extended table of games, which includes nonlocal interactions to model the trend towards an exit from a domain with walls and the attempts of pedestrians to avoid contact with the wall.

This preliminary analysis easily leads to the derivation of a `general mathematical structure` which is obtained from a balance of the particles in the space of the microstates. It corresponds to equating the transport of the particles (on the left side) to the net balance (right-hand side) of the particles in the elementary volume of the phase space:

$$\begin{aligned}
&\big[\partial_t + v_j (cos\theta_i \mathbf{i} + sin\theta_i \mathbf{j}) \cdot \nabla_{\mathbf{x}}\big] f_{ij}(t, \mathbf{x}) = \mathcal{J}_{ij}[\mathbf{f}](t, \mathbf{x}) \\
&= \sum_{h,r=1}^{n} \sum_{k,s=1}^{m} \int_{\Omega[\mathbf{x}]} \eta(\rho^a(t, \mathbf{x}^*)) \mathcal{A}_{hk}^{rs}(ij)(\rho^a(t, \mathbf{x}^*); \alpha) f_{hk}(t, \mathbf{x}) f_{rs}(t, \mathbf{x}^*) \, d\mathbf{x}^* \\
&\quad - f_{ij}(t, \mathbf{x}) \sum_{k=1}^{n} \sum_{s=1}^{m} \int_{\Omega[\mathbf{x}]} \eta(\rho^a(t, \mathbf{x}^*)) f_{ks}(t, \mathbf{x}^*) \, d\mathbf{x}^*, \quad \mathbf{f} = \{f_{ij}\}.
\end{aligned} \tag{2.6}$$

This structure is proposed as a general framework which could lead to the derivation of specific models after the micro-scale interactions have been modeled. Indeed, the examples in the previously cited papers [1, 2] refer to the aforesaid structure while further models can be derived to include additional features, at the micro-scale, of interest for the applications. The initial value problem for a specific model which is derived according to the structure given by Equation (2.6) has been studied in [2] where existence for arbitrarily large times has been proved in the Banach space $X_T = C([0,T], L^1_{M_{2n,2m}})$ of the matrix-valued functions $f = f(t,\mathbf{x}) : [0,T]\times\Omega \to M_{2n,2m}$ endowed with the norm

$$\| f \|_{X_T} = \sup_{t\in[0,T]} \| f \|_1,$$

where

$$L^1_{M_{2n,2m}} = \{ f = (f_{ij}) \in M_{2n,2m} : \| f \|_1 = \sum_{i=1}^{n}\sum_{j=1}^{m} \int_\Omega | f_{ij}(t,\mathbf{x}) | \, d\mathbf{x} < \infty \}.$$

3. From kinetic to macroscopic models

3.1. Derivation of macroscopic models. This section deals with the derivation of macroscopic equations, which should be as accurate as possible, so that such models can be used for real-time numerical simulations and provide realistic and interpretable results. Finding such macroscopic models is, therefore, a main goal in the modeling process. The process is well known and can be summarized in a few lines as follows. A good kinetic model generates an infinite set of moment equations for the infinitely many velocity moments of the density function. Good macroscopic equations will emerge from a good closure relation, which may be a constitutive relation linking some higher order moments, a projection principle or an entropy maximization principle (if the system satisfies an entropy principle). Such closure methods in rarefied gas dynamics have received much attention in recent years [18].

For the traffic flow problem, the standard approach uses closure of the moment equations in the neighborhood of a kinetic equilibrium. We refer to [6, 16, 17] for details and simply observe that other closure procedures are hard to think for the traffic, and crowds flow since there is no information about entropy functionals. This is a good reason to investigate kinetic equilibrium for crowds; they give natural closure relations for the moment equations and therefore good macroscopic models. A technical difficulty is generated by the lack of an analytic expression of an equilibrium solution which, in the classical kinetic theory is the so-called "Maxwell distribution".

The method developed in this section consists of writing conservation equations for the mass and momentum and subsequently looking for an appropriate closure. In what follows, ρ^a is kept for any expression that depends on ρ, $\rho^a(\rho)$.

`Balance Equations` can be obtained by multiplying the non-homogeneous, kinetic equation (2.6) by one, and, by summing with respect to i and j, the continuity equation is obtained:

$$\partial_t \rho(t,\mathbf{x}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{q})(t,\mathbf{x}) = 0, \tag{3.1}$$

where the density and the flow are defined in (2.3).

The momentum equation is obtained by multiplying (2.6) by v_{ij} and summing with respect to i and j:

$$\partial_t(\rho\mathbf{q})(t,\mathbf{x}) + \nabla_{\mathbf{x}} \cdot \sum_{i=1}^{n}\sum_{j=1}^{m} (v_{ij} \otimes v_{ij} f_{ij}(t,\mathbf{x})) = \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij} \mathcal{J}_{ij}[\mathbf{f}](t,\mathbf{x}), \tag{3.2}$$

where it is important to distinguish between the transport and the source term and $\nabla_{\mathbf{x}} \cdot \sum_{i=1}^{n}\sum_{j=1}^{m} (v_{ij} \otimes v_{ij} f_{ij}(t,\mathbf{x}))$ denotes the vector

$$\nabla_{\mathbf{x}} \cdot \sum_{i=1}^{n}\sum_{j=1}^{m} (v_{ij} \otimes v_{ij} f_{ij}(t,\mathbf{x})) = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{2} \partial_{x_k} (v^\ell_{ij} v^k_{ij} f_{ij}(t,\mathbf{x})) \right)_\ell ,$$

where $\ell = 1,2$ correspond to the dimension of the space variable.

