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GLOBAL SOLUTIONS OF 3D PARTIALLY DAMPED EULER-POISSON TWO-FLUID SYSTEM*

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Abstract. In this paper, we prove the global existence of small solutions to partially damped Euler-Poisson "two fluid" system in three dimensional periodic domain. Different from the previous "two fluid" Euler-Poisson systems, our model describes two fluids that obey different dynamical evolutions, one is compressible Euler and another is compressible Euler with damping. We use a carefully designed time-weighted energy frame to prove our theorem.

Keywords. Euler–Maxwell two-fluid model; dissipative analysis; global regularity.

AMS subject classifications. 35Q35; 35Q31; 76N10.

1. Introduction

In this paper, we consider the initial-boundary value problem with periodic boundary conditions for a "two-fluid" partially damped Euler-Poisson model. The system describes the dynamical evolution of the functions $\rho_\pm, u_\pm$ and Φ in the following

$$\begin{cases} \partial_t \rho_+ + \nabla\cdot(\rho_+ u_+) = 0, \\ \partial_t \rho_- + \nabla\cdot(\rho_- u_-) = 0, \\ \rho_+ m_+ (\partial_t u_+ + u_+\cdot\nabla u_+) + \rho_+ m_+ u_+ + \nabla p_+(\rho_+) = Ze\rho_+\nabla\Phi, \\ \rho_- m_- (\partial_t u_- + u_-\cdot\nabla u_-) + \nabla p_-(\rho_-) = -e\rho_-\nabla\Phi, \\ -\Delta\Phi = 4\pi e(\rho_- - Z\rho_+), \\ \nabla\times u_- = 0, \end{cases} \tag{1.1}$$

for $(t,x)\in(0,+\infty)\times\mathbb{T}^3$. These equations describe a plasma composed of electrons and ions. Here electrons have charge $-e$, density ρ_-, mass m_-, velocity u_-, pressure p_-, and the ions have charge Ze, density ρ_+, mass m_+, velocity u_+, pressure p_+. We have assumed that pressure $p_\pm$ is a smooth function depending only on the density $\rho_\pm$ and satisfies $p'_\pm(\rho_\pm)>0$, for $\rho_\pm>0$. The two fluids interact through the self-consistent electric field $E=\nabla\Phi$.

We refer to [1,3,4,23] for more physical background about the plasma which can be seen as a collection of fast-moving charged particles. We call such a system a "two-fluid" model since at high temperature and velocity, ions and electrons in a plasma tend to become two separate fluids. More exactly, two compressible Euler equations separately. When the electromagnetic field is governed by the Maxwell equation, the system becomes the famous Euler-Maxwell model. While the Maxwell equation is replaced by a simpler Poisson equation, it then turns to the situation that we are concerned with in this paper.

Extensively impressive progress had been achieved in the past decades for these well known models and some related simplified systems. In the physical case, the mass of ions m_+ is much bigger than the mass of electrons m_-. Thus, we can set $m_+=1, m_-=0$

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or $m_+=\infty, m_-=1$ which leads to so called "one fluid" models. In the case of Euler-Poisson model for the electrons, Guo [9] proved the global existence of small, neutral and irrotational solutions using the normal form method. The neutral assumption was later removed in [8]. And we refer to [14,16–18] for the two dimensional case. While in the case of Euler-Poisson model for the ions, [13] gave the similar results as in [9]. On the other hand, in the case of Euler-Maxwell model for the electrons, [7] gave the global existence of small neutral and irrotational solutions under additional generic conditions which was removed later in [15]. Recently, the result for one dimensional space is shown in [10].

For the "two-fluid" model, the global solutions of 3D Euler-Maxwell system was solved in [12] and the Euler-Poisson system was proved in [11]. Also, we refer to [2, 5, 6, 20, 21, 24–28] for the damped systems. By the line, all these results are concerned with "two-fluid" systems in which electrons and ions behave as the same equation while system (1.1) describes two fluids that obey different dynamical evolutions and interact with each other through a Poisson equation. One is compressible Euler equation and another is damped compressible Euler equation. We call this system partially damped Euler-Poisson system. And we want to derive the global existence of small, neutral solutions to this system on periodic domains in 3D.

Before dealing with our main result, we first normalize the original system (1.1) to a simpler one

$$\begin{cases} \partial_t\rho_+ + \nabla\cdot(\rho_+u_+) = 0, \\ \partial_t\rho_- + \nabla\cdot(\rho_-u_-) = 0, \\ \rho_+(\partial_t u_+ + u_+\cdot\nabla u_+) + \rho_+u_+ + \nabla p(\rho_+) = -\rho_+\nabla\Phi, \\ \rho_-(\partial_t u_- + u_-\cdot\nabla u_-) + \nabla q(\rho_-) = \rho_-\nabla\Phi, \\ -\Delta\Phi = \rho_+ - \rho_-, \\ \nabla\times u_- = 0, \end{cases} \tag{1.2}$$

here we still use the same notations to present the new quantities for convenience. This system is somehow partially dissipative and lacks symmetries. It doesn't satisfy the so-called Kawashima condition [22] even if the two fluids share the same damped Euler equation. And so far there doesn't exist a general theory to give the global existence of solutions to such partially dissipative systems without the Kawashima condition.

Now we consider the system in a neighborhood of the constant solution $(\rho_+,\rho_-,u_+,u_-) = (1,1,0,0)$. For convenience, we set

$$\rho_+ = 1+\tau_+, \qquad \rho_- = 1+\tau_-,$$

and the system (1.2) turns to be

$$\begin{cases} \partial_t\tau_+ + \nabla\cdot u_+ = -\nabla\cdot(\tau_+u_+), \\ \partial_t\tau_- + \nabla\cdot u_- = -\nabla\cdot(\tau_-u_-), \\ \partial_t u_+ + u_+ + \nabla\tau_+ = -u_+\cdot\nabla u_+ + (1-\frac{p_+'(\tau_++1)}{\tau_++1})\nabla\tau_+ + \nabla\Phi, \\ \partial_t u_- + \nabla\tau_- = -u_-\cdot\nabla u_- + (1-\frac{p_-'(\tau_-+1)}{\tau_-+1})\nabla\tau_- - \nabla\Phi, \\ -\Delta\Phi = \tau_- - \tau_+, \\ \nabla\times u_- = 0. \end{cases} \tag{1.3}$$

Without loss of generality, we can assume $p_+'(1)=1$ and $p_-'(1)=1$. The initial data are given as

$$t=0:\quad \rho_+=\tau_{+0}(x)+1,\quad \rho_-=\tau_{-0}(x)+1,\\ u_+=u_{+0}(x),\qquad u_-=\nabla\psi_0(x),\qquad x\in\mathbb{T}^3 \tag{1.4}$$

together with the neutral condition,

$$\int_{\mathbb{T}^3}\tau_{+0}(x)\,dx=\int_{\mathbb{T}^3}\tau_{-0}(x)\,dx=0. \tag{1.5}$$

Now we can state our main result concerning the partially damped Euler-Poisson system.

THEOREM 1.1. *Consider system (1.3) with smooth initial data (1.4)-(1.5). Then, there exists a small constant ϵ such that system (1.3) admits a unique global classical solution provided that*

$$\|\tau_{+0}\|_{H^s}+\|\tau_{-0}\|_{H^s}+\|u_{+0}\|_{H^s}+\|\psi_0\|_{H^{s+1}}\le\epsilon,$$

where $s\ge 25$ is an integer.

2. Time-weighted energy frame for partially dissipative systems

The inspiration of this paper comes from the authors' previous works about some partially dissipative models, such as the magnetohydrodynamics (MHD) systems [19,29] and the Oldroyd-B models [30]. These systems are usually strongly coupled and involve two entirely different phenomenon, one is no dissipation and no relaxation, another is somehow dissipative. Our model (1.1) is systematically different from all the previous related "two-fluid" models, no matter Euler-Maxwell or Euler-Poisson. Mathematically, our model lacks symmetries which leads to the main difficulty that we are concerned with.

Now, let us develop a flexible energy frame which can reveal the coupling between different variables. Like the process when we deal with MHD and Oldroyd-B models, our first step is to analyze the dissipation mechanism of all terms included, during which, the linearized system plays an important role as well as the coupling between different quantities. Then we try to build a carefully designed time-weighted energy frame in the hope that all energies will close. One key point is to notice the transmission of regularity for each term. Such energy frame not only shows the decay properties of different level energies, but also gives the way how dissipation transfers in system. Next, we turn to our system concerned in this paper.

We just deal with the system (1.3) and introduce the setting of energy frame in the following. For $s\in\mathbb{N}$, the norm $\|f\|^2_{\tilde{H}^s}$ is defined by $\|f\|^2_{\tilde{H}^s}=\sum_{0\le k\le s}\|\tilde{\nabla}^k f\|^2_{L^2}$, where $\tilde{\nabla}=(\partial_t,\nabla)$ and $\nabla=(\partial_1,\partial_2,\partial_3)$. For solutions of system (1.3) on $[0,T]\times\mathbb{T}^3$, we define the energies $\mathcal{E}_\theta,\mathcal{D}_\theta,\mathcal{H}_\theta$ $(\theta=1,2,3)$ as follows

$$\begin{aligned}\mathcal{E}_\theta(T)&=\sup_{0\le t\le T}(1+t)^{\theta-1}\Big\{\|\tau_+\|^2_{\tilde{H}^{s-3(\theta-1)}}+\|\tau_-\|^2_{\tilde{H}^{s-3(\theta-1)}}+\|u_+\|^2_{\tilde{H}^{s-3(\theta-1)}}\\&\qquad+\|u_-\|^2_{\tilde{H}^{s-3(\theta-1)}}\Big\}+\int_0^T(1+t)^\theta\|u_+\|^2_{\tilde{H}^{s-3(\theta-1)}}\,dt,\\ \mathcal{D}_\theta(T)&=\int_0^T(1+t)^{\theta-1}\|u_-\|^2_{\tilde{H}^{s-2-3(\theta-1)}}\,dt,\\ \mathcal{H}_\theta(T)&=\int_0^T(1+t)^{\theta-1}\Big(\|\tau_+\|^2_{\tilde{H}^{s-3\theta}}+\|\tau_-\|^2_{\tilde{H}^{s-3\theta}}\Big)\,dt.\end{aligned} \tag{2.1}$$

In the following, we will successively derive the estimates for the nine energies stated above.

3. *A priori* estimates

3.1. Preliminaries. In this subsection, we will introduce some useful propositions which play an important role during the proof.

PROPOSITION 3.1. *For any smooth functions f and g on $\mathbb{T}^3$, we have the following product estimate*

$$\|\tilde{\nabla}^k(fg)\|_{L^2} \lesssim \|f\|_{\tilde{H}^{\lfloor\frac{k}{2}\rfloor+2}}\|g\|_{\tilde{H}^k} + \|g\|_{\tilde{H}^{\lfloor\frac{k}{2}\rfloor+2}}\|f\|_{\tilde{H}^k}, \qquad k\in\mathbb{N}.$$

Here, notation $\lfloor\frac{k}{2}\rfloor$ means the maximum integer not larger than $\frac{k}{2}$.

Proof. The proof of this proposition is standard. For the completeness of this paper, we shall give a simple proof. For $k\in\mathbb{N}$, direct calculation yields

$$\begin{aligned}\|\tilde{\nabla}^k(fg)\|_{L^2} &= \sum_{0\le k'\le k}\|\tilde{\nabla}^{k'}f\tilde{\nabla}^{k-k'}g\|_{L^2}\\ &= \sum_{0\le k'\le\lfloor\frac{k}{2}\rfloor}\|\tilde{\nabla}^{k'}f\tilde{\nabla}^{k-k'}g\|_{L^2} + \sum_{\lfloor\frac{k}{2}\rfloor<k'\le k}\|\tilde{\nabla}^{k'}f\tilde{\nabla}^{k-k'}g\|_{L^2}.\end{aligned}$$

Then using Hölder's inequality and the Sobolev imbedding theorem, we can derive

$$\begin{aligned}\|\tilde{\nabla}^k(fg)\|_{L^2} &\lesssim \sum_{0\le k'\le\lfloor\frac{k}{2}\rfloor}\|\tilde{\nabla}^{k'}f\|_{L^\infty}\|\tilde{\nabla}^{k-k'}g\|_{L^2} + \sum_{\lfloor\frac{k}{2}\rfloor<k'\le k}\|\tilde{\nabla}^{k'}f\|_{L^2}\|\tilde{\nabla}^{k-k'}g\|_{L^\infty}\\ &\lesssim \sum_{0\le k'\le\lfloor\frac{k}{2}\rfloor}\|\tilde{\nabla}^{k'}f\|_{H^2}\|\tilde{\nabla}^{k-k'}g\|_{L^2} + \sum_{\lfloor\frac{k}{2}\rfloor<k'\le k}\|\tilde{\nabla}^{k'}f\|_{L^2}\|\tilde{\nabla}^{k-k'}g\|_{H^2}\\ &\lesssim \|f\|_{\tilde{H}^{\lfloor\frac{k}{2}\rfloor+2}}\|g\|_{\tilde{H}^k} + \|g\|_{\tilde{H}^{\lfloor\frac{k}{2}\rfloor+2}}\|f\|_{\tilde{H}^k}.\end{aligned}$$

This completes the proof of Proposition 3.1. □

PROPOSITION 3.2. *Assume that $\rho(t,x)$ is a smooth function on $[0,T]\times\mathbb{T}^3$ with $\|\rho\|_{\tilde{H}^s}<1$. Let $f(\rho)$ be a smooth function of ρ with bounded derivatives. Then if $f(0)=0$, there holds*

$$\begin{aligned}\|\tilde{\nabla}^k f(\rho)\|_{L^\infty} &\lesssim \|\rho\|_{\tilde{H}^{k+2}}, \quad \forall\, 0\le k\le s-2,\\ \|\tilde{\nabla}^k f(\rho)\|_{L^2} &\lesssim \|\rho\|_{\tilde{H}^k}, \quad \forall\, 0\le k\le s.\end{aligned}$$

Proof. For $k=0$, notice the condition $f(0)=0$ and mean value theorem, we have $f(\rho)=f(0)+f'(r(\rho))\rho=f'(r(\rho))\rho$ for some function $0\le r(\rho)\le\rho$. Then, it is easy to get,

$$\begin{aligned}\|f(\rho)\|_{L^\infty} &\lesssim \|\rho\|_{L^\infty} \lesssim \|\rho\|_{\tilde{H}^2},\\ \|f(\rho)\|_{L^2} &\lesssim \|\rho\|_{L^2}.\end{aligned}$$

For $k>0$, $\tilde{\nabla}^k f(\rho)$ can be seen as the sum of the following terms,

$$f^{\beta_1,\beta_2,\cdots,\beta_m}(\rho)\,\tilde{\nabla}^{\beta_1}\rho\tilde{\nabla}^{\beta_2}\rho\cdots\tilde{\nabla}^{\beta_m}\rho,$$

where $f^{\beta_1,\beta_2,\cdots,\beta_m}(\rho)$ are some derivatives of function $f(\cdot)$ and $\beta_1+\beta_2+\cdots+\beta_m=k$, moreover $\beta_1\le\beta_2\le\cdots\le\beta_m$. Hence,

$$\tilde{\nabla}^k f(\rho)=\sum_{\lfloor\frac{k}{2}\rfloor<\gamma\le k} c_\gamma\tilde{\nabla}^\gamma\rho,$$

here c_γ is some product function containing derivatives of f and ρ. And the highest order derivative of ρ in c_γ is not larger than $\lfloor\frac{k}{2}\rfloor$. Indeed,

$$\|\tilde{\nabla}^k f(\rho)\|_{L^\infty}\lesssim\sum_{\lfloor\frac{k}{2}\rfloor<\gamma\le k}\|c_\gamma\tilde{\nabla}^\gamma\rho\|_{L^\infty}\lesssim\sum_{\lfloor\frac{k}{2}\rfloor<\gamma\le k}\|c_\gamma\|_{L^\infty}\|\tilde{\nabla}^\gamma\rho\|_{L^\infty}.$$

Notice that $f(\rho)$ is a smooth function of ρ with bounded derivatives and $\|\rho\|_{\tilde{H}^s}<1$, it holds that,

$$\|\tilde{\nabla}^k f(\rho)\|_{L^\infty}\lesssim\sum_{\lfloor\frac{k}{2}\rfloor<\gamma\le k}\|\tilde{\nabla}^\gamma\rho\|_{L^\infty}\lesssim\|\rho\|_{\tilde{H}^{k+2}}.$$

Similarly, we get,

$$\|\tilde{\nabla}^k f(\rho)\|_{L^2}\lesssim\sum_{\lfloor\frac{k}{2}\rfloor<\gamma\le k}\|c_\gamma\|_{L^\infty}\|\tilde{\nabla}^\gamma\rho\|_{L^2}\lesssim\|\rho\|_{\tilde{H}^k}.$$

This completes the proof of Proposition 3.2. □

3.2. Basic energy estimates. Before deriving the energy estimates, we always assume that $u_\pm,\tau_\pm$ are smooth solutions of system (1.3) on $[0,T]\times\mathbb{T}^3$ and

$$\|u_\pm\|_{\tilde{H}^s}+\|\tau_\pm\|_{\tilde{H}^s}\ll 1, \tag{3.1}$$

here $s\ge 25$ is given in Theorem 1.1. Also, by the setting of initial data (1.4)-(1.5), the periodic boundary condition and system (1.3), we have

$$\begin{aligned}&\int_{\mathbb{T}^3}\tau_+(t,x)\,dx=\int_{\mathbb{T}^3}\tau_-(t,x)\,dx=0,\quad\forall t\in[0,T],\\&u_-(t,x)=\nabla\psi(t,x),\quad\forall(t,x)\in[0,T]\times\mathbb{T}^3,\end{aligned}\tag{3.2}$$

for some scalar function $\psi(t,x)$.

In the following, we first turn to the basic energies $\mathcal{E}_1,\mathcal{D}_1,\mathcal{H}_1$ and give the related lemmas.

