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WAVE PROPAGATION IN RANDOM WAVEGUIDES WITH LONG-RANGE CORRELATIONS*

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Abstract. The paper presents an analysis of acoustic wave propagation in a waveguide with random fluctuations of its sound speed profile. These random perturbations are assumed to have long-range correlation properties. In the waveguide, a monochromatic wave can be decomposed in propagating modes and evanescent modes, and the random perturbation couples all these modes. The paper presents an asymptotic analysis of the mode-coupling mechanism and uses this to characterize the transmitted wave. The paper presents the first fully rigorous characterization of wave propagation in long-range non-layered media.

Keywords. Acoustic waveguides; random media; asymptotic analysis; long-range correlations.

AMS subject classifications. 76B15; 35Q99; 60F05.

1. Introduction

Analysis of physical measurements shows that, for waves, the propagation medium may exhibit perturbations with long-range dependencies and this has stimulated interest in a mathematical understanding of how waves propagate through multiscale media [6, 18, 21, 22]. Multiscale random media with long-range correlations are used to model, for instance, the heterogenous earth crust, the turbulent atmosphere and also for biological tissue. There is a large literature about propagation in heterogeneous media which vary on a well-defined microscale [10], but for multiscale random media there are many open questions. In order to be efficient, imaging and communication algorithms require insight about how the wave is affected by the rough medium fluctuations. In view of its potential for applications, mathematical description of wave propagation in multiscale random media with long-range correlations has attracted a lot of interest over the last decade [3, 12, 15, 16, 19, 20, 23].

Multiscale random media with long-range correlations affect the wave in a way which is very different from the corruption caused by media fluctuating on a well-defined microscale and with mixing properties [10]. Wave propagation in random media with long-range correlations has already been considered in one-dimensional propagation media [12, 20] or open media under the paraxial approximation [3, 7, 8, 15, 16]. More recently, the paraxial approximation has been derived in this context from the full wave equation [17]. These works show in particular that for wave propagation in long-range media the stochastic effects appear at different propagation scales, all the stochastic effects do not appear at the same time, which is in contrast with the situation in media with mixing properties. Perturbations with long-range correlations induce first a phase modulation on the waves, a modulation that is driven by a single standard fractional Brownian motion, which does not depend on the frequency band [3, 20]. For larger propagation distances, the random phase modulation starts to oscillate very fast to produce anomalous diffusion phenomena and affects the energy propagation [12, 15, 16]. Here, we follow this line of research by considering the full wave equation in a waveguide

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with a continuous multiscale medium modeled through a stochastic process with long-range correlations.

In this paper, we consider acoustic wave propagation in a planar waveguide with a bounded cross-section $\mathcal{D}=(0,d)$, and given by the linearized conservation equations of mass and momentum (see [10])

$$\begin{aligned} \frac{1}{K(z,x)}\partial_t p+\nabla.\mathbf{u}&=0,\\ \rho(z,x)\partial_t\mathbf{u}+\nabla p&=\mathbf{F}. \end{aligned} \tag{1.1}$$

Here, p is the acoustic pressure, $\mathbf{u}$ is the acoustic velocity, ρ is the density of the medium, and K is the bulk modulus. The coordinate z represents the propagation axis along the waveguide, and the coordinate x represents the bounded transverse section $\mathcal{D}=(0,d)$ of the waveguide. The forcing term $\mathbf{F}(t,z,x)$ is given by

$$\mathbf{F}(t,z,x)=f(t)\Psi(x)\delta(z-L_S)\mathbf{e}_z, \tag{1.2}$$

where $\mathbf{e}_z$ is the unit vector pointing in the z-direction. Therefore, this term models a source located in the plan $z=L_S<0$, emitting a wave $f(t)$ in the z-direction, with a transverse profile $\Psi(x)$ (see Figure 1.1). In this paper, we assume that the medium parameters are given by:

$$\begin{aligned} \frac{1}{K(z,x)}&=\begin{cases} \frac{1}{K}(1+\sqrt{\varepsilon}V(z,x)) & \text{if } \; z\in[0,L/\varepsilon^s] \\ \frac{1}{K} & \text{if } \; z\in(-\infty,0)\cup(L/\varepsilon^s,+\infty) \end{cases}\\ \rho(z,x)&=\rho, \qquad z\in(-\infty,+\infty) \text{ and } x\in(0,d), \end{aligned}$$

where $V(z,x)$ models the fluctuations of the propagation medium and the parameter s describes the order of magnitude with respect to ε of the propagation distance in the random section. This last parameter will be chosen so that we observe nontrivial stochastic effects. The medium perturbations $V(z,x)$ are assumed to be given by

$$V(z,x):=\Theta\big(\mathcal{B}_\mathfrak{h}(z,x)\big), \tag{1.3}$$

where $\mathcal{B}_\mathfrak{h}$ is a mean-zero continuous Gaussian random field with covariance function

$$\mathbb{E}[\mathcal{B}_\mathfrak{h}(z+z',x)\mathcal{B}_\mathfrak{h}(z',y)]=r_\mathfrak{h}(z)R(x,y).$$

Here, R is assumed to be a continuous symmetric function bounded by $R(x,x)=1$, and $r_\mathfrak{h}$ is a continuous even function so that $|r_\mathfrak{h}(z)|\leq r_\mathfrak{h}(0)=1$, and

$$r_\mathfrak{h}(z)\underset{z\to+\infty}{\sim}\frac{c_\mathfrak{h}}{z^\mathfrak{h}} \qquad \text{with} \qquad \mathfrak{h}\in(0,1). \tag{1.4}$$

This last relation is the so-called long-range property. The function Θ is a bounded smooth and odd function on $\mathbb{R}$ so that the bulk modulus K takes only positive values and $\mathbb{E}[V(z,x)]=0$. Moreover, we assume that for all $l\in\mathbb{N}$,

$$\sup_{u\in\mathbb{R}}|\Theta^{(l)}(u)|\leq C^l_\Theta, \tag{1.5}$$

where $\Theta^{(l)}$ stands for the l-th derivative of Θ. This assumption is of course not optimal, but it has the advantage to provide a simple presentation. The long-range property for V then follows (see Proposition 6.1 Section 6):

$$\mathbb{E}[V(z+z',x)V(z',y)]\underset{z\to+\infty}{\sim}\frac{C_\mathfrak{h}}{z^\mathfrak{h}}R(x,y) \quad \text{with} \quad C_\mathfrak{h}=\frac{c_\mathfrak{h}}{2\pi}\Big(\int_{-\infty}^{+\infty}x\Theta(x)e^{-x^2/2}dx\Big)^2.$$

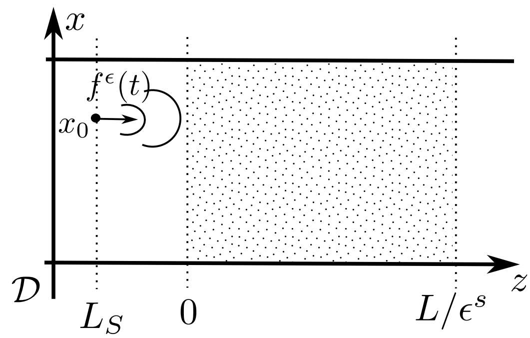

FIGURE 1.1. *Illustration of the waveguide model. The source f^ε generates a wave that is propagating in the positive z direction. The section $z\in[0,L/\varepsilon^s]$ is randomly heterogeneous with long range correlations. Our objective is to characterize the pulse as it emerges at the termination of the random section, at $z=L/\varepsilon^s$.*

The main consequence of the long-range property for the medium perturbations is that its autocorrelation function is not integrable,

$$\int_0^{+\infty} \big|\mathbb{E}[V(z+z',x)V(z',y)]\big| dz = +\infty,$$

and this is the reason why the cumulative stochastic effects for the wave propagation are very different from the ones obtained in the classical mixing case (see [10, Chapter 20] and [12, 16, 20]).

The goal of this paper is to prove that waves propagating in the random waveguide, when we let $s=1/(2-\mathfrak{h})$, exhibit a mode- and frequency-dependent random phase modulations. However, for all the frequencies generated by the source and all the propagating modes of the wave decomposition, the randomness of the phase modulations is defined in terms of the *same* standard fractional Brownian motion. This result is consistent with the ones already obtained in [3, 20] for perturbations with long-range correlations, and in contrast with the ones obtained for perturbations with mixing properties. In fact, if mixing properties are considered for the random fluctuations, the phase modulations are obtained for $s=1$ (in addition to a deterministic wave deformation affecting its energy) and are given by a *vector* of Brownian motions. The size of this vector depends on the number of propagating modes, and its correlation matrix depends on the frequency band of the source [10, Chapter 20]. In this paper, to prove the main result of this paper, we use a moment technique [3] which is very convenient in the case of long-range correlations with Gaussian underlying fluctuations.

The organization of this paper is as follows. In Section 2, we describe the wave propagation in the waveguide through a modal decomposition, moreover, the mode-coupling mechanism induced by the medium perturbations. In Section 3, we state the main result of this paper, which is used in Section 4 and Section 5 to study pulse propagation. In Section 6 we give technical results needed to deal with the long-range dependencies of the medium perturbations. These are used in Section 7 to prove the main result of this paper (Theorem 3.1). Finally, in Section 8 and Section 9, we prove respectively Theorem 4.1 and Theorem 4.2 stated in Section 4.

2. Wave propagation

In this section we describe the general strategy we use to study wave propagation in waveguides. From (1.1) we obtain the standard wave equation for the pressure wave:

$$\Delta p - \frac{1}{c^2}(1+\sqrt{\varepsilon}V(z,x)\mathbf{1}_{[0,L/\varepsilon^s]}(z))\partial_t^2 p = \nabla\cdot\mathbf{F}, \tag{2.1}$$

where $\Delta = \partial_z^2 + \partial_x^2$ and $c = \sqrt{K/\rho}$ is the sound speed. In this paper we consider mainly Dirichlet boundary conditions (Neumann boundary conditions are considered in Section 5)

$$p(t,0,z) = p(t,d,z) = 0, \qquad \forall (t,z) \in \mathbb{R}\times\mathbb{R}.$$

In (2.1) the random fluctuations of the velocity profile are modeled as a slab for $z \in [0, L/\varepsilon^s]$ with the z-direction being the propagation axis of the waveguide. The depth $z = L/\varepsilon^s$ is the position of the receivers recording the incoming signal that has propagated through the unknown medium. Assuming that no wave is incoming from the right side of the receivers, this setup provides a energy conservation relation which will be useful for the forthcoming analysis.

The wave Equation (2.1) is a linear equation so that the pressure wave can be decomposed as a superposition of monochromatic wave with the Fourier transform

$$\widehat{p}(\omega,x,z) = \int p(t,x,z)e^{i\omega t}dt \qquad \text{and} \qquad p(t,x,z) = \frac{1}{2\pi}\int \widehat{p}(\omega,x,z)e^{-i\omega t}d\omega.$$

Therefore, the pressure field $\widehat{p}(\omega,x,z)$ satisfies the following Helmholtz equation (time-harmonic wave equation) in $z\in(L_S,+\infty)$ (resp. $z\in(-\infty,L_S)$)

$$\partial_z^2\widehat{p}(\omega,x,z) + \partial_x^2\widehat{p}(\omega,x,z) + k^2(\omega)(1+\sqrt{\varepsilon}V(z,x)\mathbf{1}_{[0,L/\varepsilon^s]}(z))\widehat{p}(\omega,x,z) = 0, \tag{2.2}$$

where $k(\omega) = \omega/c$ is the wavenumber, and with Dirichlet boundary conditions,

$$\widehat{p}(\omega,0,z) = \widehat{p}(\omega,d,z) = 0, \qquad \forall z\in\mathbb{R}.$$

Moreover, according to the form of the source term (1.2) the pressure field satisfies the following jump conditions and longitudinal velocity continuity across the plane $z = L_S$

$$\widehat{p}(\omega,x,L_S^+) - \widehat{p}(\omega,x,L_S^-) = \widehat{f}(\omega)\Psi(x) \quad \text{and} \quad \partial_z\widehat{p}(\omega,x,L_S^+) = \partial_z\widehat{p}(\omega,x,L_S^-). \tag{2.3}$$

The transverse Laplacian $-\partial_x^2$ with Dirichlet boundary conditions on $\partial\mathcal{D}$ is a positive self-adjoint operator in $L^2(\mathcal{D})$, and then its spectrum is composed of a countably infinite number of positive eigenvalues $(\lambda_j)_{j\geq 1}$ since $\mathcal{D} = (0,d)$ is a bounded domain. Therefore, let us introduce for all $j\geq 1$

$$-\partial_x^2\phi_j(x) = \lambda_j\phi_j(x) \quad \forall x\in\mathcal{D}, \qquad \text{and} \qquad \phi_j(0) = \phi_j(d) = 0,$$

where $0<\lambda_1<\lambda_2<\cdots$ and the eigenvectors $(\phi_j)_{j\geq 1}$ form an orthonormal basis of $L^2(\mathcal{D})$,

$$\int_0^d \phi_j(x)\phi_l(x)dx = \delta_{jl}.$$

Here, δ_{jl} denotes the Kronecker symbol. In the planar case $\mathcal{D} = (0,d)$, we have explicit expressions for the eigenvectors and eigenvalues:

$$\phi_j(x) = \sqrt{\frac{2}{d}}\sin(j\pi x/d) \qquad \text{and} \qquad \lambda_j = \frac{j^2\pi^2}{d^2} \qquad \forall j\geq 1.$$

As a result, we have the following decomposition of the wave field

$$\widehat{p}(\omega,x,z)=\sum_{j\geq 1}\widehat{p}_j(\omega,z)\phi_j(x). \tag{2.4}$$

The two next sections present the modal decomposition of $\widehat{p}(\omega,x,z)$, first in a homogeneous waveguide ($V\equiv 0$) and then for the randomly perturbed waveguide.

2.1. Modal decomposition in homogeneous waveguides ($V\equiv 0$). This section is devoted to the modal decomposition (2.4) for a homogeneous waveguide. This describes the wave propagation from the source location $z=L_S$ to the beginning of the random section $z=0$ (see Figure 1.1). According to (2.2) and (2.4), we have for all $z\neq L_S$ and $j\geq 1$

$$\frac{d^2}{dz^2}\widehat{p}_j(\omega,z)+\beta_j^2(\omega)\widehat{p}_j(\omega,z)=0, \tag{2.5}$$

with

$$\beta_j(\omega)=\sqrt{k^2(\omega)-\lambda_j}, \qquad \text{for } j\in\{1,\dots,N(\omega)\},$$

and

$$\beta_j(\omega)=\sqrt{\lambda_j-k^2(\omega)}, \qquad \text{for } j\geq N(\omega)+1.$$

Here, $N(\omega)$ is the integer such that $\lambda_{N(\omega)}\leq k^2(\omega)<\lambda_{N(\omega)+1}$, that is for our planar waveguide

$$N(\omega)=\left[\frac{\omega d}{\pi c}\right],$$

where $[\cdot]$ stands for the integer part. As a result, according to (2.5) the pressure field can be expanded as follows

$$\begin{aligned}\widehat{p}(\omega,x,z)=&\Big[\sum_{j=1}^{N(\omega)}\frac{\widehat{a}_{j,0}(\omega)}{\sqrt{\beta_j(\omega)}}e^{i\beta_j(\omega)z}\phi_j(x)+\sum_{j\geq N(\omega)+1}\widehat{p}_{j,0}(\omega)e^{-\beta_j(\omega)z}\phi_j(x)\Big]\mathbf{1}_{(L_S,+\infty)}(z)\\&+\Big[\underbrace{\sum_{j=1}^{N(\omega)}\frac{\widehat{b}_{j,0}(\omega)}{\sqrt{\beta_j(\omega)}}e^{-i\beta_j(\omega)z}\phi_j(x)}_{\text{propagating modes}}+\underbrace{\sum_{j\geq N(\omega)+1}\widehat{q}_{j,0}(\omega)e^{\beta_j(\omega)z}\phi_j(x)}_{\text{evanescent modes}}\Big]\mathbf{1}_{(-\infty,L_S)}(z),\end{aligned} \tag{2.6}$$

and using (2.3) we find

$$\begin{aligned}\widehat{a}_{j,0}(\omega)&=-\widehat{b}_{j,0}(\omega)e^{-2i\beta_j(\omega)L_S}\\&=\frac{\sqrt{\beta_j(\omega)}}{2}\widehat{f}(\omega)e^{-i\beta_j(\omega)L_S}\big\langle\phi_j,\Psi\big\rangle_{L^2(0,d)}, \quad \text{for } j\in\{1,\dots,N(\omega)\},\end{aligned} \tag{2.7}$$

and

$$\widehat{p}_{j,0}(\omega)e^{-\beta_j(\omega)L_S}=-\widehat{q}_{j,0}(\omega)e^{\beta_j(\omega)L_S}=\frac{\sqrt{\beta_j(\omega)}}{2}\widehat{f}(\omega)\big\langle\phi_j,\Psi\big\rangle_{L^2(0,d)}, \quad \text{for } j\geq N(\omega)+1.$$

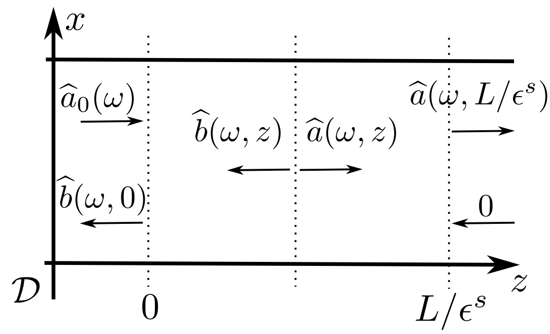

FIGURE 2.1. *Illustration of the right-going and left-going propagating mode amplitudes $\widehat{a}(\omega,z)$ and $\widehat{b}(\omega,z)$. The source generates the probing wave $\widehat{a}_0(\omega)$, the reflected wave is $\widehat{b}(\omega,0)$ and the transmitted wave, which is our interest, is $\widehat{a}(\omega,L/\varepsilon^s)$. Note that there is no energy coming from $z>L/\varepsilon^s$.*

In (2.6), we refer to the modes with $j\in\{1,\dots,N(\omega)\}$ as *propagating modes*, these are modes which can propagate over large distances. The *evanescent mode* are the modes with $j\geq N(\omega)+1$, and correspond to modes which cannot propagate over large distances. Here, $N(\omega)$ corresponds to the number of propagating modes, $\widehat{a}_{j,0}(\omega)$ (resp. $\widehat{b}_{j,0}(\omega)$) is the amplitude of the jth right-going (resp. left-going) propagating mode and $\widehat{p}_{j,0}(\omega)$ (resp. $\widehat{q}_{j,0}(\omega)$) is the amplitude of the jth right-going (resp. left-going) evanescent mode. Note that with relatively high-frequency waves, that is with short wave-lengths compared to the cross-section d, there are relatively many propagating modes.

