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THE 3D COMPRESSIBLE VISCOELASTIC FLUID IN A BOUNDED DOMAIN*

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Abstract. In this paper, we prove the global existence and uniqueness of strong solution for the 3D compressible viscoelastic fluid in a bounded domain under the condition that the initial data are close to the constant equilibrium state. Based on the standard energy estimate, the estimation of the exponential convergence rates of the strong solution is also obtained.

Keywords. Viscoelastic fluid; global existence; bounded domain; exponential convergence rates.

AMS subject classifications. 76W05; 35Q35; 35D05.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain, we are concerned with the following well-known compressible viscoelastic fluid system of Oldroyd type:

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \mathrm{div}\mathcal{T} + a\mathrm{div}(\rho F F^T), \\ F_t + \mathbf{u} \cdot \nabla F = \nabla \mathbf{u} F, \end{cases} \tag{1.1}$$

for $(x,t) \in \Omega \times \mathbb{R}^+$. Here $\rho, \mathbf{u}$ and $F \in M^{3\times 3}$ (the set of 3×3 matrices with positive determinants) stand for the density, velocity and the deformation gradient of the fluid respectively and the pressure $P(\rho)$ satisfies the γ-law that is $P(\rho) = A\rho^\gamma$, where $A > 0$ is a constant, $\gamma > 1$ is the adiabatic exponent. F^T means the transpose matrix of F. The stress tensor $\mathcal{T}$ is given by

$$\mathcal{T} = \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \lambda(\mathrm{div}\mathbf{u})\mathrm{I},$$

with the constant viscosity coefficients μ and λ satisfying the following physical condition:

$$\mu > 0, \quad 3\lambda + 2\mu > 0.$$

$a > 0$ denotes the elasticity coefficient. The corresponding elastic energy for system (1.1) is the special form of the Hookean linear elasticity:

$$W(F) = \frac{a}{2}|F|^2 + \frac{1}{\rho}\int_0^\rho P(s)ds.$$

In the context of hydrodynamics, the velocity field $\mathbf{u}(t,x)$ can be described by the flowing map, a time dependent family of orientation-preserving diff-isomorphisms $x(t,X)$, $0 \le t \le T$:

$$\frac{\partial x(t,X)}{\partial t} = \mathbf{u}(t, x(t,X)), \quad x(0,X) = X,$$

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where X is the initial labeling (Lagrangian coordinate) of the particle. Then the deformation tensor $\widetilde{F}(t,X)$ is defined as

$$\widetilde{F}(t,X) := \frac{\partial x(t,X)}{\partial X}.$$

In the Eulerian coordinate, the corresponding deformation gradient $F(t,x)$ will be defined as

$$F(t,x(t,X)) = \widetilde{F}(t,X).$$

Applying the chain rule, we see that $F(t,x)$ satisfies the following transport equation

$$\partial_t F + \mathbf{u}\cdot\nabla F = F\nabla\mathbf{u},$$

which is exactly Equation $(1.1)_3$.

In this paper, we consider the initial boundary value problem for the system (1.1), which is supplemented by the following initial and boundary conditions:

$$\begin{cases} (\rho,\mathbf{u},F)(x,0) = (\rho_0,\mathbf{u}_0,F_0), & x = (x_1,x_2,x_3)\in\Omega, \\ \mathbf{u}|_{\partial\Omega} = 0, & t\geq 0, \\ \int_\Omega \rho_0 dx/|\Omega| = \bar{\rho} > 0. \end{cases} \tag{1.2}$$

It is well-known that many fluids do not satisfy Newtonian law. There have been many attempts to capture different phenomena for non-Newtonian fluids; see the excellent survey papers [3, 6, 9, 20, 30, 35] and references therein. The fluid of Oldroyd type is one of the classical non-Newtonian fluids with elastic property. Due to the physical importance and mathematical challenges, the studies on the equations of the viscoelastic flows of Oldroyd type have attracted many physicists and mathematicians during the last decades. In the direction of incompressible case, there are many important progresses on the investigation of the global existence and large time behavior of smooth solution for sufficiently small initial data, refer to [1, 2, 5, 7, 16, 17, 19–29, 34, 40] and references therein. The global existence of strong solution were proved by Lin et al. [26], Chen and Zhang [2], Lei et al. [20–24] in Hilbert space H^ℓ, Qian [34] and Fang et al. [5, 40] in critical Besov space for the Cauchy problem. The problem is much more complicated due to the lack of damping mechanism and boundary condition on F for the initial boundary value problem. In order to bypass this difficulty, Lin and Zhang [27] used some important relations, which were first introduced by Sideris and Thomases [36], to show that some linear terms in the system are in fact high order terms and the global existence of strong solution was established. He and Xu [7] reformulated the original system along the particle trajectory and showed that in the Lagrangian coordinate, the system is decoupled and the well-posedness problem is reduced to the solvability of standard damped-wave equations with a no-slip boundary condition. They established the global existence and exponential decay estimate of strong solution for the equivalent system with the small and smooth initial data. Recently, Hu and Lin [8] studied the problem for discontinuous initial data. Although there are some progresses on the global existence of weak solutions, the global existence of weak solutions for arbitrary data is still an outstanding open question. In the direction of the compressible case, the local existence of multi-dimensional strong solution was obtained by Hu and Wang [10]. In [11, 32], the global well-posedness in the critical Besov space with the lowest regularity was established. Hu and Wu [13] provided a detailed analysis on the optimal time decay in H^2 framework. Based on [13], Jia et al. [14] obtained the optimal time decay rate under the critical Besov space framework. Wei et al. [37] and Wu et al. [38] established the optimal decay rates of the smooth solution just by energy methods. For the initial boundary value problem, global in time of the strong solution was proved to exist uniquely near the equilibrium state in [12, 33]. The main subject of this paper is to establish the global existence and uniqueness of strong solution for the system (1.1)-(1.2) under the condition that the initial data are close to the constant equilibrium state in H^2 framework. Our main results are formulated as the following theorem:

THEOREM 1.1. *Given the constant $\bar{\rho}>0$, there exists a constant ε_0, such that for any initial data $(\rho_0-\bar{\rho},\mathbf{u}_0,F_0-I)\in H^2(\Omega)$ satisfying*

$$\|(\rho_0-\bar{\rho},\mathbf{u_0},F_0-I)\|_2\le\varepsilon_0, \tag{1.3}$$

$$div(\rho_0F_0^T)=0, \tag{1.4}$$

and

$$F_0=(I+\nabla\psi_0)^{-1}, \quad \textit{for some } \psi_0\in H^3\cap H_0^1, \tag{1.5}$$

then the initial boundary value problem (1.1)-(1.2) *admits a unique solution* $(\rho-\bar{\rho},\mathbf{u},F-I)$ *globally in time with $\rho>0$, which satisfies*

$$\begin{aligned}
&\rho-\bar{\rho},F-I\in C^0([0,\infty);H^2(\Omega))\cap C^1([0,\infty);H^1(\Omega)),\\
&\mathbf{u}\in C^0([0,\infty);H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,\infty);L^2(\Omega)).
\end{aligned}$$

Moreover, the following exponential convergence rates in time hold:

$$\|(\rho-\bar{\rho},\mathbf{u},F-I)(t)\|_2+\|\partial_t(\rho,\mathbf{u},F)(t)\|\le C_0\|(\rho_0-\bar{\rho},\mathbf{u}_0,F_0-I)\|_2\exp\{-\eta_0t\},\quad\forall\ t\ge0, \tag{1.6}$$

where C_0 and η_0 are two positive constants, which are independent of t.

Notation. Let us introduce some notations for the use throughout this paper. The norms in the Sobolev spaces $H^m(\Omega)$ and $W^{m,q}(\Omega)$ are denoted respectively by $||\cdot||_m$ and $||\cdot||_{m,q}$ for $m\ge0$ and $q\ge1$. In particular, for $m=0$ we will simply use $||\cdot||$ and $||\cdot||_{L^q}$. For the sake of conciseness, we do not precise in functional space names when they are concerned with scalar-valued or vector-valued functions, $\|(f,g)\|_X$ denotes $\|f\|_X+\|g\|_X$. $C>0$ denotes the generic positive constant that only depends on the parameters coming from the problem and $f\lesssim g$ means that $f\le Cg$. We denote $\nabla=\partial_x=(\partial_1,\partial_2,\partial_3)$, where $\partial_i=\partial_{x_i}$, $\nabla_i=\partial_i$ and put $\partial_x^\ell f=\nabla^\ell f=\nabla(\nabla^{\ell-1}f)$, with an integer $\ell>0$. The integration domain Ω will be always omitted without any ambiguity and $\langle\cdot,\cdot\rangle$ denotes the inner-product in $L^2(\Omega)$.

Let us now give some comments on the key ingredients for the main difficulties and techniques in this paper. The system (1.1) is a typical example of the quasilinear hyperbolic-parabolic system, for which the local existence result of initial boundary value problems can be established by a standard contraction mapping argument; we refer to [18]. By the standard continuity argument, the global existence of the strong solution can be established by combining a priori estimates and the local existence result, thus it suffices to derive a priori energy estimates under the condition (1.3) if ε_0 is small enough. Due to the slip boundary condition, the classical energy estimates can not be applied directly to spatial derivatives since the spatial derivatives are unknown on the boundary. To overcome this difficulty, we separate the energy estimates for the spatial derivatives into that over the region away from the boundary and near the boundary in spirit of Matsumura and Nishida [31]. In other words, we establish the energy estimates for the spatial derivatives by using cutoff functions and localizations of $\partial\Omega$. Although our proofs are in the spirit of those for the Navier-Stokes equations, more intrinsic properties of the viscoelastic flow system are needed to get the dispersive estimates. It is worth mentioning that the crucial part of the proof is to obtain a Lyapunov-type energy inequality. Compared to the case of the half space in [33], in this paper, we study the initial boundary value problem of system (1.1) in a bounded domain. Moreover, the exponential convergence rates of the strong solutions are also obtained, which we believe that the result may have its own interest.

The rest of this paper is organized as follows. In Section 2, we will reformulate the system (1.1) into a new equivalent system. Section 3 is devoted to do some careful a priori estimates for the strong solutions and then the global existence of the strong solutions is established by combining a priori estimates and the local existence result.