Lemma 3.1. *Assume that energies are defined in (2.1), we then have*

$$\mathcal{E}_1(T)\lesssim\mathcal{E}_1(0)+\mathcal{E}_1^{3/2}(T)+\mathcal{E}_3^{3/2}(T)+\mathcal{D}_3^{3/2}(T)+\mathcal{H}_3^{3/2}(T).$$

Proof. Applying $\tilde{\nabla}^k$ derivative ($0\le k\le s$) on system (1.3), taking inner product with $\tilde{\nabla}^k\tau_+$ for the first equation, $\tilde{\nabla}^k\tau_-$ for the second equation, $\tilde{\nabla}^k u_+$ for the third equation and $\tilde{\nabla}^k u_-$ for the fourth equation . Adding them up, we shall get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\{\|\tau_+\|_{\tilde{H}^s}^2+\|\tau_-\|_{\tilde{H}^s}^2+\|u_+\|_{\tilde{H}^s}^2+\|u_-\|_{\tilde{H}^s}^2\Big\}+\|u_+\|_{\tilde{H}^s}^2\\=&I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8,\end{aligned}\tag{3.3}$$

where

$$
\begin{aligned}
I_1 =& -\int_{\mathbb{T}^3} \sum_{0\le k\le s} \tilde{\nabla}^k \nabla\cdot u_+ \tilde{\nabla}^k \tau_+ - \tilde{\nabla}^k \nabla \tau_+ \tilde{\nabla}^k u_+ \, dx \\
& -\int_{\mathbb{T}^3} \sum_{0\le k\le s} \tilde{\nabla}^k \nabla\cdot u_- \tilde{\nabla}^k \tau_- - \tilde{\nabla}^k \nabla \tau_- \tilde{\nabla}^k u_- \, dx, \\
I_2 =& \int_{\mathbb{T}^3} \sum_{0\le k\le s} \tilde{\nabla}^k \nabla\Phi \tilde{\nabla}^k u_+ - \tilde{\nabla}^k \nabla\Phi \tilde{\nabla}^k u_- \, dx, \\
I_3 =& -\sum_{0\le k\le s} \int_{\mathbb{T}^3} \tilde{\nabla}^k \nabla\cdot(\tau_+ u_+) \tilde{\nabla}^k \tau_+ + \tilde{\nabla}^k (\tau_+ \nabla \tau_+) \tilde{\nabla}^k u_+ \, dx, \\
I_4 =& \sum_{0\le k\le s} \int_{\mathbb{T}^3} \tilde{\nabla}^k \Big[\Big(\tau_+ + 1 - \frac{p_+'(\tau_+ + 1)}{\tau_+ + 1}\Big)\nabla\tau_+\Big] \tilde{\nabla}^k u_+ \, dx, \\
I_5 =& -\sum_{0\le k\le s} \int_{\mathbb{T}^3} \tilde{\nabla}^k (u_+ \cdot \nabla u_+) \tilde{\nabla}^k u_+ \, dx, \\
I_6 =& -\sum_{0\le k\le s} \int_{\mathbb{T}^3} \tilde{\nabla}^k \nabla\cdot(\tau_- u_-) \tilde{\nabla}^k \tau_- + \tilde{\nabla}^k (\tau_- \nabla \tau_-) \tilde{\nabla}^k u_- \, dx, \\
I_7 =& \sum_{0\le k\le s} \int_{\mathbb{T}^3} \tilde{\nabla}^k \Big[\Big(\tau_- + 1 - \frac{p_-'(\tau_- + 1)}{\tau_- + 1}\Big)\nabla\tau_-\Big] \tilde{\nabla}^k u_- \, dx, \\
I_8 =& -\sum_{0\le k\le s} \int_{\mathbb{T}^3} \tilde{\nabla}^k (u_- \cdot \nabla u_-) \tilde{\nabla}^k u_- \, dx.
\end{aligned}
$$

We shall estimate each term on the right-hand side of (3.3). First, for the term I_1, using integration by parts, we can easily derive

$$
I_1 = 0. \tag{3.4}
$$

For the second term I_2, using the evolution equations of τ_+ and τ_- in system (1.3), notice the definition of Φ, we shall get

$$
\begin{aligned}
I_2 =& \sum_{0\le k\le s} \int_{\mathbb{T}^3} \tilde{\nabla}^k \Phi (\tilde{\nabla}^k \nabla\cdot u_- - \tilde{\nabla}^k \nabla\cdot u_+) \, dx \\
=& \sum_{0\le k\le s} \int_{\mathbb{T}^3} \tilde{\nabla}^k \Phi \partial_t \tilde{\nabla}^k (\tau_+ - \tau_-) + \tilde{\nabla}^k \Phi \tilde{\nabla}^k \nabla\cdot(\tau_+ u_+) - \tilde{\nabla}^k \Phi \tilde{\nabla}^k \nabla\cdot(\tau_- u_-) \, dx \\
=& -\frac{1}{2}\frac{d}{dt} \| |\nabla|^{-1} (\tau_+ - \tau_-) \|_{\tilde{H}^s}^2 + I_{2,2}.
\end{aligned}
$$

Here and in what follows, for a function $f(x)$ on $\mathbb{T}^3$ with the integral $\int_{\mathbb{T}^3} f(x)\,dx = 0$, we use $|\nabla|^{-1} f$ to denote the negative order derivative

$$
|\nabla|^{-1} f(x) = \sum_{z\in l^3} \frac{1}{|z|} \hat{f}_z e^{iz\cdot x}.
$$

The notations for other various operations are similar. Now we turn to the second part

$I_{2,2}$, using Proposition 3.1 we have

$$
\begin{aligned}
|I_{2,2}| \lesssim & \int_{\mathbb{T}^3} \|\Phi\|_{\tilde{H}^{s+1}} \big(\|\tau_+ u_+\|_{\tilde{H}^s} + \|\tau_- u_-\|_{\tilde{H}^s}\big)\, dx \\
\lesssim & \||\nabla|^{-1}(\tau_+ - \tau_-)\|_{\tilde{H}^s} \big(\|\tau_+\|_{\tilde{H}^{\lfloor \frac{s}{2} \rfloor+2}} \|u_+\|_{\tilde{H}^s} + \|\tau_+\|_{\tilde{H}^s} \|u_+\|_{\tilde{H}^{\lfloor \frac{s}{2} \rfloor+2}} \\
& + \|\tau_-\|_{\tilde{H}^{\lfloor \frac{s}{2} \rfloor+2}} \|u_-\|_{\tilde{H}^s} + \|\tau_-\|_{\tilde{H}^s} \|u_-\|_{\tilde{H}^{\lfloor \frac{s}{2} \rfloor+2}}\big) \\
\lesssim & \||\nabla|^{-1}(\tau_+ - \tau_-)\|_{\tilde{H}^s} \big(\|\tau_+\|_{\tilde{H}^{s-9}} \|u_+\|_{\tilde{H}^s} + \|\tau_+\|_{\tilde{H}^s} \|u_+\|_{\tilde{H}^{s-6}} \\
& + \|\tau_-\|_{\tilde{H}^{s-9}} \|u_-\|_{\tilde{H}^s} + \|\tau_-\|_{\tilde{H}^s} \|u_-\|_{\tilde{H}^{s-8}}\big).
\end{aligned}
$$

Thus, by (3.2) and Poincaré's inequality, we can bound

$$
\begin{aligned}
\int_0^T |I_{2,2}(t)|\, dt \lesssim & \sup_{0\le t\le T} \||\nabla|^{-1}(\tau_+ - \tau_-)\|_{\tilde{H}^s} \big(\|u_+\|_{\tilde{H}^s} + \|u_-\|_{\tilde{H}^s} + \|\tau_+\|_{\tilde{H}^s} + \|\tau_-\|_{\tilde{H}^s}\big) \\
& \cdot \int_0^T \big(\|u_+\|_{\tilde{H}^{s-6}} + \|u_-\|_{\tilde{H}^{s-8}} + \|\tau_+\|_{\tilde{H}^{s-9}} + \|\tau_-\|_{\tilde{H}^{s-9}}\big)\, dt \\
\lesssim & \mathcal{E}_1(T)\big(\mathcal{E}_3^{1/2}(T) + \mathcal{D}_3^{1/2}(T) + \mathcal{H}_3^{1/2}(T)\big). \qquad (3.5)
\end{aligned}
$$

For the third term I_3, we have the following computation,

$$
\begin{aligned}
|I_3| \lesssim & \Big| \sum_{0\le k\le s} \int_{\mathbb{T}^3} \tilde{\nabla}^k (u_+ \cdot \nabla \tau_+) \tilde{\nabla}^k \tau_+ + \tilde{\nabla}^k (\tau_+ \nabla \cdot u_+) \tilde{\nabla}^k \tau_+ \, dx \\
& + \int_{\mathbb{T}^3} \sum_{0\le k\le s} \sum_{1\le k'\le k} \tilde{\nabla}^{k'} \tau_+ \tilde{\nabla}^{k-k'} \nabla \tau_+ \tilde{\nabla}^k u_+ + \sum_{0\le k\le s} \tau_+ \tilde{\nabla}^k \nabla \tau_+ \tilde{\nabla}^k u_+ \, dx \Big| \\
\lesssim & \Big| \int_{\mathbb{T}^3} \sum_{1\le k\le s} \sum_{1\le k_1\le k} \tilde{\nabla}^{k_1} u_+ \cdot \tilde{\nabla}^{k-k_1} \nabla \tau_+ \tilde{\nabla}^k \tau_+ + \sum_{0\le k\le s} u_+ \cdot \nabla \tilde{\nabla}^k \tau_+ \tilde{\nabla}^k \tau_+ \, dx \\
& + \sum_{0\le k\le s} \int_{\mathbb{T}^3} \tau_+ \tilde{\nabla}^k \nabla \cdot u_+ \tilde{\nabla}^k \tau_+ - \big(\tau_+ \tilde{\nabla}^k \tau_+ \tilde{\nabla}^k \nabla \cdot u_+ + \nabla \tau_+ \tilde{\nabla}^k \tau_+ \tilde{\nabla}^k u_+\big)\, dx \Big| \\
& + \Big| \sum_{0\le k\le s} \sum_{1\le k'\le k} \int_{\mathbb{T}^3} \tilde{\nabla}^{k'} \tau_+ \tilde{\nabla}^{k-k'} \nabla \cdot u_+ \tilde{\nabla}^k \tau_+ + \tilde{\nabla}^{k'} \tau_+ \tilde{\nabla}^{k-k'} \nabla \tau_+ \tilde{\nabla}^k u_+ \, dx \Big|.
\end{aligned}
$$

Hence, using Proposition 3.1 and the Sobolev imbedding theorem, it holds that

$$
\begin{aligned}
|I_3| \lesssim & \big(\|\tilde{\nabla} u_+\|_{\tilde{H}^{\lfloor \frac{s-1}{2} \rfloor+2}} \|\nabla \tau_+\|_{\tilde{H}^{s-1}} + \|\nabla \tau_+\|_{\tilde{H}^{\lfloor \frac{s-1}{2} \rfloor+2}} \|\tilde{\nabla} u_+\|_{\tilde{H}^{s-1}}\big) \|\tau_+\|_{\tilde{H}^s} \\
& + \|\nabla \cdot u_+\|_{L^\infty} \|\tau_+\|_{\tilde{H}^s}^2 + \|\nabla \tau_+\|_{L^\infty} \|\tau_+\|_{\tilde{H}^s} \|u_+\|_{\tilde{H}^s} \\
& + \big(\|\tilde{\nabla} \tau_+\|_{\tilde{H}^{\lfloor \frac{s-1}{2} \rfloor+2}} \|\nabla \cdot u_+\|_{\tilde{H}^{s-1}} + \|\nabla \cdot u_+\|_{\tilde{H}^{\lfloor \frac{s-1}{2} \rfloor+2}} \|\tilde{\nabla} \tau_+\|_{\tilde{H}^{s-1}}\big) \|\tau_+\|_{\tilde{H}^s} \\
& + \|\tilde{\nabla} \tau_+\|_{\tilde{H}^{\lfloor \frac{s-1}{2} \rfloor+2}} \|\tilde{\nabla} \tau_+\|_{\tilde{H}^{s-1}} \|u_+\|_{\tilde{H}^s} \\
\lesssim & \big(\|u_+\|_{\tilde{H}^{s-6}} + \|\tau_+\|_{\tilde{H}^{s-9}}\big) \big(\|u_+\|_{\tilde{H}^s}^2 + \|\tau_+\|_{\tilde{H}^s}^2\big).
\end{aligned}
$$

Obviously, we can directly get the estimate for I_3,

$$
\begin{aligned}
\int_0^T |I_3(t)|\, dt \lesssim & \sup_{0\le t\le T} \big(\|u_+\|_{\tilde{H}^s}^2 + \|\tau_+\|_{\tilde{H}^s}^2\big) \int_0^T \big(\|u_+\|_{\tilde{H}^{s-6}} + \|\tau_+\|_{\tilde{H}^{s-9}}\big)\, dt \\
\lesssim & \mathcal{E}_1(T)\big(\mathcal{E}_3^{1/2}(T) + \mathcal{H}_3^{1/2}(T)\big). \qquad (3.6)
\end{aligned}
$$

For the term I_4, we define $g_+(\tau_+) = \left(\tau_+ + 1 - \frac{p_+'(\tau_+ + 1)}{\tau_+ + 1}\right)$ and have the following

$$
\begin{aligned}
I_4 = & \sum_{0\le k\le s-1} \int_{\mathbb{T}^3} \tilde\nabla^k (g_+(\tau_+)\nabla\tau_+)\tilde\nabla^k u_+\,dx + \int_{\mathbb{T}^3} \tilde\nabla^s (g_+(\tau_+)\nabla\tau_+)\tilde\nabla^s u_+\,dx \\
= & \sum_{0\le k\le s-1} \int_{\mathbb{T}^3} \tilde\nabla^k (g_+(\tau_+)\nabla\tau_+)\tilde\nabla^k u_+\,dx + \sum_{1\le k_1\le s} \int_{\mathbb{T}^3} \tilde\nabla^{k_1} g_+(\tau_+)\tilde\nabla^{s-k_1}\nabla\tau_+\tilde\nabla^s u_+\,dx \\
& + \int_{\mathbb{T}^3} (g_+(\tau_+)\tilde\nabla^s\nabla\tau_+)\tilde\nabla^s u_+\,dx \\
= & \sum_{0\le k\le s-1} \int_{\mathbb{T}^3} \tilde\nabla^k (g_+(\tau_+)\nabla\tau_+)\tilde\nabla^k u_+\,dx + \sum_{1\le k_1\le s} \int_{\mathbb{T}^3} \tilde\nabla^{k_1} g_+(\tau_+)\tilde\nabla^{s-k_1}\nabla\tau_+\tilde\nabla^s u_+\,dx \\
& - \int_{\mathbb{T}^3} (\nabla g_+(\tau_+)\tilde\nabla^s\tau_+)\tilde\nabla^s u_+\,dx - \int_{\mathbb{T}^3} (g_+(\tau_+)\tilde\nabla^s\tau_+)\tilde\nabla^s\nabla\cdot u_+\,dx \\
= & I_{4,1} + I_{4,2}.
\end{aligned}
$$

Here, $I_{4,1}$ contains the first three terms on the right-hand side above and $I_{4,2}$ denotes the last term. Using Proposition 3.1 and Proposition 3.2, we have

$$
\begin{aligned}
|I_{4,1}| \lesssim & \left(\|g_+(\tau_+)\|_{\tilde H^{\lfloor\frac{s-1}{2}\rfloor+2}}\|\nabla\tau_+\|_{\tilde H^{s-1}} + \|g_+(\tau_+)\|_{\tilde H^{s-1}}\|\nabla\tau_+\|_{\tilde H^{\lfloor\frac{s-1}{2}\rfloor+2}}\right)\|u_+\|_{\tilde H^{s-1}} \\
& + \left(\|\tilde\nabla g_+(\tau_+)\|_{\tilde H^{\lfloor\frac{s-1}{2}\rfloor+2}}\|\nabla\tau_+\|_{\tilde H^{s-1}} + \|\nabla\tau_+\|_{\tilde H^{\lfloor\frac{s-1}{2}\rfloor+2}}\|\tilde\nabla g_+(\tau_+)\|_{\tilde H^{s-1}}\right)\|u_+\|_{\tilde H^s} \\
& + \|\nabla g_+(\tau_+)\|_{L^\infty}\|\tau_+\|_{\tilde H^s}\|u_+\|_{\tilde H^s} \\
\lesssim & \|\tau_+\|_{\tilde H^{\lfloor\frac{s-1}{2}\rfloor+3}}\|\tau_+\|_{\tilde H^s}\|u_+\|_{\tilde H^s} \\
\lesssim & \|\tau_+\|_{\tilde H^{s-9}}\|\tau_+\|_{\tilde H^s}\|u_+\|_{\tilde H^s}.
\end{aligned}
$$

Notice here, we use the fact that $g_+(\cdot)$ is a smooth function around zero and $g_+(0) = 0$. Indeed,

$$
\begin{aligned}
\int_0^T |I_{4,1}(t)|dt \le & \sup_{0\le t\le T}\|\tau_+\|_{\tilde H^s}\|u_+\|_{\tilde H^s}\int_0^T \|\tau_+\|_{\tilde H^{s-9}}dt \\
\le & \mathcal{E}_1(T)\mathcal{H}_3^{1/2}(T).
\end{aligned} \tag{3.7}
$$

When dealing with $I_{4,2}$, we should notice the following equality,

$$
-\nabla\cdot u_+ = \frac{\partial_t\tau_+ + u_+\cdot\nabla\tau_+}{\tau_+ + 1},
$$

then $I_{4,2}$ becomes,

$$
\begin{aligned}
I_{4,2} = & \int_{\mathbb{T}^3} (g_+(\tau_+)\tilde\nabla^s\tau_+)\tilde\nabla^s\Big(\frac{\partial_t\tau_+ + u_+\cdot\nabla\tau_+}{\tau_+ + 1}\Big)dx \\
= & \int_{\mathbb{T}^3} \frac{g_+(\tau_+)}{\tau_+ + 1}\tilde\nabla^s\tau_+\Big(\partial_t\tilde\nabla^s\tau_+ + u_+\cdot\nabla\tilde\nabla^s\tau_+ + \sum_{1\le k_1\le s}\tilde\nabla^{k_1}u_+\cdot\tilde\nabla^{s-k_1}\nabla\tau_+\Big)\,dx \\
& + \sum_{1\le k_1\le s}\int_{\mathbb{T}^3} g_+(\tau_+)\tilde\nabla^s\tau_+\tilde\nabla^{k_1}\frac{1}{\tau_+ + 1}\tilde\nabla^{s-k_1}(\partial_t\tau_+ + u_+\cdot\nabla\tau_+)dx \\
= & \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} h_+(\tau_+)|\tilde\nabla^s\tau_+|^2\,dx - \frac{1}{2}\int_{\mathbb{T}^3} h_+'(\tau_+)\partial_t\tau_+|\tilde\nabla^s\tau_+|^2\,dx
\end{aligned}
$$

$$-\frac{1}{2}\int_{\mathbb{T}^3}\big(u_+\cdot\nabla h_+(\tau_+)+h_+(\tau_+)\nabla\cdot u_+\big)|\tilde{\nabla}^s\tau_+|^2dx$$
$$+\sum_{1\leq k_1\leq s}\int_{\mathbb{T}^3}h_+(\tau_+)\tilde{\nabla}^s\tau_+\big(\tilde{\nabla}^{k_1}u_+\cdot\tilde{\nabla}^{s-k_1}\nabla\tau_+\big)dx$$
$$+\sum_{1\leq k_1\leq s}\int_{\mathbb{T}^3}g_+(\tau_+)\tilde{\nabla}^s\tau_+\tilde{\nabla}^{k_1}\frac{1}{\tau_++1}\tilde{\nabla}^{s-k_1}\big(\partial_t\tau_++u_+\cdot\nabla\tau_+\big)dx,$$

here $h_+(\tau_+)=\frac{g_+(\tau_+)}{\tau_++1}$ is a smooth bounded function. By Hölder's inequality and the Sobolev imbedding theorem, we will get

$$\Big|\int_0^T I_{4,2}(t)\,dt\Big|\lesssim\sup_{0\leq t\leq T}\Big|\int_{\mathbb{T}^3}h_+(\tau_+)|\tilde{\nabla}^s\tau_+|^2\,dx\Big|+\int_0^T\|h_+'(\tau_+)\|_{L^\infty}\|\partial_t\tau_+\|_{L^\infty}\|\tau_+\|_{\tilde{H}^s}^2\,dt$$
$$+\int_0^T\|u_+\|_{W^{1,\infty}}\|h_+(\tau_+)\|_{W^{1,\infty}}\|\tau_+\|_{\tilde{H}^s}^2+\|h_+(\tau_+)\|_{L^\infty}\|\tau_+\|_{\tilde{H}^s}^2\|u_+\|_{\tilde{H}^s}\,dt$$
$$+\int_0^T\|g_+(\tau_+)\|_{L^\infty}\|\tau_+\|_{\tilde{H}^s}\|\frac{1}{\tau_++1}\|_{\tilde{H}^s}\Big(\|\tau_+\|_{\tilde{H}^s}+\|u_+\|_{\tilde{H}^s}\|\tau_+\|_{\tilde{H}^s}\Big)\,dt.$$