2.2. Mode coupling for randomly perturbed waveguides. In this section we are interested in the modal decomposition of the pressure field over the randomly perturbed section of the waveguide $(0,L/\varepsilon^s)$. In this case, the perturbations of the propagation medium induce a mode coupling. To describe this coupling mechanism, let us define the right-going and left-going propagating mode amplitudes $\widehat{a}_j(\omega,z)$ and $\widehat{b}_j(\omega,z)$ ($j\in\{1,\dots,N(\omega)\}$) such that

$$\widehat{p}_j(z)=\frac{\widehat{a}_j(z)e^{i\beta_j z}+\widehat{b}_j(z)e^{-i\beta_j z}}{\sqrt{\beta_j}} \quad \text{and} \quad \partial_z\widehat{p}_j(z)=i\sqrt{\beta_j}(\widehat{a}_j(z)e^{i\beta_j z}-\widehat{b}_j(z)e^{-i\beta_j z}),$$

(see Figure 2.1) which give

$$\widehat{a}_j(z)=\frac{i\beta_j\widehat{p}_j(z)+\partial_z\widehat{p}_j(z)}{2i\sqrt{\beta_j}}e^{-i\beta_j z} \quad \text{and} \quad \widehat{b}_j(z)=\frac{i\beta_j\widehat{p}_j(z)-\partial_z\widehat{p}_j(z)}{2i\sqrt{\beta_j}}e^{i\beta_j z}.$$

Therefore, according to (2.2) and (2.4), we have the following coupled differential equations for the mode amplitudes

$$\begin{aligned}\frac{d}{dz}\widehat{a}_j(\omega,z)=&\sqrt{\varepsilon}\frac{ik^2(\omega)}{2}\sum_{l=1}^{N(\omega)}C_{jl}(\omega,z)\big(\widehat{a}_l(\omega,z)e^{i(\beta_l(\omega)-\beta_j(\omega))z}+\widehat{b}_l(\omega,z)e^{-i(\beta_l(\omega)+\beta_j(\omega))z}\big)\\&+\sqrt{\varepsilon}\frac{ik^2(\omega)}{2}\sum_{l\geq N(\omega)+1}C_{jl}(\omega,z)\sqrt{\beta_l(\omega)}\widehat{p}_l(\omega,z)e^{-i\beta_j(\omega)z},\end{aligned} \tag{2.8}$$

$$\frac{d}{dz}\widehat{b}_j(\omega,z) = -\sqrt{\varepsilon}\frac{ik^2(\omega)}{2}\sum_{l=1}^{N(\omega)} C_{jl}(\omega,z)\big(\widehat{a}_l(\omega,z)e^{i(\beta_l(\omega)+\beta_j(\omega))z} + \widehat{b}_l(\omega,z)e^{-i(\beta_l(\omega)-\beta_j(\omega))z}\big)$$
$$-\sqrt{\varepsilon}\frac{ik^2(\omega)}{2}\sum_{l\geq N(\omega)+1} C_{jl}(\omega,z)\sqrt{\beta_l(\omega)}\widehat{p}_l(\omega,z)e^{i\beta_j(\omega)z}, \tag{2.9}$$

for $j\in\{1,\dots,N(\omega)\}$, and

$$\frac{d^2}{dz^2}\widehat{p}_j(\omega,z) - \beta_j^2(\omega)\widehat{p}_j(\omega,z) + 2\sqrt{\varepsilon}g_j(\omega,z) = 0, \tag{2.10}$$

for $j\geq N(\omega)+1$, where

$$g_j(z) = k^2\sum_{l\geq N+1} C_{jl}(z)\sqrt{\beta_j\beta_l}\,\widehat{p}_l(z) + k^2\sum_{l=1}^{N} C_{jl}(z)\sqrt{\beta_j}\big(\widehat{a}_l(z)e^{i\beta_l z} + \widehat{b}_l(z)e^{-i\beta_l z}\big).$$

In these equations the coupling coefficients are defined by

$$C_{jl}(z) = \frac{1}{\sqrt{\beta_j\beta_l}}\int_{\mathcal{D}} V(z,x)\phi_j(x)\phi_l(x)dx. \tag{2.11}$$

We complement the systems (2.8), (2.9) and (2.10) for $z\in(0,L/\varepsilon^s)$ with the boundary conditions (for every ε):

$$\widehat{a}_j(\omega,0) = \widehat{a}_{j,0}(\omega) \qquad \text{and} \qquad \widehat{b}_j(\omega,L/\varepsilon^s) = 0, \tag{2.12}$$

(see Figure 2.1) where $\widehat{a}_{j,0}(\omega)$ is defined by (2.7)). The second condition in (2.12) means that no propagating wave is incoming from the right hand side of the perturbed section $(0,L/\varepsilon^s)$. We also introduce the following radiation condition

$$\lim_{z\to+\infty}\sum_{l\geq N(\omega)+1} |\widehat{p}_j(\omega,z)|^2 = 0, \tag{2.13}$$

Using these conditions, one can show following [11, Section 3.2] the global conservation relation

$$\|\widehat{a}(\omega,L/\varepsilon^s)\|^2_{\mathbb{C}^{N(\omega)}} + \|\widehat{b}(\omega,0)\|^2_{\mathbb{C}^{N(\omega)}} = \|\widehat{a}_0(\omega)\|^2_{\mathbb{C}^{N(\omega)}}, \tag{2.14}$$

where $\|\cdot\|_{\mathbb{C}^{N(\omega)}}$ stands for the Euclidian norm on $\mathbb{C}^{N(\omega)}$. This energy conservation can be easily derived if we neglect the coupling between the propagating and the evanescent modes since in that case the mode coupling mechanism for the propagating modes is skew Hermitian.

2.3. Propagating mode amplitude equations and propagator. The coupled mode Equations (2.8) and (2.9) for the propagating mode amplitudes are not closed, they involve also the evanescent mode amplitudes. However, it is possible to derive a closed system (2.15) for the propagating mode amplitudes taking into account the influence of the evanescent modes on the propagating modes up to an error leading to negligible effects compared to the ones discussed in the remaining of the paper (see [10, 11, 13, 14]). This process gives the following system of coupled differential equations for the mode amplitudes

$$\frac{d}{dz}\begin{bmatrix}\widehat{a}(\omega,z)\\ \widehat{b}(\omega,z)\end{bmatrix} = \Big[\sqrt{\varepsilon}\mathbf{H}(\omega,z) + \varepsilon\mathbf{G}(\omega,z)\Big]\begin{bmatrix}\widehat{a}(\omega,z)\\ \widehat{b}(\omega,z)\end{bmatrix} \tag{2.15}$$

with boundary conditions given by (2.12), and where

$$\mathbf{H}(\omega,z)=\begin{bmatrix}\mathbf{H}^a(\omega,z) & \overline{\mathbf{H}^b(\omega,z)}\\ \mathbf{H}^b(\omega,z) & \overline{\mathbf{H}^a(\omega,z)}\end{bmatrix} \quad \text{and} \quad \mathbf{G}(\omega,z)=\begin{bmatrix}\mathbf{G}^a(\omega,z) & \overline{\mathbf{G}^b(\omega,z)}\\ \mathbf{G}^b(\omega,z) & \overline{\mathbf{G}^a(\omega,z)}\end{bmatrix}$$

with

$$\begin{aligned}
\mathbf{H}^a_{jl}(z)&=\frac{ik^2}{2}C_{jl}(z)e^{i(\beta_l-\beta_j)z}, \qquad \mathbf{H}^b_{jl}(z)=-\frac{ik^2}{2}C_{jl}(z)e^{i(\beta_l+\beta_j)z}\\
\mathbf{G}^a_{jl}(z)&=\frac{ik^4}{4}\sum_{l'\geq N+1}\int_{-\infty}^{+\infty}C_{jl'}(z)C_{l'l}(z+u)e^{i\beta_l(z+u)-i\beta_j z-\beta_{l'}|u|}du,\\
\mathbf{G}^b_{jl}(z)&=-\frac{ik^4}{4}\sum_{l'\geq N+1}\int_{-\infty}^{+\infty}C_{jl'}(z)C_{l'l}(z+u)e^{i\beta_l(z+u)+i\beta_j z-\beta_{l'}|u|}du,
\end{aligned} \tag{2.16}$$

and $C_{jl}(z)$ defined by (2.11). Here, the matrices $\mathbf{H}^a$ and $\mathbf{H}^b$ describe the coupling between the propagating modes, while the matrices $\mathbf{G}^a$ and $\mathbf{G}^b$ describe the coupling between the evanescent modes with the propagating modes. Rescaling the propagating mode amplitudes according to the order of magnitude of the propagation distance, we consider

$$\widehat{a}^\varepsilon(\omega,z)=\widehat{a}(\omega,z/\varepsilon^s) \qquad \text{and} \qquad \widehat{b}^\varepsilon(\omega,z)=\widehat{b}(\omega,z/\varepsilon^s),$$

for $z\in[0,L]$, satisfying the scaled coupled mode equations

$$\frac{d}{dz}\begin{bmatrix}\widehat{a}^\varepsilon(\omega,z)\\ \widehat{b}^\varepsilon(\omega,z)\end{bmatrix}=\Big[\frac{1}{\varepsilon^{s-1/2}}\mathbf{H}\Big(\omega,\frac{z}{\varepsilon^s}\Big)+\varepsilon^{1-s}\mathbf{G}\Big(\omega,\frac{z}{\varepsilon^s}\Big)\Big]\begin{bmatrix}\widehat{a}^\varepsilon(\omega,z)\\ \widehat{b}^\varepsilon(\omega,z)\end{bmatrix} \tag{2.17}$$

with boundary conditions

$$\widehat{a}^\varepsilon(\omega,0)=\widehat{a}_0(\omega) \qquad \text{and} \qquad \widehat{b}^\varepsilon(\omega,L)=0. \tag{2.18}$$

The two-point boundary value problem (2.17) and (2.18) can be solved using the propagator matrix defined as the unique solution of

$$\frac{d}{dz}\mathbf{P}^\varepsilon(\omega,z)=\Big[\frac{1}{\varepsilon^{s-1/2}}\mathbf{H}\Big(\omega,\frac{z}{\varepsilon^s}\Big)+\varepsilon^{1-s}\mathbf{G}\Big(\omega,\frac{z}{\varepsilon^s}\Big)\Big]\mathbf{P}^\varepsilon(\omega,z) \quad \text{with} \quad \mathbf{P}^\varepsilon(\omega,0)=Id_{2N(\omega)}, \tag{2.19}$$

so that for all $z\in[0,L]$

$$\begin{bmatrix}\widehat{a}^\varepsilon(\omega,z)\\ \widehat{b}^\varepsilon(\omega,z)\end{bmatrix}=\mathbf{P}^\varepsilon(\omega,z)\begin{bmatrix}\widehat{a}^\varepsilon(\omega,0)\\ \widehat{b}^\varepsilon(\omega,0)\end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix}\widehat{a}^\varepsilon(\omega,L)\\ 0\end{bmatrix}=\mathbf{P}^\varepsilon(\omega,L)\begin{bmatrix}\widehat{a}_0(\omega,0)\\ \widehat{b}^\varepsilon(\omega,0)\end{bmatrix}, \tag{2.20}$$

according to (2.12). Because of the form of $\mathbf{H}(\omega,z)$ and $\mathbf{G}(\omega,z)$, the propagator can be expressed as

$$\mathbf{P}^\varepsilon(\omega,z)=\begin{bmatrix}\mathbf{P}^{a,\varepsilon}(\omega,z) & \overline{\mathbf{P}^{b,\varepsilon}(\omega,z)}\\ \mathbf{P}^{b,\varepsilon}(\omega,z) & \overline{\mathbf{P}^{a,\varepsilon}(\omega,z)}\end{bmatrix}, \tag{2.21}$$

where $\mathbf{P}^{a,\varepsilon}(\omega,z)$ describes the coupling mechanisms of the right-going modes (resp. left-going modes) with themselves, while $\mathbf{P}^{b,\varepsilon}(\omega,z)$ describes the coupling mechanisms between the right-going and left-going modes.

In the next section we study the asymptotic distribution as $\varepsilon\to 0$ of the propagator and we specify the propagation parameter s leading to a nontrivial asymptotic behavior.

3. Phase modulation for the propagator

This section is devoted to the main results of this paper, which are used to describe the pulse propagation in the two next sections. Note that according to (2.21) one can restrict the study of the asymptotic distribution of the propagator to the one of the two blocks

$$\mathcal{P}^{\varepsilon}(\omega,z)=\begin{bmatrix}\mathbf{P}^{a,\varepsilon}(\omega,z)\\ \mathbf{P}^{b,\varepsilon}(\omega,z)\end{bmatrix}\in\mathcal{M}_{N(\omega)}(\mathbb{C})\times\mathcal{M}_{N(\omega)}(\mathbb{C}),$$

where $\mathcal{M}_{N(\omega)}(\mathbb{C})$ stands for the set of $N(\omega)\times N(\omega)$ matrices with complex coefficients. This analysis is done in the following theorem.

THEOREM 3.1. *For $s=1/(2-\mathfrak{h})$ and for all $z\in[0,L]$, the family $(\mathcal{P}^{\varepsilon}(\omega,z))_{\varepsilon}$ converges in distribution in $\mathcal{M}_N(\mathbb{C})\times\mathcal{M}_N(\mathbb{C})$ to*

$$\begin{bmatrix}D(\omega,z)\\ \mathbf{0}\end{bmatrix},$$

with

$$D(\omega,z)=diag(e^{i\sigma_{1,H}(\omega)B_H(z)},\dots,e^{i\sigma_{N,H}(\omega)B_H(z)}),\tag{3.1}$$

where B_H is a standard fractional Brownian motion with Hurst index $H=(2-\mathfrak{h})/2\in(1/2,1)$,

$$\sigma_{j,H}(\omega)=\frac{k^2(\omega)}{2\beta_j(\omega)}\sqrt{\frac{C_{\mathfrak{h}}R_{jjjj}}{H(2H-1)}},\tag{3.2}$$

and

$$R_{mnpq}=\iint_{\mathcal{D}\times\mathcal{D}}R(x,x')\phi_m(x)\phi_n(x)\phi_p(x')\phi_q(x')dxdx'.\tag{3.3}$$

The proof of Theorem 3.1 is given in Section 7. Next, we present some remarks regarding this result. First, Theorem 3.1 implies that the first significant stochastic effects affecting the wave take place for $s=1/(2H)<1$. This is in contrast to the classical mixing case (see [10, Chapter 20]) for which all the stochastic effects appear for $s=1$. The second one concerns the convergence of $\mathbf{P}^{b,\varepsilon}(\omega,z)$ in probability to the zero matrix meaning that the coupling mechanisms between the right-going and left-going modes are negligible for ε small. In other words, the backscattering is negligible for ε small. The third one concerns the convergence in distribution of $\mathbf{P}^{a,\varepsilon}(\omega,z)$ to a diagonal matrix, which means that the coupling mechanisms between two different right-going modes is also negligible for ε small. Finally, the propagating modes are only affected by mode- and frequency-dependent phase modulations, but driven by the same fractional Brownian motion, which does not depend on the frequency. The fact that the modes depend on a common Brownian motion reflects the situation that the modes propagate through the same medium with fluctuations that are "smoother" than in the mixing case due to the long-range correlations. This effect of mode-dependent phase modulation driven by a single fractional Brownian motion has already been observed in [3, Theorem 1.2] for the random Schrödinger equation with long-range correlations.

We finish this section with two multifrequency versions of Theorem 3.1 which will be useful for the study of the pulse propagation in the next section. The proofs of the following requires only simple modifications of the one of Theorem 3.1.

THEOREM 3.2. *Let $s=1/(2-\mathfrak{h})$, $\gamma\in\mathbb{N}^*$, and $z\in[0,L]$.*

- *For all* $(\omega_1,\dots,\omega_\gamma)$ *such that*

$$N(\omega_j)=N \qquad \forall j\in\{1,\dots,\gamma\},$$

the family $(\mathcal{P}^\varepsilon(\omega_1,z),\dots,\mathcal{P}^\varepsilon(\omega_\gamma,z))_\varepsilon$ *converges in distribution in* $(\mathcal{M}_N(\mathbb{C})\times\mathcal{M}_N(\mathbb{C}))^\gamma$ *to*

$$\begin{bmatrix} D(\omega_1,z) & \cdots & D(\omega_\gamma,z) \\ \mathbf{0} & & \mathbf{0} \end{bmatrix},$$

- *For all* $M>0$, *and* $(h_1,\dots,h_\gamma)\in[-M,M]^\gamma$, *the family* $(\mathcal{P}^\varepsilon(\omega+\varepsilon^q h_1,z),\dots,\mathcal{P}^\varepsilon(\omega+\varepsilon^q h_\gamma,z))_\varepsilon$, *with* $q\in(0,s]$, *converges in distribution in* $(\mathcal{M}_{N(\omega)}(\mathbb{C})\times\mathcal{M}_{N(\omega)}(\mathbb{C}))^\gamma$ *to*

$$\begin{bmatrix} D(\omega,z) & \cdots & D(\omega,z) \\ \mathbf{0} & & \mathbf{0} \end{bmatrix},$$

where $D(\omega,z)$ *is defined by* (3.1).

This theorem characterizes the behavior of the propagator at different and nearby frequencies. It turns out that the energy and the asymptotic statistical behavior of the propagator for different frequencies are not affected on the propagation scale L/ε^s, with $s=1/(2-\mathfrak{h})$. On this scale the wave does not propagate enough and accumulate enough scattering events to affect neither the frequency coherence nor the energy. This type of phenomenon has already been observed in [12] for one-dimensional propagation media and in [16] for the random Schrödinger equation with long-range correlations.

4. Pulse propagation for $s=1/(2-\mathfrak{h})$

In this section we describe the asymptotic behavior of a pulse at the end of the random section $z=L/\varepsilon^s$ with $s=1/(2-\mathfrak{h})$. We assume that the pulse has been generated by the source (1.2) with

$$f^\varepsilon(t)=f(\varepsilon^q t)e^{-i\omega_0 t}, \qquad \text{so that} \qquad \widehat{f^\varepsilon}(\omega)=\frac{1}{\varepsilon^q}\widehat{f}\Big(\frac{\omega-\omega_0}{\varepsilon^q}\Big).$$

This profile models a source with carrier frequency ω_0 and bandwidth of order ε^q. However, for the sake of simplicity we assume that $\widehat{f}(\omega)$ is a compactly supported smooth function, and we model the carrier frequency oscillations by a complex exponential $e^{-i\omega_0 t}$ for more convenient mathematical manipulations. This later consideration leads of course to complex valued pulse profiles, but to get back to real valued pulses one only needs to add its complex conjugate which would then correspond to a source of the form $2f(\varepsilon^q t)\cos(\omega_0 t)$.

Throughout this section, we refer to a broadband pulse if $q<s$ (the order of the pulse width is small compared to the propagation distance $\varepsilon^{-q}\ll\varepsilon^{-s}$) and narrowband if $q=s$ (the order of the pulse width is comparable to the order of the propagation distance $\varepsilon^{-q}=\varepsilon^{-s}$). According to the boundary conditions (2.18) (no wave is incoming from the right) the pulse can be decomposed into two parts

$$p\Big(t,x,\frac{L}{\varepsilon^s}\Big)=p_a\Big(t,x,\frac{L}{\varepsilon^s}\Big)+p_e\Big(t,x,\frac{L}{\varepsilon^s}\Big),$$

where

$$p_a\Big(t,x,\frac{L}{\varepsilon^s}\Big)=\frac{1}{2\pi}\int d\omega e^{-i\omega t}\sum_{j=1}^{N(\omega)}\frac{\widehat{a}_j^\varepsilon(\omega,L)}{\sqrt{\beta_j(\omega)}}e^{i\beta_j(\omega)L/\varepsilon^s}\phi_j(x),$$

is the propagating part of the pulse, and

$$p_e\Big(t,x,\frac{L}{\varepsilon^s}\Big)=\frac{1}{2\pi}\int d\omega e^{-i\omega t}\sum_{j\geq N(\omega)+1}\widehat{p}_j(\omega,L/\varepsilon^s)\phi_j(x),$$

is the evanescent part of the pulse, which is small in the limit $\varepsilon\to 0$. Therefore, in what follows, we focus our attention on the propagating component of the pressure wave p_a. In order to use Theorem 3.1 we need to express the forward-propagating mode amplitudes $\widehat{a}^\varepsilon(\omega,L)$ in terms of $\mathcal{P}^\varepsilon(\omega,L)$. This can be done according to the right-hand side of (2.20), but note though that the backward-propagating mode amplitudes $\widehat{b}^\varepsilon(\omega,0)$ are not specified. However, by introducing

$$\begin{aligned}p_{pr}\Big(t,x,\frac{L}{\varepsilon^s}\Big)=&\frac{1}{4\pi\varepsilon^q}\int d\omega e^{-i\omega t}\widehat{f}\Big(\frac{\omega-\omega_0}{\varepsilon^q}\Big)\sum_{j,l=1}^{N(\omega)}\sqrt{\frac{\beta_l(\omega)}{\beta_j(\omega)}}\mathbf{P}^{a,\varepsilon}_{jl}(\omega,L)\\&\times e^{i\beta_j(\omega)L/\varepsilon^s}e^{-i\beta_l(\omega)L_S}\phi_j(x)\big\langle\phi_l,\Psi\big\rangle_{L^2(0,d)},\end{aligned}\tag{4.1}$$

we get

$$\sup_{t,x}|p_a(t,x,L/\varepsilon^s)-p_{pr}(t,x,L/\varepsilon^s)|\leq C\int|\widehat{f}(h)|\,\|\mathbf{P}^{b,\varepsilon}(\omega_0+\varepsilon^q h,L)\|_{\mathcal{M}_{N(\omega_0)}}dh,$$

using (2.14) and the change of variable

$$\omega\to\omega_0+\varepsilon^q h,$$

and where $C>0$ is a deterministic constant. Then, with simple modifications of the proof of Proposition 7.1, we obtain

$$\lim_{\varepsilon\to 0}\int|\widehat{f}(h)|\mathbb{E}\big[\|\mathbf{P}^{b,\varepsilon}(\omega_0+\varepsilon^q h,L)\|_{\mathcal{M}_{N(\omega_0)}}\big]dh=0,$$

so that

$$\lim_{\varepsilon\to 0}\mathbb{P}\Big(\sup_{t,x}|p_a(t,x,L/\varepsilon^s)-p_{pr}(t,x,L/\varepsilon^s)|>\eta\Big)=0.$$

As a result, the asymptotic behavior of the pulse $p(t,x,L/\varepsilon^s)$ is equivalent to the one of $p_{pr}(t,x,L/\varepsilon^s)$ according to [4, Theorem 3.1 pp. 27]. This is the reason why, in the next section, we only study the asymptotic behavior of $p_{pr}(t,x,L/\varepsilon^s)$. Before starting, we also remark that the nature of the source affects strongly the asymptotic shape of the transmitted pulse. Let us investigate the cases of a broadband source ($q<s$) and a narrowband source ($q=s$) in the contexts of a homogeneous medium, and then with a random medium in order to understand the effects the random perturbations have on the pulse.

4.1. Broadband ($q<s$) and Narrowband ($q=s$) Pulse in the homogenous case. This section is devoted to the study of the pulse (4.1) in a homogeneous medium. The results presented in this section will be compared to the ones obtained in Section 4.2 and Section 4.3.