2. Reformulation and preliminaries

In this section, we are going to reformulate the initial-boundary value problem (1.1)-(1.2). As [11,27,32] like to point out, the main difficulty for treating the initial boundary value problem of (1.1) lies in the lack of damping mechanism on F. To overcome this difficulty, inspired by [27,36], we introduce the following quantities:

$$G := \frac{\partial X^{-1}(t,x)}{\partial x} = F^{-1}, \quad E := G - \mathrm{I}.$$

In light of Equation $(1.1)_3$, we deduce that

$$\partial_t G + \mathbf{u}\cdot\nabla G + G\nabla\mathbf{u} = 0.$$

An important observation here is that the condition

$$\nabla_i G_0^{jk} = \nabla_k G_0^{ji} \quad \forall\ i,\ j = 1,\ 2,\ 3$$

is preserved by the flow for matrix-valued function G; which means that the matrix G is curl-free if the initial value matrix G_0 is curl-free. This property is lost if we choose to deal with F. Since the matrix G is curl-free, there exists a physically reasonable vector function $\psi = (\psi^1, \psi^2, \psi^3)$, which can actually be chosen as $\psi = X^{-1}(t,x) - x$, such that $G^{ij} = \nabla_j \psi^i$. One can easily verify that the function ψ satisfies

$$\partial_t \psi + \mathbf{u} = -\mathbf{u}\cdot\nabla\psi, \tag{2.1}$$

and $\psi|_{\partial\Omega} = 0$ since $\mathbf{u} = 0$ on $\partial\Omega$, the readers can refer to [27] for detailed discussions on ψ, which is originally due to Sideris and Thomas [36] in investigating the incompressible limit of the isotropic elastodynamics. Under the aforementioned assumptions, we have

$$F = (\nabla\psi + \mathrm{I})^{-1},$$

that is why we need the compatibility condition (1.5). It is easy to see that

$$F = \mathrm{I} - G + O(|G|^2) = \mathrm{I} - \nabla\psi + \mathcal{O}(|\nabla\psi|^2). \tag{2.2}$$

It is standard that the condition (1.4) is preserved by the flow (see [11, 32]), that is $\mathrm{div}(\rho F^T) = 0$ for any $t \geq 0$, thus we have

$$\begin{aligned}
\nabla_j(\rho F^{ji}) &= \nabla_j[\rho(\delta^{ji} - \nabla_i\psi^j + \mathcal{O}(|\nabla\psi|^2))] \\
&= \nabla_i\rho - \rho\nabla_i \mathrm{div}\psi - \nabla_j\rho\nabla_i\psi^j + \mathcal{O}(|\nabla\psi|)\nabla\mathcal{O}(|\rho\nabla\psi|) \\
&= 0,
\end{aligned}$$

which means

$$\nabla_i\rho=\rho\nabla_i\mathrm{div}\psi+\nabla_j\rho\nabla_i\psi^j+\mathcal{O}(|\nabla\psi|)\nabla\mathcal{O}(|\rho\nabla\psi|). \tag{2.3}$$

This ensures that the i-th component of the vector $\mathrm{div}(\rho FF^T)$ is

$$\begin{aligned}\nabla_j(\rho F^{ik}F^{jk})&=\nabla_j[\rho(\delta^{ik}-\nabla_k\psi^i+O(|\nabla\psi|^2))(\delta^{jk}-\nabla_k\psi^j+O(|\nabla\psi|^2))]\\&=\nabla_i\rho-\rho\Delta\psi^i-\nabla_j\rho\nabla_j\psi^i-\rho\nabla_i\mathrm{div}\psi-\nabla_j\rho\nabla_i\psi^j+\mathcal{O}(|\nabla\psi|)\nabla\mathcal{O}(|\rho\nabla\psi|)\\&=-\rho\Delta\psi^i-\nabla_j\rho\nabla_j\psi^i+\mathcal{O}(|\nabla\psi|)\nabla\mathcal{O}(|\rho\nabla\psi|).\end{aligned}$$

Under the above analysis, the system (1.1), in terms of the variables ρ, $\mathbf{u}$, ψ, can be rewritten as:

$$\begin{cases}\partial_t\rho+\mathrm{div}(\rho\mathbf{u})=0,\\ \rho\partial_t\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}+\nabla P=\mu\Delta\mathbf{u}+(\mu+\lambda)\nabla\mathrm{div}\mathbf{u}-a\rho\Delta\psi-a\nabla\rho\nabla\psi+\mathcal{O}(|\nabla\psi|)\nabla\mathcal{O}(|\rho\nabla\psi|),\\ \partial_t\psi+\mathbf{u}+\mathbf{u}\cdot\nabla\psi=0.\end{cases} \tag{2.4}$$

Without loss of generality, we assume that $\bar\rho=a/P'(\bar\rho)=1$. Set $\pi=(P'(1))^{-\frac{1}{2}}$ and take change of variables by

$$n(t,x)=\rho(\pi^2t,\pi x)-1,\quad \mathbf{v}(t,x)=\pi\mathbf{u}(\pi^2t,\pi x),\quad \phi(t,x)=\pi\psi(\pi^2t,\pi x),$$

we get from the system (2.4) that

$$\begin{cases}n_t+\mathrm{div}\mathbf{v}=-n\mathrm{div}\mathbf{v}-\mathbf{v}\nabla n,\\ \mathbf{v}_t-\mu\Delta\mathbf{v}-(\lambda+\mu)\nabla\mathrm{div}\mathbf{v}+\nabla n+\Delta\phi=-\frac{\nabla n\nabla\phi}{n+1}-\frac{n}{1+n}(\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v})\\ \qquad\qquad -\mathbf{v}\cdot\nabla\mathbf{v}-\left(\frac{P'(n+1)}{(1+n)P'(1)}-1\right)\nabla n+\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|),\\ \phi_t+\pi^2\mathbf{v}=-\mathbf{v}\cdot\nabla\phi.\end{cases} \tag{2.5}$$

The initial and boundary conditions become

$$\begin{cases}(n,\mathbf{v},\phi)(x,0)=(n_0,\mathbf{v}_0,\phi_0)(x),\\ (\mathbf{v},\phi)|_{\partial\Omega}=0.\end{cases} \tag{2.6}$$

By a standard continuity argument, the global existence of strong solutions will be proved by combining the local existence result with a priori estimates. We first state out the following local existence without proof, which can be established by using the same idea in [18].

Proposition 2.1. *Let $(n_0,\mathbf{v}_0)\in H^2(\Omega)$ and $\phi_0\in H^3(\Omega)$ be such that*

$$\inf_{x\in\bar\Omega}\{\rho_0(x)\}>0 \quad \text{and} \quad \partial_t^\ell(\mathbf{v}_0,\phi_0)|_{\partial\Omega}=0,\ell=0,1.$$

Then there exist two positive numbers T and C such that the system (2.5)-(2.6) has a unique solution $(n,\mathbf{v})\in C([0,T];H^2(\Omega))$ and $\phi\in C([0,T];H^3(\Omega))$. Moreover, the solution satisfies

$$\inf_{t\in[0,T],x\in\bar\Omega}\{\rho(t,x)\}>0, n_t\in C([0,T];H^1(\Omega)),\phi_t\in C([0,T];H^2(\Omega)),$$

$$\mathbf{v}\in L^2([0,T];H^3(\Omega)),\mathbf{v}_t\in C([0,T];L^2(\Omega))$$

and

$$\|(n,\mathbf{v},\nabla\phi)(t)\|_2 \le C\|(n_0,\mathbf{v}_0,\nabla\phi_0)\|_2.$$

Next, we establish global a priori estimates for strong solution.

PROPOSITION 2.2. *Let $(n_0,\mathbf{v}_0)\in H^2(\Omega)$ and $\phi_0\in H^3(\Omega)$ be such that*

$$\inf_{x\in\bar{\Omega}}\{\rho_0(x)\}>0 \quad and \quad \partial_t^\ell(\mathbf{v}_0,\phi_0)|_{\partial\Omega}=0, \ell=0,1.$$

Suppose that the system (2.5)-(2.6) *has a solution $(n,\mathbf{v})\in C([0,T];H^2(\Omega))$ and $\phi\in C([0,T];H^3(\Omega))$ for given $T>0$. Then there exists a small positive ε such that if*

$$\sup_{0\le t\le T}\|(n,\mathbf{v},\nabla\phi)(t)\|_2 \le \varepsilon \ll 1, \tag{2.7}$$

then for any $t\in[0,T]$, it holds that

$$\|(n,\mathbf{v},\nabla\phi)(t)\|_2+\|\partial_t(n,\mathbf{v},\nabla\phi)(t)\| \le C_1\|(n_0,\mathbf{v}_0,\nabla\phi_0)\|_2\exp\{-\eta_1 t\}, \tag{2.8}$$

where C_1 and η_1 are two positive constants, which are independent of t.

Proof. (**Proof of Theorem 1.1.**) The global existence of strong solution and large time behavior for the system (1.1)-(1.2) in Theorem 1.1 follows from Proposition 2.1 and Proposition 2.2 by standard continuity argument. ∎

3. Global existence and large-time behavior

In this section, we are devoted to prove Proposition 2.2. First, we list some elementary but useful inequalities of Sobolev type (see [4]).

LEMMA 3.1. *Let Ω be any bounded smooth domain in $\mathbb{R}^3$. Then it holds that*

$$\begin{aligned}
&(i) \quad \|f\|_{L^\infty} \le C\|f\|_2,\\
&(ii) \quad \|f\|_{L^p} \le C\|f\|_1, \quad 2\le p\le 6,\\
&(iii) \quad \|f\|_{L^6} \le C\|\nabla f\|, \quad for\ f\in H_0^1(\Omega),
\end{aligned}$$

where the positive constant C depending only on Ω.

Due to the boundary condition, the classical energy estimates can not be applied directly to spatial derivatives since the spatial derivatives are unknown on the boundary. In order to establish the estimates on the tangential derivatives of $(n,\mathbf{v},\phi)$, we recall the following lemma on the stationary Stokes equations (see [31]).

LEMMA 3.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$. If $f\in H^{k+1}(\Omega)$ and $g\in H^k(\Omega)$ $(k\ge 0)$, then the problem*

$$\begin{cases}
-\mu\Delta\mathbf{u}+\nabla p=g,\\
div\mathbf{u}=f,\\
\mathbf{u}|_{\partial\Omega}=0,
\end{cases}$$

has a solution $(p,\mathbf{u})\in H^{k+1}\times H^{k+2}\cap H_0^1$ which is unique modulo a constant of integration for p. Moreover, it holds that

$$\|\mathbf{u}\|_{k+2}^2+\|\nabla p\|_k^2 \le C(\|f\|_{k+1}^2+\|g\|_k^2). \tag{3.1}$$

We suppose that the a priori estimate (2.7) holds throughout this section, then the proof of Proposition 2.2 follows by several steps of delicate energy estimates which are stated as a sequence of lemmas. First of all, the energy estimate of lower order for $(n,\mathbf{v},\phi)$ is established in the following lemma.

LEMMA 3.3. *Under the conditions of Proposition 2.2, it holds:*

$$\frac{1}{2}\frac{d}{dt}\int |n(t)|^2+|\mathbf{v}(t)|^2+|\frac{\nabla\phi(t)}{\pi}|^2dx+\mu\int|\nabla\mathbf{v}(t)|^2dx+(\lambda+\mu)\int|div\mathbf{v}(t)|^2dx \lesssim \varepsilon\|(\nabla n,\nabla\mathbf{v},\nabla^2\phi)(t)\|^2. \tag{3.2}$$

Proof. Multiplying $(2.5)_1$, $(2.5)_2$ and $(2.5)_3$ by n, $\mathbf{v}$ and $\frac{-\Delta\phi}{\pi^2}$ respectively and then integrating over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |n|^2+|\mathbf{v}|^2+|\frac{\nabla\phi}{\pi}|^2dx+\mu\int|\nabla\mathbf{v}|^2dx+(\lambda+\mu)\int|\mathrm{div}\mathbf{v}|^2dx\\
=&\langle -n\mathrm{div}\mathbf{v}-\mathbf{v}\nabla n,n\rangle+\langle -\frac{\nabla n\nabla\phi}{n+1}-\left(\frac{P'(n+1)}{(1+n)P'(1)}-1\right)\nabla n,\mathbf{v}\rangle\\
&+\langle\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|),\mathbf{v}\rangle+\langle -\frac{n}{1+n}(\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v})-\mathbf{v}\cdot\nabla\mathbf{v},\mathbf{v}\rangle\\
&+\langle\mathbf{v}\cdot\nabla\phi,\frac{\Delta\phi}{\pi^2}\rangle.
\end{aligned}\tag{3.3}$$