In particular, in addition to the usual kinetic flux, there is a second contribution to the flux coming from the collision term due to the finite size of the interaction thresholds. To clarify this matter, we split the collision interaction term into a local interaction term and a deviation from the local term:

$$\mathcal{J}_{ij}[\mathbf{f}](t,\mathbf{x}) = |\,\Omega[\mathbf{x}]\,|\,\mathcal{J}^{\star}_{ij}[\mathbf{f}](t,\mathbf{x}) - \Big(\,|\,\Omega[\mathbf{x}]\,|\,\mathcal{J}^{\star}_{ij}[\mathbf{f}](t,\mathbf{x}) - \mathcal{J}_{ij}[\mathbf{f}](t,\mathbf{x})\Big), \tag{3.3}$$

where $\mathcal{J}^{\star}_{ij}[\mathbf{f}](t,\mathbf{x})$ is given by

$$\begin{aligned} J^{\star}_{ij}[\mathbf{f};\alpha] = &\sum_{h,r=1}^{n}\sum_{k,s=1}^{m} \eta(\rho^a(t,\mathbf{x}))\mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x});\alpha) f_{hk}(t,\mathbf{x}) f_{rs}(t,\mathbf{x}) \\ &- f_{ij}(t,\mathbf{x})\sum_{k=1}^{n}\sum_{s=1}^{m}\eta(\rho^a(t,\mathbf{x}))\, f_{ks}(t,\mathbf{x}). \end{aligned} \tag{3.4}$$

Therefore, the momentum equation (3.2) can be written as follows:

$$\partial_t(\rho\mathbf{q})(t,\mathbf{x}) + \nabla_{\mathbf{x}}\cdot(P + \rho\mathbf{q}\otimes\mathbf{q})(t,\mathbf{x}) + E(t,\mathbf{x}) = S(t,\mathbf{x}), \tag{3.5}$$

where P, E and S are given by

$$P(t,\mathbf{x}) = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}(v^{(k)}_{ij} - \mathbf{q}^{(k)}(t,\mathbf{x}))(v^{(\ell)}_{ij} - \mathbf{q}^{(\ell)}(t,\mathbf{x}))f_{ij}(t,\mathbf{x})\right)_{1\le k,\ell\le 2}, \tag{3.6}$$

$$E(t,\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}\left(|\,\Omega[\mathbf{x}]\,|\,\mathcal{J}^{\star}_{ij}[\mathbf{f}](t,\mathbf{x}) - \mathcal{J}_{ij}[\mathbf{f}](t,\mathbf{x})\right), \tag{3.7}$$

$$S(t,\mathbf{x}) = |\,\Omega[\mathbf{x}]\,|\sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}\mathcal{J}^{\star}_{ij}[\mathbf{f}](t,\mathbf{x}). \tag{3.8}$$

The closure of the conservation equations requires the computation of the source term $S(t,\mathbf{x})$ defined in (3.8) which can be obtained from an assumption of relaxation to a suitable equilibrium configuration. Moreover, $E(t,\mathbf{x})$, given by (3.7), depends on suitable assumptions consistent with the aforementioned description of the `visibility domain` on the domain $\Omega[\mathbf{x}]$.

The equations for ρ and $\mathbf{q}$ can be closed by specifying the dependence of P, E, and S on ρ and $\mathbf{q}$. In particular, the nonlocal properties of the interaction operator have to be analyzed carefully in order to approximate E. An explicit closure can be obtained from the solution of the stationary homogeneous case, which is denoted by $f^e_{ij}(\rho)$. The real motivation, from either the modeling or the analytical point of view, to close E indirectly by using the approximation $f \sim f^e$ in (3.7) is the derivation of new terms related to the presence of gradients in the macroscopic limit.

In other words, reference is made to the *local density* $\rho(t,\mathbf{x})$ which depends on time and space and on the same quality parameter used to characterize interactions at the micro-scale. Accordingly, if the stationary distribution is known, the *equilibrium quantities* can also be defined. Namely, the mean equilibrium velocity becomes

$$\mathbf{q}_e(\rho)(t,\mathbf{x}) = \frac{1}{\rho(t,\mathbf{x})}\sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij} f^e_{ij}(\rho)(t,\mathbf{x}), \tag{3.9}$$

while the equilibrium pressure is

$$P^e(\rho)(t,\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{m}(v_{ij} - \mathbf{q}_e(\rho)(t,\mathbf{x})) \otimes (v_{ij} - \mathbf{q}_e(\rho)(t,\mathbf{x}))f^e_{ij}(\rho)(t,\mathbf{x}). \tag{3.10}$$

The term $|\,\Omega[\mathbf{x}]\,|\,J^\star_{ij}[\mathbf{f}] - J_{ij}[\mathbf{f}]$ in the definition of E can be written, using (2.6) and (3.4), as

$$|\,\Omega[\mathbf{x}]\,|\,\mathcal{J}^\star_{ij}[\mathbf{f}](t,\mathbf{x}) - \mathcal{J}_{ij}[\mathbf{f}](t,\mathbf{x}) = \Psi_1[\mathbf{f}] + \Psi_2[\mathbf{f}], \tag{3.11}$$

where the terms $\Psi_1[\mathbf{f}]$, $\Psi_2[\mathbf{f}]$ are computed by using a Taylor formula, as follows:

$$\begin{aligned}
\Psi_1[\mathbf{f}] = {} & |\,\Omega[\mathbf{x}]\,| \sum_{h,r=1}^{n}\sum_{k,s=1}^{m} \eta(\rho^a(t,\mathbf{x}))\mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x});\alpha) f_{hk}(t,\mathbf{x})\, f_{rs}(t,\mathbf{x}) \\
& - \sum_{h,r=1}^{n}\sum_{k,s=1}^{m} \int_{\Omega[\mathbf{x}]} \eta(\rho^a(t,\mathbf{x}^*))\mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x}^*);\alpha) f_{hk}(t,\mathbf{x})\, f_{rs}(t,\mathbf{x}^*)\, d\mathbf{x}^* \\
= {} & \sum_{h,r=1}^{n}\sum_{k,s=1}^{m} \int_{\Omega[\mathbf{x}]} \Big(\eta(\rho^a(t,\mathbf{x}))\mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x});\alpha) f_{rs}(t,\mathbf{x}) \\
& -\eta(\rho^a(t,\mathbf{x}^*))\mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x}^*);\alpha)\, f_{rs}(t,\mathbf{x}^*)\Big) f_{hk}(t,\mathbf{x})\, d\mathbf{x}^* \\
= {} & \sum_{h,r=1}^{n}\sum_{k,s=1}^{m} \nabla_{\mathbf{x}}\big(\eta(\rho^a(t,\mathbf{x}))\mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x});\alpha)\, f_{rs}(t,\mathbf{x})\big) f_{hk}(t,\mathbf{x}) \cdot \int_{\Omega[\mathbf{x}]} (\mathbf{x} - \mathbf{x}^*) d\mathbf{x}^* \\
& + \int_{\Omega[\mathbf{x}]}\int_0^1 (1-\tau)(\mathbf{x} - \mathbf{x}^*) \cdot D^\tau \cdot (\mathbf{x} - \mathbf{x}^*) d\tau\, d\mathbf{x}^* \\
= {} & - \sum_{h,r=1}^{n}\sum_{k,s=1}^{m} (\nabla_{\mathbf{x}}(\rho^a)\partial_{\rho^a}(\eta(\rho^a(t,\mathbf{x})))\mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x});\alpha) \\
& \times f_{rs}(t,\mathbf{x})\big) \cdot V f_{hk}(t,\mathbf{x}) \\
& - \sum_{h,r=1}^{n}\sum_{k,s=1}^{m} (\nabla_{\mathbf{x}}(\rho^a)\partial_{\rho^a}(\mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x});\alpha))\,\eta(\rho^a(t,\mathbf{x})) \times f_{rs}(t,\mathbf{x})\big) \cdot V f_{hk}(t,\mathbf{x}) \\
& - \sum_{h,r=1}^{n}\sum_{k,s=1}^{m} (\nabla_{\mathbf{x}}(f_{rs}(t,\mathbf{x}))\mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x});\alpha)\,\eta(\rho^a(t,\mathbf{x}))\big) \cdot V f_{hk}(t,\mathbf{x}) \\
& + \int_{\Omega[\mathbf{x}]}\int_0^1 (1-\tau)(\mathbf{x}^* - \mathbf{x}) \cdot D^\tau \cdot (\mathbf{x}^* - \mathbf{x}) d\tau\, d\mathbf{x}^*,
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
\Psi_2[\mathbf{f}] = {} & f_{ij}(t,\mathbf{x}) \sum_{k=1}^{n}\sum_{s=1}^{m} \int_{\Omega[\mathbf{x}]} \eta(\rho^a(t,\mathbf{x}^*))\, f_{ks}(t,\mathbf{x}^*)\, d\mathbf{x}^* \\
& - |\,\Omega[\mathbf{x}]\,|\, f_{ij}(t,\mathbf{x}) \sum_{k=1}^{n}\sum_{s=1}^{m} \eta(\rho^a(t,\mathbf{x}))\, f_{ks}(t,\mathbf{x})
\end{aligned}$$