In the homogeneous case we have $\mathbf{P}^{a,\varepsilon}(\omega,L)=Id_{N(\omega)}$, and after the change of variable $\omega=\omega_0+\varepsilon^q h$ the pulse (4.1) becomes, to leading order

$$p_{pr}\Big(\frac{t}{\varepsilon^s},x,\frac{L}{\varepsilon^s}\Big)\underset{\varepsilon\to 0}{\simeq}\frac{e^{-i\omega_0 t/\varepsilon^s}}{4\pi}\sum_{j=1}^{N(\omega_0)}\phi_j(x)\langle\phi_j,\Psi\rangle_{L^2(0,d)}$$
$$\times\int e^{-ih(t-\beta_j'(\omega_0)L)/\varepsilon^{s-q}}\widehat{f}(h)e^{i\beta_j(\omega_0)L/\varepsilon^s}e^{i\Phi_j^\varepsilon(h)L/\varepsilon^{s-2q}}e^{-i\beta_j(\omega_0)L_S}dh,$$

where

$$\Phi_j^\varepsilon(h)=\sum_{n=2}^{n_q}\beta_j^{(n)}(\omega_0)\varepsilon^{q(n-2)}\frac{h^n}{n!},\tag{4.2}$$

with $\Phi^\varepsilon(h)=0$ if $s<2q$, and $n_q=[s/q]+1$. Here, we made the identification $N(\omega_0)=N(\omega_0+\varepsilon^q h)$ since we assumed that $\widehat{f}(h)$ has a compact support and ε is small. In the broadband case the source pulsewidth, which is of order $1/\varepsilon^q$, is small compared to the order of propagation distance $1/\varepsilon^s$, so that a modal dispersion can be observed. In this context, we observe the pulse for a time window of order the pulse width $1/\varepsilon^q$ and centered at t_{obs}/ε^s which is of order of the total travel time from $z=L_S$ to $z=L/\varepsilon^s$:

$$\frac{t}{\varepsilon^s}=\frac{t_{obs}}{\varepsilon^s}+\frac{u}{\varepsilon^q}\quad\text{with}\quad u\in[-T,T].$$

Proposition 4.1. *For all $j\in\{1,\dots,N(\omega_0)\}$, let us consider*

$$p_{j,pr}^\varepsilon(u,x,L)=e^{-i\beta_j(\omega_0)(L/\varepsilon^s-L_S)}e^{i\omega_0 t_j/\varepsilon^s}e^{i\omega_0 u/\varepsilon^q}p_{pr}\Big(\frac{t_j}{\varepsilon^s}+\frac{u}{\varepsilon^q},x,\frac{L}{\varepsilon^s}\Big),$$

where $s=1/(2-\mathfrak{h})$ and

$$t_j=\beta_j'(\omega_0)L.\tag{4.3}$$

- *If $q\in[s/2,s)$, we have*

$$\lim_{\varepsilon\to 0}p_{j,pr}^\varepsilon(u,x,L)=\frac{1}{2}K_{j,L}*f(u)\phi_j(x)\langle\phi_j,\Psi\rangle_{L^2(0,d)},$$

where

$$\widehat{K}_{j,L}(h)=e^{ih^2\beta_j^{(2)}(\omega_0)L/2},\quad\text{if}\quad q=\frac{s}{2},\quad\text{and}\quad K_{j,L}(t)=\delta(t),\quad\text{if}\quad q>\frac{s}{2}.\tag{4.4}$$

- *If $q\in(0,s/2)$, we have*

$$\lim_{\varepsilon\to 0}p_{j,pr}^\varepsilon(u,x,L)=0.$$

Consequently, in the broadband case, we can observe a train of coherent transmitted pulses at several well-separated observation times t_j. At these times, the pulse is a single mode traveling with the group velocity $1/\beta_j'(\omega_0)$ and dispersed through the kernel $K_{j,L}$. Note that for the case $0<q<s/2$, that is for a source with a very small pulsewidth compared to the propagation distance, the generated wave is not able to propagate. This can be understood through the stationary phase theorem and that such a pulse is carrying highly oscillating frequencies which cancel out during the propagation. However, we will see in Section 5, considering Neumann boundary conditions, that such a pulse with $q=0$ can propagate in that case.

Next we consider the narrowband case. In this case the orders of the source pulsewidth and the propagation distance are the same ($\sim 1/\varepsilon^s$) so that the propagating modes overlap, there is no modal dispersion. In this case, to describe the asymptotic behavior of the pulse, we need to compensate the rapid phase $e^{i\beta_j(\omega)L/\varepsilon^s}$. However, note that (4.1) is a superposition of $N(\omega_0)$ eigenvector ϕ_j, so that one can study the finite-dimensional vectors corresponding to the modal decomposition compensated by the fast phase as described in the following result.

Proposition 4.2. *For all $j\in\{1,\dots,N(\omega_0)\}$, let us consider the projection*

$$p^\varepsilon_{j,pr}(t,L)=e^{-i\beta_j(\omega_0)(L/\varepsilon^s-L_S)}e^{i\omega_0 t/\varepsilon^s}\big\langle p_{pr}\Big(\frac{t}{\varepsilon^s},\cdot,\frac{L}{\varepsilon^s}\Big),\phi_j\big\rangle_{L^2(0,d)}.$$

We have for $s=1/(2-\mathfrak{h})$

$$\lim_{\varepsilon\to 0}p^\varepsilon_{j,pr}(t,L)=\frac{1}{2}f(t-\beta_j'(\omega_0)L)\big\langle\phi_j,\Psi\big\rangle_{L^2(0,d)}. \tag{4.5}$$

Roughly speaking, we can then write

$$p_{pr}\Big(\frac{t}{\varepsilon^s},x,\frac{L}{\varepsilon^s}\Big)\underset{\varepsilon\to 0}{\simeq}\frac{e^{-i\omega_0 t/\varepsilon^s}}{2}\sum_{j=1}^{N(\omega_0)}f(t-\beta_j'(\omega_0)L)e^{i\beta_j(\omega_0)(L/\varepsilon^s-L_S)}\phi_j(x)\big\langle\phi_j,\Psi\big\rangle_{L^2(0,d)}.$$

The transmitted pulse is therefore a superposition of modes and each of them is centered around its travel time t_j defined by (4.3).

In the two following sections we describe how the pulse is affected by the random perturbations of the propagation medium. We consider first, the case of a pulse generated by a broadband source and second by a narrowband source.

4.2. Broadband pulse ($q<s=1/(2-\mathfrak{h})$) **in the random case.** Following the lines of Section 4.1, making the change of variable $\omega=\omega_0+\varepsilon^q h$ in (4.1) we have

$$\begin{aligned}&p_{pr}\Big(\frac{t_{obs}}{\varepsilon^s}+\frac{u}{\varepsilon^q},x,\frac{L}{\varepsilon^s}\Big)\\&=\frac{e^{-i\omega_0 t_{obs}/\varepsilon^s}e^{-i\omega_0 u/\varepsilon^q}}{4\pi}\sum_{j,l=1}^{N(\omega_0)}\sqrt{\frac{\beta_l(\omega_0)}{\beta_j(\omega_0)}}e^{\beta_j(\omega_0)L/\varepsilon^s}e^{-i\beta_l(\omega_0)L_S}\phi_j(x)\big\langle\phi_l,\Psi\big\rangle_{L^2(0,d)}\\&\quad\times\int e^{-ihu}\widehat{f}(h)e^{ih(\beta_j'(\omega_0)L-t_{obs})/\varepsilon^{s-q}}e^{i\Phi^\varepsilon_j(h)L/\varepsilon^{s-2q}}\times\mathbf{P}^{a,\varepsilon}_{jl}(\omega_0+\varepsilon^q h,L)dh,\end{aligned} \tag{4.6}$$

where Φ^ε is given by (4.2). As in the homogeneous case, we can observe a train of coherent transmitted pulses, described in the following result, at several well-separated observation times $t_{obs}=t_j$ defined by (4.3).

Theorem 4.1. *For all $j\in\{1,\dots,N(\omega_0)\}$, let us consider*

$$p^\varepsilon_{j,pr}(u,x,L)=e^{-i\beta_j(\omega_0)(L/\varepsilon^s-L_S)}e^{i\omega_0 t_j/\varepsilon^s}e^{i\omega_0 u/\varepsilon^q}p_{pr}\Big(\frac{t_j}{\varepsilon^s}+\frac{u}{\varepsilon^q},x,\frac{L}{\varepsilon^s}\Big), \tag{4.7}$$

where $s=1/(2-\mathfrak{h})$.

- *If* $q\in[s/2,s)$, *the family* $(p^\varepsilon_{1,pr}(\cdot,\cdot,L),\dots,p^\varepsilon_{N(\omega_0),pr}(\cdot,\cdot,L))_\varepsilon$ *converges in distribution on* $\mathcal{C}([-T,T]\times(0,d),\mathbb{C}^{N(\omega_0)})$ *as* ε *goes to* 0 *to* $(p^0_{1,pr}(\cdot,\cdot,L),\dots,p^0_{N(\omega_0),pr}(\cdot,\cdot,L))_\varepsilon$*, where*
$$p^0_{j,pr}(u,x,L)=\frac{e^{i\sigma_{j,H}(\omega_0)B_H(L)}}{2}K_{j,L}*f(u)\phi_j(x)\langle\phi_j,\Psi\rangle_{L^2(0,d)}. \tag{4.8}$$
Here $K_{j,L}$ *is defined by* (4.4)*,* B_H *is a standard fractional Brownian motion with Hurst index* $H=(2-\mathfrak{h})/2\in(1/2,1)$*, and* $\sigma_{j,H}(\omega_0)$ *is defined by* (3.2)*.*
- *If* $q\in(0,s/2)$*, the family* $(p^\varepsilon_{1,pr}(\cdot,\cdot,L),\dots,p^\varepsilon_{N(\omega_0),pr}(\cdot,\cdot,L))_\varepsilon$ *converges in probability on* $\mathcal{C}([-T,T]\times(0,d),\mathbb{C}^{N(\omega_0)})$ *to* 0 *as* ε *goes to* 0*.*

This result shows that the random perturbations of the propagation medium induce on the pulse, a mode-dependent and frequency-dependent phase modulation driven by a single standard fractional Brownian motion, and spread dispersively through the kernel $K_{j,L}$. This result is in contrast with the case of random perturbations with mixing properties (see [10, Section 20.4.3]) and $s=1$ in three respects. First, the random modulation is driven by a fractional Brownian motion and not by a standard Brownian motion. Second, in our case, the fractional Brownian motion is the same for all the transmitted waves and not a family of standard Brownian motion with some frequency-dependent correlation matrix. Finally, as already discussed in Section 3, on this propagation scale (L/ε^s with $s=1/(2H)$) the pulse does not accumulate enough scattering events to affect its energy [12, 16]: we have
$$\sum_{j=1}^{N(\omega_0)}\int\|p^0_{j,pr}(u,\cdot,L)\|^2_{L^2(0,d)}du=\frac{1}{2}\int|f(u)|^2du\|\Psi\|^2_{L^2(0,d)}$$
where the right-hand side is the total energy produced by the source and entering the random medium.

4.3. Narrowband pulse ($q=s=1/(2-\mathfrak{h})$) **in the random case.** As discussed in Section 4.1, in the context of a narrowband pulse, the propagating modes overlap, there is no modal dispersion, and we have from (4.1)
$$p_{pr}\Big(\frac{t}{\varepsilon^s},x,\frac{L}{\varepsilon^s}\Big)=\frac{e^{-i\omega_0 t/\varepsilon^s}}{4\pi}\sum_{j,l=1}^{N(\omega_0)}\sqrt{\frac{\beta_l(\omega_0)}{\beta_j(\omega_0)}}e^{\beta_j(\omega_0)L/\varepsilon^s}e^{-i\beta_l(\omega_0)L_S}\phi_j(x)\langle\phi_l,\Psi\rangle_{L^2(0,d)}$$
$$\times\int\widehat{f}(h)e^{ih(\beta_j'(\omega_0)L-t)}\times\mathbf{P}^{a,\varepsilon}_{jl}(\omega_0+\varepsilon^s h,L)dh. \tag{4.9}$$
As in the homogeneous case, because of the mode overlapping, we study the finite-dimensional vectors corresponding to the modal decomposition and compensated by the fast phase. The precise result is given in the following theorem.

Theorem 4.2. *For all* $j\in\{1,\dots,N(\omega_0)\}$*, let us consider the projection*
$$p^\varepsilon_{j,pr}(t,L)=e^{-i\beta_j(\omega_0)(L/\varepsilon^s-L_S)}e^{i\omega_0 t/\varepsilon^s}\Big\langle p_{pr}\Big(\frac{t}{\varepsilon^s},\cdot,\frac{L}{\varepsilon^s}\Big),\phi_j\Big\rangle_{L^2(0,d)}.$$
The family $(p^\varepsilon_{1,pr}(\cdot,L),\dots,p^\varepsilon_{N(\omega_0),pr}(\cdot,L))_\varepsilon$ *converges in distribution in* $\mathcal{C}([-T,T],\mathbb{C}^{N(\omega_0)})$ *as* ε *goes to* 0 *to* $(p^0_{1,pr}(\cdot,L),\dots,p^0_{N(\omega_0),pr}(\cdot,L))_\varepsilon$*, where for* $j\in\{1,\dots,N(\omega_0)\}$
$$p^0_{j,pr}(t,L)=\frac{1}{2}e^{i\sigma_{j,H}(\omega_0)B_H(L)}f(t-\beta_j'(\omega_0)L)\langle\phi_j,\Psi\rangle_{L^2(0,d)}.$$

Here, B_H is a standard fractional Brownian motion with Hurst index $H\in(1/2,1)$, and $\sigma_{j,H}(\omega_0)$ is defined by (3.2).

Roughly speaking, the pulse can be described as

$$p_{pr}\Big(\frac{t}{\varepsilon^s},x,\frac{L}{\varepsilon^s}\Big) \underset{\varepsilon\to 0}{\simeq} \frac{e^{-i\omega_0 t/\varepsilon^s}}{2}\sum_{j=1}^{N(\omega_0)} e^{i\sigma_{j,H}(\omega_0)B_H(L)} f(t-\beta_j'(\omega_0)L)e^{i\beta_j(\omega_0)(L/\varepsilon^s-L_S)}\phi_j(x)\big\langle\phi_j,\Psi\big\rangle_{L^2(0,d)}.$$

The transmitted pulse is therefore a superposition of modes, each of them is centered around its travel time t_j defined by (4.3), but also modulated by a mode-dependent and frequency-dependent random phase. Once again the randomness comes from the same fractional Brownian motion for all the propagating modes.

5. Pulse propagation for a single-mode waveguide with Neumann boundary conditions

In this section we study, using Theorem 3.2, the particular case of a single-mode waveguide with Neumann boundary conditions on $\partial\mathcal{D}$ to make the link with earlier works in one-dimensional propagation media [19, 20] and make a distinction with the above results. Considering Neumann conditions at the waveguide boundaries, signals with pulsewidth of order 1 ($q=0$) can propagate through the waveguide. The reason is that, compared to the context of Dirichlet boundary conditions, the Laplacian now has a null eigenvalue. Moreover, as described below, in the Neumann case the random medium fluctuations produce a random travel time correction for the pulse. In this context, the spectral analysis of the transverse Laplacian ∂_x^2 in (2.2) is slightly different: $-\partial_x^2$ with Neumann boundary conditions on $\partial\mathcal{D}$ is a nonnegative self-adjoint operator in $L^2(0,d)$. Its spectrum consists of a countably infinite number of nonnegative eigenvalues $(\lambda_j)_{j\geq 0}$ since $(0,d)$ is a bounded domain. Therefore, we have for all $j\geq 0$

$$-\partial_x^2\phi_j(x)=\lambda_j\phi_j(x)\quad\forall x\in(0,d),\qquad\text{and}\qquad \phi_j'(0)=\phi_j'(d)=0,$$

where $0=\lambda_0<\lambda_1<\cdots$ and the eigenvectors $(\phi_j)_{j\geq 0}$ form an orthonormal basis of $L^2(0,d)$. In our context, we have explicit expressions for the eigenvectors and eigenvalues

$$\lambda_0=0\qquad\text{and}\qquad \phi_0=\frac{1}{\sqrt{d}},$$

and

$$\lambda_j=\frac{j^2\pi^2}{d^2}\qquad\text{and}\qquad \phi_j(x)=\sqrt{\frac{2}{d}}\cos(j\pi x/d),\quad\text{for } j\geq 1.$$

Moreover, in this context the number of propagating modes is now given by

$$N(\omega)=1+\Big[\frac{\omega d}{\pi c}\Big],$$

so that in order to have only one propagating mode ($N(\omega)=1$), we have to take

$$\omega\in(-\omega_c,\omega_c)\qquad\text{with}\qquad \omega_c=c\sqrt{\lambda_1}.$$

Let us assume that the temporal profile of the source (1.2) is given by $f^\varepsilon(t)=f(t)$, for which the support of $\widehat{f}(\omega)$ is included in $(-\omega_c,\omega_c)$. According to the remarks at the beginning of Section 4, to study the asymptotic behavior of the pulse, we consider the following propagating part with only one mode:

$$p_a\big(t,\frac{L}{\varepsilon^s}\big)=\frac{\int_0^d\Psi(x)dx}{4\pi d}\int\widehat{f}(\omega)e^{-i\omega(t-L/(c\varepsilon^s))}\Big[\overline{\mathbf{P}^{a,\varepsilon}_{11}(\omega,L)}\Big]^{-1}d\omega.$$

Here, the coefficient $[\overline{\mathbf{P}^{a,\varepsilon}_{11}(\omega,L)}]^{-1}$ is called the transmission coefficient, and is derived from the right hand side of (2.20) and that $|\mathbf{P}^{a,\varepsilon}_{11}(\omega,L)|^2-|\mathbf{P}^{b,\varepsilon}_{11}(\omega,L)|^2=1$. Consequently, we observe the transmitted pulse in a time window of order 1 (comparable to the pulse width) and centered at $L/(c\varepsilon^s)$ (of order the total travel time)

$$p^\varepsilon_a(u,L)=p_a\big(\frac{L}{c\varepsilon^s}+u,\frac{L}{\varepsilon^s}\big)=\frac{\int_0^d\Psi(x)dx}{4\pi d}\int\widehat{f}(\omega)e^{-i\omega u}[\overline{\mathbf{P}^{a,\varepsilon}_{11}(\omega,L)}]^{-1}d\omega,$$

and we have the following result.

Theorem 5.1. *For $s=1/(2-\mathfrak{h})$ the family $(p^\varepsilon_a(\cdot,L))_\varepsilon$ converges in distribution on $\mathcal{C}([-T,T])$ as ε goes to 0 to*

$$p^0_a(t,L)=\frac{\int_0^d\Psi(x)dx}{2d}f\big(t-\sigma_H B_H(L)\big).$$

Here, B_H is a standard fractional Brownian motion with Hurst index $H=(2-\mathfrak{h})/2\in(1/2,1)$, and

$$\sigma_H^2=\frac{C_\mathfrak{h}}{4H(2H-1)d^2c^2}\int_{(0,d)^2}R(x,x')dxdx'.$$

The proof of this result follows closely the one of Theorem 4.1 and Theorem 4.2 using the first point of Theorem 3.2. The result obtained in Theorem 5.1 is similar to the ones obtained in [19,20] for one-dimensional propagation media. The transmitted pulse is the original pulse with a random time shift given by a fractional Brownian motion. In contrast to the result obtained in the same context (single-mode random waveguide), but with random perturbations having mixing properties [11, Proposition 3], the random shift is here a fractional Brownian motion, and not a standard Brownian motion. Moreover there is here no determinist deformation of the pulse and the energy is not affected because the pulse does not accumulate enough scattering events to affect it.

In the remaining part of the paper we first derive some properties on the stochastic process $V(z,x)$ which we then use in the proof of Theorem 3.1, Theorem 4.1, and Theorem 4.2.

6. The random fluctuations model

This section is devoted to some properties of the random field $V(z,x)$ (defined by (1.3)) modeling the medium fluctuations. The long-range property (1.4) is the key to observe stochastic effects driven by a fractional Brownian motion and not a standard one as in the mixing case [10, Chapter 20]. The following proposition shows that V exhibits the long-range property as well.

Proposition 6.1. *For all $s\in\mathbb{R}$ and $(x,y)\in(0,d)^2$, we have*

$$\mathbb{E}[V(z+s,x)V(s,y)]\underset{z\to+\infty}{\sim}\frac{C_\mathfrak{h}}{z^\mathfrak{h}}R(x,y)\quad\text{with}\quad C_\mathfrak{h}=\frac{c_\mathfrak{h}}{2\pi}\Big(\int_{-\infty}^{+\infty}\Theta(x)xe^{-x^2/2}dx\Big)^2.$$

The proof of this proposition follows exactly along the lines of [19, Lemma 1], but we present it in this section as preliminaries for the proof of Proposition 6.2.