Notice the fact $h_+(0)=g_+(0)=0$ and the *a priori* Assumption (3.1), using Proposition 3.2 we can obtain

$$\begin{aligned}\Big|\int_0^T I_{4,2}(t)\,dt\Big|&\lesssim\sup_{0\leq t\leq T}\|h_+(\tau_+)\|_{L^\infty}\|\tau_+\|_{\tilde{H}^s}^2+\sup_{0\leq t\leq T}\|\tau_+\|_{\tilde{H}^s}^2\int_0^T\|\partial_t\tau_+\|_{L^\infty}\,dt\\&\quad+\sup_{0\leq t\leq T}\|\tau_+\|_{\tilde{H}^s}^2\int_0^T\|u_+\|_{W^{1,\infty}}dt\\&\quad+\sup_{0\leq t\leq T}\|\tau_+\|_{\tilde{H}^s}\big(\|\tau_+\|_{\tilde{H}^s}+\|u_+\|_{\tilde{H}^s}\big)\int_0^T\|\tau_+\|_{L^\infty}dt\\&\lesssim\mathcal{E}_1^{3/2}(T)+\mathcal{E}_1(T)\mathcal{H}_3^{1/2}(T)+\mathcal{E}_1(T)\mathcal{E}_3^{1/2}(T).\end{aligned}\tag{3.8}$$

Next, we turn to I_5, by Proposition 3.1 and Proposition 3.2, it is easy to get

$$\begin{aligned}I_5=&-\sum_{1\leq k\leq s}\sum_{1\leq k_1\leq k}\int_{\mathbb{T}^3}\tilde{\nabla}^{k_1}u_+\cdot\tilde{\nabla}^{k-k_1}\nabla u_+\tilde{\nabla}^k u_+-\sum_{0\leq k\leq s}\int_{\mathbb{T}^3}u_+\cdot\nabla\tilde{\nabla}^k u_+\tilde{\nabla}^k u_+\\\lesssim&\|u_+\|_{\tilde{H}^s}^3.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^T|I_5(t)|\,dt&\lesssim\sup_{0\leq t\leq T}\|u_+\|_{\tilde{H}^s}\int_0^T\|u_+\|_{\tilde{H}^s}^2\,dt\\&\lesssim\mathcal{E}_1^{3/2}(T).\end{aligned}\tag{3.9}$$

The estimate for I_6 is similar to I_3, we have the following computation

$$\begin{aligned}|I_6|\lesssim&\Big|\sum_{0\leq k\leq s}\int_{\mathbb{T}^3}\tilde{\nabla}^k(u_-\cdot\nabla\tau_-)\tilde{\nabla}^k\tau_-+\tilde{\nabla}^k(\tau_-\nabla\cdot u_-)\tilde{\nabla}^k\tau_-\,dx\\&+\int_{\mathbb{T}^3}\sum_{0\leq k\leq s}\sum_{1\leq k'\leq k}\tilde{\nabla}^{k'}\tau_-\tilde{\nabla}^{k-k'}\nabla\tau_-\tilde{\nabla}^k u_-+\sum_{0\leq k\leq s}\tau_-\tilde{\nabla}^k\nabla\tau_-\tilde{\nabla}^k u_-\,dx\Big|\end{aligned}$$

$$\begin{aligned}
\lesssim & \Big|\int_{\mathbb{T}^3}\sum_{1\le k\le s}\sum_{1\le k_1\le k}\tilde\nabla^{k_1}u_-\cdot\tilde\nabla^{k-k_1}\nabla\tau_-\tilde\nabla^k\tau_-+\sum_{0\le k\le s}u_-\cdot\nabla\tilde\nabla^k\tau_-\tilde\nabla^k\tau_-\,dx\\
&+\sum_{0\le k\le s}\int_{\mathbb{T}^3}\tau_-\tilde\nabla^k\nabla\cdot u_-\tilde\nabla^k\tau_- -\big(\tau_-\tilde\nabla^k\tau_-\tilde\nabla^k\nabla\cdot u_-+\nabla\tau_-\tilde\nabla^k\tau_-\tilde\nabla^k u_-\big)\,dx\Big|\\
&+\sum_{0\le k\le s}\sum_{1\le k'\le k}\Big|\int_{\mathbb{T}^3}\tilde\nabla^{k'}\tau_-\tilde\nabla^{k-k'}\nabla\cdot u_-\tilde\nabla^k\tau_-+\tilde\nabla^{k'}\tau_-\tilde\nabla^{k-k'}\nabla\tau_-\tilde\nabla^k u_-\,dx\Big|.
\end{aligned}$$

Similarly, by Proposition 3.1 and the Sobolev imbedding theorem we shall derive

$$\begin{aligned}
|I_6|\lesssim&\big(\|\tilde\nabla u_-\|_{\tilde H^{\lfloor\frac{s-1}{2}\rfloor+2}}\|\nabla\tau_-\|_{\tilde H^{s-1}}+\|\nabla\tau_-\|_{\tilde H^{\lfloor\frac{s-1}{2}\rfloor+2}}\|\tilde\nabla u_-\|_{\tilde H^{s-1}}\big)\|\tau_-\|_{\tilde H^s}\\
&+\|\nabla\cdot u_-\|_{L^\infty}\|\tau_-\|^2_{\tilde H^s}+\|\nabla\tau_-\|_{L^\infty}\|\tau_-\|_{\tilde H^s}\|u_-\|_{\tilde H^s}\\
&+\big(\|\tilde\nabla\tau_-\|_{\tilde H^{\lfloor\frac{s-1}{2}\rfloor+2}}\|\nabla\cdot u_-\|_{\tilde H^{s-1}}+\|\nabla\cdot u_-\|_{\tilde H^{\lfloor\frac{s-1}{2}\rfloor+2}}\|\tilde\nabla\tau_-\|_{\tilde H^{s-1}}\big)\|\tau_-\|_{\tilde H^s}\\
&+\|\tilde\nabla\tau_-\|_{\tilde H^{\lfloor\frac{s-1}{2}\rfloor+2}}\|\tilde\nabla\tau_-\|_{\tilde H^{s-1}}\|u_-\|_{\tilde H^s}\\
\lesssim&\big(\|u_-\|_{\tilde H^{s-8}}+\|\tau_-\|_{\tilde H^{s-9}}\big)\big(\|u_-\|^2_{\tilde H^s}+\|\tau_-\|^2_{\tilde H^s}\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^T|I_6(t)|\,dt\lesssim&\sup_{0\le t\le T}\big(\|u_-\|^2_{\tilde H^s}+\|\tau_-\|^2_{\tilde H^s}\big)\int_0^T\big(\|u_-\|_{\tilde H^{s-8}}+\|\tau_-\|_{\tilde H^{s-9}}\big)\,dt\\
\lesssim&\,\mathcal{E}_1(T)\big(\mathcal{D}_3^{1/2}(T)+\mathcal{H}_3^{1/2}(T)\big).
\end{aligned}\tag{3.10}$$

The estimate for I_7 is similar to I_4, we can define $g_-(\tau_-)=\Big(\tau_-+1-\frac{p'_-(\tau_-+1)}{\tau_-+1}\Big)$, it holds that

$$\begin{aligned}
I_7=&\sum_{0\le k\le s-1}\int_{\mathbb{T}^3}\tilde\nabla^k(g_-(\tau_-)\nabla\tau_-)\tilde\nabla^k u_-\,dx+\int_{\mathbb{T}^3}\tilde\nabla^s(g_-(\tau_-)\nabla\tau_-)\tilde\nabla^s u_-\,dx\\
=&\sum_{0\le k\le s-1}\int_{\mathbb{T}^3}\tilde\nabla^k(g_-(\tau_-)\nabla\tau_-)\tilde\nabla^k u_-\,dx\\
&+\sum_{1\le k_1\le s}\int_{\mathbb{T}^3}\tilde\nabla^{k_1}g_-(\tau_-)\tilde\nabla^{s-k_1}\nabla\tau_-\tilde\nabla^s u_-\,dx+\int_{\mathbb{T}^3}(g_-(\tau_-)\tilde\nabla^s\nabla\tau_-)\tilde\nabla^s u_-\,dx\\
=&\sum_{0\le k\le s-1}\int_{\mathbb{T}^3}\tilde\nabla^k(g_-(\tau_-)\nabla\tau_-)\tilde\nabla^k u_-dx+\sum_{1\le k_1\le s}\int_{\mathbb{T}^3}\tilde\nabla^{k_1}g_-(\tau_-)\tilde\nabla^{s-k_1}\nabla\tau_-\tilde\nabla^s u_-dx\\
&-\int_{\mathbb{T}^3}(\nabla g_-(\tau_-)\tilde\nabla^s\tau_-)\tilde\nabla^s u_-\,dx-\int_{\mathbb{T}^3}(g_-(\tau_-)\tilde\nabla^s\tau_-)\tilde\nabla^s\nabla\cdot u_-\,dx\\
=&I_{7,1}+I_{7,2}.
\end{aligned}$$

Here, the definition of $I_{7,1}$ and $I_{7,2}$ is similar to $I_{4,1}$ and $I_{4,2}$. Like the previous process, we directly show that

$$|I_{7,1}|\lesssim\|\tau_-\|_{\tilde H^{s-9}}\|\tau_-\|_{\tilde H^s}\|u_-\|_{\tilde H^s}.$$

Also, we have used the fact that $g_-(\cdot)$ is a smooth function around zero and $g_-(0)=0$. Hence,

$$\int_0^T|I_{7,1}(t)|dt\le\sup_{0\le t\le T}\|\tau_-\|_{\tilde H^s}\|u_-\|_{\tilde H^s}\int_0^T\|\tau_-\|_{\tilde H^{s-9}}dt$$

$$\leq \mathcal{E}_1(T)\mathcal{H}_3^{1/2}(T). \tag{3.11}$$

For the term $I_{7,2}$, we notice the equality

$$-\nabla\cdot u_- = \frac{\partial_t\tau_- + u_-\cdot\nabla\tau_-}{\tau_-+1},$$

and then $I_{7,2}$ becomes

$$\begin{aligned}
I_{7,2} =& \int_{\mathbb{T}^3} (g_-(\tau_-)\tilde{\nabla}^s\tau_-)\tilde{\nabla}^s\Big(\frac{\partial_t\tau_- + u_-\cdot\nabla\tau_-}{\tau_-+1}\Big)dx \\
=& \int_{\mathbb{T}^3} \frac{g_-(\tau_-)}{\tau_-+1}\tilde{\nabla}^s\tau_-\Big(\partial_t\tilde{\nabla}^s\tau_- + u_-\cdot\nabla\tilde{\nabla}^s\tau_- + \sum_{1\leq k_1\leq s}\tilde{\nabla}^{k_1}u_-\cdot\tilde{\nabla}^{s-k_1}\nabla\tau_-\Big)\,dx \\
&+ \sum_{1\leq k_1\leq s}\int_{\mathbb{T}^3} g_-(\tau_-)\tilde{\nabla}^s\tau_-\tilde{\nabla}^{k_1}\frac{1}{\tau_-+1}\tilde{\nabla}^{s-k_1}\big(\partial_t\tau_- + u_-\cdot\nabla\tau_-\big)dx \\
=& \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} h_-(\tau_-)|\tilde{\nabla}^s\tau_-|^2\,dx - \frac{1}{2}\int_{\mathbb{T}^3} h_-'(\tau_-)\partial_t\tau_-|\tilde{\nabla}^s\tau_-|^2\,dx \\
&- \frac{1}{2}\int_{\mathbb{T}^3}\big(u_-\cdot\nabla h_-(\tau_-) + h_-(\tau_-)\nabla\cdot u_-\big)|\tilde{\nabla}^s\tau_-|^2dx \\
&+ \sum_{1\leq k_1\leq s}\int_{\mathbb{T}^3} h_-(\tau_-)\tilde{\nabla}^s\tau_-\big(\tilde{\nabla}^{k_1}u_-\cdot\tilde{\nabla}^{s-k_1}\nabla\tau_-\big)dx \\
&+ \sum_{1\leq k_1\leq s}\int_{\mathbb{T}^3} g_-(\tau_-)\tilde{\nabla}^s\tau_-\tilde{\nabla}^{k_1}\frac{1}{\tau_-+1}\tilde{\nabla}^{s-k_1}\big(\partial_t\tau_- + u_-\cdot\nabla\tau_-\big)dx,
\end{aligned}$$

here $h_-(\tau_-) = \frac{g_-(\tau_-)}{\tau_-+1}$ is a smooth bounded function. By Hölder's inequality and the Sobolev imbedding theorem, we have

$$\begin{aligned}
\Big|\int_0^T I_{7,2}(t)\,dt\Big| \lesssim& \sup_{0\leq t\leq T}\Big|\int_{\mathbb{T}^3} h(\tau_-)|\tilde{\nabla}^s\tau_-|^2\,dx\Big| + \int_0^T \|h_-'(\tau_-)\|_{L^\infty}\|\partial_t\tau_-\|_{L^\infty}\|\tau_-\|_{\tilde{H}^s}^2\,dt \\
&+ \int_0^T \|u_-\|_{W^{1,\infty}}\|h(\tau_-)\|_{W^{1,\infty}}\|\tau_-\|_{\tilde{H}^s}^2 + \|h(\tau_-)\|_{L^\infty}\|\tau_-\|_{\tilde{H}^s}^2\|u_-\|_{\tilde{H}^s}\,dt \\
&+ \int_0^T \|g_-(\tau_-)\|_{L^\infty}\|\tau_-\|_{\tilde{H}^s}\|\frac{1}{\tau_-+1}\|_{\tilde{H}^s}\Big(\|\tau_-\|_{\tilde{H}^s} + \|u_-\|_{\tilde{H}^s}\|\tau_-\|_{\tilde{H}^s}\Big)\,dt.
\end{aligned}$$

Notice the fact $h_-(0) = g_-(0) = 0$ and the *a priori* Assumption (3.1), we then have

$$\begin{aligned}
\Big|\int_0^T I_{7,2}(t)\,dt\Big| \lesssim& \sup_{0\leq t\leq T}\|h_-(\tau_-)\|_{L^\infty}\|\tau_-\|_{\tilde{H}^s}^2 + \sup_{0\leq t\leq T}\|\tau_-\|_{\tilde{H}^s}^2\int_0^T \|\partial_t\tau_-\|_{L^\infty}\,dt \\
&+ \sup_{0\leq t\leq T}\|\tau_-\|_{\tilde{H}^s}^2\int_0^T \|u_-\|_{W^{1,\infty}}\,dt \\
&+ \sup_{0\leq t\leq T}\|\tau_-\|_{\tilde{H}^s}\big(\|\tau_-\|_{\tilde{H}^s} + \|u_-\|_{\tilde{H}^s}\big)\int_0^T \|\tau_-\|_{L^\infty}\,dt \\
\lesssim& \mathcal{E}_1^{3/2}(T) + \mathcal{E}_1(T)\mathcal{H}_3^{1/2}(T) + \mathcal{E}_1(T)\mathcal{E}_3^{1/2}(T).
\end{aligned} \tag{3.12}$$

Finally, we turn to the last term I_8,

$$
\begin{aligned}
I_8 =& -\sum_{1\le k\le s}\sum_{1\le k_1\le k}\tilde{\nabla}^{k_1}\int_{\mathbb{T}^3} u_-\cdot\tilde{\nabla}^{k-k_1}\nabla u_-\tilde{\nabla}^k u_- -\sum_{0\le k\le s}\int_{\mathbb{T}^3} u_-\cdot\nabla\tilde{\nabla}^k u_-\tilde{\nabla}^k u_- \\
&\lesssim \big(\|u_-\|_{\tilde{H}^s}\|\nabla u_-\|_{\tilde{H}^{\lfloor \frac{s}{2}\rfloor+2}}+\|u_-\|_{\tilde{H}^{\lfloor \frac{s}{2}\rfloor+2}}\|\nabla u_-\|_{\tilde{H}^{s-1}}\big)\|u_-\|_{\tilde{H}^s} \\
&\quad +\|\nabla\cdot u_-\|_{L^\infty}\|u_-\|_{\tilde{H}^s}^2.
\end{aligned}
$$

We then obtain

$$
\begin{aligned}
\int_0^T |I_8(t)|\,dt &\lesssim \sup_{0\le t\le T}\|u_-\|_{\tilde{H}^s}^2\int_0^T\|u_-\|_{\tilde{H}^{s-8}}\,dt \\
&\lesssim \mathcal{E}_1(T)\mathcal{D}_3^{1/2}(T). \qquad (3.13)
\end{aligned}
$$

Now, integrating (3.3) with respect to time and summing up the estimates for $I_1\sim I_8$, namely (3.4)-(3.13). We then derive the estimate for energy $\mathcal{E}_1(T)$ in the following

$$
\mathcal{E}_1(T)\lesssim \mathcal{E}_1(0)+\mathcal{E}_1^{3/2}(T)+\mathcal{E}_3^{3/2}(T)+\mathcal{D}_3^{3/2}(T)+\mathcal{H}_3^{3/2}(T).
$$

Here, we have used the Young inequality and this gives rise to the lemma. ∎

LEMMA 3.2. *Assume that energies are defined in (2.1), we then have*

$$
\mathcal{D}_1(T)\lesssim \mathcal{E}_1(T).
$$

Proof. Applying $\partial_t\nabla\cdot$ to the third equation of system (1.3), we shall get

$$
\partial_t^2\nabla\cdot u_+ +\partial_t\nabla\cdot\Big(\frac{p_+'(\tau_+ +1)}{\tau_+ +1}\nabla\tau_+\Big) = -\partial_t\nabla\cdot u_+ -\partial_t\nabla\cdot(u_+\cdot\nabla u_+)+\partial_t\Delta\Phi. \qquad (3.14)
$$

Notice the first, second and fifth equations of system (1.3), we can write

$$
\begin{aligned}
\partial_t\Delta\Phi &= \partial_t(\tau_+ -\tau_-) \\
&= -\nabla\cdot u_+ -\nabla\cdot(\tau_+ u_+)+\nabla\cdot u_- +\nabla\cdot(\tau_- u_-). \qquad (3.15)
\end{aligned}
$$

Combine (3.14) and (3.15) together, define $P_+(\tau_+)=\int_0^{\tau_+}\frac{p_+'(r+1)}{r+1}\,dr$, we now have

$$
\begin{aligned}
\nabla\cdot u_- =& \partial_t^2\nabla\cdot u_+ +\partial_t\Delta P_+(\tau_+)+\partial_t\nabla\cdot u_+ +\partial_t\nabla\cdot(u_+\cdot\nabla u_+) \\
&+\nabla\cdot u_+ +\nabla\cdot(\tau_+ u_+)-\nabla\cdot(\tau_- u_-).
\end{aligned}
$$

Applying $-(-\Delta)^{-1}\nabla$ derivative to the equality above, notice the condition (3.2), i.e., $u_-=\nabla\psi$, we then have the following

$$
\begin{aligned}
-u_- =& \partial_t^2(-\Delta)^{-1}\nabla\nabla\cdot u_+ -\partial_t\nabla P_+(\tau_+)+\partial_t(-\Delta)^{-1}\nabla\nabla\cdot u_+ \\
&+\partial_t(-\Delta)^{-1}\nabla\nabla\cdot(u_+\cdot\nabla u_+)+(-\Delta)^{-1}\nabla\nabla\cdot u_+ \\
&+(-\Delta)^{-1}\nabla\nabla\cdot(\tau_+ u_+)-(-\Delta)^{-1}\nabla\nabla\cdot(\tau_- u_-). \qquad (3.16)
\end{aligned}
$$

From (3.16), we directly know that

$$
\begin{aligned}
\|u_-\|_{\tilde{H}^{s-2}}^2 \lesssim& \|u_+\|_{\tilde{H}^s}^2+\|P_+'(\tau_+)\partial_t\tau_+\|_{\tilde{H}^{s-1}}^2+\|u_+\cdot\nabla u_+\|_{\tilde{H}^{s-1}}^2 \\
&+\|\tau_+ u_+\|_{\tilde{H}^{s-2}}^2+\|\tau_- u_-\|_{\tilde{H}^{s-2}}^2. \qquad (3.17)
\end{aligned}
$$

Next, we will derive the bounds for all the right-hand side terms above.