$$
\begin{aligned}
&= f_{ij}(t,\mathbf{x}) \sum_{k=1}^{n} \sum_{s=1}^{m} \nabla_{\mathbf{x}}(\rho^a) \partial_{\rho^a} (\eta(\rho^a(t,\mathbf{x}))) f_{ks}(t,\mathbf{x}) \cdot V \\
&+ f_{ij}(t,\mathbf{x}) \sum_{k=1}^{n} \sum_{s=1}^{m} \nabla_{\mathbf{x}}(f_{ks}(t,\mathbf{x})) \eta(\rho^a(t,\mathbf{x})) \cdot V \\
&+ \int_{\Omega[\mathbf{x}]} \int_0^1 (1-\tau)(\mathbf{x}^* - \mathbf{x}) \cdot G^\tau \cdot (\mathbf{x}^* - \mathbf{x}) d\tau \, d\mathbf{x}^*, \qquad (3.13)
\end{aligned}
$$

where the vector V and matrices D^τ, G^τ are given by

$$
V = \int_{\Omega[\mathbf{x}]} (\mathbf{x}^* - \mathbf{x}) d\mathbf{x}^*, \qquad (3.14)
$$

$$
D^\tau = \sum_{h,r=1}^{n} \sum_{k,s=1}^{m} D^2_{\mathbf{x}} \big(\eta(\rho^a) \mathcal{A}_{hk}^{rs}(ij)(\rho^a) f_{rs} \big)(t, \tau\mathbf{x} + (1-\tau)\mathbf{x}^*) f_{hk}(t,\mathbf{x}),
$$

$$
G^\tau = f_{ij}(t,\mathbf{x}) \sum_{k=1}^{n} \sum_{s=1}^{m} D^2_{\mathbf{x}} (\eta(\rho^a) f_{ks})(t, \tau\mathbf{x}^* + (1-\tau)\mathbf{x}).
$$

Noting that for a matrix $A = (a_{hl})_{1 \le h,l \le 2}$ with $A = D^\tau, G^\tau$,

$$
\begin{aligned}
&\int_{\Omega[\mathbf{x}]} \int_0^1 (1-\tau)(\mathbf{x}^* - \mathbf{x}) \cdot A \cdot (\mathbf{x}^* - \mathbf{x}) d\tau \, d\mathbf{x}^* \\
&= \sum_{h=1}^{2} \sum_{l=1}^{2} \int_{\Omega[\mathbf{x}]} \int_0^1 (1-\tau) a_{hl} (\mathbf{x}_h^* - \mathbf{x}_h)(\mathbf{x}_l^* - \mathbf{x}_l) \, d\tau \, d\mathbf{x}^*,
\end{aligned}
$$

and, since $\Omega[\mathbf{x}]$ is a bounded domain,

$$
\int_{\Omega[\mathbf{x}]} (\mathbf{x}_h^* - \mathbf{x}_h)(\mathbf{x}_l^* - \mathbf{x}_l) \, d\mathbf{x}^* = O(|\,\Omega[\mathbf{x}]\,|), \quad \forall h,l = 1,2, \quad \forall \mathbf{x} \in \mathbb{R}^2.
$$

Then, assuming that D^τ and G^τ are bounded in some (weak) topological space, at least at a formal level, using (3.11)–(3.13), and considering that, at equilibrium, $f_{ij}(t,\mathbf{x}) \sim f_{ij}^e(\rho(t,\mathbf{x}))$, (3.7) yields:

$$
E^e[\mathbf{f}] = -V \cdot \nabla_{\mathbf{x}} \rho \Big(a_1(\rho) + \rho \mathbf{q}_e(\rho) a_2(\rho) \Big) + O(|\,\Omega[\mathbf{x}]\,|), \qquad (3.15)
$$

where, using the differential chain rule $\partial_x f_{hk}^e(t,\mathbf{x}) = \partial_\rho (f_{hk}^e(\rho)) \nabla_{\mathbf{x}} \rho(t,\mathbf{x})$, the a_i terms are

$$
\begin{aligned}
a_1(\rho) = &\sum_{i,h,r=1}^{n} \sum_{j,k,s=1}^{m} v_{ij} \partial_\rho \rho^a \Bigg(\partial_{\rho^a} \eta(\rho^a(t,\mathbf{x})) \mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x}); \alpha) \\
&+ \partial_{\rho^a} \mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x}); \alpha) \eta(\rho^a(t,\mathbf{x})) \Bigg) f_{hk}^e(\rho) f_{rs}^e(\rho) \\
&+ \sum_{i,h,r=1}^{n} \sum_{j,k,s=1}^{m} v_{ij} \mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x}); \alpha) \eta(\rho^a(t,\mathbf{x})) \partial_\rho f_{rs}^e(\rho) f_{hk}^e(\rho) \qquad (3.16)
\end{aligned}
$$

and

$$
a_2(\rho) = -\sum_{k=1}^{n} \sum_{s=1}^{m} \Bigg(\partial_\rho \rho^a \partial_{\rho^a} \eta[\rho^a(t,\mathbf{x})] f_{ks}^e(\rho) + \partial_\rho f_{ks}^e(\rho) \eta(\rho^a(t,\mathbf{x})) \Bigg). \qquad (3.17)
$$

As can be seen, closure of the conservation equations requires the computation of the source term $S(t,\mathbf{x})$ defined in (3.8) which can be obtained from relaxation towards a suitable equilibrium configuration. Different macro-scale models, corresponding to different relaxation models, can be derived. The aim here is to show that some classical models, see the survey [4], can be obtained when this approach is used. The approximation of the aforementioned time-relaxation term can be obtained using the following lemma.

