FAST COMMUNICATION

A SIMPLE APPROACH TO TEMPORAL CLOAKING*

GREGORY ESKIN†

Abstract. In recent years, a remarkable progress was made in the construction of spatial cloaks using the methods of transformation optics and metamaterials.

The temporal cloaking, i.e. the cloaking of an event in spacetime, was also widely studied by using transformations on spacetime domains.

We propose a simple and general method for the construction of temporal cloaking using the change of time variable only.

Keywords. wave equation; perfect cloaking; inverse problems.

Mathematics subject classification 2010. 35L05; 35Q93; 65L09.

1. Introduction

The transformation optics approach, combined with the use of metamaterials, leads to a remarkable progress in the construction of spatial cloaking devices and other problems (cf. J.B. Pendry et al. [15], U. Leonhardt [12] and others, see also [1, 16]). The mathematical analysis of the spatial cloaking was done by A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann [4–7] (see also [13]).

The temporal cloaking uses the transformation of variables in spacetime. The important works on the temporal cloaking were done by physicists (cf. McCall, Kinsler et al. [8, 10, 11, 14], M. Fridman [3] and others).

In the above works, mostly the cloaking in one direction was studied, i.e. no complete cloaking in $n \geq 2$ dimensions was achieved.

Our approach to the temporal cloaking is based on a completely new idea consisting of the change of the time variable only. This way, we get a perfect cloaking region.

To preserve the hyperbolicity of the wave equation after the change of time variable, one needs to restrict the size of the cloaked region depending on the wave speed: The size of the cloaking region is decreasing when the speed is increasing. Thus, our approach is better when the speed is not too large, as in the case of the acoustic equations and the equations of linear elasticity but for the optics problems the cloaked region is too small.

Now we shall briefly describe the content of the paper.

In §2, we apply our method to the initial boundary value problem for the classical wave equation.

In §3, we give a physical interpretation of the results of §2.

Then in §4, we extend the results of §2 to the cases of more general equations.

2. The main result

Let D be a bounded domain in $\mathbb{R}^n$. Consider the initial-boundary value problem:

$$Lu(x_0, x) \stackrel{def}{=} \frac{\partial^2 u(x_0, x)}{\partial x_0^2} - a^2 \sum_{k=1}^{n} \frac{\partial^2 u(x_0, x)}{\partial x_k^2} = 0, \quad (x_0, x) \in \mathbb{R} \times D, \qquad (2.1)$$

$$u = 0 \quad \text{for} \quad x_0 \ll 0, \ x \in D, \qquad (2.2)$$

$$u\big|_{\mathbb{R}\times\partial D} = f, \tag{2.3}$$

where $x = (x_1, \ldots, x_n)$, a is a wave speed, f has a compact support in $\mathbb{R} \times \partial D$.

Make a change of variables in $\mathbb{R} \times D$

$$y_0 = \varphi_0(x_0, x), \quad y = x. \tag{2.4}$$

Let $\tilde{L}v(y_0, y) = 0$ be the Equation (2.1) in (y_0, y)-coordinates, where $v(y_0, y) = u(x_0, x)$, (y_0, y) and (x_0, x) are related by (2.4). For the particular case when

$$\varphi_0(x_0, x) = x_0 + c(x),$$

we have

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j} + c_{y_j}(y)\frac{\partial v(y_0, y)}{\partial y_0}, \quad \frac{\partial u}{\partial x_0} = \frac{\partial v}{\partial y_0}$$

and the equation $\tilde{L}v(y_0, y) = 0$ has the following form

$$\frac{\partial^2 v}{\partial y_0^2} - a^2 \sum_{j=1}^{n} \left(\frac{\partial}{\partial y_j} + c_{y_j}(y)\frac{\partial}{\partial y_0}\right)\left(\frac{\partial}{\partial y_j} + c_{y_j}(y)\frac{\partial}{\partial y_0}\right)v = 0. \tag{2.5}$$

The symbol of (2.5) is

$$p(y, \eta) = \eta_0^2 - a^2 \sum_{j=1}^{n}(\eta_j + c_{y_j}(y)\eta_0)^2$$

or

$$p(y, \eta) = \left(1 - a^2 \sum_{j=1}^{n} c_{y_j}^2\right)\eta_0^2 - 2a^2 \sum_{j=1}^{n} a^2 c_{y_j}\eta_0\eta_j - a^2 \sum_{j=1}^{n} \eta_j^2.$$