Firstly, by Proposition 3.1, Proposition 3.2 and *a priori* Assumption (3.1), we can derive

$$\begin{aligned}\|P_+'(\tau_+)\partial_t\tau_+\|_{\tilde{H}^{s-1}}^2 \lesssim &\|\partial_t\tau_+\|_{\tilde{H}^{s-1}}^2 + \|(P_+'-1)\partial_t\tau_+\|_{\tilde{H}^{s-1}}^2\\ \lesssim &\|\partial_t\tau_+\|_{\tilde{H}^{s-1}}^2.\end{aligned}$$

Notice the first equation of system (1.3) , we then get

$$\begin{aligned}\int_0^T \|P_+'(\tau_+)\partial_t\tau_+\|_{\tilde{H}^{s-1}}^2 dt \lesssim &\int_0^T \|\nabla u_+\|_{\tilde{H}^{s-1}}^2 dt + \int_0^T \|\nabla(\tau_+ u_+)\|_{\tilde{H}^{s-1}}^2 dt\\ \lesssim &\int_0^T \|u_+\|_{\tilde{H}^s}^2 dt + \sup_{0\le t\le T}\|\tau_+\|_{\tilde{H}^s}^2 \int_0^T \|u_+\|_{\tilde{H}^s}^2 dt. \end{aligned} \tag{3.18}$$

Similarly,

$$\int_0^T \|u_+\cdot\nabla u_+\|_{\tilde{H}^{s-1}}^2 dt \lesssim \sup_{0\le t\le T}\|u_+\|_{\tilde{H}^s}^2 \int_0^T \|u_+\|_{\tilde{H}^s}^2 dt. \tag{3.19}$$

For the last two terms in (3.17), it holds that

$$\begin{aligned}&\int_0^T \Big(\|\tau_+ u_+\|_{\tilde{H}^{s-2}}^2 + \|\tau_- u_-\|_{\tilde{H}^{s-2}}^2\Big) dt\\ \le &\sup_{0\le t\le T}\big(\|\tau_+\|_{\tilde{H}^{s-2}}^2 + \|\tau_-\|_{\tilde{H}^{s-2}}^2\big)\int_0^T \big(\|u_+\|_{\tilde{H}^{s-2}}^2 + \|u_-\|_{\tilde{H}^{s-2}}^2\big) dt.\end{aligned} \tag{3.20}$$

Now, we can combine all the estimates together, i.e. (3.18), (3.19) and (3.20). Integrating (3.17) with respect to time, using Young's inequality and Assumption (3.1), we directly get

$$\begin{aligned}\mathcal{D}_1(T) \lesssim &\mathcal{E}_1(T) + \mathcal{E}_1(T)^2 + \mathcal{E}_1(T)\mathcal{D}_1(T)\\ \lesssim &\mathcal{E}_1(T),\end{aligned}$$

which completes the proof of Lemma 3.2. □

Lemma 3.3. *Assume that energies are defined in (2.1), we then have*

$$\mathcal{H}_1(T) \lesssim \mathcal{E}_1(T) + \mathcal{D}_1(T).$$

Proof. Due to the Assumption 3.2 and Poincaré's inequality, we only need to consider the highest order terms in $H_1(T)$. Applying $\tilde{\nabla}^{s-3}$ derivative on the third and fourth equations of system (1.3), taking inner product with $\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_+$ and $\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_-$ respectively, adding them up and noticing the definition of Φ, we will get

$$\begin{aligned}&\int_{\mathbb{T}^3} |\tilde{\nabla}^{s-3}\tau_+|^2 + |\tilde{\nabla}^{s-3}\tau_-|^2 dx + \int_{\mathbb{T}^3} |\tilde{\nabla}^{s-3}\nabla\Phi|^2 dx\\ = &-\int_{\mathbb{T}^3} \tilde{\nabla}^{s-3}\partial_t u_+ \tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_+ dx - \int_{\mathbb{T}^3} \tilde{\nabla}^{s-3}\partial_t u_- \tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_- dx\\ &-\int_{\mathbb{T}^3} \tilde{\nabla}^{s-3} u_+ \tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_+ dx - \int_{\mathbb{T}^3} \tilde{\nabla}^{s-3}(u_+\cdot\nabla u_+)\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_+ dx\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{T}^3}\tilde{\nabla}^{s-3}(u_-\cdot\nabla u_-)\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_-\,dx\\
&+\int_{\mathbb{T}^3}\tilde{\nabla}^{s-3}\Big[\Big(1-\frac{p_+'(\tau_++1)}{\tau_++1}\Big)\nabla\tau_+\Big]\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_+dx\\
&+\int_{\mathbb{T}^3}\tilde{\nabla}^{s-3}\Big[\Big(1-\frac{p_-'(\tau_-+1)}{\tau_-+1}\Big)\nabla\tau_-\Big]\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_-dx. \qquad (3.21)
\end{aligned}
$$

Next, we will derive the estimates for all terms on the right-hand side of (3.21). Using Assumption (3.2) and Poincaré's inequality, we directly know that

$$
\begin{aligned}
&\int_{\mathbb{T}^3}\tilde{\nabla}^{s-3}\partial_t u_+\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_+dx\\
\lesssim&\|\tilde{\nabla}^{s-3}\partial_t u_+\|_{L^2}\|\tilde{\nabla}^{s-3}|\nabla|^{-1}\tau_+\|_{L^2}\\
\lesssim&\|u_+\|_{\tilde{H}^{s-2}}\|\tau_+\|_{\tilde{H}^{s-3}}.
\end{aligned}
$$

Similarly, there holds

$$
\begin{aligned}
&\int_{\mathbb{T}^3}\tilde{\nabla}^{s-3}\partial_t u_-\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_-dx\lesssim\|u_-\|_{\tilde{H}^{s-2}}\|\tau_-\|_{\tilde{H}^{s-3}},\\
&\int_{\mathbb{T}^3}\tilde{\nabla}^{s-3}u_+\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_+dx\lesssim\|u_+\|_{\tilde{H}^{s-3}}\|\tau_+\|_{\tilde{H}^{s-3}},\\
&\int_{\mathbb{T}^3}\tilde{\nabla}^{s-3}(u_+\cdot\nabla u_+)\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_+dx\lesssim\|u_+\|^2_{\tilde{H}^{s-2}}\|\tau_+\|_{\tilde{H}^{s-3}},\\
&\int_{\mathbb{T}^3}\tilde{\nabla}^{s-3}(u_-\cdot\nabla u_-)\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_-dx\lesssim\|u_-\|^2_{\tilde{H}^{s-2}}\|\tau_-\|_{\tilde{H}^{s-3}},\\
&\int_{\mathbb{T}^3}\tilde{\nabla}^{s-3}\Big[\Big(1-\frac{p_+'(\tau_++1)}{\tau_++1}\Big)\nabla\tau_+\Big]\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_+dx\lesssim\|\tau_+\|_{\tilde{H}^{s-2}}\|\tau_+\|^2_{\tilde{H}^{s-3}},\\
&\int_{\mathbb{T}^3}\tilde{\nabla}^{s-3}\Big[\Big(1-\frac{p_-'(\tau_-+1)}{\tau_-+1}\Big)\nabla\tau_-\Big]\tilde{\nabla}^{s-3}(-\Delta)^{-1}\nabla\tau_-dx\lesssim\|\tau_-\|_{\tilde{H}^{s-2}}\|\tau_-\|^2_{\tilde{H}^{s-3}}.
\end{aligned}
$$

Integrating (3.21) with respect to time, combining all the estimates above and keeping the *a priori* Assumption (3.1) in mind, we finally get

$$
\begin{aligned}
\mathcal{H}_1(T)\lesssim&\mathcal{E}_1(T)+\mathcal{D}_1(T)+\mathcal{E}_1^{3/2}(T)+\mathcal{D}_1^{3/2}(T)\\
\lesssim&\mathcal{E}_1(T)+\mathcal{D}_1(T),
\end{aligned}
$$

which completes the proof of Lemma 3.3. □

3.3. Decay energy estimates. Since the energies (2.1) include time-space derivatives, we can not derive the estimates for $\mathcal{E}_2,\mathcal{D}_2,\mathcal{H}_2$ directly through interpolation theorem. Next, we shall turn to the decay energies $\mathcal{E}_2,\mathcal{D}_2,\mathcal{H}_2,\mathcal{E}_3,\mathcal{D}_3,\mathcal{H}_3$ respectively and give the lemmas.

Lemma 3.4. *Assume that energies are defined as before, we then have*

$$
\mathcal{E}_2(T)\lesssim\mathcal{E}_1(T)+\mathcal{D}_1(T)+\mathcal{H}_1(T)+\mathcal{E}_3^{3/2}(T)+\mathcal{D}_3^{3/2}(T)+\mathcal{H}_3^{3/2}(T).
$$

Proof. Applying $\tilde{\nabla}^k$ derivative $(0\le k\le s-3)$ on system (1.3), taking inner product with $\tilde{\nabla}^k\tau_+$ for the first equation, $\tilde{\nabla}^k\tau_-$ for the second equation, $\tilde{\nabla}^k u_+$ for the third equation and $\tilde\nabla^k u_-$ for the fourth equation. Adding the time weight $1+t$ and summing them up, we shall get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\{(1+t)\big(\|\tau_+\|^2_{\tilde H^{s-3}}+\|\tau_-\|^2_{\tilde H^{s-3}}+\|u_+\|^2_{\tilde H^{s-3}}+\|u_-\|^2_{\tilde H^{s-3}}\big)\Big\}\\&-\frac{1}{2}\big(\|\tau_+\|^2_{\tilde H^{s-3}}+\|\tau_-\|^2_{\tilde H^{s-3}}+\|u_+\|^2_{\tilde H^{s-3}}+\|u_-\|^2_{\tilde H^{s-3}}\big)+(1+t)\|u_+\|^2_{\tilde H^{s-3}}\\=&J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8,\end{aligned}\tag{3.22}$$

where

$$\begin{aligned}J_1=&-(1+t)\int_{\mathbb T^3}\sum_{0\le k\le s-3}\tilde\nabla^k\nabla\cdot u_+\tilde\nabla^k\tau_+-\tilde\nabla^k\nabla\tau_+\tilde\nabla^k u_+\,dx\\&-(1+t)\int_{\mathbb T^3}\sum_{0\le k\le s-3}\tilde\nabla^k\nabla\cdot u_-\tilde\nabla^k\tau_--\tilde\nabla^k\nabla\tau_-\tilde\nabla^k u_-\,dx,\\J_2=&(1+t)\int_{\mathbb T^3}\sum_{0\le k\le s-3}\tilde\nabla^k\nabla\Phi\tilde\nabla^k u_+-\tilde\nabla^k\nabla\Phi\tilde\nabla^k u_-\,dx,\\J_3=&-(1+t)\sum_{0\le k\le s-3}\int_{\mathbb T^3}\tilde\nabla^k\nabla\cdot(\tau_+u_+)\tilde\nabla^k\tau_++\tilde\nabla^k(\tau_+\nabla\tau_+)\tilde\nabla^k u_+\,dx,\\J_4=&(1+t)\sum_{0\le k\le s-3}\int_{\mathbb T^3}\tilde\nabla^k\Big[\Big(\tau_++1-\frac{p'_+(\tau_++1)}{\tau_++1}\Big)\nabla\tau_+\Big]\tilde\nabla^k u_+\,dx,\\J_5=&-(1+t)\sum_{0\le k\le s-3}\int_{\mathbb T^3}\tilde\nabla^k(u_+\cdot\nabla u_+)\tilde\nabla^k u_+\,dx,\\J_6=&-(1+t)\sum_{0\le k\le s-3}\int_{\mathbb T^3}\tilde\nabla^k\nabla\cdot(\tau_-u_-)\tilde\nabla^k\tau_-+\tilde\nabla^k(\tau_-\nabla\tau_-)\tilde\nabla^k u_-\,dx,\\J_7=&(1+t)\sum_{0\le k\le s-3}\int_{\mathbb T^3}\tilde\nabla^k\Big[\Big(\tau_-+1-\frac{p'_-(\tau_-+1)}{\tau_-+1}\Big)\nabla\tau_-\Big]\tilde\nabla^k u_-\,dx,\\J_8=&-(1+t)\sum_{0\le k\le s-3}\int_{\mathbb T^3}\tilde\nabla^k(u_-\cdot\nabla u_-)\tilde\nabla^k u_-\,dx.\end{aligned}$$

Like the process in Lemma 3.1, we shall estimate each term on the right-hand side of (3.22). For the first term J_1, through integration by parts we know

$$J_1=0.\tag{3.23}$$

For the second term J_2, using the equations of τ_+, τ_- and Φ, we have

$$\begin{aligned}J_2=&-(1+t)\sum_{k\le s-3}\int_{\mathbb T^3}\tilde\nabla^k\Phi(\tilde\nabla^k\nabla\cdot u_+-\tilde\nabla^k\nabla\cdot u_-)\,dx\\=&(1+t)\sum_{k\le s-3}\int_{\mathbb T^3}\tilde\nabla^k\Phi\partial_t\tilde\nabla^k(\tau_+-\tau_-)+\tilde\nabla^k\Phi\tilde\nabla^k\nabla\cdot(\tau_+u_+)-\tilde\nabla^k\Phi\tilde\nabla^k\nabla\cdot(\tau_-u_-)\,dx\\=&-\frac{1}{2}(1+t)\frac{d}{dt}\||\nabla|^{-1}(\tau_+-\tau_-)\|^2_{\tilde H^{s-3}}+J_{2,2}\\=&-\frac{1}{2}\frac{d}{dt}\Big\{(1+t)\||\nabla|^{-1}(\tau_+-\tau_-)\|^2_{\tilde H^{s-3}}\Big\}+\frac{1}{2}\||\nabla|^{-1}(\tau_+-\tau_-)\|^2_{\tilde H^{s-3}}+J_{2,2}.\end{aligned}$$

Let us focus on the term $J_{2,2}$. By Proposition 3.1 and Poincaré's inequality, we can write

$$
\begin{aligned}
|J_{2,2}| \lesssim &(1+t)\int_{\mathbb{T}^3} \|\nabla\Phi\|_{\tilde{H}^{s-3}}(\|\tau_+u_+\|_{\tilde{H}^{s-3}}+\|\tau_-u_-\|_{\tilde{H}^{s-3}})\,dx\\
\lesssim &(1+t)\||\nabla|^{-1}(\tau_+-\tau_-)\|_{\tilde{H}^{s-3}}\Big(\|\tau_+\|_{\tilde{H}^{\lfloor\frac{s-3}{2}\rfloor+2}}\|u_+\|_{\tilde{H}^{s-3}}+\|\tau_+\|_{\tilde{H}^{s-3}}\|u_+\|_{\tilde{H}^{\lfloor\frac{s-3}{2}\rfloor+2}}\\
&+\|\tau_-\|_{\tilde{H}^{\lfloor\frac{s-3}{2}\rfloor+2}}\|u_-\|_{\tilde{H}^{s-3}}+\|\tau_-\|_{\tilde{H}^{s-3}}\|u_-\|_{\tilde{H}^{\lfloor\frac{s-3}{2}\rfloor+2}}\Big)\\
\lesssim &(1+t)\|(\tau_+-\tau_-)\|_{\tilde{H}^{s-3}}\Big(\|\tau_+\|_{\tilde{H}^{s-9}}\|u_+\|_{\tilde{H}^{s-3}}+\|\tau_+\|_{\tilde{H}^{s-3}}\|u_+\|_{\tilde{H}^{s-6}}\\
&+\|\tau_-\|_{\tilde{H}^{s-9}}\|u_-\|_{\tilde{H}^{s-3}}+\|\tau_-\|_{\tilde{H}^{s-3}}\|u_-\|_{\tilde{H}^{s-8}}\Big).
\end{aligned}
$$

Obviously, we can derive the bound

$$
\begin{aligned}
\int_0^T |J_{2,2}(t)|\,dt \lesssim &\sup_{0\le t\le T}(1+t)\|(\tau_+-\tau_-)\|_{\tilde{H}^{s-3}}\Big(\|u_+\|_{\tilde{H}^{s-3}}+\|u_-\|_{\tilde{H}^{s-3}}+\|\tau_+\|_{\tilde{H}^{s-3}}\\
&+\|\tau_-\|_{\tilde{H}^{s-3}}\Big)\cdot\int_0^T(\|u_+\|_{\tilde{H}^{s-6}}+\|u_-\|_{\tilde{H}^{s-8}}+\|\tau_+\|_{\tilde{H}^{s-9}}+\|\tau_-\|_{\tilde{H}^{s-9}})\,dt\\
\lesssim &\mathcal{E}_2(T)\big(\mathcal{E}_3^{1/2}(T)+\mathcal{D}_3^{1/2}(T)+\mathcal{H}_3^{1/2}(T)\big). \qquad (3.24)
\end{aligned}
$$

Next, we turn to J_3, similar to I_3, we have the following computation

$$
\begin{aligned}
|J_3| \lesssim &(1+t)\Big|\int_{\mathbb{T}^3}\sum_{k=1}^{s-3}\sum_{k_1=1}^{k}\tilde{\nabla}^{k_1}u_+\cdot\tilde{\nabla}^{k-k_1}\nabla\tau_+\tilde{\nabla}^k\tau_+ +\sum_{0\le k\le s-3}u_+\cdot\nabla\tilde{\nabla}^k\tau_+\tilde{\nabla}^k\tau_+\,dx\\
&+\sum_{0\le k\le s-3}\int_{\mathbb{T}^3}\tau_+\tilde{\nabla}^k\nabla\cdot u_+\tilde{\nabla}^k\tau_+ -\big(\tau_+\tilde{\nabla}^k\tau_+\tilde{\nabla}^k\nabla\cdot u_+ +\nabla\tau_+\tilde{\nabla}^k\tau_+\tilde{\nabla}^k u_+\big)\,dx\Big|\\
&+(1+t)\sum_{k=0}^{s-3}\sum_{k'=1}^{k}\Big|\int_{\mathbb{T}^3}\tilde{\nabla}^{k'}\tau_+\tilde{\nabla}^{k-k'}\nabla\cdot u_+\tilde{\nabla}^k\tau_+ +\tilde{\nabla}^{k'}\tau_+\tilde{\nabla}^{k-k'}\nabla\tau_+\tilde{\nabla}^k u_+\,dx\Big|.
\end{aligned}
$$