LEMMA 3.1. *Let us assume that* $\mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x});\alpha)$ *is such that*

$$\sum_{h,r=1}^{n}\sum_{k,s=1}^{m}\mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x});\alpha)f_{hk}^e(\rho)\,f_{rs}^e(\rho)=\rho f_{ij}^e(\rho), \tag{3.18}$$

for $i=1,\dots,n,\quad j=1,\dots,m$. *Then,* $\mathcal{J}_{ij}^\star[\mathbf{f}](t,\mathbf{x})$ *is given, at equilibrium, by the relaxation*

$$\mathcal{J}_{ij}^\star[\mathbf{f}](t,\mathbf{x})\sim\nu(\rho)(f_{ij}^e(\rho)-f_{ij}),\quad \textit{where}\quad \nu(\rho)=\rho\eta(\rho^a(t,\mathbf{x})). \tag{3.19}$$

Proof. Using (3.4), the following equality is obtained:

$$\begin{aligned}\mathcal{J}_{ij}^\star[\mathbf{f};\alpha]&=\eta(\rho^a(t,\mathbf{x}))\Bigg(\sum_{h,r=1}^{n}\sum_{k,s=1}^{m}\mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x});\alpha)f_{hk}(t,\mathbf{x})\,f_{rs}(t,\mathbf{x})\\&\qquad -f_{ij}(t,\mathbf{x})\sum_{k=1}^{n}\sum_{s=1}^{m}f_{ks}(t,\mathbf{x})\Bigg)\\&=\rho\eta(\rho^a(t,\mathbf{x}))\Bigg(\frac{1}{\rho}\sum_{h,r=1}^{n}\sum_{k,s=1}^{m}\mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x});\alpha)f_{hk}(t,\mathbf{x})\,f_{rs}(t,\mathbf{x})-f_{ij}(t,\mathbf{x})\Bigg).\end{aligned}$$

Using (3.18) and the equilibrium condition $f\sim f^e$ yields

$$\begin{aligned}J_{ij}^\star[\mathbf{f};\alpha]&\sim\rho\eta(\rho^a(t,\mathbf{x}))\Bigg(\frac{1}{\rho}\sum_{h,r=1}^{n}\sum_{k,s=1}^{m}\mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x});\alpha)f_{hk}^e(\rho)\,f_{rs}^e(\rho)-f_{ij}(t,\mathbf{x})\Bigg)\\&=\rho\eta(\rho^a(t,\mathbf{x}))\Big(f_{ij}^e(\rho)-f_{ij}(t,\mathbf{x})\Big).\end{aligned}$$

This completes the proof. □

Finally, the setting of Lemma 3.1 easily leads to the following approximation:

$$S\sim\nu(\rho)\,|\,\Omega[\mathbf{x}]\,|\,\Big(\sum_{i=1}^{n}\sum_{j=1}^{m}v_{ij}(f_{ij}^e-f_{ij})\Big)=\nu(\rho)\,|\,\Omega[\mathbf{x}]\,|\,(\rho\mathbf{q_e}(\rho)-\rho\mathbf{q}). \tag{3.20}$$

A second order approximation of the density and momentum equations (3.5), together with (3.15) and (3.20), when $|\,\Omega[\mathbf{x}]\,|\to 0$ produces the following system:

$$\begin{cases}\partial_t\rho(t,x)+\nabla_x\cdot(\rho\mathbf{q})(t,x)=0,\\[2ex]\partial_t(\rho\mathbf{q})(t,x)+\nabla_{\mathbf{x}}\cdot P+\nabla_{\mathbf{x}}\cdot(\rho\mathbf{q}\otimes\mathbf{q})(t,x)+\gamma(\rho)V\cdot\nabla_{\mathbf{x}}\rho=\mu(\rho)\,(\rho\mathbf{q}_e(\rho)-\rho\mathbf{q}),\end{cases} \tag{3.21}$$

where the coefficients $\gamma(\rho)$ and $\mu(\rho)$ are computed explicitly as

$$\gamma(\rho)=-\Big(a_1(\rho)+\rho\mathbf{q}_e(\rho)\,a_2(\rho)\Big),\quad \mu(\rho)=|\,\Omega[\mathbf{x}]\,|\,\nu(\rho), \tag{3.22}$$

and the vector V and coefficients a_i, $i=1,2$, are given by (3.14) and (3.16)–(3.17), respectively.

Substituting $\mathbf{q}_e(\rho)$ with $\mathbf{q}$ in (3.6), P becomes the equilibrium pressure P^e given by (3.10). In this way, using the identity

$$\nabla_x\cdot(\rho\mathbf{q}\otimes\mathbf{q})=\rho\mathbf{q}\,div_{\mathbf{x}}\mathbf{q}+\mathbf{q}\mathbf{q}\cdot\nabla_{\mathbf{x}}\rho+\rho\mathbf{q}\cdot\nabla_{\mathbf{x}}\mathbf{q},$$

system (3.21) reduces to the density and local velocity equations

$$\begin{cases}\partial_t\rho(t,x)+\nabla_x\cdot(\rho\mathbf{q})(t,x)=0,\\[2ex] \partial_t(\mathbf{q})(t,x)+\frac{1}{\rho}\nabla_{\mathbf{x}}\cdot P^e(\rho)+\mathbf{q}\cdot\nabla_{\mathbf{x}}\mathbf{q}+d(\rho)V\cdot\nabla_{\mathbf{x}}\rho=\mu(\rho)\left(\mathbf{q}_e(\rho)-\mathbf{q}\right),\end{cases}\tag{3.23}$$

where the vector $\mathbf{q}\cdot\nabla_{\mathbf{x}}\mathbf{q}$ and the coefficient $d(\rho)$ are given by

$$\mathbf{q}\cdot\nabla_{\mathbf{x}}\mathbf{q}=\left(\sum_{k=1}^{2}(\partial_{\mathbf{x}_k}q_\ell)\mathbf{q}_k\right)_{1\le\ell\le2},\quad\text{and}\quad d(\rho)=\frac{1}{\rho}\gamma(\rho).\tag{3.24}$$

This structure provides a general reference for the derivation of macro-scale models. It corresponds to the underlying meso-scale description. Hence, it overcomes the heuristic derivation occasionally adopted in the approach of continuum mechanics. Further generalizations are reported in the following subsection.

