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GLOBAL WELL-POSEDNESS FOR N-DIMENSIONAL BOUSSINESQ SYSTEM WITH VISCOSITY DEPENDING ON TEMPERATURE*

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Abstract. In this paper, we study the global well-posedness issue for the Boussinesq system with the temperature-dependent viscosity in $\mathbb{R}^n (n \geq 2)$. With a temperature damping term, we first get a global solution in $\mathbb{R}^2$, provided the initial temperature is exponentially small compared with the initial velocity field. Then, using a weighted Chemin-Lerner-type norm, we can also give a global large solution in $\mathbb{R}^n$ if the initial data satisfies a nonlinear smallness condition. In particular, our results imply the global large solutions without any smallness conditions imposed on the initial velocity.

Keywords. Global well-posedness; Boussinesq system; Littlewood-Paley theory.

AMS subject classifications. 35Q30; 35A05; 76D03.

1. Introduction and the main results

One of the most useful models in fluid and geophysical fluid dynamics is the Boussinesq system which has the following form:

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta - \operatorname{div}(\kappa(\theta) \nabla \theta) = 0, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(\nu(\theta) \nabla u) + \nabla \Pi = \theta e_n, \\ \operatorname{div} u = 0, \\ (\theta, u)|_{t=0} = (\theta_0, u_0), \end{cases} \tag{1.1}$$

where u denotes the velocity vector field, $\Pi = \Pi(x,t)$ is the scalar pressure, the scalar function θ is the temperature and e_n is the unit vector in $\mathbb{R}^n$. The viscosity $\kappa(\theta)$ and the thermal diffusivity $\nu(\theta)$ depend on the temperature. Such dependence could be of great importance due to the large temperature contrast in certain applications.

The Boussinesq system arises from a zeroth-order approximation to the coupling between the Navier-Stokes equations and the thermodynamic equations. It can be used as a model to describe many geophysical phenomena (see [19]). Thus, it attracts the attention of many mathematicians and there have been a lot of works devoted to it. For the full Boussinesq system ($\kappa(\theta)$ and $\nu(\theta)$ are all depending on the temperature), Lorca and Boldr in [18] proved the global existence of strong solution for small data, and the global existence of weak solution and the local existence and uniqueness of strong solution for general data in [17]. By using De-Giorgi method and Harmonic analysis tools, Wang and Zhang in [21] got the global existence of smooth solutions in $\mathbb{R}^2$. A similar result in the bounded domain was obtain by Sun and Zhang in [20]. Li and Xu in [16] also generalized the result in [21] to the inviscid case (*i.e.* $\nu(\theta) = 0$). They got the global strong solution for arbitrarily large initial data in Sobolev spaces $H^s(\mathbb{R}^2), s > 2$. Jiu and Liu in [15] obtained the global well-posedness of anisotropic nonlinear Boussinesq equations with horizontal temperature-dependent viscosity and vertical thermal diffusivity in $\mathbb{R}^2$. Using $\kappa|D|\theta$ instead of $\operatorname{div}(\kappa(\theta)\nabla\theta)$ in system (1.1), Abidi and Zhang in [3] got the global solution in $\mathbb{R}^2$, provided the viscosity coefficient

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is sufficiently close to some positive constant in L^∞ norm. Zhai, Dong and Chen in [26] extended Abidi and Zhang's result into the supercritical regime for temperature dissipation. When $\kappa(\theta)$ and $\nu(\theta)$ are two positive constants which do not depend on the temperature, there are also many works, see [6,11,22]. Here, we don't report them in details.

In contrast, only few works have been dedicated to the higher dimensional case, see [12,13]. Francesco, in [13], obtained the global existence of weak solutions to the system (1.1) in $\mathbb{R}^n$, with viscosity dependent on temperature. However, to the best of our knowledge, when $\kappa(\theta)=0$ and $\nu=\nu(\theta)$, the global existence of smooth solutions for the n-dimensional Boussinesq system remains open even in $\mathbb{R}^2$. The present paper can be regarded as an attempt to this problem. With the help of the temperature damping term, we can give several different well-posedness results in $\mathbb{R}^n$. More precisely, in the present paper, we will study the following Boussinesq system:

$$\begin{cases} \partial_t\theta+u\cdot\nabla\theta+\eta\theta=0, \\ \partial_t u+u\cdot\nabla u-\operatorname{div}(\nu(\theta)\nabla u)+\nabla\Pi=\theta e_n, \\ \operatorname{div}u=0, \\ (\theta,u)|_{t=0}=(\theta_0,u_0), \end{cases} \tag{1.2}$$

where η is a positive parameter.

In the case when $\eta=0$, the system (1.2) becomes the standard viscous Boussinesq equations without thermal diffusivity; when $\eta=\nu(\theta)=0$, the system (1.2) reduces to the standard inviscid n-dimensional Boussinesq system; when $\operatorname{div}(\nu(\theta)\nabla u)$ is replaced by λu, the system (1.2) becomes the system of damped Boussinesq equations, which has been studied by Adhikar et al. in [4] and Wu, Xu and Ye in [23].

Notations. In all that follows, we denote $\eta=1$ and $\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)\stackrel{\text{def}}{=}\dot{B}^{\frac{n}{p}}_{p,1}\cap\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, where the definition of the function space $\dot{B}^s_{p,1}(\mathbb{R}^n)$ will be given in the next section. We also assume that

$$1\le\nu(\theta),\quad \nu(\cdot)\in W^{2,\infty}(\mathbb{R}^+),\quad \nu(0)=\nu. \tag{1.3}$$

Our first result in this paper is as follows:

THEOREM 1.1. *Let $1<p<4$, $1<q\le 2$ and $\nu(\theta)$ satisfy* (1.3)*. For any $\theta_0\in\dot{\mathcal{B}}^{\frac{2}{p}}_{p,1}\cap L^q(\mathbb{R}^2)$, $u_0\in\dot{B}^{-1+\frac{2}{p}}_{p,1}\cap L^2(\mathbb{R}^2)$ with $\operatorname{div}u_0=0$, the system* (1.2) *has a unique local solution (θ,u) on $[0,T]$, such that*

$$\begin{aligned} &\theta\in C([0,T];\dot{\mathcal{B}}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2))\cap\widetilde{L}^\infty_T(\dot{\mathcal{B}}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)) \\ &u\in C([0,T];\dot{B}^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))\cap\widetilde{L}^\infty_T(\dot{B}^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))\cap L^1_T(\dot{B}^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)). \end{aligned} \tag{1.4}$$

Moreover, if there exist two positive constants C_0 and ε_0 (sufficiently small), such that

$$\|\theta_0\|_{\dot{\mathcal{B}}^{\frac{2}{p}}_{p,1}}\exp\big(C_0(\|u_0\|_{\dot{B}^{-1+\frac{2}{p}}_{p,1}}+\|u_0\|^2_{L^2}+\|\theta_0\|^2_{L^q})/\nu\big)\le\varepsilon_0, \tag{1.5}$$

then the local solution can be extended to be global.

REMARK 1.1. It is obvious that the global solution obtained in the above theorem has a finite energy. In fact, we can also give a global solution which has an infinite energy, by using another method. This fact will be proved after we have completed the proof of Theorem 1.2.

In the following, we are concerned about the global well-posedness in $\mathbb{R}^n$ $(n\geq 2)$, more precisely, we have the following theorem:

THEOREM 1.2. *Let* $1\leq p<2n$, $\nu(\theta)$ *satisfy* (1.3). *For any* $\theta_0\in\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, $u_0\in\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ *with* $\operatorname{div}u_0=0$. *If there exist positive constants* c_1 *and* C_1, *such that,*

$$C_1\Big(\|e^{\nu t\Delta}u_0\cdot\nabla e^{\nu t\Delta}u_0\|_{L^1(\mathbb{R}^+;\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\theta_0\|_{\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}}\Big)\exp\Big(C_1\|u_0\|^2_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}/\nu^2\Big)\leq c_1, \tag{1.6}$$

then the system (1.2) *has a global solution* (θ,u) *with*

$$\theta\in C_b([0,\infty);\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap L^1(\mathbb{R}^+;\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)),$$
$$u\in C_b([0,\infty);\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap L^1(\mathbb{R}^+;\dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)).$$

Moreover, there holds

$$\|u-e^{\nu t\Delta}u_0\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}/\nu+\|u-e^{\nu t\Delta}u_0\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|\theta\|_{\widetilde{L}^\infty_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+\|\theta\|_{L^1_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}\leq\frac{c_1}{2}. \tag{1.7}$$

REMARK 1.2. If $\nu(\theta)$ is a positive constant, independent of θ, we can relax the condition $\theta_0\in\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ to $\theta_0\in\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$.

REMARK 1.3. As in [24], we can also construct an initial data, satisfying the nonlinear smallness condition in (1.6), but the norm of each component of the initial velocity field can be arbitrarily large in $\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ with $n<p<2n$.

The paper is organized as follows. In Section 2, we recall the Littlewood-Paley theory and give some useful lemmas about product laws and commutator's estimates in Besov spaces. In Sections 3 and 4, we present the proof of Theorems 1.1 and 1.2 respectively. In the Appendix, we give a local well-posedness result of the Boussinesq system.

Notations. Let A, B be two operators, we denote $[A,B]=AB-BA$, the commutator between A and B. For $a\lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a\leq Cb$. For X, a Banach space and I, an interval of $\mathbb{R}$, we denote by $C(I;X)$, the set of continuous functions on I with values in X. For $q\in[1,+\infty]$, the notation $L^q(I;X)$ stands for the set of measurable functions on I with values in X, such that $t\to\|f(t)\|_X$ belongs to $L^q(I)$. We always denote $(d_j)_{j\in\mathbb{Z}}$ to be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{j\in\mathbb{Z}}d_j=1$.

2. Preliminaries

Let (χ,ϕ) be two smooth radial functions, $0\leq(\chi,\phi)\leq 1$, such that χ is supported in the ball $\mathbb{B}=\{\xi\in\mathbb{R}^n,|\xi|\leq\frac{4}{3}\}$ and φ is supported in the ring $\mathbb{C}=\{\xi\in\mathbb{R}^n,\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$. Moreover, there holds

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1,\qquad\forall\xi\neq 0.$$

Let $h=\mathcal{F}^{-1}\varphi$ and $\widetilde{h}=\mathcal{F}^{-1}\chi$, then we define the dyadic blocks as follows:

$$\dot{\Delta}_jf=\varphi(2^{-j}D)f=2^{nj}\int_{\mathbb{R}^n}h(2^jy)f(x-y)dy,$$

$$\dot{S}_j f = \chi(2^{-j}D)f = 2^{nj}\int_{\mathbb{R}^n}\widetilde{h}(2^j y)f(x-y)dy.$$

Denote by $\mathcal{S}_h'(\mathbb{R}^n)$ the space of tempered distributions u, such that

$$\lim_{j\to-\infty}\dot{S}_j u = 0 \quad in \quad \mathcal{S}'.$$

Then, we have the formal decomposition

$$u = \sum_{j\in\mathbb{Z}}\dot{\Delta}_j u, \quad \forall u\in\mathcal{S}_h'(\mathbb{R}^n).$$

Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\dot{\Delta}_k\dot{\Delta}_j u \equiv 0 \quad \text{if } |k-j|\geq 2 \quad \text{and} \quad \dot{\Delta}_k(\dot{S}_{j-1}u\dot{\Delta}_j u)\equiv 0 \quad \text{if } |k-j|\geq 5.$$

Now we recall the definition of homogeneous Besov spaces.

