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THE ASYMPTOTIC BEHAVIOR OF PRIMITIVE EQUATIONS WITH MULTIPLICATIVE NOISE*

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Abstract. This article is concerned with the existence of random attractor and the existence of the invariant measure for 3D stochastic primitive equations driven by linear multiplicative noise under *non-periodic boundary conditions*. To achieve these goals, the crucial step is to establish the uniform a priori estimates in a functional space which is more regular than the solution space. But, it is very difficult because of the high nonlinearity and non-periodic boundary conditions of the stochastic primitive equations. To overcome the difficulties, we firstly obtain the existence of the absorbing ball in the solution space. Then, we use Aubin-Lions lemma and the regularity of the solution to prove that the solution operator is compact. Finally, by operating the absorbing ball with the compact solution operator, we obtain a compact absorbing ball in the solution space, which ensures the existence of the random attractor. Since the solution is Markov, the asymptotic compactness of the solution operator implies the existence of an invariant measure.

Keywords. stochastic primitive equations; random attractor; invariant measure.

AMS subject classifications. 60H15; 35Q35.

1. Introduction

The paper is concerned with the stochastic primitive equations (PEs) in a cylindrical domain

$$\mathcal{O}=M\times(-h,0)\subset\mathbb{R}^3,$$

where M is a smooth bounded domain in $\mathbb{R}^2$:

$$\partial_t\upsilon+(\upsilon\cdot\nabla)\upsilon+w\partial_z\upsilon+f\upsilon^{\perp}+\nabla p+L_1\upsilon=\sum_{k=1}^{n}\alpha_k\upsilon\circ dw_k^1, \tag{1.1}$$

$$\partial_z p+T=0, \tag{1.2}$$

$$\nabla\cdot\upsilon+\partial_z w=0, \tag{1.3}$$

$$\partial_t T+\upsilon\cdot\nabla T+w\partial_z T+L_2T=Q+\sum_{k=1}^{n}\beta_kT\circ dw_k^2. \tag{1.4}$$

The unknowns for the 3D stochastic PEs are the fluid velocity field $(\upsilon,w)=(\upsilon_1,\upsilon_2,w)\in\mathbb{R}^3$ with $\upsilon=(\upsilon_1,\upsilon_2)$ and $\upsilon^{\perp}=(-\upsilon_2,\upsilon_1)$ being horizontal, the temperature T and the pressure p. $f=f_0(\beta+y)$ is the given Coriolis parameter, Q is a given heat source. $\nabla=(\partial_x,\partial_y)$ is the horizontal gradient operator and $\Delta=\partial_x^2+\partial_y^2$ is the horizontal Laplacian. Let $\alpha_i,\beta_i\in\mathbb{R},i=1,2,\cdots,n$, $\{w_i^1,w_i^2,i=1,2,\cdots,n\}$ be a sequence of one-dimensional, independent, identically distributed Brownian motions. Here, we take $\sum_{k=1}^{n}\alpha_k\upsilon\circ dw_k^1$ and $\sum_{k=1}^{n}\beta_kT\circ dw_k^2$ to be Stratonovich multiplicative noise.

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The viscosity and the heat diffusion operators L_1 and L_2 are given by

$$L_i = -\nu_i \Delta - \mu_i \partial_{zz}, \quad i = 1, 2,$$

where ν_1 and μ_1 are the horizontal and vertical Reynolds numbers and ν_2 and μ_2 are the horizontal and vertical heat diffusivity. Without loss of generality, we assume that

$$\nu_1 = \mu_1 = \nu_2 = \mu_2 = 1.$$

The boundary of $\mathcal{O}$ is partitioned into three parts: $\Gamma_u \cup \Gamma_b \cup \Gamma_s$, where

$$\begin{aligned}
&\Gamma_u = \{(x,y,z) \in \overline{\mathcal{O}} : z = 0\},\\
&\Gamma_b = \{(x,y,z) \in \overline{\mathcal{O}} : z = -h\},\\
&\Gamma_s = \{(x,y,z) \in \overline{\mathcal{O}} : (x,y) \in \partial M, -h \le z \le 0\}.
\end{aligned}$$

Here h is a sufficiently smooth function. Without loss of generality, we assume $h = 1$.

We impose the following boundary conditions on the 3D stochastic PEs.

$$\partial_z v = \eta, \quad w = 0, \quad \partial_z T = -\gamma(T - \tau) \quad \text{on } \Gamma_u, \tag{1.5}$$

$$\partial_z v = 0, \quad w = 0, \quad \partial_z T = 0 \quad \text{on } \Gamma_b, \tag{1.6}$$

$$v \cdot \vec{n} = 0, \quad \partial_{\vec{n}} v \times \vec{n} = 0, \quad \partial_{\vec{n}} T = 0 \quad \text{on } \Gamma_s, \tag{1.7}$$

where $\eta(x,y)$ is the wind stress on the surface of the ocean, γ is a positive constant, τ is the typical temperature distribution on the top surface of the ocean and $\vec{n}$ is the norm vector to Γ_s. For the sake of simplicity, we assume that Q is independent of time and $\eta = \tau = 0$. It is worth pointing out that all results obtained in this paper are still valid for the general case by making some simple modifications.

Integrating (1.3) from -1 to z and using (1.5) and (1.6), we have

$$w(t,x,y,z) \stackrel{\text{def}}{=} \Phi(v)(t,x,y,z) = -\int_{-1}^{z} \nabla \cdot v(t,x,y,z')dz', \tag{1.8}$$

moreover,

$$\int_{-1}^{0} \nabla \cdot v dz = 0.$$

Integrating (1.2) from -1 to z, set p_s to be a certain unknown function at Γ_b satisfying

$$p(x,y,z,t) = p_s(x,y,t) - \int_{-1}^{z} T(x,y,z',t)dz'.$$

Then (1.1)-(1.4) can be rewritten as

$$\partial_t v + L_1 v + (v \cdot \nabla) v + \Phi(v)\partial_z v + \nabla p_s - \int_{-1}^{z} \nabla T dz' + f v^{\perp} = \sum_{k=1}^{n} \alpha_k v \circ dw_k^1, \tag{1.9}$$

$$\partial_t T + L_2 T + v \cdot \nabla T + \Phi(v)\partial_z T = Q + \sum_{k=1}^{n} \beta_k T \circ dw_k^2, \tag{1.10}$$

$$\int_{-1}^{0} \nabla \cdot v dz = 0. \tag{1.11}$$

The boundary value and initial conditions for (1.9)-(1.11) are given by

$$\partial_z \upsilon|_{\Gamma_u} = \partial_z \upsilon|_{\Gamma_b} = 0, \quad \upsilon \cdot \vec{n}|_{\Gamma_s} = 0, \quad \partial_{\vec{n}} \upsilon \times \vec{n}|_{\Gamma_s} = 0, \tag{1.12}$$

$$\Big(\partial_z T + \gamma T\Big)|_{\Gamma_u} = \partial_z T|_{\Gamma_b} = 0, \quad \partial_{\vec{n}} T|_{\Gamma_s} = 0; \tag{1.13}$$

$$\upsilon(x,y,z,t_0) = \upsilon_0(x,y,z), \;\; T(x,y,z,t_0) = T_0(x,y,z). \tag{1.14}$$

The primitive equations are the basic model used in the study of climate and weather prediction, which can be used to describe the motion of the atmosphere when the hydrostatic assumption is enforced (see [16, 23, 24] and the references therein). This model has been intensively investigated because of the interests stemmed from physics and mathematics. As far as we know, their mathematical study was initiated by Lions, Temam and Wang (see e.g. [28–31]). For example, the existence of global weak solutions for the primitive equations was established in [29]. Guillén-González et al. [19] obtained the global existence of strong solutions to the primitive equations with small initial data. The local existence of strong solutions to the primitive equations under the small-depth hypothesis was established by Hu et al. in [25]. Cao and Titi [9] developed a beautiful approach to dealing with the L^6-norm of the fluctuation $\tilde{\upsilon}$ of horizontal velocity and obtained the global well-posedness for the 3D viscous primitive equations. Subsequently, Kukavica and Ziane [26] developed a different method to handle non-rectangular domains and boundary conditions with physical reality. For the global well-posedness of 3D primitive equations with partial dissipation, we refer the reader to some papers, see e.g. [5–8, 10].

Along with the great successful developments of deterministic primitive equations, the random case has also been developed rapidly. Guo and Huang [17] obtained some kind of weak-type compactness properties of the strong solution under the condition that the momentum equation is driven by an additive stochastic forcing and the thermodynamical equation is driven by a deterministic heat source. When the noise is multiplicative, Gao and Sun [20] proved the global existence and uniqueness for the strong solution. Moreover, when the noise tends to zero, Gao and Sun [21] established the large deviation principle for this stochastic system. In [22], Gao and Sun studied the long-time behavior of stochastic PEs when the velocity is perturbed by an additive noise. Debussche et al. [12] concerned the global well-posedness of strong solution when the primitive equations are driven by multiplicative stochastic forcing. Under the periodic conditions, Glatt-Holtz et al. [18] constructed an invariant measure for the 3D PEs. The uniqueness of the invariant measures and large deviations for the 3D stochastic primitive equations were obtained by Dong, Zhai and Zhang in [14, 15] under the periodic conditions. Some analytical properties of weak solutions of 3D stochastic primitive equations with periodic boundary conditions were obtained in [13]. When the noise is an additive fractional noise, the long-time behavior of stochastic primitive equations is studied by one of the authors of this article in [34].

In this article, we aim to prove the existence of random attractor and the invariant measures for 3D stochastic PEs driven by linear multiplicative noise under *non-periodic boundary conditions*. The common method is to apply Sobolev compact theorem and Krylov-Bogoliubov lemma (see [11]), which requires uniform estimates with respect to the initial data in a functional space $(H^2(\mathcal{O}))^4$ that is more regular than the strong solution space $(H^1(\mathcal{O}))^4$. However, it is quite difficult because of highly nonlinear drift terms and non-periodic boundary conditions. Instead of using that method, we provide a new method to find a compact absorbing set in the strong solution space, which guarantees the existence of random attractor. The main idea is that we firstly prove that the solution operator of 3D stochastic PEs is compact in $(H^1(\mathcal{O}))^4$, $\mathbb{P}$-a.s.. Then using the compact property of solution operator and continuous dependence on initial data of the strong solution in a *good* space, we construct a compact absorbing ball. The random attractor we obtained is stronger than that in [17]. Specifically, the random attractor we obtain is compact in the strong solution space $(H^1(\mathcal{O}))^4$, $\mathbb{P}$-a.s. and attracts any orbit starting from $-\infty$ in the strong topology of $(H^1(\mathcal{O}))^4$ while the random attractor in [17] is not necessarily a compact subset in the strong solution space.

Taking into account the asymptotical compact property of the solution operator, we can prove the existence of invariant measures by showing that the one-point motions associated with the flow generated by 3D PEs define a family of Markov processes. Up to now, there are no works on the existence of invariant measures for the stochastic PEs subject to *non-periodic boundary conditions*. It may be an attempt to solve this problem by proving the asymptotic compact property of the solution operator.

The remainder of this paper is organized as follows. In Section 2, some preliminaries of 3D stochastic primitive equations are stated. In Section 3, the global well-posedness of 3D stochastic primitive equations is proved. In Section 4, we establish the existence of random attractor. Finally, in Section 5, the existence of invariant measures for 3D stochastic primitive equations is obtained. As usual, constants C may change from one line to the next, unless we give a special declaration. Denote by $C(a)$, a constant which depends on some parameter a.

2. Preliminaries

For $1\le p\le\infty$, let $L^p(\mathcal{O})$ and $L^p(M)$ be the usual Lebesgue spaces with the norm

$$|\phi|_p=\begin{cases}\left(\int_{\mathcal{O}}|\phi(x,y,z)|^p dxdydz\right)^{\frac{1}{p}}, & \phi\in L^p(\mathcal{O}),\\ \left(\int_M|\phi(x,y)|^p dxdy\right)^{\frac{1}{p}}, & \phi\in L^p(M).\end{cases}$$

In particular, $|\cdot|$ and $(\cdot,\cdot)$ represent norm and inner product of $L^2(\mathcal{O})$ (or $L^2(M)$), respectively. For $m\in\mathbb{N}_+$, $(W^{m,p}(\mathcal{O}),\|\cdot\|_{m,p})$ stands for the classical Sobolev space, see [1]. When $p=2$, we denote by $H^m(\mathcal{O})=W^{m,2}(\mathcal{O})$ with norm $\|\cdot\|_m$. Without confusion, we shall sometimes abuse notation and denote by $\|\cdot\|_m$, the norm in $H^m(M)$. Let

$$\begin{aligned}\mathcal{V}_1=\Big\{&v\in(C^\infty(\mathcal{O}))^2:\partial_z v|_{z=0}=0,\ \partial_z v|_{z=-1}=0,\ v\cdot\vec{n}|_{\Gamma_s}=0,\\ &\partial_{\vec{n}}v\times\vec{n}|_{\Gamma_s}=0,\ \int_{-1}^0\nabla\cdot v dz=0\Big\},\\ \mathcal{V}_2=\Big\{&T\in C^\infty(\mathcal{O}):\partial_z T|_{z=-1}=0,\ (\partial_z T+\gamma T)|_{z=0}=0,\ \partial_{\vec{n}}T|_{\Gamma_s}=0\Big\}.\end{aligned}$$