3.2. General approach. Let us now consider a `general approach` to determine macroscopic models and their related coefficients. One uses a general ansatz for the distribution function and an extended equilibrium function to approximate the true distribution f instead of a simple approximation by the equilibrium distribution f^e. In other words, instead of using $f^e_{ij}(\rho)$, a general ansatz can be made so that F_{ij} is an extended equilibrium distribution function.

Let us now consider $F_{ij}=F_{ij}(\rho,\mathbf{q})$ which depends not only on ρ but also on the macroscopic velocity $\mathbf{q}$. An example will be considered later.

A search is made for F_{ij} which satisfies the natural two properties; namely, that the density and momentum are given by ρ and $\rho\mathbf{q}$, respectively:

$$\rho=\sum_{i=1}^{n}\sum_{j=1}^{m}F_{ij}(\rho,\mathbf{q}),\quad\text{and}\quad\rho\mathbf{q}=\sum_{i=1}^{n}\sum_{j=1}^{m}v_{ij}F_{ij}(\rho,\mathbf{q}).\tag{3.25}$$

Moreover, F_{ij} must satisfy

$$F_{ij}(\rho,\mathbf{q}_e(\rho))=f^e_{ij}(\rho).\tag{3.26}$$

Using this approach, it is possible to include the flow $(\rho,\mathbf{q})$ in which the distribution function cannot be properly approximated by the equilibrium distributions $f^e_{ij}(\rho)$ since they do not have the correct mean value $\mathbf{q}$. With

$$\nabla_{\mathbf{x}}F_{ij}(\rho,\mathbf{q})=(\partial_\rho F_{ij})\nabla_{\mathbf{x}}\rho+(\partial_{\mathbf{q}_1}F_{ij})\nabla_{\mathbf{x}}\mathbf{q}_1+(\partial_{\mathbf{q}_2}F_{ij})\nabla_{\mathbf{x}}\mathbf{q}_2,$$

the above technique yields

$$\begin{aligned}E^F(t,\mathbf{x})=&-\Big(b_1(\rho,\mathbf{q})+\rho\mathbf{q}b_2(\rho,\mathbf{q})\Big)V\cdot\nabla_{\mathbf{x}}\rho\\&-\Big(b_3(\rho,\mathbf{q})+\rho\mathbf{q}b_4(\rho,\mathbf{q})\Big)V\cdot\nabla_{\mathbf{x}}\mathbf{q}_1\end{aligned}$$

$$-\Big(b_5(\rho,\mathbf{q})+\rho\mathbf{q}b_6(\rho,\mathbf{q})\Big)V\cdot\nabla_{\mathbf{x}}\mathbf{q}_2+O(|\,\Omega[\mathbf{x}]\,|), \tag{3.27}$$

where the coefficients b_i, $i=1,\dots,6$, which play the role of the density gradients in the asymptotic limit, are given by

$$\begin{aligned} b_1(\rho,\mathbf{q}) = &\sum_{i,h,r=1}^{n}\sum_{j,k,s=1}^{m} v_{ij}\partial_\rho\rho^a\Big(\partial_{\rho^a}\eta(\rho^a(t,\mathbf{x}))\mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x});\alpha)\\ &+\partial_{\rho^a}\mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x});\alpha)\eta(\rho^a(t,\mathbf{x}))\Big)F_{hk}(\rho,\mathbf{q})F_{rs}(\rho,\mathbf{q})\\ &+\sum_{i,h,r=1}^{n}\sum_{j,k,s=1}^{m} v_{ij}\partial_\rho F_{rs}(\rho,\mathbf{q})\mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x});\alpha)\eta(\rho^a(t,\mathbf{x}))F_{hk}(\rho,\mathbf{q}), \end{aligned} \tag{3.28}$$

$$b_2(\rho,\mathbf{q}) = -\sum_{k=1}^{n}\sum_{s=1}^{m}\Big(\partial_\rho\rho^a\partial_{\rho^a}\eta(\rho^a(t,\mathbf{x}))F_{ks}(\rho,\mathbf{q})+\partial_\rho F_{ks}(\rho,\mathbf{q})\eta(\rho^a(t,\mathbf{x}))\Big), \tag{3.29}$$

$$b_3(\rho,\mathbf{q}) = \sum_{i,h,r=1}^{n}\sum_{j,k,s=1}^{m} v_{ij}\Big(\partial_{\mathbf{q}_1}(F_{rs}(\rho,\mathbf{q}))\mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x});\alpha)\eta(\rho^a(t,\mathbf{x}))F_{hk}(\rho,\mathbf{q})\Big), \tag{3.30}$$

$$b_4(\rho,\mathbf{q}) = -\sum_{k=1}^{n}\sum_{s=1}^{m}\Big(\partial_{\mathbf{q}_1}(F_{ks}(\rho,\mathbf{q}))\eta(\rho^a(t,\mathbf{x}))\Big), \tag{3.31}$$

$$b_5(\rho,\mathbf{q}) = \sum_{i,h,r=1}^{n}\sum_{j,k,s=1}^{m} v_{ij}\Big(\partial_{\mathbf{q}_2}(F_{rs}(\rho,\mathbf{q}))\mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\mathbf{x});\alpha)\eta(\rho^a(t,\mathbf{x}))F_{hk}(\rho,\mathbf{q})\Big), \tag{3.32}$$

and

$$b_6(\rho,\mathbf{q}) = -\sum_{k=1}^{n}\sum_{s=1}^{m}\Big(\partial_{\mathbf{q}_2}(F_{ks}(\rho,\mathbf{q}))\eta(\rho^a(t,\mathbf{x}))\Big). \tag{3.33}$$