DEFINITION 2.1. *Let $(p,r)\in[1,+\infty]^2, s\in\mathbb{R}$ and $u\in\mathcal{S}_h'(\mathbb{R}^n)$. Set*

$$\|u\|_{\dot{B}^s_{p,r}} \overset{\text{def}}{=} (2^{js}\|\dot{\Delta}_j u\|_{L^p})_{l^r}.$$

Then we define $\dot{B}^s_{p,r}(\mathbb{R}^n)\overset{\text{def}}{=}\{u\in\mathcal{S}_h'(\mathbb{R}^n)\big|\|u\|_{\dot{B}^s_{p,r}}<\infty\}$.

REMARK 2.1. Let $1\leq p,r\leq\infty, s\in\mathbb{R}$, and $u\in\mathcal{S}_h'(\mathbb{R}^n)$. Then u belongs to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ if and only if there exists $\{d_{j,r}\}_{j\in Z}$ such that $\|d_{j,r}\|_{l^r}=1$ and

$$\|\dot{\Delta}_j u\|_{L^p}\leq Cd_{j,r}2^{-js}\|u\|_{\dot{B}^s_{p,r}} \qquad \text{for all } j\in\mathbb{Z}.$$

We are going to define the space of Chemin-Lerner (see [5]), in which we will work, which is a refinement of the space $L^\lambda_T(\dot{B}^s_{p,r}(\mathbb{R}^n))$.

DEFINITION 2.2. *Let $s\in\mathbb{R}$, $1\leq r,\lambda,p\leq\infty$ and $T\in(0,+\infty]$. We define $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r}(\mathbb{R}^n))$ as the completion of $C([0,T];\mathcal{S}(\mathbb{R}^n))$ by the norm*

$$\|f\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} = \left\{\sum_{q\in\mathbb{Z}}2^{rqs}\left(\int_0^T\|\dot{\Delta}_q f(t)\|^\lambda_{L^p}dt\right)^{\frac{r}{\lambda}}\right\}^{\frac{1}{r}}<\infty,$$

with the usual change if $r=\infty$. For short, we just denote this space by $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})$.

REMARK 2.2. It is easy to observe that for $0<s_1<s_2$, $\theta\in[0,1]$, $p,r,\lambda,\lambda_1,\lambda_2\in[1,+\infty]$, we have the following interpolation inequality in the Chemin-Lerner space (see [5]):

$$\|u\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})}\leq\|u\|^\theta_{\widetilde{L}^{\lambda_1}_T(\dot{B}^{s_1}_{p,r})}\|u\|^{(1-\theta)}_{\widetilde{L}^{\lambda_2}_T(\dot{B}^{s_2}_{p,r})}$$

with $\frac{1}{\lambda}=\frac{\theta}{\lambda_1}+\frac{1-\theta}{\lambda_2}$ and $s=\theta s_1+(1-\theta)s_2$.

By using the Minkowski inequality, we can easily get the following fact:

$$\|f\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})}\leq\|f\|_{L^\lambda_T(\dot{B}^s_{p,r})} \quad \text{if } \lambda\leq r, \qquad \|f\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})}\geq\|f\|_{L^\lambda_T(\dot{B}^s_{p,r})}, \quad \text{if } \lambda\geq r.$$

In order to prove the main Theorem 1.1, we need to introduce the following weighted Chemin-Lerner-type norm from [5,24]:

Definition 2.3. *Let* $f(t)\in L^1_{loc}(\mathbb{R}^+), f(t)\geq 0$. *We define*

$$\|u\|_{\widetilde{L}^q_{T,f}(\dot{B}^s_{p,r})}=\left\{\sum_{j\in\mathbb{Z}}2^{rjs}\left(\int_0^T f(t)\|\dot{\Delta}_j u(t)\|^q_{L^p}dt\right)^{\frac{r}{q}}\right\}^{\frac{1}{r}}$$

for $s\in\mathbb{R}, p\in[1,\infty], q,r\in[1,\infty)$, *and with the standard modification for* $q=\infty$ *or* $r=\infty$.

The following Bernstein's lemma will be repeatedly used throughout this paper.

Lemma 2.1. *Let* $\mathbb{B}$ *be a ball and* $\mathbb{C}$ *a ring of* $\mathbb{R}^n$. *A constant* C *exists so that for any positive real number* λ, *any non-negative integer* k, *any smooth homogeneous function* σ *of degree* m, *and any couple of real numbers* (a,b) *with* $1\leq a\leq b$, *there hold*

$$\operatorname{Supp}\hat{u}\subset\lambda\mathbb{B}\Rightarrow\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^b}\leq C^{k+1}\lambda^{k+n(1/a-1/b)}\|u\|_{L^a},$$

$$\operatorname{Supp}\hat{u}\subset\lambda\mathbb{C}\Rightarrow C^{-k-1}\lambda^k\|u\|_{L^a}\leq\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^a}\leq C^{k+1}\lambda^k\|u\|_{L^a},$$

$$\operatorname{Supp}\hat{u}\subset\lambda\mathbb{C}\Rightarrow\|\sigma(D)u\|_{L^b}\leq C_{\sigma,m}\lambda^{m+n(1/a-1/b)}\|u\|_{L^a}.$$

In the sequel, we shall frequently use Bony's decomposition from [5] in the homogeneous context:

$$uv=\dot{T}_u v+\dot{T}_v u+\dot{R}(u,v)=\dot{T}_u v+\dot{\mathcal{R}}(u,v),$$

where

$$\dot{T}_u v\stackrel{\text{def}}{=}\sum_{j\in Z}\dot{S}_{j-1}u\dot{\Delta}_j v,\qquad \dot{R}(u,v)\stackrel{\text{def}}{=}\sum_{j\in Z}\dot{\Delta}_j u\tilde{\dot{\Delta}}_j v,$$

and

$$\tilde{\dot{\Delta}}_j v\stackrel{\text{def}}{=}\sum_{|j-j'|\leq 1}\dot{\Delta}_{j'}v,\qquad \dot{\mathcal{R}}(u,v)\stackrel{\text{def}}{=}\sum_{j\in Z}\dot{S}_{j+2}v\dot{\Delta}_j u.$$

As an application of the above basic facts on Littlewood-Paley theory, we present the following product law (proofs omitted) in Besov spaces.

Lemma 2.2. *Let* $1\leq p,q\leq\infty$, $s_1\leq\frac{n}{q}$, $s_2\leq n\min\{\frac{1}{p},\frac{1}{q}\}$ *and* $s_1+s_2>n\max\{0,\frac{1}{p}+\frac{1}{q}-1\}$. *For* $\forall(a,b)\in\dot{B}^{s_1}_{q,1}(\mathbb{R}^n)\times\dot{B}^{s_2}_{p,1}(\mathbb{R}^n)$, *we have*

$$\|ab\|_{\dot{B}^{s_1+s_2-\frac{n}{q}}_{p,1}}\lesssim\|a\|_{\dot{B}^{s_1}_{q,1}}\|b\|_{\dot{B}^{s_2}_{p,1}}.$$

Lemma 2.3 (Lemma 2.100 in [5]). *Let* $1\leq p,q\leq\infty$, $s\leq 1+n\min\{\frac{1}{p},\frac{1}{q}\}$, $v\in\dot{B}^s_{q,1}(\mathbb{R}^n)$ *and* $u\in\dot{B}^{\frac{n}{p}+1}_{p,1}(\mathbb{R}^n)$. *Assume that*

$$s>-n\min\left\{\frac{1}{p},1-\frac{1}{q}\right\},\quad or\quad s>-1-n\min\left\{\frac{1}{p},1-\frac{1}{q}\right\}\quad if\quad \operatorname{div}u=0.$$

Then, there holds

$$\|[u\cdot\nabla,\dot{\Delta}_j]v\|_{L^q}\lesssim d_j 2^{-js}\|u\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|v\|_{\dot{B}^s_{q,1}}.$$

3. The proof of Theorem 1.1

Proof. Given initial data (θ_0, u_0), satisfying the assumptions of Theorem 1.1, we deduce from the Theorem A.1 in the Appendix that (1.2) has a unique local solution (θ, u) on $[0, T^*)$, such that for any $T < T^*$

$$\theta \in C([0,T]; \dot{\mathcal{B}}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)) \cap \widetilde{L}^\infty_T(\dot{\mathcal{B}}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)) \cap L^1_T(\dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)),$$
$$u \in C([0,T]; \dot{B}^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)) \cap \widetilde{L}^\infty_T(\dot{B}^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)) \cap L^1_T(\dot{B}^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)).$$

In order to prove that $T^* = \infty$ under the smallness condition (1.5), we first get, by taking L^2 inner product with $|\theta|^{r-2}\theta$ to the first equation of (1.2), that

$$\frac{1}{r}\frac{d}{dt}\|\theta\|^r_{L^r} + \|\theta\|^r_{L^r} = 0, \qquad \forall 1 < r < \infty.$$

A simple computation implies

$$\frac{d}{dt}(e^t\|\theta\|_{L^r}) = 0,$$

from which we can get

$$\|\theta\|_{L^r} = e^{-t}\|\theta_0\|_{L^r}, \qquad \forall 1 < r < \infty. \tag{3.1}$$

Similarly, taking L^2 inner product with u to the second equation of (1.2) and using (1.3), implies

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} + \|\nabla u\|^2_{L^2} = \int_{\mathbb{R}^2} \theta u_2 \, dx, \tag{3.2}$$

from which we can deduce, for any $1 < q < 2$, that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} + \|\nabla u(t)\|^2_{L^2} &\lesssim \|\theta\|_{L^q}\|u\|_{L^{\frac{q}{q-1}}} \lesssim \|\theta\|_{L^q}\|u\|^{2-\frac{2}{q}}_{L^2}\|\nabla u\|^{\frac{2}{q}-1}_{L^2} \\ &\lesssim \|\theta\|^{\frac{2q}{3q-2}}_{L^q}\|u\|^{\frac{4q-4}{3q-2}}_{L^2} + \frac{1}{2}\|\nabla u\|^2_{L^2}.\end{aligned}$$

By using the Osgood's Lemma, we have

$$\begin{aligned}\|u(t)\|^2_{L^2} + \|\nabla u(t)\|^2_{L^2_t(L^2)} &\lesssim \|u_0\|^2_{L^2} + \Big(\int_0^t \|\theta(t')\|^{\frac{2q}{3q-2}}_{L^q}\, dt'\Big)^{\frac{3q-2}{q}} \\ &\lesssim \|u_0\|^2_{L^2} + \Big(\int_0^t (e^{-t'}\|\theta_0\|_{L^q})^{\frac{2q}{3q-2}}\, dt'\Big)^{\frac{3q-2}{q}} \\ &\lesssim \|u_0\|^2_{L^2} + \|\theta_0\|^2_{L^q}.\end{aligned} \tag{3.3}$$

When $q = 2$, from (3.1) and (3.2), one has

$$\begin{aligned}\|u(t)\|_{L^2} + \|\nabla u(t)\|_{L^2_t(L^2)} &\lesssim \|u_0\|_{L^2} + \int_0^t \|\theta(t')\|_{L^2}\, dt' \lesssim \|u_0\|_{L^2} + \int_0^t (e^{-t'}\|\theta_0\|_{L^2})\, dt' \\ &\lesssim \|u_0\|_{L^2} + \|\theta_0\|_{L^2}.\end{aligned} \tag{3.4}$$

Now, we are concerned about estimates regarding θ equation of (1.2).