Proof. **(Proof of Proposition 6.1.)** The following proof is based on decompositions over the Hermite polynomials which are defined by:

$$H_l(u):=(-1)^l\frac{g^{(l)}(u)}{g(u)},\qquad \text{with}\qquad g(u):=\frac{e^{-u^2/2}}{\sqrt{2\pi}},\tag{6.1}$$

which form an orthogonal basis of $L^2(\mathbb{R},g(u)du)$ satisfying

$$\big\langle H_l,H_m\big\rangle_{L^2(\mathbb{R},g(u)du)}=l!\delta_{lm}.\tag{6.2}$$

Let us recall the Mehler formula, which is

$$\mathbb{E}[H_l(X_1)H_m(X_2)]=l!\mathbb{E}[X_1X_2]^l\delta_{lm},$$

for two centered Gaussian random variables such that $\mathbb{E}[X_1^2]=\mathbb{E}[X_2^2]=1$.

Now, decomposing Θ in this basis

$$\Theta(x)=\sum_{l\geq 1}\frac{\Theta_l}{l!}H_l(x),\qquad \text{where}\qquad \Theta_l=\big\langle H_l,\Theta\big\rangle_{L^2(\mathbb{R},w(x)dx)},$$

and using the Mehler formula, we have

$$\begin{aligned}\mathbb{E}[\Theta\big(\mathcal{B}_\mathfrak{h}(z+s,x)\big)\Theta\big(\mathcal{B}_\mathfrak{h}(s,y)\big)]&=\sum_{l,m\geq 1}\frac{\Theta_l\Theta_m}{l!m!}\mathbb{E}[H_l(\mathcal{B}_\mathfrak{h}(z+s,x))H_m(\mathcal{B}_\mathfrak{h}(s,y))]\\&=\sum_{l\geq 1}\frac{\Theta_l^2}{l!}r_\mathfrak{h}^l(z)R^l(x,y)\\&=\Theta_1^2r_\mathfrak{h}(z)R(x,y)+\sum_{l\geq 2}\frac{\Theta_l^2}{l!}r_\mathfrak{h}^l(z)R^l(x,y).\end{aligned}$$

Now, let us remark that for $l\geq 2$ we have $z^\mathfrak{h}r_\mathfrak{h}^l(z)\to 0$ as $z\to+\infty$, and for z large enough

$$\sum_{l\geq 2}\Big|\frac{\Theta_l^2}{l!}r_\mathfrak{h}^l(z)R^l(x,y)\Big|\leq C\sum_{l\geq 2}\frac{\Theta_l^2}{l!}<+\infty,$$

since for all $l\geq 1$ we have assumed (1.5). Therefore, we finally have

$$z^\mathfrak{h}\mathbb{E}[V(z+s,x)V(s,y)]\underset{z\to+\infty}{\sim}c_\mathfrak{h}\Theta_1^2R(x,y),$$

which concludes the proof of Proposition 6.1. ∎

The proof of Theorem 3.1 is based on a moment technique, and we then have to evaluate moments of the form

$$\frac{1}{\varepsilon^{n(s-1/2)}}\int_{\Delta_n(z)}\mathbb{E}\Big[\prod_{p=1}^n\Theta\Big(\mathcal{B}_\mathfrak{h}\big(\frac{z_p}{\varepsilon^s},x_p\big)\Big)\Big]\varphi_\varepsilon(z,z_1,\dots,z_n)dz_1\dots dz_n,$$

where n is an even number (otherwise this moment is 0 by symmetry), and

$$\Delta_n(z)=\big\{(z_1,\dots,z_n)\in[0,z]^n,\quad \text{s.t.}\quad 0\leq z_j\leq z_{j-1}\quad \forall j\in\{2,\dots,n\}\big\}.$$

The following proposition gives the leading order term of these moments.

PROPOSITION 6.2. *For all even numbers $n \geq 2$, and $s = 1/(2-\mathfrak{h})$, there exists a positive constant C such that*

$$\sup_{\varepsilon\in(0,1)} \sup_{(x_1,\dots,x_n)\in[0,d]^n} \frac{1}{\varepsilon^{n(s-1/2)}} \int_{[0,z]^n} \Big|\mathbb{E}\Big[\prod_{p=1}^n \Theta\Big(\mathcal{B}_{\mathfrak{h}}\Big(\frac{z_p}{\varepsilon^s}, x_p\Big)\Big)\Big]\Big| dz_1 \dots dz_n \leq C^n n^{n/2},$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{n(s-1/2)}} \int_{\Delta_n(z)} \mathbb{E}\Big[\prod_{p=1}^n \Theta\Big(\mathcal{B}_{\mathfrak{h}}\Big(\frac{z_p}{\varepsilon^s}, x_p\Big)\Big)\Big] \varphi_\varepsilon(z, z_1, \dots, z_n) dz_1 \dots dz_n$$
$$= \lim_{\varepsilon\to 0} C_{\mathfrak{h}}^{n/2} \int_{\Delta_n(z)} \sum_{\mathcal{F}} \prod_{(p,q)\in\mathcal{F}} \frac{R(x_p, x_q)}{|z_p - z_q|^{\mathfrak{h}}} \varphi_\varepsilon(z, z_1, \dots, z_n) dz_1 \dots dz_n,$$

where φ_ε is a bounded function for all ε. Here, the sum is over over all the pairings $\mathcal{F}$ of $\{1,\dots,n\}$, and the limit $\varepsilon \to 0$ is uniform with respect to $(x_1,\dots,x_n)$.

We recall that a pairing formed over vertices of $S = \{1,\dots,2l\}$ is a partition of S into l pairs of couple (p,q) such that all the elements of S appear only in one of the pairs and with $p<q$. A corresponding proposition has been proved in [17], but for the sake of completeness we provide here a version of the proof adapted to our notations.

Proof. **(Proof of Proposition 6.2.)** The proof of this proposition is inspired by some ideas developed in [24]. For the first part of the result, we decompose the function $x \mapsto \Theta(\lambda_n^{-1} x)$ over the Hermite polynomials with resulting coefficients $\Theta_{n,l}$. We then have

$$\mathbb{E}\Big[\prod_{p=1}^n \Theta\Big(\mathcal{B}_{\mathfrak{h}}\Big(\frac{z_p}{\varepsilon^s}, x_p\Big)\Big)\Big] = \mathbb{E}\Big[\prod_{p=1}^n \Theta\Big(\frac{1}{\lambda_n}\lambda_n \mathcal{B}_{\mathfrak{h}}\Big(\frac{z_p}{\varepsilon^s}, x_p\Big)\Big)\Big]$$
$$= \sum_{l_1,\dots,l_n\geq 1} \Big(\prod_{p=1}^n \frac{\Theta_{n,l_p}}{l_p!}\Big) \mathbb{E}\Big[\prod_{p=1}^n H_{l_p}\Big(\lambda_n \mathcal{B}_{\mathfrak{h}}\Big(\frac{z_p}{\varepsilon^s}, x_p\Big)\Big)\Big].$$

Here, we introduce $\lambda_n := (n-2)^{-1/2}$ for technical reason. As we will see below, this parameter allows us to enforce the convergence of a series. Now, we would like to use [24, Lemma 3.2] stating that for $n \geq 2$ and a $(X_1,\dots,X_n)$ standard Gaussian vector, that is a mean zero Gaussian vector satisfying

$$\mathbb{E}[X_j^2] = 1 \qquad \text{and} \qquad |\mathbb{E}[X_j X_l]| \leq 1 \qquad \forall (j,l) \in \{1,\dots,n\}^2 \quad \text{with} \quad j \neq l,$$

we have

$$\mathbb{E}\Big[\prod_{p=1}^n H_{l_p}(X_p)\Big] = \begin{cases} \dfrac{l_1!\cdots l_n!}{2^q(q!)} \displaystyle\sum_{I(l_1,\dots,l_n)} r_{i_1 j_1} r_{i_2 j_2} \cdots r_{i_q j_q} \\ \qquad \text{if} \quad l_1 + \cdots + l_n = 2q \text{ and } 0 \leq l_1, \dots, l_n \leq q \\ 0 \quad \text{otherwise} \end{cases} \tag{6.3}$$

with $r_{ij} = \mathbb{E}[X_i X_j]$, and

$$I(l_1,\dots,l_n) = \big\{(i_1, j_1, \dots, i_q, j_q) \in \{1,\dots,n\}^{2q}, \quad \text{s.t.} \quad i_\beta \neq j_\beta \quad \forall \beta \in \{1,\dots,q\}$$
$$\text{and all index} \quad r \in \{1,\dots,n\} \quad \text{appears } l_r \text{ times}\big\}.$$

However, we cannot apply (6.3) directly because of the parameter λ_n. To handle this term, we first make use of the following multiplication theorem [9]:

$$H_l(\lambda_n u) = \lambda_n^l \sum_{k=0}^{[l/2]} (1-\lambda_n^{-2})^k \frac{l!}{2^k (l-2k)!\,k!} H_{l-2k}(u).$$

Rewriting (6.3) to our case, we have

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{p=1}^n H_{l_p-2k_p}\Big(\mathcal{B}_{\mathfrak{h}}\Big(\frac{z_p}{\varepsilon^s},x_p\Big)\Big)\Big]\\
&=\begin{cases} \dfrac{\tilde{l}_1!\cdots\tilde{l}_n!}{2^q q!} \displaystyle\sum_{I(\tilde{l}_1,\dots,\tilde{l}_n)} \prod_{\beta=1}^q r_{\mathfrak{h}}\Big(\frac{z_{i_\beta}-z_{j_\beta}}{\varepsilon^s}\Big) R(x_{i_\beta},x_{j_\beta}) \\ \qquad \text{if} \quad \tilde{l}_1+\cdots+\tilde{l}_n = 2q \text{ and } 0\leq \tilde{l}_1,\dots,\tilde{l}_n \leq q \quad \text{with} \quad \tilde{l}_p := l_p - 2k_p, \\ 0 \quad \text{otherwise.} \end{cases}
\end{aligned}$$

Now, let us remark that all the indices l are odd since Θ is assumed to be odd ($\Theta_{n,l}=0$ for l even). Hence, $\tilde{l}_p = l_p - 2k_p \geq 1$ for all $p=1,\dots,n$, so that $q\geq n/2$. Let us consider

$$\mathcal{A}_{q,n} := \int_{[0,z]^n} \Big|\prod_{m=1}^q r_{\mathfrak{h}}\Big(\frac{z_{i_m}-z_{j_m}}{\varepsilon^s}\Big) R(x_{i_m},x_{j_m})\Big| dz_1\dots dz_n.$$

From the definition of $I(\tilde{l}_1,\dots,\tilde{l}_n)$ one can deduce that each of the $z_1,\dots,z_n$ appear at least once in the above product. Then, by keeping $n/2$ of them to integrate $r_{\mathfrak{h}}$, and bounding $r_{\mathfrak{h}}$ by $\sup_u |r_{\mathfrak{h}}(u)|=1$ for the others, we obtain

$$\mathcal{A}_{q,n} \leq (2z)^{n/2} \Big(\sup_z |r_{\mathfrak{h}}(z)|\Big)^{q-n/2} \underbrace{\Big(\sup_{x_1,x_2} |R(x_1,x_2)|\Big)^q}_{\leq 1} \Big(\int_0^z \Big|r_{\mathfrak{h}}\Big(\frac{u}{\varepsilon^s}\Big)\Big| du\Big)^{n/2}, \tag{6.4}$$

using the fact that the function $r_{\mathfrak{h}}$ is even.

Now, we want to estimate the cardinal of $I(\tilde{l}_1,\dots,\tilde{l}_n)$. For this, we apply again (6.3) with $X_1=\cdots=X_n=X$, where $X\sim\mathcal{N}(0,1)$, and with now $r_{i_m j_m}=1$. Hence, combining (6.3) and (6.4),

$$\begin{aligned}
&\int_{[0,z]^n} \Big|\mathbb{E}\Big[\prod_{p=1}^n H_{l_p-2k_p}\Big(\mathcal{B}_{\mathfrak{h}}\Big(\frac{z_p}{\varepsilon^s},x_p\Big)\Big)\Big]\Big| dz_1\dots dz_n\\
&\leq C^n \Big(\int_0^z \Big|r_{\mathfrak{h}}\Big(\frac{u}{\varepsilon^s}\Big)\Big| du\Big)^{n/2} \mathbb{E}\Big[\Big|\prod_{p=1}^n H_{l_p-2k_p}(X)\Big|\Big].
\end{aligned}$$

Moreover, according to [24, Lemma 3.1], we have

$$\mathbb{E}\Big[\Big|\prod_{j=1}^n H_{r_j}(X)\Big|\Big] \leq \prod_{j=1}^n (n-1)^{r_j/2}\sqrt{r_j!}, \tag{6.5}$$

so that

$$
\begin{aligned}
&\int_{[0,z]^n} \mathbb{E}\Big[\prod_{p=1}^{n} H_{l_p}\Big(\lambda_n \mathcal{B}_{\mathfrak{h}}\Big(\frac{z_p}{\varepsilon^s},x_p\Big)\Big)\Big]dz_1\dots dz_n \\
&\leq C^n\left(\int_0^z\Big|r_{\mathfrak{h}}\Big(\frac{u}{\varepsilon^s}\Big)\Big|du\right)^{n/2}\prod_{p=1}^{n}\frac{\lambda_n^{l_p} l_p!}{[l_p/2]!} \\
&\quad\times \sum_{\substack{p=1,\dots,n\\ k_p=0,\dots,[l_p/2]}}\prod_{p=1}^{n}(n-1)^{l_p/2-k_p}(\lambda_n^{-2}-1)^{k_p}\frac{[l_p/2]!}{2^{k_p}k_p!\sqrt{(l_p-2k_p)!}}.
\end{aligned}
$$

After standard computations, we find for l_p odd,

$$
\sqrt{(l_p-2k_p)!}\geq 2^{[l_p/2]-k_p}([l_p/2]-k_p)!, \qquad \text{and} \qquad (n-1)^{l_p/2-k_p}\leq n^{1/2}(n-1)^{[l_p/2]-k_p},
$$

and then, with the binomial theorem,

$$
\begin{aligned}
&\sum_{k_p=0}^{[l_p/2]}(n-1)^{l_p/2-k_p}(\lambda_n^{-2}-1)^{k_p}\frac{[l_p/2]!}{2^{k_p}k_p!\sqrt{(l_p-2k_p)!}} \\
&\leq \frac{n^{1/2}}{2^{[l_p/2]}}\sum_{k_p=0}^{[l_p/2]}(n-1)^{[l_p/2]-k_p}(\lambda_n^{-2}-1)^{k_p}\frac{[l_p/2]!}{k_p!([l_p/2]-k_p)!} \\
&\leq \frac{n^{1/2}}{2^{[l_p/2]}}(n+\lambda_n^{-2}-2)^{[l_p/2]}.
\end{aligned}
$$

Hence, using again that all the indices l_p are odd numbers, we obtain

$$
\begin{aligned}
&\int_{[0,z]^n}\mathbb{E}\Big[\prod_{p=1}^{n}\Theta\Big(\mathcal{B}_{\mathfrak{h}}\Big(\frac{z_p}{\varepsilon^s},x_p\Big)\Big)\Big]dz_1\dots dz_n \\
&\leq n^{n/2}C^n\left(\int_0^z\Big|r_{\mathfrak{h}}\Big(\frac{u}{\varepsilon^s}\Big)\Big|du\right)^{n/2}\sum_{\substack{l_\beta\geq 1\\ \beta\in\{1,\dots,n\}}}\prod_{p=1}^{n}\frac{\lambda_n^{l_p}|\Theta_{n,l_p}|}{2^{[l_p/2]}[l_p/2]!}(n+\lambda_n^{-2}-2)^{[l_p/2]} \\
&\leq (\lambda_n n^{1/2})^n C^n\left(\int_0^z\Big|r_{\mathfrak{h}}\Big(\frac{u}{\varepsilon^s}\Big)\Big|du\right)^{n/2}\left(\sum_{l\geq 0}\frac{|\Theta_{n,2l+1}|}{l!}\right)^n.
\end{aligned}
$$

We consider now the decomposition

$$
\begin{aligned}
\sum_{l\geq 0}\frac{|\Theta_{n,2l+1}|}{l!} &= \left(\sum_{l=0}^{[nM]-1}+\sum_{l=[nM]}^{+\infty}\right)\frac{|\Theta_{n,2l+1}|}{l!} \\
&:= I+II,
\end{aligned}
$$

where M is independent of n and will be specified later. In what follows, we consider only the case $l\geq 1$ since the bound is direct for $l=0$. For I, using definition (6.1), an integration by parts in $\Theta_{n,2l+1}$ gives

$$
\Theta_{n,2l+1}=\lambda_n^{-1}(-1)^{2l}\int\Theta^{(1)}(\lambda_n^{-1}u)g^{(2l)}(u)=\lambda_n^{-1}\Theta^{(1)}_{n,2l},
$$

and using (6.2), we obtain

$$|\Theta_{n,2l+1}| \leq \lambda_n^{-1} \|\Theta^{(1)}(u)\|_{L^2(\mathbb{R},g(u)du)} \|H_{2l}\|_{L^2(\mathbb{R},g(u)du)} \leq \lambda_n^{-1} \sup_u |\Theta^{(1)}(u)| \sqrt{(2l)!}.$$

As a result, using that $(2l)! \leq 2^{2l}(l!)^2$ we have

$$I \leq C_1 + C_2 n^{1/2} \sum_{l=1}^{[nM]-1} 2^l \leq C_1 + n^{1/2} C^{nM}.$$

For II, using again definition (6.1), we have after $2l$ integration by parts,

$$\Theta_{n,2l+1} = \lambda_n^{-2l}(-1)^1 \int \Theta^{(2l)}(\lambda_n^{-1}u) g^{(1)}(u) du.$$

Then, using (1.5) and that $l! \geq e(l/e)^l$, we obtain

$$\sum_{l\geq [nM]} \frac{|\Theta_{n,2l+1}|}{l!} \leq C \sum_{l\geq [nM]} \frac{\lambda_n^{-2l}}{l!} C_\Theta^{2l} \leq C \sum_{l\geq [nM]} \left(\frac{n}{l}\right)^l (eC_\Theta^2)^l.$$

Setting $M > eC_\Theta^2$, we obtain $II \leq C$, and then

$$\int_{[0,z]^n} \mathbb{E}\Big[\prod_{p=1}^n \Theta\Big(\mathcal{B}_\mathfrak{h}\Big(\frac{z_p}{\varepsilon^s}, x_p\Big)\Big)\Big] dz_1 \dots dz_n \leq n^{n/2} C^n \left(\int_0^z \Big|r_\mathfrak{h}\Big(\frac{u}{\varepsilon^s}\Big)\Big| du\right)^{n/2}.$$

To conclude, it remains to estimate the term involving $r_\mathfrak{h}$. According to (1.4), there exists $z_0 > 0$ such that for all $z > z_0$, we have $|r_\mathfrak{h}(z)| \leq C|z|^{-\mathfrak{h}}$, and therefore, for all $z > z_0$,

$$\int_0^z \Big|r_\mathfrak{h}\Big(\frac{u}{\varepsilon^s}\Big)\Big| du \leq C\Big(\varepsilon^s + \varepsilon^{s\mathfrak{h}} \int_{\varepsilon^s z_0}^z |u|^{-\mathfrak{h}} du\Big) \leq C\varepsilon^{2s-1}, \tag{6.6}$$

with $s = 1/(2-\mathfrak{h})$.