We denote by V_1 and V_2, the closure space of $\mathcal{V}_1$ in $(H^1(\mathcal{O}))^2$ and the closure space of $\mathcal{V}_2$ in $H^1(\mathcal{O})$, respectively. Let H_1 be the closure space of $\mathcal{V}_1$ with respect to the norm $|\cdot|_2$. Define $H_2=L^2(\mathcal{O})$. Set

$$V=V_1\times V_2,\quad H=H_1\times H_2.$$

Let $U=(v,T)$, $\tilde{U}=(\tilde{v},\tilde{T})$, V is equipped with the inner product

$$\begin{aligned}&\langle U,\tilde{U}\rangle_V\overset{\text{def}}{=}\langle v,\tilde{v}\rangle_{V_1}+\langle T,\tilde{T}\rangle_{V_2},\\ &\langle v,\tilde{v}\rangle_{V_1}\overset{\text{def}}{=}\int_{\mathcal{O}}\left(\nabla v\cdot\nabla\tilde{v}+\frac{\partial v}{\partial z}\cdot\frac{\partial\tilde{v}}{\partial z}\right)dxdydz,\end{aligned}$$

$$\langle T,\tilde{T}\rangle_{V_2} \overset{\text{def}}{=} \int_{\mathcal{O}} \left(\nabla T\cdot\nabla\tilde{T} + \frac{\partial T}{\partial z}\frac{\partial \tilde{T}}{\partial z}\right) dxdydz + \gamma\int_{\Gamma_u} T_1T_2 dxdy.$$

Subsequently, the norm of V is defined by $\|U\| = \langle U,U\rangle_V^{\frac{1}{2}}$. The inner product of H is defined by

$$\langle U,\tilde{U}\rangle_H \overset{\text{def}}{=} \langle \upsilon,\tilde{\upsilon}\rangle + \langle T,\tilde{T}\rangle,$$
$$\langle \upsilon,\tilde{\upsilon}\rangle \overset{\text{def}}{=} \int_{\mathcal{O}} \upsilon\cdot\tilde{\upsilon} dxdydz,$$
$$\langle T,\tilde{T}\rangle \overset{\text{def}}{=} \int_{\mathcal{O}} T\tilde{T} dxdydz.$$

Denote V_i' the dual space of V_i, $i=1,2$. Furthermore, we have the compact embedding relationship

$$D(A_i)\subset V_i\subset H_i\subset V_i'\subset D(A_i)',$$

and

$$\langle\cdot,\cdot\rangle_{V_i} = \langle A_i\cdot,\cdot\rangle = \langle A_i^{\frac{1}{2}}\cdot, A_i^{\frac{1}{2}}\cdot\rangle, \quad i=1,2.$$

For the sake of simplicity, in the following, we denote

$$\int_{\mathcal{O}}\cdot\, dxdydz = \int_{\mathcal{O}}\cdot\,, \quad \int_M \cdot\, dxdy = \int_M \cdot\,.$$

3. Global well-posedness of stochastic primitive equations

In this section, we aim to prove the global well-posedness of (1.9)-(1.14). Firstly, we introduce the following definition. Given $\mathcal{T}>0$, fix a single stochastic basis $(\Omega,\mathcal{F},\{\mathcal{F}_{t_0,t}\}_{t\in[t_0,\mathcal{T}]},\mathbb{P})$, where

$$\mathcal{F}_{t_0,t} \overset{\text{def}}{=} \sigma(W_k^j(s)-W_k^j(t_0), s\in[t_0,t], j=1,2). \tag{3.1}$$

Definition 3.1. *Fix $\mathcal{T}>0$, a continuous V-valued $\mathcal{F}_{t_0,t}$-adapted random field $(U(.,t))_{t\in[t_0,\mathcal{T}]} = (\upsilon(.,t),T(.,t))_{t\in[t_0,\mathcal{T}]}$ defined on $(\Omega,\mathcal{F},\mathbb{P})$ is said to be a strong (weak) solution to (1.9)-(1.14) if*

$$U\in C([t_0,\mathcal{T}];V)\cap L^2([t_0,\mathcal{T}];(H^2(\mathcal{O}))^3)\ (U\in C([t_0,\mathcal{T}];H)\cap L^2([t_0,\mathcal{T}];(H^1(\mathcal{O}))^3)\ \mathbb{P}-a.s..$$

and the following

$$\begin{aligned}
&\int_{\mathcal{O}}\upsilon(t)\cdot\phi_1 - \int_{t_0}^t ds\int_{\mathcal{O}}\{[(\upsilon\cdot\nabla)\phi_1+\Phi(\upsilon)\partial_z\phi_1]\upsilon - [(fk\times\upsilon)\cdot\phi_1 + (\int_{-1}^z Tdz')\nabla\cdot\phi_1]\}\\
&\qquad + \int_{t_0}^t ds\int_{\mathcal{O}}\upsilon\cdot L_1\phi_1\\
&= \int_{\mathcal{O}}\upsilon_0\cdot\phi_1 + \int_{t_0}^t\int_{\mathcal{O}}\sum_{k=1}^n \alpha_k\upsilon\circ dw_k^1(s,w)\cdot\phi_1,\\
&\int_{\mathcal{O}} T(t)\phi_2 - \int_{t_0}^t ds\int_{\mathcal{O}}[(\upsilon\cdot\nabla)\phi_2+\Phi(\upsilon)\partial_z\phi_2]T + \int_{t_0}^t ds\int_{\mathcal{O}} TL_2\phi_2 = \int_{\mathcal{O}} T_0\phi_2
\end{aligned}$$

$$+\int_{t_0}^{t} ds \int_{\mathcal{O}} Q\phi_2 + \int_{t_0}^{t}\int_{\mathcal{O}} \sum_{k=1}^{n} \beta_k T \circ dw_k^2(s,w)\cdot\phi_2,$$

hold $\mathbb{P}$*-a.s., for all* $t\in[t_0,\mathcal{T}]$ *and* $\phi=(\phi_1,\phi_2)\in D(A_1)\times D(A_2)$.

Consider

$$\alpha(t)=\exp(-\sum_{k=1}^{n}\alpha_k w_k^1),\ \beta(t)=\exp(-\sum_{k=1}^{n}\beta_k w_k^2).$$

Then $\alpha(t)$ and $\beta(t)$ satisfy the following Stratonovich equations

$$d\alpha(t)=-\sum_{k=1}^{n}\alpha_k\alpha(t)\circ dw_k^1(t),\quad d\beta(t)=-\sum_{k=1}^{n}\beta_k\beta(t)\circ dw_k^2(t).$$

Define

$$(u(t),\theta(t))=(\alpha(t)v(t),\ \beta(t)T(t)).$$

Then, $(u(t),\theta(t))$ satisfies

$$\partial_t u-\Delta u-\partial_{zz}u+\alpha^{-1}u\cdot\nabla u+\alpha^{-1}\Phi(u)\partial_z u+fu^{\perp}+\alpha\nabla p_s-\alpha\beta^{-1}\int_{-1}^{z}\nabla\theta dz'=0, \quad (3.2)$$

$$\partial_t\theta-\Delta\theta-\partial_{zz}\theta+\alpha^{-1}u\cdot\nabla\theta+\alpha^{-1}\Phi(u)\partial_z\theta=\beta Q, \quad (3.3)$$

$$\int_{-1}^{0}\nabla\cdot u dz=0. \quad (3.4)$$

The boundary and initial conditions for (3.2)-(3.4) are

$$\partial_z u|_{\Gamma_u}=\partial_z u|_{\Gamma_b}=0, u\cdot\vec{n}|_{\Gamma_s}=0, \partial_{\vec{n}}u\times\vec{n}|_{\Gamma_s}=0, \quad (3.5)$$

$$\Big(\partial_z\theta+\gamma\theta\Big)|_{\Gamma_u}=\partial_z\theta|_{\Gamma_b}=0,\quad \partial_{\vec{n}}\theta|_{\Gamma_s}=0, \quad (3.6)$$

$$(u\big|_{t_0},\theta\big|_{t_0})=(v_0,T_0). \quad (3.7)$$

DEFINITION 3.2. *Let* $\mathcal{T}$ *be a fixed positive time and* $(v_0,T_0)\in V$. (u,θ) *is called a strong solution of the system (3.2)-(3.7) on the time interval* $[t_0,\mathcal{T}]$ *if it satisfies (3.2)-(3.7) in the weak sense such that* $\mathbb{P}$*-a.s.*

$$u\in C([t_0,\mathcal{T}];V_1)\cap L^2([t_0,\mathcal{T}];(H^2(\mathcal{O}))^2),$$
$$\theta\in C([t_0,\mathcal{T}];V_2)\cap L^2([t_0,\mathcal{T}];H^2(\mathcal{O})).$$

THEOREM 3.1 (Existence of local solutions to (3.2)-(3.7)). *If* $Q\in L^2(\mathcal{O}), v_0\in V_1$, $T_0\in V_2$. *Then, for* $\mathbb{P}$*-a.s.,* $\omega\in\Omega$, *there exists a stopping time* $T^*>0$ *such that* (u,θ) *is a strong solution of the system (3.2)-(3.7) on the interval* $[t_0,T^*]$.

The proof of the existence of local solutions to (3.2)-(3.7) is similar to [19] and hence we omit it. Before showing the global well-posedness of the strong solution, we recall the following Lemma, a special case of a general result of Lions and Magenes [27], which will help us to show the continuity of the solution with respect to time in $(H^1(\mathcal{O}))^3$. We refer the readers to [32] for its proof.

LEMMA 3.1. *Let V,H,V' be three Hilbert spaces such that $V\subset H=H'\subset V'$, where H' and V' are the dual spaces of H and V, respectively. Suppose $u\in L^2(0,T;V)$ and $\frac{du}{dt}\in L^2(0,T;V')$. Then u is almost everywhere equal to a continuous function from $[0,T]$ into H.*

THEOREM 3.2 (Existence of global solution to (3.2)-(3.7)). *If $Q\in L^2(\mathcal{O}),v_0\in V_1$, $T_0\in V_2$, and $\mathcal{T}>0$. Then, for $\mathbb{P}$-a.s., $\omega\in\Omega$, there exists a unique strong solution (u,θ) of the system (3.2)-(3.7) or equivalently (v,T) to the system (1.9)-(1.14) on the interval $[t_0,\mathcal{T}]$.*

Proof. Let $[t_0,\tau_*)$ be the maximal interval of existence of the strong solution. For fixed $\omega\in\Omega$, we will establish various norms of this solution in the interval $[t_0,\tau_*)$. In particular, we will show that if $\tau_*<\infty$, then H^1-norm of the strong solution is bounded over the interval $[t_0,\tau_*)$.

A priori estimates: Referring to [9], define

$$\bar{\phi}(x,y)=\int_{-1}^{0}\phi(x,y,\xi)d\xi,\ \ \forall\ (x,y)\in M.$$

In particular,

$$\bar{u}(x,y)=\int_{-1}^{0}u(x,y,\xi)d\xi,\quad \text{in } M.$$

Let

$$\tilde{u}=u-\bar{u}.$$

Notice that

$$\bar{\tilde{u}}=0,\quad \nabla\cdot\bar{u}=0\ \ \text{in } M.$$

Taking the average of Equations (3.2) in the z direction over the interval $(-1,0)$, and using boundary conditions (3.5), we have

$$\partial_t\bar{u}+\alpha^{-1}\overline{u\cdot\nabla u+\Phi(u)\partial_z u}+f\bar{u}^{\perp}+\alpha\nabla p_s-\alpha\beta^{-1}\int_{-1}^{0}\int_{-1}^{z}\nabla\theta dz'dz-\Delta\bar{u}=0. \tag{3.8}$$

By the integration by parts, we get

$$\int_{-1}^{0}\Phi(u)\partial_z u dz=\int_{-1}^{0}u\nabla\cdot u dz=\int_{-1}^{0}(\nabla\cdot\tilde{u})\tilde{u}dz, \tag{3.9}$$

$$\int_{-1}^{0}u\cdot\nabla u dz=\int_{-1}^{0}\tilde{u}\cdot\nabla\tilde{u}dz+\bar{u}\cdot\nabla\bar{u}. \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), $\bar{u}$ satisfies

$$\begin{aligned}&\partial_t\bar{u}-\Delta\bar{u}+\alpha^{-1}(\overline{\tilde{u}\cdot\nabla\tilde{u}}+\overline{\tilde{u}\nabla\cdot\tilde{u}}+\bar{u}\cdot\nabla\bar{u})+f\bar{u}^{\perp}+\alpha\nabla p_s\\&\quad-\alpha\beta^{-1}\nabla\int_{-1}^{0}\int_{-1}^{z}\theta(x,y,\lambda,t)d\lambda dz=0,\end{aligned} \tag{3.11}$$

$$\nabla\cdot\bar{u}=0\quad\text{in } M, \tag{3.12}$$

$$\bar{u}\cdot\vec{n}=0,\ \partial_{\vec{n}}\bar{u}\times\vec{n}=0 \text{ on } M. \tag{3.13}$$

By subtracting (3.11) from (3.2) and using (3.5) and (3.13), we conclude that $\tilde{u}$ satisfies

$$\begin{aligned}\partial_t\tilde{u}-\Delta\tilde{u}-\partial_{zz}\tilde{u}+\alpha^{-1}\tilde{u}\cdot\nabla\tilde{u}+\alpha^{-1}\Phi(\tilde{u})\partial_z\tilde{u}+\alpha^{-1}\tilde{u}\cdot\nabla\bar{u}+\alpha^{-1}\bar{u}\cdot\nabla\tilde{u}\\ -\alpha^{-1}\overline{\tilde{u}\cdot\nabla\tilde{u}}-\alpha^{-1}\overline{\tilde{u}\nabla\cdot\tilde{u}}+f\tilde{u}^{\perp}-\alpha\beta^{-1}\int_{-1}^{z}\nabla\theta dz'+\alpha\beta^{-1}\int_{-1}^{0}\int_{-1}^{z}\nabla\theta dz'dz=0,\end{aligned} \tag{3.14}$$

$$\partial_z\tilde{u}|_{z=0}=0,\quad \partial_z\tilde{u}|_{z=-1}=0,\quad \tilde{u}\cdot\vec{n}|_{\Gamma_s}=0,\quad \partial_{\vec{n}}\tilde{u}\times\vec{n}|_{\Gamma_s}=0. \tag{3.15}$$

In the following, we will study the properties of $\bar{u}$ and $\tilde{u}$.