We first get, by applying $\dot{\Delta}_j$ to the first equation of (1.2), that

$$\partial_t\dot{\Delta}_j\theta+u\cdot\nabla\dot{\Delta}_j\theta+\dot{\Delta}_j\theta=[u\cdot\nabla,\dot{\Delta}_j]\theta.$$

Taking the L^2 inner product of the resulting equation with $|\dot{\Delta}_j\theta|^{p-2}\dot{\Delta}_j\theta$ and using $\operatorname{div} u=0$, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\dot{\Delta}_j\theta\|_{L^p}^p+\|\dot{\Delta}_j\theta\|_{L^p}^p=\int_{\mathbb{R}^2}[u\cdot\nabla,\dot{\Delta}_j]\theta\,|\dot{\Delta}_j\theta|^{p-2}\dot{\Delta}_j\theta dx.$$

From this, using Lemma 2.3, we get

$$\frac{d}{dt}\|\dot{\Delta}_j\theta\|_{L^p}+\|\dot{\Delta}_j\theta\|_{L^p}\lesssim d_j(t)2^{-\frac{2}{p}j}\|u\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}}\|\theta\|_{\dot{B}_{p,1}^{\frac{2}{p}}}.$$

Integrating the above inequality over $[0,t]$, multiplying the resulting inequality by $2^{\frac{2}{p}j}$ and taking summation for $j\in\mathbb{Z}$, we arrive at

$$\|\theta\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})}+\|\theta\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}\lesssim\|\theta_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}}+\int_0^t\|\theta\|_{\dot{B}_{p,1}^{\frac{2}{p}}}\|u\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}}d\tau. \tag{3.5}$$

By the Gronwall inequality, we get

$$\|\theta\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})}+\|\theta\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}\lesssim\|\theta_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}}\exp(C\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}). \tag{3.6}$$

With a similar method, we can also get the following estimate:

$$\|\theta\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})}+\|\theta\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})}\lesssim\|\theta_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}\exp(C\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}). \tag{3.7}$$

In the following, we give the estimates about u.

Let $\mathbb{P}=\mathrm{Id}+\nabla(-\Delta)^{-1}\operatorname{div}$ be the Leray projection operator. We first get, by applying $\mathbb{P}$ to the second equation of (1.2), that

$$\partial_t u+\mathbb{P}(u\cdot\nabla u)-\nu\Delta u-\operatorname{div}(\mathbb{P}((\nu(\theta)-\nu)\nabla u))=\mathbb{P}(\theta e_2).$$

Applying $\dot{\Delta}_j$ to the above equation and using a standard commutator's process give

$$\partial_t\dot{\Delta}_j u+u\cdot\nabla\dot{\Delta}_j u-\nu\Delta\dot{\Delta}_j u=\dot{\Delta}_j\operatorname{div}(\mathbb{P}((\nu(\theta)-\nu)\nabla u))-[\dot{\Delta}_j\mathbb{P},u\cdot\nabla]u+\dot{\Delta}_j\mathbb{P}(\theta e_2).$$

Taking the L^2 inner product of the above equation with $|\dot{\Delta}_j u|^{p-2}\dot{\Delta}_j u$ and integrating by parts, we have

$$\begin{aligned}\|u\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})}+\nu\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}\lesssim{}&\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}+\sum_{j\in\mathbb{Z}}2^{(-1+\frac{2}{p})j}\|[\dot{\Delta}_j\mathbb{P},u\cdot\nabla]u\|_{L_t^1(L^p)}\\&+\|\mathbb{P}((\nu(\theta)-\nu)\nabla u)\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})}+\|\mathbb{P}(\theta e_2)\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})}\\\lesssim{}&\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}+\sum_{j\in\mathbb{Z}}2^{(-1+\frac{2}{p})j}\|[\dot{\Delta}_j\mathbb{P},u\cdot\nabla]u\|_{L_t^1(L^p)}\\&+\|\theta\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})}\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}+\|\theta\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})}.\end{aligned} \tag{3.8}$$

Now, we have to deal with the second term on the right hand side of (3.8). In fact, by virtue of Bony's decomposition, the commutator in (3.8) may be decomposed into

$$[\dot{\Delta}_j\mathbb{P}, u\cdot\nabla]u = [\dot{\Delta}_j\mathbb{P}, \dot{T}_u\cdot\nabla]u + \dot{\Delta}_j\mathbb{P}(\dot{T}_{\nabla u}u + \dot{R}(u,\nabla u)) - \dot{T}_{\nabla\dot{\Delta}_j u}u - \dot{R}(u,\nabla\dot{\Delta}_j u). \quad (3.9)$$

We further decompose the first term in the right hand side of (3.9) into

$$[\dot{\Delta}_j\mathbb{P}, \dot{T}_u\cdot\nabla]u = \sum_{|j-k|\le 5}[\dot{\Delta}_j, \dot{S}_{k-1}u]\dot{\Delta}_k\nabla u.$$

Writing $h=\mathcal{F}^{-1}\varphi$ and applying the first-order Taylor's formula, we get, for $x\in\mathbb{R}^2$,

$$\begin{aligned}&\sum_{|j-k|\le 5}[\dot{\Delta}_j, \dot{S}_{k-1}u]\dot{\Delta}_k\nabla u\\ &=2^{2j}\sum_{|k-j|\le 5}\int_0^1\int_{\mathbb{R}^2} h(2^j z)z\cdot\dot{S}_{k-1}\nabla u(x+(\xi-1)z)\dot{\Delta}_k\nabla u(x-z)dzd\xi.\end{aligned}$$

Applying Hölder's inequality and the property of the translation invariance of the Lebesgue measure, we have

$$\begin{aligned}\|[\dot{\Delta}_j, \dot{T}_u\cdot\nabla]u\|_{L^p} &\le 2^{2j}\sum_{|k-j|\le 5}\int_0^1\int_{\mathbb{R}^2}|h_1(2^j z)|\|\dot{S}_{k-1}\nabla u(\cdot+\xi z)\|_{L^\infty}\|\dot{\Delta}_k\nabla u(\cdot - z)\|_{L^p}dzd\xi\\ &\le C2^{-j}\sum_{|k-j|\le 5}\|\dot{S}_{k-1}\nabla u\|_{L^\infty}\|\dot{\Delta}_k\nabla u\|_{L^p}\\ &\le C2^{-j}\sum_{|k-j|\le 5}\|\dot{S}_{k-1}\nabla u\|_{L^\infty}2^{(1-\frac{2}{p})k}\|\dot{\Delta}_k\nabla u\|_{L^2},\end{aligned}$$

where $h_1(z)=zh(z)$. On the one hand, we deduce from Bernstein's lemma that

$$\begin{aligned}\|\dot{S}_{k-1}\nabla u\|_{L^\infty} &\le C\sum_{\ell\le k-2}2^\ell\|\dot{\Delta}_\ell\nabla u\|_{L^2}\\ &\le Cd_k 2^k\|\nabla u\|_{L^2},\end{aligned}$$

thus, we have

$$\|[\dot{\Delta}_j\mathbb{P}, \dot{T}_u\cdot\nabla]u\|_{L^p} \le Cd_j 2^{-(-1+\frac{2}{p})j}\|\nabla u\|_{L^2}^2.$$

Along the same lines, we have

$$\begin{aligned}\|\dot{\Delta}_j\mathbb{P}(\dot{T}_{\nabla u}u)\|_{L^p} &\le C\sum_{|k-j|\le 5}\|\dot{S}_{k-1}\nabla u\|_{L^\infty}\|\dot{\Delta}_k u\|_{L^p}\\ &\le Cd_j 2^{-(-1+\frac{2}{p})j}\|\nabla u\|_{L^2}^2. \end{aligned}\quad (3.10)$$

On the other hand, we get, by applying $\operatorname{div} u=0$, that

$$\begin{aligned}\|\dot{\Delta}_j\mathbb{P}(\dot{R}(u,\nabla u))\|_{L^p} &\le C2^{(3-\frac{2}{p})j}\sum_{k\ge j-3}\|\dot{\Delta}_k u\|_{L^2}\|\tilde{\dot{\Delta}}_k u\|_{L^2}\\ &\le C2^{(3-\frac{2}{p})j}\sum_{k\ge j-3}d_k 2^{-2k}\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}\end{aligned}$$

$$\leq C d_j 2^{-(-1+\frac{2}{p})j}\|\nabla u\|_{L^2}^2. \tag{3.11}$$

The terms $\dot{T}_{\nabla\dot{\Delta}_j u}u$ and $\dot{R}(u,\nabla\dot{\Delta}_j u)$ can be estimated in the same way as (3.10), (3.11), thus we can finally get

$$\|[\dot{\Delta}_j\mathbb{P},u\cdot\nabla]u\|_{L^p}\leq C d_j 2^{-(-1+\frac{2}{p})j}\|\nabla u\|_{L^2}^2. \tag{3.12}$$

Inserting the estimates (3.12) into (3.8) and using the estimates (3.3), (3.4), (3.6), (3.7), we have

$$\begin{aligned}\|u\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})}+\nu\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}&\lesssim\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}+\|\nabla u\|_{L_t^2(L^2)}^2\\&\quad+\|\theta\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})}\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}+\|\theta\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})}\\&\lesssim\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}+\|u_0\|_{L^2}^2+\|\theta_0\|_{L^q}^2\\&\quad+(\|\theta_0\|_{\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}}}+\|\theta_0\|_{\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}}}\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})})\exp(C\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}).\end{aligned} \tag{3.13}$$

