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# TIME PERIODIC SOLUTIONS FOR A THREE-DIMENSIONAL NON-CONSERVATIVE COMPRESSIBLE TWO-FLUID MODEL*

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**Abstract.** In this paper, we consider the existence of time periodic solution to a non-conservative compressible two-fluid model with constant viscosity coefficients and unequal pressure functions $P^+\neq P^-$ in periodic domains of $\mathbb{R}^3$. Based on the topological degree theory, we obtain the existence of the time periodic solution under some smallness assumptions.

**Keywords.** Time periodic solution; Non-conservative compressible two-fluid model; Topological degree method.

**AMS subject classifications.** 76T10; 76N10; 35B10.

## 1. Introduction

**1.1. Background.** In this paper, we consider the following generic two-phase compressible system in periodic domains $\Omega_L=(-L,L)^3\subset\mathbb{R}^3(L>0$ is an arbitrary positive constant):

$$\begin{cases}\alpha^+ +\alpha^- =1,\\ \partial_t(\alpha^\pm\rho^\pm)+\mathrm{div}(\alpha^\pm\rho^\pm u^\pm)=0,\\ \partial_t(\alpha^\pm\rho^\pm u^\pm)+\mathrm{div}(\alpha^\pm\rho^\pm u^\pm\otimes u^\pm)+\alpha^\pm\nabla P^\pm(\rho^\pm)=\mathrm{div}(\alpha^\pm\tau^\pm)+\alpha^\pm\rho^\pm r^\pm,\\ P^+(\rho^+)-P^-(\rho^-)=f(\alpha^-\rho^-),\end{cases}\tag{1.1}$$

where $x\in\Omega_L$ is the spatial variable, $t$ is the time variable. $0\le\alpha^+(x,t)\le 1$ is the volume fraction of fluid $+$, and $0\le\alpha^-(x,t)\le 1$ is that of fluid $-$, which satisfies $\alpha^\pm(-x,t)=\alpha^\pm(x,t)$. Moreover $\rho^\pm(x,t)\ge 0, u^\pm(x,t)$ and $P^\pm(\rho^\pm)=A^\pm(\rho^\pm)^{\bar\gamma^\pm}$ are the densities, velocities and pressures of each phase. It is assumed that $\bar\gamma^\pm\ge 1$, and $A^\pm>0$ are constants. Without loss of generality, we set $A^+=A^-=1$ in the sequel. In addition, the function $f(\cdot)\in C^3([0,+\infty))$ is strictly decreasing near the equilibriums. Also, $\tau^\pm$ are the viscous stress tensors:

$$\tau^\pm:=\mu^\pm(\nabla u^\pm+\nabla^t u^\pm)+\lambda^\pm\mathrm{div}u^\pm\mathrm{Id},$$

where the constants $\mu^\pm$ and $\lambda^\pm$ are shear and bulk viscosity coefficients satisfying $\mu^\pm>0$, and $2\mu^\pm+3\lambda^\pm\ge 0$, which deduces $\mu^\pm+\lambda^\pm>0$; $r^\pm=(r_1^\pm,r_2^\pm,r_3^\pm)(x,t)$ are the given external forces, which are periodic in time with periodicity $T$, and satisfy $r^\pm(x,t)=-r^\pm(-x,t)$. This system is known as a two-fluid flow model. In model (1.1), we describe the capillary pressure effects in terms of $f$ in $(1.1)_4$, which is a natural way to explain the capillary effects; see [12] for details. It is worth noting that the two-fluid model is related to many other important fluid models, for instance, the compressible Navier-Stokes equations($\alpha^+\equiv 0$ or $\alpha^-\equiv 0$), and other models related to the Navier-Stokes equations. See for instance [2,4,12,17,23] and references therein for more background information about this model.

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In recent years, some efforts were made on this two-fluid model and related models. Bresch et al. [4] considered the existence of global weak solutions in the periodic domain when $1<\bar{\gamma}^{\pm}<6$, where the two pressure functions are equal and the capillary effects are considered. Precisely, they assumed a common pressure $P^{+}=P^{-}$, and included a third order derivative of $\alpha^{\pm}\rho^{\pm}$ to account for the capillary effects. Later, Bresch, Huang and Li [5] showed the existence of global weak solutions in one-dimensional case with two equal pressures, when $\bar{\gamma}^{\pm}>1$; Cui et al. [10] established the decay rate estimates of classical solutions for the same model in [4] (with two equal pressures and capillary effects). Recently, Evje et al. [12] studied the decay rates of strong solutions to (1.1). We can also refer to [20] for the result of the vanishing capillary limit of model (1.1) with capillary effect.

Since there are a lot of time periodic or almost periodic processes in the hydrodynamic applications, abundant researches have been made about time periodic solutions of evolutionary differential equations. See [6, 13, 15, 21, 22, 24, 28] for the time periodic solutions of Navier-Stokes equations. In [15], Galdi et al. got a special physically time periodic solution of the incompressible Navier-Stokes equations. The work of Ma et al. [21] was based on the spectral analysis for the optimal decay estimates on the linearized solution operator and they got the global existence of time periodic solution in $\mathbb{R}^N$ for $N\geq 5$. Farwig and Okabe [13] considered the time periodic solution of the incompressible Navier-Stokes equations in bounded domains of $\mathbb{R}^3$ with inhomogeneous boundary conditions. About the time periodic solution, there are also some researches about other related models, such as the magnetohydrodynamic equations, Navier-Stokes-Fourier systems, Euler equations etc., we refer to [7, 14, 27]. Recently, Geissert et al. [16] developed a general approach to time periodic incompressible fluid flow problems and semilinear evolution equations, which is based on a combination of interpolation and topological arguments. For more results about the time periodic solutions of the Navier-Stokes equations, we refer to [3, 29].

Our work is based on the results of Jin and Yang [18, 19] about the compressible Navier-Stokes equations, in which the topological degree method was used. In [18], they proved the existence of time periodic solutions in periodic domain $\Omega_L=(-L,L)^3$, and then obtained the solutions in the whole space $\mathbb{R}^3$ as the limit of solutions in $\Omega_L$, when $L\to\infty$. Thus, they needed the estimates of solutions independent of $L$. In [19], they got the existence of time periodic solutions in periodic domain $\Omega_L=(-L,L)^3$, and they didn't need the regularity assumption of $(\sigma_t,u_t)$($\sigma$ and $u$ denote the density and velocity respectively), which was necessary in [18]; but this conclusion can not be extended to $\mathbb{R}^3$. In this paper, we prove the existence of time periodic solutions of (1.1) by the topological degree method. Since the model (1.1) is more complicated than the compressible Navier-Stokes equations, we can just get the existence result in periodic domain. More precisely, as the coefficient of $\nabla\sigma$ is the pressure function $P(\sigma)=\sigma^{\gamma}$, Jin and Yang [18] can get the $L^2$ estimates of $\nabla u$, which is independent of $L$ by the good structure property of power function. In our framework, we can not obtain the $L^2$ estimates of $(\nabla u^{+},\nabla u^{-})$ independent of $L$, see Lemma 3.5 and Remark 3.2 for details.

**1.2. Reformulations.** In this subsection, we introduce some new variables and reformulate the model (1.1). The following reformulations are similar to that in [4, 12]. To begin with, we make some formal calculations. By the pressure equation $(1.1)_4$, we can deduce the differential identity

$$\mathrm{d}P^{+}-\mathrm{d}P^{-}=\mathrm{d}f(\alpha^{-}\rho^{-}), \tag{1.2}$$

where $P^{\pm} := P^{\pm}(\rho^{\pm})$. It is obvious that

$$\mathrm{d}P^{+} = s_{+}^{2}\mathrm{d}\rho^{+},\ \mathrm{d}P^{-} = s_{-}^{2}\mathrm{d}\rho^{-}, \quad \text{where } s_{\pm}^{2} := \frac{\mathrm{d}P^{\pm}}{\mathrm{d}\rho^{\pm}}(\rho^{\pm}) = \bar{\gamma}^{\pm}\frac{P^{\pm}(\rho^{\pm})}{\rho^{\pm}}.$$

Here $s_{\pm}$ denote the sound speed of each phase respectively. Let us introduce two variables

$$n^{\pm} := \alpha^{\pm}\rho^{\pm}, \tag{1.3}$$

which, together with $(1.1)_1$ and the above calculations allow one to write

$$\mathrm{d}\rho^{+} = \frac{1}{\alpha^{+}}(\mathrm{d}n^{+} - \rho^{+}\mathrm{d}\alpha^{+}),\ \mathrm{d}\rho^{-} = \frac{1}{\alpha^{-}}(\mathrm{d}n^{-} + \rho^{-}\mathrm{d}\alpha^{+}). \tag{1.4}$$

From (1.2) and (1.4), we obtain

$$\mathrm{d}\alpha^{+} = \frac{\alpha^{-}s_{+}^{2}}{\alpha^{-}\rho^{+}s_{+}^{2} + \alpha^{+}\rho^{-}s_{-}^{2}}\mathrm{d}n^{+} - \frac{\alpha^{+}\alpha^{-}}{\alpha^{-}\rho^{+}s_{+}^{2} + \alpha^{+}\rho^{-}s_{-}^{2}}\left(\frac{s_{-}^{2}}{\alpha^{-}} + f'\right)\mathrm{d}n^{-}. \tag{1.5}$$

Putting (1.5) into (1.4), we get the following expressions:

$$\mathrm{d}\rho^{+} = \frac{\rho^{+}\rho^{-}s_{-}^{2}}{n^{-}(\rho^{+})^{2}s_{+}^{2} + n^{+}(\rho^{-})^{2}s_{-}^{2}}\left(\rho^{-}\mathrm{d}n^{+} + \left(\rho^{+} + \rho^{+}\frac{\alpha^{-}f'}{s_{-}^{2}}\right)\mathrm{d}n^{-}\right),$$

and

$$\mathrm{d}\rho^{-} = \frac{\rho^{+}\rho^{-}s_{+}^{2}}{n^{-}(\rho^{+})^{2}s_{+}^{2} + n^{+}(\rho^{-})^{2}s_{-}^{2}}\left(\rho^{-}\mathrm{d}n^{+} + \left(\rho^{+} - \rho^{-}\frac{\alpha^{+}f'}{s_{+}^{2}}\right)\mathrm{d}n^{-}\right),$$

then we can get the expression of $\mathrm{d}P^{\pm}$:

$$\mathrm{d}P^{+} = \mathcal{C}^{2}\left(\rho^{-}\mathrm{d}n^{+} + \left(\rho^{+} + \rho^{+}\frac{\alpha^{-}f'}{s_{-}^{2}}\right)\mathrm{d}n^{-}\right),$$

and

$$\mathrm{d}P^{-} = \mathcal{C}^{2}\left(\rho^{-}\mathrm{d}n^{+} + \left(\rho^{+} - \rho^{-}\frac{\alpha^{+}f'}{s_{+}^{2}}\right)\mathrm{d}n^{-}\right),$$

where we denote

$$\mathcal{C}^{2} := \frac{s_{-}^{2}s_{+}^{2}}{\alpha^{-}\rho^{+}s_{+}^{2} + \alpha^{+}\rho^{-}s_{-}^{2}}.$$

As in [12], we focus on the case that $\inf n^{\pm} > 0$, $\inf \rho^{\pm} > 0$ and $\inf \alpha^{\pm} > 0$ in our framework. Next, we can define $\rho^{+}$ in terms of $n^{\pm}$. More precisely, from $(1.1)_1$, we get the following identity:

$$\frac{n^{+}}{\rho^{+}} + \frac{n^{-}}{\rho^{-}} = 1, \tag{1.6}$$

therefore

$$\rho^{-} = \frac{n^{-}\rho^{+}}{\rho^{+} - n^{+}}.$$

From $(1.1)_4$, we have

$$\psi(\rho^+) := P^+(\rho^+) - P^-\left(\frac{n^-\rho^+}{\rho^+ - n^+}\right) - f(n^-) = 0. \tag{1.7}$$

Differentiating $\psi(\rho^+)$ with respect to $\rho^+$ for given $n^+$ and $n^-$, we have

$$\psi'(\rho^+) = s_+^2 + s_-^2 \frac{n^- n^+}{(\rho^+ - n^+)^2} > 0.$$

Since for given $\tilde{n}^+$ and $\tilde{n}^-$, there exists $\tilde{\rho}^+ > \tilde{n}^+$ such that (1.7) holds; it follows from the implicit function theorem that in a small neighborhood of the point $(\tilde{n}^+, \tilde{n}^-, \tilde{\rho}^+)$, $\rho^+$ is a $C^1$ function of $n^+$ and $n^-$, that is $\rho^+ = \rho^+(n^+, n^-)$. We can also assume that this neighborhood is closed, which implies the compactness of the neighborhood. Thus we can get the supremum of $\rho^+$. Then we are able to define $\rho^-$ and $\alpha^\pm$ as follows:

$$\begin{aligned}
\rho^-(n^+, n^-) &= \frac{n^-\rho^+(n^+, n^-)}{\rho^+(n^+, n^-) - n^+},\\
\alpha^+(n^+, n^-) &= \frac{n^+}{\rho^+(n^+, n^-)},\\
\alpha^-(n^+, n^-) &= 1 - \frac{n^+}{\rho^+(n^+, n^-)}.
\end{aligned}$$

Based on the above analysis, system (1.1) can be rewritten as follows:

$$\begin{cases}
\quad \partial_t n^\pm + \operatorname{div}(n^\pm u^\pm) = 0,\\
\quad \partial_t(n^+u^+) + \operatorname{div}(n^+u^+\otimes u^+) + \alpha^+\mathcal{C}^2\left[\rho^-\nabla n^+ + \left(\rho^+ + \rho^+\frac{\alpha^- f'}{s_-^2}\right)\nabla n^-\right]\\
= \operatorname{div}\{\alpha^+[\mu^+(\nabla u^+ + \nabla^t u^+) + \lambda^+\operatorname{div}u^+\mathrm{Id}]\} + n^+r^+,\\
\quad \partial_t(n^-u^-) + \operatorname{div}(n^-u^-\otimes u^-) + \alpha^-\mathcal{C}^2\left[\rho^-\nabla n^+ + \left(\rho^+ - \rho^-\frac{\alpha^+ f'}{s_+^2}\right)\nabla n^-\right]\\
= \operatorname{div}\{\alpha^-[\mu^-(\nabla u^- + \nabla^t u^-) + \lambda^-\operatorname{div}u^-\mathrm{Id}]\} + n^-r^-.
\end{cases} \tag{1.8}$$

In this paper, we study the existence of time periodic solutions of (1.8) around a constant state $(\bar{n}^+, \bar{n}^-, \mathbf{0}, \mathbf{0})$. Let $m^\pm = n^\pm - \bar{n}^\pm$, we can reformulate (1.8) as

$$\begin{cases}
\quad \partial_t m^+ + \bar{n}^+\operatorname{div}u^+ = G_1,\\
\quad \partial_t m^- + \bar{n}^-\operatorname{div}u^- = G_2,\\
\quad \partial_t u^+ + \beta_1\nabla m^+ + \beta_2\nabla m^- - (\tilde{\xi} + \xi^+(m^+, m^-))(\mu^+\Delta u^+ + (\mu^+ + \lambda^+)\nabla\operatorname{div}u^+)\\
= G_3 + r^+,\\
\quad \partial_t u^- + \beta_3\nabla m^+ + \beta_4\nabla m^- - (\tilde{\xi} + \xi^-(m^+, m^-))(\mu^-\Delta u^- + (\mu^- + \lambda^-)\nabla\operatorname{div}u^-)\\
= G_4 + r^-,
\end{cases} \tag{1.9}$$

where

$$\begin{aligned}
&\beta_1 = \left(\frac{\mathcal{C}^2\rho^-}{\rho^+}\right)(\bar{n}^+, \bar{n}^-), \beta_2 = \mathcal{C}^2(\bar{n}^+, \bar{n}^-) + \frac{(\mathcal{C}^2\alpha^-)(\bar{n}^+, \bar{n}^-)f'(\bar{n}^-)}{s_-^2(\bar{n}^+, \bar{n}^-)},\\
&\beta_3 = \mathcal{C}^2(\bar{n}^+, \bar{n}^-), \beta_4 = \left(\frac{\mathcal{C}^2\rho^+}{\rho^-}\right)(\bar{n}^+, \bar{n}^-) - \frac{(\mathcal{C}^2\alpha^+)(\bar{n}^+, \bar{n}^-)f'(\bar{n}^-)}{s_+^2(\bar{n}^+, \bar{n}^-)},\\
&\tilde{\xi} = \min\left\{\inf\frac{1}{\rho^+(m^+ + \bar{n}^+, m^- + \bar{n}^-)}, \inf\frac{1}{\rho^-(m^+ + \bar{n}^+, m^- + \bar{n}^-)}\right\},\\
&\xi^\pm(m^+, m^-) = \frac{1}{\rho^\pm(m^+ + \bar{n}^+, m^- + \bar{n}^-)} - \tilde{\xi}.
\end{aligned}$$

We will assume that $-\frac{s_-^2(\bar{n}^+,\bar{n}^-)}{\alpha^-(\bar{n}^+,\bar{n}^-)} < f'(\bar{n}^-)$ to make sure $\beta_2 > 0$. Actually, by the hypotheses, the infimum can be achieved. We also have $\tilde{\xi} > 0$. The right hand side terms of the system are

$$
\begin{aligned}
G_1 &= -\mathrm{div}(m^+u^+),\ G_2 = -\mathrm{div}(m^-u^-),\\
G_3^i &= -g_1^+(m^+,m^-)\partial_i m^+ - g_2^+(m^+,m^-)\partial_i m^- - (u^+\cdot\nabla)u_i^+\\
&\quad + \mu^+ h_1^+(m^+,m^-)\partial_j m^+ \partial_j u_i^+ + \mu^+ h_2^+(m^+,m^-)\partial_j m^- \partial_j u_i^+\\
&\quad + \mu^+ h_1^+(m^+,m^-)\partial_j m^+ \partial_i u_j^+ + \mu^+ h_2^+(m^+,m^-)\partial_j m^- \partial_i u_j^+\\
&\quad + \lambda^+ h_1^+(m^+,m^-)\partial_i m^+ \partial_j u_j^+ + \lambda^+ h_2^+(m^+,m^-)\partial_i m^- \partial_j u_j^+,\\
G_4^i &= -g_1^-(m^+,m^-)\partial_i m^+ - g_2^-(m^+,m^-)\partial_i m^- - (u^-\cdot\nabla)u_i^-\\
&\quad + \mu^- h_1^-(m^+,m^-)\partial_j m^+ \partial_j u_i^- + \mu^- h_2^+(m^+,m^-)\partial_j m^- \partial_j u_i^-\\
&\quad + \mu^- h_1^+(m^+,m^-)\partial_j m^+ \partial_i u_j^- + \mu^- h_2^+(m^+,m^-)\partial_j m^- \partial_i u_j^-\\
&\quad + \lambda^- h_1^+(m^+,m^-)\partial_i m^+ \partial_j u_j^- + \lambda^- h_2^+(m^+,m^-)\partial_i m^- \partial_j u_j^-,
\end{aligned}
$$

where the summation convention is used for $1 \le i,\ j \le 3$, and $g_1^\pm$, $g_2^\pm$, $h_1^\pm$, $h_2^\pm$ are defined by

$$
\begin{cases}
g_1^+(m^+,m^-) &= \left(\frac{\mathcal{C}^2\rho^-}{\rho^+}\right)(m^+ + \bar{n}^+, m^- + \bar{n}^-) - \beta_1,\\
g_1^-(m^+,m^-) &= \mathcal{C}^2(m^+ + \bar{n}^+, m^- + \bar{n}^-) - \beta_3,
\end{cases}
$$

$$
\begin{cases}
g_2^+(m^+,m^-) = & \mathcal{C}^2(m^+ + \bar{n}^+, m^- + \bar{n}^-)\\
& + \frac{(\mathcal{C}^2\alpha^-)(m^+ + \bar{n}^+, m^- + \bar{n}^-) f'(m^- \bar{n}^-)}{s_-^2(m^+ + \bar{n}^+, m^- + \bar{n}^-)} - \beta_2,\\
g_2^-(m^+,m^-) = & \left(\frac{\mathcal{C}^2\rho^+}{\rho^-}\right)(m^+ + \bar{n}^+, m^- + \bar{n}^-)\\
& - \frac{(\mathcal{C}^2\alpha^+)(m^+ + \bar{n}^+, m^- + \bar{n}^-) f'(m^- + \bar{n}^-)}{s_+^2(m^+ + \bar{n}^+, m^- + \bar{n}^-)} - \beta_4,
\end{cases}
$$

$$
\begin{cases}
h_1^+(m^+,m^-) = & \frac{1}{m^+ + \bar{n}^+}\left(\frac{\mathcal{C}^2\alpha^-}{s_-^2}\right)(m^+ + \bar{n}^+, m^- + \bar{n}^-),\\
h_1^-(m^+,m^-) = & -\left(\frac{\mathcal{C}^2}{\rho^- s_-^2}\right)(m^+ + \bar{n}^+, m^- + \bar{n}^-),
\end{cases}
$$

$$
\begin{cases}
h_2^+(m^+,m^-) = & -\left(\frac{\mathcal{C}^2}{\rho^+ s_+^2}\right)(m^+ + \bar{n}^+, m^- + \bar{n}^-)\\
& - \frac{(\mathcal{C}^2\alpha^-)(m^+ + \bar{n}^+, m^- + \bar{n}^-) f'(m^- \bar{n}^-)}{(s_+^2 s_-^2 \rho^+)(m^+ + \bar{n}^+, m^- + \bar{n}^-)},\\
h_2^-(m^+,m^-) = & \frac{1}{m^- + \bar{n}^-}\left(\frac{\mathcal{C}^2\alpha^+}{s_+^2}\right)(m^+ + \bar{n}^+, m^- + \bar{n}^-)\\
& - \frac{(\mathcal{C}^2\alpha^+)(m^+ + \bar{n}^+, m^- + \bar{n}^-) f'(m^- \bar{n}^-)}{(s_+^2 s_-^2 \rho^-)(m^+ + \bar{n}^+, m^- + \bar{n}^-)}.
\end{cases}
$$

Notice that

$$
\zeta := \beta_1\beta_4 - \beta_2\beta_3 = -\frac{\mathcal{C}^2(\bar{n}^+,\bar{n}^-) f'(\bar{n}^-)}{\rho^+(\bar{n}^+,\bar{n}^-)} > 0,
$$

since $f \not\equiv 0$ and $f$ is a strictly decreasing function near $\bar{n}^-$.