Therefore, the macroscopic model is given by

$$\begin{cases} \partial_t\rho(t,\mathbf{x})+\nabla_{\mathbf{x}}\cdot(\rho\mathbf{q})(t,\mathbf{x})=0,\\ \partial_t(\mathbf{q})(t,\mathbf{x})+\frac{1}{\rho}\nabla_{\mathbf{x}}\cdot P^F(\rho,\mathbf{q})+\mathbf{q}\cdot\nabla_{\mathbf{x}}\mathbf{q}+\alpha_1(\rho,\mathbf{q})V\cdot\nabla_{\mathbf{x}}\rho+\alpha_2(\rho,\mathbf{q})V\cdot\nabla_{\mathbf{x}}\mathbf{q}_1\\ \qquad +\alpha_3(\rho,\mathbf{q})V\cdot\nabla_{\mathbf{x}}\mathbf{q}_2=\mu(\rho)\left(\mathbf{q}_e(\rho)-\mathbf{q}\right), \end{cases} \tag{3.34}$$

where

$$P^F(\rho,\mathbf{q}) := \sum_{i=1}^{n}\sum_{j=1}^{m}\left(v_{ij}-\mathbf{q}\right)^2 F_{ij}(\rho(t,x),\mathbf{q}(t,x)), \tag{3.35}$$

and the coefficients α_i, $i=1,2,3$ are given explicitly by

$$\alpha_1(\rho,\mathbf{q}) = -\Big(b_1(\rho,\mathbf{q})+\rho\mathbf{q}b_2(\rho,\mathbf{q})\Big),$$

$$\alpha_2(\rho,\mathbf{q}) = -\Big(b_3(\rho,\mathbf{q})+\rho\mathbf{q}b_4(\rho,\mathbf{q})\Big),$$

and

$$\alpha_3(\rho,\mathbf{q}) = -\Big(b_5(\rho,\mathbf{q})+\rho\mathbf{q}b_6(\rho,\mathbf{q})\Big).$$

3.3. Example. Finally, let us consider as a specific example a special choice for the distribution function which is given analogously to Grad's moment closure method in the kinetic theory of gases as a polynomial ansatz for the distribution function using the equilibrium function f^e_{ij}. The following lemma shows the existence of the distribution function F_{ij} that satisfies (3.25)–(3.26).

LEMMA 3.2. *Let $f^e(\rho)$ be the equilibrium and $P^e(\rho)$ the corresponding tensor given by (3.10). Let us suppose that the tensor $P^e(\rho)$ is invertible ($P^e_{11}(\rho)P^e_{22}(\rho) - P^e_{12}(\rho)P^e_{21}(\rho) \neq 0$), then there exists an $F_{ij}(\rho,\mathbf{q})$ which satisfies (3.25)–(3.26) and is given by*

$$F_{ij}(\rho,\mathbf{q},v_{ij}) = f^e_{ij}(\rho)(\gamma(\rho,\mathbf{q}) + \beta(\rho,\mathbf{q})\cdot v_{ij}), \tag{3.36}$$

where the coefficients γ and $\beta = (\beta_1,\beta_2)$ are given explicitly by

$$\gamma(\rho,\mathbf{q}) = 1 + \rho\frac{(q_2 - q_e^{(2)})(q_e^{(1)}P^e_{21} - P^e_{11}q_e^{(2)}) + (q_1 - q_e^{(1)})(q_e^{(2)}P^e_{12} - P^e_{22}q_e^{(1)})}{P^e_{11}P^e_{22} - P^e_{12}P^e_{21}}, \tag{3.37}$$

$$\beta_1(\rho,\mathbf{q}) = \rho\frac{P^e_{22}(q_1 - q_e^{(1)}) - P^e_{21}(q_2 - q_e^{(2)})}{P^e_{11}P^e_{22} - P^e_{12}P^e_{21}}, \tag{3.38}$$

$$\beta_2(\rho,\mathbf{q}) = \rho\frac{P^e_{12}(q_1 - q_e^{(1)}) - P^e_{11}(q_2 - q_e^{(2)})}{P^e_{12}P^e_{21} - P^e_{11}P^e_{22}}, \tag{3.39}$$

where $q_e^{(\ell)}(\rho)$, for $\ell = 1,2$, denote the components of the vectors $q_e(\rho)$.

Proof. A search is made for $F_{ij}(\rho,\mathbf{q})$ in the form of (3.36) where $\beta(\rho,\mathbf{q}) = (\beta_1(\rho,\mathbf{q}),\beta_2(\rho,\mathbf{q}))$. Since condition (3.25) has to be satisfied, one obtains

$$\gamma(\rho,\mathbf{q}) + \sum_{\ell=1}^{2}\beta_\ell q_e^{(\ell)} = 1, \tag{3.40}$$

$$\rho q_e^{(k)}\gamma(\rho,\mathbf{q}) + \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{\ell=1}^{2}\beta_\ell v_{ij}^{(\ell)} v_{ij}^{(k)} f^e_{ij}(\rho) = \rho q^{(k)}, \quad k = 1,2, \tag{3.41}$$

where $v_{ij}^{(k)}$, $k = 1,2$, denote the components of the vectors v_{ij}.

We have the relation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}^{(\ell)} v_{ij}^{(k)} f^e_{ij}(\rho) = P^e_{lk} + \rho q_e^{(k)} q_e^{(\ell)}, \quad k,\ell = 1,2$$

which, from (3.41), yields

$$\rho q_e^{(k)}\gamma(\rho,\mathbf{q}) + \sum_{\ell=1}^{2}\beta_\ell(P^e_{lk} + \rho q_e^{(k)} q_e^{(\ell)}) = \rho q^{(k)}. \tag{3.42}$$

Multiplying (3.40) by $\rho q_e^{(k)}$ and using (3.42) yields

$$\sum_{\ell=1}^{2}\beta_\ell P^e_{lk}(\rho) = \rho(q^{(k)} - q_e^{(k)}), \quad k = 1,2. \tag{3.43}$$

Finally, since $P^e(\rho)$ is invertible, from (3.43), it follows that

$$\beta_1(\rho,\mathbf{q})=\rho\frac{P^e_{22}(q_1-q_e^{(1)})-P^e_{21}(q_2-q_e^{(2)})}{P^e_{11}P^e_{22}-P^e_{12}P^e_{21}}$$

and

$$\beta_2(\rho,\mathbf{q})=\rho\frac{P^e_{12}(q_1-q_e^{(1)})-P^e_{11}(q_2-q_e^{(2)})}{P^e_{12}P^e_{21}-P^e_{11}P^e_{22}}.$$

Therefore, (3.40) yields the computation (3.37) of γ. Moreover, since $\beta_1(\rho,\mathbf{q}^e(\rho))=\beta_2(\rho,\mathbf{q}^e(\rho))=0$ and $\gamma(\rho,\mathbf{q}^e(\rho))=1$, the third condition, (3.26), is automatically verified. This completes the proof of the lemma. ∎