Thus, if we denote $Y(t)\stackrel{\text{def}}{=}\|u\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})}+\nu\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}$, then one can deduce from (3.6), (3.7) and (3.13) that

$$\|\theta\|_{\widetilde{L}_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}})}+\|\theta\|_{L_t^1(\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}})}\leq C\|\theta_0\|_{\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}}}\exp(CY(t)/\nu), \tag{3.14}$$

$$Y(t)\leq C\Big(\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}+\|u_0\|_{L^2}^2+\|\theta_0\|_{L^q}^2+(\|\theta_0\|_{\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}}}+\|\theta_0\|_{\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}}}Y(t)/\nu)\exp(CY(t)/\nu)\Big). \tag{3.15}$$

In particular, if we take ε_0 to be sufficiently small and C_0 to be sufficiently large in (1.5), one has

$$\|\theta_0\|_{\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}}}\exp\big\{C(\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}+\|u_0\|_{L^2}^2+\|\theta_0\|_{L^q}^2)/\nu\big\}\leq\frac{1}{4},$$

which, together with (3.14), (3.15), ensures for any $t\in(0,T^*)$ that

$$Y(t)\leq 2C(\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}+\|u_0\|_{L^2}^2+\|\theta_0\|_{L^q}^2), \tag{3.16}$$

$$\|\theta\|_{\widetilde{L}_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}})}+\|\theta\|_{L_t^1(\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}})}\leq C\|\theta_0\|_{\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}}}\exp(2C^2(\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}+\|u_0\|_{L^2}^2+\|\theta_0\|_{L^q}^2)/\nu). \tag{3.17}$$

This in turn proves that $T^*=\infty$. In fact, let us assume (by contradiction) that $T^*<\infty$. Next, applying (3.14), (3.15), (3.16) and (3.17) for all $t<T^*$ yields

$$\|u\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})}+\nu\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}+\|\theta\|_{\widetilde{L}_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}})}+\|\theta\|_{L_t^1(\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}})}\leq C<\infty. \tag{3.18}$$

Then, for all $t_0\in[0,T^*)$, one can solve (1.2) starting with data (u_0,θ_0) at time $t=t_0$ and get a solution according to Theorem A.1 in the Appendix on the interval $[t_0,T+t_0]$ with T independent of t_0. Choosing $t_0>T^*-T$ thus shows that the solution can be continued beyond T^*, which is a contradiction.

Thus, we have completed the proof of Theorem 1.1. ∎

4. Proof of Theorem 1.2

The goal of this section is to present the proof of Theorem 1.2. In fact, given $\theta_0 \in \dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, $u_0 \in \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, it deduces from the Theorem A.1 in the Appendix that there exists a positive time T so that (1.2) has a unique local solution (θ,u) on $[0,T^*)$, such that for any $T<T^*$

$$\begin{aligned}&\theta \in C([0,T];\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap \widetilde{L}^\infty_T(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)),\\ &u\in C([0,T];\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap \widetilde{L}^\infty_T(\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap L^1_T(\dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)).\end{aligned} \tag{4.1}$$

We denote T^* to be the largest possible time such that there holds (4.1). Then, the proof of Theorem 1.2 is reduced to showing that $T^*=\infty$ under the assumption of (1.6).

Towards this, we shall deal with the L^p-type energy estimate for θ and u separately. In order to do so, we denote $u(t)=u_F(t)+\bar{u}(t)$ with

$$u_F(t)\stackrel{\text{def}}{=} e^{\nu t\Delta}u_0, \quad f(t)\stackrel{\text{def}}{=}\|u_F(t)\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}}$$

and define

$$\theta_\lambda(t,x)\stackrel{\text{def}}{=}\theta(t,x)\exp\left\{-\lambda\int_0^t f(t')dt'\right\},\quad u_\lambda(t,x)\stackrel{\text{def}}{=}u(t,x)\exp\left\{-\lambda\int_0^t f(t')dt'\right\}$$

with $\lambda\ge 0$.

4.1. The estimates of θ equation.

Proposition 4.1. *Let $\theta_0\in\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ and $u\in L^\infty_T(\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap L^1_T(\dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))$ with $1\le p<2n$. Then the following equation*

$$\partial_t\theta+u\cdot\nabla\theta+\theta=0,\qquad \theta|_{t=0}=\theta_0,\qquad (t,x)\in(0,T]\times\mathbb{R}^n, \tag{4.2}$$

has a unique solution $\theta\in C([0,T];\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))$. Moreover, there exists a constant $C_1>0$ such that for $\forall t\in[0,T]$

$$\|\theta_\lambda\|_{\widetilde{L}^\infty_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+\frac{\lambda}{2}\|\theta_\lambda\|_{L^1_{t,f}(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+\|\theta_\lambda\|_{L^1_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}\le\|\theta_0\|_{\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}}+C_1\|\theta\|_{\widetilde{L}^\infty_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}. \tag{4.3}$$

Proof. Both the existence and uniqueness of a solution to (4.2) essentially follow from estimates (4.3). For simplicity, we just present the a priori estimate for a smooth enough solution of (4.2). With the notation of θ_λ, (4.2) is reduced to

$$\partial_t\theta_\lambda+\lambda f(t)\theta_\lambda+u\cdot\nabla\theta_\lambda+\theta_\lambda=0,\quad (t,x)\in[0,T]\times\mathbb{R}^n.$$

Applying $\dot{\Delta}_j$ to the above equation and taking the L^2 inner product of the resulting equation with $|\dot{\Delta}_j\theta_\lambda|^{p-2}\dot{\Delta}_j\theta_\lambda$, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\dot{\Delta}_j\theta_\lambda\|^p_{L^p}+\lambda f(t)\|\dot{\Delta}_j\theta_\lambda\|^p_{L^p}+\|\dot{\Delta}_j\theta_\lambda\|^p_{L^p}=\int_{\mathbb{R}^n}[u\cdot\nabla,\dot{\Delta}_j]\theta_\lambda|\dot{\Delta}_j\theta_\lambda|^{p-2}\dot{\Delta}_j\theta_\lambda dx.$$

From this, using Lemma 2.3, we get

$$\frac{d}{dt}\|\dot{\Delta}_j\theta_\lambda\|_{L^p}+\lambda f(t)\|\dot{\Delta}_j\theta_\lambda\|_{L^p}+\|\dot{\Delta}_j\theta_\lambda\|_{L^p}\lesssim d_j(t)2^{-\frac{n}{p}j}\|u\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}}\|\theta_\lambda\|_{\dot{B}^{\frac{n}{p}}_{p,1}}.$$

Integrating the above inequality over $[0,t]$ and using $u=u_F+\bar{u}$ leads to

$$
\begin{aligned}
&\|\dot{\Delta}_j\theta_\lambda\|_{L^\infty_t(L^p)}+\lambda\int_0^t f(\tau)\|\dot{\Delta}_j\theta_\lambda(\tau)\|_{L^p}d\tau+\int_0^t\|\dot{\Delta}_j\theta_\lambda(\tau)\|_{L^p}d\tau\\
\lesssim&\|\dot{\Delta}_j\theta_0\|_{L^p}+\int_0^t d_j(\tau)2^{-\frac{n}{p}j}\|u(\tau)\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}}\|\theta_\lambda(\tau)\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau\\
\lesssim&\|\dot{\Delta}_j\theta_0\|_{L^p}+\int_0^t d_j(\tau)2^{-\frac{n}{p}j}\|\bar{u}_\lambda(\tau)\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}}\|\theta(\tau)\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau\\
&\quad+\int_0^t d_j(\tau)2^{-\frac{n}{p}j}f(\tau)\|\theta_\lambda(\tau)\|_{\dot{B}^{\frac{n}{p}}_{p,1}}d\tau.
\end{aligned}
$$

Multiplying the above inequality by $2^{\frac{n}{p}j}$, taking summation for $j\in\mathbb{Z}$ and taking $\lambda\geq 2C$, we arrive at

$$
\|\theta_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{p}}_{p,1})}+\lambda\|\theta_\lambda\|_{L^1_{t,f}(\dot{B}^{\frac{n}{p}}_{p,1})}+\|\theta_\lambda\|_{L^1_t(\dot{B}^{\frac{n}{p}}_{p,1})}\leq\|\theta_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}}+C\|\theta\|_{L^\infty_t(\dot{B}^{\frac{n}{p}}_{p,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}. \tag{4.4}
$$

Similarly, we can obtain

$$
\begin{aligned}
&\|\theta_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\lambda\|\theta_\lambda\|_{L^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\theta_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\
\leq&\|\theta_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}+C\|\theta\|_{L^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}.
\end{aligned} \tag{4.5}
$$

Taking estimate (4.4) together with estimate (4.5) implies the estimate (4.3). Consequently, we have completed the proof of Proposition 4.1. □

4.2. The estimates of u equation. We first get, from $u(t)=u_F(t)+\bar{u}(t)$ with $u_F(t)=e^{\nu t\Delta}u_0$, that $(\bar{u},\nabla\Pi)$ solves the following Stokes system

$$
\begin{cases}
\bar{u}_t-\nu\Delta\bar{u}+\nabla\Pi=F, & (t,x)\in(0,T^*)\times\mathbb{R}^n,\\
\operatorname{div}\bar{u}=0,\\
\bar{u}|_{t=0}=0,
\end{cases} \tag{4.6}
$$

with

$$
\begin{aligned}
F=&\operatorname{div}((\nu(\theta)-\nu)\nabla u_F)+\operatorname{div}((\nu(\theta)-\nu)\nabla\bar{u})\\
&-(\bar{u}\cdot\nabla\bar{u}+\operatorname{div}(u_F\otimes\bar{u}+\bar{u}\otimes u_F)+u_F\cdot\nabla u_F)+\theta e_n.
\end{aligned}
$$

Let $\mathbb{P}$ be the Leray projection operator, applying $\mathbb{P}$ to the first equation of (4.6), then we get by Duhamel's formula that

$$
\bar{u}(t)=\int_0^t e^{\nu(t-\tau)\Delta}\mathbb{P}F(\tau)d\tau.
$$

Applying Lemma 2.1 gives rise to

$$
\|\dot{\Delta}_j\bar{u}(t)\|_{L^p}\lesssim\int_0^t e^{-c\nu 2^{2j}(t-\tau)}\|\dot{\Delta}_jF(\tau)\|_{L^p}d\tau.
$$

Multiplying the above inequality by $\exp\Big\{-\lambda\int_0^t f(t')dt'\Big\}$, we obtain

$$
\|\dot{\Delta}_j\bar{u}_\lambda(t)\|_{L^p}\lesssim\int_0^t e^{-c\nu 2^{2j}(t-\tau)}\exp\Big\{-\lambda\int_\tau^t f(t')dt'\Big\}\|\dot{\Delta}_jF_\lambda(\tau)\|_{L^p}d\tau. \tag{4.7}
$$

Then it is easy to observe that

$$\|\dot{\Delta}_j\bar{u}_\lambda\|_{L^\infty_T(L^p)}\lesssim\int_0^T\|\dot{\Delta}_jF_\lambda(\tau)\|_{L^p}d\tau,$$

from which, we deduce that

$$\|\bar{u}_\lambda\|_{\widetilde{L}^\infty_T(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\lesssim\|F_\lambda\|_{L^1_T(\dot{B}^{-1+\frac{n}{p}}_{p,1})}. \tag{4.8}$$