By (2.7), Lemma 3.1, Hölder's inequality and Poincaré's inequality, the five terms on the right hand side of the above equation can be estimated as the following:

$$|\langle -n\mathrm{div}\mathbf{v}-\mathbf{v}\nabla n,n\rangle|\lesssim|\langle n\mathbf{v},\nabla n\rangle|\lesssim\|n\|_{L^3}\|\mathbf{v}\|_{L^6}\|\nabla n\|\lesssim\varepsilon\|\nabla(n,\mathbf{v})\|^2,$$

$$\begin{aligned}
&\left|\langle -\frac{\nabla n\nabla\phi}{n+1}-\left(\frac{P'(n+1)}{(1+n)P'(1)}-1\right)\nabla n,\mathbf{v}\rangle\right|\\
\lesssim&\|\nabla\phi\|_{L^3}\|\mathbf{v}\|_{L^6}\|\nabla n\|+\|n\|_{L^3}\|\mathbf{v}\|_{L^6}\|\nabla n\|\\
\lesssim&\varepsilon\|\nabla(n,\mathbf{v})\|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\langle\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|),\mathbf{v}\rangle\right|\\
\lesssim&\|\nabla\phi\|_{L^6}^2\|\mathbf{v}\|_{L^6}\|\nabla n\|+\|\nabla\phi\|_{L^3}\|\nabla^2\phi\|\|\mathbf{v}\|_{L^6}\\
\lesssim&\varepsilon\|\nabla(n,\mathbf{v},\nabla\phi)\|^2,
\end{aligned}$$

where we have used the fact

$$\frac{P'(n+1)}{(1+n)P'(1)}-1\sim\mathcal{O}(1)n,$$

if ε is sufficiently small. For the fourth term, we have

$$\begin{aligned}
&\left|-\tfrac{n}{1+n}(\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v})-\mathbf{v}\cdot\nabla\mathbf{v},\mathbf{v}\rangle\right|\\
\lesssim&\ \|\nabla n\|\|\nabla\mathbf{v}\|\|\mathbf{v}\|_{L^\infty}+\|n\|_{L^\infty}\|\nabla\mathbf{v}\|^2+\|\mathbf{v}\|_{L^3}\|\mathbf{v}\|_{L^6}\|\nabla\mathbf{v}\|\\
\lesssim&\ \varepsilon\|\nabla(n,\mathbf{v})\|^2.
\end{aligned}$$

Taking the same argument to deal with the last term, we get

$$\big|\langle \mathbf{v}\cdot\nabla\phi, \frac{\Delta\phi}{\pi^2}\rangle\big| \lesssim \|\mathbf{v}\|_{L^6}\|\nabla\phi\|_{L^3}\|\Delta\phi\| \lesssim \varepsilon\|(\nabla\mathbf{v},\nabla^2\phi)\|^2.$$

Putting the above estimates into (3.3), we arrive at (3.2). The proof of lemma is completed. ∎

Our next goal is to derive the energy estimate of the time derivative for $(n,\mathbf{v},\nabla\phi)$.

LEMMA 3.4. *Under the conditions of Proposition 2.2, it holds:*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int |n_t(t)|^2+|\mathbf{v}_t(t)|^2+|\frac{\nabla\phi_t(t)}{\pi}|^2dx+\mu\int|\nabla\mathbf{v}_t(t)|^2dx+(\lambda+\mu)\int|div\mathbf{v}_t(t)|^2dx\\ &\lesssim\varepsilon(\|\nabla\mathbf{v}(t)\|_1^2+\|\nabla\mathbf{v}_t(t)\|^2).\end{aligned} \tag{3.4}$$

Proof. Noticing that $\mathbf{v}_t,\phi_t\big|_{\partial\Omega}=0$, then, by differentiating $(2.5)_1$, $(2.5)_2$ and $(2.5)_3$ with respect to t, multiplying them by n_t, $\mathbf{v}_t$ and $\frac{-\Delta\phi_t}{\pi^2}$ respectively and integrating over Ω, we finally arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int |n_t|^2+|\mathbf{v}_t|^2+|\frac{\nabla\phi_t}{\pi}|^2dx+\mu\int|\nabla\mathbf{v}_t|^2dx+(\lambda+\mu)\int|\mathrm{div}\mathbf{v}_t|^2dx\\ &=\langle-\mathrm{div}(n\mathbf{v})_t,n_t\rangle+\langle\Big[-\frac{\nabla n\nabla\phi}{n+1}-\Big(\frac{P'(n+1)}{(1+n)P'(1)}-1\Big)\nabla n\Big]_t,\mathbf{v}_t\rangle\\ &\quad+\langle[O(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|)]_t,\mathbf{v}_t\rangle+\langle[-\frac{n}{1+n}(\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v})-\mathbf{v}\cdot\nabla\mathbf{v}]_t,\mathbf{v}_t\rangle\\ &\quad+\langle(\mathbf{v}\cdot\nabla\phi)_t,\frac{\Delta\phi_t}{\pi^2}\rangle.\end{aligned} \tag{3.5}$$

It follows from (2.7), Lemma 3.1, Hölder's inequality and Poincaré's inequality that

$$\begin{aligned}|\langle\mathrm{div}(n\mathbf{v})_t,n_t\rangle| &\lesssim|\langle n_t\mathrm{div}\mathbf{v}+n\mathrm{div}\mathbf{v}_t+\nabla n\mathbf{v}_t+\nabla n_t\mathbf{v},n_t\rangle|\\ &\lesssim\|n_t\|_{L^3}^2\|\nabla\mathbf{v}\|_{L^3}+\|n\|_{L^6}\|\mathrm{div}\mathbf{v}_t\|\|n_t\|_{L^3}+\|\nabla n\|_{L^3}\|\mathbf{v}_t\|_{L^3}\|n_t\|_{L^3}\\ &\lesssim\varepsilon(\|\nabla\mathbf{v}\|_1^2+\|\nabla\mathbf{v}_t\|^2),\end{aligned} \tag{3.6}$$

where by $(2.5)_1$ and (2.7), we have used the following estimates:

$$\|n_t\|_{L^3}\lesssim\|\mathrm{div}\mathbf{v}\|_{L^3}+\|\mathbf{v}\nabla n\|_{L^3}\lesssim\|\mathrm{div}\mathbf{v}\|_{L^3}+\|\mathbf{v}\|_{L^6}\|\nabla n\|_{L^6}\lesssim\|\nabla\mathbf{v}\|_{L^3}. \tag{3.7}$$

Similarly,

$$\begin{aligned}&\Big|\langle\Big[-\frac{\nabla n\nabla\phi}{n+1}-\Big(\frac{P'(n+1)}{(1+n)P'(1)}-1\Big)\nabla n\Big]_t,\mathbf{v}_t\rangle\Big|+\big|\langle[O(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|)]_t,\mathbf{v}_t\rangle\big|\\ &\lesssim\big|\langle\frac{\nabla n_t\nabla\phi}{n+1},\mathbf{v}_t\rangle\big|+\|\nabla n\|_{L^3}\|\nabla\phi_t\|_{L^3}\|\mathbf{v}_t\|_{L^3}+\|\nabla n\|_{L^4}\|\nabla\phi\|_{L^4}\|n_t\|_{L^3}\|\mathbf{v}_t\|_{L^6}\\ &\quad+\Big|\langle\Big(\frac{P'(n+1)}{(1+n)P'(1)}-1\Big)\nabla n_t,\mathbf{v}_t\rangle\Big|+\|n_t\|_{L^3}\|\nabla n\|\|\mathbf{v}_t\|_{L^6}+\|\nabla\phi_t\|\|\nabla^2\phi\|_{L^3}\|\mathbf{v}_t\|_{L^6}\\ &\quad+\|\nabla\phi_t\|\|\nabla n\|_{L^6}\|\nabla\phi\|_{L^6}\|\mathbf{v}_t\|_{L^6}+\big|\langle O(|\nabla\phi|)\nabla[\mathcal{O}(|(n+1)\nabla\phi|)]_t,\mathbf{v}_t\rangle\big|\\ &\lesssim\|n_t\|_{L^3}\|\nabla\phi\|_{L^6}\|\nabla\mathbf{v}_t\|+\|n_t\|_{L^3}\|\nabla^2\phi\|\|\mathbf{v}_t\|_{L^6}+\|n_t\|_{L^3}\|\nabla n\|_{L^4}\|\nabla\phi\|_{L^4}\|\mathbf{v}_t\|_{L^6}\\ &\quad+\|\nabla n\|_{L^6}\|\nabla\phi_t\|\|\mathbf{v}_t\|_{L^3}+\|\nabla n\|_{L^4}\|\nabla\phi\|_{L^4}\|n_t\|_{L^3}\|\mathbf{v}_t\|_{L^6}+\|n\|_{L^6}\|n_t\|_{L^3}\|\nabla\mathbf{v}_t\|\end{aligned}$$

$$
\begin{aligned}
&+\|n_t\|_{L^3}\|\nabla n\|\|\mathbf{v}_t\|_{L^6}+\|\nabla\phi_t\|\|\nabla^2\phi\|_{L^3}\|\mathbf{v}_t\|_{L^6}+\|\nabla\phi_t\|\|\nabla n\|_{L^6}\|\nabla\phi\|_{L^6}\|\mathbf{v}_t\|_{L^6}\\
&+\|\nabla\phi\|_{L^\infty}^2\|n_t\|\|\nabla\mathbf{v}_t\|+\|\nabla\phi\|_{L^\infty}\|\nabla\phi_t\|\|\nabla\mathbf{v}_t\|+\|\nabla^2\phi\|_{L^3}\|n_t\|\|\nabla\phi\|_{L^\infty}\|\mathbf{v}_t\|_{L^6}\\
\lesssim&\,\varepsilon(\|\nabla\mathbf{v}\|_1^2+\|\nabla\mathbf{v}_t\|^2),
\end{aligned}\tag{3.8}
$$

where by $(2.5)_3$ and (2.7), we have used the fact

$$
\|\nabla^\ell\phi_t\|\lesssim\|\nabla^\ell\mathbf{v}\|+\|\nabla^\ell(\mathbf{v}\cdot\nabla\phi)\|\lesssim\|\nabla^\ell\mathbf{v}\|+\|\nabla\mathbf{v}\|_{\ell-1}\|\nabla\phi\|_\ell,\ \text{for } \ell=1,2.\tag{3.9}
$$

For the fourth term, we have

$$
\begin{aligned}
&\Big|\langle[-\frac{n}{1+n}(\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v})-\mathbf{v}\cdot\nabla\mathbf{v}]_t,\mathbf{v}_t\rangle\Big|\\
\lesssim&\|n_t\|_{L^3}\|\nabla^2\mathbf{v}\|\|\mathbf{v}_t\|_{L^6}+\|\nabla n\|_{L^3}\|\nabla\mathbf{v}_t\|\|\mathbf{v}_t\|_{L^6}\\
&+\|n\|_{L^\infty}\|\nabla\mathbf{v}_t\|^2+\|n\|_{L^\infty}\|\nabla\mathbf{v}\|^2+\|\nabla\mathbf{v}\|_{L^3}\|\mathbf{v}_t\|_{L^3}^2+\|\nabla\mathbf{v}\|_{L^3}\|\nabla\mathbf{v}_t\|\|\mathbf{v}_t\|_{L^6}\\
\lesssim&\,\varepsilon(\|\nabla\mathbf{v}\|_1^2+\|\nabla\mathbf{v}_t\|^2).
\end{aligned}\tag{3.10}
$$

From (3.9), the last term can be estimated as the following:

$$
\big|\langle(\mathbf{v}\cdot\nabla\phi)_t,\frac{\Delta\phi_t}{\pi^2}\rangle\big|\lesssim\|\mathbf{v}_t\|_{L^6}\|\nabla\phi\|_{L^3}\|\Delta\phi_t\|+\|\mathbf{v}\|_{L^\infty}\|\nabla\phi_t\|\|\Delta\phi_t\|\lesssim\varepsilon(\|\nabla\mathbf{v}\|_1^2+\|\nabla\mathbf{v}_t\|^2).\tag{3.11}
$$

Substituting (3.6), (3.8), (3.10) and (3.11) into (3.5) gives (3.4). The proof of the lemma is completed. ∎

We derive the dissipation on ϕ in the following lemma.