(1) Estimates of $|\theta|^2$ and $|u|^2$. Taking the inner product of equation (3.3) with θ in H_2, we get

$$\frac{1}{2}\partial_t|\theta|^2+|\nabla\theta|^2+|\theta_z|^2+\gamma|\theta(z=0)|^2=\beta\int_{\mathcal{O}}Q\theta-\alpha^{-1}\int_{\mathcal{O}}\Big(u\cdot\nabla\theta+\Phi(u)\partial_z\theta\Big)\theta.$$

By integration by parts,

$$\alpha^{-1}\int_{\mathcal{O}}\Big(u\cdot\nabla\theta+\Phi(u)\partial_z\theta\Big)\theta=0.$$

Using the Hölder's inequality, we deduce that

$$\frac{1}{2}\partial_t|\theta|^2+|\nabla\theta|^2+|\theta_z|^2+\gamma|\theta(z=0)|^2\le\beta\int_{\mathcal{O}}Q\theta\le\varepsilon|\theta|^2+C\beta^2|Q|^2.$$

Referring to equation (48) in [9], we obtain

$$|\theta|^2\le 2|\partial_z\theta|^2+2|\theta(z=0)|^2.$$

Then, we arrive at

$$\partial_t|\theta|^2+2|\nabla\theta|^2+(2-4\varepsilon)|\theta_z|^2+(2\gamma-4\varepsilon)|\theta(z=0)|^2\le C\beta^2|Q|^2. \tag{3.16}$$

Hence, there exists a positive λ such that

$$\partial_t|\theta|^2+\lambda|\theta|^2\le C\beta^2|Q|^2.$$

Applying the Gronwall's inequality, it follows that

$$|\theta(t)|^2\le|\theta_{t_0}|^2e^{-\lambda(t-t_0)}+C\int_{t_0}^{t}\beta^2e^{\lambda(s-t)}|Q|^2ds. \tag{3.17}$$

In view of (3.16) and (3.17), we obtain

$$\sup_{t\in[t_0,\tau^*)}|\theta(t)|^2+\int_{t_0}^{\tau^*}\|\theta(t)\|^2dt\le C. \tag{3.18}$$

Taking inner product of (3.2) with u in H_1, by integration by parts, we have

$$\partial_t|u|^2+(1-\varepsilon)|\nabla u|^2+|u_z|^2\le C|\theta|^2. \tag{3.19}$$

Utilizing (3.18), we get

$$\sup_{t\in[t_0,\tau^*)}|u(t)|^2+\int_{t_0}^{\tau^*}\|u(t)\|^2dt\le C. \tag{3.20}$$

(2) Estimates of $|\theta|_4^4$ and $|\tilde{u}|_4^4$. Taking the inner product of the equation (3.3) with θ^3 in H_2, we have

$$\frac{1}{4}\partial_t|\theta|_4^4+\frac{3}{4}|\nabla\theta^2|^2+\frac{3}{4}|(\theta^2)_z|^2+\gamma\int_M|\theta(z=0)|^4=\beta\int_{\mathcal{O}}Q\theta^3-\alpha^{-1}\int_{\mathcal{O}}[u\cdot\nabla\theta+\Phi(u)\partial_z\theta]\theta^3.$$

By integration by parts, we deduce that

$$\alpha^{-1}\int_{\mathcal{O}}[u\cdot\nabla\theta+\Phi(u)\partial_z\theta]\theta^3=0. \tag{3.21}$$

Applying the interpolation inequality to $|\theta^2|_3$, we obtain

$$|\theta^2|_3\le C|\theta^2|^{\frac{1}{2}}(|\nabla\theta^2|^{\frac{1}{2}}+|\partial_z\theta^2|^{\frac{1}{2}}+\alpha|\theta^2(z=0)|^{\frac{1}{2}}). \tag{3.22}$$

Using (3.22) and the Hölder's inequality, we get

$$\int_{\mathcal{O}}\beta Q\theta^3\le\varepsilon(|\nabla\theta^2|^2+|\partial_z\theta^2|^2+\alpha|\theta^2(z=0)|^2)+C\beta|Q|^{\frac{8}{5}}|\theta|_4^{\frac{12}{5}}. \tag{3.23}$$

Combining (3.21) and (3.23), we arrive at

$$\partial_t|\theta|_4^4+|\nabla\theta^2|^2+|(\theta^2)_z|^2+\alpha\int_M|\theta(z=0)|^4\le C\beta|Q|^{\frac{8}{5}}|\theta|_4^{\frac{12}{5}}. \tag{3.24}$$

Since

$$\theta^4(x,y,z)=-\int_z^0\partial_r\theta^4(x,y,r)dr+\theta^4(z=0),$$

by the Young's inequality, we have

$$\begin{aligned}|\theta|_4^4&=-\int_{\mathcal{O}}\int_z^0\partial_r\theta^4dr+\int_M\int_{-1}^0\theta^4(z=0)dz\\&\le\frac{1}{2}|\theta|_4^4+8|\partial_z(\theta^2)|^2+\int_M\theta^4(z=0),\end{aligned}$$

then

$$|\theta|_4^4\le 16|\partial_z(\theta^2)|^2+2|\theta(z=0)|_4^4.$$

From (3.24), we get

$$\begin{aligned}\partial_t|\theta|_4^4+|\theta|_4^4&\le C\beta|Q|^{\frac{8}{5}}|\theta|_4^{\frac{12}{5}},\\\partial_t|\theta|_4^2+|\theta|_4^2&\le C\beta|Q|^{\frac{8}{5}}|\theta|_4^{\frac{2}{5}}.\end{aligned}$$

Applying the Gronwall's inequality, there exists a positive number which is still denoted by λ such that

$$|\theta(t)|_4^2\le|\theta_{t_0}|_4^2e^{-\lambda(t-t_0)}+C\int_{t_0}^t\beta(s)e^{-\lambda(t-s)}|Q|^2ds \tag{3.25}$$

for $t\in[t_0,\tau_*)$.

Taking the inner product of the equation (3.14) with $|\tilde{u}|^2\tilde{u}$ in H_1, we obtain

$$\begin{aligned}
&\frac{1}{4}\partial_t|\tilde{u}|_4^4+\frac{1}{2}\int_{\mathcal{O}}\Big(|\nabla(|\tilde{u}|^2)|^2+|\partial_z(|\tilde{u}|^2)|^2\Big)+\int_{\mathcal{O}}|\tilde{u}|^2(|\nabla\tilde{u}|^2+|\partial_z\tilde{u}|^2)\\
=&-\alpha^{-1}\int_{\mathcal{O}}((\tilde{u}\cdot\nabla)\tilde{u}+\Phi(\tilde{u})\partial_z\tilde{u})\cdot|\tilde{u}|^2\tilde{u}\\
&-\alpha^{-1}\int_{\mathcal{O}}(\tilde{u}\cdot\nabla\bar{u})\cdot|\tilde{u}|^2\tilde{u}-\alpha^{-1}\int_{\mathcal{O}}(\bar{u}\cdot\nabla\tilde{u})\cdot|\tilde{u}|^2\tilde{u}\\
&+\alpha^{-1}\int_{\mathcal{O}}\overline{\tilde{u}\nabla\cdot\tilde{u}+\tilde{u}\cdot\nabla\tilde{u}}\cdot|\tilde{u}|^2\tilde{u}\\
&+\alpha\beta^{-1}\int_{\mathcal{O}}(\int_{-1}^{z}\nabla\theta dz'-\int_{-1}^{0}\int_{-1}^{z}\nabla\theta dz'dz)\cdot|\tilde{u}|^2\tilde{u}.
\end{aligned}$$

Using integration by parts and the boundary conditions (3.15), we have

$$\begin{aligned}
&\int_{\mathcal{O}}((\tilde{u}\cdot\nabla)\tilde{u}+\Phi(\tilde{u})\partial_z\tilde{u})|\tilde{u}|^2\tilde{u}=0,\\
&\int_{\mathcal{O}}(\bar{u}\cdot\nabla\tilde{u})|\tilde{u}|^2\tilde{u}=-\frac{1}{4}\int_{\mathcal{O}}|\tilde{u}|^4\nabla\cdot\bar{u}=0,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathcal{O}}[(\tilde{u}\cdot\nabla)\bar{u}]\cdot|\tilde{u}|^2\tilde{u}&=-\int_{\mathcal{O}}[(\tilde{u}\cdot\nabla)|\tilde{u}|^2\tilde{u}]\cdot\bar{u}-\int_{\mathcal{O}}(\nabla\cdot\tilde{u})|\tilde{u}|^2\tilde{u}\cdot\bar{u},\\
\int_{\mathcal{O}}\overline{\tilde{u}\nabla\cdot\tilde{u}+\tilde{u}\cdot\nabla\tilde{u}}\cdot|\tilde{u}|^2\tilde{u}&=-\int_{\mathcal{O}}\overline{\tilde{u}_k\tilde{u}_j}\partial_{x_k}(|\tilde{u}|^2\tilde{u}_j),
\end{aligned}$$

where $\tilde{u}_k$ is the k-th coordinate of $\tilde{u}$, $k=1,2$.

Based on the above equalities and by integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{4}\partial_t|\tilde{u}|_4^4+\frac{1}{2}\int_{\mathcal{O}}\Big(|\nabla(|\tilde{u}|^2)|^2+|\partial_z(|\tilde{u}|^2)|^2\Big)+\int_{\mathcal{O}}|\tilde{u}|^2(|\nabla\tilde{u}|^2+|\partial_z\tilde{u}|^2)\\
=&\alpha^{-1}\int_{\mathcal{O}}\bar{u}\cdot(\tilde{u}\cdot\nabla)|\tilde{u}|^2\tilde{u}+\alpha^{-1}\int_{\mathcal{O}}(\nabla\cdot\tilde{u})\bar{u}\cdot|\tilde{u}|^2\tilde{u}\\
&-\alpha^{-1}\int_{\mathcal{O}}\overline{\tilde{u}_k\tilde{u}_j}\partial_{x_k}(|\tilde{u}|^2\tilde{u}_j)-\alpha\beta^{-1}\int_{\mathcal{O}}\Big(\int_{-1}^{z}\theta d\lambda-\int_{-1}^{0}\int_{-1}^{z}\theta d\lambda dz\Big)\nabla\cdot|\tilde{u}|^2\tilde{u}\\
\stackrel{\text{def}}{=}&I_1+I_2+I_3+I_4. \qquad (3.26)
\end{aligned}$$

Applying the Hölder's inequality, the Minkowski's inequality and the interpolation inequalities, we obtain

$$\begin{aligned}
I_1&\le\alpha^{-1}\int_M|\bar{u}|\int_{-1}^{0}|\tilde{u}||\nabla\tilde{u}||\tilde{u}|^2dz\\
&\le\alpha^{-1}\int_M|\bar{u}|\Big(\int_{-1}^{0}|\tilde{u}|^2|\nabla\tilde{u}|^2dz\Big)^{\frac{1}{2}}\Big(\int_{-1}^{0}|\tilde{u}|^4dz\Big)^{\frac{1}{2}}\\
&\le\alpha^{-1}|\bar{u}|_{L^4(M)}|\nabla(|\tilde{u}|^2)|\Big(\int_M(\int_{-1}^{0}|\tilde{u}|^4dz)^2\Big)^{\frac{1}{4}}
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha^{-1}|\bar{u}|_{L^4(M)}|\nabla(|\tilde{u}|^2)|\Big(\int_{-1}^{0}(\int_M |\tilde{u}|^8)^{\frac{1}{2}}dz\Big)^{\frac{1}{2}}\\
&\le \alpha^{-1}|\bar{u}|_{L^4(M)}|\nabla(|\tilde{u}|^2)|\Big(\int_{-1}^{0}|(|\tilde{u}|^2)|_{L^2(M)}\|(|\tilde{u}|^2)\|_{H^1(M)}dz\Big)^{\frac{1}{2}}\\
&\le \alpha^{-1}|\bar{u}|_{L^4(M)}|\nabla(|\tilde{u}|^2)||(|\tilde{u}|^2)|^{\frac{1}{2}}[|(|\tilde{u}|^2)|^{\frac{1}{2}}+|\nabla(|\tilde{u}|^2)|^{\frac{1}{2}}+|\partial_z(|\tilde{u}|^2)|^{\frac{1}{2}}].
\end{aligned}
$$

By the Young's inequality and the interpolation inequalities, we get

$$
\begin{aligned}
I_1 &\le \varepsilon(|\nabla(|\tilde{u}|^2)|^2+|\partial_z(|\tilde{u}|^2)|^2)+C(\alpha^{-2}|\bar{u}|^2_{L^4(M)}+\alpha^{-4}|\bar{u}|^4_{L^4(M)})|\tilde{u}|_4^4\\
&\le \varepsilon(|\nabla(|\tilde{u}|^2)|^2+|\partial_z(|\tilde{u}|^2)|^2)+C(\alpha^{-2}\|u\|_1^2+\alpha^{-4}|u|^2\|u\|_1^2)|\tilde{u}|_4^4. \qquad (3.27)
\end{aligned}
$$