On the other hand, integrating (4.7) over $[0,T]$, we arrive at

$$\begin{aligned}\|\dot{\Delta}_j\bar{u}_\lambda\|_{L^1_T(L^p)}&\lesssim\int_0^T\int_0^t e^{-c\nu 2^{2j}(t-\tau)}\|\dot{\Delta}_jF_\lambda(\tau)\|_{L^p}d\tau dt\\&\lesssim\int_0^T\|\dot{\Delta}_jF_\lambda(\tau)\|_{L^p}d\tau\int_\tau^T e^{-c\nu 2^{2j}(t-\tau)}dt\lesssim\int_0^T\|\dot{\Delta}_jF_\lambda(\tau)\|_{L^p}d\tau,\end{aligned}$$

which implies

$$\nu\|u_\lambda\|_{L^1_T(\dot{B}^{1+\frac{n}{p}}_{p,1})}\lesssim\|F_\lambda\|_{L^1_T(\dot{B}^{-1+\frac{n}{p}}_{p,1})}. \tag{4.9}$$

Multiplying by $\lambda f(t)$ on both the sides of (4.7) and then integrating the resulting inequality over $[0,T]$ gives

$$\begin{aligned}\lambda\int_0^T\|\dot{\Delta}_j\bar{u}_\lambda(t)\|_{L^p}f(t)dt&\lesssim\int_0^T\int_0^t\lambda f(t)\exp\left\{-\lambda\int_\tau^t f(t')dt'\right\}\|\dot{\Delta}_jF_\lambda(\tau)\|_{L^p}d\tau dt\\&=-\int_0^T\|\dot{\Delta}_jF_\lambda(\tau)\|_{L^p}d\tau\int_\tau^T\frac{d}{dt}\exp\left\{-\lambda\int_\tau^t f(t')dt'\right\}dt\\&\lesssim\int_0^T\|\dot{\Delta}_jF_\lambda(\tau)\|_{L^p}d\tau.\end{aligned}$$

As a consequence, we have

$$\lambda\|\bar{u}_\lambda\|_{L^1_{T,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\lesssim\|F_\lambda\|_{L^1_T(\dot{B}^{-1+\frac{n}{p}}_{p,1})} \tag{4.10}$$

which, combining with estimates (4.8) and (4.9), implies that

$$\|u_\lambda\|_{\widetilde{L}^\infty_T(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\lambda\|u_\lambda\|_{L^1_{T,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\nu\|u_\lambda\|_{L^1_T(\dot{B}^{1+\frac{n}{p}}_{p,1})}\lesssim\|F_\lambda\|_{L^1_T(\dot{B}^{-1+\frac{n}{p}}_{p,1})}. \tag{4.11}$$

Applying Lemma 2.2 and using the definition of θ_λ and $\bar{u}_\lambda$, we have for any $t<T^*$ that,

$$\begin{aligned}\|\bar{u}\cdot\nabla\bar{u}_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}&\lesssim\|\bar{u}\|_{L^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})},\\\|(u_F\cdot\nabla u_F)_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}&\lesssim\|u_F\cdot\nabla u_F\|_{L^1(\mathbb{R}^+;\dot{B}^{-1+\frac{n}{p}}_{p,1})},\\\|\mathrm{div}((\nu(\theta)-\nu)\nabla u_{F,\lambda})\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}&\lesssim\int_0^t\|\theta_\lambda\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|u_F\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}}d\tau\lesssim\|\theta_\lambda\|_{L^1_{t,f}(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})},\\\|\mathrm{div}((\nu(\theta)-\nu)\nabla\bar{u}_\lambda)\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}&\lesssim\int_0^t\|\theta\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}_\lambda\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}}d\tau\lesssim\|\theta\|_{L^\infty_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}.\end{aligned}$$

Using Lemma 2.2 and interpolation inequality, we have, for any $\delta>0$, that

$$
\begin{aligned}
\|\mathrm{div}(u_F\otimes\bar{u}_\lambda+\bar{u}_\lambda\otimes u_F)\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\lesssim&\int_0^t\|\bar{u}_\lambda\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|u_F\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\,d\tau\\
\lesssim&\int_0^t\|\bar{u}_\lambda\|^{\frac12}_{\dot{B}^{1+\frac{n}{p}}_{p,1}}\|\bar{u}_\lambda\|^{\frac12}_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\|u_F\|^{\frac12}_{\dot{B}^{1+\frac{n}{p}}_{p,1}}\|u_F\|^{\frac12}_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\,d\tau\\
\lesssim&\delta\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\frac{1}{\delta}\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\|\bar{u}_\lambda\|_{L^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}.
\end{aligned}
$$

From the above estimates, one can conclude for any $0\le t<T^*$ that

$$
\begin{aligned}
\|F_\lambda\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\lesssim&\|u_F\cdot\nabla u_F\|_{L^1(\mathbb{R}^+;\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\
&+\|\theta_\lambda\|_{L^1_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+\|\theta_\lambda\|_{L^1_{t,f}(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+\frac{1}{\delta}\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\|\bar{u}_\lambda\|_{L^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\
&+\big(\delta+\|\theta\|_{L^\infty_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+\|\bar{u}\|_{L^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\big)\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}.
\end{aligned}\tag{4.12}
$$

Inserting the above estimate (4.12) into (4.11), and taking $\delta\ll\nu$, we arrive at, for any $0\le t<T^*$, that

$$
\begin{aligned}
&\|\bar{u}_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\lambda\|\bar{u}_\lambda\|_{L^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\frac{\nu}{2}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}\\
\le&C_2\|u_F\cdot\nabla u_F\|_{L^1(\mathbb{R}^+;\dot{B}^{-1+\frac{n}{p}}_{p,1})}+C_2\|\theta_\lambda\|_{L^1_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+C_2\|\theta_\lambda\|_{L^1_{t,f}(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}\\
&+\frac{C_2}{\delta}\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\|\bar{u}_\lambda\|_{L^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+C_2\big(\|\theta\|_{L^\infty_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+\|\bar{u}\|_{L^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\big)\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}.
\end{aligned}\tag{4.13}
$$

4.3. Complete the proof of Theorem 1.2.

Proof. In this subsection, we will use bootstrap's argument to complete the proof of Theorem 1.2.

Firstly, let γ be a large enough positive constant, which will be determined later on. Multiplying by γ on both the sides of estimate (4.3) gives

$$
\begin{aligned}
&\gamma\|\theta_\lambda\|_{\widetilde{L}^\infty_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+\frac{\gamma\lambda}{2}\|\theta_\lambda\|_{L^1_{t,f}(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+\gamma\|\theta_\lambda\|_{L^1_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}\\
\le&\gamma\|\theta_0\|_{\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}}+\gamma C_1\|\theta\|_{L^\infty_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})},\quad 0\le t<T^*.
\end{aligned}\tag{4.14}
$$

Summing up (4.13) and (4.14) leads to

$$
\begin{aligned}
&\|\bar{u}_\lambda\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\gamma\|\theta_\lambda\|_{\widetilde{L}^\infty_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+\frac{\gamma\lambda}{2}\|\theta_\lambda\|_{L^1_{t,f}(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+\gamma\|\theta_\lambda\|_{L^1_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}\\
&\quad+\lambda\|\bar{u}_\lambda\|_{L^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\frac{\nu}{2}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}\\
\le&C_2\|u_F\cdot\nabla u_F\|_{L^1(\mathbb{R}^+;\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\gamma\|\theta_0\|_{\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}}+C_2\|\theta_\lambda\|_{L^1_{t,f}(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}\\
&+\frac{C_2}{\delta}\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\|\bar{u}_\lambda\|_{L^1_{t,f}(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+C_2\|\theta_\lambda\|_{L^1_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}\\
&+\big(C_1\|\theta\|_{L^\infty_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+\gamma C_1\|\theta\|_{L^\infty_t(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})}+C_2\|\bar{u}\|_{L^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\big)\|\bar{u}_\lambda\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}.
\end{aligned}\tag{4.15}
$$

Choosing $\gamma\geq 2C_2$ and $\lambda\geq\max\{4C_2/\gamma, 2C_2\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}/\delta\}$ in the above inequality, we have for any $0\leq t<T^*$ that

$$\begin{aligned}&\|\bar{u}_\lambda\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\frac{\gamma}{2}\|\theta_\lambda\|_{\widetilde{L}_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}+\frac{\gamma\lambda}{4}\|\theta_\lambda\|_{L^1_{t,f}(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}+\gamma\|\theta_\lambda\|_{L^1_t(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}\\&\quad+\frac{\lambda}{2}\|\bar{u}_\lambda\|_{L^1_{t,f}(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\frac{\nu}{2}\|\bar{u}_\lambda\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})}\\&\leq C_2\|u_F\cdot\nabla u_F\|_{L^1(\mathbb{R}^+;\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\gamma\|\theta_0\|_{\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}}}\\&\quad+\big(C_1\|\theta\|_{L_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}+\gamma C_1\|\theta\|_{L_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}+C_2\|\bar{u}\|_{L_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\big)\|\bar{u}_\lambda\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})}.\end{aligned}\tag{4.16}$$

To complete the proof, we shall use the method of continuity. For this, we define

$$T^{**}\stackrel{\text{def}}{=}\sup\big\{t\in[0,T^*):\|\bar{u}\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}/\nu+\|\bar{u}\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|\theta\|_{\widetilde{L}_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}\leq c_1\big\}.\tag{4.17}$$

In what follows, we will prove that $T^{**}=T^*$ under the assumption of (1.6).