Now let now Ψ_1, Ψ_2, and Ψ_3 depend on ρ. They are given by

$$\Psi_1(\rho)=-\frac{\rho}{\det(P^e(\rho))}\left(\partial_\rho q_e^{(1)}(q_e^{(2)}P^e_{12}-P^e_{22}q_e^{(1)})+\partial_\rho q_e^{(2)}(q_e^{(1)}P^e_{21}-P^e_{11}q_e^{(2)})\right), \tag{3.44}$$

$$\Psi_2(\rho)=-\frac{\rho}{\det(P^e(\rho))}\left(P^e_{21}\partial_\rho(q_e^{(2)})-P^e_{22}\partial_\rho(q_e^{(1)})\right), \tag{3.45}$$

and

$$\Psi_3(\rho)=-\frac{\rho}{\det(P^e(\rho))}\left(P^e_{11}\partial_\rho(q_e^{(2)})-P^e_{12}\partial_\rho(q_e^{(1)})\right), \tag{3.46}$$

It follows that the derivative $\partial_\rho(F_{rs}(\rho,\mathbf{q}))$ at equilibrium $\mathbf{q}=\mathbf{q}_e(\rho)$ can be computed as

$$\partial_\rho(F_{rs})(\rho,\mathbf{q}=\mathbf{q}_e):=\Psi_{rs}(\rho)=\partial_\rho(f^e_{rs}(\rho))+f^e_{rs}(\rho)\left(\Psi_1(\rho)+\Psi_2(\rho)v_{rs}^{(1)}+\Psi_3(\rho)v_{rs}^{(2)}\right). \tag{3.47}$$

In the same way, we obtain the following computation:

$$\partial_{\mathbf{q}_1}(F_{rs})(\rho,\mathbf{q}=\mathbf{q}_e):=\Phi_{rs}=f^e_{rs}(\rho)\left(\Phi_1(\rho)+\Phi_2(\rho)v_{rs}^{(1)}+\Phi_3(\rho)v_{rs}^{(2)}\right), \tag{3.48}$$

where the terms $\Phi_i, i=1,2,3$, are given by

$$\Phi_1(\rho)=-\frac{\rho}{\det(P^e(\rho))}\left(P^e_{22}q_e^{(1)}-P^e_{12}q_e^{(2)}\right), \tag{3.49}$$

$$\Phi_2(\rho)=\frac{\rho}{\det(P^e(\rho))}P^e_{22},\quad \Phi_3(\rho)=-\frac{\rho}{\det(P^e(\rho))}P^e_{12}. \tag{3.50}$$

Moreover,

$$\partial_{\mathbf{q}_2}(F_{rs})(\rho,\mathbf{q}=\mathbf{q}_e):=\Gamma_{rs}=f^e_{rs}(\rho)\left(\Gamma_1(\rho)+\Gamma_2(\rho)v_{rs}^{(1)}+\Gamma_3(\rho)v_{rs}^{(2)}\right), \tag{3.51}$$

where Γ_i, $i=1,2,3$, are given by

$$\Gamma_1(\rho)=\frac{\rho}{\det(P^e(\rho))}\left(P^e_{21}q_e^{(1)}-P^e_{11}q_e^{(2)}\right),\qquad \Gamma_2(\rho)=-\frac{\rho}{\det(P^e(\rho))}P^e_{21},$$

and

$$\Gamma_3(\rho) = \frac{\rho}{\det(P^e(\rho))} P^e_{11}.$$

The macroscopic coefficients (3.28)–(3.33) at equilibrium are then computed as follows:

$$\begin{aligned} b_1(\rho,\mathbf{q}) := c_1(\rho) = \sum_{i,h,r=1}^{n} \sum_{j,k,s=1}^{m} v_{ij} \partial_\rho \rho^a \Big(\partial_{\rho^a} \eta(\rho^a(t,\mathbf{x})) \mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x});\alpha) \\ + \partial_{\rho^a} \mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x});\alpha) \eta(\rho^a(t,\mathbf{x})) \Big) f^e_{hk}(\rho) f^e_{rs}(\rho) \\ + \sum_{i,h,r=1}^{n} \sum_{j,k,s=1}^{m} v_{ij} \partial_\rho f^e_{rs}(\rho) \mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x});\alpha) \eta(\rho^a(t,\mathbf{x})) f^e_{hk}(\rho), \end{aligned} \tag{3.52}$$

$$b_2(\rho,\mathbf{q}) := c_2(\rho) = -\sum_{k=1}^{n} \sum_{s=1}^{m} \Big(\partial_\rho \rho^a \partial_{\rho^a} \eta(\rho^a(t,\mathbf{x})) f^e_{ks}(\rho) + \Psi_{ks}(\rho) \eta(\rho^a(t,\mathbf{x})) \Big), \tag{3.53}$$

$$b_3(\rho,\mathbf{q}) := c_3(\rho) = \sum_{i,h,r=1}^{n} \sum_{j,k,s=1}^{m} v_{ij} \Big(\Phi_{rs}(\rho) \mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x});\alpha) \eta(\rho^a(t,\mathbf{x})) f^e_{hk}(\rho) \Big), \tag{3.54}$$

$$b_4(\rho,\mathbf{q}) := c_4(\rho) = -\sum_{k=1}^{n} \sum_{s=1}^{m} \Big(\Phi_{ks}(\rho) \eta(\rho^a(t,\mathbf{x})) \Big), \tag{3.55}$$

$$b_5(\rho,\mathbf{q}) := c_5(\rho) = \sum_{i,h,r=1}^{n} \sum_{j,k,s=1}^{m} v_{ij} \Big(\Gamma_{rs}(\rho) \mathcal{A}^{rs}_{hk}(ij)(\rho^a(t,\mathbf{x});\alpha) \eta(\rho^a(t,\mathbf{x})) f^e_{hk}(\rho) \Big), \tag{3.56}$$

$$b_6(\rho,\mathbf{q}) := c_6(\rho) = -\sum_{k=1}^{n} \sum_{s=1}^{m} \Big(\Gamma_{ks}(\rho) \eta(\rho^a(t,\mathbf{x})) \Big). \tag{3.57}$$