### 1.3. Main theorem.

THEOREM 1.1. *Assume that* $r^{\pm}\in L^2((0,T);H^3(\Omega_L))$ *with* $r^{\pm}(-x,t)=-r^{\pm}(x,t)$, *and* $-\frac{s_-^2(\bar n^+,\bar n^-)}{\alpha^-(\bar n^+,\bar n^-)}<f'(\bar n^-)$. *If*

$$\int_0^T \|(r^+,r^-)\|_{H^3}^2 \mathrm{d}t\le\eta,$$

*for some appropriately small constant* $\eta>0$, *then the problem* (1.9) *has a time periodic solution* $(m^+,m^-,u^+,u^-)\in\mathcal{X}_{\eta_0}^L$, *where* $\Omega_L=(-L,L)^3$ *and* $\mathcal{X}_{\eta_0}^L$ *is defined as below.*

Define

$$\mathcal{X}^L:=\Big\{(m^+,m^-,u^+,u^-)\in L^\infty((0,T);H^3(\Omega_L))\cap L^2((0,T);H^4(\Omega_L));\\ (m^+,m^-,u^+,u^-)\text{ satisfies (a),(b),(c)}\Big\};$$

$$\begin{aligned}&\|(m^+,m^-,u^+,u^-)\|_{\mathcal{X}^L}^2\\ :=&\sup_{0\le t\le T}\Big(\|m^+\|_{H^3(\Omega_L)}^2+\|m^-\|_{H^3(\Omega_L)}^2+\|u^+\|_{H^3(\Omega_L)}^2+\|u^-\|_{H^3(\Omega_L)}^2\Big)\\ &+\int_0^T\Big(\|m^+\|_{H^4(\Omega_L)}^2+\|m^-\|_{H^4(\Omega_L)}^2+\|u^+\|_{H^4(\Omega_L)}^2+\|u^-\|_{H^4(\Omega_L)}^2\Big)\mathrm{d}t;\end{aligned}$$

$$\mathcal{X}_\eta^L:=\Big\{(m^+,m^-,u^+,u^-)\in\mathcal{X}^L;\|(m^+,m^-,u^+,u^-)\|_{\mathcal{X}^L}<\eta\Big\}.$$

REMARK 1.1. Similar to the last section of [19], by the standard energy method, we can prove that the time periodic solution in Theorem 1.1 is unique, provided that $\sup\limits_{0\le t\le T}\|(m^+,\,m^-,\,u^+,\,u^-)(t)\|_{H^4(\Omega_L)}$ is sufficiently small.

To prove Theorem 1.1, we first consider the following regularized problem

$$\begin{cases}\partial_t m^+ +\bar n^+\mathrm{div}u^+ -\epsilon\Delta m^+ =G_1,\\ \partial_t m^- +\bar n^-\mathrm{div}u^- -\epsilon\Delta m^- =G_2,\\ \partial_t u^+ +\beta_1\nabla m^+ +\beta_2\nabla m^- -(\tilde\xi+\xi^+(m^+,m^-))(\mu^+\Delta u^+ +(\mu^+ +\lambda^+)\nabla\mathrm{div}u^+)\\ =G_3+r^+,\\ \partial_t u^- +\beta_3\nabla m^+ +\beta_4\nabla m^- -(\tilde\xi+\xi^-(m^+,m^-))(\mu^-\Delta u^- +(\mu^- +\lambda^-)\nabla\mathrm{div}u^-)\\ =G_4+r^-,\\ \int_{\Omega_L} m^\pm dx=0,\end{cases}\tag{1.10}$$

where $\Omega_L=(-L,L)^3\subset\mathbb{R}^3$, $r^{\pm}$ are time periodic functions with periodic boundaries, and odd functions on the space variable $x$. We have the following proposition.

PROPOSITION 1.1. *Assume that* $r^{\pm}\in L^2((0,T);H^3(\Omega_L))$ *with* $r^{\pm}(-x,t)=-r^{\pm}(x,t)$, *and* $-\frac{s_-^2(\bar n^+,\bar n^-)}{\alpha^-(\bar n^+,\bar n^-)}<f'(\bar n^-)$. *If*

$$\int_0^T \|(r^+,r^-)\|_{H^3}^2 \mathrm{d}t\le\eta,$$

*for some appropriately small constant* $\eta > 0$, *then the regularized problem* (1.10) *admits a solution* $(m^+, m^-, u^+, u^-) \in \mathcal{X}_{\eta_0}^L$, *that satisfies*

(a) $m^+, m^-, u^+, u^-$ *are time periodic functions with the space period* $2L$ *and time period* $T$;

(b) $\int_{\Omega_L} m^{\pm} dx \equiv 0$;

(c) $m^{\pm}(-x,t) = m^{\pm}(x,t), u^{\pm}(-x,t) = -u^{\pm}(x,t)$.

Finally, we get the existence of the time periodic solution of (1.9), by a limit procedure, $\epsilon \to 0$, i.e., Theorem 1.1.

## 2. Preliminaries

Throughout this paper, $C$ denotes a generic positive constant which may vary in different estimates and $\eta_i (i = 1,2,3\cdots)$ are suitable small positive constants which will be identified in the proof of Theorem 1.1. We denote $H^k(\Omega)$, the Sobolev spaces $W^{k,2}(\Omega)$, with norm $\|\cdot\|_{H^k(\Omega)}$, and denote $L^p(\Omega)(1 \le p \le \infty)$, the $L^p$ spaces, with norm $\|\cdot\|_{L^p(\Omega)}$. When there is no ambiguity, we will write $\|\cdot\|_{H^k(\Omega)}, \|\cdot\|_{L^p(\Omega)}$ as $\|\cdot\|_{H^k}, \|\cdot\|_{L^p}$ respectively. The notation "$\langle \cdot, \cdot \rangle$" means the inner product in $L^2(\Omega)$. We also write $\|u\|_{H^p} + \|v\|_{H^p}$ as $\|(u,v)\|_{H^p}$ for short. For the readers' convenience, we list some useful lemmas as follows.

Lemma 2.1 ( [1, 18]). *Assume that* $\Omega \subset \mathbb{R}^3$ *is a bounded domain, and* $\partial\Omega$ *is locally Lipschitz continuous. If* $u|_{\partial\Omega} = 0$ (*or* $\int_\Omega u \mathrm{d}x = 0$), *then for any* $1 \le p < N, 1 \le q \le p^* = \frac{Np}{N-p}$,

$$\left(\int_\Omega |u|^q \mathrm{d}x\right)^{1/q} \le C(N,p,q) |\mathbf{meas}\Omega|^{1/q - 1/p^*} \left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)^{1/p}.$$

*In particular, if* $q = p^* = \frac{Np}{N-p}$, *then*

$$\left(\int_\Omega |u|^{p^*} \mathrm{d}x\right)^{p^*} \le C(N,p,p^*) \left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)^{1/p}.$$

Lemma 2.2 ( [1, 18]). *Assume that* $\Omega \subset \mathbb{R}^3$ *is a bounded domain, and* $\partial\Omega$ *is locally Lipschitz continuous. If* $u|_{\partial\Omega} = 0$ (*or* $\int_\Omega u \mathrm{d}x = 0$), *then*

$$\begin{aligned} \|u\|_{L^3} &\le C \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2}, \\ \|u\|_{L^4} &\le C \|u\|_{L^2}^{1/4} \|\nabla u\|_{L^2}^{3/4}, \\ \|u\|_{L^\infty} &\le C \|\nabla u\|_{H^1}, \end{aligned}$$

*where* $C$ *is independent of* $\Omega$. *Moreover, the above inequalities also hold in* $\mathbb{R}^3$ *if* $u(x) \to 0$ *as* $|x| \to \infty$.

Lemma 2.3 ( [11, 26]). *Let* $k \ge 1$ *be an integer and* $\Omega$ *be a domain as above, then we have*

$$\|\nabla^k (fg)\|_{L^p} \le C \|f\|_{L^{p_1}} \|\nabla^k g\|_{L^{p_2}} + C \|\nabla^k f\|_{L^{p_3}} \|\nabla^k g\|_{L^{p_4}},$$

*where* $p$, $p_1$, $p_2$, $p_3$, $p_4 \in [1, \infty]$ *and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

LEMMA 2.4 ( [25]). *Assume $X\subset E\subset Y$ are Banach spaces and $X\hookrightarrow\hookrightarrow E$. Then the following embeddings are compact:*

$$\Big\{\phi:\phi\in L^q(0,T;X),\frac{\partial\phi}{\partial t}\in L^1(0,T;Y)\Big\}\hookrightarrow\hookrightarrow L^q(0,T;E),\ if\ 1\le q\le\infty;$$
$$\Big\{\phi:\phi\in L^\infty(0,T;X),\frac{\partial\phi}{\partial t}\in L^r(0,T;Y)\Big\}\hookrightarrow\hookrightarrow C([0,T];E),\ if\ 1<r\le\infty.$$

LEMMA 2.5 ( [9]). *Let $E$ be a real Banach space, $\Omega\subset E$ be an open bounded set and the operator $T:\bar\Omega\to E$ be a completely continuous. The Leray-Schauder degree has the following properties:*

*(1) (Normality)* $\deg(I,\Omega,0)=1$ *if and only if* $0\in\Omega$;

*(2) (Solvability) If* $\deg(I-T,\Omega,0)\neq 0$, *then* $Tx=x$ *has a solution in* $\Omega$;

*(3) (Homotopy) Let* $T_t$: $[0,1]\times\bar\Omega\to E$ *be completely compact and* $T_tx\neq x$ *for all* $(t,x)\in[0,1]\times\partial\Omega$. *Then* $\deg(I-T_t,\Omega,0)$ *doesn't depend on* $t\in[0,1]$.

For more details about topological degree theory, please refer to [8,9].

## 3. Existence of the time periodic solutions

**3.1. Introduction of an operator $\mathcal{S}$.** To prove the existence of time periodic solutions, we define an operator

$$\mathcal{S}:\mathcal{X}^L_\eta\times[0,1]\to\mathcal{X}^L,$$
$$((n^+,\,n^-,\,w^+,\,w^-),\tau)\mapsto(m^+,\,m^-,\,u^+,\,u^-),$$

with $\eta$ appropriately small, where $(m^+,\,m^-,\,u^+,\,u^-)$ is the solution of the following linear problem with periodic boundary

$$\begin{cases}\partial_t m^+ +\bar n^+\mathrm{div}u^+ -\epsilon\Delta m^+=\tilde G_1(n^+,w^+,\tau),\\ \partial_t m^- +\bar n^-\mathrm{div}u^- -\epsilon\Delta m^-=\tilde G_2(n^-,w^-,\tau),\\ \partial_t u^+ +\beta_1\nabla m^+ +\beta_2\nabla m^- -(\tilde\xi+\tau\xi^+(n^+,n^-))(\mu^+\Delta u^+ +(\mu^+ +\lambda^+)\nabla\mathrm{div}u^+)\\ =\tilde G_3(n^+,n^-,w^+,\tau)+\tau r^+,\\ \partial_t u^- +\beta_3\nabla m^+ +\beta_4\nabla m^- -(\tilde\xi+\tau\xi^-(n^+,n^-))(\mu^-\Delta u^- +(\mu^- +\lambda^-)\nabla\mathrm{div}u^-)\\ =\tilde G_4(n^+,n^-,w^-,\tau)+\tau r^-,\\ \int_{\Omega_L} m^\pm dx=0,\end{cases}\tag{3.1}$$

where

$$\begin{aligned}&\tilde G_1=-\tau\mathrm{div}(n^+w^+),\ \tilde G_2=-\tau\mathrm{div}(n^-w^-),\\ &\tilde G_3^i=-\tau g_1^+(n^+,n^-)\partial_i n^+ -\tau g_2^+(n^+,n^-)\partial_i n^- -\tau(w^+\cdot\nabla)w_i^+\\ &\qquad +\mu^+\tau h_1^+(n^+,n^-)\partial_j n^+\partial_j w_i^+ +\mu^+\tau h_2^+(n^+,n^-)\partial_j n^-\partial_j w_i^+\\ &\qquad +\mu^+\tau h_1^+(n^+,n^-)\partial_j n^+\partial_i w_j^+ +\mu^+\tau h_2^+(n^+,n^-)\partial_j n^-\partial_i w_j^+\\ &\qquad +\lambda^+\tau h_1^+(n^+,n^-)\partial_i n^+\partial_j w_j^+ +\lambda^+\tau h_2^+(n^+,n^-)\partial_i n^-\partial_j w_j^+,\end{aligned}$$

and

$$\begin{aligned}\tilde{G}_4^i=&-\tau g_1^-(n^+,n^-)\partial_i n^+-\tau g_2^-(n^+,n^-)\partial_i n^- -\tau(w^-\cdot\nabla)w_i^-\\&+\mu^-\tau h_1^-(n^+,n^-)\partial_j n^+\partial_j w_i^-+\mu^-\tau h_2^+(n^+,n^-)\partial_j n^-\partial_j w_i^-\\&+\mu^-\tau h_1^+(n^+,n^-)\partial_j n^+\partial_i w_j^-+\mu^-\tau h_2^+(n^+,n^-)\partial_j n^-\partial_i w_j^-\\&+\lambda^-\tau h_1^+(n^+,n^-)\partial_i n^+\partial_j w_j^-+\lambda^-\tau h_2^+(n^+,n^-)\partial_i n^-\partial_j w_j^-.\end{aligned}$$

We impose the condition $\int_{\Omega_L} m^{\pm}dx=0$ to ensure the uniqueness of the solution. Since $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L} m^{\pm}\mathrm{d}x=0$, and when $(m^+,\ m^-,\ u^+,\ u^-)$ is a solution of (3.1), then $(m^++C,$ $m^-+C,\ u^+,\ u^-)$ is also a solution for any constant $C$. We should also note that when $\sup\limits_{0\le t\le T}\|m^{\pm}\|_{H^3}<\eta$, and $\eta$ is suitably small, we have

$$\frac{1}{2}\bar{n}^+\le m^++\bar{n}^+\le\frac{3}{2}\bar{n}^+,\ \frac{1}{2}\bar{n}^-\le m^++\bar{n}^-\le\frac{3}{2}\bar{n}^-,$$

and for some positive constant $C$, we have

$$|(g_i^{\pm},h_i^{\pm})(m^+,m^-)|\le C|(m^+,m^-),\ |\partial_{m^{\pm}}^k(g_i^{\pm},h_i^{\pm})(m^+,m^-)|\le C,$$

for $i=1,2$ and $k\ge 1$.

And then, we show that the operator $\mathcal{S}$ is well-defined by the following lemmas.

LEMMA 3.1. *Assume that $\eta$ is sufficiently small, then for any $(n^+,n^-,w^+,$ $w^-)\in\mathcal{X}_\eta^L,\tau\in[0,1]$, the problem (3.1) admits a unique time periodic solution $(m^+,m^-,u^+,u^-)\in\mathcal{X}^L$.*

*Proof.* Let

$$\begin{aligned}&U=(m^+,m^-,u^+,u^-)^t,W=(n^+,n^-,w^+,w^-)^t,\\&\tilde{G}(W)=(\tilde{G}_1,\tilde{G}_2,\tilde{G}_3,\tilde{G}_4)^t,R=(0,0,\tau r^+,\tau r^-)^t\end{aligned}$$

and $\mathcal{A}$ denote the matrix operator

$$\mathcal{A}=\begin{pmatrix}\epsilon\Delta & 0 & -\bar{n}^+\mathrm{div} & 0\\ 0 & \epsilon\Delta & 0 & -\bar{n}^-\mathrm{div}\\ -\beta_1\nabla & -\beta_2\nabla & \xi_1 & 0\\ -\beta_3\nabla & -\beta_4\nabla & 0 & \xi_2\end{pmatrix},$$

where $\xi_1=(\tilde{\xi}+\tau\xi^+(n^+,n^-))(\mu^+\Delta+(\mu^++\lambda^+)\nabla\mathrm{div}),\xi_2=(\tilde{\xi}+\tau\xi^-(n^+,n^-))(\mu^-\Delta+$ $(\mu^-+\lambda^-)\nabla\mathrm{div})$. Then the system (3.1) can be written as

$$U_t=\mathcal{A}U+\tilde{G}(W)+R.$$

Now, consider the initial value problem of the linear system $U_t=\mathcal{A}U$ in $\Omega_L$ with periodic boundary

$$\begin{cases}\partial_t m^++\bar{n}^+\mathrm{div}u^+-\epsilon\Delta m^+=0,\\ \partial_t m^-+\bar{n}^-\mathrm{div}u^--\epsilon\Delta m^-=0,\\ \partial_t u^++\beta_1\nabla m^++\beta_2\nabla m^--(\tilde{\xi}+\tau\xi^+(n^+,n^-))(\mu^+\Delta u^++(\mu^++\lambda^+)\nabla\mathrm{div}u^+)=0,\\ \partial_t u^-+\beta_3\nabla m^++\beta_4\nabla m^--(\tilde{\xi}+\tau\xi^-(n^+,n^-))(\mu^-\Delta u^-+(\mu^-+\lambda^-)\nabla\mathrm{div}u^-)=0,\\ (m^+,m^-,u^+,u^-)(x,0)=(m_0^+,m_0^-,u_0^+,u_0^-)(x),\end{cases}\tag{3.2}$$

where $m_0^{\pm}(x)$ are even functions with $\int_{\Omega_L} m_0^{\pm}(x)\mathrm{d}x=0$, and $u_0^{\pm}(x)$ are odd functions. It is not difficult to see that the solution $(m^+, m^-, u^+, u^-)$ has the same properties as the initial data $(m_0^+, m_0^-, u_0^+, u_0^-)$.

Multiplying $(3.2)_1, (3.2)_2, (3.2)_3, (3.2)_4$ by $\beta_1\beta_3 m^+$, $\beta_2\beta_4 m^-$, $\beta_3\bar{n}^+u^+$, $\beta_2\bar{n}^-u^-$ respectively, summing up, then integrating the result over $\Omega_L$ by parts and combining with the periodic boundary conditions, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\beta_1\beta_3|m^+|^2+\beta_2\beta_4|m^-|^2+\beta_3\bar{n}|u^+|^2+\beta_2\bar{n}^-|u^-|^2\Big)\mathrm{d}x\\
&\quad+\beta_3\bar{n}^+\int_{\Omega_L}\Big(\tilde{\xi}+\tau\xi^+(n^+,n^-)\Big)\Big(\mu^+|\nabla u^+|^2+(\mu^++\lambda^+)|\mathrm{div}u^+|^2\Big)\mathrm{d}x\\
&\quad+\beta_2\bar{n}^-\int_{\Omega_L}\Big(\tilde{\xi}+\tau\xi^-(n^+,n^-)\Big)\Big(\mu^-|\nabla u^-|^2+(\mu^-+\lambda^-)|\mathrm{div}u^-|^2\Big)\mathrm{d}x\\
&\quad+\epsilon\int_{\Omega_L}\Big(\beta_1\beta_3|\nabla m^+|^2+\beta_2\beta_4|\nabla m^-|^2\Big)\mathrm{d}x+\beta_2\beta_3\int_{\Omega_L}\Big(\bar{n}^-\nabla m^+u^-+\bar{n}^+\nabla m^-u^+\Big)\mathrm{d}x\\
&=-\beta_3\bar{n}^+\tau\int_{\Omega_L}\Big(\partial_{n^+}\xi^+\nabla n^++\partial_{n^-}\xi^+\nabla n^-\Big)\Big(\mu^+u^+\nabla u^++(\mu^++\lambda^+)u^+\mathrm{div}u^+\Big)\mathrm{d}x\\
&\quad-\beta_2\bar{n}^-\tau\int_{\Omega_L}\Big(\partial_{n^+}\xi^-\nabla n^++\partial_{n^-}\xi^-\nabla n^-\Big)\Big(\mu^-u^-\nabla u^-+(\mu^-+\lambda^-)u^-\mathrm{div}u^-\Big)\mathrm{d}x.
\end{aligned}\tag{3.3}$$

Multiplying $(3.2)_1, (3.2)_2$ by $m^-$, $m^+$ respectively, summing up, then integrating the result over $\Omega_L$ by parts and combining with the periodic boundary conditions, we obtain

$$\begin{aligned}
&\int_{\Omega_L}\big(\bar{n}^-\nabla m^+u^-+\bar{n}^+\nabla m^-u^+\big)\,\mathrm{d}x\\
&=\int_{\Omega_L}(m^-\partial_t m^++m^+\partial_t m^-)\mathrm{d}x+2\epsilon\int_{\Omega_L}\nabla m^+\nabla m^-\mathrm{d}x.
\end{aligned}\tag{3.4}$$

Note that $\zeta:=\beta_1\beta_4-\beta_2\beta_3>0$ and $\tilde{\xi}+\xi^{\pm}\geq\tilde{\xi}>0$ (cf. Section 1.2), then combining the above two equalities, we obtain

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|m^+|^2+\zeta\frac{\beta_2}{\beta_1}|m^-|^2+2\beta_3\bar{n}|u^+|^2+2\beta_2\bar{n}^-|u^-|^2\Big)\mathrm{d}x\\
&\quad+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\beta_2\beta_4|m^-+\frac{\beta_3}{\beta_4}m^+|^2+\beta_1\beta_3|m^++\frac{\beta_2}{\beta_1}m^-|^2\Big)dx\\
&\quad+\frac{1}{2}\epsilon\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla m^-|^2\Big)\mathrm{d}x\\
&\quad+\frac{1}{2}\epsilon\int_{\Omega_L}\Big(\beta_2\beta_4|\nabla m^-+\frac{\beta_3}{\beta_4}\nabla m^+|^2+\beta_1\beta_3|\nabla m^++\frac{\beta_2}{\beta_1}\nabla m^-|^2\Big)\mathrm{d}x\\
&\quad+\tilde{\xi}\int_{\Omega_L}\Big(\beta_3\bar{n}^+\mu^+|\nabla u^+|^2+\beta_2\bar{n}^-\mu^-|\nabla u^-|^2\Big)\mathrm{d}x\\
&\leq-\beta_3\bar{n}^+\tau\int_{\Omega_L}\Big(\partial_{n^+}\xi^+\nabla n^++\partial_{n^-}\xi^+\nabla n^-)(\mu^+u^+\nabla u^++(\mu^++\lambda^+)u^+\mathrm{div}u^+\Big)\mathrm{d}x\\
&\qquad-\beta_2\bar{n}^-\tau\int_{\Omega_L}\Big(\partial_{n^+}\xi^-\nabla n^++\partial_{n^-}\xi^-\nabla n^-)(\mu^-u^-\nabla u^-+(\mu^-+\lambda^-)u^-\mathrm{div}u^-\Big)\mathrm{d}x\\
&\leq C\tau\Big(\|\nabla n^+\|_{L^3}+\|\nabla n^-\|_{L^3}\Big)\Big(\|u^+\|_{L^6}\|\nabla u^+\|_{L^2}+\|u^-\|_{L^6}\|\nabla u^-\|_{L^2}\Big)
\end{aligned}$$

$$\le C\tau\eta\Big(\|\nabla u^+\|_{L^2}^2+\|\nabla u^-\|_{L^2}^2\Big), \tag{3.5}$$

then, when $\eta$ is appropriately small, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|m^+|^2+\zeta\frac{\beta_2}{\beta_1}|m^-|^2+2\beta_3\bar{n}|u^+|^2+2\beta_2\bar{n}^-|u^-|^2\Big)\mathrm{d}x\\
&\quad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\beta_2\beta_4|m^-+\frac{\beta_3}{\beta_4}m^+|^2+\beta_1\beta_3|m^++\frac{\beta_2}{\beta_1}m^-|^2\Big)dx+\epsilon\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla m^+|^2\\
&\quad+\zeta\frac{\beta_2}{\beta_1}|\nabla m^-|^2\Big)\mathrm{d}x+\epsilon\int_{\Omega_L}\Big(\beta_2\beta_4|\nabla m^-+\frac{\beta_3}{\beta_4}\nabla m^+|^2+\beta_1\beta_3|\nabla m^++\frac{\beta_2}{\beta_1}\nabla m^-|^2\Big)\mathrm{d}x\\
&\quad+\tilde{\xi}\int_{\Omega_L}\Big(\beta_3\bar{n}^+\mu^+|\nabla u^+|^2+\beta_2\bar{n}^-\mu^-|\nabla u^-|^2\Big)\mathrm{d}x\le 0.
\end{aligned}\tag{3.6}$$