If not, we assume that $T^{**}<T^*$, and for $\forall t\leq T^{**}$, taking

$$0<c_1\leq\frac{\nu}{4(C_1+\gamma C_1+C_2\nu)},$$

we deduce from (4.16) and (4.17) that

$$\begin{aligned}&\|\bar{u}_\lambda\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\gamma\|\theta_\lambda\|_{\widetilde{L}_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}+\nu\|\bar{u}_\lambda\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\gamma\|\theta_\lambda\|_{L^1_t(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}\\&\leq C\big(\|u_F\cdot\nabla u_F\|_{L^1(\mathbb{R}^+;\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|\theta_0\|_{\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}}}\big).\end{aligned}$$

As a consequence,

$$\begin{aligned}&\|\bar{u}\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\gamma\|\theta\|_{\widetilde{L}_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}+\nu\|\bar{u}\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\gamma\|\theta\|_{L^1_t(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}+\|\theta\|_{L^1_t(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}\\&\leq C\big(\|u_F\cdot\nabla u_F\|_{L^1(\mathbb{R}^+;\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|\theta_0\|_{\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}}}\big)\exp\Big(\lambda\int_0^t f(t')dt'\Big)\\&\leq C\big(\|u_F\cdot\nabla u_F\|_{L^1(\mathbb{R}^+;\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|\theta_0\|_{\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}}}\big)\exp\Big(C\|u_0\|^2_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}/\nu^2\Big).\end{aligned}\tag{4.18}$$

If the smallness condition (1.6) is satisfied, then we deduce from (4.18) that

$$\|\bar{u}\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}/\nu+\|\bar{u}\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|\theta\|_{\widetilde{L}_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}+\|\theta\|_{L^1_t(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}\leq\frac{c_1}{2}$$

for $t\leq T^{**}$, which contradicts with (4.17). Whence, we conclude that $T^{**}=\infty$ and the conclusion of Theorem 1.2 follows. $\square$

REMARK 4.1. If $n=2$, by using a similar method, we can get the following theorem:

THEOREM 4.1. *Let $1<p<4$, $\nu(\theta)$ satisfy (1.3) and $\theta_0\in\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)$, $u_0\in\dot{B}_{p,1}^{-1+\frac{2}{p}}(\mathbb{R}^2)$ with $\operatorname{div}u_0=0$. Then there exist positive constants c_1 and C_1 such that,*

$$C_1\|\theta_0\|_{\dot{\mathcal{B}}_{p,1}^{\frac{2}{p}}}\exp\bigg\{C_1(1+\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}})^2\exp(C_1\|u_0\|^2_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}/\nu^2)\bigg\}\leq c_1,\tag{4.19}$$

the system (1.2) *has a unique global solution* (θ,u), *with*

$$\theta\in C_b([0,\infty);\dot{\mathcal{B}}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2))\cap L^1(\mathbb{R}^+;\dot{\mathcal{B}}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)),$$
$$u\in C_b([0,\infty);\dot{B}^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))\cap L^1(\mathbb{R}^+;\dot{B}^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)). \tag{4.20}$$

We only outline the main steps to prove Theorem 4.1. In fact, let u_R be the solution to the following incompressible Navier-Stokes equation in $\mathbb{R}^2$:

$$\begin{cases}\partial_t u_R-\Delta u_R+u_R\cdot\nabla u_R+\nabla\pi=0,\\ \operatorname{div}u_R=0,\\ u_R|_{t=0}=u_0.\end{cases} \tag{4.21}$$

From Proposition 3.1 in [14], for any $2<p<4$, we can get (4.21) as a unique global infinite energy solution u_R with

$$u_R\in C([0,+\infty);\dot{B}^{-1+\frac{2}{p}}_{p,1})\cap\widetilde{L}^\infty((0,+\infty);\dot{B}^{-1+\frac{2}{p}}_{p,1})\cap L^1((0,+\infty);\dot{B}^{1+\frac{2}{p}}_{p,1}),$$

moreover, there holds

$$\|u_R\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{2}{p}}_{p,1})}+\|u_R\|_{L^1_t(\dot{B}^{1+\frac{2}{p}}_{p,1})}+\|\nabla\pi\|_{L^1_t(\dot{B}^{-1+\frac{2}{p}}_{p,1})}$$
$$\le C\|u_0\|_{\dot{B}^{-1+\frac{2}{p}}_{p,1}}(1+\|u_0\|_{\dot{B}^{-1+\frac{2}{p}}_{p,1}})\exp\Big\{\frac{C}{\nu^2}\|u_0\|^2_{\dot{B}^{-1+\frac{2}{p}}_{p,1}}\Big\}. \tag{4.22}$$

Thus, if we take $u_F=u_R$ in the process of proving Theorem 1.2, by using similar arguments, one can complete the proof of Theorem 4.1.

REMARK 4.2. Compared to Theorem 1.1, here, the solution to system (1.2) has infinite energy.

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Appendix A. Local well-posedness of the Boussinesq system. In this section, we mainly prove the local well-posedness of the following system:

$$\begin{cases}\partial_t\theta+u\cdot\nabla\theta=0,\\ \partial_t u+u\cdot\nabla u-\operatorname{div}(\nu(\theta)\nabla u)+\nabla\Pi=\theta e_n,\\ \operatorname{div}u=0,\\ (\theta,u)|_{t=0}=(\theta_0,u_0).\end{cases} \tag{A.1}$$

More precisely, we will prove the following theorem:

THEOREM A.1. *Let* $\nu(\theta)$ *satisfies the condition* (1.3). *For any* $1\le p<2n$ *and* $\theta_0\in\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, $u_0\in\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ *with* $\operatorname{div}u_0=0$. *The system* (A.1) *has a unique local solution* (θ,u) *on* $[0,T]$ *such that*

$$\theta\in C([0,T];\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap\widetilde{L}^\infty_T(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)),$$
$$u\in C([0,T];\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap\widetilde{L}^\infty_T(\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n))\cap L^1_T(\dot{B}^{1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)).$$

REMARK A.1. If $\nu(\theta)$ is a positive constant independent of θ, we can relax the condition $\theta_0\in\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ to $\theta_0\in\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$.

Proof of Theorem A.1. We will divide the proof into several steps. In the first step, we construct an approximate solution sequence, and then we prove the uniform estimates to the approximate solutions in the second step. The convergence of the approximate solutions will be given in the third step. In the last step, we give the unique proof of the solution by using Lagrangian approach.

Step 1. Construction of smooth approximate solutions.

Firstly, there exists a sequence $\{(\theta_0^n,\widetilde{u}_0^n)\}_{n\in\mathbb{N}}\subset(\mathcal{S}(\mathbb{R}^n))^2$ such that $(\theta_0^n,\widetilde{u}_0^n)$ converges to (θ_0,u_0) in $\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)\times\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$. Define $u_0^n\stackrel{\text{def}}{=}\mathbb{P}\widetilde{u}_0^n$ so that $\operatorname{div}u_0^n=0$. Then u_0^n belongs to $H^\infty(\mathbb{R}^n)$ and converges to u_0 in $\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$.

Therefore, applying Theorem 1.1 of [1] ensures that system (A.1) with the initial data (θ_0^n,u_0^n), admits a unique local-in-time solution (θ^n,u^n) belonging to

$$C([0,T^n);H^{\alpha+1}(\mathbb{R}^n))\times C([0,T^n);H^{\alpha}(\mathbb{R}^n))\cap\widetilde{L}^1_{loc}(0,T^n;H^{\alpha+2}(\mathbb{R}^n)),$$

with $\alpha>\frac{n}{2}$.

Step 2. Uniform estimates to the approximate solutions.

Next, we shall prove that there exists a positive time $T<\inf_{n\in\mathbb{N}}T^n$ such that (θ^n,u^n) is uniformly bounded in the space

$$E_T\stackrel{\text{def}}{=}\widetilde{L}^\infty_T(\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1})\times\widetilde{L}^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})\cap L^1_T(\dot{B}^{\frac{n}{p}+1}_{p,1}).$$

For this, we define $\big(u_F(t),u_F^n(t)\big)\stackrel{\text{def}}{=}\big(e^{\nu t\Delta}u_0,e^{\nu t\Delta}u_0^n\big)$. A simple computation implies

$$\|u_F^n\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\nu\|u_F^n\|_{L^1(\mathbb{R}^+;\dot{B}^{1+\frac{n}{p}}_{p,1})}\lesssim\|u_0^n\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\lesssim\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}},\tag{A.2}$$

and

$$\begin{aligned}\|u_F^n\|_{L^1_T(\dot{B}^{1+\frac{n}{p}}_{p,1})}&\le\|u_F\|_{L^1_T(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|u_F^n-u_F\|_{L^1_T(\dot{B}^{1+\frac{n}{p}}_{p,1})}\\&\le\|u_F\|_{L^1_T(\dot{B}^{1+\frac{n}{p}}_{p,1})}+C\|u_0^n-u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}.\end{aligned}\tag{A.3}$$

Thus, for any $\varepsilon>0$, there exist, a number $k=k(\varepsilon)\in\mathbb{N}$ and a positive time $T=T(\varepsilon,u_0)$ such that

$$\sup_{n\ge k}\|u_F^n\|_{L^1_T(\dot{B}^{1+\frac{n}{p}}_{p,1})}\le\varepsilon.\tag{A.4}$$

Denote by $\bar{u}^n\stackrel{\text{def}}{=}u^n-u_F^n$, then we can deduce from the system (A.1) that $(\theta^n,\bar{u}^n)$ satisfies the following system:

$$\begin{cases}\partial_t\theta^n+(u_F^n+\bar{u}^n)\cdot\nabla\theta^n=0,\\\partial_t\bar{u}^n-\operatorname{div}\big(\nu(\theta^n)\nabla\bar{u}^n\big)+\nabla\Pi^n=F_n,\\\operatorname{div}\bar{u}^n=0,\\(\theta^n,\bar{u}^n)|_{t=0}=(\theta_0^n,0),\end{cases}\tag{A.5}$$

where

$$F_n=\theta^n e_n-(u_F^n+\bar{u}^n)\cdot\nabla\bar{u}^n-\bar{u}^n\cdot\nabla u_F^n-u_F^n\cdot\nabla u_F^n+\operatorname{div}\big((\nu(\theta^n)-\nu(0))\nabla u_F^n\big).\tag{A.6}$$

For notational simplicity, we denote by

$$\Theta^n(t)\stackrel{\rm def}{=}\|\theta^n\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac{n}{p}}_{p,1})}+\|\theta^n\|_{L^1_t(\mathcal{B}^{\frac{n}{p}}_{p,1})},\quad U^n(t)\stackrel{\rm def}{=}\|\bar{u}^n\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\bar{u}^n\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}.$$

By using a similar derivation of (3.6), (3.7), we can obtain from the first equation of (A.5) that

$$\|\theta^n\|_{\widetilde{L}^\infty_t(\mathcal{B}^{\frac{n}{p}}_{p,1})}\le\|\theta_0\|_{\mathcal{B}^{\frac{n}{p}}_{p,1}}\exp(C(\|u^n_F\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|\bar{u}^n\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})})).\tag{A.7}$$

Now, let us turn to the uniform estimates of $\bar{u}^n$.