Lemma 3.5. *Under the conditions of Proposition 2.2, it holds:*

$$
\int|\nabla\phi(t)|^2dx\lesssim\|(\nabla\mathbf{v}_t,\nabla\mathbf{v})(t)\|^2+\varepsilon\|\nabla n(t)\|^2,\tag{3.12}
$$

$$
\frac{d}{dt}\int|\Delta\phi(t)|^2dx+\int|\Delta\phi(t)|^2dx\lesssim\varepsilon\|\nabla\mathbf{v}(t)\|_1^2+\|\Delta(\mathbf{v}-\frac{\phi}{\mu})(t)\|,\tag{3.13}
$$

$$
\frac{d}{dt}\int|\nabla\Delta\phi(t)|^2dx+\int|\nabla\Delta\phi(t)|^2dx\lesssim\varepsilon\|\nabla\mathbf{v}(t)\|_2^2+\|\nabla\Delta(\mathbf{v}-\frac{\phi}{\mu})(t)\|,\tag{3.14}
$$

for some positive constant C.

Proof. From (2.3), we have

$$
\pi^2\nabla n=\nabla\mathrm{div}\phi+n\nabla\mathrm{div}\phi+\nabla n\cdot\nabla\phi+\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|).\tag{3.15}
$$

Multiplying $(2.5)_2$ by ϕ, integrating over Ω and using (3.15), we finally arrive at

$$
\begin{aligned}
&\int|\nabla\phi|^2+\frac{|\mathrm{div}\phi|^2}{\pi^2}dx\\
=&\langle\mathbf{v}_t+\frac{\nabla n\nabla\phi}{n+1}-\frac{1}{1+n}(\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v})+\mathbf{v}\cdot\nabla\mathbf{v}+\left(\frac{P'(n+1)}{(1+n)P'(1)}-1\right)\nabla n\\
&+n\nabla\mathrm{div}\phi+\nabla n\cdot\nabla\phi+\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|),\phi\rangle\\
\lesssim&\|\mathbf{v}_t\|\|\phi\|+\|\phi\|_{L^\infty}\|\nabla n\|\|\nabla\phi\|+\|\phi\|_{L^\infty}\|\nabla n\|\|\nabla\mathbf{v}\|+\|\nabla\mathbf{v}\|\|\nabla\phi\|\\
&+\|n\|_{L^\infty}\|\nabla n\|\|\phi\|+\|n\|_{L^\infty}\|\nabla\phi\|^2+\|\phi\|_{L^\infty}\|\nabla n\|\|\nabla\phi\|+\|\phi\|_{L^\infty}\|\nabla\phi\|^2\\
\lesssim&\|(\nabla\mathbf{v}_t,\nabla\mathbf{v})\|\|\nabla\phi\|+\varepsilon\|(\nabla n,\nabla\phi)\|^2.
\end{aligned}
$$

Thus by Cauchy-Schwarz inequality we obtain (3.12).

Applying Δ to $(2.5)_3$, then multiplying it by $\Delta\phi$ and integrating over Ω, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\Delta\phi|^2dx+\frac{\pi^2}{\mu}\int|\Delta\phi|^2dx\\=&-\langle\pi^2\Delta(\mathbf{v}-\frac{\phi}{\mu}),\Delta\phi\rangle-\langle\Delta(\mathbf{v}\cdot\nabla\phi),\Delta\phi\rangle\\\lesssim&\|\Delta(\mathbf{v}-\frac{\phi}{\mu})\|\|\Delta\phi\|+\|\nabla\phi\|_{L^\infty}\|\nabla^2\mathbf{v}\|\|\Delta\phi\|+\|\nabla\mathbf{v}\|_{L^6}\|\nabla^2\phi\|_{L^3}\|\Delta\phi\|.\end{aligned}$$

Then using Cauchy-Schwarz inequality, Lemma 3.1 and (2.7), we easily get (3.13).

Similarly, applying $\nabla\Delta$ to $(2.5)_3$, then multiplying it by $\nabla\Delta\phi$ and integrating over Ω, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\nabla\Delta\phi|^2dx+\frac{\pi^2}{\mu}\int|\nabla\Delta\phi|^2dx\\=&-\langle\pi^2\nabla\Delta(\mathbf{v}-\frac{\phi}{\mu}),\nabla\Delta\phi\rangle-\langle\nabla\Delta(\mathbf{v}\cdot\nabla\phi),\nabla\Delta\phi\rangle\\\lesssim&\|\nabla\Delta(\mathbf{v}-\frac{\phi}{\mu})\|\|\nabla\Delta\phi\|+\|\nabla\phi\|_{L^\infty}\|\nabla^3\mathbf{v}\|\|\nabla\Delta\phi\|\\&+\|\nabla^2\mathbf{v}\|_{L^4}\|\nabla^2\phi\|_{L^4}\|\nabla\Delta\phi\|+\|\nabla\mathbf{v}\|_{L^\infty}\|\nabla\Delta\phi\|^2.\end{aligned}$$

Combining the above inequality, Cauchy-Schwarz inequality, Lemma 3.1 and (2.7) yields (3.14). The proof of lemma is completed. ∎

To derive the energy estimate of the spatial derivatives for $(n,\mathbf{v},\phi)$, we can not apply the classical energy estimates directly since the spatial derivatives are unknown on the boundary. To overcome these difficulties, we employ the standard technique in [31] that involves separating the estimates of the solution into that over the region away from the boundary and near the boundary. Precisely, let χ_0 be an arbitrary but fixed function in $C_0^\infty(\Omega)$, then we have the following lemma for the estimate on the region away from the boundary.

Lemma 3.6. *Under the conditions of Proposition 2.2, it holds:*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\nabla n(t)\chi_0|^2+|\nabla\mathbf{v}(t)\chi_0|^2+|\frac{\nabla^2\phi(t)\chi_0}{\pi}|^2dx\\&+\mu\int|\nabla^2\mathbf{v}(t)\chi_0|^2dx+(\lambda+\mu)\int|\nabla\, div\mathbf{v}(t)\chi_0|^2dx\\\lesssim&\varepsilon(\|\nabla\mathbf{v}(t)\|_1^2+\|\nabla n(t)\|^2+\|\nabla^2\phi(t)\|_1^2)+\|\nabla\mathbf{v}(t)\|\|(\nabla n,\nabla^2\mathbf{v},\nabla^2\phi)(t)\|.\end{aligned}\tag{3.16}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\nabla^2 n(t)\chi_0|^2+|\nabla^2\mathbf{v}(t)\chi_0|^2+|\frac{\nabla^3\phi(t)\chi_0}{\pi}|^2dx\\&+\mu\int|\nabla^3\mathbf{v}(t)\chi_0|^2dx+(\lambda+\mu)\int|\nabla^2\, div\mathbf{v}(t)\chi_0|^2dx\\\lesssim&\varepsilon(\|\nabla\mathbf{v}(t)\|_2^2+\|\nabla\mathbf{v}_t(t)\|^2+\|\nabla n(t)\|_1^2+\|\nabla^2\phi(t)\|_1^2)+\|\nabla^2\mathbf{v}(t)\|\|(\nabla^3\mathbf{v},\nabla^2 n,\nabla^3\phi)(t)\|,\end{aligned}\tag{3.17}$$

$$\int|\nabla^2\phi(t)\chi_0|^2dx\lesssim\varepsilon\|\nabla n(t)\|^2+\|(\nabla\mathbf{v}_t,\nabla\mathbf{v},\nabla^2\mathbf{v}\chi_0,\nabla\phi)(t)\|^2,\tag{3.18}$$

and

$$\int |\nabla^3\phi(t)\chi_0|^2 dx \lesssim \varepsilon\|\nabla n(t)\|_1^2 + \|\nabla^2\phi(t)\|^2 + \|\nabla \mathbf{v}_t(t)\|^2 + \|\nabla \mathbf{v}(t)\|_1^2 + \|\nabla^3 \mathbf{v}(t)\chi_0\|^2, \tag{3.19}$$

for any $t \geq 0$.

Proof. Taking the same argument in Lemma 3.4 and 3.5, we will only sketch the outline and omit the detailed calculations for simplicity. Applying ∇ to $(2.5)_1$, $(2.5)_2$ and $(2.5)_3$, then multiplying the resulting equations by $\nabla n\chi_0^2$, $\nabla \mathbf{v}\chi_0^2$ and $-\frac{\nabla\Delta\phi\chi_0^2}{\pi^2}$ respectively, and integrating over Ω, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla n\chi_0|^2 + |\nabla \mathbf{v}\chi_0|^2 + |\frac{\nabla^2\phi\chi_0}{\pi}|^2 dx + \mu\int |\nabla^2\mathbf{v}\chi_0|^2 dx + (\lambda+\mu)\int |\nabla \mathrm{div}\mathbf{v}\chi_0|^2 dx\\
=& -\mu\langle \nabla^2\mathbf{v}, 2\nabla\mathbf{v}\chi_0\nabla\chi_0\rangle - (\lambda+\mu)\langle \nabla\mathrm{div}\mathbf{v}, 2\nabla\mathbf{v}\chi_0\nabla\chi_0\rangle + \langle \nabla n, 2\nabla\mathbf{v}\chi_0\nabla\chi_0\rangle\\
&-\langle \nabla\phi_t, \frac{2\nabla^2\phi\chi_0\nabla\chi_0}{\pi^2}\rangle - \langle \nabla\mathrm{div}(n\mathbf{v}), \nabla n\chi_0^2\rangle\\
&+\langle \nabla\Big[-\frac{\nabla n\nabla\phi}{n+1} - \Big(\frac{P'(n+1)}{(1+n)P'(1)} - 1\Big)\nabla n\Big], \nabla\mathbf{v}\chi_0^2\rangle\\
&+\langle \nabla[\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|)], \nabla\mathbf{v}\chi_0^2\rangle + \langle \nabla(\mathbf{v}\cdot\nabla\phi), \frac{\nabla\Delta\phi\chi_0^2}{\pi^2}\rangle\\
&+\langle \nabla[-\frac{n}{1+n}(\mu\Delta\mathbf{v} + (\lambda+\mu)\nabla\mathrm{div}\mathbf{v}) - \mathbf{v}\cdot\nabla\mathbf{v}], \nabla\mathbf{v}\chi_0^2\rangle\\
\lesssim& \|\nabla\mathbf{v}\|\|\nabla^2\mathbf{v}\| + \|\nabla n\|\|\nabla\mathbf{v}\| + \|\nabla^2\phi\|\|\nabla\mathbf{v}\| + \varepsilon(\|\nabla\mathbf{v}\|_1^2 + \|\nabla n\|^2 + \|\nabla^2\phi\|_1^2),
\end{aligned}$$

which gives (3.16).