To prove the second result of the proposition, we just need to decompose Θ itself over the Hermite polynomials:

$$\begin{aligned}\mathbb{E}\Big[\prod_{p=1}^n \Theta\Big(\mathcal{B}_\mathfrak{h}\Big(\frac{z_p}{\varepsilon^s}, x_p\Big)\Big)\Big] &= \sum_{l_1,\dots,l_n\geq 1} \left(\prod_{p=1}^n \frac{\Theta_{l_p}}{l_p!}\right) \mathbb{E}\Big[\prod_{p=1}^n H_{l_p}\Big(\mathcal{B}_\mathfrak{h}\Big(\frac{z_p}{\varepsilon^s}, x_p\Big)\Big)\Big] \\ &= \Theta_1^n \sum_{\mathcal{F}} \prod_{(\alpha,\beta)\in\mathcal{F}} r_\mathfrak{h}\Big(\frac{z_\alpha - z_\beta}{\varepsilon^s}\Big) R(x_\alpha, x_\beta) + R_n^\varepsilon(z_1,\dots,z_n),\end{aligned}$$

with

$$R_n^\varepsilon(z_1,\dots,z_n,x_1,\dots,x_n) = \sum_{p=1}^n \sum_{S_p(l_1,\dots,l_n)} \left(\prod_{j=1}^n \frac{\Theta_{l_j}}{l_j!}\right) \mathbb{E}\Big[\prod_{j=1}^n H_{l_j}\Big(\mathcal{B}_\mathfrak{h}\Big(\frac{z_j}{\varepsilon^s}, x_j\Big)\Big)\Big],$$

and

$$S_p(l_1,\dots,l_n) = \{l_k = 1 \text{ for } k < p; \quad l_p \in \{2,\dots,n\}, \quad l_k \in \{1,\dots,n\} \text{ for } k > p\}.$$

According to (6.3), R_n^ε can be recast as

$$R_n^\varepsilon(z_1,\dots,z_n,x_1,\dots,x_n)=\sum_{p=1}^{n}\sum_{q\geq n/2+1}\sum_{\tilde S_{p,q}(l_1,\dots,l_n)}\prod_{m=1}^{n}\Big(\frac{\Theta_{l_m}}{l_m!}\Big)\mathbb{E}\Big[\prod_{m=1}^{n}H_{l_m}\big(\mathcal{B}_{\mathfrak{h}}\big(\frac{z_m}{\varepsilon^s},x_m\big)\big)\Big],$$

where $\tilde S_{p,q}(l_1,\dots,l_n)=S_p(l_1,\dots,l_n)\cap\{l_1+\cdots+l_n=2q\}$. An important point here is the following. Since there is at least one index l_j greater than 2, and that n is even, we necessarily have $q\geq n/2+1$. This is the reason why we obtain some extra powers of ε, and then obtain the convergence to the leading term. Moreover, as before, we can estimate $\mathcal{A}_{q,n}$ in the same way for $q\geq n/2+1$. Using that $r_{\mathfrak{h}}$ and R are bounded by one, we have for all $(i_1,j_1,\dots,i_q,j_q)\in I(l_1,\dots,l_n)$,

$$\mathcal{A}_{q,n}\leq\int_{[0,z]^n}\prod_{m=1}^{n/2+1}\big|r_{\mathfrak{h}}\big(\frac{z_{i'_m}-z_{j'_m}}{\varepsilon^s}\big)\big|dz_1\dots dz_n,$$

where $(i'_1,j'_1,\dots,i'_{n/2+1},j'_{n/2+1})$ repeats j twice. Moreover, since $n/2+1$ is odd, there is only one other index, denoted by j', which appears twice, and therefore two cases are possible in the multiple integral. In the first case, we have a term of the form $r_{\mathfrak{h}}^2((z_j-z_{j'})/\varepsilon)$ (if any there is only one), and

$$\int_0^z dz_j\int_0^z dz_{j'}r_{\mathfrak{h}}^2\big(\frac{z_j-z_{j'}}{\varepsilon^s}\big)\leq\begin{cases}C_1\varepsilon^{2s\mathfrak{h}} & \text{if } \mathfrak{h}\in(0,1/2),\\ C'_1\varepsilon^s\log(1/\varepsilon^s) & \text{if } \mathfrak{h}=1/2,\\ C''_1\varepsilon^s & \text{if } \mathfrak{h}\in(1/2,1),\end{cases}$$

and then, using (6.6),

$$\begin{aligned}\mathcal{A}_{q,n}&\leq\int_{[0,z]^n}\prod_{m=1}^{n/2+1}\big|r_{\mathfrak{h}}\big(\frac{z_{i'_m}-z_{j'_m}}{\varepsilon^s}\big)\big|dz_1\dots dz_n\\&=\Big(\int_0^z du\int_0^z dv\big|r_{\mathfrak{h}}\big(\frac{u-v}{\varepsilon^s}\big)\big|dudv\Big)^{n/2-1}\times\int_0^z du\int_0^z dv\, r_{\mathfrak{h}}^2\big(\frac{u-v}{\varepsilon^s}\big)dudv\\&\leq C\varepsilon^{s\mathfrak{h}n/2}\varepsilon^{s(\mathfrak{h}\wedge(1-\mathfrak{h}))}\log(1/\varepsilon^s).\end{aligned}$$

Now, if we are not in the first case, we have a term of the form $r_{\mathfrak{h}}((z_j-z_{j'})/\varepsilon^s)r_{\mathfrak{h}}((z_j-z_k)/\varepsilon^s)$, $k\neq j'$. Then, the Cauchy-Schwarz inequality with respect to z_j, a change of variable, the fact that $r_{\mathfrak{h}}$ is even, and again (6.6), lead to

$$\begin{aligned}\mathcal{A}_{q,n}&\leq\int_{[0,z]^n}\prod_{m=1}^{n/2+1}\big|r_{\mathfrak{h}}\big(\frac{z_{i'_m}-z_{j'_m}}{\varepsilon^s}\big)\big|dz_1\dots dz_n\\&\leq C\Big(\int_0^z du\int_0^z dv\big|r_{\mathfrak{h}}\big(\frac{u-v}{\varepsilon^s}\big)\big|dudv\Big)^{n/2-1}\times\int_0^{2z}r_{\mathfrak{h}}^2\Big(\frac{u}{\varepsilon^s}\Big)du\\&\leq C\varepsilon^{s\mathfrak{h}n/2}\varepsilon^{s(\mathfrak{h}\wedge(1-\mathfrak{h}))}\log(1/\varepsilon^s).\end{aligned}$$

As before, one can bound the cardinal of $I(l_1,\dots,l_n)$, and then obtain

$$\begin{aligned}&\int_{[0,z]^n}\sup_{x_1,\dots,x_n}\Big|\mathbb{E}\Big[\prod_{m=1}^{n}H_{l_m}\big(\mathcal{B}_{\mathfrak{h}}\big(\frac{z_m}{\varepsilon^s},x_m\big)\big)\Big]\Big|dz_1\dots dz_n\\&\leq C\varepsilon^{s\mathfrak{h}n/2}\varepsilon^{s(\mathfrak{h}\wedge(1-\mathfrak{h}))}\log(1/\varepsilon^s)\Big|\mathbb{E}\Big[\prod_{m=1}^{n}H_{l_m}(X)\Big]\Big|\end{aligned}$$

so that using (6.5),

$$\begin{aligned}&\frac{1}{\varepsilon^{n(s-1/2)}}\int_{[0,z]^n}\sup_{x_1,\dots,x_n}\Big|\mathbb{E}\Big[R_n^\varepsilon(z_1,\dots,z_n,x_1,\dots,x_n)\Big]\Big|dz_1\dots dz_n\\ \leq& C\varepsilon^{s(\mathfrak{h}\wedge(1-\mathfrak{h}))}\log(1/\varepsilon^s)\sum_{j=1}^n\sum_{q\geq n/2+1}\sum_{\tilde{S}_{j,q}(l_1,\cdots,l_n)}\prod_{m=1}^n\frac{|\Theta_{l_m}|(n-1)^{l_m}}{\sqrt{l_m!}}\\ \leq&\varepsilon^{s(\mathfrak{h}\wedge(1-\mathfrak{h}))}\log(1/\varepsilon^s)nC^n\left(\sum_{l\geq1}\frac{|\Theta_l|(n-1)^l}{\sqrt{l!}}\right)^n.\end{aligned}$$

Moreover, according to (1.5), we have

$$|\Theta_l|=\Big|\int\Theta^{(l)}(u)g(u)du\Big|\leq CC_{\tilde{\Theta}}^l,$$

so that the sum above is finite for n fixed, and then the error term R_n^ε converges to zero as $\varepsilon\to0$. As a result, it remains to treat the leading term. For this term, we have

$$\begin{aligned}&\sum_{\mathcal{F}}\frac{1}{\varepsilon^{n(s-1/2)}}\int_{\Delta_n(z)}\prod_{(\alpha,\beta)\in\mathcal{F}}\Big|r_\mathfrak{h}\Big(\frac{z_\alpha-z_\beta}{\varepsilon^s}\Big)-\frac{\varepsilon^{s\mathfrak{h}}c_\mathfrak{h}}{|z_\alpha-z_\beta|^\mathfrak{h}}\Big|dz_1\dots dz_n\\ \leq&\frac{(n-1)!!}{n!}\Big[\frac{1}{\varepsilon^{2s-1}}\int_0^z\int_0^z\Big|r_\mathfrak{h}\Big(\frac{u-v}{\varepsilon^s}\Big)-\frac{\varepsilon^{s\mathfrak{h}}c_\mathfrak{h}}{|u-v|^\mathfrak{h}}\Big|dudv\Big]^{n/2},\end{aligned}$$

where $(n-1)!!=n!/(2^{n/2}(n/2)!)$ is the number of pairings of $\{1,\dots,n\}$. According to (1.4), for any $\eta>0$ and z_0 such that for $z>z_0$, we have $|r_\mathfrak{h}(z)-c_\mathfrak{h}|z|^{-\mathfrak{h}}|\leq\eta c_\mathfrak{h}|z|^{-\mathfrak{h}}$, and then,

$$\begin{aligned}&\frac{1}{\varepsilon^{2s-1}}\int_0^z\int_0^z\Big|r_\mathfrak{h}\Big(\frac{u-v}{\varepsilon^s}\Big)-\frac{\varepsilon^{s\mathfrak{h}}c_\mathfrak{h}}{|u-v|^\mathfrak{h}}\Big|dudv\\ \leq&\eta c_\mathfrak{h}\int_{|u-v|>\varepsilon^s z_0}|u-v|^{-\mathfrak{h}}dudv+\varepsilon\int_{|u-v|\leq z_0}r_\mathfrak{h}(u-v)dudv+c_\mathfrak{h}\int_{|u-v|\leq\varepsilon z_0}|u-v|^{-\mathfrak{h}}dudv,\end{aligned}$$

Finally, for all $\eta>0$, we have

$$\overline{\lim_{\varepsilon\to0}}\frac{1}{\varepsilon^{2(s-2)}}\int_0^z\int_0^z\Big|r_\mathfrak{h}\Big(\frac{u-v}{\varepsilon}\Big)-\frac{\varepsilon^{\mathfrak{h}}c_\mathfrak{h}}{|u-v|^\mathfrak{h}}\Big|dudv\leq\eta c_\mathfrak{h}\int_0^z\int_0^z|u-v|^{-\mathfrak{h}}dudv,$$

which concludes the proof. ∎

7. Proof of Theorem 3.1

The proof of this theorem is based on the idea developed in [3] in which the authors study the asymptotic behavior of the solution of the random Schrödinger equation with long-range correlations. The technique is based on the characterization of the moments of the limiting process. This technique is very convenient in our case. In fact, even if the field $V(z,x)$ in (1.3) is not Gaussian, the field $\mathcal{B}_\mathfrak{h}$ is Gaussian and the moments of V can be managed as described in Proposition 6.2.

To apply the moment technique to $\mathcal{P}^\varepsilon(\omega,z)$, we iterate the integrated form of (2.19),

$$\mathcal{P}^\varepsilon(\omega,z)=\begin{bmatrix}\mathbf{P}^{a,\varepsilon}(\omega,z)\\ \mathbf{P}^{b,\varepsilon}(\omega,z)\end{bmatrix}=\begin{bmatrix}Id_N\\ \mathbf{0}\end{bmatrix}+\int_0^z\Big[\frac{1}{\varepsilon^{s-1/2}}\mathbf{H}\Big(\omega,\frac{u}{\varepsilon^s}\Big)+\varepsilon^{1-s}\mathbf{G}\Big(\omega,\frac{u}{\varepsilon^s}\Big)\Big]\mathcal{P}^\varepsilon(\omega,u)du,\tag{7.1}$$

so that the transfer operator is given in term of a series

$$\mathcal{P}^{\varepsilon}(\omega,z)=\sum_{n=0}^{+\infty}\mathcal{P}^{\varepsilon,n}(\omega,z),\tag{7.2}$$

where

$$\mathcal{P}^{\varepsilon,n}(\omega,z)=\int\cdots\int_{\Delta_n(z)}\prod_{m=1}^{n}du_m\prod_{m=1}^{n}\Big[\frac{1}{\varepsilon^{s-1/2}}\mathbf{F}^{1,\varepsilon}(\omega,u_m)+\varepsilon^{1-s}\mathbf{F}^{2,\varepsilon}(\omega,u_m)\Big]\begin{bmatrix}Id_N\\ \mathbf{0}\end{bmatrix},$$

and

$$\Delta_n(z)=\big\{(z_1,\dots,z_n)\in[0,z]^n,\quad \text{s.t.}\quad 0\leq z_j\leq z_{j-1}\quad \forall j\in\{2,\dots,n\}\big\}.$$

Here, we use the notation

$$\mathbf{F}^{1,\varepsilon}_{11}(\omega,u)=\overline{\mathbf{F}^{1,\varepsilon}_{22}(\omega,u)}=\mathbf{H}^a\Big(\omega,\frac{u}{\varepsilon^s}\Big),\qquad \mathbf{F}^{1,\varepsilon}_{21}(\omega,u)=\overline{\mathbf{F}^{1,\varepsilon}_{12}(\omega,u)}=\mathbf{H}^b\Big(\omega,\frac{u}{\varepsilon^s}\Big),$$

$$\mathbf{F}^{2,\varepsilon}_{11}(\omega,u)=\overline{\mathbf{F}^{2,\varepsilon}_{22}(\omega,u)}=\mathbf{G}^a\Big(\omega,\frac{u}{\varepsilon^s}\Big),\qquad \mathbf{F}^{2,\varepsilon}_{21}(\omega,u)=\overline{\mathbf{F}^{2,\varepsilon}_{12}(\omega,u)}=\mathbf{G}^b\Big(\omega,\frac{u}{\varepsilon^s}\Big),$$

to make the distinction in the forthcoming computations between the terms produced by the forward (resp. backward)-going propagating modes and forward (resp. backward)-going evanescent modes. As a result, we can write $\mathcal{P}^{\varepsilon,n}(\omega,z)$ as follows

$$\mathcal{P}^{\varepsilon,n}(\omega,z)=\sum_{(i_1,\dots,i_n)\in\{1,2\}^n}\int\cdots\int_{\Delta_n(z)}\prod_{m=1}^{n}du_m\Big[\prod_{m=1}^{n}\varepsilon^{i_m/2-s}\mathbf{F}^{i_m,\varepsilon}(\omega,u_m)\Big]\begin{bmatrix}Id_N\\ \mathbf{0}\end{bmatrix}\tag{7.3}$$

and

$$\prod_{m=1}^{n}\varepsilon^{i_m/2-s}\mathbf{F}^{i_m,\varepsilon}(\omega,u_m)\begin{bmatrix}Id_N\\ \mathbf{0}\end{bmatrix}=\varepsilon^{\frac{1}{2}\sum_{m=1}^{n}i_m-ns}\begin{bmatrix}\sum_{(p_1,\dots,p_n)\in\{1,2\}^n}\prod_{m=1}^{n}\mathbf{F}^{i_m,\varepsilon}_{p_{m-1},p_m}(\omega,u_m)\\ \sum_{(q_1,\dots,q_n)\in\{1,2\}^n}\prod_{m=1}^{n}\mathbf{F}^{i_m,\varepsilon}_{q_{m-1},q_m}(\omega,u_m)\end{bmatrix},\tag{7.4}$$

where $p_0=p_n=q_n=1$ and $q_0=2$. As we will see, only the component $i_l=1$ and $p_l=1$ $(l=1,\dots,n)$ in (7.4) has a nontrivial limit. All the other terms converge to 0 in probability.

Proposition 7.1. *The series (7.2) is well defined and*

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big\|\mathcal{P}^{\varepsilon}(\omega,z)-\begin{bmatrix}X^{\varepsilon}(\omega,z)\\ 0\end{bmatrix}\Big\|_{\mathcal{M}_N(\mathbb{C})\times\mathcal{M}_N(\mathbb{C})}\Big]=0,$$

where $\mathcal{P}^{\varepsilon}(\omega,z)$ is defined by (7.1), and $X^{\varepsilon}(\omega,z)$ is defined by

$$X^{\varepsilon}(\omega,z)=\sum_{n=0}^{+\infty}X^{\varepsilon,n}(\omega,z),\tag{7.5}$$

with

$$X^{\varepsilon,n}(\omega,z)=\frac{1}{\varepsilon^{n(s-1/2)}}\int\cdots\int_{\Delta_n(z)}\prod_{m=1}^{n}\mathbf{H}^a\big(\omega,\frac{u_m}{\varepsilon^s}\big)du_m.\tag{7.6}$$

Let us remark that $X^\varepsilon(\omega,z)$ corresponds to the terms $i_l=1$ and $p_l=1$ $(l=1,\dots,n)$ in (7.4). In view of (2.19) $X^\varepsilon(\omega,z)$ would correspond to the dynamic of a forward-going wave only, with no evanescent mode. Therefore, for this term we have

$$\frac{d}{dz}X^\varepsilon(\omega,z)=\frac{1}{\varepsilon^{s-1/2}}\mathbf{H}^a\big(\omega,\frac{z}{\varepsilon^s}\big)X^\varepsilon(\omega,z) \qquad \text{with} \qquad X^\varepsilon(\omega,0)=Id,$$

so that for all $z\in[0,L]$

$$\sum_{j,l=1}^{N(\omega)}|X^\varepsilon_{jl}(\omega,z)|^2=N(\omega), \tag{7.7}$$

since $\mathbf{H}^a(\omega,z)$ is skew Hermitian, meaning that $X^\varepsilon(\omega,\cdot)$ is uniformly bounded. The following result deals with the asymptotic behavior of $X^\varepsilon(\omega,\cdot)$, and then concludes the proof of Theorem 3.1 according to [4, Theorem 3.1 pp. 27].

Proposition 7.2. *For all $z\in[0,L]$, the family $(X^\varepsilon(\omega,z))_\varepsilon$ converges in distribution on $\mathcal{M}_N(\mathbb{C})$ to*

$$D(\omega,z)=diag(e^{i\sigma_{1,H}(\omega)B_H(z)},\dots,e^{i\sigma_{N,H}(\omega)B_H(z)}), \tag{7.8}$$

with $\sigma_{j,H}(\omega)$ defined by (3.2), and where B_H is a standard fractional Brownian motion with Hurst index $H=(2-\mathfrak{h})/2$, and defined on a probability space $(\tilde\Omega,\tilde{\mathcal{T}},\tilde{\mathbb{P}})$.

The remainder of this section consists of proving Proposition 7.2 and Proposition 7.1. We start with the proof of Proposition 7.2, because it allows us to illustrate all the important points arising in the proof of Proposition 7.1.

Let us note that we will prove in Proposition 7.1 (resp. Proposition 7.2) the convergence in distribution on $\mathcal{M}_N(\mathbb{C})\times\mathcal{M}_N(\mathbb{C})$ (resp. $\mathcal{M}_N(\mathbb{C})$) equipped with the weak topology. However, since the weak and the strong topology are the same on finite-dimensional vector spaces, this strategy allows lighter notations without changing the result.

7.1. Proof of Proposition 7.2. To prove the convergence of $(X^\varepsilon(\omega,z))_\varepsilon$ we only have to focus on the convergence of its moments. In fact, $(X^\varepsilon(\omega,z))_\varepsilon$ being a bounded family we directly have its tightness, that is

$$\forall\eta>0,\quad \exists\mu>0 \quad \text{such that} \quad \overline{\lim_{\varepsilon\to0}}\,\mathbb{P}\Big(|\langle X^\varepsilon(\omega,z),\lambda\rangle|^2>\mu\Big)\leq\eta.$$

Therefore, the computation of the moments allows us to characterize uniquely all the accumulation points. To compute the moments, we focus first on the first-order moment as illustration (Proposition 7.3)

$$\mathbb{E}[\langle X^\varepsilon(\omega,z),\lambda\rangle]=\sum_{j,l=1}^{N}\mathbb{E}[X^\varepsilon_{jl}(\omega,z)]\overline{\lambda_{jl}},$$

where $\lambda\in\mathcal{M}_N(\mathbb{C})$. Then, we investigate the arbitrary high order moments (Proposition 7.4)

$$\mathbb{E}\Big[\big\langle X^\varepsilon(\omega,z),\lambda\big\rangle^{M_1}\overline{\big\langle X^\varepsilon(\omega,z),\lambda\big\rangle}^{M_2}\Big]$$

$$= \sum_{p_1=1}^{M_1} \sum_{j_{1,p_1},l_{1,p_1}=1}^{N} \sum_{p_2=1}^{M_2} \sum_{j_{2,p_2},l_{2,p_2}=1}^{N} \mathbb{E}\Big[\prod_{p_1=1}^{M_1} X^{\varepsilon}_{j_{1,p_1} l_{1,p_1}}(\omega,z) \prod_{p_2=1}^{M_2} \overline{X^{\varepsilon}_{j_{2,p_2} l_{2,p_2}}}(\omega,z) \Big]$$
$$\times \prod_{p_1=1}^{M_1} \overline{\lambda_{j_{1,p_1} l_{1,p_1}}} \prod_{p_2=1}^{M_2} \lambda_{j_{2,p_2} l_{2,p_2}}. \tag{7.9}$$

7.1.1. Proof of Proposition 7.2: Moment of order one. In this section we investigate the convergence of the expectation of $X^{\varepsilon}(\omega,z)$. This step is also useful to understand more easily the computations which are similar for the high-order moments. Throughout this section let $(j,l) \in \{1,\dots,N\}^2$ be two fixed indexes. According to (7.6), we have

$$X^{\varepsilon,n}_{jl}(\omega,z)$$
$$= \frac{i^n k^{2n}(\omega)}{2^n \varepsilon^{n(s-1/2)}} \sum_{j_1,\dots,j_{n-1}=1}^{N} \int \cdots \int_{\Delta_n(z)} \prod_{m=1}^{n} C_{j_{m-1} j_m}(\omega, u_m/\varepsilon^s) e^{i(\beta_{j_m}(\omega)-\beta_{j_{m-1}}(\omega))u_m/\varepsilon^s} du_m \tag{7.10}$$

where $j_0 = j$ and $j_n = l$.