By the Hölder's inequality, the Minkowski's inequality, the interpolation inequality and the Young's inequality, we get

$$
\begin{aligned}
I_2 &= \alpha^{-1}\int_M \bar{u}\cdot\int_{-1}^{0}|\nabla\tilde{u}||\tilde{u}|^3dz\\
&\le \alpha^{-1}\int_M|\bar{u}|\Big(\int_{-1}^{0}|\nabla\tilde{u}|^2|\tilde{u}|^2dz\Big)^{\frac{1}{2}}\Big(\int_{-1}^{0}|\tilde{u}|^4dz\Big)^{\frac{1}{2}}\\
&\le \alpha^{-1}||\nabla\tilde{u}||\tilde{u}|||\bar{u}|_4\Big(\int_M\Big(\int_{-1}^{0}|\tilde{u}|^4dz\Big)^2\Big)^{\frac{1}{4}}\\
&\le \alpha^{-1}||\nabla\tilde{u}||\tilde{u}|||\bar{u}|_4\Big(\int_{-1}^{0}\Big(\int_M|\tilde{u}|^{2\times4}\Big)^{\frac{1}{2}}dz\Big)^{\frac{1}{2}}\\
&\le C\alpha^{-1}||\nabla\tilde{u}||\tilde{u}|||\bar{u}|_4\Big(\int_{-1}^{0}||\tilde{u}|^2|_{L^2(M)}(|\nabla|\tilde{u}|^2|_{L^2(M)}+|\partial_z|\tilde{u}|^2|_{L^2(M)}+||\tilde{u}|^2|_{L^2(M)})dz\Big)^{\frac{1}{2}}\\
&\le \varepsilon(||\nabla\tilde{u}||\tilde{u}||^2+|\nabla|\tilde{u}|^2|^2+|\partial_z|\tilde{u}|^2|^2)+C(\alpha^{-2}|\bar{u}|_4^2+\alpha^{-4}|\bar{u}|_4^4)||\tilde{u}|^2|^2\\
&\le \varepsilon(||\nabla\tilde{u}||\tilde{u}||^2+|\nabla|\tilde{u}|^2|^2+|\partial_z|\tilde{u}|^2|^2)+C(\alpha^{-2}\|u\|_1^2+\alpha^{-4}|u|^2\|u\|_1^2)|\tilde{u}|_4^4.
\end{aligned}
$$

Applying the Hölder's inequality, the Minkowski's inequality, the interpolation inequality and the Young's inequality, we have

$$
\begin{aligned}
I_3 &= -\alpha^{-1}\int_{\mathcal{O}}\overline{\tilde{u}_k\tilde{u}_j}\partial_{x_k}(|\tilde{u}|^2\tilde{u}_j)\\
&\le \alpha^{-1}\int_M\Big(\int_{-1}^{0}|\tilde{u}|^2dz\Big)\Big(\int_{-1}^{0}|\nabla\tilde{u}||\tilde{u}|^2dz\Big)\\
&\le \alpha^{-1}\Big(\int_{\mathcal{O}}|\nabla\tilde{u}|^2|\tilde{u}|^2\Big)^{\frac{1}{2}}\Big(\int_M\Big(\int_{-1}^{0}|\tilde{u}|^2dz\Big)^3\Big)^{\frac{1}{2}}\\
&\le \alpha^{-1}\Big(\int_{\mathcal{O}}|\nabla\tilde{u}|^2|\tilde{u}|^2\Big)^{\frac{1}{2}}\Big(\int_{-1}^{0}\Big(\int_M|\tilde{u}|^6\Big)^{\frac{1}{3}}dz\Big)^{\frac{3}{2}}\\
&\le C\alpha^{-1}\Big(\int_{\mathcal{O}}|\nabla\tilde{u}|^2|\tilde{u}|^2\Big)^{\frac{1}{2}}\Big(\int_{-1}^{0}|\tilde{u}|^{\frac{4}{3}}_{L^4(M)}\cdot\|\tilde{u}\|^{\frac{2}{3}}_{H^1(M)}dz\Big)^{\frac{3}{2}}\\
&\le C\alpha^{-1}\Big(\int_{\mathcal{O}}|\nabla\tilde{u}|^2|\tilde{u}|^2\Big)^{\frac{1}{2}}\Big(\int_{-1}^{0}|\tilde{u}|^4_{L^4(M)}dz\Big)^{\frac{1}{2}}\int_{-1}^{0}\|\tilde{u}\|_{H^1(M)}dz\\
&\le \varepsilon\int_{\mathcal{O}}|\nabla\tilde{u}|^2|\tilde{u}|^2+C\alpha^{-2}\|u\|_1^2|\tilde{u}|_4^4.
\end{aligned}
$$

Analogously, we have

$$I_4 \leq \alpha\beta^{-1}\Big(\int_{\mathcal{O}}|\nabla\tilde{u}|^2|\tilde{u}|^2\Big)^{\frac{1}{2}}\Big(\int_{\mathcal{O}}|\tilde{u}|^4\Big)^{\frac{1}{4}}\Big(\int_{\mathcal{O}}|\theta|^4\Big)^{\frac{1}{4}}$$
$$\leq \varepsilon\int_{\mathcal{O}}|\nabla\tilde{u}|^2|\tilde{u}|^2 + C\alpha^2\beta^{-2}|\tilde{u}|_4^2|\theta|_4^2.$$

Collecting all the above inequalities, we have

$$\partial_t|\tilde{u}|_4^4 + \int_{\mathcal{O}}\Big(|\nabla(|\tilde{u}|^2)|^2 + |\partial_z(|\tilde{u}|^2)|^2\Big) + \int_{\mathcal{O}}|\tilde{u}|^2(|\nabla\tilde{u}|^2 + |\partial_z\tilde{u}|^2)$$
$$\leq C\alpha^2\beta^{-2}|\theta|_4^2|\tilde{u}|_4^2 + C(\alpha^{-2}\|u\|_1^2 + \alpha^{-4}|u|^2\|u\|_1^2)|\tilde{u}|_4^4, \tag{3.28}$$

and

$$\partial_t|\tilde{u}|_4^2 \leq C\alpha^2\beta^{-2}|\theta|_4^2 + C(\alpha^{-2}\|u\|_1^2 + \alpha^{-4}|u|^2\|u\|_1^2)|\tilde{u}|_4^2. \tag{3.29}$$

Applying the Gronwall's inequality, we conclude that

$$\sup_{t\in[t_0,\tau*)}|\tilde{u}(t)|_4^4 + \int_{t_0}^{\tau*}\int_{\mathcal{O}}\Big(|\nabla(|\tilde{u}|^2)|^2 + |\partial_z(|\tilde{u}|^2)|^2\Big)ds + \int_{t_0}^{\tau*}\int_{\mathcal{O}}|\tilde{u}|^2(|\nabla\tilde{u}|^2 + |\partial_z\tilde{u}|^2)ds \leq C. \tag{3.30}$$

(3) Estimates of $|\nabla\bar{u}|^2$ and $|u_z|^2$. Taking the inner product of equation (3.11) with $-\Delta\bar{u}$ in $L^2(M)$, we arrive at

$$\frac{1}{2}\partial_t|\nabla\bar{u}|^2 + |\Delta\bar{u}|^2 = -\alpha^{-1}\int_M \bar{u}\cdot\nabla\bar{u}\cdot\Delta\bar{u} - \alpha^{-1}\int_M \overline{\tilde{u}\cdot\nabla\tilde{u}}\cdot\Delta\bar{u}$$
$$-\alpha^{-1}\int_M \overline{\tilde{u}\nabla\cdot\tilde{u}}\cdot\Delta\bar{u} - \int_M(\bar{u}^{\perp} + \alpha\nabla p_s)\cdot\Delta\bar{u}$$
$$+\alpha\beta^{-1}\int_M\nabla\int_{-1}^0\int_{-1}^z\theta(x,y,\lambda,t)d\lambda dz\cdot\Delta\bar{u}.$$

By integration by parts and (3.12)-(3.13)(for more detail, see [9]), we have

$$\int_M \bar{u}^{\perp}\cdot\Delta\bar{u} = 0, \quad \int_M \nabla p_s\cdot\Delta\bar{u} = 0,$$
$$\int_M\nabla\int_{-1}^0\int_{-1}^z\theta(x,y,\lambda,t)d\lambda dz\cdot\Delta\bar{u} = 0.$$

Applying the Hölder's inequality and the interpolation inequalities, we obtain

$$\alpha^{-1}\int_M \bar{u}\cdot\nabla\bar{u}\cdot\Delta\bar{u} \leq C\alpha^{-1}|\bar{u}|^{\frac{1}{2}}|\nabla\bar{u}||\Delta\bar{u}|^{\frac{3}{2}}$$
$$\leq \varepsilon|\Delta\bar{u}|^2 + C\alpha^{-4}|\bar{u}|^2|\nabla\bar{u}|^4.$$

Using the Hölder's inequality, the Minkowski's inequality and the Sobolev embedding theorem, we have

$$\alpha^{-1}\int_M \overline{\tilde{u}\nabla\cdot\tilde{u}}\cdot\Delta\bar{u} + \alpha^{-1}\int_M \overline{\tilde{u}\cdot\nabla\tilde{u}}\cdot\Delta\bar{u} \leq \alpha^{-1}|\Delta\bar{u}|\Big(\int_{\mathcal{O}}|\tilde{u}|^2|\nabla\tilde{u}|^2\Big)^{\frac{1}{2}}$$

$$\leq \varepsilon|\Delta\bar{u}|^2+C\alpha^{-2}\int_{\mathcal{O}}|\tilde{u}|^2|\nabla\tilde{u}|^2.$$

Collecting all the above inequalities, we conclude that

$$\partial_t|\nabla\bar{u}|^2+|\Delta\bar{u}|^2\leq C\alpha^{-4}|u|^2\|u\|_1^2|\nabla\bar{u}|^2+C\alpha^{-2}\int_{\mathcal{O}}|\tilde{u}|^2|\nabla\tilde{u}|^2. \tag{3.31}$$

Therefore, applying Gronwall's inequality and (3.20)-(3.31), we arrive at

$$\sup_{t\in[t_0,\tau*)}|\nabla\bar{u}(t)|^2\leq C. \tag{3.32}$$

Denote $u_z=\frac{\partial u}{\partial z}$, from (3.2), we get

$$\begin{aligned}&\partial_t u_z-\Delta u_z-\partial_{zz}u_z+\alpha^{-1}u\cdot\nabla u_z+\alpha^{-1}u_z\cdot\nabla u-\alpha^{-1}u_z\nabla\cdot u+\alpha^{-1}\Phi(u)u_{zz}\\&\quad+fk\times u_z-\alpha\beta^{-1}\nabla\theta=0.\end{aligned} \tag{3.33}$$

Taking the inner product of the equation (3.33) with u_z in H_1, we obtain

$$\begin{aligned}\frac{1}{2}\partial_t|u_z|^2+|\nabla u_z|^2+|u_{zz}|^2=&-\alpha^{-1}\int_{\mathcal{O}}\Big((u\cdot\nabla)u_z+\Phi(u)u_{zz}\Big)\cdot u_z-\alpha^{-1}\int_{\mathcal{O}}[(u_z\cdot\nabla)u]\cdot u_z\\&+\alpha^{-1}\int_{\mathcal{O}}(\nabla\cdot u)u_z\cdot u_z-\int_{\mathcal{O}}u_z^{\perp}\cdot u_z+\alpha\beta^{-1}\int_{\mathcal{O}}\nabla\theta\cdot u_z.\end{aligned}$$

By integration by parts, we deduce that

$$\alpha^{-1}\int_{\mathcal{O}}\Big((u\cdot\nabla)u_z+\Phi(u)u_{zz}\Big)\cdot u_z=0.$$

Thanks to the Hölder's inequality, the interpolation inequality and the Sobolev embedding theorem, we reach

$$\begin{aligned}\alpha^{-1}\int_{\mathcal{O}}[(u_z\cdot\nabla)u]\cdot u_z&\leq C\alpha^{-1}\int_{\mathcal{O}}|u||u_z||\nabla u_z|\\&\leq C\alpha^{-1}|\nabla u_z||u|_4|u_z|_4\\&\leq C\alpha^{-1}|\nabla u_z||u|_4|u_z|^{\frac14}(|\nabla u_z|^{\frac34}+|\partial_z u_z|^{\frac34}+|u_z|^{\frac34})\\&\leq\varepsilon(|\nabla u_z|^2+|\partial_z u_z|^2)+C(\alpha^{-8}|\nabla\bar{u}|^8+1)|u_z|^2.\end{aligned}$$

Similar to the above, we get

$$\int_{\mathcal{O}}(\nabla\cdot u)u_z\cdot u_z\leq\varepsilon(|\nabla u_z|^2+|u_{zz}|^2)+C(\alpha^{-8}|\nabla\bar{u}|^8+1)|u_z|^2.$$

Collecting the above inequalities, we have

$$\partial_t|u_z|^2+|\nabla u_z|^2+|u_{zz}|^2\leq C(\alpha^{-8}|\bar{u}|_4^8+\alpha^{-8}|\nabla\bar{u}|^8+1)|u_z|^2. \tag{3.34}$$

Applying Gronwall's inequality to (3.34), and by (3.32), we reach

$$\sup_{t\in[t_0,\tau*)}|u_z(t)|^2+\int_{t_0}^{\tau*}(|\nabla u_z(s)|^2+|u_{zz}(s)|^2)ds\leq C. \tag{3.35}$$

(4) Estimates of $|\nabla u|^2$ and $|\nabla\theta|^2$. Taking the inner product of equation (3.2) with $-\Delta u$ in H_1, we reach

$$\begin{aligned}&\frac{1}{2}\partial_t|\nabla u|^2+|\Delta u|^2+|\nabla\partial_z u|^2\\ =&\alpha^{-1}\int_{\mathcal{O}}[(u\cdot\nabla)u+\Phi(u)\partial_z u]\cdot\Delta u+\int_{\mathcal{O}} fk\times u\cdot\Delta u\\ &+\alpha\int_{\mathcal{O}}\nabla p_s\cdot\Delta u-\alpha\beta^{-1}\int_{\mathcal{O}}(\int_{-1}^{z}\nabla\theta dz')\cdot\Delta u.\end{aligned}$$