Similarly, applying ∇^2 to $(2.5)_1$, $(2.5)_2$ and $(2.5)_3$, then multiplying the resulting equations by $\nabla^2 n\chi_0^2$, $\nabla^2\mathbf{v}\chi_0^2$ and $-\frac{\nabla^2\Delta\phi\chi_0^2}{\pi^2}$ respectively, and integrating over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla^2 n\chi_0|^2 + |\nabla^2 \mathbf{v}\chi_0|^2 + |\frac{\nabla^3\phi\chi_0}{\pi}|^2 dx + \mu\int |\nabla^3\mathbf{v}\chi_0|^2 dx + (\lambda+\mu)\int |\nabla^2 \mathrm{div}\mathbf{v}\chi_0|^2 dx\\
=& -\mu\langle \nabla^3\mathbf{v}, 2\nabla^2\mathbf{v}\chi_0\nabla\chi_0\rangle - (\lambda+\mu)\langle \nabla^2\mathrm{div}\mathbf{v}, 2\nabla^2\mathbf{v}\chi_0\nabla\chi_0\rangle + \langle \nabla^2 n, 2\nabla^2\mathbf{v}\chi_0\nabla\chi_0\rangle\\
&-\langle \nabla^2\phi_t, \frac{2\nabla^3\phi\chi_0\nabla\chi_0}{\pi^2}\rangle - \langle \nabla^2\mathrm{div}(n\mathbf{v}), \nabla^2 n\chi_0^2\rangle\\
&+\langle \nabla^2\Big[-\frac{\nabla n\nabla\phi}{n+1} - \Big(\frac{P'(n+1)}{(1+n)P'(1)} - 1\Big)\nabla n\Big], \nabla^2\mathbf{v}\chi_0^2\rangle\\
&+\langle \nabla^2[\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|)], \nabla^2\mathbf{v}\chi_0^2\rangle + \langle \nabla^2(\mathbf{v}\cdot\nabla\phi), \frac{\nabla^2\Delta\phi\chi_0^2}{\pi^2}\rangle\\
&+\langle \nabla^2[-\frac{n}{1+n}(\mu\Delta\mathbf{v} + (\lambda+\mu)\nabla\mathrm{div}\mathbf{v}) - \mathbf{v}\cdot\nabla\mathbf{v}], \nabla^2\mathbf{v}\chi_0^2\rangle\\
\lesssim& \varepsilon(\|\nabla\mathbf{v}\|_2^2 + \|\nabla\mathbf{v}_t\|^2 + \|\nabla n\|_1^2 + \|\nabla^2\phi\|_1^2) + \|\nabla^2\mathbf{v}\|\|(\nabla^3\mathbf{v}, \nabla^2 n, \nabla^3\phi)\|,
\end{aligned}$$

which is (3.17).

By (3.15), applying ∇ to $(2.5)_2$, then multiplying the resulting equation by $\nabla\phi\chi_0^2$ and integrating over Ω, we finally arrive at

$$\begin{aligned}
&\int |\nabla^2\phi\chi_0|^2 + \frac{|\nabla\mathrm{div}\phi\chi_0|^2}{\pi^2} dx\\
=& -\langle \nabla^2\phi\nabla\phi, 2\chi_0\nabla\chi_0\rangle - \langle \nabla\mathrm{div}\phi\nabla\phi, 2\chi_0\nabla\chi_0\rangle + \langle \nabla\big[\mathbf{v}_t + \frac{\nabla n\nabla\phi}{n+1}
\end{aligned}$$

$$-\frac{1}{1+n}(\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v})+\mathbf{v}\cdot\nabla\mathbf{v}+\left(\frac{P'(n+1)}{(1+n)P'(1)}-1\right)\nabla n$$
$$+n\nabla\mathrm{div}\phi+\nabla n\cdot\nabla\phi+\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|)\big],\nabla\phi\chi_0^2\rangle$$
$$\lesssim\|\nabla\phi\|\|\nabla^2\phi\|+\|(\nabla\mathbf{v}_t,\nabla\mathbf{v},\nabla^2\mathbf{v}\chi_0)\|\|(\nabla\phi,\nabla^2\phi)\|+\varepsilon\|(\nabla n,\nabla\phi,\nabla^2\phi)\|^2.$$

Thus by Cauchy-Schwarz inequality we obtain (3.18).

Similarly, by (3.15), applying ∇^2 to $(2.5)_2$, then multiplying the resulting equation by $\nabla^2\phi\chi_0^2$ and integrating over Ω, we have

$$\int|\nabla^3\phi\chi_0|^2+\frac{|\nabla^2\mathrm{div}\phi\chi_0|^2}{\pi^2}dx$$
$$=-\langle\nabla^3\phi\nabla^2\phi,2\chi_0\nabla\chi_0\rangle-\langle\nabla^2\mathrm{div}\phi\nabla^2\phi,2\chi_0\nabla\chi_0\rangle+\langle\nabla^2\big[\mathbf{v}_t+\frac{\nabla n\nabla\phi}{n+1}$$
$$-\frac{1}{1+n}(\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v})+\mathbf{v}\cdot\nabla\mathbf{v}+\left(\frac{P'(n+1)}{(1+n)P'(1)}-1\right)\nabla n$$
$$+n\nabla\mathrm{div}\phi+\nabla n\cdot\nabla\phi+\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|)\big],\nabla^2\phi\chi_0^2\rangle$$
$$\lesssim\|\nabla^2\phi\|\|\nabla^3\phi\|+\|(\nabla\mathbf{v}_t,\nabla\mathbf{v},\nabla^2\mathbf{v},\nabla^3\mathbf{v}\chi_0)\|\|(\nabla^2\phi,\nabla^3\phi)\|+\varepsilon(\|\nabla n\|_1^2+\|\nabla^2\phi\|_1^2),$$

then by Cauchy-Schwarz inequality we get (3.19). The proof of lemma is completed. □

Finally, let us deal with the estimates near the boundary. Taking the same idea in [15, 31, 39], we need a more detailed argument using the trick of estimating the tangential derivatives and the normal derivatives separately. We choose a finite number of bounded open sets $\{\mathcal{W}_j\}_{j=1}^N$ in $\mathbb{R}^3$, such that $\partial\Omega\subset\cup_{j=1}^N\mathcal{W}_j$. In each open set $\mathcal{W}_j$ we choose the local coordinates $y=(y_1,y_2,y_3)$ as follows:

- The surface $\mathcal{W}^j\cap\partial\Omega$ is the image of a smooth vector function $z^j(y_1,y_2)=(z_1^j,z_2^j,z_3^j)(y_1,y_2)$ (e.g., take the local geodesic polar coordinate), satisfying
$$|z_{y_1}^j|=1,z_{y_1}^j\cdot z_{y_2}^j=0 \text{ and } |z_{y_2}^j|\geq\delta>0, \tag{3.20}$$
where δ is some positive constant independent of j, $1\leq j\leq N$.
- Any $x=(x_1,x_2,x_3)\in\mathcal{W}^j$ is represented by
$$x_i:=\Psi_i(y)=y_3\mathcal{N}_i^j(z^j(y_1,y_2))+z_i^j(y_1,y_2) \text{ for } i=1,\ 2,\ 3, \tag{3.21}$$
where $\mathcal{N}^j(y_1,y_2)=(\mathcal{N}_1^j,\mathcal{N}_2^j,\mathcal{N}_3^j)(z^j(y_1,y_2))$ represents the internal unit normal vector at the point $z^j(y_1,y_2)$ of the surface $\partial\Omega$.

We omit the subscript j in what follows for the simplicity of presentation. For $k=1,2$, we define the unit vectors

$$\mathrm{e}^1=z_{y_1} \text{ and } \mathrm{e}^2=\frac{z_{y_2}}{|z_{y_2}|}.$$

Then Frenet-Serret's formula gives that there exist smooth functions $(\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2,\gamma_2)$ of (y_1,y_2) satisfying

$$\frac{\partial}{\partial y_1}\begin{pmatrix}\mathrm{e}^1\\\mathrm{e}^2\\\mathcal{N}\end{pmatrix}^i=\begin{pmatrix}0&-\gamma_1&-\alpha_1\\\gamma_1&0&-\beta_1\\\alpha_1&\beta_1&0\end{pmatrix}\begin{pmatrix}\mathrm{e}^1\\\mathrm{e}^2\\\mathcal{N}\end{pmatrix}^i,$$

$$\frac{\partial}{\partial y_2}\begin{pmatrix}\mathrm{e}^1\\\mathrm{e}^2\\\mathcal{N}\end{pmatrix}^i=\begin{pmatrix}0&-\gamma_2&-\alpha_2\\\gamma_2&0&-\beta_2\\\alpha_2&\beta_2&0\end{pmatrix}\begin{pmatrix}\mathrm{e}^1\\\mathrm{e}^2\\\mathcal{N}\end{pmatrix}^i,$$

where e_i^m denote the i-th component of e^m. An elementary calculation shows that the Jacobian J of the transform (3.21) is

$$J=\Psi_{y_1}\times\Psi_{y_2}\cdot\mathcal{N}=|z_{y_2}|+(\alpha_1|z_{y_2}|+\beta_2)y_3+(\alpha_1\beta_2-\beta_1\alpha_2)y_3^2. \tag{3.22}$$

By (3.22), we have the transform (3.21) is regular by choosing y_3 so small that $J\geq\delta/2$. Therefore, the inverse function of $\Psi(y):=(\Psi_1,\Psi_2,\Psi_3)(y)$ exits, and we denote it by $y=\Psi^{-1}(x)$. Moreover $(y_1,y_2,y_3)_{x_i}(x)$ make sense and can be expressed by, using a straight forward calculation,

$$\begin{cases}\partial_{x_i}y_1=\dfrac{1}{J}(\Psi_{y_2}\times\Psi_{y_3})_i=\dfrac{1}{J}(\mathcal{A}e_i^1+\mathcal{B}e_i^2)=:a_{1i},\\ \partial_{x_i}y_2=\dfrac{1}{J}(\Psi_{y_3}\times\Psi_{y_1})_i=\dfrac{1}{J}(\mathcal{C}e_i^1+\mathcal{D}e_i^2)=:a_{2i},\\ \partial_{x_i}y_3=\dfrac{1}{J}(\Psi_{y_1}\times\Psi_{y_2})_i=\mathcal{N}_i=:a_{3i},\end{cases} \tag{3.23}$$

where $\mathcal{A}=|z_{y_2}|+\beta_2y_3$, $\mathcal{B}=-y_3\alpha_2$, $\mathcal{C}=-\beta_1y_3$, $\mathcal{D}=1+\alpha_1y_3$,

$$J=\mathcal{AD}-\mathcal{BC}\geq\delta/2. \tag{3.24}$$

Obviously, (3.23) gives

$$\sum_{i=1}^3a_{3i}^2=|\mathcal{N}|^2=1,\ a_{1i}a_{3i}=a_{2i}a_{3i}=0,\ J^2=(\mathcal{AC}+\mathcal{BD})^2-(\mathcal{A}^2+\mathcal{B}^2)(\mathcal{C}^2+\mathcal{D}^2)$$

and

$$\partial_{x_i}=a_{ki}\partial_{y_k}, \tag{3.25}$$

where we have used the Einstein convention of summing over repeated indices.