It yields that,

$$
\begin{aligned}
|J_3| \lesssim &(1+t)\big(\|\tilde{\nabla}u_+\|_{\tilde{H}^{\lfloor\frac{s-4}{2}\rfloor+2}}\|\nabla\tau_+\|_{\tilde{H}^{s-4}}+\|\nabla\tau_+\|_{\tilde{H}^{\lfloor\frac{s-4}{2}\rfloor+2}}\|\tilde{\nabla}u_+\|_{\tilde{H}^{s-4}}\big)\|\tau_+\|_{\tilde{H}^{s-3}}\\
&+(1+t)\big(\|\nabla\cdot u_+\|_{L^\infty}\|\tau_+\|^2_{\tilde{H}^{s-3}}+\|\nabla\tau_+\|_{L^\infty}\|\tau_+\|_{\tilde{H}^{s-3}}\|u_+\|_{\tilde{H}^{s-3}}\big)\\
&+(1+t)\big(\|\tilde{\nabla}\tau_+\|_{\tilde{H}^{\lfloor\frac{s-4}{2}\rfloor+2}}\|\nabla\cdot u_+\|_{\tilde{H}^{s-4}}+\|\nabla\cdot u_+\|_{\tilde{H}^{\lfloor\frac{s-4}{2}\rfloor+2}}\|\tilde{\nabla}\tau_+\|_{\tilde{H}^{s-4}}\big)\|\tau_+\|_{\tilde{H}^{s-3}}\\
&+(1+t)\|\tilde{\nabla}\tau_+\|_{\tilde{H}^{\lfloor\frac{s-4}{2}\rfloor+2}}\|\tilde{\nabla}\tau_+\|_{\tilde{H}^{s-4}}\|u_+\|_{\tilde{H}^{s-3}}\\
\lesssim &(1+t)\big(\|u_+\|_{\tilde{H}^{s-6}}+\|\tau_+\|_{\tilde{H}^{s-9}}\big)\big(\|u_+\|^2_{\tilde{H}^{s-3}}+\|\tau_+\|^2_{\tilde{H}^{s-3}}\big).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\int_0^T|J_3(t)|\,dt \lesssim &\sup_{0\le t\le T}(1+t)\big(\|u_+\|^2_{\tilde{H}^{s-3}}+\|\tau_+\|^2_{\tilde{H}^{s-3}}\big)\int_0^T\big(\|u_+\|_{\tilde{H}^{s-6}}+\|\tau_+\|_{\tilde{H}^{s-9}}\big)\,dt\\
\lesssim &\mathcal{E}_2(T)\big(\mathcal{E}_3^{1/2}(T)+\mathcal{H}_3^{1/2}(T)\big). \qquad (3.25)
\end{aligned}
$$

For the term J_4, recall the definition $g_+(\tau_+)=(\tau_++1-\frac{p'_+(\tau_++1)}{\tau_++1})$ and following the computation of I_4, we can derive

$$
J_4=(1+t)\Big\{\sum_{0\le k\le s-4}\int_{\mathbb{T}^3}\tilde{\nabla}^k(g_+(\tau_+)\nabla\tau_+)\tilde{\nabla}^k u_+\,dx
$$

$$
\begin{aligned}
&+\sum_{1\le k_1\le s-3}\int_{\mathbb{T}^3}\tilde\nabla^{k_1}g_+(\tau_+)\tilde\nabla^{s-3-k_1}\nabla\tau_+\tilde\nabla^{s-3}u_+\,dx\\
&-\int_{\mathbb{T}^3}(\nabla g_+(\tau_+)\tilde\nabla^{s-3}\tau_+)\tilde\nabla^{s-3}u_+\,dx-\int_{\mathbb{T}^3}(g_+(\tau_+)\tilde\nabla^{s-3}\tau_+)\tilde\nabla^{s-3}\nabla\cdot u_+\,dx\Big\}\\
=&J_{4,1}+J_{4,2}.
\end{aligned}
$$

Also, $J_{4,1}$ contains the first three terms above and $J_{4,2}$ denotes the remaining term. Using Proposition 3.1 and Proposition 3.2, we have

$$
\begin{aligned}
&|J_{4,1}|\\
\lesssim&(1+t)\Big\{\|g_+(\tau_+)\nabla\tau_+\|_{\tilde H^{s-4}}\|u_+\|_{\tilde H^{s-4}}+\|\nabla g_+(\tau_+)\|_{L^\infty}\|\tau_+\|_{\tilde H^{s-3}}\|u_+\|_{\tilde H^{s-3}}\\
&+\big(\|\tilde\nabla g_+(\tau_+)\|_{\tilde H^{\lfloor\frac{s-4}{2}\rfloor+2}}\|\nabla\tau_+\|_{\tilde H^{s-4}}+\|\nabla\tau_+\|_{\tilde H^{\lfloor\frac{s-4}{2}\rfloor+2}}\|\tilde\nabla g_+(\tau_+)\|_{\tilde H^{s-4}}\big)\|u_+\|_{\tilde H^{s-3}}\Big\}\\
\lesssim&(1+t)\|\tau_+\|_{\tilde H^{s-9}}\|\tau_+\|_{\tilde H^{s-3}}\|u_+\|_{\tilde H^{s-3}}.
\end{aligned}
$$

Here, we should notice the fact $g_+(\cdot)$ is a smooth function around zero and $g_+(0)=0$. Obviously,

$$
\begin{aligned}
\int_0^T|J_{4,1}(t)|dt\lesssim&\sup_{0\le t\le T}(1+t)\|\tau_+\|_{\tilde H^{s-3}}\|u_+\|_{\tilde H^{s-3}}\int_0^T\|\tau_+\|_{\tilde H^{s-9}}dt\\
\lesssim&\mathcal{E}_2(T)\mathcal{H}_3^{1/2}(T).
\end{aligned}\tag{3.26}
$$

For the term $J_{4,2}$, according to the equality

$$
-\nabla\cdot u_+=\frac{\partial_t\tau_++u_+\cdot\nabla\tau_+}{\tau_++1},
$$

and following $I_{4,2}$, we directly write

$$
\begin{aligned}
J_{4,2}=&\frac12\frac{d}{dt}(1+t)\int_{\mathbb{T}^3}h_+(\tau_+)|\tilde\nabla^{s-3}\tau_+|^2\,dx-\frac12\int_{\mathbb{T}^3}h_+(\tau_+)|\tilde\nabla^{s-3}\tau_+|^2\,dx\\
&-(1+t)\frac12\int_{\mathbb{T}^3}h_+'(\tau_+)\partial_t\tau_+|\tilde\nabla^{s-3}\tau_+|^2\,dx\\
&-(1+t)\frac12\int_{\mathbb{T}^3}\big(u_+\cdot\nabla h_+(\tau_+)+h_+(\tau_+)\nabla\cdot u_+\big)|\tilde\nabla^{s-3}\tau_+|^2dx\\
&+(1+t)\sum_{1\le k_1\le s-3}\int_{\mathbb{T}^3}h_+(\tau_+)\tilde\nabla^{s-3}\tau_+\big(\tilde\nabla^{k_1}u_+\cdot\tilde\nabla^{s-3-k_1}\nabla\tau_+\big)dx\\
&+(1+t)\sum_{1\le k_1\le s-3}\int_{\mathbb{T}^3}g_+(\tau_+)\tilde\nabla^{s-3}\tau_+\tilde\nabla^{k_1}\frac{1}{\tau_++1}\tilde\nabla^{s-3-k_1}\big(\partial_t\tau_++u_+\cdot\nabla\tau_+\big)dx.
\end{aligned}
$$

Remember here that $h_+(\tau_+)=\frac{g_+(\tau_+)}{\tau_++1}$ is a smooth bounded function. By Hölder's inequality and the Sobolev imbedding theorem, we have

$$
\begin{aligned}
&\Big|\int_0^T J_{4,2}\,dt\Big|\\
\lesssim&\sup_{0\le t\le T}(1+t)\Big|\int_{\mathbb{T}^3}h_+(\tau_+)|\tilde\nabla^{s-3}\tau_+|^2\,dx\Big|+\int_0^T(1+t)\|h_+(\tau_+)\|_{L^\infty}\|\tau_+\|_{\tilde H^{s-3}}^2\,dt
\end{aligned}
$$

$$+\int_0^T (1+t)\|h'_+(\tau_+)\|_{L^\infty}\|\partial_t\tau_+\|_{L^\infty}\|\tau_+\|^2_{\tilde{H}^{s-3}}dt$$
$$+\int_0^T (1+t)\Big(\|u_+\|_{W^{1,\infty}}\|h_+(\tau_+)\|_{W^{1,\infty}}\|\tau_+\|^2_{\tilde{H}^{s-3}}+\|h_+(\tau_+)\|_{L^\infty}\|\tau_+\|^2_{\tilde{H}^{s-3}}\|u_+\|_{\tilde{H}^{s-3}}\Big)dt$$
$$+\int_0^T (1+t)\|g_+(\tau_+)\|_{L^\infty}\|\tau_+\|_{\tilde{H}^{s-3}}\|\frac{1}{\tau_++1}\|_{\tilde{H}^{s-3}}\Big(\|\tau_+\|_{\tilde{H}^{s-3}}+\|u_+\|_{\tilde{H}^{s-3}}\|\tau_+\|_{\tilde{H}^{s-3}}\Big)\,dt.$$

Indeed, it holds that

$$\begin{aligned}\Big|\int_0^T J_{4,2}(t)\,dt\Big|\lesssim &\sup_{0\le t\le T}(1+t)\|h(\tau_+)\|_{L^\infty}\|\tau_+\|^2_{\tilde{H}^{s-3}}\\
&+\sup_{0\le t\le T}(1+t)\|\tau_+\|^2_{\tilde{H}^{s-3}}\int_0^T\|\partial_t\tau_+\|_{L^\infty}\,dt\\
&+\sup_{0\le t\le T}(1+t)\|\tau_+\|^2_{\tilde{H}^{s-3}}\int_0^T\|u_+\|_{W^{1,\infty}}\|h(\tau_+)\|_{W^{1,\infty}}dt\\
&+\sup_{0\le t\le T}(1+t)\|\tau_+\|_{\tilde{H}^{s-3}}\big(\|\tau_+\|_{\tilde{H}^{s-3}}+\|u_+\|_{\tilde{H}^{s-3}}\big)\int_0^T\|\tau_+\|_{L^\infty}dt\\
&+\sup_{0\le t\le T}(1+t)\|g(\tau_+)\|_{L^\infty}\|\tau_+\|^2_{\tilde{H}^{s-3}}\int_0^T\|\tau_+\|_{\tilde{H}^{s-9}}dt\\
\lesssim&\,\mathcal{E}_2^{3/2}(T)+\mathcal{E}_2(T)\mathcal{H}_3^{1/2}(T)+\mathcal{E}_2(T)\mathcal{E}_3^{1/2}(T).\end{aligned}\tag{3.27}$$

Now, we turn to deal with J_5 and get

$$\begin{aligned}J_5=&-(1+t)\sum_{1\le k\le s-3}\sum_{1\le k_1\le k}\tilde{\nabla}^{k_1}u_+\cdot\tilde{\nabla}^{k-k_1}\nabla u_+\tilde{\nabla}^k u_+dx\\
&-(1+t)\sum_{0\le k\le s-3}\int_{\mathbb{T}^3}u_+\cdot\nabla\tilde{\nabla}^k u_+\tilde{\nabla}^k u_+\,dx\\
\lesssim&\,(1+t)\|u_+\|^3_{\tilde{H}^{s-3}}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^T|J_5(t)|\,dt\lesssim&\sup_{0\le t\le T}\|u_+\|_{\tilde{H}^{s-3}}\int_0^T(1+t)\|u_+\|^2_{\tilde{H}^{s-3}}\,dt\\
\lesssim&\,\mathcal{E}_2^{3/2}(T).\end{aligned}\tag{3.28}$$

The estimate for J_6 is similar to J_3, we can write the following inequality directly,

$$\begin{aligned}|J_6|\lesssim&(1+t)\Big|\int_{\mathbb{T}^3}\sum_{k=1}^{s-3}\sum_{k_1=1}^{k}\tilde{\nabla}^{k_1}u_-\cdot\tilde{\nabla}^{k-k_1}\nabla\tau_-\tilde{\nabla}^k\tau_-+\sum_{0\le k\le s-3}u_-\cdot\nabla\tilde{\nabla}^k\tau_-\tilde{\nabla}^k\tau_-\,dx\\
&+\sum_{0\le k\le s-3}\int_{\mathbb{T}^3}\tau_-\tilde{\nabla}^k\nabla\cdot u_-\tilde{\nabla}^k\tau_--\big(\tau_-\tilde{\nabla}^k\tau_-\tilde{\nabla}^k\nabla\cdot u_-+\nabla\tau_-\tilde{\nabla}^k\tau_-\tilde{\nabla}^k u_-\big)\,dx\Big|\\
&+(1+t)\sum_{k=0}^{s-3}\sum_{k'=1}^{k}\Big|\int_{\mathbb{T}^3}\tilde{\nabla}^{k'}\tau_-\tilde{\nabla}^{k-k'}\nabla\cdot u_-\tilde{\nabla}^k\tau_-+\tilde{\nabla}^{k'}\tau_-\tilde{\nabla}^{k-k'}\nabla\tau_-\tilde{\nabla}^k u_-\,dx\Big|.\end{aligned}$$

Then, using Proposition 3.1 and the Sobolev imbedding theorem, we will get

$$\begin{aligned}|J_6|\lesssim&(1+t)\big(\|\tilde\nabla u_-\|_{\tilde H^{\lfloor\frac{s-4}{2}\rfloor+2}}\|\nabla\tau_-\|_{\tilde H^{s-4}}+\|\nabla\tau_-\|_{\tilde H^{\lfloor\frac{s-4}{2}\rfloor+2}}\|\tilde\nabla u_-\|_{\tilde H^{s-4}}\big)\|\tau_-\|_{\tilde H^{s-3}}\\&+(1+t)\big(\|\nabla\cdot u_-\|_{L^\infty}\|\tau_-\|^2_{\tilde H^{s-3}}+\|\nabla\tau_-\|_{L^\infty}\|\tau_-\|_{\tilde H^{s-3}}\|u_-\|_{\tilde H^{s-3}}\big)\\&+(1+t)\big(\|\tilde\nabla\tau_-\|_{\tilde H^{\lfloor\frac{s-4}{2}\rfloor+2}}\|\nabla\cdot u_-\|_{\tilde H^{s-4}}+\|\nabla\cdot u_-\|_{\tilde H^{\lfloor\frac{s-4}{2}\rfloor+2}}\|\tilde\nabla\tau_-\|_{\tilde H^{s-4}}\big)\|\tau_-\|_{\tilde H^{s-3}}\\&+(1+t)\|\tilde\nabla\tau_-\|_{\tilde H^{\lfloor\frac{s-4}{2}\rfloor+2}}\|\tilde\nabla\tau_-\|_{\tilde H^{s-4}}\|u_-\|_{\tilde H^{s-3}}\\\lesssim&(1+t)\big(\|u_-\|_{\tilde H^{s-8}}+\|\tau_-\|_{\tilde H^{s-9}}\big)\big(\|u_-\|^2_{\tilde H^{s-3}}+\|\tau_-\|^2_{\tilde H^{s-3}}\big).\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^T|J_6(t)|\,dt\lesssim&\sup_{0\le t\le T}(1+t)(\|u_-\|^2_{\tilde H^{s-3}}+\|\tau_-\|^2_{\tilde H^{s-3}})\int_0^T(\|u_-\|_{\tilde H^{s-8}}+\|\tau_-\|_{\tilde H^{s-9}})\,dt\\\lesssim&\mathcal E_2(T)\big(\mathcal D_3^{1/2}(T)+\mathcal H_3^{1/2}(T)\big).\end{aligned}\tag{3.29}$$

Next, the estimate for J_7 is similar to J_4. Notice the definition of $g_-(\tau_-)$, we have

$$\begin{aligned}J_7=&(1+t)\Big\{\sum_{0\le k\le s-4}\int_{\mathbb T^3}\tilde\nabla^k(g_-(\tau_-)\nabla\tau_-)\tilde\nabla^k u_-\,dx\\&+\sum_{k_1=1}^{s-3}\int_{\mathbb T^3}\tilde\nabla^{k_1}g_-(\tau_-)\tilde\nabla^{s-3-k_1}\nabla\tau_-\tilde\nabla^{s-3}u_-\,dx+\int_{\mathbb T^3}(g_-(\tau_-)\tilde\nabla^{s-3}\nabla\tau_-)\tilde\nabla^{s-3}u_-\,dx\Big\}\\=&(1+t)\Big\{\sum_{0\le k\le s-4}\int_{\mathbb T^3}\tilde\nabla^k(g_-(\tau_-)\nabla\tau_-)\tilde\nabla^k u_-\,dx\\&+\sum_{1\le k_1\le s-3}\int_{\mathbb T^3}\tilde\nabla^{k_1}g_-(\tau_-)\tilde\nabla^{s-3-k_1}\nabla\tau_-\tilde\nabla^{s-3}u_-\,dx\\&-\int_{\mathbb T^3}(\nabla g_-(\tau_-)\tilde\nabla^{s-3}\tau_-)\tilde\nabla^{s-3}u_-\,dx-\int_{\mathbb T^3}(g_-(\tau_-)\tilde\nabla^{s-3}\tau_-)\tilde\nabla^{s-3}\nabla\cdot u_-\,dx\Big\}\\=&J_{7,1}+J_{7,2}.\end{aligned}$$

$J_{7,1}$ contains the first three terms above and we can directly bound

$$\begin{aligned}J_{7,1}\lesssim&(1+t)\Big\{\|g_-(\tau_-)\nabla\tau_-\|_{\tilde H^{s-4}}\|u_-\|_{\tilde H^{s-4}}+\|\nabla g_-(\tau_-)\|_{L^\infty}\|\tau_-\|_{\tilde H^{s-3}}\|u_-\|_{\tilde H^{s-3}}\\&+\big(\|\tilde\nabla g_-(\tau_-)\|_{\tilde H^{\frac{s-4}{2}+2}}\|\nabla\tau_-\|_{\tilde H^{s-4}}+\|\nabla\tau_-\|_{\tilde H^{\frac{s-4}{2}+2}}\|\tilde\nabla g_-(\tau_-)\|_{\tilde H^{s-4}}\big)\|u_-\|_{\tilde H^{s-3}}\Big\}\\\lesssim&(1+t)\|\tau_-\|_{\tilde H^{s-9}}\|\tau_-\|_{\tilde H^{s-3}}\|u_-\|_{\tilde H^{s-3}}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^T|J_{7,1}(t)|dt\le&\sup_{0\le t\le T}(1+t)\|\tau_-\|_{\tilde H^{s-3}}\|u_-\|_{\tilde H^{s-3}}\int_0^T\|\tau_-\|_{\tilde H^{s-9}}dt\\\le&\mathcal E_2(T)\mathcal H_3^{1/2}(T).\end{aligned}\tag{3.30}$$