By the Hölder's inequality, the interpolation inequality and the Sobolev's inequality, we have

$$\begin{aligned}&\alpha^{-1}\int_{\mathcal{O}}[(u\cdot\nabla)u]\cdot\Delta u\\ \le& C\alpha^{-1}|\Delta u||\nabla u|_4|u|_4\\ \le&\alpha^{-1}|\Delta u||\nabla u|^{\frac14}(|\Delta u|^{\frac34}+|\nabla u_z|_4^{\frac34}+|\nabla u|^{\frac34})|u|_4\\ \le&\varepsilon(|\Delta u|^2+|\nabla u_z|^2)+C(\alpha^{-2}|u|_4^2+\alpha^{-8}|u|_4^8)|\nabla u|^2\\ \le&\varepsilon(|\Delta u|^2+|\nabla u_z|^2)+C(\alpha^{-2}|\tilde u|_4^2+\alpha^{-2}|\nabla\bar u|^2+\alpha^{-8}|\tilde u|_4^8+\alpha^{-8}|\nabla\bar u|^8)|\nabla u|^2.\end{aligned}\tag{3.36}$$

Utilizing the Hölder's inequality, the Minkowsky's inequality, the interpolation inequality and the Sobolev embedding theorem, we get

$$\begin{aligned}\alpha^{-1}\int_{\mathcal{O}}\Phi(u)u_z\cdot\Delta u\le&\alpha^{-1}\int_M\Big(\int_{-1}^0|\nabla\cdot u|dz\int_{-1}^0|u_z|\cdot|\Delta u|dz\Big)\\ \le&\alpha^{-1}|\Delta u|\Big(\int_M(\int_{-1}^0|\nabla\cdot u|dz)^4\Big)^{\frac14}\Big(\int_M(\int_{-1}^0|u_z|^2dz)^2\Big)^{\frac14}\\ \le&C\alpha^{-1}|\Delta u|\Big(\int_{-1}^0|\nabla u|^{\frac12}_{L^2(M)}(|\Delta u|^{\frac12}_{L^2(M)}+|\nabla u|^{\frac12}_{L^2(M)})dz\Big)\\ &\times\Big(\int_{-1}^0|u_z|_{L^2(M)}(|\nabla u_z|_{L^2(M)}+|u_z|_{L^2(M)})dz\Big)^{\frac12}\\ \le&\varepsilon(|\Delta u|^2+|\nabla u_z|^2)+C|\nabla u|^2(\alpha^{-2}|u_z|^2+\alpha^{-2}|u_z||\nabla u_z|\\ &+\alpha^{-4}|u_z|^4+\alpha^{-4}|u_z|^2|\nabla u_z|^2).\end{aligned}\tag{3.37}$$

We also have

$$\int_{\mathcal{O}}(fk\times u)\cdot\Delta u=0,\quad\int_{\mathcal{O}}\nabla p_s\cdot\Delta u=0.$$

Collecting all the above inequalities, we get

$$\begin{aligned}&\partial_t|\nabla u|^2+|\Delta u|^2+|\nabla\partial_z u|^2\\ \le&C(\alpha^2\beta^{-2}\|\theta\|_1^2+\alpha^{-2}|\tilde u|_4^2+\alpha^{-8}|\tilde u|_4^8+\alpha^{-2}|\nabla\bar u|^2+\alpha^{-8}|\nabla\bar u|^8\\ &+\alpha^{-2}|u_z|^2+\alpha^{-4}|u_z|^4+\alpha^{-2}|\nabla u_z|^2+\alpha^{-4}|u_z|^2|\nabla u_z|^2)|\nabla u|^2.\end{aligned}\tag{3.38}$$

Applying the Gronwall's inequality, and by (3.18), (3.30), (3.32) and (3.35), we obtain

$$\sup_{t\in[t_0,\tau*)}|\nabla u(t)|^2+\int_{t_0}^{\tau*}(|\Delta u(t)|^2+|\nabla\partial_z u|^2)dt\le C.\tag{3.39}$$

Taking the inner product of the equation (3.3) with $-\Delta\theta-\theta_{zz}$ in H_2, similar to the above, we get

$$\begin{aligned}
&\frac{1}{2}\partial_t(|\nabla\theta|^2+|\theta_z|^2+\gamma|\nabla\theta(z=0)|^2)+|\Delta\theta|^2+2(|\nabla\theta_z|^2+\gamma|\nabla\theta(z=0)|^2)+|\theta_{zz}|^2\\
&=\int_{\mathcal{O}}(\alpha^{-1}u\cdot\nabla\theta+\alpha^{-1}\Phi(u)\theta_z-\beta Q)(\Delta\theta+\theta_{zz})\\
&\leq\varepsilon(|\Delta\theta|^2+|\theta_{zz}|^2+|\nabla\theta_z|^2)+C\beta^2|Q|^2+C\alpha^{-4}|\nabla u|^2|\Delta u|^2|\theta_z|^2\\
&\quad+C(\alpha^{-2}|\tilde{u}|_4^2+\alpha^{-2}|\nabla\bar{u}|^2+\alpha^{-8}|\tilde{u}|_4^8+\alpha^{-8}|\nabla\bar{u}|^8)|\nabla\theta|^2.
\end{aligned}$$

Utilizing the Gronwall's inequality, we get

$$\begin{aligned}
&|\nabla\theta|^2+|\theta_z|^2+\gamma|\nabla\theta(z=0)|^2\\
&\quad+\int_{t_0}^t[|\Delta\theta|^2+2(|\nabla\theta_z|^2+\gamma|\nabla\theta(z=0)|^2)+|\theta_{zz}|^2]ds\leq C.
\end{aligned}\tag{3.40}$$

In the following, we will divide the proof of the global well-posedness of stochastic PEs into three steps. Concretely, firstly we prove the global existence of strong solution. Then, we show that the solution is continuous in the space V with respect to t. At last, we prove the continuity in V with respect to the initial data.

Step 1: The global existence of strong solution.

Previously, we have obtained a priori estimates in V. As we have indicated before, that $[t_0,\tau_*)$ is the maximal interval of existence of the solution of (3.2)-(3.7), we infer that $\tau_*=\infty$, a.s.. Otherwise, if there exists $A\in\mathcal{F}$ such that $\mathbb{P}(A)>0$ and for fixed $\omega\in A,\tau_*(\omega)<\infty$, it is clear that

$$\limsup_{t\to\tau_*^-(\omega)}(\|u(t)\|_1+\|\theta(t)\|_1)=\infty,\quad\text{for any }\omega\in A,$$

which contradicts a priori estimates (3.35), (3.39) and (3.40). Therefore $\tau_*=\infty$, a.s., and the strong solution (u,θ) exists globally in time a.s..

Step 2: The continuity of strong solutions with respect to t.

Multiplying (3.2) by $\eta\in V_1$, integrating with respect to the space variable, yields

$$\begin{aligned}
\langle\partial_t A_1^{\frac{1}{2}}u,\eta\rangle=\langle\partial_t u,A_1^{\frac{1}{2}}\eta\rangle=&-\langle A_1u,A_1^{\frac{1}{2}}\eta\rangle-\alpha^{-1}\langle(u\cdot\nabla)u,A_1^{\frac{1}{2}}\eta\rangle\\
&-\alpha^{-1}\langle\Phi(u)\partial_z u,A_1^{\frac{1}{2}}\eta\rangle-\langle fu^{\perp},A_1^{\frac{1}{2}}\eta\rangle\\
&+\alpha\beta^{-1}\langle\int_{-1}^z\nabla\theta dz',A_1^{\frac{1}{2}}\eta\rangle,
\end{aligned}$$

where $\langle\nabla p_s,A_1^{\frac{1}{2}}\eta\rangle=0$ is used. Taking a similar argument in (3.37), we get

$$\langle\Phi(u)\partial_z u,A_1^{\frac{1}{2}}\eta\rangle\leq\|u\|\|u\|_2|A_1^{\frac{1}{2}}\eta|.$$

By the Hölder's inequality and the Sobolev embedding theorem, we have

$$\|\partial_t(A_1^{\frac{1}{2}}u)\|_{V_1'}\leq C(\|u\|_2+\|u\|\|u\|_2+|u|+|\nabla\theta|).$$

Since

$$u\in L^\infty([t_0,\mathcal{T}];V_1)\cap L^2([t_0,\mathcal{T}];(H^2(\mathcal{O}))^2),\quad\forall\mathcal{T}>t_0,$$

we obtain

$$A_1^{\frac{1}{2}}u\in L^2([t_0,\mathcal{T}];V_1),\quad \partial_t(A_1^{\frac{1}{2}}u)\in L^2([t_0,\mathcal{T}];V_1').$$

Referring to Lemma 3.1, we deduce that

$$A_1^{\frac{1}{2}}u\in C([t_0,\mathcal{T}];H_1) \text{ or } u\in C([t_0,\mathcal{T}];V_1)\mathbb{P}-a.s..$$

To study the regularity of θ, we choose $\xi\in V_2$. By (3.3), we have

$$\begin{aligned}\langle\partial_t A_2^{\frac{1}{2}}\theta,\xi\rangle=&\langle\partial_t\theta,A_2^{\frac{1}{2}}\xi\rangle=\langle A_2\theta,A_2^{\frac{1}{2}}\xi\rangle-\alpha^{-1}\langle u\cdot\nabla\theta,A_2^{\frac{1}{2}}\xi\rangle\\&+\alpha^{-1}\langle\Phi(u)\partial_z\theta,A_2^{\frac{1}{2}}\xi\rangle+\beta\langle Q,A_2^{\frac{1}{2}}\xi\rangle.\end{aligned}$$

Taking a similar argument as above, we get

$$\langle\Phi(u)\partial_z\theta,A_2^{\frac{1}{2}}\xi\rangle\le C\|u\|^{\frac{1}{2}}\|u\|_2^{\frac{1}{2}}\|\theta\|^{\frac{1}{2}}\|\theta\|_2^{\frac{1}{2}}|A_2^{\frac{1}{2}}\xi|.$$

Then by the Hölder inequality's and the Sobolev embedding theorem, we have

$$|\partial_t A_2^{\frac{1}{2}}\theta|_{V_2'}\le C(|A_2\theta|+\alpha^{-1}\|u\|\|\theta\|_2+\alpha^{-1}\|u\|^{\frac{1}{2}}\|u\|_2^{\frac{1}{2}}\|\theta\|^{\frac{1}{2}}\|\theta\|_2^{\frac{1}{2}}+\beta|Q|).$$

In view of step one, we have

$$\theta\in L^\infty([t_0,\mathcal{T}];V_2)\cap L^2([t_0,\mathcal{T}];H^2(\mathcal{O})),\quad \forall\mathcal{T}>t_0.$$

Therefore, by the same argument as above, we get

$$A_2^{\frac{1}{2}}\theta\in L^2([t_0,\mathcal{T}];V_2),\quad \partial_t(A_2^{\frac{1}{2}}\theta)\in L^2([t_0,\mathcal{T}];V_2').$$

We deduce from Lemma 3.1 that

$$A_2^{\frac{1}{2}}\theta\in C([t_0,\mathcal{T}];H_2) \text{ or } \theta\in C([t_0,\mathcal{T}];V_2),\quad \mathbb{P}-a.s..$$

Step 3: The continuity in V with respect to the initial data.
Let (v_1,T_1) and (v_2,T_2) be two solutions of the system (3.2)-(3.7) with corresponding pressures ${p_b}'$ and ${p_b}''$, and initial data $((v_0)_1,(T_0)_1)$ and $((v_0)_2,(T_0)_2)$, respectively. Denote by $v=v_1-v_2,p_b={p_b}'-{p_b}''$ and $T=T_1-T_2$. Then we have

$$\begin{aligned}&\partial_t v-\Delta v-\partial_{zz}v+\alpha^{-1}v_1\cdot\nabla v+\alpha^{-1}(v\cdot\nabla)v_2+\alpha^{-1}\Phi(v_1)v_z+\alpha^{-1}\Phi(v)\partial_z v_2\\&\quad+fk\times v+\alpha\nabla p_b-\alpha\beta^{-1}\int_{-1}^{z}\nabla T dz'=0,\end{aligned}\tag{3.41}$$

$$\partial_t T-\Delta T-\partial_{zz}T+\alpha^{-1}v_1\cdot\nabla T+\alpha^{-1}(v\cdot\nabla)T_2+\alpha^{-1}\Phi(v_1)T_z+\alpha^{-1}\Phi(v)\partial_z T_2=0,\tag{3.42}$$

$$\int_{-1}^{0}\nabla\cdot v dz=0,\tag{3.43}$$

$$v(x,y,z,t_0)=(v_0)_1-(v_0)_2,\qquad T(x,y,z,t_0)=(T_0)_1-(T_0)_2,\tag{3.44}$$

$$(v,T)\text{ satisfies the boundary conditions }(3.5)-(3.7).\tag{3.45}$$

Multiplying $L_1 v$ in equation (3.41) and integrating with respect to the spatial variable, we have

$$\frac{1}{2}\partial_t(|\nabla v|^2+|\partial_z v|^2)+|\Delta v|^2+|\partial_z v|^2+|\nabla v_z|^2$$

$$
\begin{aligned}
&= -\alpha^{-1}\int_{\mathcal{O}}(v_1\cdot\nabla v)\cdot L_1 v - \alpha^{-1}\int_{\mathcal{O}}\Phi(v_1)v_z\cdot L_1 v\\
&\quad -\alpha^{-1}\int_{\mathcal{O}}(\Phi(v)\partial_z v_2)\cdot L_1 v - \alpha^{-1}\int_{\mathcal{O}}[(v\cdot\nabla)v_2]\cdot L_1 v\\
&\quad -\int_{\mathcal{O}}(fk\times v)\cdot L_1 v + \alpha\beta^{-1}\int_{\mathcal{O}}(\int_{-1}^{z}\nabla T dz')\cdot L_1 v\\
&\stackrel{\text{def}}{=} K_1(t)+K_2(t)+K_3(t)+K_4(t)+K_5(t)+K_6(t). \qquad (3.46)
\end{aligned}
$$