By Poincaré's inequality and Gronwall's inequality, we obtain

$$\|(m^+,m^-,u^+,u^-)(t)\|_{L^2}\le\|(m_0^+,m_0^-,u_0^+,u_0^-)\|_{L^2}e^{-C\epsilon t}.$$

Similarly, applying $\nabla^k,(1\le k\le 4)$ to $(3.2)_1,(3.2)_2,(3.2)_3,(3.2)_4$, multiplying the results by $\beta_1\beta_3\nabla^k m^+$, $\beta_2\beta_4\nabla^k m^-$, $\beta_3\bar{n}^+\nabla^k u^+$, $\beta_2\bar{n}^-\nabla^k u^-$ respectively, summing up, then integrating the result over $\Omega_L$ by parts and combining with the periodic boundary conditions and the Equations $(3.2)_1,(3.2)_2$, we obtain

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^k m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k m^-|^2+2\beta_3\bar{n}|\nabla^k u^+|^2+2\beta_2\bar{n}^-|\nabla^k u^-|^2\Big)\mathrm{d}x\\
&\quad+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\beta_2\beta_4|\nabla^k m^-+\frac{\beta_3}{\beta_4}\nabla^k m^+|^2+\beta_1\beta_3|\nabla^k m^++\frac{\beta_2}{\beta_1}\nabla^k m^-|^2\Big)dx\\
&\quad+\frac{1}{2}\epsilon\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^k\nabla m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k\nabla m^-|^2\Big)\mathrm{d}x\\
&+\frac{1}{2}\epsilon\int_{\Omega_L}\Big(\beta_2\beta_4|\nabla^k\nabla m^-+\frac{\beta_3}{\beta_4}\nabla^k\nabla m^+|^2+\beta_1\beta_3|\nabla^k\nabla m^++\frac{\beta_2}{\beta_1}\nabla^k\nabla m^-|^2\Big)\mathrm{d}x\\
&+\beta_3\bar{n}^+\int_{\Omega_L}\Big((\tilde{\xi}+\tau\xi^+(n^+,n^-))(\mu^+|\nabla^k\nabla u^+|^2+(\mu^++\lambda^+)|\nabla^k\mathrm{div}u^+|^2)\Big)\mathrm{d}x\\
&+\beta_2\bar{n}^-\int_{\Omega_L}\Big((\tilde{\xi}+\tau\xi^-(n^+,n^-))(\mu^-|\nabla^k\nabla u^-|^2+(\mu^-+\lambda^-)|\nabla^k\mathrm{div}u^-|^2)\Big)\mathrm{d}x\\
=&-\beta_3\bar{n}^+\tau\int_{\Omega_L}\Big(\partial_{n^+}\xi^+\nabla n^++\partial_{n^-}\xi^+\nabla n^-\Big)\Big(\mu^+\nabla^k\nabla u^++(\mu^++\lambda^+)\nabla^k\mathrm{div}u^+\Big)\nabla^k u^+\mathrm{d}x\\
&-\beta_2\bar{n}^-\tau\int_{\Omega_L}\Big(\partial_{n^+}\xi^-\nabla n^++\partial_{n^-}\xi^-\nabla n^-\Big)\Big(\mu^-\nabla^k\nabla u^-+(\mu^-+\lambda^-)\nabla^k\mathrm{div}u^-\Big)\nabla^k u^-\mathrm{d}x\\
&+\beta_3\bar{n}^+\tau\int_{\Omega_L}\sum_{1\le l\le k}\binom{k}{l}\nabla^l\xi^+\nabla^{k-l}\Big(\mu^+\Delta u^++(\mu^++\lambda^+)\nabla\mathrm{div}u^+\Big)\nabla^k u^+\mathrm{d}x\\
&+\beta_2\bar{n}^-\tau\int_{\Omega_L}\sum_{1\le l\le k}\binom{k}{l}\nabla^l\xi^-\nabla^{k-l}\Big(\mu^-\Delta u^-+(\mu^-+\lambda^-)\nabla\mathrm{div}u^-\Big)\nabla^k u^-\mathrm{d}x\\
:=&E,
\end{aligned}\tag{3.7}$$

and $E$ can be estimated as follows:

$$E\le C\tau\Big(\|\nabla n^+\|_{L^3}+\|\nabla n^-\|_{L^3}\Big)\Big(\|\nabla^{k+1}u^+\|_{L^2}\|\nabla^k u^+\|_{L^6}+\|\nabla^{k+1}u^-\|_{L^2}\|\nabla^k u^-\|_{L^6}\Big)$$

$$
\begin{aligned}
&+C\tau\Big(\|\nabla^k n^+\|_{L^2}+\|\nabla^k n^-\|_{L^2}\Big)\Big(\|\nabla^2 u^+\|_{L^3}\|\nabla^k u^+\|_{L^6}+\|\nabla^2 u^-\|_{L^3}\|\nabla^k u^-\|_{L^6}\Big)\\
&\le C\tau\eta\Big(\|\nabla^{k+1}u^+\|_{L^2}^2+\|\nabla^{k+1}u^-\|_{L^2}^2\Big),\ \text{for } 2\le k\le 4,\\
E&\le C\tau\Big(\|\nabla n^+\|_{L^3}+\|\nabla n^-\|_{L^3}\Big)\Big(\|\nabla^2 u^+\|_{L^2}\|\nabla u^+\|_{L^6}+\|\nabla^2 u^-\|_{L^2}\|\nabla u^-\|_{L^6}\Big)\\
&\le C\tau\eta\Big(\|\nabla^2 u^+\|_{L^2}^2+\|\nabla^2 u^-\|_{L^2}^2\Big),\ \text{for } k=1.
\end{aligned}
$$

Then, when $\eta$ is appropriately small, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^k m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k m^-|^2+2\beta_3\bar{n}|\nabla^k u^+|^2+2\beta_2\bar{n}^-|\nabla^k u^-|^2\Big)\mathrm{d}x\\
&\quad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\beta_2\beta_4|\nabla^k m^-+\frac{\beta_3}{\beta_4}\nabla^k m^+|^2+\beta_1\beta_3|\nabla^k m^++\frac{\beta_2}{\beta_1}\nabla^k m^-|^2\Big)dx\\
&\quad+\epsilon\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^{k+1}m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^{k+1}m^-|^2\Big)\mathrm{d}x\\
&\quad+\epsilon\int_{\Omega_L}\Big(\beta_2\beta_4|\nabla^k\nabla m^-+\frac{\beta_3}{\beta_4}\nabla^k\nabla m^+|^2+\beta_1\beta_3|\nabla^k\nabla m^++\frac{\beta_2}{\beta_1}\nabla^k\nabla m^-|^2\Big)\mathrm{d}x\\
&\quad+\tilde{\xi}\int_{\Omega_L}\Big(\beta_3\bar{n}^+\mu^+|\nabla^{k+1}u^+|^2+\beta_2\bar{n}^-\mu^-|\nabla^{k+1}u^-|^2\Big)\mathrm{d}x\le 0.
\end{aligned}
\tag{3.8}
$$

By Poincaré's inequality and Gronwall's inequality, we obtain

$$
\|\nabla^k(m^+,m^-,u^+,u^-)(t)\|_{L^2}\le\|\nabla^k(m_0^+,m_0^-,u_0^+,u_0^-)\|_{L^2}e^{-C\epsilon t},\ \text{for } 1\le k\le 4.
$$

Therefore, we have

$$
\|e^{t\mathcal{A}}U_0\|_{H^k}\le\|U_0\|_{H^k}e^{-C\epsilon t}.
$$

Then by Duhamel's principle, the solution to the system (3.1) can be written as

$$
U(t)=\int_{-\infty}^{t}e^{(t-\tau)\mathcal{A}}(G(W(\tau))+R(\tau))\mathrm{d}\tau,
$$

and then

$$
\begin{aligned}
\|U(t)\|_{H^k}&\le\int_{-\infty}^{t}\|e^{(t-\tau)\mathcal{A}}(G(W(\tau))+R(\tau))\|_{H^k}\mathrm{d}\tau\\
&\le\int_{-\infty}^{t}e^{-C\epsilon(t-\tau)}\|(G(W(\tau))+R(\tau))\|_{H^k}\mathrm{d}\tau\\
&=\sum_{i=0}^{\infty}\int_{t-(i+1)T}^{t-iT}e^{-C\epsilon(t-\tau)}\|(G(W(\tau))+R(\tau))\|_{H^k}\mathrm{d}\tau\\
&=\sum_{i=0}^{\infty}\int_{0}^{T}e^{-C\epsilon((i+1)T-\tau)}\|(G(W(t+\tau))+R(t+\tau))\|_{H^k}\mathrm{d}\tau\\
&\le\sum_{i=0}^{\infty}\left(\int_0^T e^{-2C\epsilon((i+1)T-\tau)}\mathrm{d}\tau\right)^{1/2}\left(\int_0^T\|(G(W(\tau))+R(\tau))\|_{H^k}^2\mathrm{d}\tau\right)^{1/2}\\
&\le C(\epsilon)\left(\int_0^T\|(G(W(t))+R(t))\|_{H^k}^2\mathrm{d}t\right)^{1/2},\ \text{for } k\le 3,
\end{aligned}
$$

where we have used the time periodic property of $W$ and $R$. By the time periodic property, we also have

$$\begin{aligned}U(t+T)&=\int_{-\infty}^{t+T} e^{(t+T-\tau)\mathcal{A}}(G(W(\tau))+R(\tau))\,\mathrm{d}\tau\\&=\int_{-\infty}^{t+T} e^{(t-(\tau-T))\mathcal{A}}(G(W(\tau-T))+R(\tau-T))\,\mathrm{d}\tau\\&=\int_{-\infty}^{t} e^{(t-\tau)\mathcal{A}}(G(W(\tau))+R(\tau))\,\mathrm{d}\tau=U(t).\end{aligned}$$

That is, $U(t)\in L^\infty((0,T);H^3)$ is a time periodic solution of (3.1) with time period T. Similar to the proof of Theorem 1.1 in Section 3, we have for any $(n^+,\, n^-,\, w^+,\, w^-)\in\mathcal{X}^L_\eta, \tau\in[0,1]$, the system (3.1) admits a time periodic solution $(m^+,\, m^-,\, u^+,\, u^-)\in\mathcal{X}^L$.

For the uniqueness, assume that there exist two solutions of system (3.1), $u_1=(m_1^+,\, m_1^-,\, u_1^+,\, u_1^-)$, $u_2=(m_2^+,\, m_2^-,\, u_2^+,\, u_2^-)$ for $(n^+,\, n^-,\, w^+,\, w^-)\in\mathcal{X}^L_\eta$, $\tau\in[0,1]$, then we have

$$\partial_t(U_1-U_2)=\mathcal{A}(U_1-U_2).$$

Similar to the above proof, let $(\bar m^+,\, \bar m^-,\, \bar u^+,\, \bar u^-)=(m_1^+-m_2^+,\, m_1^- -m_2^-,\, u_1^+-u_2^+,\, u_1^- -u_2^-)$, we have

$$\begin{aligned}&\epsilon\int_0^T\int_{\Omega_L}\left(\frac{\beta_3}{\beta_4}|\nabla\bar m^+|^2+\frac{\beta_2}{\beta_1}|\nabla\bar m^-|^2\right)\mathrm{d}x\mathrm{d}t\\&\quad+\tilde\xi\int_0^T\int_{\Omega_L}\left(\beta_3\bar n^+\mu^+|\nabla\bar u^+|^2+\beta_2\bar n^-\mu^-|\nabla\bar u^-|^2\right)\mathrm{d}x\mathrm{d}t\le 0.\end{aligned}$$

Using Poincaré's inequality, we have $(\bar m^+,\bar m^-,\bar u^+,\bar u^-)=0$, which implies the uniqueness.

Finally, we see that if $(m^+,\, m^-,\, u^+,\, u^-)(x,t)$ is the periodic solution of (3.1), then $(m^+,\, m^-,\, -u^+,\, -u^-)(-x,t)$ is also the solution of (3.1), by the uniqueness of the periodic solution, we obtain $(m^+,\, m^-,\, u^+,\, u^-)(x,t)=(m^+,\, m^-,\, -u^+,\, -u^-)(-x,t)$. The proof is completed. ∎

Remark 3.1. It is worth noting that the operator $\mathcal{A}$ defined above generates an exponentially bounded semigroup on $H^1$. One can also get the same result of Lemma 3.1 by using methods from evolution equations.

Next, we show that the operator $\mathcal{S}$ is completely continuous.

Lemma 3.2. *If $\eta$ is sufficiently small, then the operator $\mathcal{S}$ is compact.*

*Proof.* Assume that $(m^+,\, m^-,\, u^+,\, u^-)$ is the solution of system (3.1). Similar to the proof of Lemma 3.1, applying $\nabla^k(1\le k\le 4)$ to $(3.1)_1, (3.1)_2, (3.1)_3, (3.1)_4$ multiplying the results by $\beta_1\beta_3\nabla^k m^+$, $\beta_2\beta_4\nabla^k m^-$, $\beta_3\bar n^+\nabla^k u^+$, $\beta_2\bar n^-\nabla^k u^-$ respectively, summing up, then integrating the result over $\Omega_L$ by parts and combining with the periodic boundary conditions and the Equations $(3.1)_1, (3.1)_2$, we obtain

$$\begin{aligned}&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^k m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k m^-|^2+2\beta_3\bar n|\nabla^k u^+|^2+2\beta_2\bar n^-|\nabla^k u^-|^2\Big)\mathrm{d}x\\&\quad+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\beta_2\beta_4|\nabla^k m^-+\frac{\beta_3}{\beta_4}\nabla^k m^+|^2+\beta_1\beta_3|\nabla^k m^++\frac{\beta_2}{\beta_1}\nabla^k m^-|^2\Big)dx\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\epsilon\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^k\nabla m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k\nabla m^-|^2\Big)\mathrm{d}x\\
&+\frac{1}{2}\epsilon\int_{\Omega_L}\Big(\beta_2\beta_4|\nabla^k\nabla m^-+\frac{\beta_3}{\beta_4}\nabla^k\nabla m^+|^2+\beta_1\beta_3|\nabla^k\nabla m^++\frac{\beta_2}{\beta_1}\nabla^k\nabla m^-|^2\Big)\mathrm{d}x\\
&+\beta_3\bar{n}^+\int_{\Omega_L}\Big((\tilde{\xi}+\tau\xi^+(n^+,n^-))(\mu^+|\nabla^k\nabla u^+|^2+(\mu^++\lambda^+)|\nabla^k\mathrm{div}u^+|^2)\Big)\mathrm{d}x\\
&+\beta_2\bar{n}^-\int_{\Omega_L}\Big((\tilde{\xi}+\tau\xi^-(n^+,n^-))(\mu^-|\nabla^k\nabla u^-|^2+(\mu^-+\lambda^-)|\nabla^k\mathrm{div}u^-|^2)\Big)\mathrm{d}x\\
=&\Big[-\beta_3\bar{n}^+\tau\int_{\Omega_L}\Big(\partial_{n^+}\xi^+\nabla n^++\partial_{n^-}\xi^+\nabla n^-\Big)\Big(\mu^+\nabla^k\nabla u^++(\mu^++\lambda^+)\nabla^k\mathrm{div}u^+\Big)\nabla^k u^+\mathrm{d}x\\
&-\beta_2\bar{n}^-\tau\int_{\Omega_L}\Big(\partial_{n^+}\xi^-\nabla n^++\partial_{n^-}\xi^-\nabla n^-\Big)\Big(\mu^-\nabla^k\nabla u^-+(\mu^-+\lambda^-)\nabla^k\mathrm{div}u^-\Big)\nabla^k u^-\mathrm{d}x\\
&+\beta_3\bar{n}^+\tau\int_{\Omega_L}\sum_{1\le l\le k}\binom{k}{l}\nabla^l\xi^+\nabla^{k-l}\Big(\mu^+\Delta u^++(\mu^++\lambda^+)\nabla\mathrm{div}u^+\Big)\nabla^k u^+\mathrm{d}x\\
&+\beta_2\bar{n}^-\tau\int_{\Omega_L}\sum_{1\le l\le k}\binom{k}{l}\nabla^l\xi^-\nabla^{k-l}\Big(\mu^-\Delta u^-+(\mu^-+\lambda^-)\nabla\mathrm{div}u^-\Big)\nabla^k u^-\mathrm{d}x\Big]\\
&-\Big[\beta_3\langle\nabla^{k-1}\tilde{G}_1,\beta_1\nabla^{k+1}m^++\beta_2\nabla^{k+1}m^-\rangle+\beta_2\langle\nabla^{k-1}\tilde{G}_2,\beta_3\nabla^{k+1}m^++\beta_4\nabla^{k+1}m^-\rangle\Big]\\
&-\Big[\beta_3\bar{n}^+\langle\nabla^{k-1}\tilde{G}_3,\nabla^{k+1}u^+\rangle+\beta_2\bar{n}^-\langle\nabla^{k-1}\tilde{G}_4,\nabla^{k+1}u^-\rangle\Big]\\
&-\Big[\beta_3\bar{n}^+\langle\nabla^{k-1}r^+,\nabla^{k+1}u^+\rangle+\beta_2\bar{n}^-\langle\nabla^{k-1}r^-,\nabla^{k+1}u^-\rangle\Big]\\
:=&H_1+H_2+H_3+H_4.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
|H_1|\le&C\tau\Big(\|\nabla n^+\|_{L^3}+\|\nabla n^-\|_{L^3}\Big)\Big(\|\nabla^{k+1}u^+\|_{L^2}\|\nabla^k u^+\|_{L^6}+\|\nabla^{k+1}u^-\|_{L^2}\|\nabla^k u^-\|_{L^6}\Big)\\
&+C\tau\Big(\|\nabla^k n^+\|_{L^2}+\|\nabla^k n^-\|_{L^2}\Big)\Big(\|\nabla^2 u^+\|_{L^3}\|\nabla^k u^+\|_{L^6}+\|\nabla^2 u^-\|_{L^3}\|\nabla^k u^-\|_{L^6}\Big)\\
\le&C\tau\eta\Big(\|\nabla^{k+1}u^+\|_{L^2}^2+\|\nabla^{k+1}u^-\|_{L^2}^2\Big),\\
|H_2|\le&C\tau\Big(\|\nabla^k(n^+w^+)\|_{L^2}+\|\nabla^k(n^-w^-)\|_{L^2}\Big)\Big(\|\nabla^{k+1}m^+\|_{L^2}+\|\nabla^{k+1}m^-\|_{L^2}\Big)\\
\le&C\tau\Big(\|\nabla^k(n^+,n^-)\|_{L^2}\|(w^+,w^-)\|_{L^\infty}+\|\nabla^k(w^+,w^-)\|_{L^2}\|(n^+,n^-)\|_{L^\infty}\Big)\\
&\times\Big(\|\nabla^{k+1}m^+\|_{L^2}+\|\nabla^{k+1}m^+\|_{L^2}\Big)\\
\le&C\tau\Big(\|\nabla^k(n^+,n^-)\|_{L^2}^2\|\nabla(w^+,w^-)\|_{H^1}^2+\|\nabla^k(w^+,w^-)\|_{L^2}^2\|\nabla(n^+,n^-)\|_{H^1}^2\Big)\\
&+\frac{\zeta\epsilon}{4}\Big(\frac{\beta_3}{\beta_4}\|\nabla^{k+1}m^+\|_{L^2}^2+\frac{\beta_2}{\beta_1}\|\nabla^{k+1}m^-\|_{L^2}^2\Big).
\end{aligned}
$$