Proposition 7.3. *For all $(j,l) \in \{1,\dots,N\}^2$, we have*

$$\lim_{\varepsilon \to 0} \mathbb{E}[X^{\varepsilon}_{jl}(\omega,z)] = \tilde{\mathbb{E}}\Big[e^{i\sigma_{j,H}(\omega) B_H(z)} \delta_{jl} \Big],$$

where $\tilde{\mathbb{E}}$ is the expectation associated to the probability space on which the standard fractional Brownian motion B_H is defined.

Proof. **(Proof of Proposition 7.3.)** To compute the limit in ε of $\mathbb{E}[X^{\varepsilon}_{jl}(\omega,z)]$, we have the two following lemmas.

Lemma 7.1. *The series (7.5) is well defined, and we have for all $(j,l) \in \{1,\dots,N(\omega)\}^2$*

$$\mathbb{E}[X^{\varepsilon}_{jl}(\omega,z)] = \mathbb{E}\Big[\sum_{n=0}^{+\infty} X^{\varepsilon,n}_{jl}(\omega,z) \Big] = \sum_{n=0}^{+\infty} \mathbb{E}[X^{\varepsilon,n}_{jl}(\omega,z)],$$

and

$$\lim_{\varepsilon \to 0} \mathbb{E}[X^{\varepsilon}_{jl}(\omega,z)] = \sum_{n=0}^{+\infty} \lim_{\varepsilon \to 0} \mathbb{E}[X^{\varepsilon,n}_{jl}(\omega,z)].$$

Proof. **(Proof of Lemma 7.1.)** This lemma follows from a simple adaptation of the proof of the first point in Proposition 6.2. In fact, it suffices to show

$$\sum_{n \geq 0} \sup_{\varepsilon \in (0,1)} \mathbb{E}[|X^{\varepsilon,n}_{jl}(\omega,z)|^2]^{1/2} < +\infty,$$

and we have

$$\begin{aligned}\mathbb{E}[|X^{\varepsilon,n}_{jl}(\omega,z)|^2] \leq & \frac{k^{4n}(\omega)}{2^{2n}\varepsilon^{2n(s-1/2)}} \sum_{\substack{j^1_1,\dots,j^1_{n-1}=1\\ j^2_1,\dots,j^2_{n-1}=1}}^{N} \int\cdots\int_{\Delta_n(z)}\int\cdots\int_{\Delta_n(z)} \prod_{m=1}^{n} du^1_m du^2_m\\ & \times\Big|\mathbb{E}\Big[\prod_{m=1}^{n} C_{j^1_{m-1}j^1_m}(\omega,u^1_m/\varepsilon^s)C_{j^2_{m-1}j^2_m}(\omega,u^2_m/\varepsilon^s)\Big]\Big|\\ \leq & \frac{C^{2n}n^n}{(n!)^2},\end{aligned}$$

where the C_{jl} are defined by (2.11), which concludes the proof of the lemma. ∎

Lemma 7.1 concerns the inversion between the expectation and the sum with respect to n, as well as the inversion between the limit in ε and the sum. As a result, we have

$$\lim_{\varepsilon\to 0}\mathbb{E}[X^{\varepsilon}_{jl}(\omega,z)] = \sum_{n=0}^{+\infty}\frac{i^n k^{2n}(\omega)}{2^n}\sum_{j_1,\dots,j_{n-1}=1}^{N}\lim_{\varepsilon\to 0} I^{n,\varepsilon}_{j_0,\dots,j_n}(z),$$

where

$$\begin{aligned}& I^{n,\varepsilon}_{j_0,\dots,j_n}(z)\\ &= \frac{1}{\varepsilon^{n(s-1/2)}}\int\cdots\int_{\Delta_n(z)}\prod_{m=1}^{n}\frac{e^{i(\beta_{j_m}(\omega)-\beta_{j_{m-1}}(\omega))u_m/\varepsilon^s}}{\sqrt{\beta_{j_{m-1}}(\omega)\beta_{j_m}(\omega)}}\mathbb{E}\Big[\prod_{m=1}^{n} C_{j_{m-1}j_m}(\omega,u_m/\varepsilon^s)\Big]du_1\dots du_n,\end{aligned}$$

with $j_0 = j$ and $j_n = l$. However, from the Gaussian property of $\mathcal{B}_\mathfrak{h}$ and because Θ is an odd function, the product in the expectation needs to contain an even number of terms, let say $n = 2n'$, otherwise the expectation is 0. Therefore, according to the second point of Proposition 6.2, we have

$$\mathbb{E}\Big[\prod_{m=1}^{2n'} C_{j_{m-1}j_m}(\omega,u_m/\varepsilon^s)\Big] \underset{\varepsilon\to 0}{\sim} C^{n'}_{\mathfrak{h}}\varepsilon^{n's\mathfrak{h}}\sum_{\mathcal{F}_{2n'}}\prod_{(\alpha,\gamma)\in\mathcal{F}_{2n'}}\frac{R_{j_{\alpha-1}j_\alpha j_{\gamma-1}j_\gamma}}{|u_\alpha-u_\gamma|^{\mathfrak{h}}}$$

where the sum is over all the pairings of $\{1,\dots,2n'\}$. As a result, for $s = 1/(2H)$ with $H = (2-\mathfrak{h})/2$, the limit in ε of the diagonal terms is

$$\lim_{\varepsilon\to 0} I^{2n',\varepsilon}_{j,j,\dots,j}(z) = \Big(\frac{C_\mathfrak{h} R_{jjjj}}{\beta^2_j(\omega)}\Big)^{n'}\int\cdots\int_{\Delta_{2n'}(z)}\sum_{\mathcal{F}_{2n'}}\prod_{(\alpha,\gamma)\in\mathcal{F}_{2n'}}|u_\alpha-u_\gamma|^{2H-2}du_1\dots du_{2n'}.$$

However, we also have

$$\sum_{\mathcal{F}_{2n'}}\prod_{(\alpha,\gamma)\in\mathcal{F}_{2n'}}|u_\alpha-u_\gamma|^{2H-2} = c^{n'}_{1,H}\tilde{\mathbb{E}}\Big[\prod_{q=1}^{2n'}\int\frac{e^{ir_q u_q}}{|r_q|^{H-1/2}}w(dr_q)\Big],$$

where $w(dr)$ is a Gaussian white noise defined on a probability space $(\tilde\Omega,\tilde{\mathcal{T}},\tilde{\mathbb{P}})$, and

$$c_{1,H} = \Gamma(2H-1)\sin(\pi H)/\pi. \tag{7.11}$$

Therefore, we obtain

$$\lim_{\varepsilon\to 0} I^{2n',\varepsilon}_{j,j,\dots,j}(z) = \Big(C_{\mathfrak{h}}\frac{R_{jjjj}}{\beta_j^2(\omega)}\Big)^{n'}\frac{c_{1,H}^{n'}}{(2n')!}\tilde{\mathbb{E}}\Big[\prod_{q=1}^{2n'}\int\frac{e^{ir_q z}-1}{ir_q|r_q|^{H-1/2}}w(dr_q)\Big]$$
$$= \Big(C_{\mathfrak{h}}\frac{R_{jjjj}}{\beta_j^2(\omega)}\Big)^{n'}\frac{c_{1,H}^{n'}c_{2,H}^{n'}}{(2n')!}\tilde{\mathbb{E}}[B_H^{2n'}(z)],$$

with

$$c_{2,H} = \pi/(H(2H-1)\Gamma(2H-1)\sin(H\pi)), \tag{7.12}$$

and

$$B_H(z) = \frac{1}{c_{2,H}^{1/2}}\int\frac{e^{irz}-1}{ir|r|^{H-1/2}}w(dr) \tag{7.13}$$

is a standard fractional Brownian motion with Hurst index $H=(2-\mathfrak{h})/2$. As a result, we obtain

$$\lim_{\varepsilon\to 0} I^{2n',\varepsilon}_{j,j,\dots,j}(z) = \tilde{\mathbb{E}}\Big[\frac{(\tilde{\sigma}_{j,H}(\omega)B_H(z))^{2n'}}{(2n')!}\Big] \quad \text{where} \quad \tilde{\sigma}^2_{j,H}(\omega) = \frac{C_{\mathfrak{h}}R_{jjjj}}{H(2H-1)\beta_j^2(\omega)},$$

and R_{jlmn} is defined by (3.3). The following lemma deals with the offdiagonal terms, and shows that these terms converge to 0 as $\varepsilon\to 0$ because of the fast oscillating phase terms.

LEMMA 7.2. *If there exists $n_0\in\{1,\dots,2n'\}$ such that $j_{n_0-1}\neq j_{n_0}$, then*

$$\lim_{\varepsilon\to 0} I^{n,\varepsilon}_{j_0,\dots,j_{2n'}}(z) = 0.$$

Proof. **(Proof of Lemma 7.2.)** According to the second point of Proposition 6.2 we have

$$I^{n,\varepsilon}_{j_0,\dots,j_{2n'}}(z) \underset{\varepsilon\to 0}{\sim} \prod_{m=1}^{2n'}\frac{C_{\mathfrak{h}}}{\sqrt{\beta_{j_{m-1}}(\omega)\beta_{j_m}(\omega)}}\sum_{\mathcal{F}_{2n'}}\int\cdots\int_{\Delta_{2n'}(z)}\prod_{m=1}^{2n'}du_m$$
$$\times\prod_{(\alpha,\gamma)\in\mathcal{F}_{2n'}}\frac{R_{j_{\alpha-1}j_\alpha j_{\gamma-1}j_\gamma}}{|u_\alpha-u_\gamma|^{\mathfrak{h}}}e^{i(\beta_{j_\alpha}(\omega)-\beta_{j_{\alpha-1}}(\omega))u_\alpha/\varepsilon^s}e^{i(\beta_{j_\gamma}(\omega)-\beta_{j_{\gamma-1}}(\omega))u_\gamma/\varepsilon^s}du_\alpha du_\gamma.$$

For a fixed pairing $\mathcal{F}_{2n'}$ let us consider the couple (α_0,γ_0) involving n_0, let say $\alpha_0=n_0$. Using the fact that

$$|u-v|^{-\mathfrak{h}} = c_{1,H}\int\frac{e^{ir(u-v)}}{|r|^{1-\mathfrak{h}}}dr,$$

we have

$$|u_{\alpha_0}-u_{\gamma_0}|^{-\mathfrak{h}}e^{i(\beta_{j_{\alpha_0}}(\omega)-\beta_{j_{\alpha_0-1}}(\omega))u_{\alpha_0}/\varepsilon^s}e^{i(\beta_{j_{\gamma_0}}(\omega)-\beta_{j_{\gamma_0-1}}(\omega))u_{\gamma_0}/\varepsilon^s}$$
$$=c_{1,\mathfrak{h}}\int\frac{1}{|r|^{1-\mathfrak{h}}}e^{iu_{\alpha_0}(r+(\beta_{j_{\alpha_0}}(\omega)-\beta_{j_{\alpha_0-1}}(\omega))/\varepsilon^s)}\times e^{-iu_{\gamma_0}(r-(\beta_{j_{\gamma_0}}(\omega)-\beta_{j_{\gamma_0-1}}(\omega))/\varepsilon^s)}dr.$$

Using an integration by parts in the variables u_{α_0} for

$$e^{iu(r+(\beta_{\tilde{j}}(\omega)-\beta_{\tilde{l}}(\omega))/\varepsilon^s)} \qquad \text{with primitive} \qquad \frac{e^{iu(r+(\beta_{\tilde{j}}(\omega)-\beta_{\tilde{l}}(\omega))/\varepsilon^s)}-1}{i(r+(\beta_{\tilde{j}}(\omega)-\beta_{\tilde{l}}(\omega))/\varepsilon^s)},$$

we obtain

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}}|I^{n,\varepsilon}_{j_0,\dots,j_{2n'}}(z)| \leq \overline{\lim_{\varepsilon\to 0}} C_1 \int_{\Delta^{(1)}_{2n'}(z)} \prod_{\substack{m=1\\ m\neq\alpha_0\\ m\neq\gamma_0}}^{2n'} du_m \prod_{\substack{(\alpha,\gamma)\in\mathcal{F}_{2n'}\\ (\alpha,\gamma)\neq(\alpha_0,\gamma_0)}} |u_\alpha-u_\gamma|^{-\mathfrak{h}} \\ \times\int\Big|\frac{e^{iu_{\alpha_0-1}(r+(\beta_{j_{\alpha_0}}(\omega)-\beta_{j_{\alpha_0-1}}(\omega))/\varepsilon^s)}-1}{r+(\beta_{j_{\alpha_0}}(\omega)-\beta_{j_{\alpha_0-1}}(\omega))/\varepsilon^s}\Big|\frac{dr}{|r|^{1-\mathfrak{h}}}\end{aligned}$$

$$\begin{aligned}+\overline{\lim_{\varepsilon\to 0}} C_2 \int_{\Delta^{(2)}_{2n'}(z)} \prod_{\substack{m=1\\ m\neq\alpha_0+1\\ m\neq\gamma_0}}^{2n'} du_m \prod_{\substack{(\alpha,\gamma)\in\mathcal{F}_{2n'}\\ (\alpha,\gamma)\neq(\alpha_0+1,\gamma_0)}} |u_\alpha-u_\gamma|^{-\mathfrak{h}} \\ \times\int\Big|\frac{e^{iu_{\alpha_0}(r+(\beta_{j_{\alpha_0}}(\omega)-\beta_{j_{\alpha_0-1}}(\omega))/\varepsilon^s)}-1}{r+(\beta_{j_{\alpha_0}}(\omega)-\beta_{j_{\alpha_0-1}}(\omega))/\varepsilon^s}\Big|\frac{dr}{|r|^{1-\mathfrak{h}}},\end{aligned}$$

where

$$\begin{aligned}\Delta^{(1)}_{2n'}(z) = \{(u_1,\dots,u_{\alpha_0-1},u_{\alpha_0+1},\dots,u_{2n'})\in[0,z]^{n-2}, \\ \text{s.t}\quad u_j\leq u_{j-1}\quad \forall j\in\{2,\dots,2n'\}\setminus\{\alpha_0\}\} \\ \Delta^{(2)}_{2n'}(z) = \{(u_1,\dots,u_{\alpha_0},u_{\alpha_0+2},\dots,u_{2n'}), \\ \text{s.t}\quad u_j\leq u_{j-1}\quad \forall j\in\{2,\dots,2n'\}\setminus\{\alpha_0+1\}\}.\end{aligned}$$

To conclude the proof of Lemma 7.2, we have the following lemma.

LEMMA 7.3. *For all $a\neq 0$ and $u\neq 0$, we have*

$$\lim_{\varepsilon\to 0}\int\frac{|e^{iu(r-a/\varepsilon^s)}-1|^2}{|r-a/\varepsilon^s|^2|r|^{1-\mathfrak{h}}}dr = 0.$$

Proof. **(Proof of Lemma 7.3.)** Let $\mu>0$ and $\eta>0$ be small parameters. We decompose the integral into three parts as follows

$$\begin{aligned}&\frac{|e^{iu(r-a/\varepsilon^s)}-1|}{|r-a/\varepsilon^s||r|^{1-\mathfrak{h}}}dr \\ &=\Big(\int_{|r-a/\varepsilon^s|>\mu/\varepsilon^s}+\int_{\eta<|r-a/\varepsilon^s|<\mu/\varepsilon^s}+\int_{|r-a/\varepsilon^s|<\eta}\Big)\frac{|e^{iu(r-a/\varepsilon^s)}-1|}{|r-a/\varepsilon^s||r|^{1-\mathfrak{h}}}dr.\end{aligned}$$

For the last integral, making the change of variable $r\to r+a/\varepsilon^s r$ we have

$$\begin{aligned}\int_{|r-a/\varepsilon^s|<\eta}\frac{|e^{iu(r-a/\varepsilon^s)}-1|}{|r-a/\varepsilon^s||r|^{2H-1}}dr &= \int_{|r|<\eta}\frac{|e^{iur}-1|^2}{|r||r+a/\varepsilon^s|^{1-\mathfrak{h}}}dr \\ &\leq |u|\int_{|r|<\eta}\frac{dr}{|r+a/\varepsilon^s|^{1-\mathfrak{h}}} \\ &\leq |u|\varepsilon^{s(1-\mathfrak{h})}\int_{|r|<\eta}\frac{dr}{||a|-\varepsilon^s\eta|^{1-\mathfrak{h}}} \\ &\leq C\varepsilon^{s(1-\mathfrak{h})}.\end{aligned}$$

For the second one, making the change of variable $r \to r/\varepsilon^s$ we have

$$\begin{aligned}\int_{\eta<|r-a/\varepsilon^s|<\mu/\varepsilon^s} \frac{|e^{iu(r-a/\varepsilon^s)}-1|^2}{|r-a/\varepsilon^s|^2|r|^{2H-1}}dr &= \varepsilon^s \int_{\varepsilon^s\eta<|\varepsilon^s r-a|<\mu} \frac{|e^{iu(\varepsilon^s r-a)/\varepsilon^s}-1|}{|\varepsilon^s r-a||r|^{1-\mathfrak{h}}}dr \\ &= \varepsilon^{s(1-\mathfrak{h})} \int_{\varepsilon^s\eta<|r-a|<\mu} \frac{dr}{|r-a||r|^{1-\mathfrak{h}}} \\ &\le \frac{e^{s(1-\mathfrak{h})}}{(|a|-\varepsilon^s\mu)^{1-\mathfrak{h}}} \int_{\varepsilon^s\eta<|u|<\mu} \frac{dr}{|u|} \\ &\le C\varepsilon^{s(1-\mathfrak{h})}\log(1/\varepsilon).\end{aligned}$$

Finally, making the change of variable $r \to r/\varepsilon^s$, we have

$$\begin{aligned}\int_{|r-a/\varepsilon^s|>\mu/\varepsilon^s} \frac{|e^{iu(r-a/\varepsilon^s)}-1|}{|r-a/\varepsilon^s||r|^{1-\mathfrak{h}}}dr &= \varepsilon^{s(1-\mathfrak{h})} \int_{|r-a|>\mu} \frac{|e^{iu(r-a)\varepsilon^s}-1|}{|r-a||r|^{1-\mathfrak{h}}}dr \\ &\le \varepsilon^{s(1-\mathfrak{h})} \int_{|r-a|>\mu} \frac{dr}{|r-a||r|^{1-\mathfrak{h}}}.\end{aligned}$$

□ □

As a result, we obtain

$$\begin{aligned}\lim_{\varepsilon} \mathbb{E}[X^\varepsilon_{jl}(\omega,z)] &= \sum_{n'=0}^{+\infty} \frac{(ik^2(\omega)\tilde{\sigma}_{j,H}/2)^{2n'}}{(2n')!} \tilde{\mathbb{E}}[B_H^{2n'}(z)]\delta_{jl} \\ &= \tilde{\mathbb{E}}\Big[\sum_{n=0}^{+\infty} \frac{(i\sigma_{j,H}B_H(z))^n}{n!}\Big]\delta_{jl} \\ &= \tilde{\mathbb{E}}\Big[e^{i\sigma_{j,H}B_H(z)}\delta_{jl}\Big],\end{aligned}$$

which concludes the proof of Proposition 7.3. □

7.1.2. Proof of Proposition 7.2: Arbitrary order moments. To identify properly the limit in distribution of $X^\varepsilon(\omega,z)$ as $\varepsilon \to 0$, we need to identify all the moments

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[\prod_{p_1=1}^{M_1} X^\varepsilon_{j_{1,p_1}l_{1,p_1}}(\omega,z) \prod_{p_2=1}^{M_2} \overline{X^\varepsilon_{j_{2,p_2}l_{2,p_2}}(\omega,z)}\Big]. \tag{7.14}$$

However, as we will see, the computations follow the ones of the first-order moment. In the previous expression and in the forthcoming computations, all the indexes with the subscript 2 correspond to the complex conjugate terms.

Proposition 7.4. *For all* $(j_{1,1},\dots,j_{1,M_1},j_{2,1},\dots,j_{2,M_2}) \in \{1,\dots,N\}^{M_1+M_2}$, *we have*

$$\begin{aligned}&\lim_{\varepsilon\to 0} \mathbb{E}\Big[\prod_{p_1=1}^{M_1} X^\varepsilon_{j_{1,p_1}l_{1,p_1}}(\omega,z) \prod_{p_2=1}^{M_2} \overline{X^\varepsilon_{j_{2,p_2}l_{2,p_2}}(\omega,z)}\Big] \\ &= \tilde{\mathbb{E}}\Big[\prod_{p_1=1}^{M_1} e^{i\sigma_{j_{1,p_1},H}(\omega)B_H(z)}\delta_{j_{1,p_1}l_{1,p_1}} \prod_{p_2=1}^{M_2} e^{-i\sigma_{j_{2,p_2},H}(\omega)B_H(z)}\delta_{j_{2,p_2}l_{2,p_2}}\Big],\end{aligned}$$

where B_H *is defined by* (7.13).