Using the Agmon's inequality and the Hölder's inequality, we obtain

$$
\begin{aligned}
K_1(t) &\le \alpha^{-1}|v_1|_\infty|\nabla v||L_1 v|\\
&\le C\alpha^{-2}\|v_1\|^{\frac{1}{2}}\|v_1\|_2^{\frac{1}{2}}\|v\|\|v\|_2\\
&\le C\alpha^{-2}\|v_1\|\|v_1\|_2\|v\|^2+\varepsilon\|v\|_2^2.
\end{aligned}
$$

Applying the Hölder's inequality, the interpolation inequality and the Sobolev embedding theorem, we get

$$
\begin{aligned}
K_2(t) &\le \alpha^{-1}\int_{\mathcal{O}}|\int_{-1}^{0}\nabla\cdot v_1 dz|\cdot|v_z|\cdot|L_1 v|\\
&\le \alpha^{-1}|\nabla\cdot\bar{v}_1|_{L^4(M)}\int_{-1}^{0}|v_z|_{L^4(M)}|L_1 v|_{L^2(M)}dz\\
&\le C\alpha^{-1}\|v_1\|^{\frac{1}{2}}\|v_1\|_2^{\frac{1}{2}}\|v\|^{\frac{1}{2}}\|v\|_2^{\frac{1}{2}}\|v\|_2\\
&\le \varepsilon\|v\|_2^2+C\alpha^{-2}\|v_1\|^2\|v_1\|_2^2\|v\|^2.
\end{aligned}
$$

Similar to the above, we obtain

$$
\begin{aligned}
K_3(t) &\le \alpha^{-1}\int_{\mathcal{O}}|\int_{-1}^{0}\nabla\cdot v dz|\cdot|\partial_z v_2|\cdot|L_1 v|\\
&\le \alpha^{-1}|\nabla\cdot\bar{v}|_{L^4(M)}\int_{-1}^{0}|\partial_z v_2|_{L^4(M)}|L_1 v|_{L^2(M)}dz\\
&\le C\alpha^{-1}\|v\|^{\frac{1}{2}}\|v\|_2^{\frac{1}{2}}\int_{-1}^{0}\|v_2\|_{H^1(M)}^{\frac{1}{2}}\|v_2\|_{H^2(M)}^{\frac{1}{2}}\|v\|_2 dz\\
&\le \varepsilon\|v\|_2^2+C\alpha^{-4}\|v\|^2\|v_2\|^2\|v_2\|_2^2.
\end{aligned}
$$

With the help of the Hölder's inequality, we deduce that

$$
K_4(t)\le\alpha^{-1}|v|_4|\nabla v_2|_4|L_1 v|_2\le\varepsilon\|v\|_2^2+C\alpha^{-2}\|v\|^2\|v_2\|_2^2,
$$

and

$$
K_5(t)+K_6(t)\le\varepsilon\|v\|_2^2+C|v|_2^2+C\alpha^2\beta^{-2}\|T\|^2.
$$

From the boundary conditions, we know that $|T|^2$ is smaller than $|T_z|^2+|T(z=0)|_{L^2(M)}^2$, then $\|T\|^2$ is equivalent to $|\nabla T|^2+|T_z|^2+|T(z=0)|_{L^2(M)}^2$. Keeping this in mind and taking an inner product of the equation (3.42) with $L_2 T$, we have

$$
\frac{1}{2}\partial_t(|\nabla T|^2+|T_z|^2+\gamma|T(z=0)|^2)+|\Delta T|^2+|T_{zz}|^2+|\nabla T_z|^2+\gamma|\nabla T(z=0)|^2
$$

$$
\begin{aligned}
&= -\alpha^{-1}\int_{\mathcal{O}}(v_1\cdot\nabla T)L_2T - \alpha^{-1}\int_{\mathcal{O}}(v\cdot\nabla T_2)L_2T \\
&\quad -\alpha^{-1}\int_{\mathcal{O}}\Phi(v_1)T_zL_2T + \alpha^{-1}\Phi(v)\partial_zT_2L_2T \\
&\stackrel{\text{def}}{=} J_1(t)+J_2(t)+J_3(t)+J_4(t).
\end{aligned}
$$

By the Agmon's inequality, we get that

$$
\begin{aligned}
J_1(t)+J_2(t) &\le \alpha^{-1}|v_1|_\infty|\nabla T||L_2T| + \alpha^{-1}|v|_4|\nabla T_2|_4|L_2T| \\
&\le C\alpha^{-1}\|v_1\|^{\frac12}\|v_1\|_2^{\frac12}\|T\|\|T\|_2 + C\alpha^{-1}\|v\|\|T_2\|_2\|T\|_2 \\
&\le \varepsilon\|T\|_2^2 + C\alpha^{-2}\|v\|^2\|T_2\|_2^2 + C\alpha^{-2}\|v_1\|\|v_1\|_2\|T\|^2.
\end{aligned}
$$

Taking a similar argument as $K_2(t)$, we obtain

$$
\begin{aligned}
J_3(t) &\le \alpha^{-1}\int_{\mathcal{O}}\Big(\int_{-1}^0|\nabla\cdot v_1|dz\Big)\cdot|T_z|\cdot|L_2T| \\
&\le \alpha^{-1}\Big(\int_{-1}^0|L_2T|_{L^2(M)}|T_z|_{L^4(M)}dz\Big)\Big|\int_{-1}^0|\nabla\cdot v_1|dz\Big|_{L^4(M)} \\
&\le C\alpha^{-1}\int_{-1}^0\|T\|_{H^2(M)}^{\frac32}\|T\|_{H^1(M)}^{\frac12}dz\int_{-1}^0|\nabla\cdot v_1|_{L^4(M)}dz \\
&\le C\alpha^{-1}\|T\|_2^{\frac32}\|T\|^{\frac12}\int_{-1}^0\|v_1\|_{H^1(M)}^{\frac12}\|v_1\|_{H^2(M)}^{\frac12}dz \\
&\le C\alpha^{-1}\|T\|_2^{\frac32}\|T\|^{\frac12}\|v_1\|^{\frac12}\|v_1\|_2^{\frac12} \\
&\le \varepsilon\|T\|_2^2 + C\alpha^{-4}\|T\|^2\|v_1\|^2\|v_1\|_2^2.
\end{aligned}
$$

Similarly, we can prove that

$$
\begin{aligned}
J_4(t) &\le \alpha^{-1}\int_{\mathcal{O}}(\int_{-1}^0|\nabla\cdot v|)|\partial_zT_2|\cdot|L_2T| \\
&\le \alpha^{-1}\Big(\int_{-1}^0|L_2T|_{L^2(M)}|\partial_zT_2|_{L^4(M)}dz\Big)\Big|\int_{-1}^0|\nabla\cdot v|dz\Big|_{L^4(M)} \\
&\le \alpha^{-1}\Big(\int_{-1}^0\|T\|_{H^2(M)}\|T_2\|_{H^1(M)}^{\frac12}\|T_2\|_{H^2(M)}^{\frac12}dz\Big)\int_{-1}^0|\nabla\cdot v|_{L^4(M)}dz \\
&\le C\alpha^{-1}\|T\|_2\|T_2\|^{\frac12}\|T_2\|_2^{\frac12}\|v\|^{\frac12}\|v\|_2^{\frac12} \\
&\le \varepsilon\|T\|_2^2 + \varepsilon\|v\|_2^2 + C\alpha^{-4}\|T_2\|^2\|T_2\|_2^2\|v\|^2.
\end{aligned}
$$

Denote

$$
\begin{aligned}
\eta(t) &\stackrel{\text{def}}{=} \|v(t)\|^2 + \|T(t)\|^2, \\
\xi(t) &\stackrel{\text{def}}{=} \alpha^{-2}\|v_1\|\|v_1\|_2 + \alpha^{-4}\|v_1\|^2\|v_1\|_2^2 \\
&\qquad + \alpha^{-4}\|v_2\|^2\|v_2\|_2^2 + \alpha^2\|v_2\|_2^2 + \alpha^2\|T_2\|_2^2 + \alpha^{-4}\|T_2\|^2\|T_2\|_2^2 + 1.
\end{aligned}
$$

Notice that $|\nabla v|^2+|\partial_z v|^2$ is equivalent to $\|v\|^2$ and $|\nabla T|^2+|T_z|^2+|T(z=0)|_{L^2(M)}^2$ is equivalent to $\|T\|^2$; letting ε be small enough, we deduce from the above estimates of K_1-K_6 and J_1-J_4 that

$$
\frac{d\eta(t)}{dt} + \|v\|_2^2 + \|T\|_2^2 \le \eta(t)\xi(t). \tag{3.47}
$$

Since $(v_i(t),T_i(t)),i=1,2$, is the solution of stochastic PEs in the sense of Definition 3.2, we have

$$\int_{t_0}^{t}\xi(s)ds<\infty,\quad a.s., \text{ for all } t\in(t_0,\infty),$$

which implies that

$$\eta(t)\le\eta(t_0)e^{C\int_{t_0}^{t}\xi(s)ds}.$$

Therefore, we proved that for any $t\in(t_0,\infty),(u(t),\theta(t))$ is Lipschitz continuous in $V_1\times V_2$ with respect to the initial data $(u(t_0),\theta(t_0))$, which is equivalent to stating that the strong solution $(v(t),T(t))$ of (1.9)-(1.14) is Lipschitz continuous in $V_1\times V_2$ with respect to the initial data (v_{t_0},T_{t_0}), for any $t\in(t_0,\infty)$. ∎

REMARK 3.1.

(1) In the above theorem, we have obtained the continuity of the strong solution with respect to the initial data in $(H^1(\mathcal{O}))^3$. This is the key to prove the compact property of the solution operator in V. Notice that the authors only proved that the strong solution is Lipschitz continuous in the space $(L^2(\mathcal{O}))^3$ with respect to the initial data in [17], which is not enough to obtain the asymptotical behavior in $(H^1(\mathcal{O}))^3$.

(2) With the help of Lemma 3.1, we have established the continuity of the strong solution with respect to time in V and a priori estimates to prove the compact property of the solution operator in V.

(3) We release the regularity of Q from $H^1(\mathcal{O})$ to $L^2(\mathcal{O})$, which is more natural.

4. Existence of random attractor

In this section, we establish the existence of random attractor. Firstly, we recall some preliminaries from [3]. Denote by $C_0(\mathbb{R};X)$ the space of continuous functions with values in X and equal to 0 at $t=0$. Let (X,d) be a Polish space and $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ be a probability space, where $\tilde{\Omega}$ is the two-sided Wiener space $C_0(\mathbb{R};X)$.

DEFINITION 4.1. *A family of maps $S(t,s;\omega):X\to X$, $-\infty<s\le t<\infty$, parametrized by $\omega\in\tilde{\Omega}$, is said to be a stochastic flow, if $\tilde{\mathbb{P}}$-a.s.,*

(i) *$S(t,r;\omega)S(r,s;\omega)x=S(t,s;\omega)x$ for all $s\le r\le t$, $x\in X$,*

(ii) *$S(t,s;\omega)$ is continuous in X, for all $s\le t$,*

(iii) *for all $s<t$ and $x\in X$, the mapping*

$$\omega\mapsto S(t,s;\omega)x$$

is measurable from $(\tilde{\Omega},\tilde{\mathcal{F}})$ to $(X,\mathcal{B}(X))$ where $\mathcal{B}(X)$ is the Borel σ-algebra of X,

(iv) *for all $t,x\in X$, the mapping $s\mapsto S(t,s;\omega)$ is right continuous at any point.*

DEFINITION 4.2. *A set-valued map $K:\tilde{\Omega}\to 2^X$ taking values in the closed subsets of X is said to be measurable if for each $x\in X$ the map $\omega\mapsto d(x,K(\omega))$ is measurable, where*

$$d(A,B)=\sup\{\inf\{d(x,y):y\in B\}:x\in A\}\ \text{for} A,B\in 2^X,A,B\neq\emptyset,$$

and $d(x,B)=d(\{x\},B)$. Since $d(A,B)=0$ if and only if $A\subset B$, d is not a metric.

DEFINITION 4.3. *A closed set-valued measurable map $K:\tilde{\Omega}\to 2^X$ is called a random closed set.*

DEFINITION 4.4. *Given $t\in\mathbb{R}$ and $\omega\in\tilde{\Omega}, K(t,\omega)\subset X$ is called an attracting set at time t if for all bounded sets $B\subset X$,*

$$d(S(t,s;\omega)B,K(t,\omega))\to 0,\quad \textit{provided } s\to-\infty.$$

Moreover, if for all bounded sets $B\subset X$, there exists $t_B(\omega)$ such that for all $s\le t_B(\omega)$

$$S(t,s;\omega)B\subset K(t,\omega),$$

we say $K(t,\omega)$ is an absorbing set at time t.

Let $\{\vartheta_t:\tilde{\Omega}\to\tilde{\Omega}\},t\in T=\mathbb{R}$, be a family of measure-preserving transformations of the probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ such that for all $s<t$ and$\omega\in\tilde{\Omega}$, the following

(a) $(t,\omega)\to\vartheta_t\omega$ is measurable,

(b) $\vartheta_t(\omega)(s)=\omega(t+s)-\omega(t)$,

(c) $S(t,s;\omega)x=S(t-s,0;\vartheta_s\omega)x$,

hold. Then, $(\vartheta_t)_{t\in T}$ is a flow and $((\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}}),(\vartheta_t)_{t\in T})$ is a measurable dynamical system.

DEFINITION 4.5. *Given a bounded set $B\subset X$, the set*

$$\Omega(B,t,\omega)=\bigcap_{T\le t}\overline{\bigcup_{s\le T}S(t,s,\omega)B}$$

is said to be the Ω-limit set of B at time t. Obviously, if we denote $\Omega(B,0,\omega)=\Omega(B,\omega)$, we have $\Omega(B,t,\omega)=\Omega(B,\vartheta_t\omega)$.