For the term $J_{7,2}$, using the following equality

$$-\nabla\cdot u_-=\frac{\partial_t\tau_-+u_-\cdot\nabla\tau_-}{\tau_-+1},$$

it becomes

$$
\begin{aligned}
J_{7,2}=&\frac{1}{2}\frac{d}{dt}(1+t)\int_{\mathbb{T}^3} h_-(\tau_-)|\tilde{\nabla}^{s-3}\tau_-|^2\,dx-\frac{1}{2}\int_{\mathbb{T}^3} h_-(\tau_-)|\tilde{\nabla}^{s-3}\tau_-|^2\,dx\\
&-(1+t)\frac{1}{2}\int_{\mathbb{T}^3} h'_-(\tau_-)\partial_t\tau_-|\tilde{\nabla}^{s-3}\tau_-|^2\,dx\\
&-(1+t)\frac{1}{2}\int_{\mathbb{T}^3} \big(u_-\cdot\nabla h_-(\tau_-)+h_-(\tau_-)\nabla\cdot u_-\big)|\tilde{\nabla}^{s-3}\tau_-|^2dx\\
&+(1+t)\sum_{1\leq k_1\leq s-3}\int_{\mathbb{T}^3} h_-(\tau_-)\tilde{\nabla}^{s-3}\tau_-\big(\tilde{\nabla}^{k_1}u_-\cdot\tilde{\nabla}^{s-3-k_1}\nabla\tau_-\big)\,dx\\
&+(1+t)\sum_{1\leq k_1\leq s-3}\int_{\mathbb{T}^3} g_-(\tau_-)\tilde{\nabla}^{s-3}\tau_-\tilde{\nabla}^{k_1}\frac{1}{\tau_-+1}\tilde{\nabla}^{s-3-k_1}\big(\partial_t\tau_-+u_-\cdot\nabla\tau_-\big)\,dx.
\end{aligned}
$$

Recall the definition $h_-(\tau_-)=\frac{g_-(\tau_-)}{\tau_-+1}$, using Hölder's inequality and the Sobolev imbedding theorem, it yields that

$$
\begin{aligned}
&\Big|\int_0^T J_{7,2}(t)dt\Big|\\
\lesssim&\sup_{0\leq t\leq T}(1+t)\Big|\int_{\mathbb{T}^3} h_-(\tau_-)|\tilde{\nabla}^{s-3}\tau_-|^2\,dx\Big|+\int_0^T(1+t)\|h_-(\tau_-)\|_{L^\infty}\|\tau_-\|^2_{\tilde{H}^{s-3}}\,dt\\
&+\int_0^T(1+t)\|h'_-(\tau_-)\|_{L^\infty}\|\partial_t\tau_-\|_{L^\infty}\|\tau_-\|^2_{\tilde{H}^{s-3}}\,dt\\
&+\int_0^T(1+t)\big(\|u_-\|_{W^{1,\infty}}\|h_-(\tau_-)\|_{W^{1,\infty}}\|\tau_-\|^2_{\tilde{H}^{s-3}}+\|h_-(\tau_-)\|_{L^\infty}\|\tau_-\|^2_{\tilde{H}^{s-3}}\|u_-\|_{\tilde{H}^{s-3}}\big)\,dt\\
&+\int_0^T(1+t)\|g_-(\tau_-)\|_{L^\infty}\|\tau_-\|_{\tilde{H}^{s-3}}\|\frac{1}{\tau_-+1}\|_{\tilde{H}^{s-3}}\Big(\|\tau_-\|_{\tilde{H}^{s-3}}+\|u_-\|_{\tilde{H}^{s-3}}\|\tau_-\|_{\tilde{H}^{s-3}}\Big)\,dt.
\end{aligned}
$$

Similar to (3.27), we can obtain

$$
\int_0^T|J_{7,2}(t)|\,dt\lesssim\mathcal{E}_2^{3/2}(T)+\mathcal{E}_2(T)\mathcal{H}_3^{1/2}(T)+\mathcal{E}_2(T)\mathcal{E}_3^{1/2}(T). \tag{3.31}
$$

For the last term J_8 we can easily get

$$
\begin{aligned}
J_8=&-(1+t)\sum_{1\leq k\leq s-3}\sum_{1\leq k_1\leq k}\tilde{\nabla}^{k_1}u_-\cdot\tilde{\nabla}^{k-k_1}\nabla u_-\tilde{\nabla}^k u_-dx\\
&-(1+t)\sum_{0\leq k\leq s-3}\int_{\mathbb{T}^3}u_-\cdot\nabla\tilde{\nabla}^k u_-\tilde{\nabla}^k u_-\,dx\\
\lesssim&(1+t)\|u_-\|^2_{\tilde{H}^{s-3}}\|u_-\|_{\tilde{H}^{s-8}},
\end{aligned}
$$

which means,

$$
\begin{aligned}
\int_0^T|J_8(t)|\,dt\lesssim&\sup_{0\leq t\leq T}(1+t)\|u_-\|^2_{\tilde{H}^{s-3}}\int_0^T\|u_-\|_{\tilde{H}^{s-8}}dt\\
\lesssim&\mathcal{E}_2(T)\mathcal{D}_3^{1/2}(T).
\end{aligned} \tag{3.32}
$$

Next, integrating (3.22) with respect to time and summing up the estimates for $J_1 \sim J_8$, namely (3.23)-(3.32). We then derive the estimate for energy $\mathcal{E}_2(T)$ in the following

$$\mathcal{E}_2(T) \lesssim \mathcal{E}_1(T) + \mathcal{D}_1(T) + \mathcal{H}_1(T) + \mathcal{E}_3^{3/2}(T) + \mathcal{D}_3^{3/2}(T) + \mathcal{H}_3^{3/2}(T).$$

Here, we have used the Young inequality and this gives rise to the lemma. □

LEMMA 3.5. *Assume that energies are defined in* (2.1)*, we then have the following inequality*

$$\mathcal{D}_2(T) \lesssim \mathcal{E}_2(T).$$

Proof. The proof of this lemma is similar to Lemma 3.2. From (3.16), we directly know that

$$\begin{aligned}(1+t)\|u_-\|_{\tilde{H}^{s-5}}^2 \lesssim (1+t)\Big\{&\|u_+\|_{\tilde{H}^{s-3}}^2 + \|P_+'(\tau_+)\partial_t\tau_+\|_{\tilde{H}^{s-4}}^2 \\ &+ \|u_+\cdot\nabla u_+\|_{\tilde{H}^{s-4}}^2 + \|\tau_+ u_+\|_{\tilde{H}^{s-5}}^2 + \|\tau_- u_-\|_{\tilde{H}^{s-5}}^2\Big\}.\end{aligned}$$

We shall derive the bounds for all the terms on the right-hand side above. Firstly, by Proposition 3.1, Proposition 3.2 and noticing the first equation of system (1.3), we get

$$\begin{aligned}&\int_0^T (1+t)\|P_+'(\tau_+)\partial_t\tau_+\|_{\tilde{H}^{s-4}}^2 dt \le \int_0^T (1+t)\|\partial_t\tau_+\|_{\tilde{H}^{s-4}}^2 dt \\ &\lesssim \int_0^T (1+t)\|\nabla u_+\|_{\tilde{H}^{s-4}}^2 dt + \int_0^T (1+t)\|\nabla(\tau_+ u_+)\|_{\tilde{H}^{s-4}}^2 dt \\ &\lesssim \int_0^T (1+t)\|u_+\|_{\tilde{H}^{s-3}}^2 dt + \sup_{0\le t\le T}\|\tau_+\|_{\tilde{H}^{s-3}}^2 \int_0^T (1+t)\|u_+\|_{\tilde{H}^{s-3}}^2 dt.\end{aligned} \tag{3.33}$$

Similarly,

$$\int_0^T (1+t)\|u_+\cdot\nabla u_+\|_{\tilde{H}^{s-4}}^2 dt \le \sup_{0\le t\le T}\|u_+\|_{\tilde{H}^{s-4}}^2 \int_0^T (1+t)\|u_+\|_{\tilde{H}^{s-3}}^2 dt. \tag{3.34}$$

And for the last two terms,

$$\begin{aligned}&\int_0^T (1+t)\Big(\|\tau_+ u_+\|_{\tilde{H}^{s-5}}^2 + \|\tau_- u_-\|_{\tilde{H}^{s-5}}^2\Big) dt \\ &\le \sup_{0\le t\le T}\big(\|\tau_+\|_{\tilde{H}^{s-5}}^2 + \|\tau_-\|_{\tilde{H}^{s-5}}^2\big) \int_0^T (1+t)\big(\|u_+\|_{\tilde{H}^{s-5}}^2 + \|u_-\|_{\tilde{H}^{s-5}}^2\big) dt.\end{aligned} \tag{3.35}$$

Finally, we combine all the estimates together, i.e. (3.33), (3.34) and (3.35). Notice Young's inequality and Assumption (3.1), we directly get

$$\begin{aligned}\mathcal{D}_2(T) &\lesssim \mathcal{E}_2(T) + \mathcal{E}_2(T)^2 + \mathcal{E}_1(T)\mathcal{E}_2(T) \\ &\lesssim \mathcal{E}_2(T),\end{aligned}$$

which completes the proof of Lemma 3.5. □

Lemma 3.6. *Assume that energies are defined in (2.1), we then have*

$$\mathcal{H}_2(t) \lesssim \mathcal{E}_2(T) + \mathcal{D}_2(T).$$

Proof. Similar to Lemma 3.3, applying $\tilde{\nabla}^{s-6}$ derivative on the third and fourth equations of system (1.3), taking inner product with $\tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_+$ and $\tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_-$ respectively, adding them up and noticing the definition of Φ, we will get

$$\begin{aligned}
&\int_{\mathbb{T}^3} |\tilde{\nabla}^{s-6}\tau_+|^2 + |\tilde{\nabla}^{s-6}\tau_-|^2 dx + \int_{\mathbb{T}^3} |\tilde{\nabla}^{s-6}\nabla\Phi|^2 dx \\
=& -\int_{\mathbb{T}^3} \tilde{\nabla}^{s-6}\partial_t u_+ \tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_+ \, dx - \int_{\mathbb{T}^3} \tilde{\nabla}^{s-6}\partial_t u_- \tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_- \, dx \\
& -\int_{\mathbb{T}^3} \tilde{\nabla}^{s-6} u_+ \tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_+ \, dx - \int_{\mathbb{T}^3} \tilde{\nabla}^{s-6}(u_+\cdot\nabla u_+)\tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_+ \, dx \\
& -\int_{\mathbb{T}^3} \tilde{\nabla}^{s-6}(u_-\cdot\nabla u_-)\tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_- \, dx \\
& +\int_{\mathbb{T}^3} \tilde{\nabla}^{s-6}\Big[\Big(1 - \frac{p_+'(\tau_+ + 1)}{\tau_+ + 1}\Big)\nabla\tau_+\Big]\tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_+ \, dx \\
& +\int_{\mathbb{T}^3} \tilde{\nabla}^{s-6}\Big[\Big(1 - \frac{p_-'(\tau_- + 1)}{\tau_- + 1}\Big)\nabla\tau_-\Big]\tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_- \, dx.
\end{aligned} \tag{3.36}$$

Adding time weight $1+t$ to (3.36), we shall derive the estimates for all right-hand side terms in the following. Firstly,

$$\begin{aligned}
&(1+t)\int_{\mathbb{T}^3} \tilde{\nabla}^{s-6}\partial_t u_+ \tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_+ dx \\
&\lesssim (1+t)\|\tilde{\nabla}^{s-6}\partial_t u_+\|_{L^2}\|\tilde{\nabla}^{s-6}|\nabla|^{-1}\tau_+\|_{L^2} \lesssim (1+t)\|u_+\|_{\tilde{H}^{s-5}}\|\tau_+\|_{\tilde{H}^{s-6}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&(1+t)\int_{\mathbb{T}^3} \tilde{\nabla}^{s-6}\partial_t u_- \tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_- dx \lesssim (1+t)\|u_-\|_{\tilde{H}^{s-5}}\|\tau_-\|_{\tilde{H}^{s-6}}, \\
&(1+t)\int_{\mathbb{T}^3} \tilde{\nabla}^{s-6} u_+ \tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_+ dx \lesssim (1+t)\|u_+\|_{\tilde{H}^{s-6}}\|\tau_+\|_{\tilde{H}^{s-6}}, \\
&(1+t)\int_{\mathbb{T}^3} \tilde{\nabla}^{s-6}(u_+\cdot\nabla u_+)\tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_+ dx \lesssim (1+t)\|u_+\|^2_{\tilde{H}^{s-5}}\|\tau_+\|_{\tilde{H}^{s-6}}, \\
&(1+t)\int_{\mathbb{T}^3} \tilde{\nabla}^{s-6}(u_-\cdot\nabla u_-)\tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_- dx \lesssim (1+t)\|u_-\|^2_{\tilde{H}^{s-5}}\|\tau_-\|_{\tilde{H}^{s-6}}, \\
&(1+t)\int_{\mathbb{T}^3} \tilde{\nabla}^{s-6}\Big[\Big(1 - \frac{p_+'(\tau_+ + 1)}{\tau_+ + 1}\Big)\nabla\tau_+\Big]\tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_+ dx \lesssim (1+t)\|\tau_+\|^2_{\tilde{H}^{s-6}}\|\tau_+\|_{\tilde{H}^{s-5}}, \\
&(1+t)\int_{\mathbb{T}^3} \tilde{\nabla}^{s-6}\Big[\Big(1 - \frac{p_-'(\tau_- + 1)}{\tau_- + 1}\Big)\nabla\tau_-\Big]\tilde{\nabla}^{s-6}(-\Delta)^{-1}\nabla\tau_- dx \lesssim (1+t)\|\tau_-\|^2_{\tilde{H}^{s-6}}\|\tau_-\|_{\tilde{H}^{s-5}}.
\end{aligned}$$

Integrating (3.36) with time, combing all the estimates above and keeping the Assumption (3.1) in mind, we will get

$$\mathcal{H}_2(t) \lesssim \mathcal{E}_2(T) + \mathcal{D}_2(T),$$

it completes the proof of Lemma 3.6. □

LEMMA 3.7. *Assume that energies are defined in (2.1), we then have the following*

$$\mathcal{E}_3(T)\lesssim\mathcal{E}_2(T)+\mathcal{D}_2(T)+\mathcal{H}_2(T)+\mathcal{E}_3^{3/2}(T)+\mathcal{D}_3^{3/2}(T)+\mathcal{H}_3^{3/2}(T).$$

Proof. Following Lemma 3.4, we apply $\tilde{\nabla}^k$ derivative $(0\le k\le s-6)$ on system (1.3) and take inner product with $\tilde{\nabla}^k\tau_+$ for the first equation, $\tilde{\nabla}^k\tau_-$ for the second equation, $\tilde{\nabla}^k u_+$ for the third equation and $\tilde{\nabla}^k u_-$ for the fourth equation (here $k\le s-6$). Adding the time weight $(1+t)^2$ and summing them up, we then have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\{(1+t)^2\big(\|\tau_+\|^2_{\tilde{H}^{s-6}}+\big\|\tau_-\|^2_{\tilde{H}^{s-6}}+\|u_+\|^2_{\tilde{H}^{s-6}}+\|u_-\|^2_{\tilde{H}^{s-6}}\big)\Big\}\\&-(1+t)\big(\|\tau_+\|^2_{\tilde{H}^{s-6}}+\big\|\tau_-\|^2_{\tilde{H}^{s-6}}+\|u_+\|^2_{\tilde{H}^{s-6}}+\|u_-\|^2_{\tilde{H}^{s-6}}\big)+(1+t)^2\|u_+\|^2_{\tilde{H}^{s-6}}\\&=K_1+K_2+K_3+K_4+K_5+K_6+K_7+K_8,\end{aligned}\tag{3.37}$$

where

$$\begin{aligned}K_1=&-(1+t)^2\int_{\mathbb{T}^3}\sum_{k\le s-6}\tilde{\nabla}^k\nabla\cdot u_+\tilde{\nabla}^k\tau_+-\tilde{\nabla}^k\nabla\tau_+\tilde{\nabla}^k u_+\,dx\\&-(1+t)^2\int_{\mathbb{T}^3}\sum_{k\le s-6}\tilde{\nabla}^k\nabla\cdot u_-\tilde{\nabla}^k\tau_--\tilde{\nabla}^k\nabla\tau_-\tilde{\nabla}^k u_-\,dx,\\K_2=&(1+t)^2\int_{\mathbb{T}^3}\sum_{k\le s-6}\tilde{\nabla}^k\nabla\Phi\tilde{\nabla}^k u_+-\tilde{\nabla}^k\nabla\Phi\tilde{\nabla}^k u_-\,dx,\\K_3=&-(1+t)^2\sum_{k\le s-6}\int_{\mathbb{T}^3}\tilde{\nabla}^k\nabla\cdot(\tau_+u_+)\tilde{\nabla}^k\tau_++\tilde{\nabla}^k(\tau_+\nabla\tau_+)\tilde{\nabla}^k u_+\,dx,\\K_4=&(1+t)^2\sum_{k\le s-6}\int_{\mathbb{T}^3}\tilde{\nabla}^k\Big[\Big(\tau_++1-\frac{p'_+(\tau_++1)}{\tau_++1}\Big)\nabla\tau_+\Big]\tilde{\nabla}^k u_+\,dx,\\K_5=&-(1+t)^2\sum_{k\le s-6}\int_{\mathbb{T}^3}\tilde{\nabla}^k(u_+\cdot\nabla u_+)\tilde{\nabla}^k u_+\,dx,\\K_6=&-(1+t)^2\sum_{k\le s-6}\int_{\mathbb{T}^3}\tilde{\nabla}^k\nabla\cdot(\tau_-u_-)\tilde{\nabla}^k\tau_-+\tilde{\nabla}^k(\tau_-\nabla\tau_-)\tilde{\nabla}^k u_-\,dx,\\K_7=&(1+t)^2\sum_{k\le s-6}\int_{\mathbb{T}^3}\tilde{\nabla}^k\Big[\Big(\tau_-+1-\frac{p'_-(\tau_-+1)}{\tau_-+1}\Big)\nabla\tau_-\Big]\tilde{\nabla}^k u_-\,dx,\\K_8=&-(1+t)\sum_{k\le s-3}\int_{\mathbb{T}^3}\tilde{\nabla}^k(u_-\cdot\nabla u_-)\tilde{\nabla}^k u_-\,dx.\end{aligned}$$

Like before, we shall estimate each term on the right-hand side of (3.37). Using integration by parts, it is easy to get

$$K_1=0.$$

For the term K_2, we still have the following

$$
\begin{aligned}
K_2 =& -(1+t)^2 \sum_{k\leq s-6} \int_{\mathbb{T}^3} \tilde{\nabla}^k \Phi (\tilde{\nabla}^k \nabla\cdot u_+ - \tilde{\nabla}^k \nabla \cdot u_-)\, dx \\
=&(1+t)^2 \sum_{k\leq s-6} \int_{\mathbb{T}^3} \tilde{\nabla}^k \Phi \partial_t \tilde{\nabla}^k (\tau_+ - \tau_-) + \tilde{\nabla}^k \Phi \tilde{\nabla}^k \nabla\cdot(\tau_+ u_+) - \tilde{\nabla}^k \Phi \tilde{\nabla}^k \nabla\cdot(\tau_- u_-)\, dx \\
=& -\frac{1}{2}\frac{d}{dt}\Big\{(1+t)^2 \||\nabla|^{-1}(\tau_+ - \tau_-)\|^2_{\tilde{H}^{s-6}}\Big\} + (1+t)\||\nabla|^{-1}(\tau_+ - \tau_-)\|^2_{\tilde{H}^{s-6}} + K_{2,2}
\end{aligned}
$$