It is strictly hyperbolic with respect to y_0 (cf. [9], §23.2, or [2], §48) if

$$1 - a^2 \sum_{j=1}^{n} c_{y_j}^2 > 0. \tag{2.6}$$

Indeed, $p(y, \eta_0, \eta) = 0$ has two distinct real roots

$$\eta_0 = \frac{a^2\, c_y \cdot \eta \pm \sqrt{a^4(c_y \cdot \eta)^2 + (1 - a^2|c_y|^2)a^2|\eta|^2}}{1 - a^2|c_y|^2}, \tag{2.7}$$

when (2.6) holds. Here $c_y \cdot \eta = \sum_{j=1}^{n} c_{y_j}\eta_j$. Note that, when $n \geq 2$ and $c_y \cdot \eta = 0$, the roots in (2.7) will be not real when $1 - a^2|c_y|^2 < 0$, i.e. (2.5) is not strictly hyperbolic when (2.6) is not satisfied.

Thus, the initial boundary value problems are well-posed when (2.6) holds.

We specify that

$$\varphi_0(x_0, x) = x_0 + c(x) \quad \text{for} \quad x_0 \geq 0, \tag{2.8}$$
$$\varphi_0(x_0, x) = x_0 \quad \text{for} \quad x_0 < 0, \tag{2.9}$$

where

$$c(x) = c_0 \chi(x), \quad c_0 > 0, \tag{2.10}$$

$\chi(x) \in C_0^\infty(\mathbb{R}^n), \chi(x) = 0$ for $|x| > c_1, \chi(x) = 1$ for $|x| < \frac{c_1}{2}, 0 < \chi(x) < 1$ for $\frac{c_1}{2} < |x| < c_1$, the ball $|x| \leq c_1$ is inside D. Therefore, $y_0 = \varphi_0(x_0, x)$ is strictly increasing in $x_0 \in \mathbb{R}$ and has a jump $\varphi_0(+0, x) - \varphi_0(-0, x) = c(x)$ at $x_0 = 0$.

Denote by Y^+ and Y^- the sets $\{y_0 \geq c(y), y \in \mathbb{R}^n\}$ and $\{y_0 < 0, y \in \mathbb{R}^n\}$, respectively.

Let

$$Y_0 = \{(y_0, y) : 0 < y_0 < c(y), |y| < c_1\}. \tag{2.11}$$

Note that

$$Y^+ \cup \overline{Y^-} = \mathbb{R}^{n+1} \setminus Y_0, \tag{2.12}$$

and $y_0 = \varphi(x_0, x), y = x$ maps $\mathbb{R} \times \mathbb{R}^n$ onto $Y^- \cup Y^+$.

Consider the equation

$$\tilde{L}^- v^-(y_0, y) = 0 \quad \text{in} \quad Y^- \cap (\mathbb{R} \times D), \tag{2.13}$$

where $\tilde{L}^- = L, y_0 = x_0 < 0$, with the initial condition

$$v^- = 0 \quad \text{for} \quad y_0 \ll 0, y \in D, \tag{2.14}$$

and the boundary condition

$$v^- = f \quad \text{on} \quad (\mathbb{R} \times \partial D) \cap Y^-. \tag{2.15}$$

This problem is well-posed and has a unique smooth solution $v^-(y_0, y)$. Now consider the initial-boundary value problem for

$$\tilde{L}^+ v^+(y_0, y) = 0 \quad \text{in} \quad Y^+ \cap (\mathbb{R} \times D), \tag{2.16}$$

corresponding to the change of variable $y_0 = \varphi_0(x_0, x) = x_0 + c(x), \; x_0 > 0$, with the boundary condition

$$v^+ = f \quad \text{on} \quad (\mathbb{R} \times \partial D) \cap Y^+, \tag{2.17}$$

and the initial conditions

$$v^+\Big|_{\partial Y^+ \cap (\mathbb{R} \times D)} = v^-\Big|_{\partial Y^- \cap (\mathbb{R} \times D)}, \quad \frac{\partial v^+}{\partial y_0}\Big|_{\partial Y^+ \cap (\mathbb{R} \times D)} = \frac{\partial v^-}{\partial y_0}\Big|_{\partial Y^- \cap (\mathbb{R} \times D)}. \tag{2.18}$$

Note that we assume that $v^-\big|_{\partial Y^- \cap D}$ and $\frac{\partial v^-}{\partial y_0}\big|_{\partial Y^- \cap D}$ are already known.