Thus, in each $\mathcal{W}_j$, the first two equations of system (2.5) and (3.15) can be rewritten in the local coordinates (y_1,y_2,y_3) as following

$$\begin{aligned}\mathcal{L}^n:=&\frac{dn}{dt}+\frac{1}{J}[(\mathcal{A}e^1+\mathcal{B}e^2)\cdot\mathbf{v}_{y_1}+(\mathcal{C}e^1+\mathcal{D}e^2)\cdot\mathbf{v}_{y_2}+J\mathcal{N}\cdot\mathbf{v}_{y_3}]=\mathfrak{F},\\ \mathcal{L}^{\mathbf{v}}:=&\mathbf{v}_t-\frac{\mu}{J^2}[(\mathcal{A}^2+\mathcal{B}^2)\mathbf{v}_{y_1y_1}+2(\mathcal{AC}+\mathcal{BD})\mathbf{v}_{y_1y_2}+(\mathcal{C}^2+\mathcal{D}^2)\mathbf{v}_{y_2y_2}+J^2\mathbf{v}_{y_3y_3}]\\ &+\text{ one order terms of }\mathbf{v}+\frac{1}{J}(\mathcal{A}e^1+\mathcal{B}e^2)[(\mu+\lambda)\frac{dn}{dt}+n]_{y_1}\\ &+\frac{1}{J}(\mathcal{C}e^1+\mathcal{D}e^2)[(\mu+\lambda)\frac{dn}{dt}+n]_{y_2}+\mathcal{N}[(\mu+\lambda)\frac{dn}{dt}+n]_{y_3}\\ &+\frac{1}{J^2}[(\mathcal{A}^2+\mathcal{B}^2)\phi_{y_1y_1}+2(\mathcal{AC}+\mathcal{BD})\phi_{y_1y_2}+(\mathcal{C}^2+\mathcal{D}^2)\phi_{y_2y_2}+J^2\phi_{y_3y_3}]\\ &+\text{ one order terms of }\phi=\mathfrak{G},\\ \mathcal{L}^\phi:=&\frac{1}{J^2}(\mathcal{A}e^1+\mathcal{B}e^2)[(\mathcal{A}e^1+\mathcal{B}e^2)\phi_{y_1y_1}+(\mathcal{C}e^1+\mathcal{D}e^2)\phi_{y_2y_1}+J\mathcal{N}\phi_{y_3y_1}]\\ &+\frac{1}{J^2}(\mathcal{C}e^1+\mathcal{D}e^2)[(\mathcal{A}e^1+\mathcal{B}e^2)\phi_{y_1y_2}+(\mathcal{C}e^1+\mathcal{D}e^2)\phi_{y_2y_2}+J\mathcal{N}\phi_{y_3y_2}]\\ &+\frac{\mathcal{N}}{J}[(\mathcal{A}e^1+\mathcal{B}e^2)\phi_{y_3y_1}+(\mathcal{C}e^1+\mathcal{D}e^2)\phi_{y_3y_2}+J\mathcal{N}\phi_{y_3y_3}]\end{aligned}$$

$$-[\frac{\pi^2}{J}(\mathcal{A}\mathrm{e}^1+\mathcal{B}\mathrm{e}^2)n_{y_1}+\frac{\pi^2}{J}(\mathcal{C}\mathrm{e}^1+\mathcal{D}\mathrm{e}^2)n_{y_2}+\pi^2\mathcal{N}n_{y_3}]$$
$$+\text{ one order terms of }\phi=\mathfrak{H},$$

where

$$\frac{d}{dt}:=\partial_t+\mathbf{v}\cdot\nabla \text{ denotes the material derivative,}$$
$$\mathfrak{F}:=-n\mathrm{div}\mathbf{v},$$
$$\mathfrak{G}:=-\frac{\nabla n\nabla\phi}{n+1}-\frac{n}{1+n}(\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v})-\mathbf{v}\cdot\nabla\mathbf{v}-\left(\frac{P'(n+1)}{(1+n)P'(1)}-1\right)\nabla n$$
$$+\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|)+(\mu+\lambda)\nabla\mathfrak{F},$$
$$\mathfrak{H}=-[n\nabla\phi+\nabla n\cdot\nabla\phi+\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|)].$$

If we denote the tangential derivatives by $\partial=(\partial_{y_1},\partial_{y_2})$ and χ_j be arbitrary but fixed function in $C_0^\infty(\mathcal{W}_j)$, then $\chi_j\partial^k\mathbf{v}=\chi_j\partial^k\phi=0$ on $\partial\Omega_j^{-1}$, where $0\le k\le 2$ and $\Omega_j^{-1}:=\{y|y=\Psi^{-1}(x),x\in\Omega_j=\mathcal{W}_j\cap\Omega\}$. Taking the similar idea as in Lemma 3.6, we have the following estimates for the tangential derivatives.

Lemma 3.7. *Under the conditions of Proposition 2.2, it holds:*

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega_j^{-1}}|\partial n(t)\chi_j|^2+|\partial\mathbf{v}(t)\chi_j|^2+|\frac{\partial\nabla\phi(t)\chi_j}{\pi}|^2dy\\&\quad+\int_{\Omega_j^{-1}}|\partial\nabla\mathbf{v}(t)\chi_j|^2dy+\int_{\Omega_j^{-1}}|\partial\frac{dn(t)}{dt}\chi_j|^2dy\\&\lesssim\varepsilon(\|\nabla\mathbf{v}(t)\|_1^2+\|\nabla n(t)\|^2+\|\nabla^2\phi(t)\|_1^2)+\|\nabla\mathbf{v}(t)\|\|(\nabla n,\nabla^2\mathbf{v},\nabla^2\phi)(t)\|.\end{aligned}\tag{3.26}$$

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega_j^{-1}}|\partial^2 n(t)\chi_j|^2+|\partial^2\mathbf{v}(t)\chi_j|^2+|\frac{\partial^2\nabla\phi(t)\chi_j}{\pi}|^2dy\\&\quad+\int_{\Omega_j^{-1}}|\partial^2\nabla\mathbf{v}(t)\chi_j|^2dy+\int_{\Omega_j^{-1}}|\partial^2\frac{dn(t)}{dt}\chi_j|^2dy\\&\lesssim\varepsilon(\|\nabla\mathbf{v}(t)\|_2^2+\|\nabla\mathbf{v}_t(t)\|^2+\|\nabla n(t)\|_1^2+\|\nabla^2\phi(t)\|_1^2)+\|\nabla^2\mathbf{v}(t)\|\|(\nabla^3\mathbf{v},\nabla^2 n,\nabla^3\phi)(t)\|,\end{aligned}\tag{3.27}$$

$$\int_{\Omega_j^{-1}}|\partial\nabla\phi(t)\chi_j|^2dy\lesssim\varepsilon\|\nabla n(t)\|^2+\|(\nabla\mathbf{v}_t,\nabla\mathbf{v},\partial\nabla\mathbf{v}\chi_j,\nabla\phi)(t)\|^2,\tag{3.28}$$

$$\int_{\Omega_j^{-1}}|\partial^2\nabla\phi(t)\chi_j|^2dy\lesssim\varepsilon\|\nabla n(t)\|_1^2+\|\nabla^2\phi(t)\|^2+\|\nabla\mathbf{v}_t\|^2+\|\nabla\mathbf{v}(t)\|_1^2+\|\partial^2\nabla\mathbf{v}(t)\chi_j\|^2,\tag{3.29}$$

for any $t\ge 0$.

Next, we turn to estimates of derivatives in the normal directions.

Lemma 3.8. *Under the conditions of Proposition 2.2, it holds:*

$$\frac{d}{dt}\int_{\Omega_j^{-1}}|n_{y_3}(t)\chi_j|^2dy+\int_{\Omega_j^{-1}}|(\frac{dn(t)}{dt})_{y_3}\chi_j|^2dy+\int_{\Omega_j^{-1}}|n_{y_3}(t)\chi_j|^2dy$$

$$\lesssim \|(\nabla \mathbf{v}, \nabla \phi, \mathbf{v}_t)(t)\|^2 + \varepsilon \|(\nabla^2 \mathbf{v}, \nabla^2 \phi)(t)\|^2 + \int_{\Omega_j^{-1}} |\partial \nabla \mathbf{v}(t) \chi_j|^2 dy + \int_{\Omega_j^{-1}} |\partial \nabla \phi(t) \chi_j|^2 dy, \tag{3.30}$$

$$\frac{d}{dt} \int_{\Omega_j^{-1}} |\partial^k \partial_{y_3}^{\ell+1} n(t) \chi_j|^2 dy + \int_{\Omega_j^{-1}} |\partial^k \partial_{y_3}^{\ell+1} (\frac{dn(t)}{dt}) \chi_j|^2 dy + \int_{\Omega_j^{-1}} |\partial^k \partial_{y_3}^{\ell+1} n(t) \chi_j|^2 dy$$
$$\lesssim \|(\nabla \mathbf{v}, \mathbf{v}_t, \nabla \phi)(t)\|_1^2 + \varepsilon \|(\nabla n, \nabla^2 \mathbf{v}, \nabla^2 \phi)(t)\|_1^2 + \int_{\Omega_j^{-1}} |\partial^{k+1} \partial_{y_3}^{\ell} \nabla \mathbf{v}(t) \chi_j|^2$$
$$+ |\partial^{k+1} \partial_{y_3}^{\ell} \nabla \phi(t) \chi_j|^2 dy, \tag{3.31}$$

$$\int_{\Omega_j^{-1}} |\partial_{y_3} div \phi(t) \chi_j|^2 dy \lesssim \varepsilon \|(\nabla n, \nabla^2 \phi)(t)\|^2 + \|\nabla \phi(t)\|^2 + \int_{\Omega_j^{-1}} |\partial_{y_3} n(t) \chi_j|^2 dy, \tag{3.32}$$

$$\int_{\Omega_j^{-1}} |\partial^k \partial_{y_3}^{\ell+1} div \phi(t) \chi_j|^2 dy$$
$$\lesssim \varepsilon \|(\nabla^2 n, \nabla^3 \phi)(t)\|^2 + \|(\nabla n, \nabla^2 \phi)(t)\|^2 + \int_{\Omega_j^{-1}} |\partial^k \partial_{y_3}^{\ell+1} n(t) \chi_j|^2 dy, \tag{3.33}$$

for any $t \geq 0$, $k + \ell = 1$.

Proof. First, we use the equations $\partial_{y_3}(\mathcal{L}^n - \mathfrak{F}) = 0$, $\mathcal{N}(\mathcal{L}^{\mathbf{v}} - \mathfrak{G}) = 0$, and $\mathcal{N}(\mathcal{L}^{\phi} - \mathfrak{H}) = 0$, we obtain

$$(\frac{dn}{dt})_{y_3} + \frac{1}{J}[(\mathcal{A}\mathrm{e}^1 + \mathcal{B}\mathrm{e}^2) \cdot \mathbf{v}_{y_1 y_3} + (\mathcal{C}\mathrm{e}^1 + \mathcal{D}\mathrm{e}^2) \cdot \mathbf{v}_{y_2 y_3} + J\mathcal{N} \cdot \mathbf{v}_{y_3 y_3}]$$
$$+ \text{ one order terms of } \mathbf{v} = \mathfrak{F}_{y_3}, \tag{3.34}$$

$$\mathcal{N}\mathbf{v}_t - \frac{\mu}{J^2}[(\mathcal{A}^2 + \mathcal{B}^2)\mathcal{N}\mathbf{v}_{y_1 y_1} + 2(\mathcal{A}\mathcal{C} + \mathcal{B}\mathcal{D})\mathcal{N}\mathbf{v}_{y_1 y_2} + (\mathcal{C}^2 + \mathcal{D}^2)\mathcal{N}\mathbf{v}_{y_2 y_2} + J^2 \mathcal{N}\mathbf{v}_{y_3 y_3}]$$
$$+ \text{ one order terms of } \mathbf{v} + [(\mu + \lambda)\frac{dn}{dt} + n]_{y_3}$$
$$+ \frac{1}{J^2}[(\mathcal{A}^2 + \mathcal{B}^2)\mathcal{N}\phi_{y_1 y_1} + 2(\mathcal{A}\mathcal{C} + \mathcal{B}\mathcal{D})\mathcal{N}\phi_{y_1 y_2} + (\mathcal{C}^2 + \mathcal{D}^2)\mathcal{N}\phi_{y_2 y_2} + J^2 \mathcal{N}\phi_{y_3 y_3}]$$
$$+ \text{ one order terms of } \phi = \mathcal{N}\mathfrak{G}, \tag{3.35}$$

$$\frac{1}{J}[(\mathcal{A}\mathrm{e}^1 + \mathcal{B}\mathrm{e}^2)\phi_{y_3 y_1} + (\mathcal{C}\mathrm{e}^1 + \mathcal{D}\mathrm{e}^2)\phi_{y_3 y_2} + J\mathcal{N}\phi_{y_3 y_3}] - \pi^2 n_{y_3}$$
$$+ \text{ one order terms of } \phi = \mathcal{N}\mathfrak{H}. \tag{3.36}$$