We first get, by applying $\dot{\Delta}_j\mathbb{P}$ to the second equation of (A.5) and using a standard commutator's argument, that

$$\begin{aligned}\partial_t\dot{\Delta}_j\bar{u}^n-\operatorname{div}\big(\nu(\dot{S}_m\theta^n)\dot{\Delta}_j\nabla\bar{u}^n\big)=&\dot{\Delta}_j\mathbb{P}F_n+\operatorname{div}([\dot{\Delta}_j\mathbb{P},\nu(\dot{S}_m\theta^n)])\nabla\bar{u}^n\\&+\dot{\Delta}_j\mathbb{P}\operatorname{div}(\nu(\theta^n)-(\nu(\dot{S}_m\theta^n)))\nabla\bar{u}^n.\end{aligned}\tag{A.8}$$

From (1.3), we know that $\nu(\dot{S}_m\theta^n)$ has a positive lower bound. Thus, applying Lemma 2.1 and the Hölder inequality, we get, for some positive constant c, that

$$\begin{aligned}\frac{d}{dt}\|\dot{\Delta}_j\bar{u}^n\|_{L^p}+c2^{2j}\|\dot{\Delta}_j\bar{u}^n\|_{L^p}\lesssim&\|\dot{\Delta}_j\mathbb{P}F_n\|_{L^p}+\|\operatorname{div}([\dot{\Delta}_j\mathbb{P},\nu(\dot{S}_m\theta^n)]\nabla\bar{u}^n\|_{L^p}\\&+\|\dot{\Delta}_j\mathbb{P}\operatorname{div}(\nu(\theta^n)-(\nu(\dot{S}_m\theta^n)))\nabla\bar{u}^n\|_{L^p}.\end{aligned}\tag{A.9}$$

After time integration, multiplying $2^{(-1+\frac{n}{p})j}$ and summing up over j, we infer

$$\begin{aligned}&\|\bar{u}^n\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\bar{u}^n\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}\\\lesssim&\|F_n\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\sum_{j\in\mathbb{Z}}2^{(-1+\frac{n}{p})j}\|\operatorname{div}([\dot{\Delta}_j\mathbb{P},\nu(\dot{S}_m\theta^n)])\nabla\bar{u}^n\|_{L^1_t(L^p)}\\&+\sum_{j\in\mathbb{Z}}2^{(-1+\frac{n}{p})j}\|\dot{\Delta}_j\mathbb{P}\operatorname{div}(\nu(\theta^n)-(\nu(\dot{S}_m\theta^n)))\nabla\bar{u}^n\|_{L^1_t(L^p)}.\end{aligned}\tag{A.10}$$

Then using product laws in Besov spaces and (A.2), we obtain

$$\begin{aligned}&\|\bar{u}^n\cdot\nabla\bar{u}^n+u^n_F\cdot\nabla u^n_F\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\\lesssim&\|\bar{u}^n\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\|\bar{u}^n\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\|u^n_F\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}.\end{aligned}$$

Thanks to $\operatorname{div}u^n_F=\operatorname{div}\bar{u}^n=0$, we get, by using product laws and interpolation inequality in Besov spaces, that

$$\begin{aligned}&\|\bar{u}^n\cdot\nabla\bar{u}^n+u^n_F\cdot\nabla u^n_F+\bar{u}^n\cdot\nabla u^n_F+u^n_F\cdot\nabla\bar{u}^n\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\\lesssim&\|\bar{u}^n\otimes\bar{u}^n+u^n_F\otimes u^n_F+\bar{u}^n\otimes u^n_F+u^n_F\otimes\bar{u}^n\|_{L^1_t(\dot{B}^{\frac{n}{p}}_{p,1})}\\\lesssim&\int_0^t\|\bar{u}^n\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\bar{u}^n\|_{\dot{B}^{\frac{n}{p}}_{p,1}}+\|u^n_F\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|u^n_F\|_{\dot{B}^{\frac{n}{p}}_{p,1}}+\|\bar{u}^n\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|u^n_F\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\,d\tau\\\lesssim&\int_0^t\|\bar{u}^n\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\|\bar{u}^n\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}}+\|u^n_F\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\|u^n_F\|_{\dot{B}^{1+\frac{n}{p}}_{p,1}}\,d\tau\end{aligned}$$

$$\lesssim \|\bar{u}^n\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\|\bar{u}^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}\|u_F^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}.$$

Along the same lines, one has

$$\begin{aligned}\|\operatorname{div}\big(\nu(\theta^n)-\nu(0)\nabla u_F^n\big)\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}&\lesssim\|\theta^n\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\|u_F^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}\\&\lesssim\Theta^n(t)\|u_F^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}.\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}\|F_n\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{n}{p}})}\lesssim&\|\theta^n\|_{L_t^1(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}\\&+(U^n(t))^2+\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}\|u_F^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\Theta^n(t)\|u_F^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}.\end{aligned}\tag{A.11}$$

By Lemmas 2.2 and 2.3, one has

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}2^{(-1+\frac{n}{p})j}\|\dot{\Delta}_j\mathbb{P}\operatorname{div}(\nu(\theta^n)-(\nu(\dot{S}_m\theta^n)))\nabla\bar{u}^n\|_{L_t^1(L^p)}\\&\lesssim\|\theta^n-\dot{S}_m\theta^n\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})}\|\bar{u}^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})},\end{aligned}\tag{A.12}$$

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}2^{(-1+\frac{n}{p})j}\|\operatorname{div}([\dot{\Delta}_j\mathbb{P},\nu(\dot{S}_m\theta^n)])\nabla\bar{u}^n\|_{L_t^1(L^p)}\\&\lesssim\|\nabla\nu(\dot{S}_m\theta^n)\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})}\|\bar{u}^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})}\\&\lesssim2^m\|\theta^n\|_{\widetilde{L}_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}\|\bar{u}^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{n}{p}})}\\&\lesssim t^{\frac12}2^m\|\theta^n\|_{\widetilde{L}_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}(\|\bar{u}^n\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})}+\|\bar{u}^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}).\end{aligned}\tag{A.13}$$

Inserting the estimates (A.11)-(A.13) into (A.10) leads to

$$\begin{aligned}&\|\bar{u}^n\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{n}{p}})}+\|\bar{u}^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}\\&\lesssim\|\theta^n\|_{L_t^1(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}+(U^n(t))^2+\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}\|u_F^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}\\&\quad+\Theta^n(t)\|u_F^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\|\theta^n-\dot{S}_m\theta^n\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}})}\|\bar{u}^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}\\&\quad+t^{\frac12}2^m\|\theta^n\|_{\widetilde{L}_t^\infty(\dot{\mathcal{B}}_{p,1}^{\frac{n}{p}})}(\|\bar{u}^n\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{n}{p}-1})}+\|\bar{u}^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}),\end{aligned}\tag{A.14}$$

which implies

$$\begin{aligned}U^n(t)\lesssim&t\Theta^n(t)+(U^n(t))^2+t^{\frac12}2^m\Theta^n(t)U^n(t)\\&+\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{n}{p}}}\|u_F^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}+\Theta^n(t)\|u_F^n\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{n}{p}})}\end{aligned}\tag{A.15}$$

provided that

$$\|(\mathrm{Id}-\dot{S}_m)\theta^n\|_{\widetilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{n}{p}})}\le c_0\tag{A.16}$$

for some sufficiently small positive constant c_0 and some integer $m\in\mathbb{Z}$.

By using a similar derivation of (3.5), we can infer

$$\|\theta^n-\dot S_m\theta^n\|_{\widetilde L^\infty_t(\dot B^{\frac np}_{p,1})}\le\sum_{j\ge m}2^{\frac{nj}{p}}\|\dot\Delta_j\theta_0\|_{L^p}+\|\theta_0\|_{\dot B^{\frac np}_{p,1}}\big(\exp\big(CU^n(t)+C\|u^n_F\|_{L^1_t(\dot B^{1+\frac np}_{p,1})}\big)-1\big). \tag{A.17}$$

Next, for any $n\in\mathbb{N}$, we define

$$T^n_*\overset{\mathrm{def}}{=}\sup\{t\in(0,T^n):\ U^n(t)\le2\varepsilon_0\}, \tag{A.18}$$

with $\varepsilon_0\in(0,\frac12)$ to be determined. We shall prove $\inf_{n\in\mathbb{N}}T^n_*>0$.

Firstly, we deduce from (A.7) that

$$\begin{aligned}\Theta^n(t)&\le C\|\theta_0\|_{\dot{\mathcal B}^{\frac np}_{p,1}}\exp\big(C\big(U^n(t)+\|u_0\|_{\dot B^{-1+\frac np}_{p,1}}\big)\big)\\&\le C\|\theta_0\|_{\dot{\mathcal B}^{\frac np}_{p,1}}\exp\big(C\big(1+\|u_0\|_{\dot B^{-1+\frac np}_{p,1}}\big)\big)\\&\overset{\mathrm{def}}{=}\Theta_0.\end{aligned} \tag{A.19}$$

Secondly, noticing that $\theta_0\in\dot B^{\frac np}_{p,1}(\mathbb{R}^n)$, thus, there exists $m=m(c_0)\in\mathbb{Z}$ large enough such that

$$\sum_{j\ge m}2^{\frac{nj}{p}}\|\dot\Delta_j\theta_0\|_{L^p}\le\frac12c_0. \tag{A.20}$$

Taking ε_0 and T_0 small enough such that

$$C\|\theta_0\|_{\dot B^{\frac np}_{p,1}}\big(\exp\big(2C\varepsilon_0+C\|u^n_F\|_{L^1_{T_0}(\dot B^{1+\frac np}_{p,1})}\big)-1\big)\le\frac12c_0. \tag{A.21}$$

Combining with (A.20) and (A.21), we can deduce from (A.17) that (A.16) is fulfilled for $T=\min(T^n_*,T_0)$. Without loss of generality, we may assume $T^n_*\le T_0$. Then, for any $t\le T^n_*$, we deduce from (A.15) that

$$U^n(t)\lesssim t\Theta_0+2\varepsilon_0U^n(t)+t^{\frac12}2^m\Theta_0U^n(t)+\|u_0\|_{\dot B^{-1+\frac np}_{p,1}}\|u^n_F\|_{L^1_t(\dot B^{1+\frac np}_{p,1})}+\Theta_0\|u^n_F\|_{L^1_t(\dot B^{1+\frac np}_{p,1})}. \tag{A.22}$$

Finally, taking ε_0 and T_1 small enough ensures

$$C\big(2\varepsilon_0+t^{\frac12}2^m\Theta_0\big)\le\frac34,$$

$$CT_1\Theta_0+C\big(\|u_0\|_{\dot B^{-1+\frac np}_{p,1}}+\Theta_0\big)\|u^n_F\|_{L^1_{T_1}(\dot B^{1+\frac np}_{p,1})}\le\frac{\varepsilon_0}{4},$$

which together with (A.22) implies

$$U^n(t)\le\varepsilon_0,\quad\forall t\le\min(T^n_*,T_1).$$

However, by the definition of T_*^n, we eventually infer $T_*^n \geq T_1$ and $\sup U^n(T_1) \leq \varepsilon_0$, which along with (A.2) and (A.19) ensures that (θ^n, u^n) is uniformly bounded in E_{T_1}.

Step 3. Convergence.

The convergence of u_F^n to u_F readily stems from the definition of Besov spaces. The convergence of $(\theta^n, \bar{u}^n)$ is obtained by Ascoli's compactness theorem. The proof for the sequence satisfying the theorem is standard. A similar argument can be found on pages 1330-1331 of [8].

Step 4. Uniqueness of the solution.

As in [10, 26], by using Lagrangian approach to prove the uniqueness of the solution for the inhomogeneous Navier-Stokes equations; here, we also apply a Lagrangian approach to show the uniqueness part of Theorem A.1. Let us first derive algebraic relations involving changes of coordinates.