Note that

$$
H_3=-\beta_3\bar{n}^+\langle\nabla^{k-1}\tilde{G}_3,\nabla^{k+1}u^+\rangle-\beta_2\bar{n}^-\langle\nabla^{k-1}\tilde{G}_4,\nabla^{k+1}u^-\rangle:=H_3^1+H_3^2,
$$

then for $H_3^1$ we have

$$
|H_3^1|\le C\tau\Big(|\langle\nabla^{k-1}(g_1^+(n^+,n^-)\nabla n^++g_2^+(n^+,n^-)\nabla n^-),\nabla^{k+1}u^+\rangle|
$$

$$
\begin{aligned}
&+|\langle\nabla^{k-1}[h_1^+(n^+,n^-)(\nabla n^+\cdot\nabla)w^+ + h_2^+(n^+,n^-)(\nabla n^-\cdot\nabla)w^+],\nabla^{k+1}u^+\rangle|\\
&+|\langle\nabla^{k-1}[h_1^+(n^+,n^-)\nabla n^+\nabla^t w^+ + h_2^+(n^+,n^-)\nabla n^-\nabla^t w^+],\nabla^{k+1}u^+\rangle|\\
&+|\langle\nabla^{k-1}[h_1^+(n^+,n^-)\nabla n^+\mathrm{div}w^+ + h_2^+(n^+,n^-)\nabla n^-\mathrm{div}w^+],\nabla^{k+1}u^+\rangle|\\
&+|\langle\nabla^{k-1}[(w^+\cdot\nabla)w^+],\nabla^{k+1}u^+\rangle|\Big),
\end{aligned}
$$

that is

$$
\begin{aligned}
|H_3^1|\le& C\tau\Big(\|\nabla^{k-1}g_1^+(n^+,n^-)\|_{L^6}\|\nabla n^+\|_{L^3}+\|g_1^+(n^+,n^-)\|_{L^\infty}\|\nabla^k n^+\|_{L^2}\Big)\|\nabla^{k+1}u^+\|_{L^2}\\
&+C\tau\Big(\|\nabla^{k-1}g_2^+(n^+,n^-)\|_{L^6}\|\nabla n^-\|_{L^3}+\|g_2^+(n^+,n^-)\|_{L^\infty}\|\nabla^k n^-\|_{L^2}\Big)\|\nabla^{k+1}u^+\|_{L^2}\\
&+C\tau\Big(\|\nabla^{k-1}[h_1^+(n^+,n^-)\nabla w^+]\|_{L^2}\|\nabla n^+\|_{L^\infty}+\|h_1^+(n^+,n^-)\nabla w^+\|_{L^\infty}\|\nabla^k n^+\|_{L^2}\\
&+\|\nabla^{k-1}[h_2^+(n^+,n^-)\nabla w^+]\|_{L^2}\|\nabla n^-\|_{L^\infty}+\|h_2^+(n^+,n^-)\nabla w^+\|_{L^\infty}\|\nabla^k n^-\|_{L^2}\\
&+\|\nabla^{k-1}w^+\|_{L^6}\|\nabla w^+\|_{L^3}+\|w^+\|_{L^\infty}\|\nabla^k w^+\|_{L^2}\Big)\|\nabla^{k+1}u^+\|_{L^2}\\
\le& C\tau\Big(\|\nabla^k(n^+,n^-)\|_{L^2}^2\|\nabla(n^+,n^-)\|_{H^1}^2+(\|\nabla^k(n^+,n^-)\|_{L^2}^2\|\nabla w^+\|_{H^1}^2\\
&+\|\nabla(n^+,n^-)\|_{H^1}^2\|\nabla^k w^+\|_{L^2}^2)\|\nabla(n^+,n^-)\|_{H^1}^2+\|\nabla w\|_{H^2}^2\|\nabla^k(n^+,n^-)\|_{L^2}^2\\
&+\|\nabla^k w^+\|_{L^2}^2\|\nabla w^+\|_{H^1}^2\Big)+\frac{\beta_3\bar n^+\tilde\xi}{4}\|\nabla^{k+1}u^+\|_{L^2}^2\\
\le& C\tau\Big(\|\nabla^k(n^+,n^-)\|_{L^2}^2+\|\nabla^k w^+\|_{L^2}\Big)\Big(\|\nabla(n^+,n^-)\|_{H^1}^2+\|\nabla w^+\|_{H^2}^2\\
&+\|\nabla(n^+,n^-)\|_{H^1}^4+\|\nabla(n^+,n^-)\|_{H^1}^2\|\nabla w^+\|_{H^1}^2\Big)+\frac{\beta_3\bar n^+\tilde\xi}{4}\|\nabla^{k+1}u^+\|_{L^2}^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|H_3^2|\le& C\tau\Big(\|\nabla^k(n^+,n^-)\|_{L^2}^2+\|\nabla^k w^-\|_{L^2}\Big)\Big(\|\nabla(n^+,n^-)\|_{H^1}^2+\|\nabla w^-\|_{H^2}^2\\
&+\|\nabla(n^+,n^-)\|_{H^1}^4+\|\nabla(n^+,n^-)\|_{H^1}^2\|\nabla w^-\|_{H^1}^2\Big)+\frac{\beta_3\bar n^+\tilde\xi}{4}\|\nabla^{k+1}u^+\|_{L^2}^2,
\end{aligned}
$$

then

$$
\begin{aligned}
|H_3|\le& C\tau\Big(\|\nabla^k(n^+,n^-)\|_{L^2}^2+\|\nabla^k(w^+,w^-)\|_{L^2}\Big)\Big(\|\nabla(n^+,n^-)\|_{H^1}^2+\|\nabla(w^+,w^-)\|_{H^2}^2\\
&+\|\nabla(n^+,n^-)\|_{H^1}^4+\|\nabla(n^+,n^-)\|_{H^1}^2\|\nabla(w^+,w^-)\|_{H^1}^2\Big)+\frac{\beta_3\bar n^+\tilde\xi}{4}\|\nabla^{k+1}u^+\|_{L^2}^2\\
&+\frac{\beta_2\bar n^-\tilde\xi}{4}\|\nabla^{k+1}u^-\|_{L^2}^2.
\end{aligned}
$$

For the last term, we have

$$
|H_4|\le C\tau\Big(\|\nabla^{k-1}r^+\|_{L^2}^2+\|\nabla^{k-1}r^+\|_{L^2}^2\Big)+\frac{\tilde\xi}{4}\Big(\beta_3\bar n^+\|\nabla^{k+1}u^+\|_{L^2}^2+\beta_2\bar n^-\|\nabla^{k+1}u^-\|_{L^2}^2\Big).
$$

Let $k=4$, note that $\tau\in[0,1]$ for sufficiently small $\eta>0$, combining the above estimates, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^4 m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^4 m^-|^2+2\beta_3\bar n|\nabla^4 u^+|^2+2\beta_2\bar n^-|\nabla^4 u^-|^2\Big)\mathrm{d}x
$$

$$
\begin{aligned}
&+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\beta_2\beta_4|\nabla^4 m^- + \frac{\beta_3}{\beta_4}\nabla^4 m^+|^2 + \beta_1\beta_3|\nabla^4 m^+ + \frac{\beta_2}{\beta_1}\nabla^4 m^-|^2\Big)dx + \epsilon\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}\\
&\times|\nabla^5 m^+|^2 + \zeta\frac{\beta_2}{\beta_1}|\nabla^5 m^-|^2\Big)\mathrm{d}x + \tilde{\xi}\int_{\Omega_L}\Big(\beta_3\bar{n}^+\mu^+|\nabla^5 u^+|^2 + \beta_2\bar{n}^-\mu^-|\nabla^5 u^-|^2\Big)\mathrm{d}x\\
\le{}& C\tau\Big(\|\nabla^4(n^+,n^-)\|_{L^2}^2 + \|\nabla^4(w^+,w^-)\|_{L^2}\Big)\Big(\|\nabla(n^+,n^-)\|_{H^1}^2 + \|\nabla(w^+,w^-)\|_{H^2}^2\\
&+\|\nabla(n^+,n^-)\|_{H^1}^4 + \|\nabla(n^+,n^-)\|_{H^1}^2\|\nabla(w^+,w^-)\|_{H^1}^2\Big) + C\tau\|\nabla^3(r^+,r^-)\|_{L^2}^2. \qquad (3.9)
\end{aligned}
$$

Integrating (3.9) from 0 to $T$, we have

$$
\begin{aligned}
&\int_0^T \epsilon\Big(\zeta\frac{\beta_3}{\beta_4}\|\nabla^5 m^+\|_{L^2}^2 + \zeta\frac{\beta_2}{\beta_1}\|\nabla^5 m^-\|_{L^2}^2\Big)\mathrm{d}t + \int_0^T\tilde{\xi}\Big(\beta_3\bar{n}^+\mu^+\|\nabla^5 u^+\|_{L^2}^2\\
&\quad+\beta_2\bar{n}^-\mu^-\|\nabla^5 u^-\|_{L^2}^2\Big)\mathrm{d}t\\
\le{}& C\tau\sup_{0\le t\le T}\Big(\|\nabla(n^+,n^-)\|_{H^1}^2 + \|\nabla(w^+,w^-)\|_{H^2}^2 + \|\nabla(n^+,n^-)\|_{H^1}^4\\
&\quad+\|\nabla(n^+,n^-)\|_{H^1}^2\|\nabla(w^+,w^-)\|_{H^1}^2\Big)\int_0^T\Big(\|\nabla(n^+,n^-)\|_{H^3}^2 + \|\nabla(w^+,w^-)\|_{H^3}^2\Big)\mathrm{d}t\\
&\quad+C\tau\int_0^T\|\nabla^3(r^+,r^-)\|_{L^2}^2\mathrm{d}t \le K^*. \qquad (3.10)
\end{aligned}
$$

Then, there exists a time $t^*\in(0,T)$ such that

$$
\begin{aligned}
&\epsilon T\Big(\zeta\frac{\beta_3}{\beta_4}\|\nabla^5 m^+(\cdot,t^*)\|_{L^2}^2 + \zeta\frac{\beta_2}{\beta_1}\|\nabla^5 m^-(\cdot,t^*)\|_{L^2}^2\Big) + \tilde{\xi}T\Big(\beta_3\bar{n}^+\mu^+\|\nabla^5 u^+(\cdot,t^*)\|_{L^2}^2\\
&\quad+\beta_2\bar{n}^-\mu^-\|\nabla^5 u^-(\cdot,t^*)\|_{L^2}^2\Big) \le K^*.
\end{aligned}
$$

Using Poincaré's inequality, we have

$$
\|\nabla^4 m^+(\cdot,t^*)\|_{L^2}^2 + \|\nabla^4 m^-(\cdot,t^*)\|_{L^2}^2 + \|\nabla^4 u^+(\cdot,t^*)\|_{L^2}^2 + \|\nabla^4 u^-(\cdot,t^*)\|_{L^2}^2 \le CK^*.
$$

Then, integrating (3.9) from $t^*$ to $t$ for any $t\in(t^*,T]$, we have

$$
\|\nabla^4 m^+(\cdot,t)\|_{L^2}^2 + \|\nabla^4 m^-(\cdot,t)\|_{L^2}^2 + \|\nabla^4 u^+(\cdot,t)\|_{L^2}^2 + \|\nabla^4 u^-(\cdot,t)\|_{L^2}^2 \le CK^*.
$$

Moreover, by the time periodic assumption, we have

$$
\|\nabla^4 m^+(\cdot,0)\|_{L^2}^2 + \|\nabla^4 m^-(\cdot,0)\|_{L^2}^2 + \|\nabla^4 u^+(\cdot,0)\|_{L^2}^2 + \|\nabla^4 u^-(\cdot,0)\|_{L^2}^2 \le CK^*.
$$

Repeating the above process for $t\in(0,t^*)$, we obtain

$$
\sup_{0\le t\le T}\Big(\|m^+(\cdot,t)\|_{H^4}^2 + \|m^-(\cdot,t)\|_{H^4}^2 + \|u^+(\cdot,t)\|_{H^4}^2 + \|u^-(\cdot,t)\|_{H^4}^2\Big) \le CK^*. \qquad (3.11)
$$

Similarly, applying $\nabla^3$ to $(3.1)_1,(3.1)_2,(3.1)_3,(3.1)_4$, multiplying the results by $\partial_t(\nabla^3 m^+)$, $\partial_t(\nabla^3 m^-)$, $\partial_t(\nabla^3 u^+)$, $\partial_t(\nabla^3 u^-)$ respectively, summing up, then integrating the result over $\Omega_L$ by parts and combining with the periodic boundary conditions and the Equations $(3.1)_1,(3.1)_2$, finally integrating the result over $[0,T]$ and combining with (3.10), we obtain

$$
\int_0^T\Big(\|\partial_t(\nabla^3 m^+)\|_{L^2}^2 + \|\partial_t(\nabla^3 m^-)\|_{L^2}^2 + \|\partial_t(\nabla^3 u^+)\|_{L^2}^2 + \|\partial_t(\nabla^3 u^-)\|_{L^2}^2\Big)\mathrm{d}t
$$

$$
\begin{aligned}
\le & C\int_0^T \Big(\|\nabla^4(u^+,u^-)\|_{L^2}^2+\epsilon\|\nabla^4(m^+,m^-)\|_{L^2}^2+\|\nabla^4(m^+,m^-)\|_{L^2}^2\\
&+\tau\|\nabla^3(n^+,n^-)\|_{L^6}^2\|\nabla^2u^+\|_{L^3}^2+\|\nabla^5(u^+,u^-)\|_{L^2}^2+\|\nabla^3\tilde G_1\|_{L^2}^2+\|\nabla^3\tilde G_2\|_{L^2}^2\\
&+\|\nabla^3\tilde G_3\|_{L^2}^2+\|\nabla^3\tilde G_4\|_{L^2}^2+\|\nabla^3 r^+\|_{L^2}^2+\|\nabla^3 r^-\|_{L^2}^2\Big)\mathrm{d}t\\
\le & C\tau\sup_{0\le t\le T}\Big(\|\nabla(n^+,n^-)\|_{H^1}^2+\|\nabla(w^+,w^-)\|_{H^2}^2+\|\nabla(n^+,n^-)\|_{H^1}^2\|\nabla(w^+,w^-)\|_{H^1}^2\\
&+\|\nabla(n^+,n^-)\|_{H^1}^4\Big)\int_0^T\Big(\|\nabla(n^+,n^-)\|_{H^3}^2+\|\nabla(w^+,w^-)\|_{H^3}^2\Big)\mathrm{d}t\\
&+C\tau\int_0^T\|\nabla^3(r^+,r^-)\|_{L^2}^2\mathrm{d}t. \qquad (3.12)
\end{aligned}
$$

From (3.10), (3.11) and (3.12), using the Lemma 2.4, we get the compactness of operator $\mathcal{S}$. The proof is complete. ∎

The next lemma shows the continuity of the operator $\mathcal{S}$.

LEMMA 3.3. *If $\eta$ is sufficiently small, then the operator $\mathcal{S}$ is continuous.*

*Proof.* Assume that $(n_i^+,n_i^-,w_i^+,w_i^-)\in\mathcal{X}_\eta^L,\tau_i\in[0,1],(n^+,n^-,w^+,w^-)\in\mathcal{X}_\eta^L,\tau\in[0,1]$, and

$$
\lim_{i\to\infty}\|(n_i^+-n^+,n_i^--n^-,w_i^+-w^+,w_i^--w^-)\|_{\mathcal{X}^L}=0,\ \text{as}\ \lim_{i\to\infty}\tau_i=\tau.
$$

Let

$$
\begin{aligned}
(m_i^+,m_i^-,u_i^+,u_i^-)&=\mathcal{S}\left((n_i^+,n_i^-,w_i^+,w_i^-),\tau_i\right),\\
(m^+,m^-,u^+,u^-)&=\mathcal{S}\left((n^+,n^-,w^+,w^-),\tau\right).
\end{aligned}
$$

Then $(\tilde m_i^+,\ \tilde m_i^-,\ \tilde u_i^+,\ \tilde u_i^-):=(m_i^+-m^+,\ m_i^--m^-,\ u_i^+-u^+,\ u_i^--u^-)$ is a periodic solution of the following system

$$
\begin{cases}
\partial_t\tilde m^++\bar n^+\mathrm{div}\tilde u^+-\epsilon\Delta\tilde m^+=I_1,\\
\partial_t\tilde m^-+\bar n^-\mathrm{div}\tilde u^--\epsilon\Delta\tilde m^-=I_2,\\
\partial_t\tilde u^++\beta_1\nabla\tilde m^++\beta_2\nabla\tilde m^--(\tilde\xi+\tau_i\xi^+(n_i^+,n_i^-))(\mu^+\Delta\tilde u^++(\mu^++\lambda^+)\nabla\mathrm{div}\tilde u^+)\\
=I_3+(\tau_i-\tau)r^+,\\
\partial_t\tilde u^-+\beta_3\nabla\tilde m^++\beta_4\nabla\tilde m^--(\tilde\xi+\tau_i\xi^-(n_i^+,n_i^-))(\mu^-\Delta\tilde u^-+(\mu^-+\lambda^-)\nabla\mathrm{div}\tilde u^-)\\
=I_4+(\tau_i-\tau)r^-,
\end{cases}
\qquad (3.13)
$$

where

$$
\begin{aligned}
I_1=&(\tau-\tau_i)\mathrm{div}(n^+w^+)-\tau_i\mathrm{div}((n_i^+-n^+)w^++n_i^+(w_i^+-w^+)),\\
I_2=&(\tau-\tau_i)\mathrm{div}(n^-w^-)-\tau_i\mathrm{div}((n_i^--n^-)w^-+n_i^-(w_i^--w^-)),\\
I_3=&(\tau_i-\tau)\xi^+(n^+,n^-)(\mu^+\Delta u^++(\mu^++\lambda^+)\nabla\mathrm{div}u^+)\\
&+\tau_i(\xi^+(n_i^+,n_i^-)-\xi^+(n^+,n^-))(\mu^+\Delta u^++(\mu^++\lambda^+)\nabla\mathrm{div}u^+)\\
&-(\tau_i-\tau)g_1^+(n_i^+,n_i^-)\nabla n_i^+-\tau(g_1^+(n_i^+,n_i^-)-g_1^+(n^+,n^-))\nabla n_i^+\\
&-\tau g_1^+(n^+,n^-)\nabla(n_i^+-n^+)-(\tau_i-\tau)g_2^+(n_i^+,n_i^-)\nabla n_i^-\\
&-\tau(g_2^+(n_i^+,n_i^-)-g_2^+(n^+,n^-))\nabla n_i^--\tau g_2^+(n^+,n^-)\nabla(n_i^--n^-)
\end{aligned}
$$

$$
\begin{aligned}
&-(\tau_i-\tau)(w_i^+\cdot\nabla)w_i^+-\tau(((w_i^+-w^+)\cdot\nabla)w_i^++(w^+\cdot\nabla)(w_i^+-w^+))\\
&+\mu^+(\tau_i-\tau)h_1^+(n_i^+,n_i^-)(\nabla n_i^+\cdot\nabla)w_i^+\\
&+\mu^+\tau(h_1^+(n_i^+,n_i^-)-h_1^+(n^+,n^-))(\nabla n_i^+\cdot\nabla)w_i^+\\
&+\mu^+\tau h_1^+(n^+,n^-)((\nabla(n_i^+-n^+)\cdot\nabla)w_i^++(\nabla n^+\cdot\nabla)(w_i^+-w^+))\\
&+\mu^+(\tau_i-\tau)h_2^+(n_i^+,n_i^-)(\nabla n_i^-\cdot\nabla)w_i^+\\
&+\mu^+\tau(h_2^+(n_i^+,n_i^-)-h_2^+(n^+,n^-))(\nabla n_i^-\cdot\nabla)w_i^+\\
&+\mu^+\tau h_2^+(n^+,n^-)((\nabla(n_i^--n^-)\cdot\nabla)w_i^++(\nabla n^-\cdot\nabla)(w_i^+-w^+))\\
&+\mu^+(\tau_i-\tau)h_1^+(n_i^+,n_i^-)\nabla n_i^+\nabla^t w_i^+\\
&+\mu^+\tau(h_1^+(n_i^+,n_i^-)-h_1^+(n^+,n^-))\nabla n_i^+\nabla^t w_i^+\\
&+\mu^+\tau h_1^+(n^+,n^-)(\nabla(n_i^+-n^+)\nabla^t w_i^++\nabla n^+\nabla^t(w_i^+-w^+))\\
&+\mu^+(\tau_i-\tau)h_2^+(n_i^+,n_i^-)\nabla n_i^-\nabla^t w_i^+\\
&+\mu^+\tau(h_2^+(n_i^+,n_i^-)-h_2^+(n^+,n^-))\nabla n_i^-\nabla^t w_i^+\\
&+\mu^+\tau h_2^+(n^+,n^-)(\nabla(n_i^--n^-)\nabla^t w_i^++\nabla n^-\nabla^t(w_i^+-w^+))\\
&+\mu^+(\tau_i-\tau)h_1^+(n_i^+,n_i^-)\nabla n_i^+\mathrm{div}w_i^+\\
&+\mu^+\tau(h_1^+(n_i^+,n_i^-)-h_1^+(n^+,n^-))\nabla n_i^+\mathrm{div}w_i^+\\
&+\mu^+\tau h_1^+(n^+,n^-)(\nabla(n_i^+-n^+)\mathrm{div}w_i^++\nabla n^+\mathrm{div}(w_i^+-w^+))\\
&+\mu^+(\tau_i-\tau)h_2^+(n_i^+,n_i^-)\nabla n_i^-\mathrm{div}w_i^+\\
&+\mu^+\tau(h_2^+(n_i^+,n_i^-)-h_2^+(n^+,n^-))\nabla n_i^-\mathrm{div}w_i^+\\
&+\mu^+\tau h_2^+(n^+,n^-)(\nabla(n_i^--n^-)\mathrm{div}w_i^++\nabla n^-\mathrm{div}(w_i^+-w^+)),
\end{aligned}
$$

and

$$
\begin{aligned}
I_4=&(\tau_i-\tau)\xi^-(n^+,n^-)(\mu^-\Delta u^-+(\mu^-+\lambda^-)\nabla\mathrm{div}u^-)\\
&+\tau_i(\xi^-(n_i^+,n_i^-)-\xi^-(n^+,n^-))(\mu^-\Delta u^-+(\mu^-+\lambda^-)\nabla\mathrm{div}u^-)\\
&-(\tau_i-\tau)g_1^+(n_i^+,n_i^-)\nabla n_i^+-\tau(g_1^-(n_i^+,n_i^-)-g_1^-(n^+,n^-))\nabla n_i^+\\
&-\tau g_1^-(n^+,n^-)\nabla(n_i^+-n^+)-(\tau_i-\tau)g_2^-(n_i^+,n_i^-)\nabla n_i^-\\
&-\tau(g_2^-(n_i^+,n_i^-)-g_2^-(n^+,n^-))\nabla n_i^--\tau g_2^-(n^+,n^-)\nabla(n_i^--n^-)\\
&-(\tau_i-\tau)(w_i^-\cdot\nabla)w_i^--\tau(((w_i^--w^-)\cdot\nabla)w_i^-+(w^-\cdot\nabla)(w_i^--w^-))\\
&+\mu^-(\tau_i-\tau)h_1^-(n_i^+,n_i^-)(\nabla n_i^+\cdot\nabla)w_i^-\\
&+\mu^-\tau(h_1^-(n_i^+,n_i^-)-h_1^-(n^+,n^-))(\nabla n_i^+\cdot\nabla)w_i^-\\
&+\mu^-\tau h_1^-(n^+,n^-)((\nabla(n_i^+-n^+)\cdot\nabla)w_i^-+(\nabla n^+\cdot\nabla)(w_i^--w^-))\\
&+\mu^-(\tau_i-\tau)h_2^-(n_i^+,n_i^-)(\nabla n_i^-\cdot\nabla)w_i^-\\
&+\mu^-\tau(h_2^-(n_i^+,n_i^-)-h_2^-(n^+,n^-))(\nabla n_i^-\cdot\nabla)w_i^-\\
&+\mu^-\tau h_2^-(n^+,n^-)((\nabla(n_i^--n^-)\cdot\nabla)w_i^-+(\nabla n^-\cdot\nabla)(w_i^--w^-))\\
&+\mu^-(\tau_i-\tau)h_1^-(n_i^+,n_i^-)\nabla n_i^+\nabla^t w_i^-\\
&+\mu^-\tau(h_1^-(n_i^+,n_i^-)-h_1^-(n^+,n^-))\nabla n_i^+\nabla^t w_i^-\\
&+\mu^-\tau h_1^-(n^+,n^-)(\nabla(n_i^+-n^+)\nabla^t w_i^-+\nabla n^+\nabla^t(w_i^--w^-))\\
&+\mu^-(\tau_i-\tau)h_2^-(n_i^+,n_i^-)\nabla n_i^-\nabla^t w_i^-
\end{aligned}
$$

$$
\begin{aligned}
&+\mu^-\tau(h_2^-(n_i^+,n_i^-)-h_2^-(n^+,n^-))\nabla n_i^-\nabla^t w_i^-\\
&+\mu^-\tau h_2^-(n^+,n^-)(\nabla(n_i^- - n^-)\nabla^t w_i^- + \nabla n^-\nabla^t(w_i^- - w^-))\\
&+\mu^-(\tau_i-\tau)h_1^-(n_i^+,n_i^-)\nabla n_i^+ \mathrm{div} w_i^-\\
&+\mu^-\tau(h_1^-(n_i^+,n_i^-)-h_1^-(n^+,n^-))\nabla n_i^+ \mathrm{div} w_i^-\\
&+\mu^-\tau h_1^-(n^+,n^-)(\nabla(n_i^+ - n^+)\mathrm{div} w_i^- + \nabla n^+\mathrm{div}(w_i^- - w^-))\\
&+\mu^-(\tau_i-\tau)h_2^-(n_i^+,n_i^-)\nabla n_i^- \mathrm{div} w_i^-\\
&+\mu^-\tau(h_2^-(n_i^+,n_i^-)-h_2^-(n^+,n^-))\nabla n_i^- \mathrm{div} w_i^-\\
&+\mu^-\tau h_2^-(n^+,n^-)(\nabla(n_i^- - n^-)\mathrm{div} w_i^- + \nabla n^-\mathrm{div}(w_i^- - w^-)).
\end{aligned}
$$

Similar to the proof of Lemma 3.2, we have

$$\lim_{n\to\infty}\|(m_i^+ - m^+, m_i^- - m^-, u_i^+ - u^+, u_i^- - u^-)\|_{\mathcal{X}^L} = 0.$$

Therefore, we get the continuity of the operator $\mathcal{S}$. □

**3.2. Energy estimates.** In this subsection, we deduce some energy estimates, and interpret why we can not get the existence of time periodic solutions in $\mathbb{R}^3$, see Remark 3.2.