It is easy to identify

$$\Omega(B,t,\omega)=\{x\in X:\text{there exists } s_n\to-\infty \text{ and } x_n\in B \text{ such that } \lim_{n\to\infty}S(t,s_n,\omega)x_n=x\}.$$

Furthermore, if there exists a compact attracting set $K(t,\omega)$ at time t, it is not difficult to check that $\Omega(B,t,\omega)$ is a nonempty compact subset of X and $\Omega(B,t,\omega)\subset K(t,\omega)$.

DEFINITION 4.6. *For all $t\in\mathbb{R}$ and $\omega\in\tilde{\Omega}$, a random closed set $\omega\to\mathcal{A}(t,\omega)$ is called the random attractor, if $\tilde{\mathbb{P}}-a.s.$,*

(1) *$\mathcal{A}(t,\omega)$ is a nonempty compact subset of X,*

(2) *$\mathcal{A}(t,\omega)$ is the minimal closed attracting set, i.e., if $\tilde{\mathcal{A}}(t,\omega)$ is another closed attracting set, then $\mathcal{A}(t,\omega)\subset\tilde{\mathcal{A}}(t,\omega)$,*

(3) *it is invariant, in the sense that, for all $s\le t$, $S(t,s;\omega)\mathcal{A}(s,\omega)=\mathcal{A}(t,\omega)$.*

Let $\mathcal{A}(\omega)=\mathcal{A}(0,\omega)$, then the invariance property can be written as

$$S(t,s;\omega)\mathcal{A}(\vartheta_s\omega)=\mathcal{A}(\vartheta_t\omega).$$

We will prove the existence of the random attractor by applying Theorem 2.2 in [3]. For the readers' convenience, we state it here.

THEOREM 4.1. *Let $(S(t,s;\omega))_{t\ge s,\omega\in\tilde{\Omega}}$ be a stochastic dynamical system satisfying **(i)-(iv)** in Definition 4.1. Assume that there exists a family of measure-preserving* *mappings $\vartheta_t, t\in\mathbb{R}$ such that* ***(a)-(c)*** *in Definition 4.4 hold and there exists a compact attracting set $K(\omega)$ at time 0, for $\tilde{\mathbb{P}}$-a.s.. Set*

$$\mathcal{A}(\omega)=\overline{\bigcup_{B\subset X, B \text{ bounded}} \Omega(B,\omega)}$$

where the union is taken over all the bounded subsets of X. Then we have $\tilde{\mathbb{P}}$-a.s.,

(1) *$\mathcal{A}(\omega)$ is a nonempty compact subset of X. If X is connected, it is a connected subset of $K(\omega)$.*

(2) *The family $\mathcal{A}(\omega)$, $\omega\in\Omega$, is measurable.*

(3) *$\mathcal{A}(\omega)$ is invariant in the sense that $S(t,s;\omega)\mathcal{A}(\vartheta_s\omega)=\mathcal{A}(\vartheta_t\omega)$, $s\leq t$.*

(4) *It attracts all bounded sets from $-\infty$: for bounded $B\subset X$ and $\omega\in\tilde{\Omega}$*

$$d(S(t,s;\omega)B,\mathcal{A}(\vartheta_t\omega))\rightarrow 0, \quad \text{when } s\rightarrow -\infty.$$

Moreover, it is the minimal closed set with this property: if $\tilde{\mathcal{A}}(\vartheta_t\omega)$ is a closed attracting set, then $\mathcal{A}(\vartheta_t\omega)\subset\tilde{\mathcal{A}}(\vartheta_t\omega)$.

(5) *For any bounded set $B\subset X$, $d(S(t,s;\omega)B,\mathcal{A}(\vartheta_t\omega))\rightarrow 0$ in probability when $t\rightarrow\infty$.*

And if the time shift $\vartheta_t, t\in\mathbb{R}$ is ergodic,

(6) *There exists a bounded set $B\subset X$ such that*

$$\mathcal{A}(\omega)=\mathcal{A}(B,\omega),$$

(7) *$\mathcal{A}(\omega)$ is the largest compact measurable set which is invariant.*

Before showing the existence of random attractor, we recall the Aubin-Lions lemma from [33].

LEMMA 4.1. *Let B_0,B,B_1 be Banach spaces such that B_0,B_1 are reflexive and $B_0\overset{c}{\subset} B\subset B_1$. For $0<T<\infty$, set*

$$X\overset{\text{def}}{=}\left\{h\middle| h\in L^2([0,T];B_0),\frac{dh}{dt}\in L^2([0,T];B_1)\right\}.$$

Then X is a Banach space equipped with the norm $|h|_{L^2([0,T];B_0)}+|h'|_{L^2([0,T];B_1)}$. Moreover,

$$X\overset{c}{\subset} L^2([0,T];B).$$

The main result in this paper reads as

THEOREM 4.2. *Let $Q\in L^2(\mathcal{O}), \upsilon_0\in V_1, T_0\in V_2$. Then the solution operator $(S(t,s;\omega))_{t\geq s,\omega\in\tilde{\Omega}}$ of 3D stochastic PEs (1.9)-(1.14): $S(t,s;\omega)(\upsilon_s,T_s)=(\upsilon(t),T(t))$ satisfies* ***(i)-(iv)*** *in Definition 4.1 and possesses a compact absorbing ball $\mathcal{B}(0,\omega)$ in V at time 0. Furthermore, for $\tilde{\mathbb{P}}$-a.s. $\omega\in\Omega$, set*

$$\mathcal{A}(\omega)=\overline{\bigcup_{B\subset V} \Omega(B,\omega)}$$

where the union is taken over all the bounded subsets of V. Then $\mathcal{A}(\omega)$ is the random attractor of stochastic PEs (1.9)-(1.14) and possesses the properties ***(1)-(7)*** *of Theorem 4.1 with space X replaced by space V.*

Proof. Denote by $w=(w_1 \overset{\text{def}}{=} \sum_{k=1}^n \alpha_k w_k^1, w_2 \overset{\text{def}}{=} \sum_{k=1}^n \beta_k w_k^2)$ the $\mathbb{R}^2-$valued Brownian motion, which has a version ω in $C_0(\mathbb{R},\mathbb{R}^2) \overset{\text{def}}{=} \tilde{\Omega}$, the space of continuous functions which are zero at $t=0$. In the following, we consider a canonical version of w given by the probability space $(C_0(\mathbb{R},\mathbb{R}^2),\mathcal{B}(C_0(\mathbb{R},\mathbb{R}^2)),\tilde{\mathbb{P}})$, where $\tilde{\mathbb{P}}$ is the Wiener-measure generated by w and $\mathcal{B}(C_0(\mathbb{R},\mathbb{R}^2))$ is the family of Borel subsets of $C_0(\mathbb{R},\mathbb{R}^2)$. Now, define the stochastic flow $(S(t,s;\omega))_{t\geq s,\ \omega\in\tilde{\Omega}}$ by

$$S(t,s;\omega)(v_s,T_s)=(\alpha^{-1}(t)u(t,\omega_1),\beta^{-1}(t)\theta(t,\omega_2)), \tag{4.1}$$

where (v,T) is the strong solution to (1.9)-(1.14) with $(v_s,T_s)=(\alpha^{-1}(s)u_s(s,\omega_1), \beta^{-1}(s)\theta_s(s,\omega_2))$ and (u,θ) is the strong solution to (3.2)-(3.7). It can be checked that assumptions **(i)-(iv)** and **(a)-(c)** of stochastic dynamics are satisfied with $X=V$. Indeed, properties **(i), (ii), (iv)** of the solution operator $(S(t,s;\omega))_{t\geq s,\omega\in\tilde{\Omega}}$ follows by Theorem 3.2 and property **(iii)** of the solution operator also holds with the help of the global existence of strong solution to (1.9)-(1.14) resting upon Faedo-Galerkin method. Furthermore, $(\tilde{\Omega},\mathcal{B}(C_0(\mathbb{R},\mathbb{R}^2)),\tilde{\mathbb{P}},\vartheta)$ is an ergodic metric dynamical system.

In the following, we will prove the existence of the random attractor. Let $(u(t,\omega;t_0,u_0),\theta(t,\omega;t_0,\theta_0))$ be the solution to (3.2)-(3.7) with initial value $u(t_0)=u_0$ and $\theta(t_0)=\theta_0$. By the law of the iterated logarithm, we have

$$\lim_{t\to-\infty} \frac{\sum_{k=1}^n \alpha_k w_k^1}{t} = \lim_{t\to-\infty} \frac{\sum_{k=1}^n \beta_k w_k^2}{t} = 0. \tag{4.2}$$

Obviously, $t\to\beta^2(t)e^{\lambda t}$ is pathwise integrable over $(-\infty,0]$, where λ is positive. And we have

$$\lim_{t\to-\infty} \beta^2(t)e^{\lambda t}=0, \quad \tilde{\mathbb{P}}-a.e.. \tag{4.3}$$

In view of (3.16), (3.17) and (4.3), for $\tilde{\mathbb{P}}$-a.e. $\omega\in\tilde{\Omega}$, there exists a random variable $r_1(\omega)$, depending only on λ, such that for arbitrary $\rho>0$, there exists $t(\omega)\leq -4$ such that for all $t_0\leq t(\omega)$ and $(u_0,\theta_0)\in V$ with $\|u_0\|+\|\theta_0\|\leq\rho$, $\theta(t,\omega;t_0,\theta_0)$ satisfies

$$\sup_{t\in[-4,0]} |\theta(t,\omega;t_0,\theta_0)|^2 + \int_{-4}^0 \|\theta(s)\|^2 ds \leq r_1(\omega). \tag{4.4}$$

In view of (3.19) and (4.4), taking a similar argument as (4.4), for $\tilde{\mathbb{P}}$-a.e. $\omega\in\tilde{\Omega}$, we deduce that there exists random variable $r_2(\omega)$, depending only on λ, such that for arbitrary $\rho>0$, there exists $t(\omega)\leq -4$ such that for all $t_0\leq t(\omega)$ and $(u_0,\theta_0)\in V$ with $\|u_0\|+\|\theta_0\|\leq\rho$, $u(t,\omega;t_0,u_0)$ satisfies

$$\sup_{t\in[-4,0]} |u(t,\omega;t_0,u_0)|^2 + \int_{-4}^0 \|u(s)\|^2 ds \leq r_2(\omega). \tag{4.5}$$

By (3.25), repeating the argument as in (4.4), for $\tilde{\mathbb{P}}$-a.e. $\omega\in\tilde{\Omega}$, there exists random variable $r_3(\omega)$, depending only on λ, such that for arbitrary $\rho>0$, there exists $t(\omega)\leq -4$ such that for all $t_0\leq t(\omega)$ and $(u_0,\theta_0)\in V$ with $\|u_0\|+\|\theta_0\|\leq\rho$, $\theta(t,\omega;t_0,\theta_0)$ satisfies

$$\sup_{t\in[-4,0]} |\theta(t,\omega;t_0,\theta_0)|_4^2 \leq r_3(\omega). \tag{4.6}$$

Integrating (3.29) with respect to time over $[t,-3]$ yields,

$$|\tilde{u}(-3)|_4^2 \leq \Big(|\tilde{u}(t)|_4^2 + C\int_t^{-3} \alpha^2(s)\beta^{-2}(s)|\theta(s)|_4^2 ds\Big) \times \mathrm{e}^{C\int_t^{-3}\alpha^{-2}(s)\|u(s)\|^2(1+\alpha^{-2}(s)|u(s)|^2)ds}, \tag{4.7}$$

Integrating (4.7) with respect to t over $[-4,-3]$, we obtain

$$|\tilde{u}(-3)|_4^2 \leq \Big(\int_{-4}^{-3} |\tilde{u}(t)|_4^2 ds + C\int_{-4}^{-3} \alpha^2(s)\beta^{-2}(s)|\theta(s)|_4^2 ds\Big) \times \mathrm{e}^{C\int_{-4}^{-3}\alpha^{-2}(s)\|u(s)\|^2(1+\alpha^{-2}(s)|u(s)|^2)ds}. \tag{4.8}$$

Therefore, by virtue of (4.4)-(4.6), we conclude that for $\tilde{\mathbb{P}}$-a.e. $\omega\in\tilde{\Omega}$, there exists random variable $C_1(\omega)$, depending only on λ, such that for arbitrary $\rho>0$, there exists $t(\omega)\leq -4$ such that for all $t_0\leq t(\omega)$ and $(u_0,\theta_0)\in V$ with $\|u_0\|+\|\theta_0\|\leq\rho$, $\tilde{u}(-3,\omega;t_0,u_0)$ satisfies

$$|\tilde{u}(-3)|_4^2 \leq C_1(\omega). \tag{4.9}$$

Integrating (3.29) with respect to time over $[-3,t]$, we get

$$|\tilde{u}(t)|_4^2 \leq \Big(|\tilde{u}(-3)|_4^2 + C\int_{-3}^{t} \alpha^2(s)\beta^{-2}(s)|\theta(s)|_4^2 ds\Big) \times \mathrm{e}^{C\int_{-3}^{t}\alpha^{-2}(s)\|u(s)\|^2(1+\alpha^{-2}(s)|u(s)|^2)ds}.$$

In view of (4.4)-(4.6) and (4.9), for $\tilde{\mathbb{P}}$-a.e. $\omega\in\tilde{\Omega}$, we deduce that there exists random variable $r_4(\omega)$, such that for arbitrary $\rho>0$, there exists $t(\omega)\leq -3$ such that for all $t_0\leq t(\omega)$ and $(u_0,\theta_0)\in V$ with $\|u_0\|+\|\theta_0\|\leq\rho$, $u(t,\omega;t_0,u_0)$ satisfies

$$\sup_{t\in[-3,0]} |\tilde{u}(t,\omega;t_0,\tilde{u}_0)|_4^2 \leq r_4(\omega). \tag{4.10}$$