This initial-boundary value problem also has a unique smooth solution when (2.6) holds. Therefore we can determine a function $v(y_0, y)$ such that $v = v^+$ in $Y^+ \cap (\mathbb{R} \times D)$, $v = v^-$ in $Y^- \cap (\mathbb{R} \times D)$, $\tilde{L}^+ v^+ = 0$ in $Y^+ \cap \mathbb{R} \times D$, $\tilde{L}^- v^- = 0$ in $Y^- \cap (\mathbb{R} \times D)$, $v = v^- = 0$ for $y_0 \ll 0, \; y \in D$, $v\big|_{\mathbb{R} \times \partial D} = f$ and v^+ and v^- satisfy the conditions (2.18).

Once $v(y_0, y)$ is given, we determine $u^+(x_0, x)$ for $x_0 > 0$, such that $u^+(x_0, x) = v^+(y_0, y)$, where $x_0 = y_0 - c(y), x = y, (y_0, y) \in Y^+$. Analogously, $u^-(x_0, x) = v^-(y_0, y)$, where $x_0 = y_0, x = y, (y_0, y) \in Y^-$.

Then $u^+(x_0, x)$ and $u^-(x_0, x)$ satisfy the wave Equation (2.1) for $x_0 > 0$ and $x_0 < 0$, respectively. Since the conditions (2.18) are satisfied, we have that

$$\lim_{\substack{x \to 0 \\ x_0 < 0}} u^-(x_0, x) = \lim_{\substack{x_0 \to 0 \\ x_0 > 0}} u^+(x_0, x) \tag{2.19}$$

$$\lim_{\substack{x \to 0 \\ x_0 < 0}} \frac{\partial u^-(x_0, x)}{\partial x_0} = \lim_{\substack{x \to 0 \\ x_0 > 0}} \frac{\partial u^+}{\partial x_0}(x_0, x)$$

Therefore $u(x_0, x) = u^+(x_0, x)$ for $x_0 > 0$ and $u(x_0, x) = u^-(x_0, x)$ for $x_0 < 0$ satisfies (2.1) in $\mathbb{R} \times D$ and also satisfies the initial and boundary conditions (2.2) and (2.3).

Since $c(x) = 0$ in $D \setminus \overline{B}$, where $B = \{x : |x| < c_1\}$, we have that $x_0 = y_0, x = y$ in $\mathbb{R} \times (D \setminus \overline{B})$. Thus

$$u(x_0, x) = v(x_0, x) \quad \text{in} \quad \mathbb{R} \times (D \setminus \overline{B}). \tag{2.20}$$

Therefore the boundary data of $u(x_0, x)$ and $v(y_0, y)$ on $\mathbb{R} \times \partial D$ are the same. This implies that one can not distinguish between $u(x_0, x)$ in $\mathbb{R} \times D$ and $v(y_0, y)$ in $(\mathbb{R} \times D) \cap (Y^+ \cup Y^-)$ using the boundary measurements. Since the domain Y_0 is outside of $Y^+ \cup \overline{Y}^-$, Y_0 is a temporal cloaking domain and the observer on $\mathbb{R} \times \partial D$ does not suspect its existence.

REMARK 2.1. Note that the change of variables (2.6) is not singular as it happens in the case of a spatial cloaking domain.

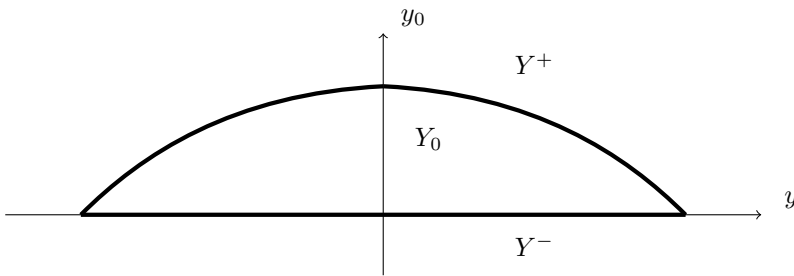Summarizing the results of this section, we get the following theorem:

FIG. 2.1. *Domain Y_0 is a temporal cloaked region.*

THEOREM 2.1. *Let domains Y_0, Y^+, Y^- be the same as in (2.11) and (2.12) and let operators $\tilde{L}^-$ and $\tilde{L}^+$ be the same as in (2.13) and (2.16). Let the condition (2.6) hold. Consider the solution $v^-(y_0, y)$ of the initial-boundary value problem (2.13), (2.14) and (2.15) in $Y^- \cap (\mathbb{R} \times D)$.*

Let $v^+(y_0, y)$ be the solution in $Y^+ \cap (\mathbb{R} \times D)$ of the initial-boundary value problem (2.16), (2.17) and (2.18).