Finally, the following general macro-scale structure is obtained from (3.34):

$$\begin{cases} \partial_t \rho(t,\mathbf{x}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{q})(t,\mathbf{x}) = 0, \\ \partial_t(\mathbf{q})(t,\mathbf{x}) + \frac{1}{\rho} \nabla_{\mathbf{x}} \cdot P^e(\rho) + \mathbf{q} \cdot \nabla_{\mathbf{x}} \mathbf{q} + \beta_1(\rho) V \cdot \nabla_{\mathbf{x}} \rho + \beta_2(\rho) V \cdot \nabla_{\mathbf{x}} \mathbf{q}_1 \\ \qquad + \beta_3(\rho) V \cdot \nabla_{\mathbf{x}} \mathbf{q}_2 = \mu(\rho)(\mathbf{q}_e(\rho) - \mathbf{q}), \end{cases} \tag{3.58}$$

where the coefficients β_i, for $i = 1,2,3$ are given explicitly by

$$\beta_1(\rho) = -(c_1(\rho) + \rho \mathbf{q}_e(\rho) c_2(\rho)), \quad \beta_2(\rho) = -(c_3(\rho) + \rho \mathbf{q}_e(\rho) c_4(\rho)),$$

and

$$\beta_3(\rho) = -(c_5(\rho) + \rho \mathbf{q}_e(\rho) c_6(\rho)),$$

and $P^e(\rho)$ and μ are still given, respectively, by (3.10) and (3.22).

This new structure enriches, by adding new terms related to the presence of gradients, the previous ones, (3.23) and (3.24). The aim again consists of obtaining macro-scale structures corresponding to the underlying descriptions at the micro-scale. Hopefully, these structures will offer the background for the derivation of specific models. The critical analysis of the literature is often based on heuristic approaches which should, according to the authors bias, be referred to appropriate frameworks also according to the hints of [14].

The approach is developed in a general framework which allows one to understand under which specific assumptions at the micro-scale models taken from the literature can be derived from the underlying description at the micro-scale. The derivation requires that the local equilibrium equation is known. Therefore, since it is not analytically known as in the classical kinetic theory, the modeling approach can be delivered through a hybrid system consisting of a macroscopic description plus a system of kinetic ordinary differential equations which can be used to compute the pseudo-Maxwellian distribution function, from the time asymptotic of (2.6), where the space gradients are made equal to zero. A problem that remains open is that of developing an approach to the case of crowds in bounded domains where pedestrians interact with walls and obstacles according to nonlocal interaction rules.

Another problem that remains open concerns the validity of the derivation of a macroscopic model. Any proof concerning the derivation of a macroscopic limit for a kinetic model will, as a by-product, give an existence proof for the corresponding macroscopic equation. However, uniform regularity estimates would likely be needed to obtain the limit of the nonlinear term. These estimates, if they exist, must be sharp because it is known that the solutions of models like (3.21) become singular after a finite time.

REFERENCES

[1] J-P. Agnelli, F. Colasuonno, and D. Knopoff, *A kinetic theory approach to the dynamics of crowd evacuation from bounded domains*, Math. Models Meth. Appl. Sci., 25(1), 109–129, 2015.

[2] N. Bellomo, A. Bellouquid, and D. Knopoff, *From the micro-scale to collective crowd dynamics*, Multiscale Mod. Sim., 11, 943–963, 2013.

[3] N. Bellomo, D. Knopoff, and J. Soler, *On the difficult interplay between life, "complexity", and mathematical sciences*, Math. Models Meth. Appl. Sci., 23(10), 1861–1913, 2013.

[4] N. Bellomo, B. Piccoli, and A. Tosin, *Modeling crowd dynamics from a complex system viewpoint*, Math. Models Meth. Appl. Sci., 22, No.1230004, 2012.

[5] A. Bellouquid, E. De Angelis, and L. Fermo, *Towards the modeling of vehicular traffic as a complex system: A kinetic theory approach*, Math. Models Meth. Appl. Sci., 22, No.1140003, 2012.

[6] A. Bellouquid and M. Delitala, *Asymptotic limits of a discrete kinetic theory model of vehicular traffic*, Appl. Math. Lett., 24(5), 672–678, 2011.

[7] R. Borsche, A. Klar, S. Köhn, and A. Meurer, *Coupling traffic flow networks to pedestrian motion*, Math. Models Meth. Appl. Sci., 24, 359–380, 2014.

[8] S. Buchmuller and U. Weidman, *Parameters of pedestrians, pedestrian traffic and walking facilities*, ETH Report Nr.132, October, 2006.

[9] A. Chertock, A. Kurganov, A. Polizzi, and I. Timofeyev, *Pedestrian flow models with slowdown interactions*, Math. Models Meth. Appl. Sci., 24, 249–275, 2014.

[10] C. Cercignani, R. Illner, and M. Pulvirenti, *The Mathematical Theory of Dilute Gases*, Springer Verlag, Berlin, Heidelberg, 1994.

[11] E. De Angelis, *Nonlinear hydrodynamic models of traffic flow modelling and mathematical problems*, Math. Comp. Modelling., 29, 83–95, 1999.

[12] P. Degond, C. Appert-Rolland, M. Moussaïd, J. Pettré, and G. Theraulaz, *A hierarchy of heuristic-based models of crowd dynamics*, J. Stat. Phys., 152(6), 1033–1068, 2013.

[13] L. Fermo and A. Tosin, *A fully-discrete-state kinetic theory approach to modeling vehicular traffic*, SIAM J. Appl. Math., 73, 1533–1556, 2013.

[14] M. Gromov, *In a search for a structure, Part 1: On entropy*, preprint, July 2012. WEB repository http://www.ihes.fr/ gromov/.

[15] D. Helbing, *Traffic and related self-driven many-particle systems*, Rev. Mod. Phy., 73, 1067–1141, 2001.

[16] A. Klar and R. Wegener, *A hierarchy of models for multilane vehicular traffic. I. modeling, and II. numerical invetigations*, SIAM J. Appl. Math., 59, 983–1001, 1999.

[17] A. Klar and R. Wegener, *Kinetic derivation of macroscopic anticipation models for vehicular traffic*, SIAM J. Appl. Math., 60(5), 1749–1766, 2000.

[18] D. Levermore, *Moment closure hierarchies for kinetic theories*, J. Stat. Phys., 83, 1021–1065, 1996.

[19] M. Moussaid, D. Helbing, S. Garnier, A. Johanson, M. Combe, and G. Theraulaz, *Experimental study of the behavioural underlying mechanism underlying self-organization in human crowd*, Proc. Royal Society B: Biol. Sci., 276, 2755–2762, 2009.

[20] A. Schadschneider and A. Seyfried, *Empirical results for pedestrian dynamics and their implications for modeling*, Netw. Heterog. Media., 6, 545–560, 2011.