Taking integration of (3.28) with respect to time over $[-3,0]$ yields,

$$\int_{-3}^{0} (|\nabla(|\tilde{u}|^2)|^2 + |\partial_z(|\tilde{u}|^2)|^2 + |\tilde{u}|\nabla\tilde{u}||^2 + |\tilde{u}|\partial_z\tilde{u}||^2)dt \leq |\tilde{u}(-3)|_4^2 + C\int_{-3}^{0}\alpha^2\beta^{-2}|\theta|_4^2|\tilde{u}|_4^2 + C\int_{-3}^{0}(\alpha^{-2}\|u\|^2+\alpha^{-4}|u|^2\|u\|^2)|\tilde{u}|_4^2. \tag{4.11}$$

By (4.5), (4.6) and (4.11), for $\tilde{\mathbb{P}}$-a.e. $\omega\in\tilde{\Omega}$, we infer that that there exists random variable $C_2(\omega)$, such that for $\rho>0$, there exists $t(\omega)\leq -3$ such that for all $t_0\leq t(\omega)$ and $(u_0,\theta_0)\in V$ with $\|u_0\|+\|\theta_0\|\leq\rho$, $\tilde{u}(t,\omega;t_0,u_0)$ satisfies

$$\int_{-3}^{0} (|\nabla(|\tilde{u}|^2)|^2 + |\partial_z(|\tilde{u}|^2)|^2 + |\tilde{u}|\nabla\tilde{u}||^2 + |\tilde{u}|\partial_z\tilde{u}||^2)dt \leq C_2(\omega). \tag{4.12}$$

By (3.31), (4.6) and (4.12), proceeding as (4.10), for $\tilde{\mathbb{P}}$-a.e. $\omega\in\tilde{\Omega}$, there exists random variable $C_3(\omega)$, such that for arbitrary $\rho>0$, there exists $t(\omega)\leq -2$ such that for all $t_0\leq t(\omega)$ and $(u_0,\theta_0)\in V$ with $\|u_0\|+\|\theta_0\|\leq\rho$, $\bar{u}(t,\omega;t_0,\bar{u}_0)$ satisfies

$$\sup_{t\in[-2,0]} |\nabla\bar{u}(t,\omega;t_0,\bar{u}_0)|^2 \leq C_3(\omega). \tag{4.13}$$

In view of (3.34) and (4.13), following the steps in (4.10), for $\tilde{\mathbb{P}}$-a.e. $\omega\in\tilde{\Omega}$, there exists a random variable $r_5(\omega)$, such that for arbitrary $\rho>0$, there exists $t(\omega)\leq -1$ such that for all $t_0\leq t(\omega)$ and $(u_0,\theta_0)\in V$ with $\|u_0\|+\|\theta_0\|\leq\rho$, satisfies

$$\sup_{t\in[-1,0]}|\partial_z u(t,\omega;t_0,u_0)|^2+\int_{-1}^{0}|\nabla u_z(s,\omega;t_0,u_0)|^2ds\leq r_5(\omega). \tag{4.14}$$

Regarding (3.38), (4.4), (4.5) and (4.14), we repeat the procedures of deriving (4.10) and (4.11). For $\tilde{\mathbb{P}}$-a.e. $\omega\in\tilde{\Omega}$, there exists a random variable $r_6(\omega)$, such that for arbitrary $\rho>0$, there exists $t(\omega)\leq -1$ such that for all $t_0\leq t(\omega)$ and $(u_0,\theta_0)\in V$ with $\|u_0\|+\|\theta_0\|\leq\rho$, $u(t,\omega;t_0,u_0)$ satisfies

$$\sup_{t\in[-1,0]}|\nabla u(t,\omega;t_0,u_0)|^2+\int_{-1}^{0}|\Delta u(s,\omega;t_0,u_0)|^2ds\leq r_6(\omega). \tag{4.15}$$

By (3.40), (4.4), (4.14) and (4.15), proceeding as above, for $\tilde{\mathbb{P}}$-a.e. $\omega\in\tilde{\Omega}$, there exists a random variable $r_7(\omega)$, such that for arbitrary $\rho>0$, there exists $t(\omega)\leq -1$ such that for all $t_0\leq t(\omega)$ and $(u_0,\theta_0)\in V$ with $\|u_0\|+\|\theta_0\|\leq\rho$, $\theta(t,\omega;t_0,\theta_0)$ satisfies

$$\sup_{t\in[-1,0]}\|\theta(t,\omega;t_0,\theta_0)\|^2\leq r_7(\omega).$$

Now we are ready to prove the desired compact result. Let $r(\omega)=r_5(\omega)+r_6(\omega)+r_7(\omega)$, then $B(-1,r(\omega))$, the ball of center $0\in V$ and radius $r(\omega)$, is an absorbing set at time -1 for $(S(t,s;\omega))_{t\geq s,\omega\in\tilde{\Omega}}$. According to Theorem 4.1, in order to prove the existence of the random attractor in the space V, we need to to construct a compact absorbing set at time 0 in V. Let $\mathcal{B}$ be a bounded subset of V, set

$$\mathcal{C}_T\overset{\text{def}}{=}\{(A_1^{\frac12}v,A_2^{\frac12}T)\Big|(v(-1),T(-1))\in\mathcal{B},(v(t),T(t))=S(t,-1;\omega)(v(-1),T(-1)),t\in[-1,0]\}.$$

We claim that $\mathcal{C}_T$ is compact in $L^2([-1,0];H)$. Indeed, the space $V_1\times V_2\subset H_1\times H_2$ is compact as $V_i\subset H_i$ is compact. Let $(v(-1),T(-1))\in\mathcal{B}$; by the argument of step 2 in the proof of Theorem 4.1, we have

$$(A_1^{\frac12}u,A_2^{\frac12}\theta)\in L^2([-1,0];V_1\times V_2),\quad (\partial_tA_1^{\frac12}u,\partial_tA_2^{\frac12}\theta)\in L^2([-1,0];V_1'\times V_2').$$

Therefore, we deduce the result from Lemma 4.1 with

$$B_0=V_1\times V_2,\quad B=H_1\times H_2,\quad B_1=V_1'\times V_2'.$$

Now, we aim to show that for any fixed $t\in(-1,0],\omega\in\tilde{\Omega},S(t,-1;\omega)$ is a compact operator in V. Taking any bounded sequences $\{(\nu_{0,n},\tau_{0,n})\}_{n\in\mathbb{N}}$ in $\mathcal{B}$, for any fixed $t\in(-1,0]$, $\omega\in\tilde{\Omega}$, we devote to extracting a convergent subsequence from $\{S(t,-1;\omega)(\nu_{0,n},\tau_{0,n})\}$. Since $\{(A_1^{\frac12}v,A_2^{\frac12}T)\}\subset\mathcal{C}_T$, by Lemma 4.1, there is a function $(\nu_*,\theta_*)\in L^2([-1,0];V)$ and a subsequence of $\{S(t,-1;\omega)(\nu_{0,n},\tau_{0,n})\}_{n\in\mathbb{N}}$ still denoted by $\{S(t,-1;\omega)(\nu_{0,n},\tau_{0,n})\}_{n\in\mathbb{N}}$, such that

$$\lim_{n\to\infty}\int_{-1}^{0}\|S(t,-1;\omega)(\nu_{0,n},\tau_{0,n})-(\nu_*(t),\theta_*(t))\|^2dt=0. \tag{4.16}$$

By the measure theory, we know that the convergence in mean square implies almost sure convergence. Therefore, it follows from (4.16) that there exists a subsequence $\{S(t,-1;\omega)(\nu_{0,n},\tau_{0,n})\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\|S(t,-1;\omega)(\nu_{0,n},\tau_{0,n})-(\nu_*(t),\theta_*(t))\|=0,\quad a.e.\ t\in(-1,0]. \tag{4.17}$$

Fix any $t\in(-1,0]$. By (4.17), we can select a $t_0\in(-1,t)$ such that

$$\lim_{n\to\infty}\|S(t_0,-1,\omega)(\nu_{0,n},\tau_{0,n})-(\nu_*(t_0),\theta_*(t_0))\|=0.$$

Then, by the continuity of $S(t-t_0,t_0;\omega)$ in V with respect to the initial value, we have

$$\begin{aligned}S(t,-1;\omega)(\nu_{0,n},\tau_{0,n})&=S(t-t_0,t_0;\omega)S(t_0,-1;\omega)(\nu_{0,n},\tau_{0,n})\\&\to S(t-t_0,t_0;\omega)(\nu_*(t_0),\theta_*(t_0)),\quad \text{in } V.\end{aligned}$$

Hence, for any $t\in(-1,0]$, we can always find a convergent subsequence of $\{S(t,-1;\omega)(\nu_{0,n},\tau_{0,n})\}_{n\in\mathbb{N}}$ in V, which implies that for any fixed $t\in(-1,0],\omega\in\tilde{\Omega},S(t,-1;\omega)$ is a compact operator in V. Set

$$\mathcal{B}(0,\omega)=\overline{S(0,-1;\omega)B(-1,r(\omega))},$$

then, $\mathcal{B}(0,\omega)$ is a closed set of $S(0,-1;\omega)B(-1,r(\omega))$ in V. Using the above argument, we know that $\mathcal{B}(0,\omega)$ is a random compact set in V. Precisely, $\mathcal{B}(0,\omega)$ is a compact absorbing set in V at time 0. Indeed, for $(\nu_{0,n},\tau_{0,n})\in\mathcal{B}$, there exists $s(\mathcal{B})\in\mathbb{R}_-$ such that for any $s\leq s(\mathcal{B})$, we have

$$\begin{aligned}S(0,s;\omega)(\nu_{0,n},\tau_{0,n})&=S(0,-1;\omega)S(-1,s;\omega)(\nu_{0,n},\tau_{0,n})\\&\subset S(0,-1;\omega)B(-1,r(\omega))\subset\mathcal{B}(0,\omega).\end{aligned}$$

Therefore, we conclude the result from Theorem 4.1. □

5. Existence of invariant measure

Now, we are ready to prove the existence of invariant measure of the system (1.9)-(1.14).

Let $U_0=(v_0,T_0)\in V$, $U(t,\omega;U_0)\overset{\text{def}}{=}(v(t,\omega;t_0,v_0),T(t,\omega;t_0,T_0))$ is the solution to (3.2)-(3.7) with the initial value U_0. Following the standard argument, we can show that $U(t,\omega;U_0),t\in[t_0,\mathcal{T}]$ is a Markov process in the sense that, for every bounded, $\mathcal{B}(V)$-measurable $F:V\to\mathbb{R}$, and all $s,t\in[t_0,\mathcal{T}]$, $t_0\leq s\leq t\leq\mathcal{T}$,

$$\mathbb{E}(F(U(t,\omega;U_0))|\mathcal{F}_s)(\omega)=\mathbb{E}(F(U(t,s,U(s))))\ \text{ for } \tilde{\mathbb{P}}-a.e.\ \omega\in\Omega,$$

where $\mathcal{F}_s=\mathcal{F}_{t_0,s}$ (see (3.1)), $U(t,s,U(s))$ is the solution to (1.9)-(1.14) at time t with initial data $U(s)$.

For $B\in\mathcal{B}(V)$, define

$$\tilde{\mathbb{P}}_t(U_0,B)=\tilde{\mathbb{P}}((U(t,\omega;U_0)\in B).$$

For any probability measure ν defined on $\mathcal{B}(V)$, denote the distribution at time t of the solution to (1.9)-(1.14) with initial distribution ν by

$$(\nu\tilde{\mathbb{P}}_t)(\cdot)=\int_V\tilde{\mathbb{P}}_t(x,\cdot)\nu(dx).$$

For $t\geq t_0$ and any continuous and bounded function $f\in C_b(V;\mathbb{R})$, we have

$$\tilde{\mathbb{P}}_tf(U_0)=\mathbb{E}[f(U(t,\omega;U_0)]=\int_V f(x)\tilde{\mathbb{P}}_t(U_0,dx).$$

Definition 5.1. *Let ρ be a probability measure on $\mathcal{B}(V)$. ρ is called an invariant measure for $\tilde{\mathbb{P}}_t$, if*

$$\int_V f(x)\rho(dx) = \int_V \tilde{\mathbb{P}}_t f(x)\rho(dx)$$

for all $f \in C_b(V;\mathbb{R})$ and $t \geq 0$.

Let $\mu_.$ be a transition probability from $\tilde{\Omega}$ to V, i.e., $\mu_.$ is a Borel probability measure on V and $\omega \to \mu_.(B)$ is measurable for every Borel set $B \subset V$. Denote by $\mathcal{P}_{\tilde{\Omega}}(V)$ the set of transition probabilities with $\mu_.$ and $\nu_.$ identified if $\tilde{\mathbb{P}}\{\omega : \mu_\omega \neq \nu_\omega\} = 0$.

In view of Proposition 4.5 in [4], the existence of random attractor obtained in Theorem 4.2 implies the existence of invariant Markov measure $\mu_. \in \mathcal{P}_{\tilde{\Omega}}(V)$ for S such that $\mu_\omega(\mathcal{A}(\omega)) = 1$, $\tilde{\mathbb{P}}$-a.e.. Therefore, referring to [2], there exists an invariant measure for the Markov semigroup $\tilde{\mathbb{P}}_t$ and it is given by

$$\rho(B) = \int_{\tilde{\Omega}} \mu_\omega(B)\tilde{\mathbb{P}}(d\omega),$$

where $B \subseteq V$ is a Borel set. If the invariant measure ρ for $\tilde{\mathbb{P}}$ is unique, the invariant Markov measure $\mu_.$ for S is unique and given by

$$\mu_\omega = \lim_{t\to\infty} S(0,-t,\omega)\rho.$$

Based on the above, we arrive at

Theorem 5.1. *The Markov semigroup $(\tilde{\mathbb{P}}_t)_{t\geq 0}$ induced by the solution $(U(t,\omega;U_0))_{t\geq 0}$ to (1.9)-(1.14) has an invariant measure ρ with $\rho(\mathcal{A}(\omega)) = 1$ $\tilde{\mathbb{P}}-a.e.$.*

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