Let $v(y_0, y) = v^-(y_0, y)$ for $y_0 \leq 0$, $v(y_0, y) = v^+(y_0, y)$ for $y_0 \geq c(y)$, i.e. $v(y_0, y)$ is the solution in $\overline{Y}^+ \cup Y^- = (\mathbb{R}^n \times D) \setminus Y_0$.

Let $u^+(x_0, x) = v^+(y_0, y)$ for $x_0 > 0$, where $x = y, x_0 = y_0 - c(y)$ and let $u^-(x_0, x) = v^-(y_0, y)$ for $x_0 < 0$ where $x = y, x_0 = y_0$. It follows from (2.18) that $u(x_0, x) = u^+(x_0, x)$ for $x_0 > 0, u(x_0, x) = u^-(x_0, x)$ for $x_0 < 0$ extends to a smooth function

in $\mathbb{R} \times D$ that satisfies (2.1), (2.2) and (2.3). The boundary measurements of $v(y_0, y)$ defined in $(\mathbb{R} \times D) \setminus Y_0$ and $u(x_0, x)$ defined in $\mathbb{R} \times D$ are equal.

Thus Y_0 is a perfect temporal cloak.

3. Physical interpretation of Theorem 2.1

Assume, for the definiteness, that (2.1) describes the vibration of a membrane where $a = \sqrt{\frac{T}{\rho}}$ is the speed, T is the tension and ρ is the density. Let $v^-(y_0, y)$ be the solution of (2.1), (2.2) and (2.3) for $y_0 \le 0$. We shall call $V^-(y_0, y)$ the physical solution of (2.1) (to distinguish from the numerical solution $v^-(y_0, y)$), i.e. $V^-(y_0, y)$ is the actual vibrating membrane where $V^-(y_0, y)$ is the position of the membrane at point y and at time $y_0 \le 0$.

Consider now the equation $\tilde{L}^+ v^+(y_0, y) = 0$ in Y^+ with the boundary condition (2.3) and the initial conditions (2.18). Denote by $w(x_0, x_1)$ the function $v^+(y_0 - c(y), y)$ where $x_0 = y_0 - c(y) \ge 0, y = x$. Function $w(x_0, x)$ is the solution of (2.1) for $x_0 \in [0, +\infty)$ with the boundary condition (2.3) and the initial conditions

$$w(0, x) = v^-(0, x), \quad \frac{\partial w(0, x)}{\partial x_0} = \frac{\partial v^-(0, x)}{\partial x_0}, \tag{3.1}$$

(cf. (2.18)). Knowing $w(x_0, x)$, we can recover $v^+(y_0, y)$ by the formula

$$v^+(y_0, y) = w(y_0 - c(y), y), \quad (y_0, y) \in Y^+. \tag{3.2}$$

Let $W(x_0, x)$ be the physical solution of (2.1) for $x_0 > 0$, i.e. the actual membrane vibrationg on $(0, +\infty)$.

It follows from (3.2) that $V^+(y_0, y) = W(y_0 - c(y), y)$ is the physical solution of (2.5) on Y^+, i.e. $V^+(y_0, y)$ is the actual vibrating membrane shifted in time.

It follows from (2.1) that the physical initial data of $V^+(y_0, y)$ on $y_0 - c(y) = 0$ are equal to the physical final data of $V^-(y_0, y)$ on $y_0 = 0$, i.e.

$$V^+(y_0, y)\Big|_{y_0 - c(y) = 0} = V^-(0, y), \quad \frac{\partial V^+}{\partial y_0}\Big|_{y_0 - c(y) = 0} = \frac{\partial V^-(0, y)}{\partial y_0}, \tag{3.3}$$

since $W(0, y) = V^-(0, y), \frac{\partial W(0,y)}{\partial y_0} = \frac{\partial V^-(0,y)}{\partial y_0}$.