LEMMA 3.4. *Assume that $\tau\in(0,1], |m^\pm|\le\frac{\bar n^\pm}{2}$, and $(m^+,m^-,u^+,u^-)\in\mathcal{X}^L$ is the solution of the following problem*

$$
\begin{cases}
\partial_t m^+ + \bar n^+ \mathrm{div} u^+ - \epsilon\Delta m^+ = \tilde G_1(m^+,u^+,\tau),\\
\partial_t m^- + \bar n^- \mathrm{div} u^- - \epsilon\Delta m^- = \tilde G_2(m^-,u^-,\tau),\\
\partial_t u^+ + \beta_1\nabla m^+ + \beta_2\nabla m^- - (\tilde\xi + \tau\xi^+(m^+,m^-))(\mu^+\Delta u^+ + (\mu^+ + \lambda^+)\nabla \mathrm{div} u^+)\\
= \tilde G_3(m^+,m^-,u^+,\tau) + \tau r^+,\\
\partial_t u^- + \beta_3\nabla m^+ + \beta_4\nabla m^- - (\tilde\xi + \tau\xi^-(m^+,m^-))(\mu^-\Delta u^- + (\mu^- + \lambda^-)\nabla \mathrm{div} u^-)\\
= \tilde G_4(m^+,m^-,u^-,\tau) + \tau r^-,\\
\int_{\Omega_L} m^\pm dx = 0,
\end{cases}
\tag{3.14}
$$

*then we have*

$$
\begin{aligned}
&\frac14\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^k m^+|^2 + \zeta\frac{\beta_2}{\beta_1}|\nabla^k m^-|^2\Big)\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\nabla^k m^+\nabla^{k-1}u^+ + \nabla^k m^-\nabla^{k-1}u^-\Big)\mathrm{d}x\\
&\le C\tau\Big(\|\nabla(m^+,m^-)\|_{H^3}^2 + \|\nabla(u^+,u^-)\|_{H^3}^2\Big)\Big(\|\nabla(m^+,m^-)\|_{H^1} + \|\nabla(m^+,m^-)\|_{H^2}^2\\
&\quad + \|\nabla(u^+,u^-)\|_{H^2}^2 + \|\nabla(u^+,u^-)\|_{H^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2 + \|\nabla(m^+,m^-)\|_{H^2}^4\Big)\\
&\quad + C\tau\|\nabla^{k-1}(r^+,r^-)\|_{L^2}^2 + \eta_1\epsilon\Big(\|\nabla^{k+1}m^+\|_{L^2}^2 + \|\nabla^{k+1}m^-\|_{L^2}^2\Big)\\
&\quad + C_1\Big(\|\nabla^k u^+\|_{L^2}^2 + \|\nabla^k u^-\|_{L^2}^2\Big) + \eta_2\Big(\|\nabla^{k+1}u^+\|_{L^2}^2 + \|\nabla^{k+1}u^-\|_{L^2}^2\Big),\\
&\quad \textit{for } k=1,2,3,4,
\end{aligned}
\tag{3.15}
$$

*where $C, C_1, \eta_1, \eta_2$ are constants independent of $L$ and $\epsilon$, and $\eta_1,\eta_2$ are sufficiently small.*

*Proof.* Assume that $(m^+, m^-, u^+, u^-)$ is the solution of system (3.14). Similar to Lemma 3.1, applying $\nabla^k, (k\ge1)$ to $(3.14)_1, (3.14)_2$, and $\nabla^{k-1}$ to $(3.14)_3, (3.14)_4$

multiplying the results by $\beta_3\nabla^{k-1}u^+$, $\beta_2\nabla^{k-1}u^-$, $\beta_3\nabla^k m^+$, $\beta_2\nabla^k m^-$ respectively, summing up, then integrating the result over $\Omega_L$ by parts and combining with the periodic boundary conditions and the Equations $(3.14)_1,(3.14)_2$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^k m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k m^-|^2+\beta_2\beta_4|\nabla^k m^-+\frac{\beta_3}{\beta_4}\nabla^k m^+|^2\\
&\quad+\beta_1\beta_3|\nabla^k m^+ +\frac{\beta_2}{\beta_1}\nabla^k m^-|^2\Big)\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\beta_3\nabla^k m^+\nabla^{k-1}u^+ +\beta_2\nabla^k m^-\nabla^{k-1}u^-\Big)\mathrm{d}x\\
&=\int_{\Omega_L}\Big(\beta_3\bar n^+|\nabla^{k-1}\mathrm{div}u^+|^2+\beta_2\bar n^-|\nabla^{k-1}\mathrm{div}u^-|^2\Big)\mathrm{d}x\\
&\quad-\epsilon\int_{\Omega_L}\Big(\beta_3\nabla^{k+1}m^+\nabla^k u^+ +\beta_2\nabla^{k+1}m^-\nabla^k u^-\Big)\mathrm{d}x\\
&\quad+\int_{\Omega_L}\Big[\beta_3\nabla^{k-1}\big((\tilde\xi+\tau\xi^+(m^+,m^-))(\mu^+\Delta u^+ +(\mu^+ +\lambda^+)\nabla\mathrm{div}u^+)\big)\nabla^k m^+\\
&\quad+\beta_2\nabla^{k-1}\big((\tilde\xi+\tau\xi^-(m^+,m^-))(\mu^-\Delta u^- +(\mu^- +\lambda^-)\nabla\mathrm{div}u^-)\big)\nabla^k m^-\Big]\mathrm{d}x\\
&\quad+\Big[\beta_3\tau\langle\nabla^k(m^+u^+),\nabla^k u^+\rangle+\beta_2\tau\langle\nabla^k(m^-u^-),\nabla^k u^-\rangle\Big]\\
&\quad+\Big[\beta_3\langle\nabla^{k-1}\tilde G_3,\nabla^k m^+\rangle+\beta_2\langle\nabla^{k-1}\tilde G_4,\nabla^k m^-\rangle\Big]\\
&\quad+\Big[\beta_3\langle\nabla^{k-1}r^+,\nabla^k m^+\rangle+\beta_2\bar n^-\langle\nabla^{k-1}r^-,\nabla^k m^-\rangle\Big]:=J_1+J_2+J_3+J_4+J_5+J_6.
\end{aligned}$$

Next, we estimate the terms $J_i(i=1,2,\cdots,6)$. Since we want the estimates independent of the periodic domain $\Omega_L$, the following estimates will be more complicated compared to Lemma 3.1; precisely, we have

$$|J_1|\le\frac{1}{2}C_1\|\nabla^k u^+\|_{L^2}^2+\frac{1}{2}C_1\|\nabla^k u^-\|_{L^2}^2.$$

Without loss of generality, assume $\epsilon\le 1$, then

$$|J_2|\le\frac{1}{2}C_1\|\nabla^k u^+\|_{L^2}^2+\frac{1}{2}C_1\|\nabla^k u^-\|_{L^2}^2+\eta_1\epsilon\Big(\|\nabla^{k+1}m^+\|_{L^2}^2+\|\nabla^{k+1}m^-\|_{L^2}^2\Big).$$

$$\begin{aligned}
|J_3|\le& C\tau\Big(\|(m^+,m^-)\|_{L^\infty}\|\nabla^{k+1}(u^+,u^-)\|_{L^2}\|\nabla^k(m^+,m^-)\|_{L^2}\\
&+\|\nabla(m^+,m^-)\|_{L^3}\|\nabla^k(u^+,u^-)\|_{L^6}\|\nabla^k(m^+,m^-)\|_{L^2}\\
&+\|\nabla^2(m^+,m^-)\|_{L^3}\|\nabla^{k-1}(u^+,u^-)\|_{L^6}\|\nabla^k(m^+,m^-)\|_{L^2}\\
&+\sum_{3\le l\le k-1}\|\nabla^l(m^+,m^-)\|_{L^3}\|\nabla^{k+1-l}(u^+,u^-)\|_{L^6}\|\nabla^k(m^+,m^-)\|_{L^2}\Big)\\
\le&\frac{\eta_2}{2}\Big(\|\nabla^{k+1}u^+\|_{L^2}^2+\|\nabla^{k+1}u^-\|_{L^2}^2\Big)+\frac{\zeta}{16}\Big(\frac{\beta_3}{\beta_4}\|\nabla^k m^+\|_{L^2}^2+\frac{\beta_2}{\beta_1}\|\nabla^k m^-\|_{L^2}^2\Big)\\
&+C\tau\Big(\|\nabla^k(m^+,m^-)\|_{L^2}^2+\|\nabla^k(u^+,u^-)\|_{L^2}^2\Big)\|\nabla(m^+,m^-)\|_{H^2}^2\\
&+\|\nabla(u^+,u^-)\|_{H^{k-2}}^2\|\nabla(m^+,m^-)\|_{H^{k-1}}^2,\ \text{for } k\ge 4,
\end{aligned}$$

where the Leibniz's formula is used. And

$$|J_3|\le C\tau\|(m^+,m^-)\|_{L^\infty}\|\nabla^2(u^+,u^-)\|_{L^2}\|\nabla(m^+,m^-)\|_{L^2}$$

$$
\begin{aligned}
&\leq \frac{\eta_2}{2}\Big(\|\nabla^2 u^+\|_{L^2}^2+\|\nabla^2 u^-\|_{L^2}^2\Big)+\frac{\zeta}{16}\Big(\frac{\beta_3}{\beta_4}\|\nabla m^+\|_{L^2}^2+\frac{\beta_2}{\beta_1}\|\nabla m^-\|_{L^2}^2\Big)\\
&\quad +C\tau\Big(\|\nabla(m^+,m^-)\|_{L^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2+\|\nabla^2(u^+,u^-)\|_{L^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2\Big),\\
&\quad \text{for } k=1,\\
|J_3|&\leq C\tau\Big(\|(m^+,m^-)\|_{L^\infty}\|\nabla^3(u^+,u^-)\|_{L^2}\|\nabla^2(m^+,m^-)\|_{L^2}\\
&\quad +\|\nabla(m^+,m^-)\|_{L^3}\|\nabla^2(u^+,u^-)\|_{L^6}\|\nabla^2(m^+,m^-)\|_{L^2}\Big)\\
&\leq \frac{\eta_2}{2}\Big(\|\nabla^3 u^+\|_{L^2}^2+\|\nabla^3 u^-\|_{L^2}^2\Big)+\frac{\zeta}{16}\Big(\frac{\beta_3}{\beta_4}\|\nabla^2 m^+\|_{L^2}^2+\frac{\beta_2}{\beta_1}\|\nabla^2 m^-\|_{L^2}^2\Big)\\
&\quad +C\tau\|\nabla^2(m^+,m^-)\|_{L^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2+C\tau\|\nabla^3(u^+,u^-)\|_{L^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2,\\
&\quad \text{for } k=2,\\
|J_3|&\leq C\tau\Big(\|(m^+,m^-)\|_{L^\infty}\|\nabla^4(u^+,u^-)\|_{L^2}\|\nabla^3(m^+,m^-)\|_{L^2}\\
&\quad +\|\nabla(m^+,m^-)\|_{L^3}\|\nabla^3(u^+,u^-)\|_{L^6}\|\nabla^3(m^+,m^-)\|_{L^2}\\
&\quad +\|\nabla^2(m^+,m^-)\|_{L^3}\|\nabla^2(u^+,u^-)\|_{L^6}\|\nabla^3(m^+,m^-)\|_{L^2}\Big)\\
&\leq \frac{\eta_2}{2}\Big(\|\nabla^4 u^+\|_{L^2}^2+\|\nabla^4 u^-\|_{L^2}^2\Big)+\frac{\zeta}{16}\Big(\frac{\beta_3}{\beta_4}\|\nabla^3 m^+\|_{L^2}^2+\frac{\beta_2}{\beta_1}\|\nabla^3 m^-\|_{L^2}^2\Big)\\
&\quad +C\tau\|\nabla^3(m^+,m^-)\|_{L^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2+C\tau\|\nabla^3(u^+,u^-)\|_{L^2}^2\|\nabla(m^+,m^-)\|_{H^2}^2,\\
&\quad \text{for } k=3,
\end{aligned}
$$

Now, we have the estimate

$$
\begin{aligned}
|J_3|&\leq \frac{\eta_2}{2}\Big(\|\nabla^{k+1} u^+\|_{L^2}^2+\|\nabla^{k+1} u^-\|_{L^2}^2\Big)+\frac{\zeta}{16}\Big(\frac{\beta_3}{\beta_4}\|\nabla^k m^+\|_{L^2}^2+\frac{\beta_2}{\beta_1}\|\nabla^k m^-\|_{L^2}^2\Big)\\
&\quad +C\tau\Big(\|\nabla(m^+,m^-)\|_{H^{k-1}}^2+\|\nabla(u^+,u^-)\|_{H^{k-1}}^2\Big)\Big(|\nabla(m^+,m^-)\|_{H^{k-2}}^2\\
&\quad +\|\nabla(u^+,u^-)\|_{H^{k-2}}^2\Big),\ \text{for } k\geq 4,\\
|J_3|&\leq \frac{\eta_2}{2}\Big(\|\nabla^{k+1} u^+\|_{L^2}^2+\|\nabla^{k+1} u^-\|_{L^2}^2\Big)+\frac{\zeta}{16}\Big(\frac{\beta_3}{\beta_4}\|\nabla^k m^+\|_{L^2}^2+\frac{\beta_2}{\beta_1}\|\nabla^k m^-\|_{L^2}^2\Big)\\
&\quad +C\tau\Big(\|\nabla(m^+,m^-)\|_{H^3}^2+\|\nabla(u^+,u^-)\|_{H^3}^2\Big)\Big(\|\nabla(m^+,m^-)\|_{H^2}^2+\|\nabla(u^+,u^-)\|_{H^2}^2\Big)\\
&\quad \text{for } k=1,2,3,4.
\end{aligned}
$$

Similarly, for the other terms, we have

$$
\begin{aligned}
|J_4|&\leq C\tau\|(m^+,m^-)\|_{L^\infty}\Big(\|\nabla^k u^+\|_{L^2}\|\nabla^k u^+\|_{L^2}+\|\nabla^k u^-\|_{L^2}\|\nabla^k u^-\|_{L^2}\Big)\\
&\quad +C\tau\sum_{1\leq l\leq k-1}\|\nabla^l(m^+,m^-)\|_{L^2}\|\nabla^{k-l}(u^+,u^-)\|_{L^3}\|\nabla^k(u^+,u^-)\|_{L^6}\\
&\quad +C\tau\|\nabla^k(m^+,m^-)\|_{L^2}\|(u^+,u^-)\|_{L^\infty}\|\nabla^k(u^+,u^-)\|_{L^2}\\
&\leq \frac{\eta_2}{2}\Big(\|\nabla^{k+1} u^+\|_{L^2}^2+\|\nabla^{k+1} u^-\|_{L^2}^2\Big)+\frac{\zeta}{16}\Big(\frac{\beta_3}{\beta_4}\|\nabla^k m^+\|_{L^2}^2+\frac{\beta_2}{\beta_1}\|\nabla^k m^-\|_{L^2}^2\Big)\\
&\quad +C\tau\Big(\|\nabla(m^+,m^-)\|_{H^1}\|\nabla^k(u^+,u^-)\|_{L^2}^2+\|\nabla(m^+,m^-)\|_{H^{k-2}}^2\|\nabla^k(u^+,u^-)\|_{H^{k-1}}^2\\
&\quad +\|\nabla(u^+,u^-)\|_{H^1}^2\|\nabla^k(u^+,u^-)\|_{L^2}^2\Big),\ \text{for } k\geq 4,
\end{aligned}
$$

$$
\begin{aligned}
|J_4| \le & \frac{\eta_2}{2}\Big(\|\nabla^{k+1}u^+\|_{L^2}^2+\|\nabla^{k+1}u^-\|_{L^2}^2\Big)+\frac{\zeta}{16}\Big(\frac{\beta_3}{\beta_4}\|\nabla^k m^+\|_{L^2}^2+\frac{\beta_2}{\beta_1}\|\nabla^k m^-\|_{L^2}^2\Big)\\
&+C\tau\Big(\|\nabla(m^+,m^-)\|_{H^1}\|\nabla(u^+,u^-)\|_{H^3}^2+\|\nabla(m^+,m^-)\|_{H^2}^2\|\nabla^k(u^+,u^-)\|_{H^3}^2\\
&+\|\nabla(u^+,u^-)\|_{H^1}^2\|\nabla(u^+,u^-)\|_{H^3}^2\Big), \text{ for } k=1,2,3,4.
\end{aligned}
$$

For $J_5$,

$$
J_5=\Big[\beta_3\langle\nabla^{k-1}\tilde{G}_3,\nabla^k m^+\rangle+\beta_2\langle\nabla^{k-1}\tilde{G}_4,\nabla^k m^-\rangle\Big]:=J_5^1+J_5^2.
$$

$$
\begin{aligned}
|J_5^1| \le C\tau\Big(&|\langle\nabla^{k-1}(g_1^+(m^+,m^-)\nabla m^+),\nabla^k m^+\rangle|+|\langle\nabla^{k-1}(g_2^+(m^+,m^-)\nabla m^-),\nabla^k m^+\rangle|\\
&+|\langle\nabla^{k-1}[(u^+\cdot\nabla)u^+],\nabla^k m^+\rangle|+|\langle\nabla^{k-1}[h_1^+(m^+,m^-)(\nabla m^+\cdot\nabla)u^+],\nabla^k m^+\rangle|\\
&+|\langle\nabla^{k-1}[h_2^+(m^+,m^-)(\nabla m^-\cdot\nabla)u^+],\nabla^k m^+\rangle|\\
&+|\langle\nabla^{k-1}[h_1^+(m^+,m^-)\nabla m^+\nabla^t u^+],\nabla^k m^+\rangle|\\
&+|\langle\nabla^{k-1}[h_2^+(m^+,m^-)\nabla m^-\nabla^t u^+],\nabla^k m^+\rangle|\\
&+|\langle\nabla^{k-1}[h_1^+(m^+,m^-)\nabla m^+\mathrm{div}u^+],\nabla^k m^+\rangle|\\
&+|\langle\nabla^{k-1}[h_2^+(m^+,m^-)\nabla m^-\mathrm{div}u^+],\nabla^k m^+\rangle|\Big).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
|J_5^1| \le & C\tau\Big(\|(m^+,m^-)\|_{L^\infty}\|\nabla^k(m^+,m^-)\|_{L^2}+\|\nabla^{k-1}(m^+,m^-)\|_{L^6}\|\nabla(m^+,m^-)\|_{L^3}\\
&+\sum_{1\le l\le k-2}\|\nabla^l(m^+,m^-)\|_{L^3}\|\nabla^{k-l}(m^+,m^-)\|_{L^6}\Big)\|\nabla^k m^+\|_{L^2}\\
&+C\tau\Big(\|u^+\|_{L^\infty}\|\nabla^k u^+\|_{L^2}+\sum_{1\le l\le k-2}\|\nabla^l u^+\|_{L^3}\|\nabla^{k-l}u^+\|_{L^6}\\
&+\|\nabla^{k-1}u^+\|_{L^6}\|\nabla u^+\|_{L^3}\Big)\|\nabla^k m^+\|_{L^2}+C\tau\Big[\Big(\|\nabla(m^+,m^-)\|_{L^\infty}\|\nabla^k u^+\|_{L^2}\\
&+\sum_{1\le l\le k-3}\|\nabla^{l+1}(m^+,m^-)\|_{L^3}\|\nabla^{k-l}u^+\|_{L^6}+\|\nabla^{k-1}(m^+,m^-)\|_{L^6}\|\nabla^2 u^+\|_{L^3}\\
&+\|\nabla^k(m^+,m^-)\|_{L^2}\|\nabla u^+\|_{L^\infty}\Big)\|(m^+,m^-)\|_{L^\infty}\\
&+C\tau\Big(\|\nabla(m^+,m^-)\|_{L^6}\|\nabla^{k-1}u^+\|_{L^6}+\sum_{1\le l\le k-3}\|\nabla^{l+1}(m^+,m^-)\|_{L^6}\|\nabla^{k-l}u^+\|_{L^6}\\
&+\|\nabla^{k-1}(m^+,m^-)\|_{L^6}\|\nabla u^+\|_{L^6}\Big)\|\nabla(m^+,m^-)\|_{L^6}\\
&+\sum_{2\le l\le k-2}\|\nabla^l(m^+,m^-)\|_{L^6}\|\nabla^{k-1-l}(\nabla(m^+,m^-)\nabla u^+)\|_{L^3}\\
&+\|\nabla^{k-1}(m^+,m^-)\|_{L^6}\|\nabla(m^+,m^-)\|_{L^6}\|\nabla u^+\|_{L^6}\Big]\|\nabla^k m^+\|_{L^2}\\
\le & C\tau\Big(\|\nabla(m^+,m^-)\|_{H^1}^2\|\nabla^k(m^+,m^-)\|_{L^2}^2+\|\nabla(m^+,m^-)\|_{H^{k-2}}^2\|\nabla(m^+,m^-)\|_{H^{k-1}}^2\\
&+\|\nabla u^+\|_{H^1}^2\|\nabla^k u^+\|_{L^2}^2+\|\nabla u^+\|_{H^{k-2}}^2\|\nabla u^+\|_{H^{k-1}}^2+\|\nabla(m^+,m^-)\|_{H^2}^4\|\nabla^k u^+\|_{L^2}^2
\end{aligned}
$$

$$+\|\nabla(m^+,m^-)\|_{H^{k-2}}^4\|\nabla u^+\|_{H^{k-1}}^2+\|\nabla^k(m^+,m^-)\|_{L^2}^2\|\nabla u^+\|_{H^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2$$
$$+\|\nabla(m^+,m^-)\|_{H^{k-2}}^4\|\nabla u^+\|_{H^{k-2}}^2\Big)+\frac{\zeta}{16}\frac{\beta_3}{\beta_4}\|\nabla^k m^+\|_{L^2}^2,\ \text{for } k\geq 4;$$