Proof. Using (7.10) we have

$$\begin{aligned}
&\prod_{p_1=1}^{M_1} X^{\varepsilon}_{j_{1,p_1} l_{1,p_1}}(\omega,z) \prod_{p_2=1}^{M_2} \overline{X^{\varepsilon}_{j_{2,p_2} l_{2,p_2}}(\omega,z)}\\
&=\sum_{p_1=1}^{M_1}\sum_{p_2=1}^{M_2}\sum_{n_{1,p_1}=0}^{+\infty}\sum_{n_{2,p_2}=0}^{+\infty}\sum_{j_{1,p_1,1},\dots,j_{1,p_1,n_{1,p_1}-1}=1}^{N}\sum_{j_{2,p_2,q_2,1},\dots,j_{2,p_2,n_{2,p_2}-1}=1}^{N}\mathbf{X}^{\varepsilon}_{\mathbf{n},\mathbf{j}}\\
&=\sum_{\mathbf{J}_{\mathbf{n},\mathbf{j}}}\mathbf{X}^{\varepsilon}_{\mathbf{n},\mathbf{j}},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{X}^{\varepsilon}_{\mathbf{n},\mathbf{j}}=&\frac{i^{n_1-n_2}k^{2n}(\omega)}{2^n\varepsilon^{n(s-1/2)}}\prod_{p_1=1}^{M_1}\int\cdots\int_{\Delta_{n_{1,p_1}}(z)}\prod_{p_2=1}^{M_2}\int\cdots\int_{\Delta_{n_{2,p_2}}(z)}\\
&\times\prod_{p_1=1}^{M_1}\prod_{m_{1,p_1}=1}^{n_{1,p_1}} C_{j_{1,p_1,m_{1,p_1}-1}j_{1,p_1,m_{1,p_1}}}(\omega,u_{1,p_1,m_{1,p_1}}/\varepsilon^s)\\
&\times e^{i(\beta_{j_{1,p_1,m_{1,p_1}}}(\omega)-\beta_{j_{1,p_1,m_{1,p_1}-1}}(\omega))u_{1,p_1,m_{1,p_1}}/\varepsilon^s}du_{1,p_1,m_{1,p_1}}\\
&\times\prod_{p_2=1}^{M_2}\prod_{m_{2,p_2}=1}^{n_{2,p_2}} C_{j_{2,p_2,m_{2,p_2}-1}j_{2,p_2,m_{2,p_2}}}(\omega,u_{2,p_2,m_{2,p_2}}/\varepsilon^s)\\
&\times e^{-i(\beta_{j_{2,p_2,m_{2,p_2}}}(\omega)-\beta_{j_{2,p_2,m_{2,p_2}-1}}(\omega))u_{2,p_2,m_{2,p_2}}/\varepsilon^s}du_{2,p_2,m_{2,p_2}},
\end{aligned}$$

with

$$n_1=\sum_{p_1=1}^{M_1}n_{1,p_1},\qquad n_2=\sum_{p_2=1}^{M_2}n_{2,p_2},\qquad\text{and}\qquad n=n_1+n_2.$$

Moreover, we have $j_{1,p_1,0}=j_{1,p_1}$, $j_{1,p_1,n_{1,p_1}}=l_{1,p_1}$, $j_{2,p_2,0}=j_{2,p_2}$, and $j_{2,p_2,n_{2,p_2}}=l_{2,p_2}$. To obtain the limit in ε of the expectation of the previous expression, we need first to exchange the infinite sums with the expectation. To do so, we use an adapted version of Lemma 7.1.

LEMMA 7.4. *We have*

$$\mathbb{E}\Big[\prod_{p_1=1}^{M_1} X^{\varepsilon}_{j_{1,p_1} l_{1,p_1}}(\omega,z) \prod_{p_2=1}^{M_2} \overline{X^{\varepsilon}_{j_{2,p_2} l_{2,p_2}}(\omega,z)}\Big]=\sum_{\mathbf{J}_{\mathbf{n},\mathbf{j}}}\mathbb{E}[\mathbf{X}^{\varepsilon}_{\mathbf{n},\mathbf{j}}],$$

and

$$\lim_{\varepsilon\to0}\mathbb{E}\Big[\prod_{p_1=1}^{M_1} X^{\varepsilon}_{j_{1,p_1} l_{1,p_1}}(\omega,z) \prod_{p_2=1}^{M_2} \overline{X^{\varepsilon}_{j_{2,p_2} l_{2,p_2}}(\omega,z)}\Big]=\sum_{\mathbf{J}_{\mathbf{n},\mathbf{j}}}\lim_{\varepsilon\to0}\mathbb{E}[\mathbf{X}^{\varepsilon}_{\mathbf{n},\mathbf{j}}].$$

Proof. **(Proof of Lemma 7.4.)** To prove this lemma, it suffices to show,

$$\sum_{\mathbf{J}_{\mathbf{n},\mathbf{j}}}\sup_{\varepsilon\in(0,1)}\mathbb{E}[|\mathbf{X}^{\varepsilon}_{\mathbf{n},\mathbf{j}}|^2]^{1/2}<+\infty.$$

Following the proof of Lemma 7.1, we have

$$\mathbb{E}[|\mathbf{X}^\varepsilon_{\mathbf{n},\mathbf{j}}|^2] \leq \frac{k^{4n}(\omega)}{2^{2n}} \prod_{p_1=1}^{M_1} \frac{1}{(n_{1,p_1}!)^2} \prod_{p_2=1}^{M_2} \frac{1}{(n_{2,p_2}!)^2} |\mathbf{E}^\varepsilon_{\mathbf{n},\mathbf{j}}|,$$

where

$$\begin{aligned}
\mathbf{E}^\varepsilon_{\mathbf{n},\mathbf{j}} = \frac{1}{\varepsilon^{2n(s-1/2)}} \int_{[0,z]^{2n}} \mathbb{E}\Big[& \prod_{p_1=1}^{M_1} \prod_{m_{1,p_1}=1}^{n_{1,p_1}} C_{j_{1,p_1,m_{1,p_1}-1} j_{1,p_1,m_{1,p_1}}}(\omega, u^1_{1,p_1,m_{1,p_1}}/\varepsilon^s) \\
& \times C_{j_{1,p_1,m_{1,p_1}-1} j_{1,p_1,m_{1,p_1}}}(\omega, u^2_{1,p_1,m_{1,p_1}}/\varepsilon^s) \\
\times & \prod_{p_2=1}^{M_2} \prod_{m_{2,p_2}=1}^{n_{2,p_2}} C_{j_{2,p_2,m_{2,p_2}-1} j_{2,p_2,m_{2,p_2}}}(\omega, u^1_{2,p_2,m_{2,p_2}}/\varepsilon^s) \\
& \times C_{j_{p_2,m_{2,p_2}-1} j_{p_2,m_{2,p_2}}}(\omega, u^2_{2,p_2,m_{2,p_2}}/\varepsilon^s)\Big] \\
& \prod_{m_{1,p_1}=1}^{n_{1,p_1}} du^1_{1,p_1,m_{1,p_1}} du^2_{1,p_1,m_{1,p_1}} \prod_{m_{2,p_2}=1}^{n_{2,p_2}} du^1_{2,p_2,m_{2,p_2}} du^2_{2,p_2,m_{2,p_2}},
\end{aligned}$$

so that for $s = 1/(2H)$ we obtain by following the proof of the first point in Proposition 6.2

$$|\mathbf{E}^\varepsilon_{\mathbf{n},\mathbf{j}}| \leq C^{2n}.$$

Consequently, we have

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}[|\mathbf{X}^\varepsilon_{\mathbf{n},\mathbf{j}}(\omega,z)|^2] \leq \prod_{p_1=1}^{M_1} \frac{C^{n_{1,p_1}} n_{1,p_1}^{n_{1,p_1}}}{(n_{1,p_1}!)^2} \prod_{p_2=1}^{M_2} \frac{C^{n_{2,p_2}} n_{2,p_2}^{n_{2,p_2}}}{(n_{2,p_2}!)^2},$$

which concludes the proof of Lemma 7.4. ∎

According to Lemma 7.4, to compute (7.14), it suffices to compute termwise $\lim_{\varepsilon\to 0}\mathbb{E}[\mathbf{X}^\varepsilon_{\mathbf{n},\mathbf{j}}]$, which is done in the two following lemmas. The first one deals with the "diagonal" terms while the second one deals with the "offdiagonal" terms. In the first lemma, we need to have $n = 2n'$ to obtain a nontrivial limit.

Lemma 7.5. *If for all* $(i, p_i, m_{i,p_i}) \in \{1,2\} \times \{1,\dots,M_i\} \times \{1,\dots,n_{i,p_i}\}$,

$$j_{i,p_i,m_{p_i}} = j_{i,p_i},$$

then we have

$$\lim_{\varepsilon\to 0}\mathbb{E}[\mathbf{X}^\varepsilon_{\mathbf{n},\mathbf{j}}] = \tilde{\mathbb{E}}\Big[\prod_{p_1=1}^{M_1}\big[i\sigma_{j_{1,p_1},H}(\omega)B_H(z)\big]^{n_{1,p_1}} \frac{1}{n_{1,p_1}!} \prod_{p_2=1}^{M_2}\big[-i\sigma_{j_{2,p_2},H}B_H(z)\big]^{n_{2,p_2}} \frac{1}{n_{2,p_2}!}\Big],$$

with $\sigma_{j,H}(\omega)$ *defined by* (3.2).

Lemma 7.6. *If there exists* $(i, p_i, m_{i,p_i}) \in \{1,2\} \times \{1,\dots,M_i\} \times \{1,\dots,n_{i,p_i}\}$ *such that,*

$$j_{i,p_i,m_{i,p_i}} \neq j_{i,p_i,m_{i,p_i}-1},$$

then we have

$$\lim_{\varepsilon\to 0}\mathbb{E}[\mathbf{X}^{\varepsilon}_{\mathbf{n},\mathbf{j}}]=0.$$

These two lemmas imply that

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\prod_{p_1=1}^{M_1} X^{\varepsilon}_{j_{1,p_1}l_{1,p_1}}(\omega,z)\prod_{p_2=1}^{M_2}\overline{X^{\varepsilon}_{j_{2,p_2}l_{2,p_2}}(\omega,z)}\Big]=\sum_{\mathbf{J}_{\mathbf{n},\mathbf{j}}}\lim_{\varepsilon\to 0}\mathbb{E}[\mathbf{X}^{\varepsilon}_{\mathbf{n},\mathbf{j}}]$$
$$=\tilde{\mathbb{E}}\Big[\prod_{p_1=1}^{M_1} e^{i\sigma_{j_{1,p_1},H}(\omega)B_H(z)}\delta_{j_{1,p_1}l_{1,p_1}}\prod_{p_2=1}^{M_2} e^{-i\sigma_{j_{2,p_2},H}(\omega)B_H(z)}\delta_{j_{2,p_2}l_{2,p_2}}\Big],$$

which concludes the proof of Proposition 7.4.

Proof. **(Proof of Lemma 7.5.)** Adapting the second point of Proposition 6.2, we have for $s=1/(2H)$ and $n=2n'$

$$\lim_{\varepsilon\to 0}\mathbb{E}[\mathbf{X}^{\varepsilon}_{\mathbf{n},\mathbf{j}}]$$
$$=\frac{i^{n_1-n_2}k^{4n'}(\omega)}{2^{2n'}}\prod_{p_1=1}^{M_1}\Big[\frac{R^{1/2}_{j_{1,p_1}j_{1,p_1}j_{1,p_1}j_{1,p_1}}}{\beta_{j_{1,p_1}}(\omega)}\Big]^{n_{1,p_1}}\frac{1}{n_{1,p_1}!}\prod_{p_2=1}^{M_2}\Big[\frac{R^{1/2}_{j_{2,p_2}j_{2,p_2}j_{2,p_2}j_{2,p_2}}}{\beta_{j_{2,p_2}}(\omega)}\Big]^{n_{2,p_2}}\frac{1}{n_{2,p_2}!}$$
$$\times\int_{[0,z]^{2n'}}\prod_{p_1=1}^{M_1}\prod_{m_{1,p_1}=1}^{n_{1,p_1}}du_{1,p_1,q_1,m_{1,p_1}}\prod_{p_2=1}^{M_2}\prod_{m_{2,p_2}=1}^{n_{2,p_2}}du_{2,p_2,m_{2,p_2}}\sum_{\mathcal{F}_{\mathbf{n},\mathbf{j}}}\prod_{(\alpha,\gamma)\in\mathcal{F}_{\mathbf{n},\mathbf{j}}}\frac{C_{\mathfrak{h}}}{|u_\alpha-u_\gamma|^{\mathfrak{h}}},$$

where the sum is over all the pairings of $\mathbf{I}_{\mathbf{n},\mathbf{j}}$ defined by

$$\mathbf{I}_{\mathbf{n},\mathbf{j}}=\big\{(i,p_i,m_{i,p_i})\in\{1,2\}\times\{1,\dots,M_i\}\times\{1,\dots,n_{i,p_i}\}\big\}. \tag{7.15}$$

Moreover, we have

$$\sum_{\mathcal{F}_{\mathbf{n},\mathbf{j}}}\prod_{(\alpha,\gamma)\in\mathcal{F}_{\mathbf{n},\mathbf{j}}}|u_\alpha-u_\gamma|^{-\mathfrak{h}}=c_{1,H}^{n'}\tilde{\mathbb{E}}\Big[\prod_{p_1=1}^{M_1}\prod_{m_{1,p_1}=1}^{n_{1,p_1}}\int\frac{e^{ir_{1,p_1,m_{1,p_1}}u_{1,p_1,m_{1,p_1}}}}{|r_{1,p_1,m_{1,p_1}}|^{H-1/2}}w(dr_{1,p_1,m_{1,p_1}})$$
$$\times\prod_{p_2=1}^{M_2}\prod_{m_{2,p_2}=1}^{n_{2,p_2}}\int\frac{e^{ir_{2,p_2,m_{2,p_2}}u_{2,p_2,m_{2,p_2}}}}{|r_{2,p_2,m_{2,p_2}}|^{H-1/2}}w(dr_{2,p_2,m_{2,p_2}})\Big],$$

where $w(dr)$ is a Gaussian white noise on the probability space $(\tilde{\Omega},\tilde{\mathcal{T}},\tilde{\mathbb{P}})$, $c_{1,H}$ is given by (7.11). Then, we finally have

$$\lim_{\varepsilon\to 0}\mathbb{E}[\mathbf{X}^{\varepsilon}_{\mathbf{n},\mathbf{j}}]=\tilde{\mathbb{E}}\Big[\prod_{p_1=1}^{M_1}\Big[\frac{ik^2(\omega)}{2\beta_{j_{1,p_1}}(\omega)}\sqrt{c_{1,H}c_{2,H}C_{\mathfrak{h}}R_{j_{1,p_1}j_{1,p_1}j_{1,p_1}j_{1,p_1}}}B_H(z)\Big]^{n_{1,p_1}}\frac{1}{n_{1,p_1}!}$$
$$\times\prod_{p_2=1}^{M_2}\Big[\frac{-ik^2(\omega)}{2\beta_{j_{2,p_2}}(\omega)}\sqrt{c_{1,H}c_{2,H}C_{\mathfrak{h}}R_{j_{2,p_2}j_{2,p_2}j_{2,p_2}j_{2,p_2}}}B_H(z)\Big]^{n_{2,p_2}}\frac{1}{n_{2,p_2}!}\Big].$$

where $c_{2,H}$ is given by (7.12), and B_H by (7.13). ∎

Proof. **(Proof of Lemma 7.6.)** First, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}[\mathbf{X}^{\varepsilon}_{\mathbf{n},\mathbf{j}}]\\
&=\frac{k^{2n}(\omega)}{2^n}\prod_{p_1=1}^{M_1}\int\cdots\int_{\Delta_{n_{1,p_1}}(z)}\prod_{m_{1,p_1}=1}^{n_{1,p_1}}du_{1,p_1,m_{1,p_1}}\prod_{p_2=1}^{M_2}\int\cdots\int_{\Delta_{n_{1,p_1}}(z)}\prod_{m_{2,p_2}=1}^{n_{2,p_2}}du_{2,p_2,m_{2,p_2}}\\
&\quad\times\prod_{p_1=1}^{M_1}\prod_{m_{1,p_1}=1}^{n_{1,p_1}}e^{i(\beta_{j_{1,p_1,m_{1,p_1}}}(\omega)-\beta_{j_{1,p_1,m_{1,p_1}-1}}(\omega))u_{1,p_1,m_{1,p_1}}/\varepsilon^s}\\
&\quad\times\prod_{p_2=1}^{M_2}\prod_{m_{2,p_2}=1}^{n_{2,p_2}}e^{-i(\beta_{j_{2,p_2,m_{2,p_2}}}(\omega)-\beta_{j_{2,p_2,m_{2,p_2}-1}}(\omega))u_{2,p_2,m_{2,p_2}}/\varepsilon^s}\\
&\quad\times\tilde{\mathbf{E}}_{\mathbf{n},\mathbf{j}},
\end{aligned}$$

with

$$\begin{aligned}
\tilde{\mathbf{E}}_{\mathbf{n},\mathbf{j}}&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{n(s-1/2)}}\mathbb{E}\Big[\prod_{p_1=1}^{M_1}\prod_{m_{1,p_1}=1}^{n_{1,p_1}}C_{j_{1,p_1,m_{1,p_1}-1}j_{1,p_1,m_{1,p_1}}}(\omega,u_{1,p_1,m_{1,p_1}}/\varepsilon^s)\\
&\qquad\times\prod_{p_2=1}^{M_2}\prod_{m_{2,p_2}=1}^{n_{2,p_2}}C_{j_{2,p_2,m_{2,p_2}-1}j_{2,p_2,m_{2,p_2}}}(\omega,u_{2,p_2,m_{2,p_2}}/\varepsilon^s)\Big]\\
&=\sum_{\mathcal{F}_{\mathbf{n},\mathbf{j}}}\prod_{(\alpha,\gamma)\in\mathcal{F}_{\mathbf{n},\mathbf{j}}}\frac{C_{\mathfrak{h}}}{|u_\alpha-u_\gamma|^{\mathfrak{h}}}\frac{R_{j_{\alpha-(0,0,1)}j_\alpha j_{\gamma-(0,0,1)}j_\gamma}}{\sqrt{\beta_{j_{\alpha-(0,0,1)}}(\omega)\beta_{j_\alpha}(\omega)\beta_{j_{\gamma-(0,0,1)}}(\omega)\beta_{j_\gamma}(\omega)}},
\end{aligned}$$

by adapting the proof of the second point in Proposition 6.2, and where the sum is over all the pairings of $\mathbf{I}_{\mathbf{n},\mathbf{j}}$ defined by (7.15). Let us fix a pairing $\mathcal{F}_{\mathbf{n},\mathbf{j}}$ and denote $\mathbf{i}_0=(i,p_i,m_{i,p_i})$ given in the statement of the lemma such that

$$j_{i,p_i,m_{i,p_i}}\neq j_{i,p_i,m_{p_i}-1}.$$

Following the proof of Lemma 7.2 and using Lemma 7.3, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{n(s-1/2)}}\int\cdots\int_{\Delta_{i,p_i}(z)}\prod_{m_{i,p_i}=1}^{n_{i,p_i}}du_{\alpha_0}\\
&\qquad\times\prod_{(\alpha,\gamma)\in\mathcal{F}_{\mathbf{n},\mathbf{j}}}|u_\alpha-u_\gamma|^{-\mathfrak{h}}e^{i(\beta_{j_\alpha}(\omega)-\beta_{j_{\alpha-1}}(\omega))u_\alpha/\varepsilon^s}e^{i(\beta_{j_\gamma}(\omega)-\beta_{j_{\gamma-1}}(\omega))u_\gamma/\varepsilon^s}=0,
\end{aligned}$$

and then

$$\lim_{\varepsilon\to 0}\mathbb{E}[\mathbf{X}^{\varepsilon}_{\mathbf{n},\mathbf{j}}]=0,$$

since all the pairings $\mathcal{F}_{\mathbf{n},\mathbf{j}}$ contain a term (α_0,γ_0) involving $\mathbf{i}_0$. That concludes the proof of Lemma 7.6 and at the same time the one of Proposition 7.4. □□

7.2. Proof of Proposition 7.1. The first point of the proposition follows the idea of Lemma 7.1, and for the second point (the convergence in probability) we only need to prove that

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big|\Big\langle\mathcal{P}^\varepsilon(\omega,z)-\begin{bmatrix}X^\varepsilon(\omega,z)\\0\end{bmatrix},\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}\Big\rangle\Big|\Big]=0,$$

thanks to the Markov's inequality, where

$$\Big\langle \mathcal{P}^\varepsilon(\omega,z)-\begin{bmatrix} X^\varepsilon(\omega,z)\\ 0\end{bmatrix},\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}\Big\rangle=\sum_{j,l=1}^{N}(\mathbf{P}^{a,\varepsilon}_{jl}(\omega,z)-X^\varepsilon_{jl}(\omega,z))\overline{\lambda_{1,jl}}+\mathbf{P}^{b,\varepsilon}_{jl}(\omega,z)\overline{\lambda_{2,jl}}.$$

However, both points need the following lemma.