Summarizing, we have a physical solution $V^-(y_0, y)$ (membrane) on Y^- satisfying the initial condition (2.2) and the boundary condition (2.3), and we have another physical solution $V^+(y_0, y)$ on Y^+ satisfying the boundary condition (2.3). The initial physical condition of $V^+(y_0, y)$ are equal to the final physical condition of $V^-(y_0, y)$. Note that the cloaking region Y_0 is outside of domains Y^- and Y^+ of these two physical solutions V^- and V^+. If one takes the physical measurement of the force $T\frac{\partial V^+}{\partial n}$ on the boundary, one will get the same result as when we measure the force on the boundary for the initial boundary value problem (2.1), (2.2) and (2.3), i.e. it is impossible to find out whether the cloaking region exists. Here $\frac{\partial}{\partial n}$ is the normal derivative to ∂D.

4. More general equations

Results of §2 can be easily extended to the case of more general equations.

Consider a strictly hyperbolic equation in $\mathbb{R} \times D$ of the form

$$Lu \overset{def}{=} \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n g(x_0, x)}} \frac{\partial}{\partial x_j}\left(\sqrt{(-1)^n g(x_0, x)} \; g^{jk}(x_0, x)\frac{\partial u(x_0, x)}{\partial x_k}\right) = 0, \tag{4.1}$$

where $g^{-1}(x_0, x) = \det[g^{jk}(x_0, x)]_{j,k=0}^n$, $g^{00}(x,t) > 0$, $\det[g^{jk}(x_0, x)]_{j,k=1}^n \neq 0$.

We assume that the initial and boundary conditions (2.2) and (2.3) are satisfied. Note that we allow the metric to be time-dependent.

We assume also that the boundary $\mathbb{R} \times \partial D$ is time-like, i.e.

$$\sum_{j,k=0}^n g^{jk}(x_0, x)\nu_j \nu_k < 0 \quad \text{on } \mathbb{R} \times \partial D, \tag{4.2}$$

where $(\nu_0, \nu_1, ..., \nu_n)$ is the normal to $\mathbb{R} \times \partial D$. Then the initial-boundary value problem (4.1), (2.2) and (2.3) is well-posed (cf. [9], §23.2).

Consider the change of variables

$$y_0 = \varphi_0(x_0, x), \quad y_k = x_k, \ 1 \leq k \leq n. \tag{4.3}$$

The Equation (4.1) has the following form in (y_0, y_n) coordinates (cf. (2.5)):

$$\hat{L}v \stackrel{def}{=} \sum_{j,k=0}^n \frac{1}{\sqrt{(-1)^n \hat{g}(y_0, y)}} \frac{\partial}{\partial y_j}\left(\sqrt{(-1)^n \hat{g}(y_0, y)} \ \hat{g}^{jk}(y_0, y)\frac{\partial v(y_0, y)}{\partial y_k}\right) = 0, \tag{4.4}$$

where

$$\hat{g}^{jk}(y_0, y) = g^{jk}(x_0, x), \quad 1 \leq j, k \leq n, \tag{4.5}$$

$$\hat{g}^{00}(y_0, y) = \sum_{p,r=0}^n g^{pr}(x_0, x)\frac{\partial\varphi_0}{\partial x_p}\frac{\partial\varphi_0}{\partial x_r}, \tag{4.6}$$

$$\hat{g}^{0j}(y_0, y) = \sum_{p=0}^n g^{pj}(x_0, x)\frac{\partial\varphi_0}{\partial x_p}. \tag{4.7}$$

We consider the case when $\varphi_0(x_0, x)$ is arbitrarily strictly increasing in x_0 piecewise smooth function having a jump at $x_0 = 0$.

We get from $u(x_0, x) = v(y_0, y)$ as in §2

$$u_{x_j} = v_{y_j} + \varphi_{0x_j}v_{y_0}, \ 1 \leq j \leq n, \quad u_{x_0} = \varphi_{0x_0}v_{y_0}. \tag{4.8}$$

Therefore Equation (4.4) can be written in the form similar to (2.5).

Let $x_0 = \psi(y_0, y)$ be the inverse to $\varphi_0(x_0, y)$, i.e. $\varphi_0(\psi(y_0, y), y) = y_0$. As in (2.6) the Equation (4.4) will be hyperbolic with respect to y_0 if (cf. [9], §23.2)

$$\tilde{g}^{00}(y_0, y_0) = \sum_{p,r=0}^n g^{pr}\frac{\partial\varphi_0}{\partial x_r}\frac{\partial\varphi_0}{\partial x_p} > 0. \tag{4.9}$$

Note that (4.9) coincides with (2.6) when (4.1) has the form (2.1).