To eliminate $\mathcal{N}\mathbf{v}_{y_3 y_3}$ and $\mathcal{N}\phi_{y_3 y_3}$ in (3.35), we take the summation $\mu \times (3.34) + (3.35) - (3.36)$ and obtain

$$(2\mu + \lambda)(\frac{dn}{dt})_{y_3} + (\pi^2 + 1) n_{y_3}$$
$$= \frac{\mu}{J^2}[(\mathcal{A}^2 + \mathcal{B}^2)\mathcal{N}\mathbf{v}_{y_1 y_1} + 2(\mathcal{A}\mathcal{C} + \mathcal{B}\mathcal{D})\mathcal{N}\mathbf{v}_{y_1 y_2} + (\mathcal{C}^2 + \mathcal{D}^2)\mathcal{N}\mathbf{v}_{y_2 y_2}] - \mathcal{N}\mathbf{v}_t$$

$$
\begin{aligned}
&-\frac{\mu}{J}[(\mathcal{A}\mathrm{e}^1+\mathcal{B}\mathrm{e}^2)\cdot\mathbf{v}_{y_1y_3}+(\mathcal{C}\mathrm{e}^1+\mathcal{D}\mathrm{e}^2)\cdot\mathbf{v}_{y_2y_3}]+\text{ one order terms of }\mathbf{v}\\
&-\frac{1}{J^2}[(\mathcal{A}^2+\mathcal{B}^2)\mathcal{N}\phi_{y_1y_1}+2(\mathcal{A}\mathcal{C}+\mathcal{B}\mathcal{D})\mathcal{N}\phi_{y_1y_2}+(\mathcal{C}^2+\mathcal{D}^2)\mathcal{N}\phi_{y_2y_2}]\\
&-\frac{1}{J}[(\mathcal{A}\mathrm{e}^1+\mathcal{B}\mathrm{e}^2)\phi_{y_3y_1}+(\mathcal{C}\mathrm{e}^1+\mathcal{D}\mathrm{e}^2)\phi_{y_3y_2}]+\text{ one order terms of }\phi\\
&+\mu\mathfrak{F}_{y_3}+\mathcal{N}\mathfrak{G}-\mathcal{N}\mathfrak{H}=\mathfrak{K}.
\end{aligned}\tag{3.37}
$$

Multiplying the above equation by $(\frac{dn}{dt}+n)_{y_3}\chi_j^2$ and integrating on Ω_j^{-1}, we obtain the following estimate on one order derivative in the normal direction to the boundary:

$$
\begin{aligned}
&\frac{\pi^2+1+2\mu+\lambda}{2}\frac{d}{dt}\int_{\Omega_j^{-1}}|n_{y_3}\chi_j|^2dy+\int_{\Omega_j^{-1}}(2\mu+\lambda)|(\frac{dn}{dt})_{y_3}\chi_j|^2+(\pi^2+1)|n_{y_3}\chi_j|^2dy\\
=&-(\pi^2+1+2\mu+\lambda)\int_{\Omega_j^{-1}}(\mathbf{v}\cdot\nabla n)_{y_3}n_{y_3}\chi_j^2dy+\int_{\Omega_j^{-1}}(\frac{dn}{dt}+n)_{y_3}\mathfrak{K}\chi_j^2dy.
\end{aligned}\tag{3.38}
$$

It follows from Lemma 3.1, Hölder's inequality and Cauchy's inequality, for the first term on the right hand side of (3.38), we can deduce that

$$
\begin{aligned}
|\int_{\Omega_j^{-1}}(\mathbf{v}\cdot\nabla n)_{y_3}n_{y_3}\chi_j^2dy|\lesssim&|\int_{\Omega_j^{-1}}\mathbf{v}_{y_3}\cdot\nabla n n_{y_3}\chi_j^2dy|+\frac{1}{2}|\int_{\Omega_j^{-1}}(n_{y_3})^2\mathrm{div}(\mathbf{v}\chi_j^2)dy|\\
\lesssim&\|\nabla\mathbf{v}\|_1\|\nabla n\|_1^2\lesssim\varepsilon\|\nabla n\|_1^2.
\end{aligned}\tag{3.39}
$$

The second term in (3.38) can be bounded as the following:

$$
\begin{aligned}
&|\int_{\Omega_j^{-1}}(\frac{dn}{dt}+n)_{y_3}\mathfrak{K}\chi_j^2dy|\\
\le&\frac{2\mu+\lambda}{2}\int_{\Omega_j^{-1}}|(\frac{dn}{dt})_{y_3}\chi_j|^2dy+\frac{\pi^2+1}{2}\int_{\Omega_j^{-1}}|n_{y_3}\chi_j|^2dy+C\int_{\Omega_j^{-1}}|\mathfrak{K}\chi_j|^2dy\\
\le&\frac{2\mu+\lambda}{2}\int_{\Omega_j^{-1}}|(\frac{dn}{dt})_{y_3}\chi_j|^2dy+\frac{\pi^2+1}{2}\int_{\Omega_j^{-1}}|n_{y_3}\chi_j|^2dy\\
&+C(\|(\nabla\mathbf{v},\nabla\phi,\mathbf{v}_t)\|^2+\varepsilon\|(\nabla\mathbf{v},\nabla\phi)\|_1^2+\int_{\Omega_j{}^{-1}}|\partial\nabla\mathbf{v}\chi_j|^2dy+\int_{\Omega_j{}^{-1}}|\partial\nabla\phi\chi_j|^2dy).
\end{aligned}\tag{3.40}
$$

Substituting (3.39) and (3.40) into (3.38), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_j^{-1}}|n_{y_3}\chi_j|^2dy+\int_{\Omega_j^{-1}}|(\frac{dn}{dt})_{y_3}\chi_j|^2+|n_{y_3}\chi_j|^2dy\\
&\lesssim\|(\nabla\mathbf{v},\nabla\phi,\mathbf{v}_t)\|^2+\varepsilon\|(\nabla\mathbf{v},\nabla\phi)\|_1^2+\int_{\Omega_j{}^{-1}}|\partial\nabla\mathbf{v}\chi_j|^2dy+\int_{\Omega_j{}^{-1}}|\partial\nabla\phi\chi_j|^2dy,
\end{aligned}
$$

which is (3.30).

Applying $\partial^k\partial_{y_3}^\ell(k+\ell=1)$ to (3.37), multiplying the resultant equation by $\partial^k\partial_{y_3}^{\ell+1}(\frac{dn}{dt}+n)\chi_j^2$, integrating on Ω_j^{-1} and as in the proof of (3.30), we conclude

$$
\frac{d}{dt}\int_{\Omega_j^{-1}}|\partial^k\partial_{y_3}^{\ell+1}n\chi_j|^2dy+\int_{\Omega_j^{-1}}|\partial^k\partial_{y_3}^{\ell+1}(\frac{dn}{dt})\chi_j|^2dy+\int_{\Omega_j^{-1}}|\partial^k\partial_{y_3}^{\ell+1}n\chi_j|^2dy
$$

$$\lesssim \|(\nabla \mathbf{v},\mathbf{v}_t,\nabla\phi)\|_1^2+\delta\|(\nabla n,\nabla^2\mathbf{v},\nabla^2\phi)\|_1^2+\int_{\Omega_j{}^{-1}}|\partial^{k+1}\partial_{y_3}^{\ell}\nabla\mathbf{v}\chi_j|^2+|\partial^{k+1}\partial_{y_3}^{\ell}\nabla\phi\chi_j|^2dy,$$

which gives (3.31).

We rewrite (3.15) in the local coordinates (y_1,y_2,y_3) as following

$$\frac{1}{J}(\mathcal{A}\mathrm{e}^1+\mathcal{B}\mathrm{e}^2)(\mathrm{div}\phi-\pi^2 n)_{y_1}+\frac{1}{J}(\mathcal{C}\mathrm{e}^1+\mathcal{D}\mathrm{e}^2)(\mathrm{div}\phi-\pi^2 n)_{y_2}$$
$$+\mathcal{N}(\mathrm{div}\phi-\pi^2 n)_{y_3}+\text{one order terms of }\phi=\mathfrak{H}.$$

Applying $\partial^k\partial_{y_3}^{\ell}(k+\ell=0,1)$ to the above equation, multiplying the resultant equation by by $\chi_j^2\mathcal{N}\partial^k\partial_{y_3}^{\ell+1}\mathrm{div}\phi$, integrating on Ω_j^{-1} and as in the proof of (3.30), we yield (3.32) and (3.33). This completes the proof of lemma. ∎

Finally, we apply Lemma 3.2 to get the estimates on the tangential derivatives of n and $\mathbf{v}-\frac{\phi}{\mu}$.

Lemma 3.9. *Under the conditions of Proposition 2.2, it holds:*

$$\|\nabla^2(\mathbf{v}-\frac{\phi}{\mu})(t)\|^2+\|\nabla n(t)\|^2$$
$$\lesssim\|(\frac{dn}{dt},div\phi)(t)\|_1^2+\|\mathbf{v}_t(t)\|^2+\|\nabla\mathbf{v}(t)\|_1^2\|\nabla^2\mathbf{v}(t)\|^2+\|\nabla\phi(t)\|_1^2\|\nabla^2\phi(t)\|_1^2, \tag{3.41}$$

$$\int_{\Omega_j^{-1}}|\partial\nabla^2(\mathbf{v}-\frac{\phi}{\mu})(t)\chi_j|^2dy+\int_{\Omega_j^{-1}}|\partial\nabla n(t)\chi_j|^2dy$$
$$\lesssim\|(\nabla\mathbf{v},\mathbf{v}_t)(t)\|_1^2+\|\nabla n(t)\|^2+\|\nabla^2\phi(t)\|^2+\int_{\Omega_j^{-1}}|\partial\nabla\frac{dn(t)}{dt}\chi_j|^2dy$$
$$+\int_{\Omega_j^{-1}}|\partial\nabla\, div\phi(t)\chi_j|^2dy+\|\nabla n(t)\|\|\nabla\frac{dn(t)}{dt}\|+\|\nabla\mathbf{v}(t)\|_1^2\|\nabla^3\mathbf{v}(t)\|^2$$
$$+\|\nabla n(t)\|_1^2\|\nabla^2\phi(t)\|_1^2+\|\nabla\phi(t)\|_1^2\|\nabla^2\phi(t)\|_1^2, \tag{3.42}$$

for any $t\geq 0$.

Proof. We rewrite Equations $(2.5)_{1,2}$ in the form of Stokes equations as:

$$\begin{cases}\mathrm{div}\mathbf{v}=-\frac{\frac{dn}{dt}}{1+n},\\ -\mu\Delta(\mathbf{v}-\frac{\phi}{\mu})+\nabla n=-\mathbf{v}_t+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v}-\frac{\nabla n\nabla\phi}{n+1}-\frac{n}{1+n}(\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v})\\ \qquad\qquad -\mathbf{v}\cdot\nabla\mathbf{v}-\left(\frac{P'(n+1)}{(1+n)P'(1)}-1\right)\nabla n+\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|),\\ \mathbf{v}|_{\partial\Omega}=0,\ \phi|_{\partial\Omega}=0.\end{cases} \tag{3.43}$$

Hence applying Lemma 3.2 together with Lemma 3.1 to (3.43), we can obtain (3.41).