Lemma 7.7. *We have*

$$\sum_{n\geq 0}\sup_{\varepsilon\in(0,1)}\mathbb{E}[|\mathcal{P}^{1,\varepsilon,n}_{jl}(\omega,z)|^2]^{1/2}+\mathbb{E}[|\mathcal{P}^{2,\varepsilon,n}_{jl}(\omega,z)|^2]^{1/2}<+\infty.$$

According to this result and the proof of Lemma 7.1, we have

$$\lim_{\varepsilon\to 0}\mathbb{E}[|\mathbf{P}^{a,\varepsilon}_{jl}(\omega,z)-X^\varepsilon_{jl}(\omega,z))|]\leq\sum_{n=0}^{+\infty}\lim_{\varepsilon\to 0}\mathbb{E}[|\mathcal{P}^{1,\varepsilon,n}_{jl}(\omega,z)-X^{\varepsilon,n}_{jl}(\omega,z)|^2]^{1/2}. \tag{7.16}$$

and

$$\lim_{\varepsilon\to 0}\mathbb{E}[|\mathbf{P}^{b,\varepsilon}_{jl}(\omega,z)|]\leq\sum_{n=0}^{+\infty}\lim_{\varepsilon\to 0}\mathbb{E}[|\mathcal{P}^{2,\varepsilon,n}_{jl}(\omega,z)|^2]^{1/2}. \tag{7.17}$$

Proof. **(Proof of Lemma 7.7.)** The proof of this result consists of adapting the one of Lemma 7.1. In fact, according to (7.3) and following the proof of the first point in Proposition 6.2, the terms

$$\mathbb{E}[|\mathcal{P}^{1,\varepsilon,n}_{jl}(\omega,z)|^2]\qquad\text{and}\qquad\mathbb{E}[|\mathcal{P}^{2,\varepsilon,n}_{jl}(\omega,z)|^2]$$

can be bounded by sums and products of terms of the form

$$\int_0^z\int_0^z|u-v|^{-\mathfrak{h}}dudv<+\infty,$$

and

$$\begin{aligned}&\frac{2}{\varepsilon^{2s-3/2}}\int_0^{+\infty}dwe^{-\beta_l(\omega)|w|}\int_0^z\int_0^u\Big|\frac{u}{\varepsilon^s}+w-\frac{v}{\varepsilon^s}\Big|^{-\mathfrak{h}}dudv\\ &\leq\frac{\sqrt{\varepsilon}C}{\beta_l(\omega)}\int_0^z\int_0^z|u-v|^{-\mathfrak{h}}dudv<+\infty,\end{aligned} \tag{7.18}$$

coming from $\mathbf{H}^a$, $\mathbf{H}^b$, $\mathbf{G}^a$, and $\mathbf{G}^b$ defined by (2.16). Let us remark that we get rid of the w in the right hand side of (7.18) using the fact $w\to|u-v+\varepsilon^s w|^{-\mathfrak{h}}$ is a decreasing function with respect to $|w|$. Moreover, we have to remark that the infinite sums in the definition of $\mathbf{G}^a$, and $\mathbf{G}^b$ give rise to finite terms in all these estimates. In fact, thanks to (7.18), these infinite sums involve

$$\sum_{l\geq N(\omega)+1}\frac{1}{\beta_l^2(\omega)}<+\infty, \tag{7.19}$$

since we consider a planar waveguide. Finally, with all these estimates we obtain

$$\sum_{n\geq 0}\sup_{\varepsilon\in(0,\varepsilon_0)}\mathbb{E}[|\mathcal{P}^{1,\varepsilon,n}_{jl}(\omega,z)|^2]^{1/2}+\mathbb{E}[|\mathcal{P}^{2,\varepsilon,n}_{jl}(\omega,z)|^2]^{1/2}\leq\sum_{n\geq 0}\frac{C^n n^{n/2}}{n!}<+\infty,$$

which concludes the proof of Lemma 7.7. ∎

Now, to compute (7.16) and (7.17), we remark that according to (7.3) and Proposition 6.2, the terms

$$\mathbb{E}[|\mathcal{P}^{1,\varepsilon,n}_{jl}(\omega,z)-X^{\varepsilon,n}_{jl}(\omega,z)|^2] \qquad \text{and} \qquad \mathbb{E}[|\mathcal{P}^{2,\varepsilon,n}_{jl}(\omega,z)|^2]$$

can be expressed by sums and products of terms of the form

$$|u-v|^{-\mathfrak{h}} e^{i(\beta_{l_1}(\omega)-\nu\beta_{j_1}(\omega))u/\varepsilon^s} e^{i(\beta_{l_2}(\omega)+\beta_{j_2}(\omega))v/\varepsilon^s},$$

where $\nu\in\{-1,1\}$ and

$$\frac{1}{\varepsilon^{2s-3/2}}\int dw e^{i\beta_{l_1}(\omega)w} e^{-\beta_{l'}(\omega)|w|} \times |u-v+\varepsilon^s w|^{-\mathfrak{h}} e^{i(\beta_{l_1}(\omega)-\nu_1\beta_{j_1}(\omega))u/\varepsilon^s} e^{i(\beta_{l_2}(\omega)-\nu_2\beta_{j_2}(\omega))v/\varepsilon^s},$$

where $(\nu_1,\nu_2)\in\{-1,1\}^2$. The contribution of these two terms is 0 in the limit $\varepsilon\to 0$. It is easy to see according to (7.18) that the contribution of the second term is 0 in the limit $\varepsilon\to0$. For the first one, it suffices to follow exactly the proof of Lemma 7.2. Consequently, we have

$$\lim_{\varepsilon\to0}\mathbb{E}[|\mathcal{P}^{1,\varepsilon,n}_{jl}(\omega,z)-X^{\varepsilon,n}_{jl}(\omega,z)|^2]=0 \qquad \text{and} \qquad \lim_{\varepsilon\to0}\mathbb{E}[|\mathcal{P}^{2,\varepsilon,n}_{jl}(\omega,z)|^2]=0,$$

which concludes the proof of Proposition 7.1.

8. Proof of Theorem 4.1

First of all, let us remark that according to Proposition 7.1, we have

$$\lim_{\varepsilon\to0}\mathbb{P}\Big(\sup_{t,x}|p_{pr}(t,x,L/\varepsilon^s)-\tilde{p}_{pr}(t,x,L/\varepsilon^s)|>\eta\Big)=0, \tag{8.1}$$

where p_{pr} is defined by (4.1), and

$$\tilde{p}_{pr}\Big(t,x,\frac{L}{\varepsilon^s}\Big):=\frac{1}{4\pi\varepsilon^q}\int d\omega e^{-i\omega t}\widehat{f}\Big(\frac{\omega-\omega_0}{\varepsilon^q}\Big)\sum_{j,l=1}^{N(\omega)}\sqrt{\frac{\beta_l(\omega)}{\beta_j(\omega)}}X^{\varepsilon}_{jl}(\omega,L) \times e^{i\beta_j(\omega)L/\varepsilon^s}e^{-i\beta_l(\omega)L_S}\phi_j(x)\langle\phi_l,\Psi\rangle_{L^2(0,d)}.$$

Then, according to [4, Theorem 3.1 pp. 27] we just have to prove Theorem 4.1 by replacing p_{pr} with $\tilde{p}_{pr}$. Second, let us remark that $\mathcal{C}([-T,T]\times(0,d),\mathbb{C}^{N(\omega_0)})$ equipped with the supremum norm on $[-T,T]\times(0,d)$ is a separable Banach space, so that the tightness and the relative compactness are the same (see [4, Theorem 5.2 pp. 60]). Consequently, according to the Arzelà–Ascoli theorem, we only need to prove the following result.

Lemma 8.1. *We have*

$$\lim_{M\to+\infty}\overline{\lim_{\varepsilon\to0}}\mathbb{P}\Big(\sup_{u,x}\sum_{j=1}^{N(\omega_0)}\Big|\tilde{p}^{\varepsilon}_{j,pr}(u,x,L)\Big|>M\Big)=0,$$

and for all $\eta>0$

$$\lim_{\tau\to0}\overline{\lim_{\varepsilon\to0}}\mathbb{P}\Big(\sup_{|x_1-x_2|+|u_1-u_2|\le\tau}\sum_{j=1}^{N(\omega_0)}\Big|\tilde{p}^{\varepsilon}_{j,pr}(u_1,x_1,L)-\tilde{p}^{\varepsilon}_{j,pr}(u_2,x_2,L)\Big|>\eta\Big)=0.$$

Consequently, using that the family $(\tilde{p}_{pr}(\cdot,\cdot,L/\varepsilon^s))_\varepsilon$ is uniformly bounded, we just have to characterize all the possible limits through their moments as is done in the two following lemmas.

LEMMA 8.2. *We have*

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\prod_{m_1=1}^{\gamma_1}\tilde{p}^{\varepsilon}_{\tilde{j}_{1,m_1},pr}(u_{1,m_1},x_{1,m_1},L)\prod_{m_2=1}^{\gamma_2}\overline{\tilde{p}^{\varepsilon}_{\tilde{j}_{2,m_2},pr}(u_{2,m_2},x_{2,m_2},L)}\Big]$$
$$=\tilde{\mathbb{E}}\Big[\prod_{m_1=1}^{\gamma_1}p^{0}_{\tilde{j}_{2,m_2},pr}(u_{1,m_1},x_{1,m_1},L)\prod_{m_2=1}^{\gamma_2}\overline{p^{0}_{\tilde{j}_{2,m_2},pr}(u_{2,m_2},x_{2,m_2},L)}\Big],$$

for all $(\gamma_1,\gamma_2)\in(\mathbb{N}^*)^2$, $(u_{n,m})\in[-T,T]^{\gamma_1+\gamma_2}$, $(x_{n,m})\in(0,d)^{\gamma_1+\gamma_2}$, *and* $(\tilde{j}_{n,m})\in\{1,\dots,N(\omega_0)\}^{\gamma_1+\gamma_2}$. *Here,* $p^0_{\tilde{j},pr}$ *is defined by* (4.8) *and* $\tilde{\mathbb{E}}$ *is the expectation associated to the probability space on which the standard fractional Brownian motion is defined.*

Proof. **(Proof of Lemma 8.1.)** This lemma is a direct consequence of (7.7). For the first point we have

$$\overline{\lim_{\varepsilon\to 0}}\mathbb{E}\Big[\sup_{j,u,x}\big|p^{\varepsilon}_{j,pr}(u,x,L)\big|\Big]\leq C_{N(\omega_0)}\int|\widehat{f}(h)|dh<+\infty,$$

in addition to the Markov's inequality. For the second point, in the same way and using the regularity of the eigenvectors ϕ_j we have for all $\tau>0$

$$\overline{\lim_{\varepsilon\to 0}}\mathbb{E}\Big[\sup_{j}\sup_{|x_1-x_2|+|u_1-u_2|\leq\tau}\big|p^{\varepsilon}_{j,pr}(u_1,x_1,L)-p^{\varepsilon}_{j,pr}(u_2,x_2,L)\big|\leq C_{N(\omega_0)}\tau\int|h\widehat{f}(h)|dh,$$

which concludes the proof of Lemma 8.1. □

Proof. **(Proof of Lemma 8.2.)** Expanding the product

$$\mathbf{M}_\varepsilon=\mathbb{E}\Big[\prod_{m_1=1}^{\gamma_1}\tilde{p}^{\varepsilon}_{\tilde{j}_{1,m_1},pr}(u_{1,m_1},x_{1,m_1},L)\prod_{m_2=1}^{\gamma_2}\overline{\tilde{p}^{\varepsilon}_{\tilde{j}_{2,m_2},pr}(u_{2,m_2},x_{2,m_2},L)}\Big],$$

according to the definition (4.7) (with $X^\varepsilon(\omega,L)$ instead of $\mathbf{P}^{a,\varepsilon}(\omega,L)$), gives

$$\begin{aligned}\mathbf{M}_\varepsilon=&\frac{1}{(4\pi)^{\gamma_1+\gamma_2}}\sum_{\substack{1\leq m_1\leq\gamma_1\\1\leq m_2\leq\gamma_2}}\sum_{\substack{1\leq j_{1,m_1},l_{1,m_1}\leq N(\omega_0)\\1\leq j_{2,m_2},l_{2,m_2}\leq N(\omega_0)}}\prod_{m_1,m_2}\sqrt{\frac{\beta_{l_{1,m_1}}(\omega_0)\beta_{l_{2,m_2}}(\omega_0)}{\beta_{j_{1,m_1}}(\omega_0)\beta_{j_{2,m_2}}(\omega_0)}}\\&\times e^{i(\beta_{j_{1,m_1}}(\omega_0)-\beta_{\tilde{j}_{1,m_1}}(\omega_0))L/\varepsilon^s}e^{-i(\beta_{j_{2,m_2}}(\omega_0)-\beta_{\tilde{j}_{2,m_2}}(\omega_0))L/\varepsilon^s}\\&\times e^{i(\beta_{\tilde{j}_{1,m_1}}(\omega_0)-\beta_{l_{1,m_1}}(\omega_0))L_S}e^{-i(\beta_{\tilde{j}_{2,m_2}}(\omega_0)-\beta_{l_{2,m_2}}(\omega_0))L_S}\\&\times\phi_{j_{1,m_1}}(x_{1,m_1})\phi_{l_{1,m_1}}(x_0)\phi_{j_{2,m_2}}(x_{2,m_2})\phi_{l_{2,m_2}}(x_0)\\&\times\int\cdots\int\prod_{m_1,m_2}dh_{1,m_1}dh_{2,m_2}\widehat{f}(h_{1,m_1})\overline{\widehat{f}(h_{2,m_2})}e^{-i(h_{1,m_1}u_{1,m_1}-h_{2,m_2}u_{2,m_2})}\\&\times e^{ih_{1,m_1}(\beta'_{j_{1,m_1}}(\omega_0)-\beta'_{\tilde{j}_{1,m_1}}(\omega_0))L/\varepsilon^{s-q}}e^{-ih_{2,m_2}(\beta'_{j_{2,m_2}}(\omega_0)-\beta'_{\tilde{j}_{2,m_2}}(\omega_0))L/\varepsilon^{s-q}}\\&\times e^{i(\Phi^{\varepsilon}_{j_{1,m_1}}(h_{1,m_1})-\Phi^{\varepsilon}_{j_{2,m_2}}(h_{2,m_2}))L/\varepsilon^{s-2q}}\\&\times\mathbb{E}\Big[\prod_{m_1,m_2}X^{\varepsilon}_{j_{1,m_1}l_{1,m_1}}(\omega_0+\varepsilon^q h_{1,m_1},L)\overline{X^{\varepsilon}_{j_{2,m_2}l_{2,m_2}}(\omega_0+\varepsilon^q h_{2,m_2},L)}\Big].\end{aligned}$$

According to Proposition 7.4 (with slight adaptations regarding the frequency) we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\prod_{m_1,m_2} X^{\varepsilon}_{j_{1,m_1}l_{1,m_1}}(\omega_0+\varepsilon^q h_{1,m_1},L)\overline{X^{\varepsilon}_{j_{2,m_2}l_{2,m_2}}(\omega_0+\varepsilon^q h_{2,m_2},L)}\Big]$$
$$=\tilde{\mathbb{E}}\Big[\prod_{m_1,m_2} D_{j_{1,m_1}j_{1,m_1}}(\omega_0,L)\overline{D_{j_{2,m_2}j_{2,m_2}}(\omega_0,L)}\Big]\prod_{m_1,m_2}\delta_{j_{1,m_1}l_{1,m_1}}\delta_{j_{2,m_2}l_{2,m_2}}.$$

Moreover, using the Riemann-Lebesgue lemma and the terms of the form $e^{ih(\beta'_j(\omega_0)-\beta'_{\tilde{j}}(\omega_0))L/\varepsilon^{s-q}}$ in $\mathbf{M}_\varepsilon$, the only nontrivial term in the limit $\varepsilon\to 0$ is obtained for $j_{1,m_1}=\tilde{j}_{1,m_1}$ and $j_{2,m_2}=\tilde{j}_{2,m_2}$. Therefore, in the case $q>s/2$, we have

$$\lim_{\varepsilon\to 0}\mathbf{M}_\varepsilon=\tilde{\mathbb{E}}\Big[\prod_{m_1=1}^{\gamma_1}\frac{D_{\tilde{j}_{1,m_1}\tilde{j}_{1,m_1}}(\omega_0,L)}{2}f(u_{1,m_1})\phi_{\tilde{j}_{1,m_1}}(x_{1,m_1})\phi_{\tilde{j}_{1,m_1}}(x_0)$$
$$\times\prod_{m_2=1}^{\gamma_2}\frac{\overline{D_{\tilde{j}_{2,m_2}\tilde{j}_{2,m_2}}(\omega_0,L)}}{2}f(u_{2,m_2})\phi_{\tilde{j}_{2,m_2}}(x_{2,m_2})\phi_{\tilde{j}_{2,m_2}}(x_0)\Big],$$

and in the case $q=s/2$

$$\lim_{\varepsilon\to 0}\mathbf{M}_\varepsilon=\tilde{\mathbb{E}}\Big[\prod_{m_1=1}^{\gamma_1}\frac{D_{\tilde{j}_{1,m_1}\tilde{j}_{1,m_1}}(\omega_0,L)}{2}K_{\tilde{j}_{1,m_1},L}*f(u_{1,m_1})\phi_{\tilde{j}_{1,m_1}}(x_{1,m_1})\phi_{\tilde{j}_{1,m_1}}(x_0)$$
$$\times\prod_{m_2=1}^{\gamma_2}\frac{\overline{D_{\tilde{j}_{2,m_2}\tilde{j}_{2,m_2}}(\omega_0,L)}}{2}\overline{K_{\tilde{j}_{2,m_2},L}*f(u_{2,m_2})}\phi_{\tilde{j}_{2,m_2}}(x_{2,m_2})\phi_{\tilde{j}_{2,m_2}}(x_0)\Big].$$

Finally, the case $q\in(0,s/2)$ is a consequence of the stationary phase method, which therefore concludes the proof of Lemma 8.2. ∎

9. Proof of Theorem 4.2

The proof of this theorem follows closely the one of Theorem 4.1. Using (8.1) we just need to prove Theorem 4.2 for $\tilde{p}_{pr}$ instead of p_{pr}. The tightness of the family $(\tilde{p}^{\varepsilon}_{j,pr}(\cdot,L))_{j,\varepsilon}$ follows along the same lines, and for the identification of the moments we also have using Proposition 7.4 (with a slight adaptations regarding the frequency) for all $(\gamma_1,\gamma_2)\in(\mathbb{N}^*)^2$, $(t_{n,m})\in[-T,T]^{\gamma_1+\gamma_2}$, and $(\tilde{j}_{n,m})\in(0,d)^{\gamma_1+\gamma_2}$

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\prod_{m_1=1}^{\gamma_1}\tilde{p}^{\varepsilon}_{\tilde{j}_{1,m_1},pr}(t_{1,m_1},L)\prod_{m_2=1}^{\gamma_2}\overline{\tilde{p}^{\varepsilon}_{\tilde{j}_{2,m_2},pr}(t_{2,m_2},L)}\Big]$$
$$=\tilde{\mathbb{E}}\Big[\prod_{m_1=1}^{\gamma_1}p^{0}_{\tilde{j}_{1,m_1},pr}(t_{1,m_1},L)\prod_{m_2=1}^{\gamma_2}\overline{p^{0}_{\tilde{j}_{2,m_2},pr}(t_{2,m_2},L)}\Big].$$

Conclusion. We have considered wave propagation in a random medium that exhibits long-range correlations. The waves propagate in a waveguide with a random speed of propagation. In Theorem 3.1 and Theorem 3.2 we described the case of monochromatic waves, while in Theorem 4.1 and Theorem 4.2 we investigated the cases of broadband and narrowband pulses. In all these cases the propagating wave is affected by a random mode-dependent and frequency-dependent phase modulation driven by the same fractional Brownian motion for all the propagating modes, and without affecting its energy. Moreover, the shapes of the modes are not affected by the ramdomness. Finally, in Theorem 5.1 we have investigated the case of a single-mode waveguide. In this case, the wave propagation is affected by a random time shift given by a fractional Brownian motion without affecting the pulse shape nor its energy. In the notation of Theorem 3.1 the phase modulation appears at depths L/ε^s with $s=1/(2H)$ for H the Hurst exponent characterizing the random medium. We assume here that $H>1/2$ so that the medium fluctuations are persistent and then the phase shift appears before the shape of the pulse starts to be modified due to scattering. The smoother the medium fluctuations are (the larger the Hurst index H is), the earlier the onset of the random phase correction. As also observed for one-dimensional propagation media and the random Schrödinger equation [12, 16], the wave has not experienced enough scattering to affect its energy at the depth regime of the onset of the random phase. To observe significant effects on the energy propagation, the waves need to go deeper in the random medium, with the depth depending on the Hurst coefficient. These aspects will be addressed in future works. Note however also that in such a context the forward scattering regime is not necessarily valid and some significant back scattering can occur. Note finally that the results presented in this paper are in contrast with the case of mixing random fluctuations when the pulse transformation and travel time correction appear simultaneously and for larger propagation distance (L/ε^s with $s=1$).

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