Similar to the estimate of $J_3$, we can obtain the estimates of $J_5^1$ for $k=1,2,3$ and $J_5^2$, thus we have

$$\begin{aligned}|J_5|\leq& C\tau\Big(\|\nabla(m^+,m^-)\|_{H^{k-1}}^2+\|\nabla(u^+,u^-)\|_{H^{k-1}}^2\Big)\Big(\|\nabla(m^+,m^-)\|_{H^{k-2}}^2\\&+\|\nabla(u^+,u^-)\|_{H^{k-2}}^2+\|\nabla(u^+,u^-)\|_{H^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2+\|\nabla(m^+,m^-)\|_{H^{k-2}}^4\Big)\\&+\frac{\zeta}{16}\Big(\frac{\beta_3}{\beta_4}\|\nabla^k m^+\|_{L^2}^2+\frac{\beta_2}{\beta_1}\|\nabla^k m^-\|_{L^2}^2\Big),\ \text{for } k\geq 4.\end{aligned}$$

And

$$\begin{aligned}|J_5|\leq& C\tau\Big(\|\nabla(m^+,m^-)\|_{H^3}^2+\|\nabla(u^+,u^-)\|_{H^3}^2\Big)\Big(\|\nabla(m^+,m^-)\|_{H^2}^2+\|\nabla(u^+,u^-)\|_{H^2}^2\\&+\|\nabla(u^+,u^-)\|_{H^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2+\|\nabla(m^+,m^-)\|_{H^2}^4\Big)+\frac{\zeta}{16}\Big(\frac{\beta_3}{\beta_4}\|\nabla^k m^+\|_{L^2}^2\\&+\frac{\beta_2}{\beta_1}\|\nabla^k m^-\|_{L^2}^2\Big),\ \text{for } k=1,2,3,4.\end{aligned}$$

Finally,

$$|J_6|\leq\frac{\zeta}{16}\Big(\frac{\beta_3}{\beta_4}\|\nabla^k m^+\|_{L^2}^2+\frac{\beta_2}{\beta_1}\|\nabla^k m^-\|_{L^2}^2\Big)+C\tau\|\nabla^{k-1}(r^+,r^-)\|_{L^2}^2.$$

Combining these estimates, we get (3.15). ∎

Lemma 3.5. *Assume that $\tau\in(0,1],|m^\pm|\leq\frac{\bar n^\pm}{2}$, and $(m^+,m^-,u^+,u^-)\in\mathcal{X}^L$ is the solution of (3.14), then we have*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|m^+|^2+\zeta\frac{\beta_2}{\beta_1}|m^-|^2+2\beta_3\bar n|u^+|^2+2\beta_2\bar n^-|u^-|^2\Big)\mathrm{d}x\\&+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\beta_2\beta_4|m^-+\frac{\beta_3}{\beta_4}m^+|^2+\beta_1\beta_3|m^++\frac{\beta_2}{\beta_1}m^-|^2\Big)\mathrm{d}x\\&+\epsilon\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla m^-|^2\Big)\mathrm{d}x+\tilde\xi\int_{\Omega_L}\big(\beta_3\bar n^+\mu^+|\nabla u^+|^2+\beta_2\bar n^-\mu^-|\nabla u^-|^2\big)\mathrm{d}x\\\leq& C\tau\Big(\|\nabla(m^+,m^-)\|_{H^1}+\|\nabla(u^+,u^-)\|_{H^1}\Big)\Big(\|\nabla(u^+,u^-)\|_{L^2}^2+\|\nabla(m^+,m^-)\|_{L^2}^2\Big)\\&+C\tau\|(r^+,r^-)\|_{L^{\frac{6}{5}}}^2,\end{aligned}\tag{3.16}$$

*where $C$ is a constant indenpendent of $\epsilon$, but dependent on $L$.*

*Proof.* Assume that $(m^+,\ m^-,\ u^+,\ u^-)$ is the solution of (3.14). Multiplying $(3.14)_1,(3.14)_2$, $(3.14)_3,(3.14)_4$ by $\beta_1\beta_3m^+$, $\beta_2\beta_4m^-$, $\beta_3\bar n^+u^+$, $\beta_2\bar n^-u^-$ respectively, summing up, then integrating the result over $\Omega_L$ by parts and combining with the periodic boundary conditions and the Equations $(3.14)_1,(3.14)_2$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|m^+|^2+\zeta\frac{\beta_2}{\beta_1}|m^-|^2+2\beta_3\bar n|u^+|^2+2\beta_2\bar n^-|u^-|^2\Big)\mathrm{d}x$$

$$
\begin{aligned}
&+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\left(\beta_2\beta_4|m^-+\frac{\beta_3}{\beta_4}m^+|^2+\beta_1\beta_3|m^++\frac{\beta_2}{\beta_1}m^-|^2\right)\mathrm{d}x\\
&+\frac{1}{2}\epsilon\int_{\Omega_L}\left(\zeta\frac{\beta_3}{\beta_4}|\nabla m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla m^-|^2\right)\mathrm{d}x\\
&+\frac{1}{2}\epsilon\int_{\Omega_L}\left(\beta_2\beta_4|\nabla m^-+\frac{\beta_3}{\beta_4}\nabla m^+|^2+\beta_1\beta_3|\nabla m^++\frac{\beta_2}{\beta_1}\nabla m^-|^2\right)\mathrm{d}x\\
&+\beta_3\bar{n}^+\int_{\Omega_L}\left((\tilde{\xi}+\tau\xi^+(m^+,m^-))(\mu^+|\nabla u^+|^2+(\mu^++\lambda^+)|\mathrm{div}u^+|^2)\right)\mathrm{d}x\\
&+\beta_2\bar{n}^-\int_{\Omega_L}\left((\tilde{\xi}+\tau\xi^-(m^+,m^-))(\mu^-|\nabla u^-|^2+(\mu^-+\lambda^-)|\mathrm{div}u^-|^2)\right)\mathrm{d}x\\
=&-\beta_3\bar{n}^+\tau\int_{\Omega_L}(\partial_{m^+}\xi^+\nabla m^++\partial_{m^-}\xi^+\nabla m^-)(\mu^+u^+\nabla u^++(\mu^++\lambda^+)u^+\mathrm{div}u^+)\mathrm{d}x\\
&-\beta_2\bar{n}^-\tau\int_{\Omega_L}(\partial_{m^+}\xi^-\nabla m^++\partial_{m^-}\xi^-\nabla m^-)(\mu^-u^-\nabla u^-+(\mu^-+\lambda^-)u^-\mathrm{div}u^-)\mathrm{d}x\\
&+\left[\beta_3\langle m^+u^+,\beta_1\nabla m^++\beta_2\nabla m^-\rangle+\beta_2\langle m^-u^-,\beta_3\nabla m^++\beta_4\nabla m^-\rangle\right]\\
&+\left[\beta_3\bar{n}^+\langle\tilde{G}_3,u^+\rangle+\beta_2\bar{n}^-\langle\tilde{G}_4,u^-\rangle\right]+\left[\beta_3\bar{n}^+\langle r^+,u^+\rangle-\beta_2\bar{n}^-\langle r^-,u^-\rangle\right]\\
\le& C\tau\left(\|\nabla m^+\|_{L^3}+\|\nabla m^-\|_{L^3}\right)\left(\|u^+\|_{L^6}\|\nabla u^+\|_{L^2}+\|u^-\|_{L^6}\|\nabla u^-\|_{L^2}\right)\\
&+C\tau\left(\|m^+\|_{L^6}\|u^+\|_{L^6}+\|m^-\|_{L^6}\|u^-\|_{L^6}\right)\left(\|\nabla m^+\|_{L^{\frac{3}{2}}}+\|\nabla m^-\|_{L^{\frac{3}{2}}}\right)\\
&+C\tau\Big(\|u^+\|_{L^6}\|\nabla u^+\|_{L^{\frac{3}{2}}}\|u^+\|_{L^6}+\|u^-\|_{L^6}\|\nabla u^-\|_{L^{\frac{3}{2}}}\|u^-\|_{L^6}+\|r^+\|_{L^{\frac{6}{5}}}\|u^+\|_{L^6}\\
&+\|r^-\|_{L^{\frac{6}{5}}}\|u^-\|_{L^6}\Big)\\
\le& C\tau\left(\|\nabla(m^+,m^-)\|_{H^1}^2\|\nabla(u^+,u^-)\|_{L^2}^2+\|\nabla(m^+,m^-)\|_{L^2}^4+C\tau\|\nabla(u^+,u^-)\|_{L^2}^4\right)\\
&+\|(r^+,r^-)\|_{L^{\frac{6}{5}}}^2+\frac{\tilde{\xi}}{2}\Big(\frac{\beta_3\bar{n}^-\mu^+}{2}\|\nabla u^+\|_{L^2}^2+\frac{\beta_2\bar{n}^-\mu^-}{2}\|\nabla u^-\|_{L^2}^2\Big)\\
\le& C\tau\left(\|\nabla(m^+,m^-)\|_{H^1}+\|\nabla(u^+,u^-)\|_{H^1}\right)\left(\|\nabla(u^+,u^-)\|_{L^2}^2+\|\nabla(m^+,m^-)\|_{L^2}^2\right)\\
&+C\tau\|(r^+,r^-)\|_{L^{\frac{6}{5}}}^2+\frac{\tilde{\xi}}{2}\Big(\frac{\beta_3\bar{n}^-\mu^+}{2}\|\nabla u^+\|_{L^2}^2+\frac{\beta_2\bar{n}^-\mu^-}{2}\|\nabla u^-\|_{L^2}^2\Big)
\end{aligned}
$$

which implies (3.16). □

Remark 3.2. We see that the constant $C$ depends on $L$ in (3.16). In the above proof, we need to estimate the term $\int_{\Omega_L} m^\pm u^\pm\nabla m^\pm\mathrm{d}x$. Since there are no terms like $(m^\pm)^p,(u^\pm)^p$ in (3.14), we can not get the $L^p$ estimates of $m^\pm$ or $u^\pm$. If we want the estimate to be independent of $L$, we can just use the inequality $\|v\|_{L^6}\le C\|\nabla v\|_{L^2}$, then we need to estimate $\|\nabla m^\pm\|_{L^{3/2}}$. What we use is the inequality $\|v\|_{L^p}\le|\mathbf{meas}\Omega|^{1/p-1/q}\|v\|_{L^q}$, then the estimate depends on $L$.

Lemma 3.6. *Assume that* $\tau\in(0,1],|m^\pm|\le\frac{\bar{n}^\pm}{2}$, *and* $(m^+,m^-,u^+,u^-)\in\mathcal{X}^L$ *is the solution of* (3.14), *then we have*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\left(\beta_1|\nabla^k m^+|^2+\beta_4|\nabla^k m^-|^2+\bar{n}^+|\nabla^k u^+|^2+\bar{n}^-|\nabla^k u^-|^2\right)\mathrm{d}x+2\epsilon\int_{\Omega_L}\Big(\beta_1
$$

$$\times|\nabla^{k+1}m^+|^2+\beta_4|\nabla^{k+1}m^-|^2\Big)\mathrm{d}x+\tilde{\xi}\int_{\Omega_L}\Big(\bar{n}^+\mu^+|\nabla^{k+1}u^+|^2+\bar{n}^-\mu^-|\nabla^{k+1}u^-|^2\Big)\mathrm{d}x$$
$$\le C\tau\Big(\|\nabla(m^+,m^-)\|_{H^3}^2+\|\nabla(u^+,u^-)\|_{H^3}^2\Big)\Big(\|\nabla(m^+,m^-)\|_{H^2}^2+\|\nabla(u^+,u^-)\|_{H^2}$$
$$+\|\nabla(u^+,u^-)\|_{H^2}^2+\|\nabla(u^+,u^-)\|_{H^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2+\|\nabla(m^+,m^-)\|_{H^2}^4\Big)$$
$$+C\tau\|\nabla^{k-1}(r^+,r^-)\|_{L^2}^2+C_2\Big(\|\nabla^k m^+\|_{L^2}^2+\|\nabla^k m^-\|_{L^2}^2\Big),\ \textit{for } k=1,2,3,4, \tag{3.17}$$

*where $C,C_2$ are constants independent of $L$ and $\epsilon$.*

*Proof.* Assume that $(m^+,\ m^-,\ u^+,\ u^-)$ is the solution of (3.14). Similar to the proof of Lemma 3.1, applying $\nabla^k(k\ge 1)$ to $(3.14)_1,(3.14)_2,(3.14)_3,(3.14)_4$, multiplying the results by $\beta_1\nabla^k m^+$, $\beta_4\nabla^k m^-$, $\bar{n}^+\nabla^k u^+$, $\bar{n}^-\nabla^k u^-$ respectively, summing up, then integrating the result over $\Omega_L$ by parts and combining with the periodic boundary conditions and the Equations $(3.14)_1,(3.14)_2$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\beta_1|\nabla^k m^+|^2+\beta_4|\nabla^k m^-|^2+\bar{n}^+|\nabla^k u^+|^2+\bar{n}^-|\nabla^k u^-|^2\Big)\mathrm{d}x$$
$$+\epsilon\int_{\Omega_L}\Big(\beta_1|\nabla^k\nabla m^+|^2+\beta_4|\nabla^k\nabla m^-|^2\Big)\mathrm{d}x$$
$$+\bar{n}^+\int_{\Omega_L}\Big(\tilde{\xi}+\tau\xi^+(m^+,m^-)\Big)\Big(\mu^+|\nabla^k\nabla u^+|^2+(\mu^++\lambda^+)|\nabla^k\mathrm{div}u^+|^2\Big)\mathrm{d}x$$
$$+\bar{n}^-\int_{\Omega_L}\Big(\tilde{\xi}+\tau\xi^-(m^+,m^-)\Big)\Big(\mu^-|\nabla^k\nabla u^-|^2+(\mu^-+\lambda^-)|\nabla^k\mathrm{div}u^-|^2\Big)\mathrm{d}x$$
$$=\int_{\Omega_L}\Big[-\bar{n}^+\tau(\partial_{m^+}\xi^+\nabla m^++\partial_{m^-}\xi^+\nabla m^-)(\mu^+\nabla^k\nabla u^++(\mu^++\lambda^+)\nabla^k\mathrm{div}u^+)\nabla^k u^+$$
$$-\bar{n}^-\tau(\partial_{m^+}\xi^-\nabla m^++\partial_{m^-}\xi^-\nabla m^-)(\mu^-\nabla^k\nabla u^-+(\mu^-+\lambda^-)\nabla^k\mathrm{div}u^-)\nabla^k u^-$$
$$+\bar{n}^+\tau\sum_{1\le l\le k}\binom{k}{l}\nabla^l\xi^+\nabla^{k-l}(\mu^+\Delta u^++(\mu^++\lambda^+)\nabla\mathrm{div}u^+)\nabla^k u^+$$
$$+\bar{n}^-\tau\sum_{1\le l\le k}\binom{k}{l}\nabla^l\xi^-\nabla^{k-l}(\mu^-\Delta u^-+(\mu^-+\lambda^-)\nabla\mathrm{div}u^-)\nabla^k u^-$$
$$+\beta_2\bar{n}^+\nabla^k m^-\mathrm{div}\nabla^k u^++\beta_3\bar{n}^+\nabla^k m^+\mathrm{div}\nabla^k u^-\Big]\mathrm{d}x+\Big[\beta_1\langle\nabla^k\tilde{G}_1,\nabla^k m^+\rangle$$
$$+\beta_4\langle\nabla^k\tilde{G}_2,\nabla^k m^-\rangle\Big]-\Big[\beta_3\bar{n}^+\langle\nabla^{k-1}\tilde{G}_3,\nabla^{k+1}u^+\rangle+\beta_2\bar{n}^-\langle\nabla^{k-1}\tilde{G}_4,\nabla^{k+1}u^-\rangle\Big]$$
$$-\Big[\beta_3\bar{n}^+\langle\nabla^{k-1}r^+,\nabla^{k+1}u^+\rangle-\beta_2\bar{n}^-\langle\nabla^{k-1}r^-,\nabla^{k+1}u^-\rangle\Big]:=K_1+K_2+K_3+K_4. \tag{3.18}$$

Similar to the proof of Lemma 3.4, we have

$$|K_1|\le C\tau\Big(\|\nabla(m^+,m^-)\|_{L^\infty}\|\nabla^{k+1}(u^+,u^-)\|_{L^2}\|\nabla^k(u^+,u^-)\|_{L^2}$$
$$+\sum_{2\le l\le k-2}\|\nabla^l(m^+,m^-)\|_{L^3}\|\nabla^{k-l+2}(u^+,u^-)\|_{L^2}\|\nabla^k(u^+,u^-)\|_{L^6}$$
$$+\|\nabla^{k-1}(m^+,m^-)\|_{L^2}\|\nabla^3(u^+,u^-)\|_{L^3}\|\nabla^k(u^+,u^-)\|_{L^6}$$
$$+\|\nabla^k(m^+,m^-)\|_{L^2}\|\nabla^2(u^+,u^-)\|_{L^3}\|\nabla^k(u^+,u^-)\|_{L^6}$$

$$
\begin{aligned}
&+\|\nabla^k m^-\|_{L^2}\|\nabla^{k+1}u^+\|_{L^2}+\|\nabla^k m^+\|_{L^2}\|\nabla^{k+1}u^-\|_{L^2}\Big)\\
\le& C\tau\Big(\|\nabla(m^+,m^-)\|_{H^2}^2\|\nabla^k(u^+,u^-)\|_{L^2}^2+\|\nabla(m^+,m^-)\|_{H^{k-2}}^2\|\nabla(u^+,u^-)\|_{H^{k-1}}^2\\
&+\|\nabla^{k-1}(m^+,m^-)\|_{L^2}^2\|\nabla^3(u^+,u^-)\|_{H^1}^2+\|\nabla^k(m^+,m^-)\|_{L^2}^2\|\nabla^2(u^+,u^-)\|_{H^1}^2\Big)\\
&+\frac{1}{4}C_2\|\nabla^k(m^+,m^-)\|_{L^2}^2+\frac{\tilde{\xi}}{8}\Big(\bar{n}^+\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2+\bar{n}^-\mu^-\|\nabla^{k+1}u^-\|_{L^2}^2\Big)\\
\le& C\tau\Big(\|\nabla(m^+,m^-)\|_{H^{k-2}}^2+\|\nabla(u^+,u^-)\|_{H^{k-2}}^2\Big)\Big(\|\nabla(m^+,m^-)\|_{H^{k-1}}^2\\
&+\|\nabla(u^+,u^-)\|_{H^{k-1}}^2\Big)+\frac{1}{4}C_2\|\nabla^k(m^+,m^-)\|_{L^2}^2+\frac{\tilde{\xi}}{8}\Big(\bar{n}^+\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2\\
&+\bar{n}^-\mu^-\|\nabla^{k+1}u^-\|_{L^2}^2\Big),\ \text{for } k\ge 4;\\
|K_1|\le& C\tau\Big(\|\nabla(m^+,m^-)\|_{H^2}^2+\|\nabla(u^+,u^-)\|_{H^2}^2\Big)\Big(\|\nabla(m^+,m^-)\|_{H^3}^2+\|\nabla(u^+,u^-)\|_{H^3}^2\Big)\\
&+\frac{1}{4}C_2\|\nabla^k(m^+,m^-)\|_{L^2}^2+\frac{\tilde{\xi}}{8}\Big(\bar{n}^+\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2+\bar{n}^-\mu^-\|\nabla^{k+1}u^-\|_{L^2}^2\Big),\\
&\text{for } k=1,2,3,4.
\end{aligned}
$$