By using the operator $\chi_j\partial$ to Stokes Equation $(3.43)_2$, then together with $(3.43)_{1,3}$, we get the following Stokes problem:

$$\begin{cases}\mathrm{div}(\chi_j\partial\mathbf{v})=-\chi_j\partial(\frac{\frac{dn}{dt}}{1+n})+\nabla\chi_j\partial\mathbf{v},\\ -\mu\Delta[\chi_j\partial(\mathbf{v}-\frac{\phi}{\mu})]+\nabla(\chi_j\partial n)=-2\mu\nabla\chi_j\nabla[\partial(\mathbf{v}-\frac{\phi}{\mu})]-\mu\Delta\chi_j\partial(\mathbf{v}-\frac{\phi}{\mu})+\nabla\chi_j\partial n\\ \qquad +\chi_j\partial\big[-\mathbf{v}_t+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v}-\frac{\nabla n\nabla\phi}{n+1}-\frac{n}{1+n}(\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\mathrm{div}\mathbf{v})\\ \qquad\quad -\mathbf{v}\cdot\nabla\mathbf{v}-\left(\frac{P'(n+1)}{(1+n)P'(1)}-1\right)\nabla n+\mathcal{O}(|\nabla\phi|)\nabla\mathcal{O}(|(n+1)\nabla\phi|)\big],\\ \chi_j\partial(\mathbf{v}-\frac{\phi}{\mu})|_{\partial\Omega_j^{-1}}=0.\end{cases} \tag{3.44}$$

Thus applying Lemma 3.2 together with Lemma 3.1 to (3.44), we easily deduce (3.42). This completes the proof of lemma. ∎

Now we are in a position to prove Proposition 2.2.

Proof. (**Proof of Proposition 2.2.**) We divide the proof of Proposition 2.2 into several steps.

Step 1: We first estimate the lower order derivatives for $(n,\mathbf{v},\phi)$. Let D be a fixed but large positive constant. By summing up

$$D^5\times[(3.2)+(3.4)]+D^4\times(3.12)+D^2\times[(3.16)+(3.26)]$$
$$+D\times[(3.18)+(3.28)]+D^{\frac{1}{2}}\times(3.30)+(3.32),$$

there exists a function $H_1(\mathbf{v},n,\phi,\mathbf{v}_t,n_t,\phi_t,\nabla n)$, which is equivalent to $\|(\mathbf{v},n,\nabla\phi,\mathbf{v}_t,n_t,\nabla\phi_t,\nabla n)\|^2$, such that

$$\begin{aligned}&\frac{d}{dt}\{H_1(t)+\|(\nabla\mathbf{v}\chi_0,\nabla^2\phi\chi_0)(t)\|^2+\sum_{j=1}^N\int_{\Omega_j^{-1}}|\partial\mathbf{v}(t)\chi_j|^2+|\partial\nabla\phi(t)\chi_j|^2dy\}\\&\quad+D^2\|(\nabla\mathbf{v},\nabla\mathbf{v}_t)(t)\|+\|\nabla\frac{dn(t)}{dt}\|^2+D\|\nabla^2\mathbf{v}(t)\chi_0\|^2\\&\quad+\sum_{j=1}^N\int_{\Omega_j^{-1}}D|\partial\nabla\mathbf{v}(t)\chi_j|^2+|n_{y_3}(t)\chi_j|^2dy+D\|\nabla\phi(t)\|^2+\|\nabla\mathrm{div}\phi(t)\|^2\\&\le\frac{1}{D^{1/3}}\|(\nabla n,\nabla^2\mathbf{v},\nabla^2\phi)(t)\|^2+C\varepsilon\|(\nabla^2n,\nabla^3\phi)(t)\|^2.\end{aligned}\tag{3.45}$$

Plugging (3.41) and (3.13) into (3.45), by the largeness of D and smallness of ε, we deduce that

$$\begin{aligned}&\frac{d}{dt}\{H_1(t)+\|(\nabla\mathbf{v}\chi_0,\nabla^2\phi\chi_0,\Delta\phi)(t)\|^2\}+\|\nabla\mathbf{v}(t)\|_1^2+\|(\nabla n,\nabla\mathbf{v}_t)(t)\|^2+\|\nabla\phi\|_1^2\\&\le C\varepsilon\|(\nabla^2n,\nabla^3\phi(t))(t)\|^2.\end{aligned}\tag{3.46}$$

Step 2: Next, we deal with the higher order derivatives for $(n,\mathbf{v},\phi)$. Let $\ell=0$ in (3.31) and (3.33), then summing up

$$2D^4\times[(3.17)+(3.27)]+D^3\times[(3.19)+(3.29)]+D\times(3.31)+(3.33),$$

we get

$$\begin{aligned}&\frac{d}{dt}\{D^4\|(\nabla^2n\chi_0,\nabla^2\mathbf{v}\chi_0,\frac{\nabla^3\phi\chi_0}{\pi})(t)\|^2+D^4\sum_{j=1}^N\int_{\Omega_j^{-1}}|\partial^2n(t)\chi_j|^2+|\partial^2\mathbf{v}(t)\chi_j|^2\\&\quad+|\frac{\partial^2\nabla\phi(t)\chi_j}{\pi}|^2dy+D\sum_{j=1}^N\int_{\Omega_j^{-1}}|\partial\partial_{y_3}n(t)\chi_j|^2dy\}+\|\nabla^3\mathbf{v}(t)\chi_0\|^2\\&\quad+\sum_{j=1}^N\int_{\Omega_j^{-1}}|\partial^2\nabla\mathbf{v}(t)\chi_j|^2dy+\|\nabla^2\frac{dn(t)}{dt}\chi_0\|^2+\sum_{j=1}^N\int_{\Omega_j^{-1}}|\partial\nabla\frac{dn(t)}{dt}\chi_j|^2dy\\&\quad+\|\nabla^3\phi(t)\chi_0\|^2+\sum_{j=1}^N\int_{\Omega_j^{-1}}|\partial^2\nabla\phi\chi_j|^2+|\partial\nabla\mathrm{div}\phi\chi_j|^2dy\end{aligned}$$

$$\leq C\big[\varepsilon D^4(\|\nabla\mathbf{v}(t)\|_2^2+\|\nabla\mathbf{v}_t(t)\|^2+\|\nabla n(t)\|_1^2+\|\nabla^2\phi(t)\|_1^2)
+D^4\|\nabla^2\mathbf{v}(t)\|\|(\nabla^3\mathbf{v},\nabla^2 n,\nabla^3\phi)(t)\|
+D^3(\|\nabla\phi(t)\|_1^2+\|\nabla\mathbf{v}_t(t)\|^2+\|\nabla\mathbf{v}(t)\|_1^2)\big]. \tag{3.47}$$

Taking $\ell=1$ in (3.31) and (3.33), then substituting (3.42) and (3.33) into (3.31), we obtain

$$\begin{aligned}
&\frac{d}{dt}D\int_{\Omega_j^{-1}}|\partial_{y_3}^2 n(t)\chi_j|^2dy+\int_{\Omega_j^{-1}}|\partial_{y_3}^2(\frac{dn(t)}{dt})\chi_j|^2dy+\int_{\Omega_j^{-1}}|\partial_{y_3}^2 n(t)\chi_j|^2dy\\
&\quad+\int_{\Omega_j^{-1}}|\partial\nabla^2(\mathbf{v}-\frac{\phi}{\mu})(t)\chi_j|^2dy+\int_{\Omega_j^{-1}}|\partial\nabla n(t)\chi_j|^2dy+\int_{\Omega_j^{-1}}|\partial_{y_3}\nabla\mathrm{div}\phi(t)\chi_j|^2dy\\
&\leq C\big[\|\nabla n(t)\|^2+\|(\nabla\mathbf{v},\mathbf{v}_t,\nabla\phi)(t)\|_1^2+\varepsilon\|(\nabla^2 n,\nabla^3\mathbf{v},\nabla^3\phi)(t)\|^2+\int_{\Omega_j^{-1}}|\partial\nabla\frac{dn(t)}{dt}\chi_j|^2dy\\
&\quad+\int_{\Omega_j^{-1}}|\partial\nabla\mathrm{div}\phi(t)\chi_j|^2dy\big].
\end{aligned}\tag{3.48}$$

Adding $D\times(3.47)$ to (3.48), there exists a function $H_2(\nabla^2 n)$ which is equivalent to $\|\nabla^2 n\|^2$ and satisfies

$$\begin{aligned}
&\frac{d}{dt}\{D^5\|(\nabla^2\mathbf{v}\chi_0,\frac{\nabla^3\phi\chi_0}{\pi})(t)\|^2+D^5\sum_{j=1}^N\int_{\Omega_j^{-1}}|\partial^2\mathbf{v}(t)\chi_j|^2+|\frac{\partial^2\nabla\phi(t)\chi_j}{\pi}|^2dy+H_2(t)\}\\
&\quad+\|\nabla^3\mathbf{v}(t)\chi_0\|^2+\sum_{j=1}^N\int_{\Omega_j^{-1}}|\partial^2\nabla\mathbf{v}(t)\chi_j|^2dy+\|\nabla^2\frac{d}{dt}n(t)\|^2\\
&\quad+\|\nabla^3\phi(t)\chi_0\|^2+\sum_{j=1}^N\int_{\Omega_j^{-1}}|\partial^2\nabla\phi\chi_j|^2dy+\|\nabla^2\mathrm{div}\phi\|^2\\
&\leq C\big[\|\nabla n(t)\|^2+D^4\|(\nabla\mathbf{v},\mathbf{v}_t,\nabla\phi)(t)\|_1^2+D^5\varepsilon\|(\nabla^2 n,\nabla^3\mathbf{v},\nabla^3\phi)(t)\|^2\\
&\quad+D^5\|\nabla^2\mathbf{v}(t)\|\|(\nabla^3\mathbf{v},\nabla^2 n,\nabla^3\phi)(t)\|\big].
\end{aligned}\tag{3.49}$$

Applying Lemma 3.2 together with Lemma 3.1 to (3.43) as (3.42), we have

$$\begin{aligned}
&\|\nabla^3(\mathbf{v}-\frac{\phi}{\mu})(t)\|^2+\|\nabla^2 n(t)|^2\\
&\leq C\big[\|(\nabla\mathbf{v}(t),\mathbf{v}_t(t))\|_1^2+\|\nabla n(t)\|^2+\|\nabla^2\phi(t)\|^2+\|\nabla^2\frac{dn(t)}{dt}\|^2\\
&\quad+\|\nabla^2\mathrm{div}\phi(t)\|^2+\|\nabla n(t)\|\|\nabla\frac{dn(t)}{dt}\|+\|\nabla\mathbf{v}(t)\|_1^2\|\nabla^3\mathbf{v}(t)\|^2\\
&\quad+\|\nabla n(t)\|_1^2\|\nabla^2\phi(t)\|_1^2+\|\nabla\phi(t)\|_1^2\|\nabla^2\phi(t)\|_1^2\big].
\end{aligned}\tag{3.50}$$

Step 3: Now, we are in a position to establish the energy estimate of Gronwall-type. First of all, applying the classical L^p-estimates of elliptic system to $(2.5)_2$, it holds that

$$\|\nabla^2\mathbf{v}(t)\|^2\leq C\|(\mathbf{v}_t,\nabla p,\nabla\mathbf{v},\nabla^2\phi)(t)\|^2. \tag{3.51}$$

Due to the conservation of total mass and the Poincaré's inequality implies that

$$\|n(t)\|^2\leq C\|\nabla n(t)\|^2.$$

Then by summing up

$$D^{15}\times(3.46)+D^2\times(3.49)+D\times(3.50)+(3.14),$$

and using the largeness of D and smallness of ε, there exists a function $H_3(n,\mathbf{v},\phi)$ which is equivalent to $\|(n,\mathbf{v})(t)\|_2^2+\|\phi(t)\|_3^2+\|(n_t,\mathbf{v}_t,\nabla\phi_t)(t)\|^2$ and satisfies

$$\frac{d}{dt}H_3(t)+C(H_3(t)+\|\nabla^3\mathbf{v}(t)\|^2)\leq 0, \tag{3.52}$$

By Gronwall's inequality, (3.52) yields the exponential decaying of $H_3(t)$. Thus we prove (2.8) and the proof of Proposition 2.2 is completed. ∎

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