Now, we focus on the estimate of $K_{2,2}$ and derive

$$
\begin{aligned}
|K_{2,2}| \lesssim& (1+t)^2 \int_{\mathbb{T}^3} \|\Phi\|_{\tilde{H}^{s-5}} (\|\tau_+ u_+\|_{\tilde{H}^{s-6}} + \|\tau_- u_-\|_{\tilde{H}^{s-6}})\, dx \\
\lesssim& (1+t)^2 \||\nabla|^{-1}(\tau_+ - \tau_-)\|_{\tilde{H}^{s-6}} \big(\|\tau_+\|_{\tilde{H}^{\frac{s-6}{2}+2}} \|u_+\|_{\tilde{H}^{s-6}} + \|\tau_+\|_{\tilde{H}^{s-6}} \|u_+\|_{\tilde{H}^{\frac{s-6}{2}+2}} \\
&+ \|\tau_-\|_{\tilde{H}^{\frac{s-6}{2}+2}} \|u_-\|_{\tilde{H}^{s-6}} + \|\tau_-\|_{\tilde{H}^{s-6}} \|u_-\|_{\tilde{H}^{\frac{s-6}{2}+2}} \big) \\
\lesssim& (1+t)^2 \||\nabla|^{-1}(\tau_+ - \tau_-)\|_{\tilde{H}^{s-6}} \big(\|\tau_+\|_{\tilde{H}^{s-9}} \|u_+\|_{\tilde{H}^{s-6}} + \|\tau_+\|_{\tilde{H}^{s-6}} \|u_+\|_{\tilde{H}^{s-6}} \\
&+ \|\tau_-\|_{\tilde{H}^{s-9}} \|u_-\|_{\tilde{H}^{s-6}} + \|\tau_-\|_{\tilde{H}^{s-6}} \|u_-\|_{\tilde{H}^{s-8}} \big),
\end{aligned}
$$

which yields,

$$
\begin{aligned}
\int_0^T |K_{2,2}(t)|\, dt \lesssim& \sup_{0\leq t\leq T} (1+t)^2 \||\nabla|^{-1}(\tau_+ - \tau_-)\|_{\tilde{H}^{s-6}} \\
&\cdot \big(\|u_+\|_{\tilde{H}^{s-6}} + \|u_-\|_{\tilde{H}^{s-6}} + \|\tau_+\|_{\tilde{H}^{s-6}} + \|\tau_-\|_{\tilde{H}^{s-6}} \big) \\
&\cdot \int_0^T (\|u_+\|_{\tilde{H}^{s-6}} + \|u_-\|_{\tilde{H}^{s-8}} + \|\tau_+\|_{\tilde{H}^{s-9}} + \|\tau_-\|_{\tilde{H}^{s-9}})\, dt \\
\lesssim& \mathcal{E}_3(T) \big(\mathcal{E}_3^{1/2}(T) + \mathcal{D}_3^{1/2}(T) + \mathcal{H}_3^{1/2}(T) \big).
\end{aligned}
$$

Next, we turn to handle K_3, similar to J_3 we have

$$
\begin{aligned}
|K_3| \lesssim& (1+t)^2 \Big| \int_{\mathbb{T}^3} \sum_{k=1}^{s-6} \sum_{k_1=1}^{k} \tilde{\nabla}^{k_1} u_+ \cdot \tilde{\nabla}^{k-k_1} \nabla \tau_+ \tilde{\nabla}^k \tau_+ + \sum_{0\leq k\leq s-6} u_+ \cdot \nabla \tilde{\nabla}^k \tau_+ \tilde{\nabla}^k \tau_+ \, dx \\
&+ \sum_{0\leq k\leq s-6} \int_{\mathbb{T}^3} \tau_+ \tilde{\nabla}^k \nabla\cdot u_+ \tilde{\nabla}^k \tau_+ - \big(\tau_+ \tilde{\nabla}^k \tau_+ \tilde{\nabla}^k \nabla\cdot u_+ + \nabla \tau_+ \tilde{\nabla}^k \tau_+ \tilde{\nabla}^k u_+ \big)\, dx \Big| \\
&+ (1+t)^2 \sum_{k=1}^{s-6} \sum_{k'=1}^{k} \Big| \int_{\mathbb{T}^3} \tilde{\nabla}^{k'} \tau_+ \tilde{\nabla}^{k-k'} \nabla\cdot u_+ \tilde{\nabla}^k \tau_+ + \tilde{\nabla}^{k'} \tau_+ \tilde{\nabla}^{k-k'} \nabla \tau_+ \tilde{\nabla}^k u_+ \, dx \Big|,
\end{aligned}
$$

which yields,

$$
\begin{aligned}
|K_3| \lesssim& (1+t)^2 \big(\|\tilde{\nabla} u_+\|_{\tilde{H}^{\lfloor \frac{s-7}{2} \rfloor + 2}} \|\nabla \tau_+\|_{\tilde{H}^{s-7}} + \|\nabla \tau_+\|_{\tilde{H}^{\lfloor \frac{s-7}{2} \rfloor + 2}} \|\tilde{\nabla} u_+\|_{\tilde{H}^{s-7}} \big) \|\tau_+\|_{\tilde{H}^{s-6}} \\
&+ (1+t)^2 \big(\|\nabla\cdot u_+\|_{L^\infty} \|\tau_+\|^2_{\tilde{H}^{s-6}} + \|\nabla \tau_+\|_{L^\infty} \|\tau_+\|_{\tilde{H}^{s-6}} \|u_+\|_{\tilde{H}^{s-6}} \big) \\
&+ (1+t)^2 \big(\|\tilde{\nabla} \tau_+\|_{\tilde{H}^{\lfloor \frac{s-7}{2} \rfloor + 2}} \|\nabla\cdot u_+\|_{\tilde{H}^{s-7}} + \|\nabla\cdot u_+\|_{\tilde{H}^{\lfloor \frac{s-7}{2} \rfloor + 2}} \|\tilde{\nabla} \tau_+\|_{\tilde{H}^{s-7}} \big) \|\tau_+\|_{\tilde{H}^{s-6}} \\
&+ (1+t)^2 \|\tilde{\nabla} \tau_+\|_{\tilde{H}^{\lfloor \frac{s-7}{2} \rfloor + 2}} \|\tilde{\nabla} \tau_+\|_{\tilde{H}^{s-7}} \|u_+\|_{\tilde{H}^{s-6}} \\
\lesssim& (1+t)^2 \big(\|u_+\|_{\tilde{H}^{s-6}} + \|\tau_+\|_{\tilde{H}^{s-9}} \big) \big(\|u_+\|^2_{\tilde{H}^{s-6}} + \|\tau_+\|^2_{\tilde{H}^{s-6}} \big).
\end{aligned}
$$

We then have

$$\begin{aligned}\int_0^T |K_3(t)|\,dt \lesssim& \sup_{0\le t\le T}(1+t)^2(\|u_+\|^2_{\tilde H^{s-6}}+\|\tau_+\|^2_{\tilde H^{s-6}})\int_0^T(\|u_+\|_{\tilde H^{s-6}}+\|\tau_+\|_{\tilde H^{s-9}})\,dt\\ \lesssim& \mathcal{E}_3(T)\big(\mathcal{E}_3^{1/2}(T)+\mathcal{H}_3^{1/2}(T)\big).\end{aligned}$$

For the estimate of K_4, we still use the function $g_+(\tau_+)=\big(\tau_++1-\frac{p'_+(\tau_++1)}{\tau_++1}\big)$ and have

$$\begin{aligned}K_4=&(1+t)^2\Big\{\sum_{0\le k\le s-7}\int_{\mathbb{T}^3}\tilde\nabla^k(g_+(\tau_+)\nabla\tau_+)\tilde\nabla^k u_+\,dx\\ &+\sum_{1\le k_1\le s-6}\int_{\mathbb{T}^3}\tilde\nabla^{k_1}g_+(\tau_+)\tilde\nabla^{s-6-k_1}\nabla\tau_+\tilde\nabla^{s-6}u_+\,dx\\ &-\int_{\mathbb{T}^3}(\nabla g_+(\tau_+)\tilde\nabla^{s-6}\tau_+)\tilde\nabla^{s-6}u_+\,dx-\int_{\mathbb{T}^3}(g_+(\tau_+)\tilde\nabla^{s-6}\tau_+)\tilde\nabla^{s-6}\nabla\cdot u_+\,dx\Big\}\\ =&K_{4,1}+K_{4,2}.\end{aligned}$$

$K_{4,1}$ stands for the first three terms and we can bound

$$\begin{aligned}&|K_{4,1}|\\ \lesssim&(1+t)^2\Big\{\|g_+(\tau_+)\nabla\tau_+\|_{\tilde H^{s-7}}\|u_+\|_{\tilde H^{s-7}}+\|\nabla g_+(\tau_+)\|_{L^\infty}\|\tau_+\|_{\tilde H^{s-6}}\|u_+\|_{\tilde H^{s-6}}\\ &+\big(\|\tilde\nabla g_+(\tau_+)\|_{\tilde H^{\lfloor\frac{s-7}{2}\rfloor+2}}\|\nabla\tau_+\|_{\tilde H^{s-7}}+\|\nabla\tau_+\|_{\tilde H^{\lfloor\frac{s-7}{2}\rfloor+2}}\|\tilde\nabla g_+(\tau_+)\|_{\tilde H^{s-7}}\big)\|u_+\|_{\tilde H^{s-6}}\Big\}\\ \lesssim&(1+t)^2\|\tau_+\|_{\tilde H^{\lfloor\frac{s-7}{2}\rfloor+2}}\|\tau_+\|_{\tilde H^{s-6}}\|u_+\|_{\tilde H^{s-6}}\\ \lesssim&(1+t)^2\|\tau_+\|_{\tilde H^{s-9}}\|\tau_+\|_{\tilde H^{s-6}}\|u_+\|_{\tilde H^{s-6}}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^T|K_{4,1}(t)|dt\le&\sup_{0\le t\le T}(1+t)^2\|\tau_+\|_{\tilde H^{s-6}}\|u_+\|_{\tilde H^{s-6}}\int_0^T\|\tau_+\|_{\tilde H^{s-9}}dt\\ \le&\mathcal{E}_3(T)\mathcal{H}_3^{1/2}(T).\end{aligned}$$

For the term $K_{4,2}$, according to the equality

$$-\nabla\cdot u_+=\frac{\partial_t\tau_++u_+\cdot\nabla\tau_+}{\tau_++1},$$

it becomes

$$\begin{aligned}K_{4,2}=&\frac12\frac{d}{dt}(1+t)^2\int_{\mathbb{T}^3}h_+(\tau_+)|\tilde\nabla^{s-6}\tau_+|^2\,dx-(1+t)\int_{\mathbb{T}^3}h_+(\tau_+)|\tilde\nabla^{s-6}\tau_+|^2\,dx\\ &-(1+t)^2\frac12\int_{\mathbb{T}^3}h'_+(\tau_+)\partial_t\tau_+|\tilde\nabla^{s-6}\tau_+|^2\,dx\\ &-(1+t)^2\frac12\int_{\mathbb{T}^3}\big(u_+\cdot\nabla h_+(\tau_+)+h_+(\tau_+)\nabla\cdot u_+\big)|\tilde\nabla^{s-6}\tau_+|^2\,dx\\ &+(1+t)^2\sum_{1\le k_1\le s-6}\int_{\mathbb{T}^3}h_+(\tau_+)\tilde\nabla^{s-6}\tau_+\big(\tilde\nabla^{k_1}u_+\cdot\tilde\nabla^{s-6-k_1}\nabla\tau_+\big)\,dx\end{aligned}$$

$$+(1+t)^2\sum_{1\le k_1\le s-6}\int_{\mathbb{T}^3} g_+(\tau_+)\tilde\nabla^{s-6}\tau_+\tilde\nabla^{k_1}\frac{1}{\tau_++1}\tilde\nabla^{s-6-k_1}\big(\partial_t\tau_++u_+\cdot\nabla\tau_+\big)\,dx.$$

Recall that $h_+(\tau_+)=\frac{g_+(\tau_+)}{\tau_++1}$, by Hölder's inequality and the Sobolev imbedding theorem, we have

$$\begin{aligned}
&\Big|\int_0^T K_{4,2}(t)\,dt\Big|\\
\lesssim&\sup_{0\le t\le T}(1+t)^2\Big|\int_{\mathbb{T}^3} h_+(\tau_+)|\tilde\nabla^{s-6}\tau_+|^2\,dx\Big|+\int_0^T(1+t)^2\|h_+(\tau_+)\|_{L^\infty}\|\tau_+\|^2_{\tilde H^{s-6}}\,dt\\
&+\int_0^T(1+t)^2\|h_+'(\tau_+)\|_{L^\infty}\|\partial_t\tau_+\|_{L^\infty}\|\tau_+\|^2_{\tilde H^{s-6}}\,dt\\
&+\int_0^T(1+t)^2\big(\|u_+\|_{W^{1,\infty}}\|h_+(\tau_+)\|_{W^{1,\infty}}\|\tau_+\|^2_{\tilde H^{s-6}}+\|h_+(\tau_+)\|_{L^\infty}\|\tau_+\|^2_{\tilde H^{s-6}}\|u_+\|_{\tilde H^{s-6}}\big)\,dt\\
&+\int_0^T(1+t)^2\|g_+(\tau_+)\|_{L^\infty}\|\tau_+\|_{\tilde H^{s-6}}\Big\|\frac{1}{\tau_++1}\Big\|_{\tilde H^{s-6}}\Big(\|\tau_+\|_{\tilde H^{s-6}}+\|u_+\|_{\tilde H^{s-6}}\|\tau_+\|_{\tilde H^{s-6}}\Big)dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Big|\int_0^T K_{4,2}(t)\,dt\Big|\\
\lesssim&\sup_{0\le t\le T}(1+t)^2\|h(\tau_+)\|_{L^\infty}\|\tau_+\|^2_{\tilde H^{s-6}}+\sup_{0\le t\le T}(1+t)^2\|\tau_+\|^2_{\tilde H^{s-6}}\int_0^T\|\partial_t\tau_+\|_{L^\infty}\,dt\\
&+\sup_{0\le t\le T}(1+t)^2\|\tau_+\|^2_{\tilde H^{s-6}}\int_0^T\|u_+\|_{W^{1,\infty}}\|h(\tau_+)\|_{W^{1,\infty}}dt\\
&+\sup_{0\le t\le T}(1+t)^2\|\tau_+\|_{\tilde H^{s-6}}\big(\|\tau_+\|_{\tilde H^{s-6}}+\|u_+\|_{\tilde H^{s-6}}\big)\int_0^T\|\tau_+\|_{L^\infty}dt\\
&+\sup_{0\le t\le T}(1+t)^2\|g(\tau_+)\|_{L^\infty}\|\tau_+\|^2_{\tilde H^{s-6}}\int_0^T\|\tau_+\|_{\tilde H^{s-9}}dt\\
\lesssim&\,\mathcal{E}_3^{3/2}(T)+\mathcal{E}_3(T)\mathcal{H}_3^{1/2}(T).
\end{aligned}$$

Now, we turn to K_5 and directly give

$$\begin{aligned}
K_5=&-(1+t)^2\sum_{1\le k\le s-6}\sum_{1\le k_1\le k}\tilde\nabla^{k_1}u_+\cdot\tilde\nabla^{k-k_1}\nabla u_+\tilde\nabla^k u_+dx\\
&-(1+t)^2\sum_{0\le k\le s-6}\int_{\mathbb{T}^3}u_+\cdot\nabla\tilde\nabla^k u_+\tilde\nabla^k u_+\,dx\\
\lesssim&(1+t)^2\|u_+\|^3_{\tilde H^{s-6}},
\end{aligned}$$

which means

$$\begin{aligned}
\int_0^T|K_5(t)|\,dt\lesssim&\sup_{0\le t\le T}(\|u_+\|_{\tilde H^{s-6}}+\|\tau_+\|_{\tilde H^{s-6}})\int_0^T(1+t)^2\|u_+\|^2_{\tilde H^{s-6}}\,dt\\
\lesssim&\,\mathcal{E}_3^{3/2}(T).
\end{aligned}$$

Next, we turn to K_6 and similar to K_3, we have

$$
\begin{aligned}
|K_6| \lesssim & (1+t)^2 \Big| \int_{\mathbb{T}^3} \sum_{k=1}^{s-6} \sum_{k_1=1}^{k} \tilde{\nabla}^{k_1} u_- \cdot \tilde{\nabla}^{k-k_1} \nabla \tau_- \tilde{\nabla}^k \tau_- + \sum_{0\le k\le s-6} u_- \cdot \nabla \tilde{\nabla}^k \tau_- \tilde{\nabla}^k \tau_- \, dx \\
& + \sum_{0\le k\le s-6} \int_{\mathbb{T}^3} \tau_- \tilde{\nabla}^k \nabla \cdot u_- \tilde{\nabla}^k \tau_- - \big(\tau_- \tilde{\nabla}^k \tau_- \tilde{\nabla}^k \nabla \cdot u_- + \nabla \tau_- \tilde{\nabla}^k \tau_- \tilde{\nabla}^k u_- \big) \, dx \Big| \\
& + (1+t)^2 \sum_{k=0}^{s-6} \sum_{k'=1}^{k} \Big| \int_{\mathbb{T}^3} \tilde{\nabla}^{k'} \tau_- \tilde{\nabla}^{k-k'} \nabla \cdot u_- \tilde{\nabla}^k \tau_- + \tilde{\nabla}^{k'} \tau_- \tilde{\nabla}^{k-k'} \nabla \tau_- \tilde{\nabla}^k u_- \, dx \Big|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|K_6| \lesssim & (1+t)^2 \big(\|\tilde{\nabla} u_-\|_{\tilde{H}^{\lfloor \frac{s-7}{2} \rfloor +2}} \|\nabla \tau_-\|_{\tilde{H}^{s-7}} + \|\nabla \tau_-\|_{\tilde{H}^{\lfloor \frac{s-7}{2} \rfloor +2}} \|\tilde{\nabla} u_-\|_{\tilde{H}^{s-7}} \big) \|\tau_-\|_{\tilde{H}^{s-6}} \\
& + (1+t)^2 \big(\|\nabla \cdot u_-\|_{L^\infty} \|\tau_-\|^2_{\tilde{H}^{s-6}} + \|\nabla \tau_-\|_{L^\infty} \|\tau_-\|_{\tilde{H}^{s-6}} \|u_-\|_{\tilde{H}^{s-6}} \big) \\
& + (1+t)^2 \big(\|\tilde{\nabla} \tau_-\|_{\tilde{H}^{\lfloor \frac{s-7}{2} \rfloor +2}} \|\nabla \cdot u_-\|_{\tilde{H}^{s-7}} + \|\nabla \cdot u_-\|_{\tilde{H}^{\lfloor \frac{s-7}{2} \rfloor +2}} \|\tilde{\nabla} \tau_-\|_{\tilde{H}^{s-7}} \big) \|\tau_-\|_{\tilde{H}^{s-6}} \\
& + (1+t)^2 \|\tilde{\nabla} \tau_-\|_{\tilde{H}^{\lfloor \frac{s-7}{2} \rfloor +2}} \|\tilde{\nabla} \tau_-\|_{\tilde{H}^{s-7}} \|u_-\|_{\tilde{H}^{s-6}} \\
\lesssim & (1+t)^2 \big(\|u_-\|_{\tilde{H}^{s-9}} + \|\tau_-\|_{\tilde{H}^{s-9}} \big) \big(\|u_-\|^2_{\tilde{H}^{s-6}} + \|\tau_-\|^2_{\tilde{H}^{s-6}} \big),
\end{aligned}
$$

which yields

$$
\begin{aligned}
\int_0^T |K_6(t)| \, dt \lesssim & \sup_{0\le t\le T} (1+t)^2 (\|u_-\|^2_{\tilde{H}^{s-6}} + \|\tau_-\|^2_{\tilde{H}^{s-6}}) \\
& \cdot \int_0^T (\|u_-\|_{\tilde{H}^{s-8}} + \|\tau_-\|_{\tilde{H}^{s-9}}) \, dt \\
\lesssim & \mathcal{E}_3(T) \big(\mathcal{D}_3^{1/2}(T) + \mathcal{H}_3^{1/2}(T) \big).
\end{aligned}
$$