And

$$K_2=\beta_1\langle\nabla^k\tilde{G}_1,\nabla^k m^+\rangle+\beta_4\langle\nabla^k\tilde{G}_2,+\nabla^k m^-\rangle:=K_2^1+K_2^2.$$

$$
\begin{aligned}
K_2^1=&-\beta_1\tau\langle\nabla^k(\mathrm{div}(m^+u^+)),\nabla^k m^+\rangle\\
=&-\beta_1\tau\int_{\Omega_L}\Big(u^+\nabla^{k+1}m^+\nabla^k m^++\sum_{1\le l\le k}\binom{k}{l}\nabla^l u^+\nabla^{k-l}\nabla m^+\nabla^k m^+\\
&+\mathrm{div}u^+|\nabla^k m^+|^2+\sum_{1\le l\le k}\binom{k}{l}\nabla^l\mathrm{div}u^+\nabla^{k-l}m^+\nabla^k m^+\Big)\mathrm{d}x\\
=&-\beta_1\tau\int_{\Omega_L}\Big(\frac{1}{2}\mathrm{div}u^+|\nabla^k m^+|^2+\binom{k}{1}\nabla u^+|\nabla^k m^+|^2+\sum_{2\le l\le k-2}\binom{k}{l}\nabla^l u^+\nabla^{k-l}\nabla m^+\nabla^k m^+\\
&+\binom{k}{k-1}\nabla^{k-1}u^+\nabla^2 m^+\nabla^k m^++\nabla^k u^+\nabla m^+\nabla^k m^++\binom{k}{1}\nabla\mathrm{div}u^+\nabla^{k-1}m^+\nabla^k m^+\\
&+\sum_{2\le l\le k-2}\binom{k}{l}\nabla^l\mathrm{div}u^+\nabla^{k-l}m^+\nabla^k m^++\binom{k}{k-1}\nabla^{k-1}\mathrm{div}u^+\nabla m^+\nabla^k m^+\\
&+\nabla^k\mathrm{div}u^+m^+\nabla^k m^+\Big)\mathrm{d}x,
\end{aligned}
$$

then

$$
\begin{aligned}
|K_2^1|\le& C\tau\Big(\|\nabla u^+\|_{L^\infty}\|\nabla^k m^+\|_{L^2}^2+\sum_{2\le l\le k-2}\|\nabla^l u^+\|_{L^4}\|\nabla^{k-l}\nabla m^+\|_{L^4}\|\nabla^k m^+\|_{L^2}\\
&+\|\nabla^{k-1}u^+\|_{L^4}\|\nabla^2 m^+\|_{L^4}\|\nabla^k m^+\|_{L^2}+\|\nabla^k u^+\|_{L^6}\|\nabla m^+\|_{L^3}\|\nabla^k m^+\|_{L^2}\\
&+\|\nabla^2 u^+\|_{L^4}\|\nabla^{k-1}m^+\|_{L^4}\|\nabla^k m^+\|_{L^2}+\sum_{2\le l\le k-2}\|\nabla^l\mathrm{div}u^+\|_{L^6}\|\nabla^{k-l}m^+\|_{L^3}\\
&\times\|\nabla^k m^+\|_{L^2}+\|\nabla^k u^+\|_{L^2}\|\nabla m^+\|_{L^\infty}\|\nabla^k m^+\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\nabla^{k+1}u^+\|_{L^2}\|m^+\|_{L^\infty}\|\nabla^k m^+\|_{L^2}\Big)\\
\le& C\tau\Big(\|\nabla u^+\|_{H^2}\|\nabla^k m^+\|_{L^2}^2+\|\nabla u^+\|_{H^{k-2}}^2\|\nabla m^+\|_{H^{k-1}}^2\\
&+\|\nabla u^+\|_{H^{k-1}}^2\|\nabla m^+\|_{H^{k-2}}^2+\|\nabla m^+\|_{H^1}^2\|\nabla^k m^+\|_{L^2}^2\Big)\\
&+\frac{1}{4}C_2\Big(\|\nabla^k m^+\|_{L^2}^2+\|\nabla^k m^-\|_{L^2}^2\Big)+\frac{\tilde{\xi}}{8}\bar{n}^+\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2, \text{ for } k\ge 4.
\end{aligned}
$$

and the estimats of $K_2^1$ for $k=1,2,3$ and $K_2^2$ are similar, thus we have

$$
\begin{aligned}
|K_2|\le& C\tau\Big(\|\nabla(u^+,u^-)\|_{H^2}\|\nabla(m^+,m^-)\|_{H^3}^2+\|\nabla(u^+,u^-)\|_{H^2}^2\|\nabla(m^+,m^-)\|_{H^3}^2\\
&+\|\nabla(m^+,m^-)\|_{H^1}^2\|\nabla(m^+,m^-)\|_{H^3}^2+\|\nabla(u^+,u^-)\|_{H^3}^2\|\nabla(m^+,m^-)\|_{H^2}^2\Big)\\
&+\frac{1}{4}C_2\Big(\|\nabla^k m^+\|_{L^2}^2+\|\nabla^k m^-\|_{L^2}^2\Big)+\frac{\tilde{\xi}}{8}\Big(\bar{n}^+\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2\\
&+\bar{n}^-\mu^-\|\nabla^{k+1}u^-\|_{L^2}^2\Big), \text{ for } k=1,2,3,4.
\end{aligned}
$$

For $K_3$,

$$
\begin{aligned}
K_3=&-\beta_3\bar{n}^+\langle\nabla^{k-1}\tilde{G}_3,\nabla^{k+1}u^+\rangle+\beta_2\bar{n}^-\langle\nabla^{k-1}\tilde{G}_4,\nabla^{k+1}u^-\rangle:=K_3^1+K_3^2.\\
|K_3^1|\le& C\tau\Big(|\langle\nabla^{k-1}(g_1^+(m^+,m^-)\nabla m^++g_2^+(m^+,m^-)\nabla m^-),\nabla^{k+1}u^+\rangle|\\
&+|\langle\nabla^{k-1}[h_1^+(m^+,m^-)(\nabla m^+\cdot\nabla)u^++h_2^+(m^+,m^-)(\nabla m^-\cdot\nabla)u^+],\nabla^{k+1}u^+\rangle|\\
&+|\langle\nabla^{k-1}[h_1^+(m^+,m^-)\nabla m^+\nabla^t u^++h_2^+(m^+,m^-)\nabla m^-\nabla^t u^+],\nabla^{k+1}u^+\rangle|\\
&+|\langle\nabla^{k-1}[h_1^+(m^+,m^-)\nabla m^+\mathrm{div}u^++h_2^+(m^+,m^-)\nabla m^-\mathrm{div}u^+],\nabla^{k+1}u^+\rangle|\\
&+|\langle\nabla^{k-1}[(u^+\cdot\nabla)u^+],\nabla^{k+1}u^+\rangle|\Big),
\end{aligned}
$$

that is

$$
\begin{aligned}
|K_3^1|\le& C\tau\Big[\Big(\|(m^+,m^-)\|_{L^\infty}\|\nabla^k(m^+,m^-)\|_{L^2}+\|\nabla^{k-1}(m^+,m^-)\|_{L^3}\|\nabla(m^+,m^-)\|_{L^6}\\
&+\sum_{1\le l\le k-2}\|\nabla^l(m^+,m^-)\|_{L^3}\|\nabla^{k-l}(m^+,m^-)\|_{L^6}\Big)+\Big(\|u^+\|_{L^\infty}\|\nabla^k u^+\|_{L^2}\\
&+\sum_{1\le l\le k-2}\|\nabla^l u^+\|_{L^3}\|\nabla^{k-l}u^+\|_{L^6}+\|\nabla^{k-1}u^+\|_{L^6}\|\nabla u^+\|_{L^3}\Big)\\
&+\Big(\|\nabla(m^+,m^-)\|_{L^\infty}\|\nabla^k u^+\|_{L^2}+\sum_{1\le l\le k-3}\|\nabla^{l+1}(m^+,m^-)\|_{L^3}\|\nabla^{k-l}u^+\|_{L^6}\\
&+\|\nabla^{k-1}(m^+,m^-)\|_{L^6}\|\nabla^2 u^+\|_{L^3}+\|\nabla^k(m^+,m^-)\|_{L^2}\|\nabla u^+\|_{L^\infty}\Big)\|(m^+,m^-)\|_{L^\infty}\\
&+\Big(\|\nabla(m^+,m^-)\|_{L^6}\|\nabla^{k-1}u^+\|_{L^6}+\sum_{1\le l\le k-3}\|\nabla^{l+1}(m^+,m^-)\|_{L^6}\|\nabla^{k-l-1}u^+\|_{L^6}\\
&+\|\nabla^{k-1}(m^+,m^-)\|_{L^6}\|\nabla u^+\|_{L^6}\Big)\|\nabla(m^+,m^-)\|_{L^6}\\
&+\sum_{2\le l\le k-2}\|\nabla^l(m^+,m^-)\|_{L^6}\|\nabla^{k-1-l}(\nabla(m^+,m^-)\nabla u^+)\|_{L^3}\\
&+\|\nabla^{k-1}(m^+,m^-)\|_{L^6}\|\nabla(m^+,m^-)\|_{L^6}\|\nabla u^+\|_{L^6}\Big]\|\nabla^{k+1}u^+\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\tau\Big(\|\nabla(m^+,m^-)\|_{H^1}^2\|\nabla^k(m^+,m^-)\|_{L^2}^2+\|\nabla(m^+,m^-)\|_{H^{k-2}}^2\|\nabla(m^+,m^-)\|_{H^{k-1}}^2\\
&\quad+\|\nabla u^+\|_{H^1}^2\|\nabla^k u^+\|_{L^2}^2+\|\nabla u^+\|_{H^{k-2}}^2\|\nabla u^+\|_{H^{k-1}}^2+\|\nabla(m^+,m^-)\|_{H^2}^4\|\nabla^k u^+\|_{L^2}^2\\
&\quad+\|\nabla^k(m^+,m^-)\|_{H^{k-2}}^4\|\nabla u^+\|_{H^{k-1}}^2+\|\nabla^k(m^+,m^-)\|_{L^2}^2\|\nabla u^+\|_{H^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2\\
&\quad+\|\nabla(m^+,m^-)\|_{H^{k-2}}^4\|\nabla u^+\|_{H^{k-2}}^2\Big)+\frac{\tilde{\xi}}{8}\bar{n}^+\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2,\ \text{for } k\ge 4;
\end{aligned}
$$

Similarly, we can get the estimates of $K_3^1$ for $k=1,2,3$, and $K_3^2$, thus we have

$$
\begin{aligned}
|K_3|&\le C\tau\Big(\|\nabla(m^+,m^-)\|_{H^{k-1}}^2+\|\nabla(u^+,u^-)\|_{H^{k-1}}^2\Big)\Big(\|\nabla(m^+,m^-)\|_{H^{k-2}}^2\\
&\quad+\|\nabla(u^+,u^-)\|_{H^{k-2}}^2+\|\nabla(u^+,u^-)\|_{H^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2\\
&\quad+\|\nabla(m^+,m^-)\|_{H^{k-2}}^4\Big)+\frac{\tilde{\xi}}{8}\Big(\bar{n}^+\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2+\bar{n}^-\mu^-\|\nabla^{k+1}u^-\|_{L^2}^2\Big),\ \text{for } k\ge 4,\\
|K_3|&\le C\tau\Big(\|\nabla(m^+,m^-)\|_{H^3}^2+\|\nabla(u^+,u^-)\|_{H^3}^2\Big)\Big(\|\nabla(m^+,m^-)\|_{H^2}^2\\
&\quad+\|\nabla(u^+,u^-)\|_{H^2}^2+\|\nabla(u^+,u^-)\|_{H^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2\\
&\quad+\|\nabla(m^+,m^-)\|_{H^2}^4\Big)+\frac{\tilde{\xi}}{8}\Big(\bar{n}^+\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2+\bar{n}^-\mu^-\|\nabla^{k+1}u^-\|_{L^2}^2\Big),\\
&\quad\text{for } k=1,2,3,4.
\end{aligned}
$$

And

$$
\begin{aligned}
|K_4|&\le\frac{\tilde{\xi}}{8}\Big(\bar{n}^+\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2+\bar{n}^-\mu^-\|\nabla^{k+1}u^-\|_{L^2}^2\Big)+C\tau\|\nabla^{k-1}(r^+,r^-)\|_{L^2}^2,\\
&\quad\text{for } k\ge 1.
\end{aligned}
$$

Combining these estimates, we get (3.17). ∎

LEMMA 3.7. *Assume that* $\tau\in(0,1],|m^{\pm}|\le\frac{\bar{n}^{\pm}}{2}$, *and* $(m^+,m^-,u^+,u^-)\in\mathcal{X}^L$ *is the solution of* (3.14), *then we have*

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^{k-1}m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^{k-1}m^-|^2+2\beta_3\bar{n}+|\nabla^{k-1}u^+|^2+2\beta_2\bar{n}^-|\nabla^{k-1}u^-|^2\Big)\mathrm{d}x\\
&\quad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\beta_2\beta_4|\nabla^{k-1}m^-+\frac{\beta_3}{\beta_4}\nabla^{k-1}m^+|^2+\beta_1\beta_3|\nabla^{k-1}m^++\frac{\beta_2}{\beta_1}\nabla^{k-1}m^-|^2\Big)\mathrm{d}x\\
&\quad+\epsilon\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^k m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k m^-|^2\Big)\mathrm{d}x+\tilde{\xi}\int_{\Omega_L}\Big(\beta_3\bar{n}^+\mu^+|\nabla^k u^+|^2\\
&\quad+\beta_2\bar{n}^-\mu^-|\nabla^k u^-|^2\Big)\mathrm{d}x\\
&\le C\tau\Big(\|\nabla(m^+,m^-)\|_{H^2}^2+\|\nabla(u^+,u^-)\|_{H^2}^2\Big)\Big(\|\nabla(m^+,m^-)\|_{H^2}^2+\|\nabla(m^+,m^-)\|_{H^2}^4\\
&\quad+\|\nabla(u^+,u^-)\|_{H^2}^2\Big)+C\tau\Big(\|\nabla(u^+,u^-)\|_{H^2}\|\nabla(m^+,m^-)\|_{H^3}^2+\|\nabla^{k-2}(r^+,r^-)\|_{L^2}^2\Big),\\
&\quad\textit{for } k=2,3,4,
\end{aligned}\tag{3.19}
$$

*where* $C$ *is a constant independent of* $L$ *and* $\epsilon$.

*Proof.* Assume that $(m^+,\ m^-,\ u^+,\ u^-)$ is the solution of (3.14). Similar to the proof of Lemma 3.1, applying $\nabla^k(k=1,2,3)$ to $(3.14)_1,(3.14)_2,(3.14)_3,(3.14)_4$ multiplying the results by $\beta_1\beta_3\nabla^k m^+$, $\beta_2\beta_4\nabla^k m^-$, $\beta_3\bar{n}^+\nabla^k u^+$, $\beta_2\bar{n}^-\nabla^k u^-$ respectively, summing up, then integrating the result over $\Omega_L$ by parts and combining with the periodic boundary conditions and the Equations $(3.14)_1,(3.14)_2$, we obtain

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\left(\zeta\frac{\beta_3}{\beta_4}|\nabla^k m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k m^-|^2+2\beta_3\bar{n}^+|\nabla^k u^+|^2+2\beta_2\bar{n}^-|\nabla^k u^-|^2\right)\mathrm{d}x\\
&+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\left(\beta_2\beta_4|\nabla^k m^-+\frac{\beta_3}{\beta_4}\nabla^k m^+|^2+\beta_1\beta_3|\nabla^k m^++\frac{\beta_2}{\beta_1}\nabla^k m^-|^2\right)dx\\
&+\frac{1}{2}\epsilon\int_{\Omega_L}\left(\zeta\frac{\beta_3}{\beta_4}|\nabla^k\nabla m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k\nabla m^-|^2\right)\mathrm{d}x\\
&+\frac{1}{2}\epsilon\int_{\Omega_L}\left(\beta_2\beta_4|\nabla^k\nabla m^-+\frac{\beta_3}{\beta_4}\nabla^k\nabla m^+|^2+\beta_1\beta_3|\nabla^k\nabla m^++\frac{\beta_2}{\beta_1}\nabla^k\nabla m^-|^2\right)\mathrm{d}x\\
&+\beta_3\bar{n}^+\int_{\Omega_L}\left((\tilde{\xi}+\tau\xi^+(m^+,m^-))(\mu^+|\nabla^k\nabla u^+|^2+(\mu^++\lambda^+)|\nabla^k\mathrm{div}u^+|^2)\right)\mathrm{d}x\\
&+\beta_2\bar{n}^-\int_{\Omega_L}\left((\tilde{\xi}+\tau\xi^-(m^+,m^-))(\mu^-|\nabla^k\nabla u^-|^2+(\mu^-+\lambda^-)|\nabla^k\mathrm{div}u^-|^2)\right)\mathrm{d}x\\
=&\Big[-\beta_3\bar{n}^+\tau\int_{\Omega_L}(\partial_{m^+}\xi^+\nabla m^++\partial_{m^-}\xi^+\nabla m^-)(\mu^+\nabla^k\nabla u^++(\mu^++\lambda^+)\nabla^k\mathrm{div}u^+)\nabla^k u^+\\
&-\beta_2\bar{n}^-\tau\int_{\Omega_L}(\partial_{m^+}\xi^-\nabla m^++\partial_{m^-}\xi^-\nabla m^-)(\mu^-\nabla^k\nabla u^-+(\mu^-+\lambda^-)\nabla^k\mathrm{div}u^-)\nabla^k u^-\\
&+\beta_3\bar{n}^+\tau\int_{\Omega_L}\sum_{1\le l\le k}\binom{k}{l}\nabla^l\xi^+\nabla^{k-l}(\mu^+\Delta u^++(\mu^++\lambda^+)\nabla\mathrm{div}u^+)\nabla^k u^+\\
&+\beta_2\bar{n}^-\tau\int_{\Omega_L}\sum_{1\le l\le k}\binom{k}{l}\nabla^l\xi^-\nabla^{k-l}(\mu^-\Delta u^-+(\mu^-+\lambda^-)\nabla\mathrm{div}u^-)\nabla^k u^-\Big]\mathrm{d}x\\
&-\Big[\beta_3\langle\nabla^{k-1}\tilde{G}_1,\beta_1\nabla^{k+1}m^++\beta_2\nabla^{k+1}m^-\rangle+\beta_2\langle\nabla^{k-1}\tilde{G}_2,\beta_3\nabla^{k+1}m^++\beta_4\nabla^{k+1}m^-\rangle\Big]\\
&-\Big[\beta_3\bar{n}^+\langle\nabla^{k-1}\tilde{G}_3,\nabla^{k+1}u^+\rangle+\beta_2\bar{n}^-\langle\nabla^{k-1}\tilde{G}_4,\nabla^{k+1}u^-\rangle\Big]\\
&-\Big[\beta_3\bar{n}^+\langle\nabla^{k-1}r^+,\nabla^{k+1}u^+\rangle+\beta_2\bar{n}^-\langle\nabla^{k-1}r^-,\nabla^{k+1}u^-\rangle\Big]:=M_1+M_2+M_3+M_4.
\end{aligned}\tag{3.20}$$

Similar to the proof of Lemma 3.4, we have

$$\begin{aligned}
|M_1|\le&C\tau\|\nabla(m^+,m^-)\|_{L^\infty}\|\nabla^4(u^+,u^-)\|_{L^2}\|\nabla^3(u^+,u^-)\|_{L^2}\\
&+C\tau\|\nabla^2(m^+,m^-)\|_{L^3}\|\nabla^3(u^+,u^-)\|_{L^2}\|\nabla^3(u^+,u^-)\|_{L^6}\\
&+C\tau\|\nabla^3(m^+,m^-)\|_{L^2}\|\nabla^2(u^+,u^-)\|_{L^3}\|\nabla^3(u^+,u^-)\|_{L^6}\\
\le&C\tau\Big(\|\nabla(m^+,m^-)\|_{H^2}^2\|\nabla^3(u^+,u^-)\|_{L^2}^2+\|\nabla^2(m^+,m^-)\|_{H^1}^2\|\nabla^3(u^+,u^-)\|_{L^2}^2\\
&+\|\nabla^3(m^+,m^-)\|_{L^2}^2\|\nabla^2(u^+,u^-)\|_{H^1}^2\Big)+\frac{\tilde{\xi}}{6}\Big(\bar{n}^+\mu^+\|\nabla^4u^+\|_{L^2}^2+\bar{n}^-\mu^-\|\nabla^4u^-\|_{L^2}^2\Big)\\
&\text{for } k=3.
\end{aligned}$$

Similar estimates can be established for $k=1,2$, then we have

$$|M_1|\le C\tau\|\nabla(m^+,m^-)\|_{H^2}^2\|\nabla(u^+,u^-)\|_{H^2}^2+\frac{\tilde{\xi}}{6}\Big(\bar{n}^+\beta_3\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2$$

$$+\bar{n}^-\beta_2\mu^-\|\nabla^{k+1}u^-\|_{L^2}^2\Big), \text{ for } k=1,2,3.$$

For $M_2$,

$$\begin{aligned}
M_2 =& -\beta_3\langle\nabla^{k-1}\tilde{G}_1,\beta_1\nabla^{k+1}m^+ +\beta_2\nabla^{k+1}m^-\rangle-\beta_2\langle\nabla^{k-1}\tilde{G}_2,\beta_3\nabla^{k+1}m^+ +\beta_4\nabla^{k+1}m^-\rangle\\
:=& M_2^1+M_2^2.\\
M_2^1 =&\beta_3\tau\langle\nabla^{k-1}(\mathrm{div}(m^+u^+)),\beta_1\nabla^{k+1}m^+ +\beta_2\nabla^{k+1}m^-\rangle\\
=&\beta_3\tau\langle\nabla^{2}(\mathrm{div}(m^+u^+)),\beta_1\nabla^{4}m^+ +\beta_2\nabla^{4}m^-\rangle\\
=&-\beta_3\tau\int_{\Omega_L}\Big(u^+\nabla^3m^+ +2\nabla u^+\nabla^2m^+ +\nabla^2u^+\nabla m^+ +\mathrm{div}u^+\nabla^2m^+ +2\nabla\mathrm{div}u^+\nabla m^+\\
&+\nabla^2\mathrm{div}u^+m^+\Big)\Big(\beta_1\nabla^4m^+ +\beta_2\nabla^4m^-\Big)\mathrm{d}x\\
\le& C\tau\Big(\|u^+\|_{L^\infty}\|\nabla^3m^+\|_{L^2}+\|\nabla u^+\|_{L^6}\|\nabla^2m^+\|_{L^3}+\|\nabla^2u^+\|_{L^6}\|\nabla m^+\|_{L^3}\\
&+\|\nabla^3u^+\|_{L^2}\|m^+\|_{L^\infty}\Big)\|\nabla^4(m^+,m^-)\|_{L^2}\\
\le& C\tau\|\nabla u^+\|_{H^2}\|\nabla(m^+,m^-)\|_{H^3}^2, \text{ for } k=3,
\end{aligned}$$

then

$$M_2\le C\tau\|\nabla(u^+,u^-)\|_{H^2}\|\nabla(m^+,m^-)\|_{H^3}^2, \text{ for } k=1,2,3.$$