We are given a C^1-diffeomorphism X over $\mathbb{R}^n$. For $H:\mathbb{R}^n \to \mathbb{R}^m$, we agree that $\overline{H}(y) = H(x)$ with $x = X(y)$. With this convention, the chain rule writes

$$D_y\overline{H}(y) = D_x H(X(y)) \cdot D_y X(y) \quad \text{with} \quad (D_y X)_{ij} = \partial_{y_j} X^i. \tag{A.23}$$

or, denoting $\nabla_y = {}^T D_y$,

$$\nabla_y \overline{H}(y) = (\nabla_y X(y)) \cdot \nabla_x H(X(y)).$$

Hence we have

$$D_x H(x) = D_y \overline{H}(y) \cdot A(y) \quad \text{with} \quad A(y) \stackrel{\text{def}}{=} (D_y X(y))^{-1} = D_x X^{-1}(x). \tag{A.24}$$

Let us first derive formally the Lagrangian equations corresponding to (A.1). Let X_u be the flow associated to the vector-field u, that is the solution to

$$X_u(t,y) = y + \int_0^t u(\tau, X_u(\tau, y))d\tau.$$

Now, denoting

$$(\rho, v, P)(t,y) \stackrel{\text{def}}{=} (\theta, u, \Pi)\big(t, X_u(t,y)\big). \tag{A.25}$$

Thanks to the transport equation of (A.1) and the chain rule, we infer $\rho(t,\cdot) \equiv \theta_0$. Applying Proposition 8 in [9] implies that $(v, \nabla P)$ belongs to the same functional space as $(u, \nabla\Pi)$. Moreover, using the chain rule leads to

$$\big(\partial_t u + u \cdot \nabla_x u\big)\big(t, X_u(t,y)\big) = 0, \quad \nabla_x \Pi\big(t, X_u(t,y)\big) = A^T \nabla_y P(t,y).$$

By using the Lemma A.1 in the appendix of [10] gives

$$\text{div}_x u\big(t, X_u(t,y)\big) = \text{Tr}(D_y v \cdot A)(t,y) = \text{div}_y(Av)(t,y),$$

and

$$\text{div}_x\big(\nu \nabla u\big)\big(t, X_u(t,y)\big) = \text{div}_y\big(\nu(\theta_0) A \nabla v\big)(t,y).$$

Whence transform (A.1) into the formulation in the Lagrangian coordinates

$$\begin{cases} \partial_t v - \text{div}\big(\nu(\theta_0) A \nabla v\big) + A^T \nabla P & = \theta_0 e_n, \\ \text{div}(Av) = 0, \\ v|_{t=0} = u_0. \end{cases} \tag{A.26}$$

Now let $(\theta_i,u_i,\nabla\Pi_i)$, $i=1,2$, be two solutions to (A.26) and (ρ_i,v_i,P_i) be determined by (A.25).

Denote $(\delta v,\delta P,\delta A)\stackrel{\text{def}}{=}(v_2-v_1,P_2-P_1,A_2-A_1)$, where

$$A_i(t,y)\stackrel{\text{def}}{=}\left(\mathrm{Id}+\int_0^t Dv_i(\tau,y)d\tau\right)^{-1},\quad \text{for } i=1,2. \tag{A.27}$$

Then the system for $(\delta v,\nabla\delta P)$ reads

$$\begin{cases}\partial_t\delta v-\mathrm{div}\big(\nu(\theta_0)\nabla\delta v\big)+\nabla\delta P=\delta F,\\ \mathrm{div}\,\delta v=g,\\ \delta v|_{t=0}=0,\end{cases} \tag{A.28}$$

where

$$\begin{aligned}g&\stackrel{\text{def}}{=}\mathrm{div}\big((\mathrm{Id}-A_2)\delta v-\delta A v_1\big),\\ R&\stackrel{\text{def}}{=}-\partial_t A_2\delta v+(\mathrm{Id}-A_2)\partial_t\delta v-\partial_t\delta A v_1-\delta A\partial_t v_1,\\ \delta F&\stackrel{\text{def}}{=}\mathrm{div}\big(\nu(\theta_0)(A_2-\mathrm{Id})\nabla\delta v\big)+\mathrm{div}\big(\nu(\theta_0)\delta A\nabla v_1\big).\end{aligned}$$

In the sequel, we shall take T to be so small that

$$\int_0^T\|Dv_i(t)\|_{\dot B^{\frac{n}{p}}_{p,1}}dt\le c,\quad i=1,2,$$

for a small enough constant c. Then we deduce from (A.27) that

$$A_i(t,y)=\mathrm{Id}+\sum_{k=1}^{\infty}(-1)^k\left(\int_0^t Dv_i(\tau,y)d\tau\right)^k,\quad i=1,2.$$

Moreover, as proved in the appendix of [10], we have the following estimates:

$$\begin{aligned}\|\partial_t A_i\|_{\dot B^{\frac{n}{p}}_{p,1}}&\lesssim\|Dv_i\|_{\dot B^{\frac{n}{p}}_{p,1}},\quad i=1,2,\\ \|\delta A\|_{L^\infty_t(\dot B^{\frac{n}{p}}_{p,1})}&\lesssim\|D\delta v\|_{L^1_t(\dot B^{\frac{n}{p}}_{p,1})},\\ \|A_i-\mathrm{Id}\|_{L^\infty_t(\dot B^{\frac{n}{p}}_{p,1})}&\lesssim\|Dv_i\|_{L^1_t(\dot B^{\frac{n}{p}}_{p,1})},\quad i=1,2,\\ \|\partial_t\delta A\|_{L^2_t(\dot B^{-1+\frac{n}{p}}_{p,1})}&\lesssim\|v_1,v_2\|_{L^2_t(\dot B^{\frac{n}{p}}_{p,1})}\|D\delta v\|_{L^1_t(\dot B^{\frac{n}{p}}_{p,1})}+\|\delta v\|_{L^2_t(\dot B^{\frac{n}{p}}_{p,1})}.\end{aligned} \tag{A.29}$$

Choosing m to be large enough such that there holds

$$\|(\mathrm{Id}-\dot S_m)\theta_0\|_{\dot B^{\frac{n}{p}}_{p,1}}\le c_0. \tag{A.30}$$

A simple computation implies that

$$\partial_t\delta v-\mathrm{div}\big(\nu(\dot S_m\theta_0)\nabla\delta v\big)+\nabla\delta P=\delta F+\mathrm{div}\big((\nu(\theta_0)-\nu(\dot S_m\theta_0))\nabla\delta v\big). \tag{A.31}$$

Applying $\dot\Delta_j\mathbb{P}$ to (A.31) and using the fact that $\mathrm{div}\,\delta v=g$, $\partial_t g=\mathrm{div}\,R$, we arrive at

$$\partial_t\dot\Delta_j\delta v-\mathrm{div}\big(\nu(\dot S_m\theta_0)\dot\Delta_j\nabla\delta v\big)$$

$$=\dot{\Delta}_j\mathbb{Q}R+\dot{\Delta}_j\mathbb{P}\delta F+\operatorname{div}([\dot{\Delta}_j\mathbb{P},\nu(\dot{S}_m\theta_0)]\nabla\delta v+\dot{\Delta}_j\mathbb{P}\operatorname{div}(\nu(\theta_0)-\nu(\dot{S}_m\theta_0))\nabla\delta v.$$

From the above equation and (A.30), we can get by a similar derivation of (A.10) that

$$\begin{aligned}\|\delta v\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\|\delta v\|_{L^1_t(\dot{B}^{\frac{n}{p}+1}_{p,1})}\lesssim&\|R\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\|\delta F\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\&+2^m\|\theta_0\|_{\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}}t^{\frac12}\|\delta v\|^{\frac12}_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}\|\delta v\|^{\frac12}_{L^1_t(\dot{B}^{\frac{n}{p}+1}_{p,1})}.\end{aligned}\tag{A.32}$$

Define

$$\delta E(t)\stackrel{\text{def}}{=}\|\delta v\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\|\delta v\|_{L^1_t(\dot{B}^{\frac{n}{p}+1}_{p,1})},$$

then we deduce from estimate (A.32) that

$$\delta E(t)\le C_{(m,\|\theta_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}})}(\|R\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|\delta F\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+t^{\frac12}\delta E(t)).\tag{A.33}$$

To get the uniqueness of the solutions, we only need to prove $\delta E(t)=0$ for small enough t.
Now applying (A.29), product laws and interpolation in Besov spaces, we arrive at

$$\begin{aligned}\|\operatorname{div}\big(\nu(\theta_0)(A_2-\mathrm{Id})\nabla\delta v\big)\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\lesssim&\|\nu(\theta_0)\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|A_2-\mathrm{Id}\|_{L^\infty_t(\dot{B}^{\frac{n}{p}}_{p,1})}\|\delta v\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\\\lesssim&\|\theta_0\|_{\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}}\|Dv_2\|_{L^1_t(\dot{B}^{\frac{n}{p}}_{p,1})}\delta E(t).\end{aligned}$$

Similarly, one has

$$\begin{aligned}\|\operatorname{div}\big(\nu(\theta_0)\delta A\nabla v_1\big)\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\lesssim&\|\nu(\theta_0)\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\delta A\|_{L^\infty_t(\dot{B}^{\frac{n}{p}}_{p,1})}\|v_1\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}\\\lesssim&\|\theta_0\|_{\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}}\|D\delta v\|_{L^1_t(\dot{B}^{\frac{n}{p}}_{p,1})}\|v_1\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}\\\lesssim&\|\theta_0\|_{\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}}\|v_1\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}E(t).\end{aligned}$$

Thus,

$$\|\delta F\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\lesssim\|\theta_0\|_{\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}}(\|v_1\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|v_2\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})})\delta E(t).\tag{A.34}$$

Using the same argument, we deduce from (A.29) that

$$\|R\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}\lesssim\big(\|\partial_t v_1\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|v_2\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|v_1\|_{L^2_t(\dot{B}^{\frac{n}{p}}_{p,1})}\big)\delta E(t).\tag{A.35}$$

Plugging (A.34), (A.35) into (A.33), we eventually get

$$\begin{aligned}\delta E(t)\lesssim\Big\{&\|\partial_t v_1\|_{L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})}+\|v_1\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}+\|v_1\|_{L^2_t(\dot{B}^{\frac{n}{p}}_{p,1})}\\&+2^mt^{\frac12}+(1+\|\theta_0\|_{\dot{\mathcal{B}}^{\frac{n}{p}}_{p,1}})\|v_2\|_{L^1_t(\dot{B}^{1+\frac{n}{p}}_{p,1})}\Big\}\delta E(t).\end{aligned}\tag{A.36}$$

From the Equation (A.26), we can easily get $\partial_t v_1\in L^1_t(\dot{B}^{-1+\frac{n}{p}}_{p,1})$, from which and taking t small enough in (A.36) implies that $\delta E(t)=0$. The uniqueness on $[0,T]$ can be obtained by a standard argument.

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