The estimate for K_7 is similar to K_4 and we can write

$$
\begin{aligned}
K_7 = & (1+t)^2 \Big\{ \sum_{0\le k\le s-7} \int_{\mathbb{T}^3} \tilde{\nabla}^k (g_-(\tau_-) \nabla \tau_-) \tilde{\nabla}^k u_- \, dx \\
& + \sum_{1\le k_1\le s-6} \int_{\mathbb{T}^3} \tilde{\nabla}^{k_1} g_-(\tau_-) \tilde{\nabla}^{s-6-k_1} \nabla \tau_- \tilde{\nabla}^{s-6} u_- \, dx \\
& - \int_{\mathbb{T}^3} (\nabla g_-(\tau_-) \tilde{\nabla}^{s-6} \tau_-) \tilde{\nabla}^{s-6} u_- \, dx - \int_{\mathbb{T}^3} (g_-(\tau_-) \tilde{\nabla}^{s-6} \tau_-) \tilde{\nabla}^{s-6} \nabla \cdot u_- \, dx \Big\} \\
= & K_{7,1} + K_{7,2}.
\end{aligned}
$$

Here $K_{7,1}$ is similar to $K_{4,1}$ and yields,

$$
\begin{aligned}
K_{7,1} \lesssim & (1+t)^2 \Big\{ \|g_-(\tau_-) \nabla \tau_-\|_{\tilde{H}^{s-7}} \|u_-\|_{\tilde{H}^{s-7}} + \|\nabla g_-(\tau_-)\|_{L^\infty} \|\tau_-\|_{\tilde{H}^{s-6}} \|u_-\|_{\tilde{H}^{s-6}} \\
& + \big(\|\tilde{\nabla} g_-(\tau_-)\|_{\tilde{H}^{\frac{s-7}{2}+2}} \|\nabla \tau_-\|_{\tilde{H}^{s-7}} + \|\nabla \tau_-\|_{\tilde{H}^{\frac{s-7}{2}+2}} \|\tilde{\nabla} g_-(\tau_-)\|_{\tilde{H}^{s-7}} \big) \|u_-\|_{\tilde{H}^{s-6}} \Big\} \\
\lesssim & (1+t)^2 \|\tau_-\|_{\tilde{H}^{s-9}} \|\tau_-\|_{\tilde{H}^{s-6}} \|u_-\|_{\tilde{H}^{s-6}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_0^T |K_{7,1}(t)| dt \le & \sup_{0\le t\le T} (1+t)^2 \|\tau_-\|_{\tilde{H}^{s-6}} \|u_-\|_{\tilde{H}^{s-6}} \int_0^T \|\tau_-\|_{\tilde{H}^{s-9}} dt \\
\le & \mathcal{E}_3(T) \mathcal{H}_3^{1/2}(T).
\end{aligned}
$$

Like before, the estimate for $K_{7,2}$ is based on the following equality

$$
\begin{aligned}
K_{7,2}=&\frac{1}{2}\frac{d}{dt}(1+t)^2\int_{\mathbb{T}^3} h_-(\tau_-)|\tilde{\nabla}^{s-6}\tau_-|^2\,dx-(1+t)\int_{\mathbb{T}^3} h_-(\tau_-)|\tilde{\nabla}^{s-6}\tau_-|^2\,dx\\
&-(1+t)^2\frac{1}{2}\int_{\mathbb{T}^3} h'_-(\tau_-)\partial_t\tau_-|\tilde{\nabla}^{s-6}\tau_-|^2\,dx\\
&-(1+t)^2\frac{1}{2}\int_{\mathbb{T}^3}\big(u_-\cdot\nabla h_-(\tau_-)+h_-(\tau_-)\nabla\cdot u_-\big)|\tilde{\nabla}^{s-6}\tau_-|^2dx\\
&+(1+t)^2\sum_{1\le k_1\le s-6}\int_{\mathbb{T}^3} h_-(\tau_-)\tilde{\nabla}^{s-6}\tau_-\big(\tilde{\nabla}^{k_1}u_-\cdot\tilde{\nabla}^{s-6-k_1}\nabla\tau_-\big)dx\\
&+(1+t)^2\sum_{1\le k_1\le s-6}\int_{\mathbb{T}^3} g_-(\tau_-)\tilde{\nabla}^{s-6}\tau_-\tilde{\nabla}^{k_1}\frac{1}{\tau_-+1}\tilde{\nabla}^{s-6-k_1}\big(\partial_t\tau_-+u_-\cdot\nabla\tau_-\big)dx.
\end{aligned}
$$

And similarly,

$$
\begin{aligned}
&\Big|\int_0^T K_{7,2}(t)\,dt\Big|\\
\lesssim&\sup_{0\le t\le T}(1+t)^2\Big|\int_{\mathbb{T}^3} h_-(\tau_-)|\tilde{\nabla}^{s-6}\tau_-|^2\,dx\Big|+\int_0^T(1+t)^2\|h_-(\tau_-)\|_{L^\infty}\|\tau_-\|^2_{\tilde{H}^{s-6}}\,dt\\
&+\int_0^T(1+t)^2\|h'_-(\tau_-)\|_{L^\infty}\|\partial_t\tau_-\|_{L^\infty}\|\tau_-\|^2_{\tilde{H}^{s-6}}\,dt\\
+&\int_0^T(1+t)^2\big(\|u_-\|_{W^{1,\infty}}\|h_-(\tau_-)\|_{W^{1,\infty}}\|\tau_-\|^2_{\tilde{H}^{s-6}}+\|h_-(\tau_-)\|_{L^\infty}\|\tau_-\|^2_{\tilde{H}^{s-6}}\|u_-\|_{\tilde{H}^{s-6}}\big)\,dt\\
+&\int_0^T(1+t)^2\|g_-(\tau_-)\|_{L^\infty}\|\tau_-\|_{\tilde{H}^{s-6}}\|\frac{1}{\tau_-+1}\|_{\tilde{H}^{s-6}}\Big(\|\tau_-\|_{\tilde{H}^{s-6}}+\|u_-\|_{\tilde{H}^{s-6}}\|\tau_-\|_{\tilde{H}^{s-6}}\Big)\,dt.
\end{aligned}
$$

It yields,

$$
\begin{aligned}
&\Big|\int_0^T K_{7,2}(t)\,dt\Big|\\
\lesssim&\sup_{0\le t\le T}(1+t)^2\|h(\tau_-)\|_{L^\infty}\|\tau_-\|^2_{\tilde{H}^{s-6}}+\sup_{0\le t\le T}(1+t)^2\|\tau_-\|^2_{\tilde{H}^{s-6}}\int_0^T\|\partial_t\tau_-\|_{L^\infty}\,dt\\
&+\sup_{0\le t\le T}(1+t)^2\|\tau_-\|^2_{\tilde{H}^{s-6}}\int_0^T\|u_-\|_{W^{1,\infty}}\|h(\tau_-)\|_{W^{1,\infty}}dt\\
&+\sup_{0\le t\le T}(1+t)^2\|\tau_-\|_{\tilde{H}^{s-6}}\big(\|\tau_-\|_{\tilde{H}^{s-6}}+\|u_-\|_{\tilde{H}^{s-6}}\big)\int_0^T\|\tau_-\|_{L^\infty}dt\\
&+\sup_{0\le t\le T}(1+t)^2\|g(\tau_-)\|_{L^\infty}\|\tau_-\|^2_{\tilde{H}^{s-6}}\int_0^T\|\tau_-\|_{\tilde{H}^{s-9}}dt\\
\lesssim&\,\mathcal{E}_3^{3/2}(T)+\mathcal{E}_3(T)\mathcal{H}_3^{1/2}(T).
\end{aligned}
$$

Finally, we turn to the last term K_8 and give

$$\begin{aligned}
K_8 =& -(1+t)^2 \sum_{1\le k\le s-6}\sum_{1\le k_1\le k} \tilde\nabla^{k_1} u_- \cdot \tilde\nabla^{k-k_1}\nabla u_- \tilde\nabla^k u_- dx\\
&-(1+t)^2 \sum_{0\le k\le s-6}\int_{\mathbb{T}^3} u_-\cdot\nabla\tilde\nabla^k u_- \tilde\nabla^k u_-\, dx\\
&\lesssim (1+t)^2\|u_-\|^2_{\tilde H^{s-6}}\|u_-\|_{\tilde H^{s-8}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^T |K_8(t)|\,dt &\lesssim \sup_{0\le t\le T}(1+t)^2\|u_-\|^2_{\tilde H^{s-6}}\int_0^T \|u_-\|_{\tilde H^{s-8}}dt\\
&\lesssim \mathcal{E}_3(T)\mathcal{D}_3^{1/2}(T).
\end{aligned}$$

Following the same process as in Lemma 3.1 and Lemma 3.4, we then finish the proof of this lemma. ∎

Lemma 3.8. *Assume that energies are defined in (2.1), we then have*

$$\mathcal{D}_3(T)\lesssim \mathcal{E}_3(T).$$

Proof. Like before, from (3.16) we directly have

$$\begin{aligned}
(1+t)^2\|u_-\|^2_{\tilde H^{s-8}} \lesssim& (1+t)^2\big\{\|u_+\|^2_{\tilde H^{s-6}} + \|P'_+(\tau_+)\partial_t\tau_+\|^2_{\tilde H^{s-7}}\\
&+\|u_+\cdot\nabla u_+\|^2_{\tilde H^{s-7}} + \|\tau_+u_+\|^2_{\tilde H^{s-8}} + \|\tau_-u_-\|^2_{\tilde H^{s-8}}\big\}.
\end{aligned}$$

Firstly, using Proposition 3.1 and Proposition 3.2, noticing the first equation of system (1.3), we can get

$$\begin{aligned}
\int_0^T (1+t)^2\|P'_+(\tau_+)\partial_t\tau_+\|^2_{\tilde H^{s-7}}dt &\lesssim \int_0^T (1+t)^2\|\partial_t\tau_+\|^2_{\tilde H^{s-7}}dt\\
\lesssim \int_0^T (1+t)^2\|\nabla u_+\|^2_{\tilde H^{s-7}}dt &+ \int_0^T (1+t)^2\|\nabla(\tau_+u_+)\|^2_{\tilde H^{s-7}}dt\\
\lesssim \int_0^T (1+t)^2\|u_+\|^2_{\tilde H^{s-6}}dt &+ \sup_{0\le t\le T}\|\tau_+\|^2_{\tilde H^{s-6}}\int_0^T (1+t)^2\|u_+\|^2_{\tilde H^{s-6}}dt.
\end{aligned}$$

Similarly,

$$\int_0^T (1+t)^2\|u_+\cdot\nabla u_+\|^2_{\tilde H^{s-7}}dt \le \sup_{0\le t\le T}\|u_+\|^2_{\tilde H^{s-7}}\int_0^T (1+t)^2\|u_+\|^2_{\tilde H^{s-6}}dt.$$

At last,

$$\begin{aligned}
&\int_0^T (1+t)^2\Big(\|\tau_+u_+\|^2_{\tilde H^{s-8}} + \|\tau_-u_-\|^2_{\tilde H^{s-8}}\Big)dt\\
\le& \sup_{0\le t\le T}\big(\|\tau_+\|^2_{\tilde H^{s-8}} + \|\tau_-\|^2_{\tilde H^{s-8}}\big)\int_0^T (1+t)^2\big(\|u_+\|^2_{\tilde H^{s-8}} + \|u_-\|^2_{\tilde H^{s-8}}\big)dt.
\end{aligned}$$

Combining the above estimates together and using Young's inequality, we then get

$$\mathcal{D}_3(T)\lesssim \mathcal{E}_3(T) + \mathcal{E}_3(T)^2 + \mathcal{E}_1(T)\mathcal{E}_3(T) + \mathcal{E}_3(T)^3 \lesssim \mathcal{E}_3(T),$$

which completes the proof of this lemma. ∎

Lemma 3.9. *Assume that energies are defined in (2.1), we then have*

$$\mathcal{H}_3(t)\lesssim \mathcal{E}_3(T)+\mathcal{D}_3(T).$$

Proof. Like before, we directly apply $\tilde\nabla^{s-9}$ derivative on the third and fourth equations of system (1.3), take inner product with $\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_+$ and $\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_-$ respectively. Adding them up,

$$\begin{aligned}
&\int_{\mathbb{T}^3}|\tilde\nabla^{s-9}\tau_+|^2+|\tilde\nabla^{s-9}\tau_-|^2dx+\int_{\mathbb{T}^3}|\tilde\nabla^{s-9}\nabla\Phi|^2dx\\
=&-\int_{\mathbb{T}^3}\tilde\nabla^{s-9}\partial_t u_+\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_+dx-\int_{\mathbb{T}^3}\tilde\nabla^{s-9}\partial_t u_-\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_-dx\\
&-\int_{\mathbb{T}^3}\tilde\nabla^{s-9}u_+\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_+dx-\int_{\mathbb{T}^3}\tilde\nabla^{s-9}\big(u_+\cdot\nabla u_+\big)\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_+\,dx\\
&-\int_{\mathbb{T}^3}\tilde\nabla^{s-9}\big(u_-\cdot\nabla u_-\big)\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_-\,dx\\
&+\int_{\mathbb{T}^3}\tilde\nabla^{s-9}\Big[\Big(1-\frac{p_+'(\tau_++1)}{\tau_++1}\Big)\nabla\tau_+\Big]\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_+\,dx\\
&+\int_{\mathbb{T}^3}\tilde\nabla^{s-9}\Big[\Big(1-\frac{p_-'(\tau_-+1)}{\tau_-+1}\Big)\nabla\tau_-\Big]\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_-\,dx.
\end{aligned}\tag{3.38}$$

Adding the time weight $(1+t)^2$, we then have the following estimates

$$\begin{aligned}
&(1+t)^2\int_{\mathbb{T}^3}\tilde\nabla^{s-9}\partial_t u_+\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_+dx\lesssim(1+t)^2\|u_+\|_{\tilde H^{s-8}}\|\tau_+\|_{\tilde H^{s-9}},\\
&(1+t)^2\int_{\mathbb{T}^3}\tilde\nabla^{s-9}\partial_t u_-\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_-dx\lesssim(1+t)^2\|u_-\|_{\tilde H^{s-8}}\|\tau_-\|_{\tilde H^{s-9}},\\
&(1+t)^2\int_{\mathbb{T}^3}\tilde\nabla^{s-9}u_+\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_+dx\lesssim(1+t)^2\|u_+\|_{\tilde H^{s-9}}\|\tau_+\|_{\tilde H^{s-9}},\\
&(1+t)^2\int_{\mathbb{T}^3}\tilde\nabla^{s-9}\big(u_+\cdot\nabla u_+\big)\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_+dx\lesssim(1+t)^2\|u_+\|^2_{\tilde H^{s-8}}\|\tau_+\|_{\tilde H^{s-9}},\\
&(1+t)^2\int_{\mathbb{T}^3}\tilde\nabla^{s-9}\big(u_-\cdot\nabla u_-\big)\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_-dx\lesssim(1+t)^2\|u_-\|^2_{\tilde H^{s-8}}\|\tau_-\|_{\tilde H^{s-9}},\\
&(1+t)^2\int_{\mathbb{T}^3}\tilde\nabla^{s-9}\Big[(1-\frac{p_+'(\tau_++1)}{\tau_++1})\nabla\tau_+\Big]\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_+dx\\
\lesssim&(1+t)^2\|\tau_+\|^2_{\tilde H^{s-9}}\|\tau_+\|_{\tilde H^{s-8}},\\
&(1+t)^2\int_{\mathbb{T}^3}\tilde\nabla^{s-9}\Big[(1-\frac{p_-'(\tau_-+1)}{\tau_-+1})\nabla\tau_-\Big]\tilde\nabla^{s-9}(-\Delta)^{-1}\nabla\tau_-dx\\
\lesssim&(1+t)^2\|\tau_-\|^2_{\tilde H^{s-9}}\|\tau_-\|_{\tilde H^{s-8}}.
\end{aligned}$$

Integrating (3.38) with time and combing all the estimates above, we finally get

$$\mathcal{H}_3(t)\lesssim\mathcal{E}_3(T)+\mathcal{D}_3(T)+\mathcal{E}_1^{1/2}(T)\big(\mathcal{E}_3(T)+\mathcal{D}_3(T)\big)\lesssim\mathcal{E}_3(T)+\mathcal{D}_3(T).$$

It then completes the proof of Lemma 3.9. ∎

4. Proof of the Theorem 1.1

In this section we will combine the above *a priori* estimates for all energies defined in (2.1) together and give the proof of Theorem 1.1. First, we define the total energy as follows,

$$\mathcal{E}_{\text{total}}(t) = \sum_{\theta=1}^{3} \mathcal{E}_\theta(t) + \mathcal{D}_\theta(t) + \mathcal{H}_\theta(t).$$

Now, multiplying each inequality in Lemma 3.1 to Lemma 3.9 by a suitable number respectively and summing them up, we then obtain the following

$$\mathcal{E}_{\text{total}}(t) \leq C_1 \mathcal{E}_{\text{total}}(0) + C_1 \mathcal{E}_{\text{total}}^{3/2}(t), \tag{4.1}$$

for some positive constant C_1.

Although $E_{\text{total}}(0)$ contains some space-time derivatives at time $t=0$, we can derive $E_{\text{total}}(t)\big|_{t=0} \lesssim \|u_\pm(0,\cdot)\|_{H^s}^2 + \|\tau_\pm(0,\cdot)\|_{H^s}^2$ from system (1.3) easily. Hence, the setting of initial data in Theorem 1.1 yields that $C_1 E_{\text{total}}(0) \leq C_2 \epsilon$ for some positive constant C_2. Since the local well-posedness result can be achieved through basic energy method, there exists a positive time T such that

$$\mathcal{E}_{\text{total}}(t) \leq 2C_2\epsilon, \quad \forall t \in [0,T]. \tag{4.2}$$

Let T^* be the largest possible time T for which (4.2) holds, we only need to show $T^* = \infty$. Notice the estimate (4.1), we can use standard continuation argument to show $T^* = \infty$ provided that ϵ is small enough. We omit the details here and finish the proof of Theorem 1.1.

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2