To simplify the condition (4.9), consider a particular case when $g^{00} > 0, g^{0j} = g^{j0} = 0, \ 1 \leq j \leq n$, and

$$\sum_{j,k=1}^n g^{jk}\xi_j\xi_k \leq -C_0(\xi_1^2 + \xi_2^2 + \xi_n^2), \tag{4.10}$$

i.e. the case when the spatial part of the Equation (4.1) is elliptic. Let $\varphi_0 = y_o - c(y)$ as in §2. Then the inequality (4.9) has the form

$$g^{00} + \sum_{j,k=0}^n g^{jk}c_{y_j}c_{y_k} > 0. \tag{4.11}$$

Thus, we get from (4.10) and (4.11) that $|c_y|^2 \leq \frac{1}{C_0} g^{00}$.

For the well-posedness of the initial-boundary value problem, the boundary $\mathbb{R} \times \partial D$ must be time-like, i.e.

$$\sum_{j,k=0}^{n} \hat{g}^{jk}(y) \nu_j \nu_k < 0 \quad \text{on} \quad \mathbb{R} \times \partial D. \tag{4.12}$$

Note that (4.12) follows from (4.2) since $\nu_0 = 0$ and $\hat{g}^{jk} = g^{jk}$ for $1 \leq j, k \leq n$. Therefore Theorem 2.1 holds also for the Equation (4.1), assuming that (4.9) holds.

Similar results hold for the hyperbolic systems, for example, the elasticity equations.

REFERENCES

[1] T. Ergin, N. Stenger, P. Brenner, J. Pendry, and M. Wegener, *The three-dimensional invisibility cloak at optical wave lengths*, Science, 328(5976):337–339, 2010. 1
[2] G. Eskin, *Lectures on Linear Partial Differential Equations*, AMS, GSM, 123, 2011. 2
[3] M. Fridman et al, *Demonstration of temporal cloaking*, Nature, 481:62–65, 2012.
[4] A. Greenleaf, M. Lassas, and G. Uhlmann, *On nonuniqueness for Calderon's inverse problem*, Math. Res. Lett., 10:685–693, 2003. 1
[5] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, *Full-wave invisibility of active devices at all frequencies*, Comm. Math. Phys., 275(3):749–789, 2007. 1
[6] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, *Invisibility and inverse problems*, Bull. Amer. Math. Soc., 46:55–97, 2009. 1
[7] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, *Cloaking devices, electromagnetic wormholes and transformation optics*, SIAM Review, 51:3–33, 2009. 1
[8] J. Gratus, P. Kinsler, M. McCall, and R. Thompson, *On spacetime transformation Optics in Temporal and Spatial Dispersion*, New J. Phys., 18:1367–2630, 2016. 1
[9] L. Hormander, *The Analysis of Linear Partial Differential Operators I*, Springer Verlag, 3(24), 1985. 2
[10] P. Kinsler and M. McCall, *Cloaks, editors and bubbles: application of spacetime transformation theory*, Ann. Phys. (Berlin), 526(1-2):51–62, 2014. 1
[11] P. Kinsler and M. McCall, *Generalized transformation design: metrics, speeds and diffusion*, Wave Motion, 77:91–106, 2018. 1
[12] U. Leonhardt, *Optimal conformal mapping*, Science, 312:1777–1780, 2006. 1
[13] H. Liu and G. Uhlmann, *Regularized transformation optics cloaking in acoustic and electromagnetic scattering*, Inverse problems and imaging, Panor Syntheses, Soc. Math. France, Paris, 44:111–136, 2015. 1
[14] M. McCall et al, *A spacetime cloak, or a history editor*, Journal of Optics, 13(2):024003, 2011. 1
[15] J.B. Pendry, D. Schurig, and D.R. Smith, *Controlling electromagnetic fields*, Science, 312:1780–1782, 2006. 1
[16] D. Schurig, J. Mock, B. Justice, S. Cummer, J. Pendry, A. Starr, and D. Smith, *Metamaterial electromagnetic cloak at microwave frequencies*, Science, 314:977–980, 2006. 1