For $M_3$,

$$\begin{aligned}
M_3 =& -\beta_3\bar{n}^+\langle\nabla^{k-1}\tilde{G}_3,\nabla^{k+1}u^+\rangle-\beta_2\bar{n}^-\langle\nabla^{k-1}\tilde{G}_4,\nabla^{k+1}u^-\rangle:=M_3^1+M_3^2.\\
|M_3^1|\le& C\tau\Big(|\langle\nabla^{k-1}(g_1^+(m^+,m^-)\nabla m^+ +g_2^+(m^+,m^-)\nabla m^-),\nabla^{k+1}u^+\rangle|\\
&+|\langle\nabla^{k-1}[h_1^+(m^+,m^-)(\nabla m^+\cdot\nabla)u^+ +h_2^+(m^+,m^-)(\nabla m^-\cdot\nabla)u^+],\nabla^{k+1}u^+\rangle|\\
&+|\langle\nabla^{k-1}[h_1^+(m^+,m^-)\nabla m^+\nabla^t u^+ +h_2^+(m^+,m^-)\nabla m^-\nabla^t u^+],\nabla^{k+1}u^+\rangle|\\
&+|\langle\nabla^{k-1}[h_1^+(m^+,m^-)\nabla m^+\mathrm{div}u^+ +h_2^+(m^+,m^-)\nabla m^-\mathrm{div}u^+],\nabla^{k+1}u^+\rangle|\\
&+|\langle\nabla^{k-1}[(u^+\cdot\nabla)u^+],\nabla^{k+1}u^+\rangle|\Big).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|M_3^1|\le& C\tau\Big[\Big(\|(m^+,m^-)\|_{L^\infty}\|\nabla^3(m^+,m^-)\|_{L^2}+\|\nabla^2(m^+,m^-)\|_{L^3}\|\nabla(m^+,m^-)\|_{L^6}\Big)\\
&+\Big(\|u^+\|_{L^\infty}\|\nabla^3u^+\|_{L^2}+\|\nabla u^+\|_{L^3}\|\nabla^2u^+\|_{L^6}\Big)+\Big(\|\nabla(m^+,m^-)\|_{L^\infty}\|\nabla^3u^+\|_{L^2}\\
&+\|\nabla^2(m^+,m^-)\|_{L^3}\|\nabla^2u^+\|_{L^6}+\|\nabla^3(m^+,m^-)\|_{L^2}\|\nabla u^+\|_{L^\infty}\Big)+\|\nabla(m^+,m^-)\|_{L^\infty}\\
&\times\Big(\|\nabla(m^+,m^-)\|_{L^3}\|\nabla^2u^+\|_{L^6}+\|\nabla^2(m^+,m^-)\|_{L^3}\|\nabla u^+\|_{L^6}\Big)\Big]\|\nabla^4u^+\|_{L^2}\\
\le& C\tau\Big(\|\nabla(m^+,m^-)\|_{H^1}^2\|\nabla^3(m^+,m^-)\|_{L^2}^2+\|\nabla^2(m^+,m^-)\|_{H^1}^2\|\nabla^2(m^+,m^-)\|_{L^2}^2\\
&+\|\nabla u^+\|_{H^1}^2\|\nabla^3u^+\|_{L^2}^2+\|\nabla(m^+,m^-)\|_{H^2}^2\|\nabla^3u^+\|_{L^2}^2\\
&+\|\nabla^2(m^+,m^-)\|_{H^1}^2\|\nabla^3u^+\|_{L^2}^2+\|\nabla^3(m^+,m^-)\|_{L^2}^2\|\nabla u^+\|_{H^2}^2\\
&+\|\nabla(m^+,m^-)\|_{H^2}^2\|\nabla(m^+,m^-)\|_{H^1}^2\|\nabla^3u^+\|_{L^2}^2\\
&+\|\nabla(m^+,m^-)\|_{H^2}^2\|\nabla^2(m^+,m^-)\|_{H^1}^2\|\nabla^2u^+\|_{L^2}^2\Big)+\frac{\tilde{\xi}}{6}\bar{n}^+\beta_3\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2
\end{aligned}$$

$$\leq C\tau\Big[\|\nabla(m^+,m^-)\|_{H^1}^2\|\nabla(m^+,m^-)\|_{H^2}^2+\|\nabla u^+\|_{H^1}^2\|\nabla u^+\|_{H^2}^2$$
$$+\Big(\|\nabla(m^+,m^-)\|_{H^2}^2+\|\nabla(m^+,m^-)\|_{H^2}^4\Big)\|\nabla u^+\|_{H^2}^2\Big]+\frac{\tilde{\xi}}{6}\bar{n}^+\beta_3\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2$$
for $k=3$.

Similarly, we can get the estimates of $M_3^1$ for $k=1,2$ and $M_3^2$, thus we have

$$|M_3|\leq\frac{\tilde{\xi}}{6}\Big(\bar{n}^+\beta_3\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2+\bar{n}^-\beta_2\mu^-\|\nabla^{k+1}u^-\|_{L^2}^2\Big)+C\tau\Big(\|\nabla(m^+,m^-)\|_{H^2}^2$$
$$+\|\nabla(u^+,u^-)\|_{H^2}^2\Big)\Big(\|\nabla(m^+,m^-)\|_{H^2}^2+\|\nabla(m^+,m^-)\|_{H^2}^4+\|\nabla(u^+,u^-)\|_{H^1}^2\Big),$$
for $k=1,2,3$.

And

$$|M_4|\leq\frac{\tilde{\xi}}{6}\Big(\bar{n}^+\beta_3\mu^+\|\nabla^{k+1}u^+\|_{L^2}^2+\bar{n}^-\beta_2\mu^-\|\nabla^{k+1}u^-\|_{L^2}^2\Big)+C\tau\|\nabla^{k-1}(r^+,r^-)\|_{L^2}^2,$$
for $k=1,2,3$.

Combining these estimates and taking $k$ as $k-1$, we get the (3.19). □

**3.3. Existence in bounded domain.** In this subsection, we deduce the existence of the time periodic solution in periodic domain. To begin with, we discuss the case $\tau=0$, then we prove Proposition 1.1 and Theorem 1.1.

LEMMA 3.8. *When $\tau=0$, we have $\mathcal{S}((n^+,n^-,w^+,w^-),0)\equiv 0$.*

*Proof.* Applying $\nabla^k,(k\geq 1)$ to $(3.1)_1,(3.1)_2,(3.1)_3,(3.1)_4$, multiplying the results by $\beta_1\beta_3\nabla^k m^+$, $\beta_2\beta_4\nabla^k m^-$, $\beta_3\bar{n}^+\nabla^k u^+$, $\beta_2\bar{n}^-\nabla^k u^-$ respectively, summing up, then integrating the result over $\Omega_L$ by parts and combining with the periodic boundary conditions and the Equations $(3.1)_1,(3.1)_2$, we obtain

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^k m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k m^-|^2+2\beta_3\bar{n}|\nabla^k u^+|^2+2\beta_2\bar{n}^-|\nabla^k u^-|^2\Big)\mathrm{d}x$$
$$+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_L}\Big(\beta_2\beta_4|\nabla^k m^-+\frac{\beta_3}{\beta_4}\nabla^k m^+|^2+\beta_1\beta_3|\nabla^k m^++\frac{\beta_2}{\beta_1}\nabla^k m^-|^2\Big)dx$$
$$+\frac{1}{2}\epsilon\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^k\nabla m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k\nabla m^-|^2\Big)\mathrm{d}x$$
$$+\frac{1}{2}\epsilon\int_{\Omega_L}\Big(\beta_2\beta_4|\nabla^k\nabla m^-+\frac{\beta_3}{\beta_4}\nabla^k\nabla m^+|^2+\beta_1\beta_3|\nabla^k\nabla m^++\frac{\beta_2}{\beta_1}\nabla^k\nabla m^-|^2\Big)\mathrm{d}x$$
$$+\beta_3\bar{n}^+\int_{\Omega_L}\Big(\tilde{\xi}(\mu^+|\nabla^k\nabla u^+|^2+(\mu^++\lambda^+)|\nabla^k\mathrm{div}u^+|^2)\Big)\mathrm{d}x$$
$$+\beta_2\bar{n}^-\int_{\Omega_L}\Big(\tilde{\xi}(\mu^-|\nabla^k\nabla u^-|^2+(\mu^-+\lambda^-)|\nabla^k\mathrm{div}u^-|^2)\Big)\mathrm{d}x=0.$$

Integrating from 0 to $T$, we have

$$\frac{1}{2}\epsilon\int_0^T\int_{\Omega_L}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^k\nabla m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k\nabla m^-|^2\Big)\mathrm{d}x+\int_0^T\int_{\Omega_L}\Big(\tilde{\xi}(\beta_3\bar{n}^+\mu^+|\nabla^k\nabla u^+|^2$$
$$+\beta_2\bar{n}^-\mu^-|\nabla^k\nabla u^-|^2\Big)\mathrm{d}x\leq 0.$$

Then, by the Poincaré's inequality, we obtain $(m^+,\ m^-,\ u^+,\ u^-)\equiv 0$. The proof is complete. ∎

*Proof.* **(Proof of Proposition 1.1.)** Note that, to solve the problem (1.9) is equivalent to solving the equation

$$U-\mathcal{S}(U,1)=0,\ U=(m^+,m^-,u^+,u^-)\in\mathcal{X}^L.$$

In what follows, we apply the topological degree theory to solve this problem. To begin with, we show that there exists $\eta_0>0$, such that

$$(I-\mathcal{S}(\cdot,\tau))(\partial B_{\eta_0}(0))\neq 0,\ \text{for any}\ \tau\in[0,1], \tag{3.21}$$

where $B_{\eta_0}(0)$ is the ball of radius $\eta_0$ centered at 0 in $\mathcal{X}^L$. Then, if (3.21) holds, to prove the existence of the solution, we just need to show

$$\deg(I-\mathcal{S}(\cdot,1),B_{\eta_0},0)\neq 0. \tag{3.22}$$

Notice that when $\eta_0$ is suitably small, we have, $\|m^\pm\|_{L^\infty}\leq \sup_{0\leq t\leq T}\|\nabla m^\pm\|_{H^1}\leq\eta_0\leq\frac{n^\pm}{2}$. Let $M_1\times(3.15)+(3.17)+M_2\times((3.16)+(3.19))$, for appropriate $M_1,M_2$, and integrating from 0 to $T$, then suming up the result for $k(k=1,2,3,4)$. Since $\eta_1,\eta_2$ are suitable small, we can take

$$\begin{aligned}&C_2\leq\frac{M_1\zeta}{8}\min(\frac{\beta_3}{\beta_4},\ \frac{\beta_2}{\beta_1}),\ \eta_1M_1\leq\min(\beta_1,\beta_4),\eta_2M_1\leq\frac{\tilde{\xi}}{2}\min(\bar{n}^+\mu^+,\bar{n}^-\mu^-),\\&M_1C_1\leq\frac{M_2\tilde{\xi}}{2}\min(\beta_3\bar{n}^+\mu^+,\beta_2\bar{n}^-\mu^-).\end{aligned}$$

We have

$$\begin{aligned}&\frac{M_1}{8}\int_0^T\int_{\Omega_L}\sum_{1\leq k\leq 4}\Big(\zeta\frac{\beta_3}{\beta_4}|\nabla^k m^+|^2+\zeta\frac{\beta_2}{\beta_1}|\nabla^k m^-|^2\Big)\mathrm{d}x\mathrm{d}t+\frac{M_2\tilde{\xi}}{2}\int_0^T\int_{\Omega_L}\Big(\beta_3\bar{n}^+\mu^+|\nabla u^+|^2\\&\quad+\beta_2\bar{n}^-\mu^-|\nabla u^-|^2\Big)\mathrm{d}x\mathrm{d}t+\frac{\tilde{\xi}}{2}\int_0^T\int_{\Omega_L}\sum_{1\leq k\leq 4}\Big(\bar{n}^+\mu^+|\nabla^{k+1}u^+|^2\\&\quad+\bar{n}^-\mu^-|\nabla^{k+1}u^-|^2\Big)\mathrm{d}x\mathrm{d}t\\&\leq C\tau\sup_{0\leq t\leq T}\Big(\|\nabla(m^+,m^-)\|_{H^1}+\|\nabla(m^+,m^-)\|_{H^2}^2+\|\nabla(u^+,u^-)\|_{H^2}^2+\|\nabla(m^+,m^-)\|_{H^2}^4\\&\quad+\|\nabla(u^+,u^-)\|_{H^2}+\|\nabla(m^+,m^-)\|_{H^2}^2\|\nabla(u^+,u^-)\|_{H^2}^2\Big)\int_0^T\Big(\|\nabla(m^+,m^-)\|_{H^3}^2\\&\quad+\|\nabla(u^+,u^-)\|_{H^3}^2\Big)\mathrm{d}t+C\tau\int_0^T\Big(\|(r^+,r^-)\|_{L^{6/5}}^2+\|\nabla(r^+,r^-)\|_{H^2}^2\Big)\mathrm{d}t\\&\leq C\tau(\eta_0^3+\eta_0^4+\eta_0^6)+C\tau\int_0^T\|(r^+,r^-)\|_{H^3}^2\mathrm{d}t.\end{aligned}$$

Then, by the Poincaré's inequality, we have

$$\int_0^T\Big(\|m^+\|_{H^4}^2+\|m^-\|_{H^4}^2+\|u^+\|_{H^5}^2+\|u^-\|_{H^5}^2\Big)\mathrm{d}t$$

$$\leq C\tau\left(\eta_0^3+\eta_0^4+\eta_0^6\right)+C\tau\int_0^T\|(r^+,r^-)\|_{H^3}^2\mathrm{d}t. \tag{3.23}$$

By the Mean Value Theorem, there exists a $t^*\in(0,T)$, such that

$$\begin{aligned}&\left(\|m^+\|_{H^4}^2+\|m^-\|_{H^4}^2+\|u^+\|_{H^5}^2+\|u^-\|_{H^5}^2\right)(x,t^*)\\&\leq C\tau\left(\eta_0^3+\eta_0^4+\eta_0^6\right)+C\tau\int_0^T\|(r^+,r^-)\|_{H^3}^2\mathrm{d}t.\end{aligned} \tag{3.24}$$

Integrating (3.17) from $t^*$ to $t(t\in(t^*,T])$, summing up the results for $k$, using the Poincaré's inequality and (3.24), we have

$$\begin{aligned}&\left(\|m^+\|_{H^4}^2+\|m^-\|_{H^4}^2+\|u^+\|_{H^4}^2+\|u^-\|_{H^4}^2\right)(\cdot,t)\\&\leq C\tau\left(\eta_0^3+\eta_0^4+\eta_0^6\right)+C\tau\int_0^T\|(r^+,r^-)\|_{H^3}^2\mathrm{d}t.\end{aligned}$$

Using the time periodicity, we have

$$\begin{aligned}&\left(\|m^+\|_{H^4}^2+\|m^-\|_{H^4}^2+\|u^+\|_{H^4}^2+\|u^-\|_{H^4}^2\right)(\cdot,0)\\&\leq C\tau\left(\eta_0^3+\eta_0^4+\eta_0^6\right)+C\tau\int_0^T\|(r^+,r^-)\|_{H^3}^2\mathrm{d}t.\end{aligned}$$

Repeating the above process yields

$$\begin{aligned}&\sup_{0\leq t\leq T}\left(\|m^+\|_{H^4}^2+\|m^-\|_{H^4}^2+\|u^+\|_{H^4}^2+\|u^-\|_{H^4}^2\right)\\&\leq C\tau\left(\eta_0^3+\eta_0^4+\eta_0^6\right)+C\tau\int_0^T\|(r^+,r^-)\|_{H^3}^2\mathrm{d}t.\end{aligned} \tag{3.25}$$

Suming up (3.23) and (3.25), when $\int_0^T\|(r^+,r^-)\|_{H^3}^2\mathrm{d}t$ and $\eta_0$ are appropriately small, we have

$$\|(m^+,m^-,u^+,u^-)\|_{X^L}^2\leq C\tau\left(\eta_0^3+\eta_0^4+\eta_0^6\right)+C\tau\int_0^T\|(r^+,r^-)\|_{H^3}^2\mathrm{d}t\leq\frac{1}{2}\eta_0^2. \tag{3.26}$$

Therefore, (3.21) holds. By Lemma 3.8, we have $\mathcal{S}(\cdot,0)\equiv 0$, then using Lemma 2.5, we have

$$\deg(I-\mathcal{S}(\cdot,1),B_{\eta_0},0)=\deg(I-\mathcal{S}(\cdot,0),B_{\eta_0},0)=\deg(I,B_{\eta_0},0)=1,$$

which implies that problem (1.10) has a solution $U=(m^+,m^-,u^+,u^-)$ with $\|U\|_{\mathcal{X}^L}\leq\eta_0$. The proof is complete. □

Finally, we give the proof of Theorem 1.1 by taking limit of the regularized problem (1.10).

*Proof.* **(Proof of Theorem 1.1.)** Assume $U_\epsilon=(m_\epsilon^+,m_\epsilon^-,u_\epsilon^+,u_\epsilon^-)$ is the solution of the regularized problem (1.10). From the proof of Proposition 1.1, we have

$$\sup_{0\leq t\leq T}\|U_\epsilon(\cdot,t)\|_{H^4}^2+\int_0^T\|U_\epsilon(\cdot,t)\|_{H^4}^2\mathrm{d}t\leq C\eta_0, \tag{3.27}$$

where $\eta_0$ is independent of $\epsilon$. Then integrating (3.17) from $t$ to $t+h$, then integrating the result from 0 to $T$, we have

$$\begin{aligned}&\int_0^T \Big[\big(\beta_1\|\nabla^k m_\epsilon^+\|_{L^2}^2+\beta_4\|\nabla^k m_\epsilon^-\|_{L^2}^2+\bar{n}^+\|\nabla^k u_\epsilon^+\|_{L^2}^2+\bar{n}^-\|\nabla^k u_\epsilon^-\|_{L^2}^2\big)(\cdot,t+h)\\&\quad-\big(\beta_1\|\nabla^k m_\epsilon^+\|_{L^2}^2+\beta_4\|\nabla^k m_\epsilon^-\|_{L^2}^2+\bar{n}^+\|\nabla^k u_\epsilon^+\|_{L^2}^2+\bar{n}^-\|\nabla^k u_\epsilon^-\|_{L^2}^2\big)(\cdot,t)\Big]\mathrm{d}t\le Ch,\\&\text{for } k=4, \qquad (3.28)\end{aligned}$$

where $C$ is independent of $\epsilon$. Therefore, there exists a subsequence of $U_\epsilon$, denoted by $U_{\epsilon_i}$, such that

$$(m_{\epsilon_i}^+,m_{\epsilon_i}^-,u_{\epsilon_i}^+,u_{\epsilon_i}^-)\overset{*}{\rightharpoonup}(m^+,m^-,u^+,u^-) \text{ in } L^\infty((0,T);H^4(\Omega_L));$$

$$(m_{\epsilon_i}^+,m_{\epsilon_i}^-,u_{\epsilon_i}^+,u_{\epsilon_i}^-)\to(m^+,m^-,u^+,u^-) \text{ in } L^2((0,T);H^4(\Omega_L)).$$

Using the Sobolev imbedding theorem to $m_\epsilon^+(x,t)\in L^\infty((0,T);H^4(\Omega_L))$, we have $m_\epsilon^+(x,t)\in C^\alpha(\Omega_L)$ for $\alpha\in(0,1)$ and any $t$. Next we show that there exists $\beta\in(0,1)$ such that $m_\epsilon^+(x,t)\in C^\beta(0,T)$, precisely

$$|m_\epsilon^+(x,t_1)-m_\epsilon^+(x,t_2)|\le C|t_1-t_2|^\beta,$$

holds for any $t_1,\ t_2\in(0,T), x\in\Omega_L$. Take a ball $B_r$ of radius $r$ centered at $x$, with $r=|t_1-t_2|^\eta$, $\eta=\frac{1}{2\alpha+3}$. Recalling (3.12), by Poincaré's inequality, we have

$$\begin{aligned}\int_{B_r}|m_\epsilon^+(y,t_1)-m_\epsilon^+(y,t_2)|\mathrm{d}y&=\int_{B_r}\Big|\int_{t_1}^{t_2}\frac{\partial m_\epsilon^+(y,t)}{\partial t}\mathrm{d}t\Big|\mathrm{d}y\\&\le C\Big(\int_{t_1}^{t_2}\int_{B_r}\Big|\frac{\partial m_\epsilon^+(y,t)}{\partial t}\Big|^2\mathrm{d}y\mathrm{d}t\Big)^{1/2}|t_1-t_2|^{1/2}r^{3/2}\\&\le C|t_1-t_2|^{1/2}r^{3/2}=C|t_1-t_2|^{(1+3\eta)/2}.\end{aligned}$$

Then, there exists $x^*\in B_r$ such that

$$|m_\epsilon^+(x^*,t_1)-m_\epsilon^+(x^*,t_2)|\le C|t_1-t_2|^{(1-3\eta)/2}.$$

Moreover, we have

$$\begin{aligned}&|m_\epsilon^+(x,t_1)-m_\epsilon^+(x,t_2)|\\&\le|m_\epsilon^+(x,t_1)-m_\epsilon^+(x^*,t_1)|+|m_\epsilon^+(x^*,t_1)-m_\epsilon^+(x^*,t_2)|+|m_\epsilon^+(x^*,t_2)-m_\epsilon^+(x,t_2)|\\&\le C(|t_1-t_2|^{\eta\alpha}+|t_1-t_2|^{(1-3\eta)/2})\le C|t_1-t_2|^{\alpha/(2\alpha+3)}.\end{aligned}$$

Taking $\beta=\frac{\alpha}{2\alpha+3}\in(0,1)$, we have $m_\epsilon^+(x,t)\in C^\beta(0,T)$. In the same way, we have

$$\begin{aligned}&|m_\epsilon^-(x_1,t_1)-m_\epsilon^-(x_2,t_2)|\le C(|x_1-x_2|^\alpha+|t_1-t_2|^\beta),\\&|u_\epsilon^+(x_1,t_1)-u_\epsilon^+(x_2,t_2)|\le C(|x_1-x_2|^\alpha+|t_1-t_2|^\beta),\\&|u_\epsilon^-(x_1,t_1)-u_\epsilon^-(x_2,t_2)|\le C(|x_1-x_2|^\alpha+|t_1-t_2|^\beta).\end{aligned}$$

for $\alpha,\beta\in(0,1)$, where $C$ is independent of $\epsilon$. Thus $U_\epsilon\in C^{\alpha,\beta}(\Omega_L\times(0,T))$. Using the Arzela-Ascoli Theorem, we have

$$(m_{\epsilon_i}^+,m_{\epsilon_i}^-,u_{\epsilon_i}^+,u_{\epsilon_i}^-)\to(m^+,m^-,u^+,u^-), \text{ uniformly as } \epsilon\to 0.$$

We deduce that $(m^+,m^-,u^+,u^-)$ is a time periodic solution of (1.9). This completes the proof. ∎

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